Linear Algebra

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Contents

***Abstract*—This book provides solved examples on Linear Algebra.**

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1.1. Consider the vector space $\mathbb{P}_n$ of real polynomials in x of degree $\leq n$. Define

$$T : \mathbb{P}_2 \to \mathbb{P}_3 \tag{1.1.1}$$

by

$$(Tf)(x) = \int_0^x f(t)\,dt + f'(x). \tag{1.1.2}$$

Then find the matrix representation of T with respect to the bases

$$\left\{1, x, x^2\right\} \text{ and } \left\{1, x, x^2, x^3\right\} \tag{1.1.3}$$

1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A. Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \tag{1.2.1}$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{1.4.1}$$

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1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0, 1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over $\mathbb{R}$.

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over $\mathbb{R}$. An inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t)\,dt, \quad f, g \in V. \tag{1.6.1}$$

Let

$$W = span\left\{1 - t^2, 1 + t^2\right\} \tag{1.6.2}$$

and $W^\perp$ be the orthogonal complement of W in V. Which of the following conditions is satisfied for all $h \in W^\perp$?

a) h is an even function
b) h is an odd function
c) $h(t) = 0$ has a real solution
d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix}\begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.7.1}$$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n satisfy

a) $m + n = 3$
b) $m > n$
c) $m < n$
d) $m = n$

1.8. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \tag{1.8.1}$$

Then find

$$\lim_{n\to\infty} p_{23}^{(n)} \tag{1.8.2}$$

1.9. Let $L(\mathbb{R})^n$ be the space of $\mathbb{R}$−linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $Ker(T)$ denotes the kernel of T then which of the following are true?

a) There exists $T \in L\left(\mathbb{R}^5\right)\ \{0\}$ such that $Range(T) = Ker(T)$
b) There does not exist $T \in L\left(\mathbb{R}^5\right)\ \{0\}$ such that $Range(T) = Ker(T)$
c) There exists $T \in L\left(\mathbb{R}^6\right)\ \{0\}$ such that $Range(T) = Ker(T)$
d) There does not exist $T \in L\left(\mathbb{R}^6\right)\ \{0\}$ such that $Range(T) = Ker(T)$

1.10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \to W, T : W \to V, \tag{1.10.1}$$

where W is some finite dimensional vector space such that

a) both T_1 and T_2 are onto
b) both T_1 and T_2 are one to one
c) T_1 is onto, T_2 is one to one
d) T_1 is one to one, T_2 is onto

1.11. Let $A = \left[a_{ij}\right]$ be a 3×3 complex matrix. Identify the correct statements

a) $det\left[(-1)^{i+j} a_{ij}\right] = det(A)$
b) $det\left[(-1)^{i+j} a_{ij}\right] = -det(A)$
c) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = det(A)$
d) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = -det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1.12.1}$$

be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t)\,dt, r(x) = \frac{d}{dx}p(x) \tag{1.12.2}$$

Let V denote the real vector space of all polynomials in x. Then which of the following are true?

a) q and r are linearly independent in V
b) q and r are linearly dependent in V
c) x^n belongs to the linear span of q and r
d) x^{n+1} belongs to the linear span of q and r.

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.

b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over $\mathbb{R}$ if and only if B is diagonalisable over $\mathbb{R}$.
c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same minimal polynomial.
d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in $\mathbb{R}$.

1.14. Consider a matrix

$$A = \left[a_{ij}\right], 1 \le i, j \le 5 \tag{1.14.1}$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \tag{1.14.2}$$

Then in which of the following cases A is a positive definite matrix?
a) $n_i = 1 \forall i = 1, 2, 3, 4, 5.$
b) $n_1 < n_2 < \cdots < n_5.$
c) $n_1 = n_2 = \cdots = n_5.$
d) $n_1 > n_2 > \cdots > n_5.$

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \to \mathbb{R}^n \tag{1.15.1}$$

by

$$T_w = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \tag{1.15.2}$$

Which of the following are true?
a) $det(T_w) = 1$
b) $T_w(v_1)_w^T(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
c) $T_w = T_w^{-1}$
d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.16.1}$$

over the field $\mathbb{Q}$ of rationals. Which of the following matrices are of the form $P^T A P$ for suitable 2×2 invertible matrix P over $\mathbb{Q}$?

a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$
b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \tag{1.17.1}$$

Then which of the following are true?
a) $\lim_{n\to\infty} p_{12}^{(n)} = 0$
b) $\lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{21}^{(n)}$
c) $\lim_{n\to\infty} p_{22}^{(n)} = \frac{1}{8}$
d) $\lim_{n\to\infty} p_{21}^{(n)} = \frac{1}{3}$

2 December 2018

2.1. Consider the subspaces W_1 and W_2 of $\mathbb{R}^3$ given by

$$W_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 0\right\} \tag{2.1.1}$$

$$W_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 0\right\}. \tag{2.1.2}$$

If $W \subseteq \mathbb{R}^3$, such that
a) $W \cap W_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$
b) $\{W \cap W_1\} \perp \{W \cap W_2\}$,
then
a) $W = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$
b) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$
c) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$
d) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

2.2. Let

$$C = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\} \tag{2.2.1}$$

be a basis of $\mathbb{R}^2$ and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}. \tag{2.2.2}$$

If $T[C]$ represents the matrix of T with respect to the basis C then which among the following is true?

a) $T[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$
b) $T[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$
c) $T[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$
d) $T[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$

Solution: See Tables 2.2.1 and 2.2.2

Linear Transformation and change of Basis	If matrix $\mathbf{A}$ represents Linear Transformation with respect to standard ordered basis and matrix $\mathbf{B}$ represents same transformation with respect to basis $\mathbf{V}$,Then $$\mathbf{B} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$$

TABLE 2.2.1: Linear Transformation and change of basis

In above question $\mathbf{A} = \mathbf{T}$,$\mathbf{B} = \mathbf{T}[\mathbf{C}]$,$\mathbf{V} = \mathbf{C}$.

2.3. Let $W_1 = \left\{\mathbf{x} \in \mathbb{R}^4 :\right\}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \mathbf{x} = 0 \qquad (2.3.1)$$
$$\begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \qquad (2.3.2)$$
$$\begin{pmatrix} 2 & 0 & 2 & -1 \end{pmatrix} \mathbf{x} = 0 \qquad (2.3.3)$$

and $W_2 = \left\{\mathbf{x} \in \mathbb{R}^4 :\right\}$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \qquad (2.3.4)$$
$$\begin{pmatrix} 1 & 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \qquad (2.3.5)$$
$$\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} \mathbf{x} = 0. \qquad (2.3.6)$$

Then which among the following is true?

a) $\dim(W_1) = 1$
b) $\dim(W_2) = 2$
c) $\dim(W_1 \cap W_2) = 1$
d) $\dim(W_1 + W_2) = 3$

2.4. Let A be an $n \times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + \jmath I$. Then all eigenvalues of $(A - \jmath I)B$ are

a) purely imaginary
b) of modulus one
c) real
d) of modulus less than one

2.5. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ as column vectors.Let

$$\mathbf{M} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k \end{pmatrix}, \qquad (2.5.1)$$
$$\mathbf{N} = \begin{pmatrix} \mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \ldots & \mathbf{u}_n \end{pmatrix} \qquad (2.5.2)$$

and $\mathbf{P}$ be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. Then which of the following is true?

a) rank($\mathbf{MPM}^*$) $= k$ whenever $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$.

Evaluate $\mathbf{T}$	For linear transformation $\mathbf{T}$ we have $$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-2y \end{pmatrix}$$ $$\mathbf{T}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ $$\implies \mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$
Evaluate inverse of basis $\mathbf{C}$	To find inverse of matrix $\mathbf{C}$ we row reduce augmented matrix $\mathbf{CI}$ $$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2=-\frac{1}{3}R_2]{R_2=R_2-2R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$ $$\xleftrightarrow{R_1=R_1-2R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$ $$\therefore \mathbf{C}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$
Evaluate $\mathbf{TC}$	$$\mathbf{TC} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$ $$= \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$$
Evaluate $\mathbf{T}[\mathbf{C}]$=$\mathbf{C}^{-1}\mathbf{TC}$	$$\mathbf{T}[\mathbf{C}] = \mathbf{C}^{-1}\mathbf{TC}$$ $$= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$$ $$\implies \mathbf{T}[\mathbf{C}] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$$
Conclusion	Option 3) is correct.Options 1),2) and 4) are incorrect

TABLE 2.2.2: Calculation of $\mathbf{T}[\mathbf{C}]$

Orthonormal Basis	$B = \{u_1, u_2, ..., u_n\}$ is the Orthonormal basis for C^n if it generates every vector C^n and the inner product $< u_i, u_j > = 0$ if $i \neq j$. That is the vectors are mutually perpendicular and $< u_i, u_j > = 1$ otherwise.
Trace	Trace of a square matrix A, denoted by $\mathbf{tr}(\mathbf{A})$ is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of A Some useful properties of Trace : $\mathbf{tr}(\mathbf{AB}) = \mathbf{tr}(\mathbf{BA})$, where A is the $m \times n$ matrix and B is the $n \times m$ matrix
Basis Theorem	A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if A matrix columns are linearly independent,then it is invertible.

TABLE 2.5.1: Definitions

b) $\text{tr}(\mathbf{MPM}^*) = \sum_{i=1}^{k} \alpha_i$
c) $\text{rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n-k)$
d) $\text{rank}(\mathbf{MM}^* + \mathbf{NN}^*) < n$.

Solution: See Tables 2.5.1 2.5.2 and 2.5.3

Rank($\mathbf{MPM}^*$) = $\mathbf{k}$	Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $\mathbf{u_2} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $\mathbf{u_4} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $\mathbf{k} = 2$, then $\mathbf{M} = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$ $\mathbf{P} = \begin{pmatrix}\alpha_1 & 0\\0 & \alpha_2\end{pmatrix}$ $\mathbf{MPM}^* = \begin{pmatrix}\alpha_1 & 0 & 0 & 0\\0 & \alpha_2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$ $\implies$ Rank($\mathbf{MPM}^*$) ≤ 2 (which is the value of k) (It depends on diagonal values α_1 and α_2) Rank($\mathbf{MPM}^*$) is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, ..., \alpha_k$ is 0. Thus, Rank($\mathbf{MPM}^*$) $\neq \mathbf{k}$ Thus, the given statement is false
Trace($\mathbf{MPM}^*$) = $\sum_{i=1}^{k} \alpha_i$	Consider $\mathbf{MP} = \mathbf{A}$ and $\mathbf{M}^* = \mathbf{B}$ Using Properties, Trace($\mathbf{AB}$) = Trace($\mathbf{BA}$) We can say, Trace($\mathbf{MPM}^*$) = Trace($\mathbf{M}^*\mathbf{MP}$) $\mathbf{M} = \begin{pmatrix}u_1 & u_2 & u_3 & .... & u_k\end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix}\bar{u_1}\\\bar{u_2}\\\bar{u_3}\\.\\.\\.\\\bar{u_k}\end{pmatrix}$ $\mathbf{M}^*\mathbf{M} = \begin{pmatrix}\bar{u_1}u_1 & 0 & 0 & ... & 0\\0 & \bar{u_2}u_2 & 0 & ... & 0\\0 & 0 & \bar{u_3}u_3 & ... & 0\\. & . & . & ... & .\\0 & 0 & 0 & ... & \bar{u_k}u_k\end{pmatrix}$

	(Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 0$ if $i \neq j$) $\mathbf{M}^*\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 1$ if $i = j$) $\mathbf{M}^*\mathbf{M} = \mathbf{I^k}$ $\mathbf{M}^*\mathbf{M}\mathbf{P} = \mathbf{I^k}\mathbf{P} = \mathbf{P}$ $\text{Trace}(\mathbf{M}^*\mathbf{M}\mathbf{P}) = \text{Trace}(\mathbf{I^k}\mathbf{P}) = \text{Trace}(\mathbf{P}) = \sum_{i=1}^{k} \alpha_i$ (Refer Definition section of Trace, it is sum of elements on the main diagonal) So, the given statement is true
$\text{Rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n-k)$	$\mathbf{M} = \{u_1, u_2, ..., u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $\mathbf{u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Consider $k = 2$, then $\mathbf{M} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{N} = \begin{pmatrix} u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{M}^*\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{Rank}(\mathbf{M}^*\mathbf{N}) = 0$ But, $\min(k, n-k) = (2, 2) = 2$ And, this is clear from above that $\text{Rank}(\mathbf{M}^*\mathbf{N}) \neq \min(k, n-k)$ Thus, above statement is false
$\text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$	$\text{Rank}(\mathbf{M}) = \text{Rank}(\mathbf{M}^*)$ $\text{Rank}(\mathbf{N}) = \text{Rank}(\mathbf{N}^*)$ $\text{Rank}(\mathbf{M}+\mathbf{N}) \leq \text{Rank}(\mathbf{M}) + \text{Rank}(\mathbf{N})$

	$\mathbf{M} = \{u_1, u_2, ..., u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \mathbf{u_2} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \mathbf{u_4} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $k = 2$, then $\mathbf{M} = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ $\text{Rank}(\mathbf{M}) = 2 = k$ $\mathbf{N} = \begin{pmatrix}u_3 & u_4\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\\1 & 0\\0 & 1\end{pmatrix}$ $\text{Rank}(\mathbf{N}) = 2 = n - k$ Thus, $\text{Rank}(\mathbf{MM}^* + \mathbf{NN}^*) = \text{Rank}(\mathbf{M} + \mathbf{N}) = 4 = n$ Thus, above statement is false

TABLE 2.5.2: Finding of True and False Statements

$\text{Rank}(\mathbf{MPM}^*) = \mathbf{k}$	False
$\text{Trace}(\mathbf{MPM}^*) = \sum_{i=1}^{k} \alpha_i$	True
$\text{Rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n - k)$	False
$\text{Rank}(\mathbf{MM}^* + \mathbf{NN}^*) < n$	False

TABLE 2.5.3: Conclusion of above Solutions

2.6. Let $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function

$$B(a, b) = ab \tag{2.6.1}$$

Which of the following is true?

a) B is a linear transformation
b) B is a positive definite bilinear form
c) B is symmetric but not positive definite
d) B is neither linear nor bilinear

2.7. Let $\mathbf{A}$ be an invertible real $n \times n$ matrix. Define a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{2.7.1}$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T \mathbf{y} \tag{2.7.2}$$

Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivate of F at $(\mathbf{x}, \mathbf{y})$ which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{2.7.3}$$

Then, if

a) $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
b) $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
c) $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
d) $\mathbf{x} = 0$ or $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

2.8. Let

$$T : \mathbb{R}^n \to \mathbb{R}^n \tag{2.8.1}$$

be a linear map that satisfies

$$T^2 = T - I. \tag{2.8.2}$$

Then which of the following is true?

a) T is invertible.
b) $T - I$ is not invertible.
c) T has a real eigenvalue.
d) $T^3 = -I$.

2.9. Let

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{2.9.1}$$

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix}. \tag{2.9.2}$$

Then which of the following are true?

a) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are inconsistent.
b) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are consistent.
c) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is consistent.
d) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is inconsistent.

2.10. Let

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix}. \tag{2.10.1}$$

Given that 1 is an eigenvalue of $\mathbf{M}$, then which among the following are correct?

a) The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
b) The minimal polynomial of $\mathbf{M}$ is $(x-1)^2(x+4)$
c) $\mathbf{M}$ is not diagonalizable.
d) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I})$.

Solution:

a) Given,

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix} \tag{2.10.2}$$

$$\lambda = 1 \tag{2.10.3}$$

Characteristic equation is given as:

$$|M - \lambda I| = 0 \tag{2.10.4}$$

$$\begin{pmatrix} 1-\lambda & -1 & 1 \\ 2 & 1-\lambda & 4 \\ -2 & 1 & -4-\lambda \end{pmatrix} = 0 \tag{2.10.5}$$

$$\lambda^3 + 2\lambda^2 - 7\lambda + 4 = 0 \tag{2.10.6}$$

$$\implies (\lambda - 1)^2(\lambda + 4) = 0 \tag{2.10.7}$$

Characteristic equation is given as,

$$p(x) = (X-1)^2(X+4) \tag{2.10.8}$$

Minimal polynomial will be,

$$p(x) = (X-1)^a(X+4)^b : a \leq 2, b \leq 1 \tag{2.10.9}$$

For $a = 1,\ b = 1$,

$$p(\mathbf{M}) = (\mathbf{M} - 1)(\mathbf{M} + 4) \neq 0 \tag{2.10.10}$$

$\therefore (X-1)(X+4)$ is not a minimal polynomial. Thus option a is not correct.

b) For $a = 2,\ b = 1$,

$$p(\mathbf{M}) = (\mathbf{M} - 1)^2(\mathbf{M} + 4) = 0 \tag{2.10.11}$$

$\therefore (X-1)^2(X+4)$ is minimal polynomial.

Thus option b is correct.

c) We need to find all the eigenvalues and corresponding eigenvectors to it.

$$\implies (\lambda - 1)^2(\lambda + 4) = 0 \quad (2.10.12)$$
$$\therefore \lambda = 1, -4 \quad (2.10.13)$$

For $\lambda = 1$, corresponding eigenvector is $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$.

For $\lambda = -4$, corresponding eigenvector is $\begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}$.

For 3×3 matrix only 2 independent eigenvectors are there. Hence, P^{-1} does not exist and M can't be diagonalizable.
Thus, option c is correct.

d) Using Cayley-Hamilton

$$p(\mathbf{M}) = \mathbf{M}^3 + 2\mathbf{M}^2 - 7\mathbf{M} + 4\mathbf{I} = 0 \quad (2.10.14)$$
$$\mathbf{I} = -\frac{1}{4}(\mathbf{M}^3 + 2\mathbf{M}^2 - 7\mathbf{M}) \quad (2.10.15)$$
$$\mathbf{M}^{-1} = -\frac{1}{4}(\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I}) \quad (2.10.16)$$

Thus option d is not correct.

Hence, the correct options are b and c.

2.11. Let $\mathbf{A}$ be a real matrix with characteristic polynomial $(x-1)^3$. Pick the correct statements from below:

a) $\mathbf{A}$ is necessarily diagonalizable.
b) If the minimal polynomial of $\mathbf{A}$ is $(x-1)^3$, then $\mathbf{A}$ is diagonalizable.
c) The characteristic polynomial of $\mathbf{A}^2$ is $(x-1)^3$
d) If $\mathbf{A}$ has exactly two Jordan blocks, then $(\mathbf{A} - \mathbf{I})^2$ is diagonalizable.

Solution: See Tables 2.11.1 and 2.11.2

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 2.11.1: Definitions

Statement	Solution
1.	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ Therefore, $p(x) = (x-1)^3$ Soving $(\mathbf{A}-\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $\mathbf{A}-\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}-\mathbf{I} \neq \mathbf{0}$ Therefore, $m(x) = (x-1)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 1$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 1$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ A matrix is diagonalizable iff its jordan form is a diagonal matrix. Since $\mathbf{J}$ is not diagonizable therefore $\mathbf{A}$ is not diagonizable.
Conclusion	Therefore the statement is false.

2.	Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$
	Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ Therefore, $p(x) = (x-1)^3$ Soving $(\mathbf{A}-\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $(\mathbf{A}-\mathbf{I})^2 \neq \mathbf{0}$ Therefore, $m(x) = (x-1)^3$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of only one block with principal diagonal values as $\lambda_1 = 1$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{J}$ is not diagonizable therefore $\mathbf{A}$ is not diagonizable.
Conclusion	Therefore the statement is false.
3.	Give that, $p(x)$ of $\mathbf{A} = (x-1)^3$ Hence the eigen values of $\mathbf{A} = 1, 1, 1$ Hence the eigen values of $\mathbf{A}^2 = 1^2, 1^2, 1^2$ or $1, 1, 1$ Therefore $p(x)$ of $\mathbf{A}^2 = (x-1)^3$
Conclusion	Therefore the statement is True.

4.	We know that jordan form of a matrix is similar to the original matrix Let **J** be the jordan form of the matrix **A** then, $$\mathbf{A} = \mathbf{PJP}^{-1}$$ $$\mathbf{A} - \mathbf{I} = \mathbf{PJP}^{-1} - \mathbf{I}$$ $$\mathbf{A} - \mathbf{I} = \mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}$$ $$(\mathbf{A} - \mathbf{I})^2 = \mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}\mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}$$ $$(\mathbf{A} - \mathbf{I})^2 = \mathbf{P}(\mathbf{J} - \mathbf{I})^2\mathbf{P}^{-1}$$ Therefore $(\mathbf{A} - \mathbf{I})^2$ is similar to $(\mathbf{J} - \mathbf{I})^2$ Since **A** has exactly two jordan blocks and order of **A** is 3. $$\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $$\mathbf{J} - \mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ $$(\mathbf{J} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Since $(\mathbf{J} - \mathbf{I})^2$ is diagonal matrix. Therefore $(\mathbf{A} - \mathbf{I})^2$ is diagonalizable.
Conclusion	Therefore the statement is True.

TABLE 2.11.2: Solution summary

2.12. Let P_3 be the vector space of polynomails with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \to P_3 \tag{2.12.1}$$

defined by

$$T\left(p(x)\right) = p(x-1) + p(x+1) \tag{2.12.2}$$

Which of the following properties does the matrix of T with respect to the standard basis $B = \left\{1, x, x^2, x^3\right\}$ of P_3 satisfy?

a) $detT = 0$.
b) $(T - 2I)^4 = 0$ but $(T - 2I)^3 \neq 0$.
c) $(T - 2I)^3 = 0$ but $(T - 2I)^2 \neq 0$.
d) 2 is an eigenvalue with multiplicity 4.

2.13. Let $\mathbf{M}$ be an $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq= 0$is an eigenvalue of $\mathbf{M}$ with corresponding unit column vector $\mathbf{u}$, then which of the following are true?

a) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k - 1$.
b) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k$.
c) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k + 1$.
d) $(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$.

2.14. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$B\left(\mathbf{x}, \mathbf{y}\right) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \tag{2.14.1}$$

Let $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$W = \left\{\mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0\right\} \tag{2.14.2}$$

Then W

a) is not a subspace of $\mathbb{R}^2$.
b) equals $\mathbf{0}$.
c) is the y axis
d) is the line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2.15. Consider the Quadratic forms

$$Q_1(x, y) = xy \tag{2.15.1}$$
$$Q_2(x, y) = x^2 + 2xy + y^2 \tag{2.15.2}$$
$$Q_3(x, y) = x^2 + 3xy + 2y^2 \tag{2.15.3}$$

on $\mathbb{R}^2$. Choose the correct statements from below

a) Q_1 and Q_2 are equivalent.
b) Q_1 and Q_3 are equivalent.
c) Q_2 and Q_3 are equivalent.
d) all are equivalent.

Solution: See Tables 2.15.1 2.15.2

Matrix representation	The Matrix representation of quadratic forms $$Q(x,y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{X}^T\mathbf{A}\mathbf{X} \tag{2.15.4}$$ The symmetric matrix of the quadratic form is $$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.15.5}$$
Equivalent condition	Two quadratic forms $\mathbf{X}^T\mathbf{A}\mathbf{X}$ and $\mathbf{Y}^T\mathbf{B}\mathbf{Y}$ are called equivalent if their matrices, A and B are congruent. Two real quadratic forms are equivalent over the real field iff they have the same rank and the same index.
Rank	The rank of a quadratic form is the rank of its associated matrix.
Index	The index of the quadratic form is equal to the number of positive eigen values of the matrix of quadratic form.

TABLE 2.15.1: Definitions and results used

	Matrix	**Rank**	**Eigen Values**	**Index**
$Q_1(x,y)$	$\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_1]{R_1 \leftarrow R_2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\operatorname{rank}(\mathbf{A}_1) = 2$	$\left\lvert \mathbf{A}_1 - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0$ $\implies \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = 0$ $\implies \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$	Index of $\mathbf{A}_1 = 1$
$Q_2(x,y)$	$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\operatorname{rank}(\mathbf{A}_2) = 1$	$\left\lvert \mathbf{A}_2 - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$ $\implies (\lambda)(\lambda - 2) = 0$ $\implies \lambda_1 = 0, \lambda_2 = 2$	Index of $\mathbf{A}_2$=2
$Q_3(x,y)$	$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}$ $\operatorname{rank}(\mathbf{A}_3) = 2$	$\left\lvert \mathbf{A}_3 - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} 1-\lambda & \frac{3}{2} \\ \frac{3}{2} & 2-\lambda \end{vmatrix} = 0$ $\implies \left(\lambda - \frac{\sqrt{10}+3}{2}\right)\left(\lambda + \frac{\sqrt{10}-3}{2}\right) = 0$ $\implies \lambda_1 = \frac{3+\sqrt{10}}{2}, \lambda_2 = \frac{3-\sqrt{10}}{2}$	Index of $\mathbf{A}_3 = 1$
Conclusion	We can say that $Q_1(x,y)$ and $Q_3(x,y)$ are equivalent as the rank and index are same.			

TABLE 2.15.2: Finding which quadratic forms are equivalent

2.16. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array} \tag{2.16.1}$$

For any two states i and j, let $p_{ij}^{(n)}$ denote the n-step transition probability of going from i to j. Identify correct statements.

a) $\lim_{n\to\infty} p_{11}^{(n)} = \frac{2}{9}$
b) $\lim_{n\to\infty} p_{21}^{(n)} = 0$
c) $\lim_{n\to\infty} p_{32}^{(n)} = \frac{1}{3}$
d) $\lim_{n\to\infty} p_{13}^{(n)} = \frac{1}{3}$

3 June 2018

3.1. Let $\mathbf{A}$ be a $(m \times n)$ matrix and $\mathbf{B}$ be a $(n \times m)$ matrix over real numbers with $m < n$. Then

a) $\mathbf{AB}$ is always nonsingular.
b) $\mathbf{AB}$ is always singular.
c) $\mathbf{BA}$ is always nonsingular.
d) $\mathbf{BA}$ is always singular.

Solution: See Table 3.1.1

$$rank(\mathbf{A}) \leq \min(m, n) \tag{3.1.1}$$
$$\implies \leq m, \because m < n \tag{3.1.2}$$
$$rank(\mathbf{B}) \leq \min(n, m) \tag{3.1.3}$$
$$\implies \leq m, \because m < n \tag{3.1.4}$$

We also know that $\mathbf{AB}$ will be a $m \times m$ matrix and $\mathbf{BA}$ will be a $n \times n$ matrix.

$$rank(\mathbf{AB}) \leq \min(rank(\mathbf{A}), rank(\mathbf{B})) \tag{3.1.5}$$
$$\implies \leq m \tag{3.1.6}$$
$$rank(\mathbf{BA}) \leq \min(rank(\mathbf{B}), rank(\mathbf{A})) \tag{3.1.7}$$
$$\implies \leq m \tag{3.1.8}$$

3.2. If $\mathbf{A}$ is a (2×2) matrix over $\mathbb{R}$ with $det(\mathbf{A} + \mathbf{I}) = 1 + det(\mathbf{A})$. Then we can conclude that

a) $det(\mathbf{A}) = 0$.
b) $\mathbf{A} = 0$.
c) $tr(\mathbf{A}) = 0$.
d) $\mathbf{A}$ is nonsingular.

3.3. The system of equations

$$x + 2x^2 + 3xy = 6 \tag{3.3.1}$$
$$x + x^2 + 3xy + y = 5 \tag{3.3.2}$$
$$x - x^2 + y = 7 \tag{3.3.3}$$

a) has solutions in rational numbers.
b) has solutions in real numbers.
c) has solutions in complex numbers.
d) has no solutions.

3.4. The trace of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \tag{3.4.1}$$

is

a) 7^{20}.
b) $2^{20} + 3^{20}$.
c) $2^{21} + 3^{20}$.
d) $2^{20} + 3^{20} + 1$.

Solution: Let,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{3.4.2}$$

To find the eigen values of $\mathbf{A}$:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{3.4.3}$$
$$\implies \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 03-\lambda & \end{vmatrix} = 0 \tag{3.4.4}$$
$$\implies (2-\lambda)(2-\lambda)(3-\lambda) = 0 \tag{3.4.5}$$
$$\implies \lambda = 2, 2, 3 \tag{3.4.6}$$

Eigen values of $\mathbf{A}$ are 2,2,3.
Hence, the eigen values of $\mathbf{A}^{20}$ are: $2^{20}, 2^{20}$ and 3^{20} respectively.
As we know that the sum of eigen values of a matrix equals the trace of the matrix, hence, the trace of $\mathbf{A}^{20}$ is:

$$tr = 2^{20} + 2^{20} + 3^{20} \tag{3.4.7}$$
$$= 2.2^{20} + 3^{20} \tag{3.4.8}$$

Therefore, option 3 is the required answer.

3.5. Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2, \quad \forall x, y \in \mathbb{R} \tag{3.5.1}$$

Options	Explanation
AB is always nonsingular	$rank(\mathbf{AB}) \leq m$ Let,$rank(\mathbf{AB}) = k,k < m$. So,there are $m-k$ linearly dependent columns or rows So,**AB** will be singular Hence,incorrect
Example	$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\end{pmatrix},\mathbf{B}=\begin{pmatrix}1 & 3\\2 & 6\\5 & 6\end{pmatrix}$ $\mathbf{AB}=\begin{pmatrix}20 & 33\\40 & 66\end{pmatrix},rank(\mathbf{AB}) = 1$ 2^{nd} row is linearly dependent on 1^{st} row. **AB** is singular
AB is always singular	$rank(\mathbf{AB}) \leq m$ Let,$rank(\mathbf{AB}) = m$ So,there are 0 linearly dependent columns or rows So,**AB** will be nonsingular Hence,incorrect
Example	$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 5\end{pmatrix},\mathbf{B}=\begin{pmatrix}1 & 3\\2 & 4\\5 & 6\end{pmatrix}$ $\mathbf{AB}=\begin{pmatrix}20 & 29\\35 & 52\end{pmatrix},rank(\mathbf{AB}) = 2$ **AB** is nonsingular
BA is always nonsingular	$rank(\mathbf{BA}) \leq m.rank(\mathbf{BA})$ can be atmost m **BA** is $n \times n$ matrix.$n > m$. So,there are atleast $n-m$ linearly dependent columns or rows. So,**BA** will be singular always. Hence,incorrect
Example	$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 5\end{pmatrix},\mathbf{B}=\begin{pmatrix}1 & 3\\2 & 4\\5 & 6\end{pmatrix}$ $\mathbf{BA}=\begin{pmatrix}7 & 14 & 18\\10 & 20 & 26\\17 & 34 & 45\end{pmatrix},rank(\mathbf{BA}) = 2$ 2^{nd} column is linearly dependent on 1^{st} column **BA** is singular
BA is always singular	$rank(\mathbf{BA}) \leq m.rank(\mathbf{BA})$ can be atmost m **BA** is $n \times n$ matrix.$n > m$. So,there are atleast $n-m$ linearly dependent columns or rows. So,**BA** will be singular always. Hence,correct
Example	Same example as above. **BA** is always singular.

TABLE 3.1.1: Finding Correct Option

This implies that

a) $\lambda = -5$

b) $\lambda \geq 1$

c) $0 < \lambda < 1$

d) There is no such $\lambda \in \mathbb{R}$

3.6. Let $\mathbb{R}, n \geq 2$, be equipped with the standard inner product. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be n column vectors forming an orthonormal basis of $\mathbb{R}^n$. Let A be the $n \times n$ matrix formed by the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then

a) $\mathbf{A} = \mathbf{A}^{-1}$
b) $\mathbf{A} = \mathbf{A}^\top$
c) $\mathbf{A}^{-1} = \mathbf{A}^\top$
d) $det(\mathbf{A}) = 1$

3.7. Consider a Markov Chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.7.1}$$

Then,

a) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
b) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$
c) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
d) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

3.8. Let V denote the vector space of all sequences $\mathbf{a} = (a_1, a_2, \ldots)$ of real numbers such that

$$\sum_n 2^n |a|_n \tag{3.8.1}$$

converges. Define

$$\|\cdot\| : V \to \mathbb{R} \tag{3.8.2}$$

by

$$\|\mathbf{a}\| = \sum_n 2^n |a|_n . \tag{3.8.3}$$

Which of the following are true?

a) V contains onlythe sequence $(0, 0, \ldots)$
b) V is finite dimensional
c) V has a countable linear basis
d) V is a complete normed space

3.9. Let V be a vector space over $\mathbb{C}$ with dimension n. Let $T : V \to V$ be a linear transformation with only1 as eigenvalue. Then which of the following must be true?

a) $T - I = 0$
b) $(T - I)^{n-1} = 0$
c) $(T - I)^n = 0$
d) $(T - I)^{2n} = 0$

3.10. If $\mathbf{A}$ is a 5×5 matrix and the dimension of the solution space of $\mathbf{Ax} = 0$ is at least two, then

a) $\operatorname{rank}\left(\mathbf{A}^2\right) \leq 3$
b) $\operatorname{rank}\left(\mathbf{A}^2\right) \geq 3$
c) $\operatorname{rank}\left(\mathbf{A}^2\right) = 3$
d) $\det\left(\mathbf{A}^2\right) = 0$

3.11. Let $\mathbf{A} \in M_3(\mathbb{R})$ be such that $\mathbf{A}^3 = \mathbf{I}_{3\times3}$. Then

a) minimal polynomial of $\mathbf{A}$ can only be of degree 2
b) minimal polynomial of $\mathbf{A}$ can only be of degree 3
c) either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A} = -\mathbf{I}$
d) there can be uncountably many $\mathbf{A}$ satisfying the above.

3.12. Let $\mathbf{A}$ be an $n \times n, n > 1$ matrix satisfying

$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I} = \mathbf{0} \tag{3.12.1}$$

Then which of the following statements is true?

a) $\mathbf{A}$ is invertible
b) $t^2 - 7t + 12n = 0$ where $t = tr(\mathbf{A})$
c) $d^2 - 7d + 12 = 0$ where $d = det(\mathbf{A})$
d) $\lambda^2 - 7\lambda + 12 = 0$ where λ is an eigenvalue of $\mathbf{A}$

Solution: See Table 3.12.1

<table>
<tr><td>Given</td><td>$\mathbf{A}$ be the $n \times n$ matrix where $n > 1$ satisfying the following equation
$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I}_{n\times n} = \mathbf{0}_{n\times n} \tag{3.12.2}$$</td></tr>
<tr><td>Explanation</td><td>The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation.
Using this theorem the given equation (3.12.2) can be written as ,
$$\lambda^2 - 7\lambda + 12 = 0 \tag{3.12.3}$$
$$(\lambda - 4)(\lambda - 3) = 0 \tag{3.12.4}$$
$$\lambda_1 = 3 \tag{3.12.5}$$
$$\lambda_2 = 4 \tag{3.12.6}$$
Here λ_1 and λ_2 were eigen values of matrix $\mathbf{A}$
We know that determinant is product of eigen values.
$$d = Det(\mathbf{A}) \tag{3.12.7}$$
$$\implies d = \lambda_1\lambda_2 \tag{3.12.8}$$
$$\implies d = 12 \neq 0 \tag{3.12.9}$$</td></tr>
<tr><td>Statement 1</td><td>$\mathbf{A}$ is invertible</td></tr>
<tr><td></td><td>From equation (3.12.9), since $d \neq 0$ the given matrix $\mathbf{A}$ is Invertible.
True Statement</td></tr>
<tr><td>Statement 2</td><td>$$t^2 - 7t + 12n = 0 \tag{3.12.10}$$</td></tr>
<tr><td></td><td>We know that the trace is the sum of the eigen values.
$$t = Tr(\mathbf{A}) \tag{3.12.11}$$
$$\implies t = \lambda_1 + \lambda_2 \tag{3.12.12}$$
$$\implies t = 7 \tag{3.12.13}$$
Substituting the equation (3.12.13) in (3.12.10) we get,
$$7^2 - 7(7) + 12n = 0 \tag{3.12.14}$$
$$12n = 0 \tag{3.12.15}$$
Since given that $n > 1$ the equation (3.12.15) is not possible i.e $12n \neq 0$.
Therefore, $t^2 - 7t + 12n = 0$ is a False Statement</td></tr>
<tr><td>Statement 3</td><td>$$d^2 - 7d + 12 = 0 \tag{3.12.16}$$</td></tr>
<tr><td></td><td>Substituting the equation (3.12.9) in (3.12.16), we get,
$$12^2 - 7(12) + 12 = 0 \tag{3.12.17}$$
$$72 = 0 \tag{3.12.18}$$
From equation (3.12.15) it is clear that the above statement 3 is invalid.
False Statement</td></tr>
</table>

Statement 4	$$\lambda^2 - 7\lambda + 12 = 0 \qquad (3.12.19)$$
	By Cayley Hamilton Theorem, equation (3.12.3) shows that the above statement 4 is valid. **True Statement**

TABLE 3.12.1: Explanation

3.13. Let **A** be a 6×6 matrix over $\mathbb{R}$ with characteristic polynomial

$$(x-3)^2\,(x-2)^4 \tag{3.13.1}$$

and minimal polynomial

$$(x-3)\,(x-2)^2 \tag{3.13.2}$$

Then the Jordan canonical form of **A** can be

a) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

Solution: See Tables 3.13.1 and 3.13.1

Jordan canonical form	If **A** is a matrix of order n×n ,then the Jordan canonical form of **A** is a matrix of order n×n expressed as $$\mathbf{J} = \begin{pmatrix} \mathbf{J_1} & & \\ & \ddots & \\ & & \mathbf{J_k} \end{pmatrix} \quad (3.13.3)$$ where $\mathbf{J_1}, ..., \mathbf{J_k}$ are the Jordan blocks.
Algebraic multiplicity A_M	Algebraic multiplicity of characteristic value λ in the characteristic polynomial determines the size of Jordan block for that eigen value $$A_M = \text{Size of Jordan block for that } \lambda \quad (3.13.4)$$
Geometric multiplicity G_M	Geometric multiplicity determines the number of Jordan sub-blocks in a Jordan block for λ
Minimal Polynomial	The multiplicity of λ in the minimal polynomial determines the size of the largest sub-block.

TABLE 3.13.1: Definition and Properties used

Characteristic polynomial	$$p(x) = (x-3)^2 (x-2)^4 \quad (3.13.5)$$
Algebraic Multiplicity	$$\text{For } \lambda = 3, A_M = 2 \quad (3.13.6)$$ $$\text{For } \lambda = 2, A_M = 4 \quad (3.13.7)$$
Minimal polynomial	$$m(x) = (x-3)(x-2)^2 \quad (3.13.8)$$
Finding Jordan blocks for λ_1=3	For λ_1=3,We can write from table3.13.1 that $$\text{The highest order of Jordan block} = 1$$ $$\text{Size of Jordan block} = A_M = 2$$ The Jordan blocks for λ_1=3

	$$\mathbf{J_1} = \begin{pmatrix}3\end{pmatrix}, \mathbf{J_2} = \begin{pmatrix}3\end{pmatrix} \tag{3.13.9}$$
Finding Jordan blocks for λ_1=2	For λ_1=2,We can write from table3.13.1 that $$\text{The highest order of Jordan block} = 2$$ $$\text{Size of Jordan block} = A_M = 4$$ The Jordan blocks for λ_1=3 $$\mathbf{J_3} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}, \mathbf{J_4} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix} \tag{3.13.10}$$ or $$\mathbf{J_3} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}, \mathbf{J_4} = \begin{pmatrix}2\end{pmatrix}, \mathbf{J_5} = \begin{pmatrix}2\end{pmatrix} \tag{3.13.11}$$
Jordan canonical form	Jordan canonical form of $\mathbf{A}$ is $$\mathbf{J} = \begin{pmatrix}\mathbf{J_1} & & & \\ & \mathbf{J_2} & & \\ & & \mathbf{J_3} & \\ & & & \mathbf{J_4}\end{pmatrix} \text{or} \begin{pmatrix}\mathbf{J_1} & & & & \\ & \mathbf{J_2} & & & \\ & & \mathbf{J_3} & & \\ & & & \mathbf{J_4} & \\ & & & & \mathbf{J_5}\end{pmatrix} \tag{3.13.12}$$ $$\begin{pmatrix}3 & 0 & 0 & 0 & 0 & 0\\0 & 3 & 0 & 0 & 0 & 0\\0 & 0 & 2 & 1 & 0 & 0\\0 & 0 & 0 & 2 & 0 & 0\\0 & 0 & 0 & 0 & 2 & 1\\0 & 0 & 0 & 0 & 0 & 2\end{pmatrix} \text{or} \begin{pmatrix}3 & 0 & 0 & 0 & 0 & 0\\0 & 3 & 0 & 0 & 0 & 0\\0 & 0 & 2 & 1 & 0 & 0\\0 & 0 & 0 & 2 & 0 & 0\\0 & 0 & 0 & 0 & 2 & 0\\0 & 0 & 0 & 0 & 0 & 2\end{pmatrix} \tag{3.13.13}$$
Conclusion	From above,we can say that options 2) and 3) are correct.

TABLE 3.13.2: Finding Jordan canonical form

3.14. Let V be an inner product space and S be a subset of V. Let $\bar{S}$ denote the closure of S in V with respect to the topology induced by the metric given by the inner product. Which of the following statements is true?

a) $S = (S^\perp)^\perp$
b) $\bar{S} = (S^\perp)^\perp$
c) $\overline{span(S)} = (S^\perp)^\perp$
d) $S^\perp = \left((S^\perp)^\perp\right)^\perp$

3.15. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.15.1}$$

and

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A}\mathbf{x} \tag{3.15.2}$$

Which of the following statements is true?

a) The matrix of second order partial derivatives of the quadratic form Q is $2\mathbf{A}$
b) The rank of the quadratic form Q is 2
c) The signature of the quadratic form Q is $++0$
d) The quadratic form Q take the value 0 for some non-zero vector $\mathbf{x}$

3.16. Assume that a non-singular matrix

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \tag{3.16.1}$$

where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices respectively with all diagonal entries are zero, and $\mathbf{D}$ si a diagonal matrix. Let $\mathbf{x}^*$ be the solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Then the Gauss-Seidel iteration method

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k + \mathbf{c}, k = 0, 1, 2, \dots \tag{3.16.2}$$

with $\|\mathbf{H}\| < 1$ converges to $\mathbf{x}^*$ provided $\mathbf{H}$ is equal to

a) $-\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$
b) $-(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$
c) $-\mathbf{D}(\mathbf{L} + \mathbf{U})^{-1}$
d) $-(\mathbf{L} - \mathbf{D})^{-1}\mathbf{U}$

3.17. Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.17.1}$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

a) $d(1) = 1$
b) $d(1) = 2$
c) $\pi_1 = \frac{1}{2}$
d) $\pi_1 = \frac{1}{3}$

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4.1. Let $\mathbf{A}$ be a real symmetric matrix and $\mathbf{B} = \mathbf{I} + i\mathbf{A}$, where $i^2 = -1$. Then choose the correct option.

a) $\mathbf{B}$ is invertible if and only if $\mathbf{A}$ is invertible.
b) All Eigenvalues of $\mathbf{B}$ are necessarily real.
c) $\mathbf{B} - \mathbf{I}$ is necessarily invertible.
d) $\mathbf{B}$ is necessarily invertible.

Solution: See Table 4.1.1.

Statement 1.	**B** is invertible if and only if **A** is invertible.
False statement	Matrix **B** is invertible even if **A** is non invertible.
Example:	Consider a matrix $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.1.1)$$ a real non invertible,symmetric matrix. $$\implies \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix} \quad (4.1.2)$$ is invertible even if **A** is non invertible.
Statement 2.	All Eigenvalues of **B** are necessarily real.
False statement	Matrix **B** can have complex Eigenvalues.
Proof :	Eigen values of **B** = Eigen values of (**I**) + i (Eigen values of **A**). Clearly from (4.1.2) above Eigen values of **B** are 1 and $1 + i$ respectively. Hence **B** can also have complex Eigen value.
Statement 3.	$\mathbf{B} - \mathbf{I}$ is necessarily invertible.
False statement	$\mathbf{B} - \mathbf{I} = i\mathbf{A}$ will be invertible if **A**, is invertible.
Proof:	We have $\mathbf{B} - \mathbf{I} = i\mathbf{A}$ $\implies \mathbf{B} - \mathbf{I} = i\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$,from (4.1.1) Hence $\mathbf{B} - \mathbf{I}$ is not invertible,unless **A** is invertible.
Statement 4.	**B** is necessarily invertible.
Correct Statement:	Matrix **B** has non zero Eigen values corresponding to Eigenvector X .
Proof:	Let X be an Eigen vector of **A** corresponding to Eigen value λ also,$\lambda \epsilon \mathbb{R}$ $\implies \mathbf{A}X = \lambda X$ $\therefore \mathbf{B}X = (\mathbf{I} + i\mathbf{A})X = \mathbf{I}X + i\mathbf{A}X = X + i\lambda X$ $\implies \mathbf{B}X = (1 + i\lambda)X$ Therefore, $1 + i\lambda$ is an Eigen value of **B**, corresponding to Eigen vector X,which are non zero. Hence, **B** is necessarily invertible.

TABLE 4.1.1: Solution summary

4.2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then the smallest positive integer n such that $\mathbf{A}^n = \mathbf{I}$ is

Solution: *Property of eigen values of A:* Let $\mathbf{A}$ be an arbitary $n \times n$ matrix of complex numbers with eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the eigen values of kth power of $\mathbf{A}$, that is the eigen values of $\mathbf{A}^k$, for any positive integer k are $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$. Let us calculate the eigen values of $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.2.1}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{4.2.2}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \tag{4.2.3}$$

$$-\lambda(1-\lambda) + 1 = 0 \tag{4.2.4}$$

$$\lambda^2 - \lambda + 1 = 0 \tag{4.2.5}$$

$$\implies \lambda = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.2.6}$$

From the above property, the eigen values of $\mathbf{A}^n$ are λ^n. Also as it is given that $\mathbf{A}^n = \mathbf{I}$,

$$\implies \lambda^n = 1 \tag{4.2.7}$$

$$\implies \left(\frac{-1 \pm \sqrt{3}i}{2}\right)^n = 1 \tag{4.2.8}$$

Clearly $n \neq 1$. For $n = 2$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^2 = \frac{-1 \mp \sqrt{3}i}{2} \tag{4.2.9}$$

For $n = 4$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^4 = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.2.10}$$

For $n = 6$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^6 = 1 \tag{4.2.11}$$

Hence $n = 6$ is the smallest positive integer.

4.3. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$. Then the system $\mathbf{AX} = \mathbf{b}$ over the real numbers has

a) No solution when $\beta \neq 7$
b) Infinite number of solutions when $\alpha \neq 2$
c) Infinite number of solutions when $\alpha = 2$ and $\beta \neq 7$
d) A unique solution if $\alpha \neq 2$

Solution: First we derive the Row Reduced Echelon Form (RREF) of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ as follows,

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{pmatrix} \xleftrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.3.1}$$

$$\xleftrightarrow{R_2 = \frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.3.2}$$

$$\xleftrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha - 2 & \beta - 2 \end{pmatrix} \tag{4.3.3}$$

$$\xleftrightarrow{R_3 = R_3 - 5R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{pmatrix} \tag{4.3.4}$$

From the RREF of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ in (4.3.4) we make the following observations for different values of α and β in Table 4.3.1. ,

Values	Observations
$\beta \neq 7$	Then the existence of solution and the number of solutions will entirely depend on value of α
$\alpha = 2$ $\beta \neq 7$	Then RREF in (4.3.4) will contain Zero Row in R_3. Moreover solvability condition will not satisfy. $\implies$ system will have Zero solutions
$\alpha \neq 2$	RREF in (4.3.4) will have all pivots $\implies$ RREF in (4.3.4) will be fullrank $\implies$ $\mathbf{AX} = \mathbf{b}$ have unique solution.

TABLE 4.3.1

Hence, if $\alpha \neq 2$ then the system $\mathbf{AX} = \mathbf{b}$ has unique solution.

4.4. Consider a Markov chain $\{X_n | n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then, $P(X_3 = 1 \mid X_0 = 1)$ equals

Solution: The three step transitional probabilities are given as,

$$P(X_3 = j \mid X_0 = i) = P(X_{n+3} = j \mid X_n = i) = (\mathbf{P}^3)_{ij} \; for\; any\; n \tag{4.4.1}$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \tag{4.4.2}$$

From (4.4.2),

$$P(X_3 = 1 \mid X_0 = 1) = (\mathbf{P}^3)_{11} = \frac{1}{4} \tag{4.4.3}$$

4.5. Let $\mathbf{A}$ be an $m \times n$ matrix with rank r. If the linear system $\mathbf{AX} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

a) $m = r$
b) the column space of $\mathbf{A}$ is a proper subspace of $\mathbf{R}^m$
c) the null space of $\mathbf{A}$ is a non-trivial subspace of $\mathbf{R}^n$ whenever $m = n$
d) $m \geq n$ implies $m = n$

Solution: *Theorem*

Theorem 4.1. *Consider the $m \times n$ system $Ax = b$, with either $b \neq 0$ or $b = 0$. We distinguish the following cases:*

a) ***Unique Solution****: If rank[A,b] = rank(A) = $n \leq m$, then and only then the system has a unique solution. In this case, indeed as many as $m - n$ equations are redundant. And the solution* $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$. *This is called as* ***Exactly Determined****.*

b) ***No Solution****: If rank[A,b] > rank(A) which necessarily implies* $\mathbf{b} \neq 0$ *and $m >$ rank(A), then and only then the system has no solution. This is called as* ***Overdetermined****.*

See Table 4.5.1 If the columns of an $m \times n$ matrix $\mathbf{A}$ span $\mathbf{R}^m$ then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of $\mathbf{A}$ is defined to be

$$Null(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} = 0\} \tag{4.5.1}$$

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \tag{4.5.2}$$

Reduced Row Echelon form is

$$RREF(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.5.3}$$

$\therefore$ the only possible nullspace of the matrix $\mathbf{A}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $\mathbf{B}$ be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \tag{4.5.4}$$

Reduced Row Echelon form is

$$RREF(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.5.5}$$

$\therefore$ the rank of matrix $\mathbf{B} = 3$.

4.6. Let $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and eigen values of} \mathbf{A} \in \mathbb{Q} \right.$

a) $\mathbf{M}$ is empty
b) $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$
c) If $\mathbf{A} \in \mathbf{M}$ then the eigen values of $\mathbf{A} \in \mathbb{Z}$
d) If $\mathbf{A},\mathbf{B} \in \mathbf{M}$ such that $\mathbf{AB}=\mathbf{I}$ then $\left|\mathbf{A}\right| \in \{+1,-1\}$

Solution: See Table 4.6.1.

Options	Observations
$m = r$	The rank of any matrix **A** is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of $\mathbf{R}^m$. $\therefore$ This statement is **True**.
the column space of **A** is a proper subspace of $\mathbf{R}^m$	Any subspace of a vector space **V** other than **V** itself is considered a proper subspace of **V**. Which means that linear combination of **A** will span less than m. That will make the resultant **b** span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting. $\therefore$ This statement is **False**.
the null space of **A** is a non-trivial subsapce of $\mathbf{R}^n$ whenever $m = n$	From (4.5.2) we see that even when $m = n$ then also we are getting a trivial nullspace. $\therefore$ This statement is **False**.
$m \geq n$ implies $m = n$	It is given that the number of rows are greater than the column, and it is given that there exists a solution. If we refer to theorem (4.1) we see that the corresponding system will be **Exactly Determined** system. As an example, it will look like (4.5.4). $\therefore$ This statement is **True**.

TABLE 4.5.1: Solution

M is empty	Consider **A**=**I**=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.The elements of **A** $\in \mathbb{Z}$ and it's eigen values $1 \in \mathbb{Q}$. So, **M** is not empty.
M $=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$	Let **A**=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where elements of **A** $\in \mathbb{Z}$.The characteristic equation can be written as : $$\lambda^2 + 1 = 0 \implies \lambda = \pm i$$

	We see that $\lambda \in \mathbb{C}$ which is contradicting the main definition of **M** .So,this is not correct.
Eigen values of $\mathbf{A} \in \mathbb{Z}$	Given ,$\mathbf{A} \in \mathbf{M}$.Let λ_1,λ_2 be the eigen values of **A**.The characteristic polynomial can be written as: $$\lambda^2 - tr(\mathbf{A})\lambda + \det \mathbf{A} = 0 \text{ where } tr(\mathbf{A}) = \lambda_1 + \lambda_2, \det \mathbf{A} = \lambda_1\lambda_2$$ Given the eigen values λ_1,$\lambda\_2 \in \mathbb{Q}$,For this to be possible the discriminant of above equation should $\in \mathbb{Z}$ $$\sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1\lambda_2} \in \mathbb{Z}$$ $$\implies \sqrt{(\lambda_1 - \lambda_2)^2} \in \mathbb{Z}$$ $\implies \lambda_1 - \lambda_2 \in \mathbb{Z}$ This is possible when both λ_1,$\lambda_2 \in \mathbb{Z}$.
If **AB**=**I** then $\lvert\mathbf{A}\rvert \in \{+1,-1\}$	As **A**,**B** $\in$ **M**, $\implies \lvert\mathbf{A}\rvert,\lvert\mathbf{B}\rvert \in \mathbb{Z}$ Given **AB**=**I** $\implies \lvert\mathbf{A}\rvert\lvert\mathbf{B}\rvert=1$ This is possible only when $\lvert\mathbf{A}\rvert=\lvert\mathbf{B}\rvert=\pm 1$
Conclusion	options 3) and 4) are correct.

TABLE 4.6.1: Solution

4.7. Let A be a 3×3 matrix with real entries.Identify the correct statements.

a) A is necessarily diagonalizable over **R**
b) If A has distinct real eigen values than it is diagonalizable over**R**
c) If A has distinct eigen values than it is diagonalizable over **C**
d) If all eigen values are non zero than it is diagonalizable over **C**

Solution: See Table 4.7.1.

Statement 1.	A is necessarily diagonalizable over **R**
False statement Example:	Matrix A is diagonalizable if and only if there is a basis of $\mathbf{R}^3$consisting of eigenvectors of A. Consider a matrix $$\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 4\end{pmatrix} \quad (4.7.1)$$ Eigen values are: $$\begin{pmatrix}1-\lambda & 1 & 0\\0 & 1-\lambda & 1\\0 & 0 & 4-\lambda\end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \quad (4.7.2)$$ $$\lambda_1 = 1 \text{ has eigen vector} \begin{pmatrix}1\\0\\0\end{pmatrix} \text{and} \lambda_2 = 4 \text{ has eigen vector} \begin{pmatrix}1\\3\\9\end{pmatrix} \quad (4.7.3)$$ We have found only two linearly independent eigenvectors for A,not diagonalisable
Statement 2.	If A has distinct real eigen values than it is diagonalizable over**R**
True statement	Distinct real eigenvalues implies linearly independent eigenvectors . and if a matrix has n linearly independent vectors than it is diagonalizable.
Proof 1: Proof 2:	**Distinct eigen values implies linearly independent vectors that spans entire space.** Consider 2 eigen vectors **v**,**w** with eigen values λ,μ respectively. such that $\lambda \neq \mu$ $$\alpha(\mathbf{v}) + \beta(\mathbf{w}) = 0 \quad (4.7.4)$$ $$\alpha A(\mathbf{v}) + \beta A(\mathbf{w}) = 0 \quad (4.7.5)$$ $$\alpha\lambda\mathbf{v} + \beta\mu\mathbf{w} = 0 \quad (4.7.6)$$ Multiplying (4.7.4)with -λ and subtracting from (4.7.6) we have, $$\beta(\mu - \lambda)\mathbf{w} = 0 \quad (4.7.7)$$ eigen values are distinct $(\mu - \lambda) \neq 0$. From equation(4.7.7) we have, $\beta = 0$ substituting $\beta = 0$ in equation (4.7.4)we have,$\alpha = 0$.As, $\mathbf{v} \neq 0$ **which proves that vectors are linearly independent**. **If a matrix has n linearly independent vectors than it is diagonalizable** If$\begin{pmatrix}\mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n}\end{pmatrix}$are n independent eigen vectors then, $A\mathbf{p_1} = \lambda\mathbf{p_1}, \cdots, A\mathbf{p_n} = \lambda\mathbf{p_n}$ $$D = \begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\0 & \lambda_2 & \cdots & 0\\\vdots & \vdots & \vdots & \vdots\\0 & 0 & \cdots & \lambda_n\end{pmatrix} P = \begin{pmatrix}\mathbf{P_1} & \mathbf{P_2} & \cdots & \mathbf{P_n}\end{pmatrix} \quad (4.7.8)$$ Now, $A\mathbf{P_i} = \lambda_i\mathbf{P_i} \implies AP = PD$

	so,$P^{-1}AP = D$ is a diagonal matrix.
Statement 3.	If A has distinct real eigen values than it is diagonalizable over**C**
True statement . Proof: Example:	If A is an $N \times N$ complex matrix with n distinct eigenvalues, then any set of n corresponding eigenvectors form a basis for $\mathbf{C}^n$ It is sufficient to prove that the set of eigenvectors is linearly independent which is proved in statement 2. $$A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad (4.7.9)$$ Eigen values of A are: $$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6 \quad (4.7.10)$$
	Eigen vectors are: $$x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad (4.7.11)$$ Matrix A is diagonalizable because there is a basis of $\mathbf{C}^3$consisting of eigenvectors of A.
Statement 4.	If all eigen values are non zero than it is diagonalizable over **C**
False Statement:	Matrix would be diagonalizable if and only if it has linearly independent eigenvectors .
Example:	Consider a matrix $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.7.12)$$ Eigen values are: $$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \neq 0 \quad (4.7.13)$$ $$\lambda_1 = 1 \text{ has eigen vector} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{and} \lambda_2 = 4 \text{ has eigen vector} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \quad (4.7.14)$$ We have found only two linearly independent eigenvectors for A,not diagonalisable.

TABLE 4.7.1: Solution summary

Given	V be a vector space over C of all the polynomials in a variable X of degree atmost 3 $D : P_3 \to P_3$ $D : V \to V$ be the linear operator given by differentiation wrt X $D(P(x)) \to P'(x)$ A be the matrix of D wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$

TABLE 4.8.1

4.8. Let V be a vector space over C of all the polynomials in a variable X of degree atmost 3. Let $D : V \to V$ be the linear operator given by differentiation with respect to X. Let A be the matrix of D with respect to some basis for V. Which of the following are true?

a) A is nilpotent matrix

b) A is diagonalizable matrix

c) the rank of A is 2

d) the Jordan canonical form of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: See Tables 4.8.1, 4.8.2 and 4.8.3

4.9. For every 4×4 real symmetric non-singular matrix $\mathbf{A}$ there exists a positive integer p such that

a) $p\mathbf{I} + \mathbf{A}$ is positive definite
b) $\mathbf{A}^p$ is positive definite
c) $\mathbf{A}^{-p}$ is positive definite
d) $\exp(p\mathbf{A}) - \mathbf{I}$ is positive definite

Solution: A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal,that is its eigen value decomposition is

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.9.1}$$

$\mathbf{D}$ is the diagonal matrix containing the real eigen values of $\mathbf{A}$

$\mathbf{P}$ has the corresponding eigen vectors

$$\mathbf{P}\mathbf{P}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{4.9.2}$$

A real matrix is positive definite if

$$\mathbf{x}^T\mathbf{A}\mathbf{x} > 0 \tag{4.9.3}$$

$$\implies \mathbf{x}^T\lambda\mathbf{x} > 0 \tag{4.9.4}$$

$$\implies \lambda\mathbf{x}^T\mathbf{x} > 0 \tag{4.9.5}$$

$$\implies \lambda > 0 \tag{4.9.6}$$

In other words, all the eigen values of A are positive See Table 4.9.1
Let $\mathbf{A}$ be

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.9.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{4.9.8}$$

From the table,the choices would be option 1,2,3

4.10. Let $\mathbf{A}$ be an $m \times n$ matrix of rank m with $n > m$. If for some non-zero real number α, we have $\mathbf{x^T A A^T x} = \alpha \mathbf{x^T x}$, for all $x \in \mathbf{R^m}$, then $\mathbf{A^T A}$ has,

a) exactly two distinct eigenvalues.
b) 0 as an eigenvalue with multiplicity $n - m$.
c) α as a non-zero eigenvalue.
d) exactly two non-zero distinct eigenvalues.

Solution: Refer Table 4.10.1.
Refer Table 4.10.2.

4.11. Consider a Markov chain with five states

$\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \tag{4.11.1}$$

Which of the following are true?

a) 3 and 1 are in the same communicating class
b) 1 and 4 are in the same communicating class
c) 4 and 2 are in the same communicating class
d) 2 and 5 are in the same communicating class

Solution: See Tables 4.11.1 and 4.11.2

Matrix	$D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$ $D(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$ $D(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$ $D(x^2) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ $D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$ $D(x^3) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ Matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Inference	An $n \times n$ matrix with λ as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial $(A - \lambda I)^n$
Nilpotent	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ All eigen values of matrix A is 0 Thus, above matrix is nilpotent matrix Thus, above statement is true

TABLE 4.8.2

Diagonalizable	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $Rank(A) + nullity(A) =$ no of column $Rank(A) = 3$, no of column $= 4$ $nullity(A) = 4 - 3 = 1$ means there exists only one linearly independent eigen vector corresponding to 0 eigen values Thus,matrix A is not Diagonalizable. Thus, above statement is false
Rank	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Rank of matrix A is 3 Thus, above statement is false
Jordan CF	Assume characteristic polynomial of matrix A is $c_A(x)$ $c_A(x) = x^4$ Assume minimal polynomial of A is $m_A(x)$ $m_A(x)$ always divide $c_A(x)$ $m_A(x) = \{x, x^2, x^3, x^4\}$ Minimal polynomial always annihilates its matrix.Thus,we see that $m_A(A) = \{A = 0, A^2 = 0, A^3 = 0, A^4 = 0\}$ But we see that neither A is zero matrix nor A^2 and A^3 equal to zero but A^4 is equal to zero.Thus,x^4 is minimal polynomial. Algebraic Multiplicity $= a_M(\lambda = 0) = 4$ Geometric Multiplicity $= g_M(\lambda = 0) =$ $nullity(A - 0I) = nullity(A) = 1$ Hence, Jordan form of block size 4 Using Inference, $\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ $\lambda = 0$ $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is same as given in the question. Thus,statement is true

OPTIONS	DERIVATIONS
Choice 1	$p\mathbf{I} + \mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{PDP}^T \quad (4.9.9)$ $= \mathbf{PD}_1\mathbf{P}^T \quad (4.9.10)$ $\mathbf{D}_1 = \begin{pmatrix} \lambda_1 + p & 0 & 0 & 0 \\ 0 & \lambda_2 + p & 0 & 0 \\ 0 & 0 & \lambda_3 + p & 0 \\ 0 & 0 & 0 & \lambda_4 + p \end{pmatrix} \quad (4.9.11)$ Some of the eigen values of $\mathbf{A}$ may be negative. All the eigen values in $\mathbf{D}_1$ are positive only if $p > \lvert\lambda_i\rvert \; \forall i \in [1, 4] \quad (4.9.12)$
Choice 2	$\mathbf{A}^2 = \mathbf{AA} \quad (4.9.13)$ $= (\mathbf{PDP}^T)(\mathbf{PDP}^T) \quad (4.9.14)$ $= \mathbf{PD}^2\mathbf{P}^T \quad (4.9.15)$ Similarly, $\mathbf{A}^p = \mathbf{PD}^p\mathbf{P}^T \quad (4.9.16)$ $\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \quad (4.9.17)$ $\mathbf{A}^p$ is positive definite only if p is even.
Choice 3	$\mathbf{A}^{-p} = \mathbf{PD}^{-p}\mathbf{P}^T \quad (4.9.18)$ $\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \quad (4.9.19)$ $\mathbf{A}^{-p}$ is positive definite only if p is even.
Choice 4	$\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \quad (4.9.20)$ $\implies \exp(p\mathbf{A}) - \mathbf{I} = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{PIP}^T \quad (4.9.21)$ $= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \quad (4.9.22)$ $\exp(p\mathbf{D}) - \mathbf{I} = \begin{pmatrix} e^{\lambda_1} - 1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} - 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} - 1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4} - 1 \end{pmatrix} \quad (4.9.23)$ $\mathbf{A}$ is non-singular $\implies \forall i \in [1, 4], \lambda_i \neq 0 \quad (4.9.24)$ $e^{\lambda_i} < 1 \quad (4.9.25)$ So, $\exp(p\mathbf{A}) - \mathbf{I}$ is not positive definite.

TABLE 4.9.1: Solution

Given	Derivation
Given	$\mathbf{A}$ is a $m \times n$ matrix of rank m with $n > m$. A non-zero real number α. To find eigenvalues of $\mathbf{A^TA}$.
Eigenvalues of $\mathbf{AA^T}$	$\mathbf{AA^T}$ is a $m \times m$ matrix and $\mathbf{A^TA}$ is a $n \times n$ matrix. Let, λ be a non-zero eigen value of $\mathbf{A^TA}$. $\mathbf{A^TAv} = \lambda\mathbf{v} \quad \mathbf{v} \in \mathbf{R^n} \qquad (4.10.1)$ $\mathbf{AA^TAv} = \lambda\mathbf{Av} \qquad (4.10.2)$ Let, $\mathbf{x} = \mathbf{Av} \quad \mathbf{x} \in \mathbf{R^m} \qquad (4.10.3)$ $\mathbf{AA^Tx} = \lambda\mathbf{x} \qquad (4.10.4)$ $\mathbf{x^TAA^Tx} = \lambda\mathbf{x^Tx} \qquad (4.10.5)$ Given, $\mathbf{x^TAA^Tx} = \alpha\mathbf{x^Tx} \qquad (4.10.6)$ $\implies \alpha\mathbf{x^Tx} = \lambda\mathbf{x^Tx} \qquad (4.10.7)$ From equation (4.10.7), $\lambda = \alpha$ as $\lVert\mathbf{x}\rVert \neq 0$ As rank($\mathbf{A^TA}$) = rank($\mathbf{A}$) = m and equation (4.10.7) satisfies the condition in question. Therefore the only non-zero eigen value is α $\mathbf{A^TA}$ has an eigenvalue α with multiplicity m.
Eigenvalues of $\mathbf{A^TA}$	$\mathbf{A^TA}$ is a $n \times n$ matrix. Given $n > m$, We know that, $\mathbf{A^TA}$ and $\mathbf{AA^T}$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero. 1. From above, as α is non-zero, $\mathbf{A^TA}$ has α as its eigenvalue with multiplicity m 2. $\mathbf{A^TA}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A^TA}$ are α and 0.

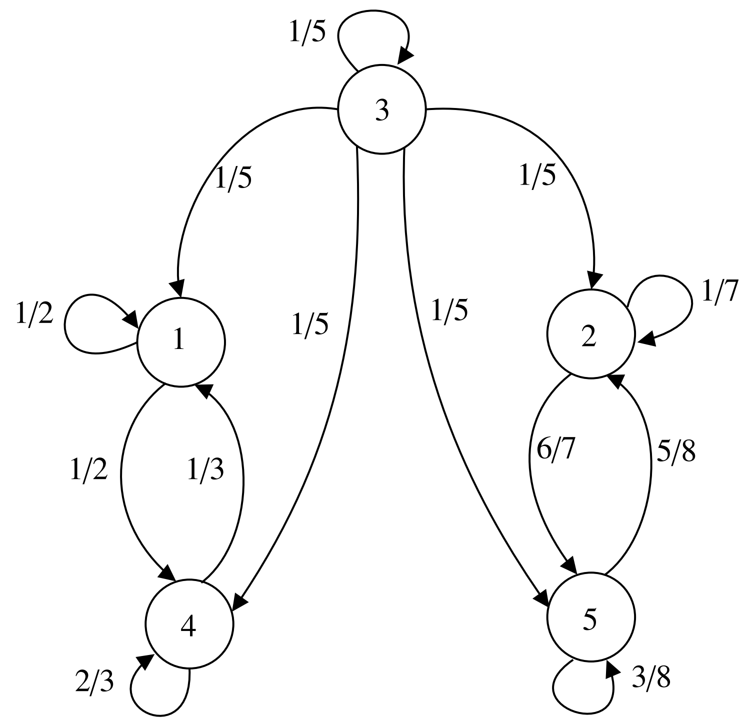
TABLE 4.10.1: Explanation

$\mathbf{A^TA}$ has exactly two distinct eigenvalues.	True statement
$\mathbf{A^TA}$ has 0 as an eigenvalue with multiplicity $n - m$	True statement
$\mathbf{A^TA}$ has α as a non-zero eigenvalue	True statement
$\mathbf{A^TA}$ has exactly two non-zero distinct eigenvalues.	False statement

TABLE 4.10.2: Solution

Accessibility of states in Markov's chain	We say that state j is accessible from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. Every state is accessible from itself since $p_{ii}^{(0)} = 1$
Communication between states	Two states i and j are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words, $i \leftrightarrow j$ means $i \to j$ and $j \to i$.
Communicating class	For each Markov chain, there exists a unique decomposition of the state space S into a sequence of disjoint subsets $C_1, C_2, ...,$ $S = \bigcup_{i=1}^{\infty} C_i$ in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.

TABLE 4.11.1: Definition and Result used

<table>
<tr>
<td>Drawing Transition diagram</td>
<td>


</td>
</tr>
<tr>
<td>Checking whether the states 3 and 1 are in the same communicating class</td>
<td>Here,
State 1 is accessible from the state 3.
But, State 3 is not accessible from the state 1
i.e. $3 \rightarrow 1,\ 1 \nrightarrow 3$
$\implies \boxed{3 \nleftrightarrow 1}$

Therefore, 3 and 1 are not in the same communicating class.</td>
</tr>
<tr>
<td>Checking whether the states 1 and 4 are in the same communicating class</td>
<td>Here,
State 1 is accessible from the state 4.
Also, State 4 is accessible from the state 1
i.e. $3 \rightarrow 1,\ 1 \rightarrow 3$
$\implies \boxed{3 \leftrightarrow 1}$

Therefore, 1 and 4 are in the same communicating class.</td>
</tr>
<tr>
<td>Checking whether the states 4 and 2 are in the same communicating class</td>
<td>Here,
State 2 is not accessible from the state 4.
Also, State 4 is not accessible from the state 2
i.e. $4 \nrightarrow 2,\ 2 \nrightarrow 4$</td>
</tr>
</table>

	$\implies \boxed{4 \nleftrightarrow 2}$ Therefore, 4 and 2 are not in the same communicating class.
Checking whether the states 2 and 5 are in the same communicating class	Here, State 2 is accessible from the state 5. Also, State 5 is accessible from the state 2 i.e. $5 \to 2,\ 2 \to 5$ $\implies \boxed{2 \leftrightarrow 5}$ Therefore, 2 and 5 are in the same communicating class.
Conclusion	Communication classes are: $\boxed{S = \{1, 4\} \cup \{3\} \cup \{2, 5\}}$ Option 2) and 4) are true.

TABLE 4.11.2: Solution

5 June 2017

5.1. Let $\mathbf{A}$ be a 4×4 matrix. Suppose that the null space $N(\mathbf{A})$ of $\mathbf{A}$ is

$$\left\{(x,y,z,w)\in\mathbf{R}^4 : x+y+z=0, x+y+w=0\right\} \tag{5.1.1}$$

Then which one of the following is correct

a) dim(column space($\mathbf{A}$)) = 1
b) dim(column space($\mathbf{A}$)) = 2
c) rank($\mathbf{A}$) = 1
d) $\mathbf{S}=\{(1,1,1,0),(1,1,0,1)\}$ is a basis of $N(\mathbf{A})$

Solution: The nullspace is given by

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{5.1.2}$$

Row reducing the above matrix we get,

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\xleftrightarrow[R_2\leftarrow R_2\times-1]{R_2\leftarrow R_2-R_1}\begin{pmatrix}1&1&1&0\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{5.1.3}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2}\begin{pmatrix}1&1&0&1\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{5.1.4}$$

See Table 5.1.1

5.2. Let $\mathbf{A}$ and $\mathbf{B}$ be real invertible matrices such that

$$\mathbf{AB}=-\mathbf{BA}. \tag{5.2.1}$$

Then

a) trace$\mathbf{A}$ = trace($\mathbf{B}$) = 0
b) trace$\mathbf{A}$ = trace($\mathbf{B}$) = 1
c) trace$\mathbf{A}$ = 0, trace($\mathbf{B}$) = 1
d) trace($\mathbf{A}$) = 1, trace($\mathbf{B}$) = 0

Solution: See Tables 5.2.1 and 5.2.2

5.3. Let $\mathbf{A}$ be an n × n self-adjoint matrix with eigenvalues $\lambda_1,\cdots,\lambda_2$. Let,

$$\|\mathbf{X}\|_2=\sqrt{|\mathbf{X}_1^2|+\cdots+|\mathbf{X}_n^2|} \tag{5.3.1}$$

for $\mathbf{X}$=$(\mathbf{X}_1,\cdots,\mathbf{X}_n)\in\mathbb{C}^n$. If

$$p(\mathbf{A})=a_0\mathbf{I}+a_1\mathbf{A}+\cdots+a_n\mathbf{A}^n \tag{5.3.2}$$

then $sup_{\|\mathbf{X}\|_2=1}\|p(\mathbf{A})\mathbf{X}\|_2$ is equal to

Solution: We know that $\mathbf{A}$ is a self adjoint matrix and hence $\mathbf{A}=\mathbf{A}^*$ with eigen values $\lambda_1,\lambda_2,\cdots,\lambda_n$.Now as we are given,

$$p(\mathbf{A})=a_0\mathbf{I}+a_1\mathbf{A}+\cdots+a_n\mathbf{A}^n \tag{5.3.3}$$

then,

$$(p(\mathbf{A}))^*=a_0\mathbf{I}^*+a_1\mathbf{A}^*+\cdots+a_n(\mathbf{A}^*)^n \tag{5.3.4}$$

Since, $\mathbf{A}=\mathbf{A}^*$ we can state that,

$$p(\mathbf{A})(p(\mathbf{A}))^*=(p(\mathbf{A}))^*p(\mathbf{A}) \tag{5.3.5}$$

Hence p($\mathbf{A}$) is a normal matrix. Now using spectral theorem for a normal matrix,

$$\|p(\mathbf{A})\|_2=\rho(p(\mathbf{A})) \tag{5.3.6}$$

sup refers to the smallest element that is greater than or equal to every number in the set.Hence, sup of $\|p(\mathbf{A})\|_2$ will be,

$$=\max\{|\alpha| : \alpha \text{ is the eigen value of p(A)}\} \tag{5.3.7}$$

$$=\max\{|p(\lambda_j)| : j=1,2,\cdots n\} \tag{5.3.8}$$

$$=\max\{|a_0+a_1\lambda_j+\cdots+a_n{\lambda_j}^n| : j=1,2,\cdots n\} \tag{5.3.9}$$

Now, to find $sup\,\|p(\mathbf{A})\mathbf{X}\|_2$,

$$=max\{|a_0+a_1\lambda_j+\cdots+a_n{\lambda_j}^n| : j=1,2,\cdots n\}\,\|\mathbf{X}\|_2 \tag{5.3.10}$$

Since, we have to find $sup_{\|\mathbf{X}\|_2=1}$ i.e,

$$\|\mathbf{X}\|_2=\sqrt{|\mathbf{X}_1^2|+\cdots+|\mathbf{X}_n^2|}=1 \tag{5.3.11}$$

Hence the final answer will be,

$$=max\{|a_0+a_1\lambda_j+\cdots+a_n{\lambda_j}^n| : j=1,2,\cdots n\} \tag{5.3.12}$$

5.4. Let $p(x)=\alpha x^2+\beta x+\gamma$ be a polynomial, where $\alpha,\beta,\gamma\epsilon R$. Fix $X_0\epsilon R$. Let $S=\{(a,b,c)\,\epsilon R^3 : p(x)=a(x-x_0)^2+b(x-x_0)+c\}$ for all $x\epsilon R$. Find the number of elements in S is

a) 0
b) 1
c) Strictly greater than 1 but finite
d) Infinite

dim(C(**A**)) = 1	**False**. Because the number of pivot variables are 2 as obtained in (5.1.4)
dim(C(**A**)) = 2	**True**. Since the number of pivot variables are 2, the rank of **A** is 2. $\therefore dim(C(\mathbf{A})) = 2 \quad [\because dim(C(\mathbf{A})) = rank(\mathbf{A})]$
rank(**A**) = 1	**False**. Because the rank(**A**) = 2, as the number of pivot variables are 2
$\mathbf{S} = \{(1,1,1,0),(1,1,0,1)\}$ is a basis of $N(\mathbf{A})$	**False**. Let, $\mathbf{u} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \mathbf{v} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ Consider, $\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\2\\0\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ Similarly, $\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\3\\0\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ Hence, the given vectors do not form the basis.

TABLE 5.1.1

Definition	Matrix **A** is said to be similar to matrix **B** if there exists matrix **P** such that $\mathbf{A} = \mathbf{PBP}^{-1}$
Properties	Similar matrices have same eigenvalues Sum of eigenvalue of a matrix equals its trace From above two properties we can conclude that similar matrices have same trace

TABLE 5.2.1: Similar matrices and Properties

Solution:

$$S = \{(a,b,c)\,\epsilon R^3 : p(x) = a(x-x_0)^2 + b(x-x_0) + c\},$$

$$\forall \mathbf{x} \epsilon R\,(FixX_0) \tag{5.4.3}$$

$$p(x) = \alpha x^2 + \beta x + \gamma \tag{5.4.1}$$

$$\implies p(x) = (\alpha\beta\gamma)\left(x^2 x 1\right)^T \tag{5.4.2}$$

$$p(x) = (abc)\left((x - x_0)^2 (x - x_0) 1\right)^T \quad (5.4.4)$$
$$= a\left(x^2 - 2x_0 x + x_0^2\right) + b(x - x_0) + c \quad (5.4.5)$$
$$= ax^2 + (b - 2ax_0)x + \left(ax_0{}^2 - bx_0 + c\right) \quad (5.4.6)$$

Refer (5.4.2) and (5.4.6) and comparing the co-coefficients of powers of x,

$$\alpha = a, \beta = b - 2ax_0, \gamma = ax_0{}^2 - bx_0 + c \quad (5.4.7)$$

$$a = \alpha, b = \beta + 2\alpha x_0, c = \gamma - \alpha x_0{}^2 + (\beta + 2\alpha x_0)x_0 \quad (5.4.8)$$

Here α, β, γ and x_0 are the real fixed numbers. So a, b, c have unique values.
Hence S contain only 1 element.So option 2 is correct

5.5. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (5.5.1)$$

and $\mathbf{I}$ be the 3×3 identity matrix. If

$$6\mathbf{A}^{-1} = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} \quad (5.5.2)$$

for $a, b, c \in \mathbb{R}$ then (a,b,c) equals

a) (1,2,1)

b) (1,-1,2)

c) (4,1,1)

d) (1,4,1)

Solution: Finding the characteristic equation,

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} \quad (5.5.3)$$

$$\implies (1-\lambda)(-2-\lambda)(-3-\lambda) = 0 \quad (5.5.4)$$
$$\implies (\lambda^2 + \lambda - 2)(-3-\lambda) = 0 \quad (5.5.5)$$
$$\implies \lambda^3 + 4\lambda^2 + \lambda - 6 = 0 \quad (5.5.6)$$

Using Cayley-Hamilton Theorem we get,

$$\mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} - 6\mathbf{I} = 0 \quad (5.5.7)$$
$$\implies \mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} = 6\mathbf{I} \quad (5.5.8)$$
$$\implies \mathbf{A}(\mathbf{A}^2 + 4\mathbf{A} + \mathbf{I}) = 6\mathbf{I} \quad (5.5.9)$$

$\left|\mathbf{A}\right| = 6 \neq 0$ hence inverse exists. Hence (5.5.9) we get,

$$6\mathbf{A}^{-1} = \mathbf{A}^2 + 4\mathbf{A} + \mathbf{I} \quad (5.5.10)$$

Comparing (5.5.2) and (5.5.10) we get,

$$a = 1 \quad b = 4 \quad c = 1 \quad (5.5.11)$$

Hence $(a, b, c) = (1, 4, 1)$

5.6. Find the Eigenvalues of the matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 5 \\ 2 & 5 & -3 \end{pmatrix} \quad (5.6.1)$$

a) -4, 3, -3
b) 4, 3, 1
c) 4, -4$\pm\sqrt{13}$
d) 4, -2$\pm\sqrt{7}$

Solution: Using the characteristic equation of the matrix can find the Eigenvalues,

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \quad (5.6.2)$$

$$\implies \begin{vmatrix} \lambda-1 & -1 & -2 \\ -1 & \lambda+2 & -5 \\ -2 & -5 & \lambda+3 \end{vmatrix} = 0 \quad (5.6.3)$$

The expression that is obtained after expanding the determinant and simplifying it is,

$$(\lambda - 1)(\lambda^2 + 5\lambda - 19) - (5\lambda + 31) = 0 \quad (5.6.4)$$

Further simplifying this we obtain the cubic equation,

$$\lambda^3 + 4\lambda^2 - 29\lambda - 12 = 0 \quad (5.6.5)$$

Solving this equation, the Eigenvalues obtained are,

$$\lambda_1 = -7.605, \ \lambda_2 = -0.394 \ and \ \lambda_3 = 4 \quad (5.6.6)$$

Therefore, the Eigenvalues of the given matrix are 4, -4 $\pm\sqrt{13}$ (Option 3)

5.7. Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers $a_0, a_1, \cdots, a_k$. For $p \in V$

$$max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} \quad (5.7.1)$$

defines a norm on V

a) only if $k < n$
b) only if $k \geq n$
c) if $k + 1 \leq n$

d) if $k \geq n+1$

Solution: Options 2 and 4 are correct as verified in the table 5.7.2

The scalar multiplication and triangle inequality properties holds true for all k.

$$max\left\{\left|\alpha p(a_j)\right|\right\} = |\alpha|\, max\left\{\left|p(a_j)\right|\right\} \tag{5.7.4}$$

$$max\left\{\left|p(a_i) + p(a_j)\right|\right\} \leq max\left\{|p(a_i)|\right\} + max\left\{\left|p(a_j)\right|\right\} \tag{5.7.5}$$

The positivity property holds true only if $k \geq n$ as more than n roots are possible when,

$$p(x) = 0 \implies \left|p(a_j)\right|_{0 \leq j \leq k} = 0 \tag{5.7.6}$$

$$\implies max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} = 0 \tag{5.7.7}$$

5.8. Let **V** be the vector space of polynomials of degree at most 3 in a variable x with coefficients in $\mathbb{R}$. Let **T**=d/dx be the linear transformation of **V** to itself given by differentiation.

Which of the following are correct?

a) **T** is invertible
b) 0 is an eigenvalue of **T**
c) There is a basis with respect to which the matrix of **T** is nilpotent.
d) The matrix of **T** with respect to the basis $\left(1, 1+x, 1+x+x^2, 1+x+x^2+x^3\right)$ is diagonal.

Solution: See Tables 5.8.1 , 5.8.2 and 5.8.3.

trace(**A**) = 0 trace(**B**) = 0	From (5.2.1) we have $$\mathbf{AB} = -\mathbf{BA}$$ $$\implies \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$$ So, matrix **A** and (-**A**) are similar..∴ $$trace(\mathbf{A}) = trace(-\mathbf{A})$$ $$\implies trace(\mathbf{A}) = 0$$ Similarly From (5.2.1) we have $$\mathbf{AB} = -\mathbf{BA}$$ $$\implies \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$$ So, matrix **B** and (-**B**) are similar..∴ $$trace(\mathbf{B}) = trace(-\mathbf{B})$$ $$\implies trace(\mathbf{B}) = 0$$ So this statement is true
trace(**A**) = 1 trace(**B**) = 1	From (5.2.1) we have $$\mathbf{AB} = -\mathbf{BA}$$ $$\implies \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$$ So, matrix **A** and (-**A**) are similar..∴ $$trace(\mathbf{A}) = trace(-\mathbf{A})$$ $$\implies trace(\mathbf{A}) = 0.$$ As trace(**A**) = 0 this statement is false
trace(**A**) = 0 trace(**B**) = 1	From (5.2.1) we have $$\mathbf{AB} = -\mathbf{BA}$$ $$\implies \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$$ So, matrix **B** and (-**B**) are similar..∴ $$trace(\mathbf{B}) = trace(-\mathbf{B})$$ $$\implies trace(\mathbf{B}) = 0.$$ As trace(**B**) = 0 this statement is false
trace(**A**) = 1 trace(**B**) = 0	From (5.2.1) we have $$\mathbf{AB} = -\mathbf{BA}$$ $$\implies \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$$ So, matrix **A** and (-**A**) are similar..∴ $$trace(\mathbf{A}) = trace(-\mathbf{A})$$ $$\implies trace(\mathbf{A}) = 0.$$ As trace(**A**) = 0 this statement is false

TABLE 5.2.2: Calculation of trace

Properties	**Norm** $\forall x \in V$
Positivity	$\|\|x\|\| \geq 0, \|\|x\|\| = 0 \iff x = 0$
Scalar Multiplication	$\|\|\alpha x\|\| = \|\alpha\| \, \|\|x\|\|, \alpha \in F$
Triangle Inequality	$\|\|x + y\|\| \leq \|\|x\|\| + \|\|y\|\|$

TABLE 5.7.1: Properties of Norm

For $p \in V$ then the norm, $max\left\{\left\|p(a_j)\right\| : 0 \leq j \leq k\right\} = 0 \iff \left\|p(a_j)\right\|_{0 \leq j \leq k} = 0$	
Conditions	**Explanation**
only if $k < n$ Example:	A polynomial doesn't necessarily have k distinct real roots, i.e., it may have repeated, complex roots. let p be polynomial of degree $n = 2$ and $k = 1$ given by:- $$p(x) = x^2 + 4x + 4 \qquad (5.7.2)$$ $$\left\|p(a_j)\right\|_{0 \leq j \leq 1} = 0 \implies a_0 = -2, a_1 = -2 \qquad (5.7.3)$$ but $a_0, a_1, \cdots, a_k$ should be distinct real numbers. This contradicts the property of Norm. Thus condition fails.
only if $k \geq n$	p is a polynomial of degree ≤n, it can't have more than n roots and is only possible when, $p(x) = 0 \implies \left\|p(a_j)\right\|_{0 \leq j \leq k} = 0$ hence p is identically zero. Thus condition satisfies.
if $k + 1 \leq n$	Not a norm for $k < n$. Hence incorrect.
if $k \geq n + 1$	Norm for $k \geq n$. Hence correct.

TABLE 5.7.2: Verifying Positivity Property of Norm

Nilpotent Matrix	1. If all the eigen values of matrix is zero then it is said to nilpotent matrix 2. Determinant and trace of nilpotent matrix are always zero.
Invertible Matrix	A matrix is said to be invertible matrix if its determinant is non zero.
Diagonal matrix	diagonal matrix is a matrix in which the entries outside the main diagonal are all zero.

TABLE 5.8.1: Definition

Given	$T\,:\,P_3 \rightarrow P_3$ $T : V \rightarrow V$ be the linear operator given by differentiation wrt x $T(P(x)) \rightarrow P'(x)$ A be the matrix of T wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$

TABLE 5.8.2: Result used

Checking whether matrix of T is nilpotent	$T : V \rightarrow V$ $TP(x) = P'(x)$ Differentiating wrt x to find matrix A; $T(1) = 0 = a_1 x + b_1 x + c_1 x^2 + d_1 x^3$ $T(x)= 1 = a_2 + b_2 x + c_2 x^2 + d_2 x^3$ $T(x^2) = 2x = a_3 + b_3 x + c_3 x^2 + d_3 x^3$ $T(x^3) = 3x^2 = a_4 + b_4 x + c_4 x^2 + d_4 x^3$ Representing A in matrix form ; $A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ from the above matrix of T we can say it is nilpotent matrix.
Checking eigen value of matrix T	$A = \begin{pmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 2 & 0 \\ 0 & 0 & 0-\lambda & 3 \\ 0 & 0 & 0 & 0-\lambda \end{pmatrix}$ $\implies \lambda = 0$
Checking whether matrix of T is invertible	Since $\det A= 0$. Therefore matrix of T is not invertible
Checking whether Matrix of T is diagonal matrix	Let basis be $B' = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ Differentiating wrt x ;

	$T(1) = 0 = a_1 x + b_1(1+x) + c_1(1+x+x^2) + d_1(1+x+x^2+x^3)$ $T(1+x) = 1 = a_2 + b_2(1+x) + c_2(1+x+x^2) + d_2(1+x+x^2x^3)$ $T(1+x+x^2) = 1+2x = a_3 + b_3(1+x) + c_3(1+x+x^2)$ $+d_3(1+x+x^2+x^3)$ $T(1+x+x^2+x^3) = 1+2x+3x^2 = a_4 + b_4(1+x) + c_4(1+x+x^2)$ $+d_4(1+x+x^2+x^3)$ $B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ above matrix is not a diagonal matrix
Conclusion	Thus we can conclude Option 2) and 3) are correct.

TABLE 5.8.3: Solution

5.9. Let m, n, r be natural numbers. Let A be an $m \times n$ matrix with real entries such that $(AA^t)^r = I$, where I is the $m \times m$ is identity matrix and A^t is the transpose of the matrix A. We can conclude that

a) $m = n$

b) AA^t is invertible

c) A^tA is invertible

d) if $m = n$, then A is invertible

Solution: Options 2) and 4) are correct. See Table 5.9.1

5.10. Let $\mathbf{A}$ be a $n \times n$ real matrix with $\mathbf{A}^2 = \mathbf{A}$. Then

a) the eigenvalues of $\mathbf{A}$ are either 0 or 1
b) $\mathbf{A}$ is a diagonal matrix with diagonal entries 0 or 1
c) $rank(\mathbf{A}) = trace(\mathbf{A})$
d) if $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A})$

Solution: See Table 5.10.1

5.11. For any $n \times n$ matrix B, let $N(B) = \{X \in \mathbb{R}^n : BX = 0\}$ be the null space of B. Let A be a 4×4 matrix with $dim(N(A - 4I)) = 2, dim(N(A - 2I)) = 1$ and $rank(A) = 3$ Which of the following are true?

a) 0,2 and 4 are eigenvalues of A
b) determinant(A)=0
c) A is not diagonalizable
d) trace(A)=8

Option	Answer
1) $m = n$	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $r = 1$ $(\mathbf{A}\mathbf{A}^{\mathbf{T}})^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ Since $m \neq n$ Option 1 is False.
2) AA^t is invertible	w.k.t $\det(A^n) = (\det(A))^n$ Since $(AA^t)^r = I$ So $\det((AA^T)^r) = \det(I)$ $(\det(AA^T))^r = 1$ $\implies \det(AA^T) \neq 0$ Hence AA^T is invertible Option 2 is True.
3) A^tA is invertible	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $r = 1$ $(\mathbf{A}^{\mathbf{T}}\mathbf{A})^r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ But $\det(AA^T) = 0$. $\implies AA^T$ is not invertible. Hence Option 3 is False
4) if $m = n$ then A is invertible	Since $\det(AA^T) \neq 0$ det(A).det(A^T) $\neq 0$ det(A).det(A) $\neq 0$ $\implies A$ is invertible. Hence Option 4 is True

TABLE 5.9.1

Solution: See Table 5.11.1.

Given	A is a 4×4 matrix. $dim\,(N\,(A - 2I)) = 2$, $dim\,(N\,(A - 4I)) = 1$, and $rank\,(A) = 3$
Eigenvalues of a matrix	The number λ is an eigenvalue of a matrix A if and only if$A - \lambda I$ is singular,

	i.e. $\lvert A - \lambda I \rvert = 0$ For $\lambda = 2$ Given, $dim\,(N\,(A - 2I)) = 2$ $\implies nullity(A - 2I) = 2$ $rank(A) + nullity(A) = n$ $\implies rank\,(A - 2I) = 4 - 2 = 2$ $\implies$ $(A - 2I)$ is not a full rank matrix Therefore $\lvert A - 2I \rvert = 0$ Also, $\implies N\,(A - 2I) = \{X \in \mathbb{R}^4 : (A - 2I)X = 0\}$ $\implies$ $(A - 2I)X = 0$ gives two eigen vectors $\implies$ 2 is an eigenvalue of A with multiplicity 2. Similarly, for $\lambda = 4$ Given, $dim\,(N\,(A - 4I)) = 1$ $\implies rank\,(A - 4I) = 4 - 1 = 3$ $\implies$ $(A - 4I)$ is not a full rank matrix

	Therefore $\left\|A - 4I\right\| = 0$ $\implies$ 4 is an eigenvalue of A with multiplicity 1. For $\lambda = 0$ Given that $rank(A) = 3$ $\implies$ A is not a full rank matrix Therefore $\left\|A\right\| = 0$ $\implies$ 0 is an eigenvalue of A with multiplicity 1.
Determinant	Given that $rank(A) = 3$ $\implies$ A is not a full rank matrix Therefore $\left\|A\right\| = 0$
Diagonalizability	An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen vectors. $rank(A) + nullity(A) = n$ $\implies$ for $\lambda = 0$, $nullity(A - \lambda I) = nullity(A) = 4 - 3 = 1$ $\implies$ There exists only one linearly independent eigen vector corresponding to 0 eigen value Thus, matrix A is not diagonalizable.
Trace	$Trace(A)$=sum of eigen values $\implies$ $Trace(A) = 0 + 2 + 2 + 4 = 8$
Conclusion	Option (1), (2) and (4) are correct

TABLE 5.11.1: Solution

5.12. Which of the following 3x3 matrices are diagonizable over $\mathbb{R}$?

a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: See Tables 5.12.1 and 5.12.2

Objective	Explanation
Eigenvalues of $\mathbf{A}$	Since $$\mathbf{A}^2 = \mathbf{A} \quad (5.10.1)$$ $$\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \quad (5.10.2)$$ From Cayley-Hamilton Theorem we have, $$\lambda^2 - \lambda = 0 \quad (5.10.3)$$ $$\implies \lambda(\lambda - 1) = 0 \quad (5.10.4)$$ $$\implies \lambda = 0, 1 \quad (5.10.5)$$ A matrix $\mathbf{A}$ satisfying $\mathbf{A}^2 = \mathbf{A}$ is an idempotent matrix with eigen values equal to 0 or 1.
Check if $\mathbf{A}$ is necessary diagonal	Consider $$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.6)$$ $$(5.10.7)$$ Then, $$\mathbf{A}^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.8)$$ $$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.9)$$ $$= \mathbf{A} \quad (5.10.10)$$ Hence $\mathbf{A}$ is idempotent but not diagonal.
Relation between rank and trace of $\mathbf{A}$	Rank of matrix is defined as the number of non-zero eigenvalues. Since number of non-zero eigenvalues is 1, $$rank(\mathbf{A}) = 1 \quad (5.10.11)$$ $$trace(\mathbf{A}) = \sum_i \lambda_i = 0 + 1 = 1 \quad (5.10.12)$$ $$\implies rank(\mathbf{A}) = trace(\mathbf{A}) \quad (5.10.13)$$
Relation between rank and trace of $\mathbf{I} - \mathbf{A}$	Now for the matrix $\mathbf{I} - \mathbf{A}$ we have, $$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \quad (5.10.14)$$ $$= \mathbf{I}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{A}^2 \quad (5.10.15)$$ $$= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \quad (5.10.16)$$ $$= \mathbf{I} - \mathbf{A} \quad (5.10.17)$$ Hence $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Therefore we conclude, $$rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \quad (5.10.18)$$
Answer	(1),(3) and (4)

TABLE 5.10.1

Test for diagonalizability	Let $\mathbf{W}_i$ be the eigenspace corresponding to eigenvalue λ_i of $\mathbf{A}$ 1)$\mathbf{A}$ is diagonalizable 2) characteristic polynomial of $\mathbf{A}$ is $\mathrm{f} = (\mathbf{x} - \lambda_1)^{d_1} \ldots.(\mathbf{x} - \lambda_k)^{d_k}$ and $dim(\mathbf{W}_i) = d_i$ 3) $\sum_{i=1}^{k} \mathbf{W_i} = n$
Concept for diagonalization	A linear operator $\mathbf{A}$ on a n-dimensional space $\mathbb{V}$ is diagonalizable , if and only if $\mathbf{A}$ has n distinct characteristic vectors or null spaces corresponding to the characteristic values

TABLE 5.12.1: Illustration of theorem.

Option A	Given matrix is $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $\mathbf{A}$ is $det(x\mathbf{I} - \mathbf{A})$ $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-1) & -3 & -2 \\ 0 & (x-4) & -5 \\ 0 & 0 & x-6 \end{vmatrix}$ Characteristic Polynomial $= (x-1)(x-4)(x-6)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(xI - A) = 0$ which gives $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6$ Thus over $\mathbb{R}$ matrix $\mathbf{A}$ has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $dim\mathbf{W}_i = d_i$ 3) $\sum_i \mathbf{W_i} = n = 3$, which is equal to dim of $\mathbf{A}$.

Conclusion on Option A	Option A satisfy all three condition of Diagonalizability over $\mathbb{R}$.
Option B	Given matrix is $\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $det(x\mathbf{I}-\mathbf{A})$ $det(x\mathbf{I}-\mathbf{A}) = \begin{vmatrix} x & -1 & 0 \\ 1 & x & 0 \\ 0 & 0 & x-1 \end{vmatrix}$ Characteristic Polynomial = $(x-1)(x+i)(x-i)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is not the product of linear factors over $\mathbb{R}$ beacuse roots of characteristic eq are complex . Thus $\mathbf{A}$ is not diagonalizable over $\mathbb{R}$.
Conclusion on Option B	Option B does not satisfy condition 1.
Option C	Given matrix is $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $\mathbf{A}$ is $det(x\mathbf{I}-\mathbf{A})$ $det(x\mathbf{I}-\mathbf{A})=\begin{vmatrix} (x-1) & -2 & -3 \\ -2 & (x-1) & -4 \\ -3 & -4 & x-1 \end{vmatrix}$ Characteristic Polynomial = $(x+3.19)(x+0.877)(x-7.07)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial are product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $det(x\mathbf{I}-\mathbf{A})=0$ which gives $\lambda_1=-3.19, \lambda_2=-0.887, \lambda_3=7.07$

	Thus over $\mathbb{R}$ matrix $\mathbf{A}$ has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $dim\mathbf{W}_i = d_i$ 3) $\sum_i \mathbf{W_i} = n = 3$, which is equal to dim of $\mathbf{A}$.
Conclusion on Option C	Option C satisfy all three condition of Diagonalizability over $\mathbb{R}$.
Option D	Given matrix is $\mathbf{A}=\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $\mathbf{A}$ is $det(x\mathbf{I} - \mathbf{A})$ $det(x\mathbf{I} - \mathbf{A})= \begin{vmatrix} x & -1 & -2 \\ 0 & x & -1 \\ 0 & 0 & x \end{vmatrix}$ Characteristic Polynomial $= (x)(x)(x) = x^3$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(x\mathbf{I} - \mathbf{A}) = 0$ $\lambda_1 = 0$ $d_1 = 3$ $\mathbf{W}_1 = \mathbf{A} - \lambda_1\mathbf{I} \implies \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $dim\mathbf{W}_1 = 2$ $dim\mathbf{W}_i \neq d_i$ Algebric Multiplicity is not equal to Geometric Multiplicity.
Conclusion on Option D	Option D does not satisfy second condition of Diagonalizability.
Answer	Option A and Option C are Diagonalizable over $\mathbb{R}$.

TABLE 5.12.2: Option Checking Table

Positive Semi Definite Matrix	A $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive semi definite if $\mathbf{x^T M x} \geq 0$ for all non-zero $\mathbf{x}$ in $\mathbb{R}^n$. Formally $\mathbf{M}$ is positive semi-definite $\iff \mathbf{x^T M x} \geq 0\ \forall\ \mathbf{x} \in \mathbb{R}^n \backslash \{0\}$
Theorem	For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L(V)}$, following are equivalent. 1). $\mathbf{x^T M x} \geq 0\ \forall\ \mathbf{x} \in \mathbf{V}$. 2). All the eigenvalues of $\mathbf{M}$ are non-negative.

TABLE 5.13.1: Definition and Result used

Calculating eigen values of $\mathbf{A}$	Given $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Calculating, eigen values of $\mathbf{A}$, ie $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ $\implies \left\lvert \begin{pmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{pmatrix} \right\rvert = 0$ $\implies (3-\lambda)((2-\lambda)(1-\lambda)-9) - 1(1-\lambda-6) + 2(3-2(2-\lambda)) = 0$ $\implies \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$ $\implies \lambda_1 = 6,\ \lambda_2 = \sqrt{3}$ and $\lambda_3 = -\sqrt{3}$ Hence, $\mathbf{A}$ has exactly two positive eigen values.
Proving $\mathbf{x^T A x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ using contradiction	Suppose $\mathbf{x^T A x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Then, by theorem above in definition section, matrix $\mathbf{A}$ is positive semi definite. Hence, all the eigen values of $\mathbf{A}$ non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$. So, $\mathbf{x^T A x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Similarly, as $\lambda_i \leq 0, \forall i$ is also not true, so $\mathbf{x^T A x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Thus, $\mathbf{x^T A x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$.
Correct Options	Hence, correct options are (1) and (4).

TABLE 5.13.2: Solution

5.13. Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q(X)} = \mathbf{X^T A X}$ for $\mathbf{X} \in \mathbb{R}^3$. Then

a) $\mathbf{A}$ has exactly two positive eigen values.
b) all the eigen values of $\mathbf{A}$ are positive.
c) $\mathbf{Q(X)} \geq 0\ \forall\ \mathbf{X} \in \mathbb{R}^3$
d) $\mathbf{Q(X)} < 0$ for some $\mathbf{X} \in \mathbb{R}^3$

Solution: See Tables 5.13.1 and 5.13.2

5.14. Consider the matrix

$$A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}; x \in \mathbf{R}. \tag{5.14.1}$$

Then,

a) A(x) has eigenvalue 0 for some $x \in \mathbf{R}$.
b) 0 is not an eigenvalue of A(x) for any $x \in \mathbf{R}$.
c) A(x) has eigenvalue 0 $\forall x \in \mathbf{R}$.
d) A(x) is invertible $\forall x \in \mathbf{R}$.

Solution: Let $\lambda = 0$ be an eigenvalue. Hence,

$$\left\lvert A - \lambda I \right\rvert = 0 \tag{5.14.2}$$

$$\implies \left\lvert A \right\rvert = 0 \tag{5.14.3}$$

$$\implies \left\lvert A \right\rvert = \begin{vmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{vmatrix} = 0 \tag{5.14.4}$$

Performing row reduction we get,

$$\begin{vmatrix} 1+x^2 & 7 & 11 \\ 0 & \frac{2x^3-19x}{1+x^2} & \frac{4x^2-33x+4}{1+x^2} \\ 0 & 0 & \frac{26x^3-244x^2+538x-68}{2x^3-19x} \end{vmatrix} = 0 \tag{5.14.5}$$

$$\implies 26x^3 - 244x^2 + 538x - 68 = 0 \tag{5.14.6}$$

$$\implies x_1 = 6.01, x_2 = 3.23, x_3 = 0.13 \tag{5.14.7}$$

See Table 5.14.1

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6.1. The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \tag{6.1.1}$$

is

a) positive definite.
b) non-negative definite but not positive definite.
c) negative definite.
d) neither negative definite nor positive definite.

.

Solution:

a) For a real symmetric matrix to be positive definite the eigen values of the matrix should

OPTIONS	Explanation
Option (b)	At the Values of x given by (5.14.7), eigen value $\lambda = 0$. Hence option (b) can't be correct.
Option (c)	If one of the eigenvalue is 0 for A(x) then, $\lvert A(x)\rvert = 0 \forall x \in R$. But from (5.14.7) we have concluded that $\lvert A\rvert = 0$ only for, $x_1 = 6.01, x_2 = 3.23, x_3 = 0.13$. Hence, Option (c) is incorrect.
Option (d)	Now for the values of x given by (5.14.7), $\lvert A\rvert = 0$. Hence it is not invertible $\forall x \in \mathbf{R}$ Hence Option (d) is incorrect.
Option (a)	Now clearly from above arguments A(x) has eigenvalue 0 for some $x \in R$ Hence Option (a) is Correct.

TABLE 5.14.1

be positive.

b) For a real symmetric matrix to be negative definite the eigen values of the matrix should be negative.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

The characteristic equation of the matrix **A**is given by

$$\left|V - \lambda\mathbf{I}\right| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\implies \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0 \tag{6.1.2}$$

The Eigen values of **A** are:

$$\lambda_1 = 5/2$$
$$\lambda_2 = 3/2 \tag{6.1.3}$$
$$\lambda_3 = 4$$

Since all the eigen values of matrix **A** are positive, Therefore the matrix **A** is positive definite.

6.2. Let J denote the matrix of order $n \times n$ with all entries 1 and let B be a $3n \times 3n$ matrix given by $B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$.
Find rank of matrix B. **Solution:** See Tables 6.2.1 and 6.2.2

6.3. Which of the following sets of functions from $\mathfrak{R}$ to $\mathfrak{R}$ is a vector space over $\mathfrak{R}$?

$$S_1 = \{f \mid \lim_{x\to 3} f(x) = 0\} \tag{6.3.1}$$
$$S_2 = \{g \mid \lim_{x\to 3} g(x) = 1\} \tag{6.3.2}$$
$$S_3 = \{h \mid \lim_{x\to 3} h(x)\ exists\} \tag{6.3.3}$$

is

a) Only S_1
b) Only S_2
c) S_1 and S_3 but not S_2
d) All the three are vector spaces

Solution: Let S be a set of functions. Let $f_1, f_2 \in S$ and $\alpha, \beta \in \mathfrak{R}$

For a set of functions to be considered as a vector space:

a) The linear combination of f_1 and f_2 should be in S.
i.e. $\alpha f_1(x) + \beta f_2(x) \in S$
b) The $\mathbf{0}$ should belong to S
i.e. $\mathbf{0} \in S$

Case1: Test for S_1

a) Let $f_1, f_2 \in S_1$ and $\alpha, \beta \in \mathfrak{R}$

$$\lim_{x\to 3} f_1(x) = 0$$
$$\lim_{x\to 3} f_2(x) = 0 \tag{6.3.4}$$

Given	a) Matrix J of $n\times n$ dimension with all entries 1. b) Matrix B of $3n\times 3n$ dimension $$B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$$
Transforming matrix B into Block diagonal matrix using transformation Matrix	$$M = \mathbf{T}(B)$$ $$M = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$$ $$M = \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}$$
Rank of Block Diagonal matrix M	It is equal to the sum of rank of individual blocks in diagonal $$r(J) = 1$$ $$\therefore r(M) = 1 + 1 + 1 = 3$$
Rank of a matrix and its transformation are same.	$\therefore$ rank of matrix B is $$r(B) = r(M) = 3$$

TABLE 6.2.1

Then Using (6.3.4)

$$\begin{aligned} &\lim_{x\to 3}(\alpha f_1(x) + \beta f_2(x)) \\ &= \alpha\left(\lim_{x\to 3} f_1(x)\right) + \beta\left(\lim_{x\to 3} f_2(x)\right) \\ &= \alpha \times 0 + \beta \times 0 \\ &= 0 \end{aligned}$$

$$\therefore \alpha f_1(x) + \beta f_2(x) \in S_1$$

b) Let $f(x) = 0$
then

$$\lim_{x\to 3} f(x) = 0$$

$$\therefore \mathbf{0} \in S_1$$

Hence, S_1 is a vector space.

Case2: Test for S_2

a) Let $g_1, g_2 \in S_2$ and $\alpha, \beta \in \Re$

$$\begin{aligned} \lim_{x\to 3} g_1(x) = 1 \\ \lim_{x\to 3} g_2(x) = 1 \end{aligned} \tag{6.3.5}$$

Then Using (6.3.5)

$$\begin{aligned} &\lim_{x\to 3}(\alpha g_1(x) + \beta g_2(x)) \\ &= \alpha\left(\lim_{x\to 3} g_1(x)\right) + \beta\left(\lim_{x\to 3} g_2(x)\right) \\ &= \alpha \times 1 + \beta \times 1 \\ &= \alpha + \beta \end{aligned}$$

$$\therefore \alpha g_1(x) + \beta g_2(x) \in S_1 \quad iff \quad \alpha + \beta = 1$$

b) Let $g(x) = 0$

Example	Let $n = 2$ $$J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ $$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
Transforming matrix B into Block diagonal matrix using transformation Matrix	$$M = \mathbf{T}(B)$$ $$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$ $$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$
Rank of Block Diagonal matrix M	It is equal to the sum of rank of individual blocks in diagonal $$r(J) = 1$$ $$\therefore r(M) = 1 + 1 + 1 = 3$$
Rank of a matrix and its transformation are same.	$\therefore$ rank of matrix B is $$r(B) = r(M) = 3$$

TABLE 6.2.2

then

$$\lim_{x\to 3} g(x) = 1$$

$$\therefore \mathbf{0} \notin S_1$$

Hence, S_2 is not a vector space.

Case3: Test for S_3

a) Let $h_1, h_2 \in S_3$ and $\alpha, \beta \in \Re$

$$\begin{aligned} &\lim_{x\to 3} h_1(x)\ exists \\ &\lim_{x\to 3} h_2(x)\ exists \end{aligned} \tag{6.3.6}$$

Then Using (6.3.6)

$$\lim_{x\to 3}(\alpha h_1(x) + \beta h_2(x)) \ exists$$

$$\therefore \alpha h_1(x) + \beta h_2(x) \in S_3$$

b) Let $h(x) = 0$
then

$$\lim_{x\to 3^-} h(x) = 0 = \lim_{x\to 3^+} h(x)$$

$$\therefore \mathbf{0} \in S_1$$

Hence, S_3 is a vector space.

Therefore, Option (3) is correct.

6.4. Let **A** be an n × m matrix with each entry equal to +1,-1 or 0 such that every column has exactly one +1 and exactly one -1. We can conclude that

1. Rank $\mathbf{A} \leq n - 1$ (6.4.1)
2. Rank $\mathbf{A} = m$ (6.4.2)
3. $n \leq m$ (6.4.3)
4. $n - 1 \leq m$ (6.4.4)

Solution: See Table 6.4.1

option	Solution
1.	Let us consider **A** as follows and let s be the summation of all column entries: $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$ $$\left\|\mathbf{A} - \lambda \mathbf{I}\right\| = \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{pmatrix} = 0$$ $$= \begin{pmatrix} a_{11} + a_{21} + \cdots + an1 - \lambda & a_{11} + a_{21} + \cdots + an1 - \lambda & \dots & a_{11} + a_{21} + \cdots + an1 - \lambda \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{pmatrix}$$ $$\implies (s - \lambda) \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{pmatrix} = 0$$
Example	Since s=0 according to question, Therefore $\lambda = 0$ is an eigen value of **A**. Since $\lambda = 0$, Hence **A** is singular. Which means at least two rows are linearly dependent. Therefore, $$\text{Rank}(\mathbf{A}) < n$$ $$\text{Rank}(\mathbf{A}) \leq n - 1$$ Let us Consider **A** as follows,where n=4 and m=3 $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$ Calculating Row Reduced Echelon Form of **A** as follows:

	$$\xleftrightarrow[R_4 \leftarrow R_2 + R_4]{R_4 \leftarrow R_1 + R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$ $$\xleftrightarrow{R_4 \leftarrow R_3 + R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
Conclusion	Since the Rank **A**=3 and n=4, Therefore the Rank **A** $\leq n-1$ statement is true.
2.	Let us Consider **A** as follows,where n=2 and m=2 $$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$ Applying elementary transformations on **A** as follows: $$\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$
Conclusion	Since the Rank **A**=1 and m=2, Therefore the Rank **A** $\neq m$, Hence the statement is false.
3.	Let us Consider **A** as follows,where n=3 and m=2 $$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \quad (6.4.5)$$
Conclusion	Since there exists a matrix **A** when n>m, Therefore the statement is false.
4	Let us Consider **A** as follows,where n=4 and m=2 $$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.4.6)$$
Conclusion	Since there exists a matrix **A** when n-1>m, Therefore the statement is false.

TABLE 6.4.1: Solution summary

Options	Solutions	True/False
1.	Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ Now lets find the eigen values of matrix $\mathbf{A}$ $\left\lvert \mathbf{A} - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$ $\implies \lambda^2 - \lambda - 1 = 0$ On solving we get 2 eigen values $\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \beta_1 = \frac{1-\sqrt{5}}{2}$ We know that if eigenvalue of $\mathbf{A}$ is λ then eigenvalue of $\mathbf{A}^n$ is λ^n. In this problem we can say that the eigenvalues α_n and β_n of $\mathbf{A}^n$ are $\alpha_n = \alpha_1^n \quad \beta_n = \beta_1^n$ Since $\alpha_1 > 1$ we can say that $\alpha_n \to \infty$ as $n \to \infty$.	True
2.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since $-1 < \beta_1 < 0$,we can say that $\beta_n \to 0$ as $n \to \infty$.	True
3.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since β_1 is negative because $-1 < \beta_1 < 0$, if n is even then β_n is positive.	True
4.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since β_1 is negative, if n is odd then β_n is negative.	True

TABLE 6.6.1

6.5. Let m, n and r be natural numbers. Let $\mathbf{A}$ be an m × n matrix with real entries such that $(\mathbf{AA^t})^r = \mathbf{I}$, where $\mathbf{I}$ is the m × m identity matrix and $\mathbf{A^t}$ is the transpose of the matrix $\mathbf{A}$. We can conclude that

Options:

a) m = n
b) $\mathbf{AA^t}$ is invertible
c) $\mathbf{A^tA}$ is invertible
d) if m = n, then $\mathbf{A}$ is invertible

6.6. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let α_n and β_n denote the two eigenvalues of $\mathbf{A}^n$ such that $|\alpha_n| \geq |\beta_n|$. Then

a) $\alpha_n \to \infty$ as $n \to \infty$
b) $\beta_n \to 0$ as $n \to \infty$
c) β_n is positive if n is even.
d) β_n is negative if n is odd.

Solution: See Table 6.6.1.

6.7. Let M_n denote the vector space of all $n \times n$ real matrices. Which of the following is a linear subspaces of M_n :-

a) $V_1 = \{A \in M_n :$ A is nonsingular$\}$
b) $V_2 = \{A \in M_n : det(A) = 0\}$
c) $V_3 = \{A \in M_n : trace(A) = 0\}$
d) $V_4 = \{BA : A \in M_n\}$, where B is some fixed matrix in M_n

Solution: See Table 6.7.1

6.8. If $\mathbf{P}$ and $\mathbf{Q}$ are invertible matrices such that $\mathbf{PQ} = -\mathbf{QP}$,then we can conclude that

a) $Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 0$
b) $Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 1$
c) $Tr(\mathbf{P}) = -Tr(\mathbf{Q})$
d) $Tr(\mathbf{P}) \neq Tr(\mathbf{Q})$

Solution: See Table 6.8.1

Vector space	Is it subspace to M_n?
1)V_1: All non-singular matrices of $n \times n$	The matrices $I_{n\times n}$ and $-I_{n\times n}$ are non-singular matrices, but the sum $I_{n\times n} - I_{n\times n}$ is zero matrix and it is singular. $\therefore$ V_1 does not form subspace of M_n.
2)V_2: All singular matrices of $n \times n$	The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} and \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are singular matrices, but the sum is a non-singular matrix. $\therefore$ V_2 does not form subspace M_n.
3)V_3: All matrices of $n \times n$ with trace =0	Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be matrices with Trace = 0. $Tr(\mathbf{v_1} + \alpha \mathbf{v_2}) = Tr(\mathbf{v_1}) + \alpha Tr(\mathbf{v_2}) = 0$. $\therefore$ the vector space V_3 forms linear subspace of M_n.
4)V_4: F_A = BA, where B is some fixed matrix in M_n	Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be matrices in the vector space V_4. $F_{v_1+\alpha v_2} = B(\mathbf{v_1} + \alpha \mathbf{v_2})$ $=B\mathbf{v_1} + \alpha B\mathbf{v_2} =$ $F_{v_1} + \alpha F_{v_2}$. $\therefore$ V_4 forms linear subspace of M_n.

TABLE 6.7.1

Given	**P** and **Q** are invertible matrices. Therefore $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ exists. $$\mathbf{PQ} = -\mathbf{QP} \quad (6.8.1)$$
To Prove	Tr(**P**)=0
Proof 1	Post multiplying equation (6.8.1) by $\mathbf{Q}^{-1}$ we get, $$\mathbf{PQQ}^{-1} = -\mathbf{QPQ}^{-1} \quad (6.8.2)$$ $$\implies \mathbf{PI} = -\mathbf{QPQ}^{-1} \quad (6.8.3)$$ $$\implies \mathbf{P} = -\mathbf{QPQ}^{-1} \quad (6.8.4)$$ Taking trace on both sides for the equation (6.8.4), $$Tr(\mathbf{P}) = Tr(-\mathbf{QPQ}^{-1}) \quad (6.8.5)$$ $$\implies Tr(\mathbf{P}) = -Tr(\mathbf{QPQ}^{-1}) \quad (6.8.6)$$ We know that Tr(**AB**)=Tr(**BA**) Let **A**=**Q** and **B**=$\mathbf{PQ}^{-1}$
	From the above property of trace equation (6.8.6) can be modified as $$Tr(\mathbf{P}) = -Tr(\mathbf{PQ}^{-1}\mathbf{Q}) \quad (6.8.7)$$ $$\implies Tr(\mathbf{P}) = -Tr(\mathbf{PI}) \quad (6.8.8)$$ $$\implies Tr(\mathbf{P}) = -Tr(\mathbf{P}) \quad (6.8.9)$$ $$\implies 2Tr(\mathbf{P}) = 0 \quad (6.8.10)$$ $$\implies Tr(\mathbf{P}) = 0 \quad (6.8.11)$$
To Prove	Tr(**Q**)=0
Proof 2	Post multiplying equation (6.8.1) by $\mathbf{P}^{-1}$ we get, $$\mathbf{PQP}^{-1} = -\mathbf{QPP}^{-1} \quad (6.8.12)$$ $$\implies \mathbf{PQP}^{-1} = -\mathbf{QI} \quad (6.8.13)$$ $$\implies \mathbf{PQP}^{-1} = -\mathbf{Q} \quad (6.8.14)$$ Taking trace on both sides for the equation (6.8.14),

	$$Tr(\mathbf{PQP}^{-1}) = Tr(-\mathbf{Q}) \quad (6.8.15)$$ $$\implies Tr(\mathbf{PQP}^{-1}) = -Tr(\mathbf{Q}) \quad (6.8.16)$$ We know that Tr(**AB**)=Tr(**BA**) Let **A**=**P** and **B**=$\mathbf{QP}^{-1}$ From the above property of trace equation (6.8.16) can be modified as $$Tr(\mathbf{QP}^{-1}\mathbf{P}) = -Tr(\mathbf{Q}) \quad (6.8.17)$$ $$\implies Tr(\mathbf{QI}) = -Tr(\mathbf{Q}) \quad (6.8.18)$$ $$\implies Tr(\mathbf{Q}) = -Tr(\mathbf{Q}) \quad (6.8.19)$$ $$\implies 2Tr(\mathbf{Q}) = 0 \quad (6.8.20)$$ $$\implies Tr(\mathbf{Q}) = 0 \quad (6.8.21)$$
Statement 1	Tr(**P**)=Tr(**Q**)=0
Explanation	From equation (6.8.11) and (6.8.21) we could say that, $$Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 0 \quad (6.8.22)$$ Valid Conclusion
Statement 2	Tr(**P**) = $Tr(\mathbf{Q}) = 1$
Explanation	From equation (6.8.11) and (6.8.21) we could say that, $$Tr(\mathbf{P}) = Tr(\mathbf{Q}) \neq 1 \quad (6.8.23)$$ Invalid Conclusion
Statement 3	Tr(**P**) = $-Tr(\mathbf{Q})$
Explanation	Substituting the conclusion 1 result equation (6.8.22) in equation (6.8.9) we get, $$Tr(\mathbf{P}) = -Tr(\mathbf{Q}) \quad (6.8.24)$$
	Valid Conclusion
Statement 4	Tr(**P**) $\neq Tr(\mathbf{Q})$
Explanation	From equation (6.8.11) and (6.8.21) we could say that, $$Tr(\mathbf{P}) = Tr(\mathbf{Q}) \quad (6.8.25)$$ Invalid Conclusion

TABLE 6.8.1: Explanation with Proofs

Let n be an odd number ≥ 7.Let,

$$\mathbf{A} = [a_{ij}] \quad (6.8.26)$$

be and $n \times n$ matrix with,

$$a_{i,i+1} = 1, \forall (i = 1, 2, ...n - 1) \quad (6.8.27)$$

and $a_{n,1} = 1$. Let $a_{ij} = 0$ for all the other pairs (i, j). Then we can conclude that,

a) $\mathbf{A}$ has 1 as an eigenvalue
b) $\mathbf{A}$ has -1 as an eigenvalue
c) $\mathbf{A}$ has at least one eigenvalue with multiplicity ≥ 2
d) $\mathbf{A}$ has no real eigenvalues

Solution: We can represent our matrix as:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & ..... & 0 & 0 \\ 0 & 0 & 1 & ..... & 0 & 0 \\ . \\ . \\ 0 & 0 & 0 & ..... & 0 & 1 \\ 1 & 0 & 0 & ..... & 0 & 0 \end{pmatrix} \quad (6.8.28)$$

$$\mathbf{A^T} = \begin{pmatrix} 0 & 0 & 0 & ..... & 0 & 1 \\ 1 & 0 & 0 & ..... & 0 & 0 \\ 0 & 1 & 0 & ..... & 0 & 0 \\ . \\ . \\ 0 & 0 & 0 & .... & 1 & 0 \end{pmatrix} \quad (6.8.29)$$

$\mathbf{A}$ is our given matrix.We know that Characteristic Equation of $\mathbf{A}$ and $\mathbf{A^T}$ is same.Consider the minimal polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0 \quad (6.8.30)$$

We can represent it in $n \times n$ matrix with 1's on sub-diagonals and in last column it has negative of the coefficient,and rest all 0.We represent it using $\mathbf{C}$.It is known as the companion matrix.

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & .... & 0 & -a_0 \\ 1 & 0 & 0 & .... & 0 & -a_1 \\ 0 & 1 & 0 & .... & 0 & -a_2 \\ . \\ . \\ 0 & 0 & 0 & .... & 1 & -a_{n-1} \end{pmatrix} \quad (6.8.31)$$

(6.8.30) is also the characteristic equation of $\mathbf{C}$
Comparing (6.8.29) with (6.8.31) we get:

$$a_0 = -1, a_1 = a_2 = a_3 = a_4 = ... = a_{n-1} = 0 \quad (6.8.32)$$

Substituting (6.8.32) into (6.8.30) we get:

$$x^n - 1 = 0 \quad (6.8.33)$$

By Cayley-Hamilton Theorem:

$$\lambda^n - 1 = 0 \quad (6.8.34)$$

(6.8.35)

λ=n^{th} roots of unity. See Table 6.8.2.

6.9. Let $\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ be 3 distinct subspaces of $\mathbf{R}^{10}$ such that each $\mathbf{W_i}$ has dimension of 9. Let $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$. Then we can conclude that

a) $\mathbf{W}$ may not be a subspace of $\mathbf{R}^{10}$

b) dim $\mathbf{W} \leq 8$

c) dim $\mathbf{W} \geq 7$

d) dim $\mathbf{W} \leq 3$

Solution: See Table 6.9.1

Options	Explanation
A has 1 as an eigen value	One value out of the n^{th} roots of unity is 1.So,correct
A has -1 as an eigen value	Since,n is odd.So,-1 cannot be one of the value of n^{th} roots of unity. Hence,incorrect
A has atleast one eigenvalue with multiplicity ≥ 2	All values of n^{th} roots of unity are distinct. So there is no eigenvalue with multiplicity ≥ 2. Hence,incorrect.
A has no real eigen values	One of the value is 1,which is real. Hence,incorrect.

TABLE 6.8.2: Finding Correct Option

Given	$\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ are 3 distinct subspaces of $\mathbf{R}^{10}$ Each $\mathbf{W_i}$ has dimension 9 $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$
Statement1	$\mathbf{W}$ may not be a subspace of $\mathbf{R}^{10}$
Explanation	As $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$ and $\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ are subspaces of $\mathbf{W}$,then $\mathbf{W}$ must be a subspace of $\mathbf{R}^{10}$. So the first option is false.
Statement2	dim $\mathbf{W} \leq 8$
Explanation	As $\mathbf{W}$ be a subspace of a finite dimension vector space $\mathbf{R}^{10}$ and dim $\mathbf{R}^{10}$= 10, so $\mathbf{W}$ is finite dimension and dim $\mathbf{W} \leq 10$
Theorem	dim $(\mathbf{W_1} \cap \mathbf{W_2})$ $= \dim(\mathbf{W_1})+\dim(\mathbf{W_2})-\dim(\mathbf{W_1} + \mathbf{W_2})$ and $\mathbf{W_1} \cap \mathbf{W_2}$ is also a subspace of $\mathbf{R}^{10}$
Proof	The minimum dimension of $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$
Explanation	Let us consider $\mathbf{V} = \mathbf{R}^{10}$ and $dim(\mathbf{V}) = 10$ and $\mathbf{U} = \mathbf{W_1} \cap \mathbf{W_2}$

	So, $dim(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = dim(\mathbf{U})$ $+dim(\mathbf{W_3})$ - $dim(\mathbf{U} + \mathbf{W_3})$ **or,** $dim(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = dim(\mathbf{W_1})$ $+dim(\mathbf{W_2})+dim(\mathbf{W_3})$ - $dim(\mathbf{W_1} + \mathbf{W_1})$ $-dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3})$
	Now, $(\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3} \subseteq \mathbf{V}$ $\implies dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3}) \leq dim(\mathbf{V})$ $\implies -dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3}) \geq -dim(\mathbf{V})$ Similarly, $(\mathbf{W_1} + \mathbf{W_2}) \subseteq \mathbf{V}$ $\implies dim(\mathbf{W_1} + \mathbf{W_2}) \leq dim(\mathbf{V})$ $\implies -dim(\mathbf{W_1} + \mathbf{W_2}) \geq -dim(\mathbf{V})$
	Considering these two inequations, $-dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3})-dim(\mathbf{W_1} + \mathbf{W_2})$ $\geq -2dim(\mathbf{V})$ **or,** $dim(\mathbf{W_1}) + dim(\mathbf{W_2}) + dim(\mathbf{W_3})$ $-dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3})-dim(\mathbf{W_1} + \mathbf{W_2})$ $\geq dim(\mathbf{W_1}) + dim(\mathbf{W_2}) + dim(\mathbf{W_3}) - 2dim(\mathbf{V})$ **or,** $dim(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3})$ $\geq dim(\mathbf{W_1}) + dim(\mathbf{W_2}) + dim(\mathbf{W_3}) - 2dim(\mathbf{V})$ $\implies dim(\mathbf{W}) \geq dim(\mathbf{W_1}) + dim(\mathbf{W_2})$ $+dim(\mathbf{W_3}) - 2dim(\mathbf{V})$
Statement 3	dim $\mathbf{W} \geq 7$
Explanation	As $dim(\mathbf{W}) \geq dim(\mathbf{W_1}) + dim(\mathbf{W_2})$ $+dim(\mathbf{W_3}) - 2dim(\mathbf{V})$ $\implies dim(\mathbf{W}) \geq (9+9+9) - (2\times10)$ $\implies dim(\mathbf{W}) \geq 7$
Answer	$7 \leq dim(\mathbf{W}) \leq 10$

TABLE 6.9.1: **Solution summary**

Hence, we can conclude that $dim(\mathbf{W}) \geq 7$.