Linear Algebra

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CONTENTS

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***Abstract*—This book provides solved examples on Linear Algebra.**

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1.1. Consider the vector space $\mathbb{P}_n$ of real polynomials in x of degree $\leq n$. Define

$$T : \mathbb{P}_2 \to \mathbb{P}_3 \tag{1.1.1}$$

by

$$(Tf)(x) = \int_0^x f(t)\,dt + f'(x). \tag{1.1.2}$$

Then find the matrix representation of T with respect to the bases

$$\left\{1, x, x^2\right\} \text{ and } \left\{1, x, x^2, x^3\right\} \tag{1.1.3}$$

1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A. Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \tag{1.2.1}$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

Solution: Let $P_A(x)$ denote the characteristic polynomial of a matrix $\mathbf{A}$, then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

The characteristic polynomial of a matrix **A** is defined as

$$P_A(x) = det(xI - A) \tag{1.2.2}$$

Let matrix A be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.2.3}$$

$$\implies P_A(x) = det(xI - A) \tag{1.2.4}$$
$$= det\begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \tag{1.2.5}$$
$$= x^2 - (a+d)x + (ad-bc) \tag{1.2.6}$$

From Cayley Hamilton theorem, we can write:

$$A^2 - (a+d)A + (ad-bc) = 0 \tag{1.2.7}$$

Multiplying both sides with A^{-2} :

$$(ad-bc)A^{-2} - (a+d)A^{-1} + I = 0 \tag{1.2.8}$$

Dividing with $(ad-bc)$ on both sides:

$$(A^{-1})^{-2} - \left(\frac{a+d}{ad-bc}\right)A^{-1} + \left(\frac{1}{ad-bc}\right)I = 0$$

From above equation, we can write $P_{A^{-1}}(x)$ as:

$$x^2 - \left(\frac{a+d}{ad-bc}\right)x + \left(\frac{1}{ad-bc}\right) \tag{1.2.9}$$

So, $P_A(x) - P_{A^{-1}}(x)$ becomes:

$$\left(\frac{a+d}{ad-bc} - (a+d)\right)x + \left((ad-bc) - \frac{1}{ad-bc}\right)$$

Hence it can be observed that $P_A(x) - P_{A^{-1}}(x)$ becomes a constant when either $a+d=0$ or $ad-bc=1$.
From the given options it is easy to see that option 3 is the correct answer as its determinant $(ad-bc)=1$.

From (1.2.9), eigenvalues of A^{-1} can be calculated as

$$x^2 - 6x + 1 = 0 \tag{1.2.10}$$
$$\implies x = 3 + \sqrt{8} \text{ or } 3 - \sqrt{8} \tag{1.2.11}$$

1.3. Which of the following matrices is not diagonalizable over $\mathbb{R}$?

a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
c) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
d) $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{1.4.1}$$

1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0,1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over $\mathbb{R}$.

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over $\mathbb{R}$. An inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t)\,dt, \quad f,g \in V. \tag{1.6.1}$$

Let

$$W = span\left\{1-t^2, 1+t^2\right\} \tag{1.6.2}$$

and $W^\perp$ be the orthogonal complement of W in V. Which of the following conditions is satisfied for all $h \in W^\perp$?

a) h is an even function
b) h is an odd function
c) $h(t) = 0$ has a real solution
d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix}\begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \tag{1.7.1}$$

where $m,n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m,n satisfy

a) $m + n = 3$
b) $m > n$
c) $m < n$
d) $m = n$

1.8. Consider a Markov Chain with state space $\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 2 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 3 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 4 & 0 & 0 & 0 & 0 & 1 \end{array} \tag{1.8.1}$$

Then find

$$\lim_{n\to\infty} p_{23}^{(n)} \tag{1.8.2}$$

1.9. Let $L(\mathbb{R})^n$ be the space of $\mathbb{R}-$linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $Ker(T)$ denotes the kernel of T then which of the following are true?

a) There exists $T \in L\left(\mathbb{R}^5\right)$ $\{0\}$ such that $Range(T) = Ker(T)$
b) There does not exist $T \in L\left(\mathbb{R}^5\right)$ $\{0\}$ such that $Range(T) = Ker(T)$
c) There exists $T \in L\left(\mathbb{R}^6\right)$ $\{0\}$ such that $Range(T) = Ker(T)$
d) There does not exist $T \in L\left(\mathbb{R}^6\right)$ $\{0\}$ such that $Range(T) = Ker(T)$

1.10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \to W, T : W \to V, \tag{1.10.1}$$

where W is some finite dimensional vector space such that

a) both T_1 and T_2 are onto
b) both T_1 and T_2 are one to one
c) T_1 is onto, T_2 is one to one
d) T_1 is one to one, T_2 is onto

1.11. Let $A = \left[a_{ij}\right]$ be a 3×3 complex matrix. Identify the correct statements

a) $det\left[(-1)^{i+j} a_{ij}\right] = det(A)$
b) $det\left[(-1)^{i+j} a_{ij}\right] = -det(A)$
c) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = det(A)$
d) $det\left[\left(\sqrt{-1}\right)^{i+j} a_{ij}\right] = -det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1.12.1}$$

be a non-constant polynomial of degree $n \geq 1$. Consider the polynomial

$$q(x) = \int_0^x p(t)\,dt, r(x) = \frac{d}{dx}p(x) \tag{1.12.2}$$

Let V denote the real vector space of all polynomials in x. Then which of the following are true?

a) q and r are linearly independent in V
b) q and r are linearly dependent in V
c) x^n belongs to the linear span of q and r
d) x^{n+1} belongs to the linear span of q and r.

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.
b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over $\mathbb{R}$ if and only if B is diagonalisable over $\mathbb{R}$.
c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same minimal polynomial.
d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in $\mathbb{R}$.

1.14. Consider a matrix

$$A = \left[a_{ij}\right], 1 \leq i, j \leq 5 \tag{1.14.1}$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \tag{1.14.2}$$

Then in which of the following cases A is a positive definite matrix?

a) $n_i = 1 \forall i = 1, 2, 3, 4, 5.$
b) $n_1 < n_2 < \cdots < n_5.$
c) $n_1 = n_2 = \cdots = n_5.$
d) $n_1 > n_2 > \cdots > n_5.$

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \to \mathbb{R}^n \tag{1.15.1}$$

by

$$T_w = v - \frac{2v^T w}{w^T w}w, \quad v \in \mathbb{R}^n \tag{1.15.2}$$

Which of the following are true?

a) $det(T_w) = 1$
b) $T_w(v_1)_w^T(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
c) $T_w = T_w^{-1}$

d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.16.1}$$

over the field $\mathbb{Q}$ of rationals. Which of the following matrices are of the form P^TAP for suitable 2×2 invertible matrix P over $\mathbb{Q}$?

a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \end{array} \tag{1.17.1}$$

Then which of the following are true?

a) $\lim_{n\to\infty} p_{12}^{(n)} = 0$

b) $\lim_{n\to\infty} p_{12}^{(n)} = \lim_{n\to\infty} p_{21}^{(n)}$

c) $\lim_{n\to\infty} p_{22}^{(n)} = \frac{1}{8}$

d) $\lim_{n\to\infty} p_{21}^{(n)} = \frac{1}{3}$

2 December 2018

2.1. Consider the subspaces W_1 and W_2 of $\mathbb{R}^3$ given by

$$W_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \mathbf{x} = 0\right\} \tag{2.1.1}$$

$$W_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \mathbf{x} = 0\right\}. \tag{2.1.2}$$

If $W \subseteq \mathbb{R}^3$, such that

a) $W \cap W_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

b) $\{W \cap W_1\} \perp \{W \cap W_2\}$,

then

a) $W = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

b) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$

c) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

d) $W = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

Solution: Using (2.1.1),

$$\mathbf{W_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{2.1.3}$$

From (2.1.2),

$$\mathbf{W_2} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \tag{2.1.4}$$

From (2.1a), we can say that, both the subspaces $\mathbf{W}$ and $\mathbf{W_2}$ of $\mathbf{R^3}$ contains the column vector as follows: .

$$\mathbf{W} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.1.5}$$

$$\mathbf{W_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{2.1.6}$$

From (2.1.4) and (2.1.6),

$$\mathbf{W_2} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.1.7}$$

$$Rank(\mathbf{W_2}) = 2 \tag{2.1.8}$$

Since, rank < 3 and the vectors are linearly independent they span a subspace of $\mathbf{R^3}$.

Consider the vector,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{W} \cap \mathbf{W_1} \tag{2.1.9}$$

From (2.1a) and (2.1b),

The vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\implies \begin{pmatrix} x & y & z \end{pmatrix} . \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \tag{2.1.10}$$

$$\implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{2.1.11}$$

Since, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{W} \cap \mathbf{W_1}$:

From (2.1.3) and (2.1.11),

$$\mathbf{W_1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.1.12}$$

Also from (2.1.5) and (2.1.11),

$$\boxed{\mathbf{W} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}} \tag{2.1.13}$$

Using (2.1.13),
The vectors linearly independent and rank($\mathbf{W}$)=2 (< 3), then the vector span subspace of $\mathbf{R^3}$.

Hence,

$$\boxed{\mathbf{W} = span\{(0, 1, -1), (0, 1, 1)\} \implies \textbf{Ans : 1}} \tag{2.1.14}$$

2.2. Let

$$C = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \tag{2.2.1}$$

be a basis of $\mathbb{R}^2$ and

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}. \tag{2.2.2}$$

If $T\,[C]$ represents the matrix of T with respect to the basis C then which among the following is true?

a) $T\,[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$

b) $T\,[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$

c) $T\,[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$

d) $T\,[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$

Solution: See Tables 2.2.1 and 2.2.2
In above question $\mathbf{A} = \mathbf{T}, \mathbf{B} = \mathbf{T[C]}, \mathbf{V} = \mathbf{C}$.

Linear Transformation and change of Basis	If matrix **A** represents Linear Transformation with respect to standard ordered basis and matrix **B** represents same transformation with respect to basis **V**,Then $\mathbf{B} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$

TABLE 2.2.1: Linear Transformation and change of basis

2.3. Let $W_1 = \left\{ \mathbf{x} \in \mathbb{R}^4 : \right\}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.1}$$
$$\begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.2}$$
$$\begin{pmatrix} 2 & 0 & 2 & -1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.3}$$

and $W_2 = \left\{ \mathbf{x} \in \mathbb{R}^4 : \right\}$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.4}$$
$$\begin{pmatrix} 1 & 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \tag{2.3.5}$$
$$\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} \mathbf{x} = 0. \tag{2.3.6}$$

Then which among the following is true?

a) $\dim(W_1) = 1$
b) $\dim(W_2) = 2$
c) $\dim(W_1 \cap W_2) = 1$
d) $\dim(W_1 + W_2) = 3$

2.4. Let A be an $n \times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + \jmath I$. Then all eigenvalues of $(A - \jmath I)\,B$ are

a) purely imaginary
b) of modulus one
c) real
d) of modulus less than one

Solution:

a) If $\mathbf{A}$ is a self-adjoint matrix, then it satisfies

$$\mathbf{A}^* = \mathbf{A} \tag{2.4.1}$$

where $\mathbf{A}^*$ is the complex conjugate of $\mathbf{A}$

b) For a self-adjoint(Hermitian) matrix the eigen values are real.

c) Let $\mathbf{A}$ be an $n \times n$ matrix, λ_A be its eigen values and $\mathbf{X}$ be its eigen vector.

$$\mathbf{A}\mathbf{X} = \lambda_A \mathbf{X} \tag{2.4.2}$$

d) If λ_A be the eigen value of $\mathbf{A}$, then
   i) Eigen value of $\mathbf{A} + k\mathbf{I}$ is $\lambda_A + k$
   ii) Eigen value of $\mathbf{A}^p$ is λ_A^p
   iii) Eigen value of $\mathbf{A}^{-1}$ is $1/\lambda_A$

Since $\mathbf{A}$ is an $n \times n$ complex matrix and a self-adjoint matrix. Hence, eigen values of $\mathbf{A}$ are

Evaluate **T**	For linear transformation **T** we have $$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+y\\x-2y\end{pmatrix}$$ $$\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&1\\1&-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$ $$\implies \mathbf{T}=\begin{pmatrix}1&1\\1&-2\end{pmatrix}$$
Evaluate inverse of basis **C**	To find inverse of matrix **C** we row reduce augmented matrix **CI** $$\begin{pmatrix}1&2&1&0\\2&1&0&1\end{pmatrix}\xleftrightarrow[R_2=-\frac{1}{3}R_2]{R_2=R_2-2R_1}\begin{pmatrix}1&2&1&0\\0&1&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$$ $$\xleftrightarrow{R_1=R_1-2R_2}\begin{pmatrix}1&0&-\frac{1}{3}&\frac{2}{3}\\0&1&\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$$ $$\therefore \mathbf{C}^{-1}=\begin{pmatrix}-\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&-\frac{1}{3}\end{pmatrix}$$
Evaluate **TC**	$$\mathbf{TC}=\begin{pmatrix}1&1\\1&-2\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}$$ $$=\begin{pmatrix}3&3\\-3&0\end{pmatrix}$$
Evaluate $\mathbf{T}[\mathbf{C}]=\mathbf{C}^{-1}\mathbf{TC}$	$$\mathbf{T}[\mathbf{C}]=\mathbf{C}^{-1}\mathbf{TC}$$ $$=\begin{pmatrix}-\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&-\frac{1}{3}\end{pmatrix}\begin{pmatrix}3&3\\-3&0\end{pmatrix}$$ $$\implies \mathbf{T}[\mathbf{C}]=\begin{pmatrix}-3&-1\\3&2\end{pmatrix}$$
Conclusion	Option 3) is correct.Options 1),2) and 4) are incorrect

TABLE 2.2.2: Calculation of $\mathbf{T}[\mathbf{C}]$

real. Let λ_A be the eigen value of **A** and **X** be its eigen vector.

$$\mathbf{AX}=\lambda_A\mathbf{X} \tag{2.4.3}$$

The eigen value of **B**

$$\mathbf{B}=(\mathbf{A}+i\mathbf{I})^{-1}$$

Eigen value of $\mathbf{A}+i\mathbf{I}$ is λ_A+i
Eigen value of **B** i.e. $(\mathbf{A}+i\mathbf{I})^{-1}$ is $\frac{1}{\lambda_A+i}$
Eigen value of $\mathbf{A}-i\mathbf{I}$ is λ_A-i
Now Using (2.4.3)

$$(\mathbf{A}+i\mathbf{I})^{-1}\mathbf{X}=\frac{1}{\lambda_A+i}\mathbf{X} \tag{2.4.4}$$

$$(\mathbf{A}-i\mathbf{I})\mathbf{X}=(\lambda_A-i)\mathbf{X} \tag{2.4.5}$$

Multiplying (2.4.4) by $\mathbf{A}-i\mathbf{I}$

$$(\mathbf{A}-i\mathbf{I})(\mathbf{A}+i\mathbf{I})^{-1}\mathbf{X}=(\mathbf{A}-i\mathbf{I})\frac{1}{\lambda_A+i}\mathbf{X} \tag{2.4.6}$$

Using (2.4.5) in (2.4.6)

$$(\mathbf{A}-i\mathbf{I})(\mathbf{A}+i\mathbf{I})^{-1}\mathbf{X}=(\lambda_A-i)\frac{1}{\lambda_A+i}\mathbf{X}$$

$$(\mathbf{A}-i\mathbf{I})\mathbf{BX}=\left(\frac{\lambda_A-i}{\lambda_A+i}\right)\mathbf{X} \tag{2.4.7}$$

From (2.4.7) the eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ are:

a) $\frac{\lambda_A-i}{\lambda_A+i}$
b) not real
c) Magnitude:

$$\left|\frac{\lambda_A-i}{\lambda_A+i}\right|=\frac{\sqrt{\lambda_A^2+1}}{\sqrt{\lambda_A^2+1}}=1 \tag{2.4.8}$$

Therefore, option (2) is correct.

What happens when the eigen values of **A** are complex?
If λ_A is complex i.e.

$$\lambda_A=x+iy \tag{2.4.9}$$

from (2.4.7) Eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ are:

$$EV=\frac{\lambda_A-i}{\lambda_A+i} \tag{2.4.10}$$

Using (2.4.9) in (2.4.10) we get,

$$EV=\frac{x+i(y-1)}{x+i(y+1)} \tag{2.4.11}$$

Rationalizing (2.4.11) we get,

$$EV=\frac{x^2-2xi+y^2-1}{x^2+(y+1)^2} \tag{2.4.12}$$

From (2.4.12)
The eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ are complex.
They can be real only if the eigen values of **A** are purely imaginary.
Verification of the result using a 2×2 matrix.

Eigen values of **A**	Eigen Values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$
(1) If eigen values of **A** are real	(a) $\frac{\lambda_A-i}{\lambda_A+i}$ (b) not real (c) Magnitude = 1
(2) If eigen values of **A** are complex	(a) $\frac{x^2-2xi+y^2-1}{x^2+(y+1)^2}$ (b) complex
(3) If eigen values of **A** are purely imaginary	(a) $\frac{y^2-1}{(y+1)^2}$ (b) real (c) Magnitude $\leqslant 1$

TABLE 2.4.1

Let

$$\mathbf{A} = \begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix} \tag{2.4.13}$$

Characteristic equation of **A**:

$$\begin{aligned} |\mathbf{A}-\lambda\mathbf{I}| &= 0 \\ \implies \lambda^2-\lambda-i &= 0 \end{aligned} \tag{2.4.14}$$

Eigen values of **A**:

$$\begin{aligned} \lambda_1 &= -0.3-0.625i \\ \lambda_2 &= 1.3+0.625i \end{aligned} \tag{2.4.15}$$

Let α be the eigen values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$
Using (2.4.12) we get

$$\begin{aligned} \alpha_1 &= -2.25+2.6i \\ \alpha_2 &= 0.25-0.6i \end{aligned} \tag{2.4.16}$$

Now let's verify (2.4.16)

$$(\mathbf{A}-i\mathbf{I})\mathbf{B} = \begin{pmatrix} -1 & 2 \\ -2i & -1+2i \end{pmatrix} \tag{2.4.17}$$

Characteristic equation of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$:

$$\begin{aligned} |\mathbf{A}-\alpha\mathbf{I}| &= 0 \\ \alpha^2+(2-2i)\alpha+1+2i &= 0 \end{aligned} \tag{2.4.18}$$

Eigen Values of $(\mathbf{A}-i\mathbf{I})\mathbf{B}$ using (2.4.18)

$$\begin{aligned} \alpha_1 &= -2.25+2.6i \\ \alpha_2 &= 0.25-0.6i \end{aligned} \tag{2.4.19}$$

Since (2.4.16) and (2.4.19) are equal.
Hence the result is verified. See Table 2.4.1

2.5. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathbb{C}^n$ as column vectors.Let

$$\mathbf{M} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_k \end{pmatrix}, \tag{2.5.1}$$

$$\mathbf{N} = \begin{pmatrix} \mathbf{u}_{k+1} & \mathbf{u}_{k+2} & \ldots & \mathbf{u}_n \end{pmatrix} \tag{2.5.2}$$

and **P** be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$. Then which of the following is true?

a) $\text{rank}(\mathbf{MPM}^*) = k$ whenever $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$.
b) $\text{tr}(\mathbf{MPM}^*) = \sum_{i=1}^{k} \alpha_i$
c) $\text{rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n-k)$
d) $\text{rank}(\mathbf{MM}^* + \mathbf{NN}^*) < n$.

Solution: See Tables 2.5.1 2.5.2 and 2.5.3

Orthonormal Basis	$B = \{u_1, u_2, ..., u_n\}$ is the Orthonormal basis for C^n if it generates every vector C^n and the inner product $< u_i, u_j > = 0$ if $i \neq j$. That is the vectors are mutually perpendicular and $< u_i, u_j > = 1$ otherwise.
Trace	Trace of a square matrix A, denoted by $\mathbf{tr}(\mathbf{A})$ is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of A Some useful properties of Trace : $\mathbf{tr}(\mathbf{AB}) = \mathbf{tr}(\mathbf{BA})$, where A is the $m \times n$ matrix and B is the $n \times m$ matrix
Basis Theorem	A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if A matrix columns are linearly independent,then it is invertible.

TABLE 2.5.1: Definitions

Rank($\mathbf{MPM}^*$) = $\mathbf{k}$	Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}; \mathbf{u_2} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}; \mathbf{u_4} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ Consider $\mathbf{k} = 2$, then $\mathbf{M} = \begin{pmatrix}u_1 & u_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\\0 & 0\end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{pmatrix}$ $\mathbf{P} = \begin{pmatrix}\alpha_1 & 0\\0 & \alpha_2\end{pmatrix}$ $\mathbf{MPM}^* = \begin{pmatrix}\alpha_1 & 0 & 0 & 0\\0 & \alpha_2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$ $\implies$ Rank($\mathbf{MPM}^*$) ≤ 2 (which is the value of k) (It depends on diagonal values α_1 and α_2) Rank($\mathbf{MPM}^*$) is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, ..., \alpha_k$ is 0.

	Thus, Rank($\mathbf{MPM}^*$) $\neq$ $\mathbf{k}$ Thus, the given statement is false
Trace($\mathbf{MPM}^*$) $= \sum_{i=1}^{k} \alpha_i$	Consider $\mathbf{MP} = \mathbf{A}$ and $\mathbf{M}^* = \mathbf{B}$ Using Properties, Trace($\mathbf{AB}$) = Trace($\mathbf{BA}$) We can say, Trace($\mathbf{MPM}^*$) = Trace($\mathbf{M}^*\mathbf{MP}$) $\mathbf{M} = \begin{pmatrix} u_1 & u_2 & u_3 & .... & u_k \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} \bar{u_1} \\ \bar{u_2} \\ \bar{u_3} \\ . \\ . \\ . \\ \bar{u_k} \end{pmatrix}$ $\mathbf{M}^*\mathbf{M} = \begin{pmatrix} \bar{u_1}u_1 & 0 & 0 & ... & 0 \\ 0 & \bar{u_2}u_2 & 0 & ... & 0 \\ 0 & 0 & \bar{u_3}u_3 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \bar{u_k}u_k \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 0$ if $i \neq j$) $\mathbf{M}^*\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$ (Refer to Properties mentioned in Orthonormal Basis in Definition section that is $< u_i, u_j > = 1$ if $i = j$) $\mathbf{M}^*\mathbf{M} = \mathbf{I^k}$ $\mathbf{M}^*\mathbf{MP} = \mathbf{I^kP} = \mathbf{P}$ Trace($\mathbf{M}^*\mathbf{MP}$) = Trace($\mathbf{I^kP}$) = Trace($\mathbf{P}$) $= \sum_{i=1}^{k} \alpha_i$ (Refer Definition section of Trace, it is sum of elements on the main diagonal) So, the given statement is true
Rank($\mathbf{M}^*\mathbf{N}$) = min(k,$n-k$)	$\mathbf{M} = \{u_1, u_2, ..., u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $\mathbf{u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Consider $k = 2$, then

	$\mathbf{M} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{N} = \begin{pmatrix} u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{M}^*\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{Rank}(\mathbf{M}^*\mathbf{N}) = 0$ But, $\min(k,n-k) = (2,2) = 2$ And, this is clear from above that $\text{Rank}(\mathbf{M}^*\mathbf{N}) \neq \min(k,n-k)$ Thus, above statement is false
$\text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$	$\text{Rank}(\mathbf{M}) = \text{Rank}(\mathbf{M}^*)$ $\text{Rank}(\mathbf{N}) = \text{Rank}(\mathbf{N}^*)$ $\text{Rank}(\mathbf{M}+\mathbf{N}) \leq \text{Rank}(\mathbf{M}) + \text{Rank}(\mathbf{N})$ $\mathbf{M} = \{u_1, u_2, ..., u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, ..., u_n\}$ Consider orthogonal vectors, $\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ Consider $k = 2$, then $\mathbf{M} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\text{Rank}(\mathbf{M}) = 2 = k$ $\mathbf{N} = \begin{pmatrix} u_3 & u_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\text{Rank}(\mathbf{N}) = 2 = n - k$ Thus, $\text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) = \text{Rank}(\mathbf{M} + \mathbf{N}) = 4 = n$ Thus, above statement is false

TABLE 2.5.2: Finding of True and False Statements

$\text{Rank}(\mathbf{MPM}^*) = \mathbf{k}$	False

Trace($\mathbf{MPM}^*$) $= \sum_{i=1}^{k} \alpha_i$	True
Rank($\mathbf{M}^*\mathbf{N}$) $= \min(k, n-k)$	False
Rank($\mathbf{MM}^* + \mathbf{NN}^*$) $< n$	False

TABLE 2.5.3: Conclusion of above Solutions

2.6. Let $B : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the function $B(a,b) = ab$. Which of the following is true-

a) B is a linear transformation
b) B is a positive definite bilinear form
c) B is symmetric but not positive definite
d) B neither linear nor bilinear

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^T \tag{2.6.1}$$

Then

$$B(x,y) = \mathbf{x}^T \frac{\mathbf{R}}{2} \mathbf{x} \tag{2.6.2}$$

where $\mathbf{R}$ is the reflection matrix defined as:-

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.6.3}$$

(2.6.2) represent Quadratic form of B(x,y). See Table 2.6.1

2.7. Let $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function

$$B(a,b) = ab \tag{2.7.1}$$

Which of the following is true?

a) B is a linear transformation
b) B is a positive definite bilinear form
c) B is symmetric but not positive definite
d) B is neither linear nor bilinear

2.8. Let $\mathbf{A}$ be an invertible real $n \times n$ matrix. Define a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{2.8.1}$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T \mathbf{y} \tag{2.8.2}$$

Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivate of F at $(\mathbf{x}, \mathbf{y})$ which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{2.8.3}$$

Then, if

a) $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
b) $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
c) $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
d) $\mathbf{x} = 0$ or $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

Solution: See Tables 2.8.1 and 2.8.2

Options	Explanation
B is a linear transformation	Let the transformation be $B : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that, $B(\mathbf{x}) = xy$ where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ Now $B(\mathbf{e}) = ab$ where $\mathbf{e} = \begin{pmatrix} a \\ b \end{pmatrix}$ Hence, $B(c\mathbf{e}) = c^2 B(\mathbf{e})$ Hence B is not a linear transformation. Hence incorrect.
B is a positive definite bilinear form Bilinear Form Symmetric Positive Definite	$f : \mathbb{V}\times\mathbb{V}\to\mathbb{F}$ where $\mathbb{V}$ is a vector space and $\mathbb{F}$ is a field f is a bilinear if the following holds true - If one variable is fixed then other should be linear Let's say x is fixed,x=c (2.6.2) becomes $B(x, y) = cy$,y is linear Let's say y is fixed,y=c (2.6.2) becomes $B(x, y) = cx$,x is linear Hence B is a bilinear form. Again a bilinear form f is symmetric if $f(\alpha,\beta) = f(\beta,\alpha)$ Here, $B(a, b) = ab$,from (2.6.2) $B(b, a) = ba$,from (2.6.2) $ba = ab$,Hence B is symmetric. A symmetric bilinear f is positive definite if $f(\alpha,\alpha) > 0 \ \forall\alpha \neq 0$ Here, $B(a, a) = a^2$ from (2.6.2) $a^2 > 0 \ \forall a \neq 0$ **Conclusion:** B is symmetric and positive definite bilinear form. Hence Correct.
B is symmetric but not positive definite	From previous proof it is obvious that B is both symmetric as well as positive definite Hence incorrect
B neither linear nor bilinear	From previous proofs it is obvious that B is bilinear. Hence incorrect.
Result	B is symmetric and positive definite bilinear form

TABLE 2.6.1: Finding Correct Option

Invertible	A square matrix is invertible if and only if it does not have a zero eigenvalue. So, from the definition of eigen vector we can write that $$\mathbf{Ax} \neq 0 \tag{2.8.4}$$ The transpose of an invertible matrix is also invertible with inverse $(\mathbf{A}^{-1})^T$. $$\mathbf{AA}^{-1} = \mathbf{I} \implies (\mathbf{A}^{-1})^T \mathbf{A}^T = \mathbf{I}^T = \mathbf{I} \tag{2.8.5}$$ So,similarly we can say that $$\mathbf{A}^T\mathbf{y} \neq 0 \tag{2.8.6}$$
Derivative of F	Suppose F: $\mathbb{R}^n \to \mathbb{R}^m$,the derivative of a function F is given by the Jacobian matrix $$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \tag{2.8.7}$$
Inner product	The inner product of $\mathbf{x}$ and $\mathbf{y}$ is given by $$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{x} \tag{2.8.8}$$

TABLE 2.8.1: Definition and Properties used

Given	$$F(\mathbf{x}, \mathbf{y}) = \langle \mathbf{Ax}, \mathbf{y} \rangle \tag{2.8.9}$$
using inner product definition	$$F(\mathbf{x}, \mathbf{y}) = (\mathbf{Ax})^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \tag{2.8.10}$$ $$F(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T\mathbf{Ax} \tag{2.8.11}$$
Derivative of F	using (2.8.7), We can write that $$DF(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} = \begin{pmatrix} \mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T \end{pmatrix} \tag{2.8.12}$$
If $\mathbf{x} \neq 0$,then $DF(\mathbf{x}, 0) \neq 0$	using (2.8.12),

	$$DF(\mathbf{x},0) = \begin{pmatrix} 0 & \mathbf{x}^T\mathbf{A}^T \end{pmatrix} \quad (2.8.13)$$ From (2.8.4),we know that $$\mathbf{A}\mathbf{x} \neq 0 \quad (2.8.14)$$ $$\implies \mathbf{x}^T\mathbf{A}^T \neq 0 \quad (2.8.15)$$ So, We can say that $$DF(\mathbf{x},0) \neq 0 \quad (2.8.16)$$
If $\mathbf{y} \neq 0$,then $DF(0,\mathbf{y}) \neq 0$	using (2.8.12), $$DF(0,\mathbf{y}) = \begin{pmatrix} \mathbf{y}^T\mathbf{A} & 0 \end{pmatrix} \quad (2.8.17)$$ From (2.8.6),we know that $$\mathbf{A}^T\mathbf{y} \neq 0 \quad (2.8.18)$$ $$\implies \mathbf{y}^T\mathbf{A} \neq 0 \quad (2.8.19)$$ So, We can say that $$DF(0,\mathbf{y}) \neq 0 \quad (2.8.20)$$
If $(\mathbf{x},\mathbf{y}) \neq 0$,then $DF(\mathbf{x},\mathbf{y}) \neq 0$	using (2.8.12), $$DF(\mathbf{x},\mathbf{y}) = \begin{pmatrix} \mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T \end{pmatrix} \quad (2.8.21)$$ As $(\mathbf{x},\mathbf{y}) \neq 0, DF(\mathbf{x},\mathbf{y}) = 0$ iff $\mathbf{A}$=0 From (2.8.4),we know that $$\mathbf{A} \neq 0 \quad (2.8.22)$$ So, We can say that $$DF(\mathbf{x},\mathbf{y}) \neq 0 \quad (2.8.23)$$
If $\mathbf{x} = 0$ or $\mathbf{y} = 0$,then $DF(\mathbf{x},\mathbf{y}) = 0$	From (2.8.20), $$DF(0,\mathbf{y}) \neq 0 \quad (2.8.24)$$ From (2.8.16), $$DF(\mathbf{x},0) \neq 0 \quad (2.8.25)$$ So, if $\mathbf{x} = 0$ or $\mathbf{y} = 0$, $$DF(\mathbf{x},\mathbf{y}) \neq 0 \quad (2.8.26)$$
Conclusion	

	From above,we can say that options 1),2),3) are correct.

TABLE 2.8.2: Finding derivative of linear transformation

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\|x\mathbf{I} - \mathbf{A}\right\|$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 2.9.1: Definitions

2.9. Let

$$T : \mathbb{R}^n \to \mathbb{R}^n \tag{2.9.1}$$

be a linear map that satisfies

$$T^2 = T - I. \tag{2.9.2}$$

Then which of the following is true?

a) T is invertible.
b) $T - I$ is not invertible.
c) T has a real eigenvalue.
d) $T^3 = -I$.

Solution: See Tables 2.9.1 and 2.9.2

Statement	Solution
1.	Given that $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ Since $\mathbf{T}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ therefore the matrix corresponding to it is of order $n \times n$. $\text{Since } \mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$ $\therefore \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$ $\implies p(x) = x^2 - x + 1$ will be annihilating polynomial. $\therefore p(\mathbf{T}) = \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$ We know that minimal polynomial always divides annihilating polynomial. $\therefore$ The roots of minimal polynomial are as follows: $x = \frac{1 \pm \sqrt{3}i}{2} \qquad (2.9.3)$ Therefore any eigenvalue of $\mathbf{T}$ is a root of its minimal polynomial. Since 0 is not a root of p(x), Therefore 0 is not an eigen value for $\mathbf{T}$. Since $\mathbf{T}$ is not invertible iff there exists an eigen value which is zero. $\therefore \mathbf{T} \text{ is invertible.} \qquad (2.9.4)$
Conclusion	Therefore the statement is true.
2.	From equation (2.9.3) , Since 1 is not a root of p(x), Therefore 1 is not an eigen value for $\mathbf{T}$. Therefore, 0 is not an eigen values of $\mathbf{T} - \mathbf{I}_n$. $\therefore \mathbf{T} - \mathbf{I}_n \text{ is invertible.} \qquad (2.9.5)$
Conclusion	Therefore the statement is false.

3.	From equation (2.9.3) , Therefore any eigenvalue of $\mathbf{T}$ is a root of its minimal polynomial. But the roots of minimal polynomial are not real. Therefore $\mathbf{T}$ cant have a real eigen value.
Conclusion	Therefore the statement is false.
4.	Since $\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$ (2.9.6) $\mathbf{T}^3 = \mathbf{T}(\mathbf{T} - \mathbf{I}_n)$ (2.9.7) $\therefore \mathbf{T}^3 = \mathbf{T}^2 - \mathbf{T}$ (2.9.8) $\therefore \mathbf{T}^3 = -\mathbf{I}_n$ (2.9.9)
Conclusion	Therefore the statement is true.

TABLE 2.9.2: Solution summary

2.10. Let

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \qquad (2.10.1)$$

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix}. \qquad (2.10.2)$$

Then which of the following are true?

a) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are inconsistent.
b) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are consistent.
c) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is consistent.
d) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is inconsistent.

Solution: See Table 2.10.1

Given	$\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \mathbf{b_1} = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b_2} = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix} \quad (2.10.3)$
Solution	Solving for $\mathbf{Mx} = \mathbf{b_1}$, Row Reducing the augmented matrix $$\begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 4 \end{pmatrix} \quad (2.10.4)$$ $$\xleftrightarrow[R_4 \leftarrow R_4 - 2R_2]{R_4 \leftarrow 2R_4 - R_1} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 \\ 0 & 0 & -1 & 0 & -4 & -4 & 1 \end{pmatrix} \quad (2.10.5)$$ $$\xleftrightarrow{R_4 \leftarrow R_4 + R_3} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.10.6)$$ $$\implies Rank\left(M\right) = 3, Rank\left(\mathbf{M}\vert\mathbf{b_1}\right) = 4 \quad (2.10.7)$$ $$\implies Rank\left(M\right) \neq Rank\left(\mathbf{M}\vert\mathbf{b_1}\right) \quad (2.10.8)$$
	Solving for $\mathbf{Mx} = \mathbf{b_2}$, Row Reducing the augmented matrix $$\begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 3 \\ 1 & 1 & 1 & 0 & 1 & 1 & 3 \end{pmatrix} \quad (2.10.9)$$ $$\xleftrightarrow[R_4 \leftarrow R_4 + 2R_2]{R_4 \leftarrow 2R_4 - R_1} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 3 \\ 0 & 0 & -1 & 0 & -4 & -4 & -1 \end{pmatrix} \quad (2.10.10)$$ $$\xleftrightarrow{R_4 \leftarrow R_4 + R_3} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.10.11)$$ $$\implies Rank\left(M\right) = 3, Rank\left(\mathbf{M}\vert\mathbf{b_2}\right) = 4 \quad (2.10.12)$$ $$\implies Rank\left(M\right) \neq Rank\left(\mathbf{M}\vert\mathbf{b_2}\right) \quad (2.10.13)$$
	Solving for $\mathbf{Mx} = (\mathbf{b_1} - \mathbf{b_2})$, Row Reducing the augmented matrix

	$\begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & -2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \quad (2.10.14)$ $\xleftrightarrow[R_4 \leftarrow R_4 - 2R_2]{R_4 \leftarrow 2R_4 - R_1} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & -1 & 0 & -4 & -4 & 2 \end{pmatrix} \quad (2.10.15)$ $\xleftrightarrow{R_4 \leftarrow R_4 + R_3} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.10.16)$ $\implies Rank\left(M\right) = 3, Rank\left(\mathbf{M} \mid (\mathbf{b_1} - \mathbf{b_2})\right) = 3 \quad (2.10.17)$ $\implies Rank\left(M\right) = Rank\left(\mathbf{M} \mid (\mathbf{b_1} - \mathbf{b_2})\right) \quad (2.10.18)$
Statement 1	Both systems $\mathbf{Mx} = \mathbf{b_1}$ and $\mathbf{Mx} = \mathbf{b_2}$ are inconsistent
	Eq.(2.10.8) and (2.10.13) violate the condition of consistency (2.10.19) **True Statement**
Statement 2	Both systems $\mathbf{Mx} = \mathbf{b_1}$ and $\mathbf{Mx} = \mathbf{b_2}$ are consistent
	Eq.(2.10.8) and (2.10.13) violate the condition of consistency (2.10.20) **False Statement**
Statement 3	Systems $\mathbf{Mx} = \mathbf{b_1} - \mathbf{b_2}$ are consistent
	Eq.(2.10.18) satisfy the condition of consistency (2.10.21) **True Statement**
Statement 4	Systems $\mathbf{Mx} = \mathbf{b_1} - \mathbf{b_2}$ are inconsistent
	Eq.(2.10.18) satisfy the condition of consistency (2.10.22) **False Statement**

TABLE 2.10.1: Explanation

2.11. Let

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix}. \tag{2.11.1}$$

Given that 1 is an eigenvalue of $\mathbf{M}$, then which among the following are correct?

a) The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
b) The minimal polynomial of $\mathbf{M}$ is $(x-1)^2(x+4)$
c) $\mathbf{M}$ is not diagonalizable.
d) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I})$.

Solution: See Table 2.11.1

Given	$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix} \quad (2.11.2)$ One of the eigenvalue of $\mathbf{M}$ is 1
Solution	Let the eigenvalues of matrix $\mathbf{M}$ of order 3×3 be $\lambda_1, \lambda_2, \lambda_3$ From given , let $\lambda_1 = 1$. We know that sum of the eigenvalues of matrix is Trace of the matrix and product of eigenvalues of matrix is Determinant of the matrix. Trace of the square matrix(Tr($\mathbf{M}$)) is the sum of the elements in the main diagonal of $\mathbf{M}$. $Tr(\mathbf{M}) = 1 + 1 - 4 \quad (2.11.3)$ $\implies Tr(\mathbf{M}) = -2 \quad (2.11.4)$ $\implies \lambda_1 + \lambda_2 + \lambda_3 = -2 \quad (2.11.5)$ $\implies \lambda_2 + \lambda_3 = -3 \quad (2.11.6)$ $\implies \lambda_2 = -3 - \lambda_3 \quad (2.11.7)$ By row reducing the matrix $\mathbf{M}$, we get , $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} \quad (2.11.8)$
	$Det(\mathbf{M}) = 1\left(3\left(-\frac{4}{3}\right)\right) = -4 \quad (2.11.9)$ $\implies \lambda_1\lambda_2\lambda_3 = -4 \quad (2.11.10)$ $\implies \lambda_2\lambda_3 = -4 \quad (2.11.11)$ Solving equations (2.11.7) and (2.11.11) one of the possibilities we get, $\lambda_1 = 1 \quad (2.11.12)$ $\lambda_2 = 1 \quad (2.11.13)$ $\lambda_3 = -4 \quad (2.11.14)$
	Using the eigenvalues the characteristic polynomial of matrix $\mathbf{M}$ is given by, $c(x) = x^3 + 2x^2 - 7x + 4 = 0 \quad (2.11.15)$ The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation. Using the above theorem, the equation (2.11.15) can be written as, $\mathbf{M}^3 + 2\mathbf{M}^2 - 7\mathbf{M} + 4\mathbf{I} = 0 \quad (2.11.16)$ $\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I} + 4\mathbf{M}^{-1} = 0 \quad (2.11.17)$ $\implies \mathbf{M}^{-1} = -\frac{1}{4}(\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I}) \quad (2.11.18)$
Statement 1	The minimal polynomial of $\mathbf{M}$ is $(x-1)(x+4)$
	If (x-1)(x+4) is a minimal polynomial of $\mathbf{M}$ then,

	$$(\mathbf{M}-\mathbf{I})(\mathbf{M}+4\mathbf{I}) = \mathbf{0}_{3\times3} \quad (2.11.19)$$ But, $$(\mathbf{M}-\mathbf{I})(\mathbf{M}+4\mathbf{I}) = \begin{pmatrix} -4 & -4 & -4 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \neq \mathbf{0}_{3\times3} \quad (2.11.20)$$ **False Statement**
Statement 2	The minimal polynomial of **M** is $(x-1)^2(x+4)$
	Let m(x) be the minimal polynomial $$m(x) = (x-1)^2(x+4) \quad (2.11.21)$$ $$= x^3+2x^2-7x+4 \quad (2.11.22)$$ $$= c(x)$$ In this case both minimal polynomial and characteristic polynomial were same. Therefore wecould say that equation (2.11.21) is the minimal polynomial of **M** as it satisfies equation (2.11.16) by Cayley Hamilton Theorem. **True Statement**
Statement 3	**M** is not diagonalizable.
	M is diagonalizable if and only if its minimal polynomial is a product of distinct monic linear
	factors. From equation (2.11.21) we could see that one of the factor of minimal polynomial is repeated and it is not a linear factor. Therefore, Matrix **M** is not diagonalizable. **True Statement**
Statement 4	$$\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M}+3\mathbf{I}) \quad (2.11.23)$$
	Comparing equation (2.11.18) and (2.11.23) we could say that the given statement is **False Statement**.

TABLE 2.11.1: Explanation

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\|x\mathbf{I} - \mathbf{A}\right\|$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 2.12.1: Definitions

2.12. Let $\mathbf{A}$ be a real matrix with characteristic polynomial $(x-1)^3$. Pick the correct statements from below:

a) $\mathbf{A}$ is necessarily diagonalizable.
b) If the minimal polynomial of $\mathbf{A}$ is $(x-1)^3$, then $\mathbf{A}$ is diagonalizable.
c) The characteristic polynomial of $\mathbf{A}^2$ is $(x-1)^3$
d) If $\mathbf{A}$ has exactly two Jordan blocks, then $(\mathbf{A}-\mathbf{I})^2$ is diagonalizable.

Solution: See Tables 2.12.1 and 2.12.2

Statement	Solution
1.	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ Therefore, $p(x) = (x-1)^3$ Soving $(\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $(\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} - \mathbf{I} \neq \mathbf{0}$ Therefore, $m(x) = (x-1)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 1$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 1$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ A matrix is diagonalizable iff its jordan form is a diagonal matrix. Since $\mathbf{J}$ is not diagonizable therefore $\mathbf{A}$ is not diagonizable.
Conclusion	Therefore the statement is false.

2.	Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ Therefore, $p(x) = (x-1)^3$ Soving $(\mathbf{A}-\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $(\mathbf{A}-\mathbf{I})^2 \neq \mathbf{0}$ Therefore, $m(x) = (x-1)^3$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of only one block with principal diagonal values as $\lambda_1 = 1$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{J}$ is not diagonizable therefore $\mathbf{A}$ is not diagonizable.
Conclusion	Therefore the statement is false.
3.	Give that, $p(x)$ of $\mathbf{A} = (x-1)^3$ Hence the eigen values of $\mathbf{A} = 1, 1, 1$ Hence the eigen values of $\mathbf{A}^2 = 1^2, 1^2, 1^2$ or $1, 1, 1$ Therefore $p(x)$ of $\mathbf{A}^2 = (x-1)^3$
Conclusion	Therefore the statement is True.

4.	We know that jordan form of a matrix is similar to the original matrix Let $\mathbf{J}$ be the jordan form of the matrix $\mathbf{A}$ then, $$\begin{aligned} \mathbf{A} &= \mathbf{PJP}^{-1} \\ \mathbf{A}-\mathbf{I} &= \mathbf{PJP}^{-1}-\mathbf{I} \\ \mathbf{A}-\mathbf{I} &= \mathbf{P}(\mathbf{J}-\mathbf{I})\mathbf{P}^{-1} \\ (\mathbf{A}-\mathbf{I})^2 &= \mathbf{P}(\mathbf{J}-\mathbf{I})\mathbf{P}^{-1}\mathbf{P}(\mathbf{J}-\mathbf{I})\mathbf{P}^{-1} \\ (\mathbf{A}-\mathbf{I})^2 &= \mathbf{P}(\mathbf{J}-\mathbf{I})^2\mathbf{P}^{-1} \end{aligned}$$ Therefore $(\mathbf{A}-\mathbf{I})^2$ is similar to $(\mathbf{J}-\mathbf{I})^2$ Since $\mathbf{A}$ has exactly two jordan blocks and order of $\mathbf{A}$ is 3. $$\begin{aligned} \therefore \mathbf{J} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{J}-\mathbf{I} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ (\mathbf{J}-\mathbf{I})^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$ Since $(\mathbf{J}-\mathbf{I})^2$ is diagonal matrix. Therefore $(\mathbf{A}-\mathbf{I})^2$ is diagonalizable.
Conclusion	Therefore the statement is True.

TABLE 2.12.2: Solution summary

2.13. Let P_3 be the vector space of polynomails with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \to P_3 \quad (2.13.1)$$

defined by

$$T\left(p(x)\right) = p(x-1) + p(x+1) \quad (2.13.2)$$

Which of the following properties does the matrix of T with respect to the standard basis $B = \left\{1, x, x^2, x^3\right\}$ of P_3 satisfy?

a) $detT = 0$.
b) $(T-2I)^4 = 0$ but $(T-2I)^3 \neq 0$.
c) $(T-2I)^3 = 0$ but $(T-2I)^2 \neq 0$.
d) 2 is an eigenvalue with multiplicity 4.

Solution: Given

$$T(p(x)) = p(x+1) + p(x-1). \quad (2.13.3)$$

The matrix of T with respect to the standard basis $\mathbf{B} = \{1, x, x^2, x^3\}$ is given by:

$$\begin{aligned} p(x) = 1 &\implies T(1) = 1 + 1 \\ &\quad = 2 && (2.13.4) \\ p(x) = x &\implies T(x) = x + 1 + x - 1 \\ &\quad = 2x && (2.13.5) \\ p(x) = x^2 &\implies T(x^2) = (x+1)^2 + (x-1)^2 \\ &\quad = 2 + 2x^2 && (2.13.6) \\ p(x) = x^3 &\implies T(x^3) = (x+1)^3 + (x-1)^3 \\ &\quad = 6x + 2x^3 && (2.13.7) \end{aligned}$$

Hence, matrix of T is:

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.13.8)$$

See Table 2.13.1

$\det(T) = 0$	**False**. From (2.13.8), it is found that the determinant is not zero as the eigenvalues are nonzero.
$(T-2\mathbf{I})^4 = 0$ but $(T-2\mathbf{I})^3 \neq 0$	**False**. $(T - 2\mathbf{I}) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\implies (T-2\mathbf{I})^2 = 0$ and hence $(T-2\mathbf{I})^4 = 0$ and $(T-2\mathbf{I})^3 = 0$
$(T-2\mathbf{I})^3 = 0$ but $(T-2\mathbf{I})^2 \neq 0$	**False**. Because $(T-2\mathbf{I})^3 = 0$ and $(T-2\mathbf{I})^2 = 0$
2 is an eigenvalue with multiplicity 4.	**True**. It is noted that the matrix of T is an upper triangular matrix having the value 2 along its principal diagonal and hence 2 is an eigenvalue with algebraic multiplicity 4.

TABLE 2.13.1

2.14. Let $\mathbf{M}$ be an $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq= 0$is an eigenvalue of $\mathbf{M}$ with corresponding unit column vector $\mathbf{u}$, then which of the following are true?

a) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k - 1$.
b) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k$.
c) $\text{rank}(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*) = k + 1$.
d) $(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$.

Solution: See Tables 2.14.1 and 2.14.2

2.15. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$B(\mathbf{x}, \mathbf{y}) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \quad (2.15.1)$$

Let $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$W = \left\{\mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0\right\} \quad (2.15.2)$$

Then W

a) is not a subspace of $\mathbb{R}^2$.
b) equals $\mathbf{0}$.
c) is the y axis
d) is the line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution: See Tables 2.15.1 , 2.15.2 and 2.15.3.

<table>
<tr><th>Objective</th><th>Explanation</th></tr>
<tr><td>Rank of $\mathbf{M} - \lambda\mathbf{uu}^*$</td><td>

Since

$$rank\,(\mathbf{A} - \mathbf{B}) \geq rank\,(\mathbf{A}) - rank\,(\mathbf{B}) \tag{2.14.1}$$
$$\implies rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) \geq rank\,(\mathbf{M}) - rank\,(\mathbf{uu}^*) \tag{2.14.2}$$
$$\implies rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) \geq k - rank\,(\mathbf{uu}^*) \tag{2.14.3}$$

If $\mathbf{A}$ is a non-zero column vector of order $m \times 1$ and $\mathbf{B}$ is a non-zero row vector of order $1 \times n$ then $rank\,(AB) = 1$. So,

$$rank\,(\mathbf{uu}^*) = 1 \tag{2.14.4}$$
$$\implies rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) \geq k - 1 \tag{2.14.5}$$

Also since,

$$\mathbf{M} - \lambda\mathbf{uu}^* = \mathbf{M} - \mathbf{Muu}^* = \mathbf{M}\,(I - \mathbf{uu}^*) \tag{2.14.6}$$

and

$$rank\,(\mathbf{M}\,(\mathbf{I} - \mathbf{uu}^*)) \leq min\,(rank\,(\mathbf{M})\,, rank\,(\mathbf{I} - \mathbf{uu}^*)) \tag{2.14.7}$$
$$\implies rank\,(\mathbf{M}\,(\mathbf{I} - \mathbf{uu}^*)) \leq k \tag{2.14.8}$$

Thus we have from (2.14.5) and (2.14.8) that

$$rank\,(\mathbf{M} - \lambda\mathbf{uu}^*) = k - 1 \text{ or } k \tag{2.14.9}$$

Consider a matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.14.10}$$

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TABLE 2.14.1

Objective	Explanation
	such that $rank\left(M\right) = 1$. The eigenvalue of $\mathbf{M}$ is $\lambda = 1$ and the corresponding eigenvector is $$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.14.11}$$ Thus we have, $$\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \tag{2.14.12}$$ $$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.14.13}$$ $$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.14.14}$$ $$\implies rank\left(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*\right) = 0 \tag{2.14.15}$$ Hence if $rank\left(\mathbf{M}\right) = k$ then $rank\left(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*\right) = k - 1$.
$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$	Let the given statement be P(n):$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n\mathbf{u}\mathbf{u}^*$. It can be seen that P(1) is true. Assume P(n) is true for some $k \in \mathbf{N}$ such that $$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^k = \mathbf{M}^k - \lambda^k\mathbf{u}\mathbf{u}^* \tag{2.14.16}$$ Now to prove that P(k+1) is true we have $$(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^{k+1} = (\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)(\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)^k \tag{2.14.17}$$ $$= (\mathbf{M} - \lambda\mathbf{u}\mathbf{u}^*)\left(\mathbf{M}^k - \lambda^k\mathbf{u}\mathbf{u}^*\right) \tag{2.14.18}$$ $$= \mathbf{M}^{k+1} - \lambda^k\mathbf{M}\mathbf{u}\mathbf{u}^* - \lambda\mathbf{M}^k\mathbf{u}\mathbf{u}^* + \lambda^{k+1}\mathbf{u}\mathbf{u}^*\mathbf{u}\mathbf{u}^* \tag{2.14.19}$$ $$= \mathbf{M}^{k+1} - \lambda^{k+1}\mathbf{u}\mathbf{u}^* - \lambda^{k+1}\mathbf{u}\mathbf{u}^* + \lambda^{k+1}\mathbf{u}\left\|\mathbf{u}\right\|^2\mathbf{u}^* \tag{2.14.20}$$ $$= \mathbf{M}^{k+1} - 2\lambda^{k+1}\mathbf{u}\mathbf{u}^* + \lambda^{k+1}\mathbf{u}\mathbf{u}^* \tag{2.14.21}$$ $$= \mathbf{M}^{k+1} - \lambda^{k+1}\mathbf{u}\mathbf{u}^* \tag{2.14.22}$$ Hence, by the Principle of Mathematical Induction P(n) is true for all n.
Answer	(1) and (4)

TABLE 2.14.2

Subspace	A non-empty subset $\mathbf{W}$ of $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if for each pair of vectors α, β in $\mathbf{W}$ and each scalar c in $\mathbf{F}$ the vector $c\alpha+\beta$ is again in $\mathbf{W}$.

TABLE 2.15.1: Definitions and theorem used

Statement	**Observations**
Given	$$\mathbf{W} = \left\{\mathbf{v} \in \mathbb{R}^2 : \mathbf{B}(\mathbf{v_0}, \mathbf{v}) = 0\right\} \tag{2.15.3}$$ $$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.15.4}$$ $$\mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.15.5}$$ $$\mathbf{v_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.15.6}$$ $$\mathbf{B}(\mathbf{v}, \mathbf{w}) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \tag{2.15.7}$$ we will express (2.15.7) in quadratic form. $$\mathbf{B}(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{w} \tag{2.15.8}$$ From (2.15.4), (2.15.6), (2.15.8) we will calculate $\mathbf{B}(\mathbf{v_0}, \mathbf{v})$ $$\implies \mathbf{B}(\mathbf{v_0}, \mathbf{v}) = \mathbf{v_0}^T \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{v} \tag{2.15.9}$$ $$\implies \mathbf{B}(\mathbf{v_0}, \mathbf{v}) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.15.10}$$ $$\implies \mathbf{B}(\mathbf{v_0}, \mathbf{v}) = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.15.11}$$ Now we find the basis vector for $\mathbf{W}$, which is the basis vector of null space of $\mathbf{B}(\mathbf{v_0}, \mathbf{v})$. $$\implies \mathbf{B}(\mathbf{v_0}, \mathbf{v}) = 0 \tag{2.15.12}$$ $$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2.15.13}$$ $$\implies \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \tag{2.15.14}$$ $$\implies x_1 = x_2 \tag{2.15.15}$$ Therefore, the basis vector for $\mathbf{W}$ is $$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.15.16}$$ Therefore $$\mathbf{W} = \{k\mathbf{b} : \forall k \in \mathbb{R}\} \tag{2.15.17}$$

TABLE 2.15.2: Observations

Option	Solution	True/False
1.	Now we will see whether $\mathbf{W}$ is a subspace or not. Let α,β be two pair of vectors in $\mathbf{W}$ where $$\alpha = m\mathbf{b} \quad (2.15.18)$$ $$\beta = n\mathbf{b} \quad (2.15.19)$$ Here $m,n \in \mathbb{R}$ and now we will see whether the vector $c\alpha + \beta$ is in $\mathbf{W}$ or not where c is a scalar value in $\mathbb{R}$. Here $$c\alpha + \beta = cm\mathbf{b} + n\mathbf{b} \quad (2.15.20)$$ $$\implies c\alpha + \beta = (cm + n)\mathbf{b} \quad (2.15.21)$$ From (2.15.21), $(cm + n) \in \mathbb{R}$ and we can say that the vector $c\alpha + \beta \in \mathbf{W}$. Therefore, $\mathbf{W}$ is a subspace of $\mathbb{R}^2$	
2.	From Table 2.15.2, we got $\mathbf{W}$ contains the vectors which are all linear combination of basis vector $\mathbf{b}$ as shown in (2.15.17). Therefore, $$\mathbf{W} \neq \{(0, 0)\} \quad (2.15.22)$$	False
3.	Let us consider a vector on y-axis $$\mathbf{p} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.15.23)$$ Here $$\mathbf{p} \neq k\mathbf{b} \quad (2.15.24)$$ for any $k \in \mathbb{R}$ The vector $\mathbf{p}$ can not be written in terms of the basis vector $\mathbf{b}$. Then $\mathbf{p} \notin \mathbf{W}$. Therefore, the vectors in $\mathbf{W}$ is not y-axis.	False
4.	There is only one basis vector $\mathbf{b}$ for $\mathbf{W}$. Therefore the vectors in $\mathbf{W}$ forms a straight line in vector space $\mathbb{R}^2$. Since, $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\mathbf{b} \quad (2.15.25)$$ $$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\mathbf{b} \quad (2.15.26)$$ Therefore, the line passes through (0,0) and (1,1).	True

TABLE 2.15.3: Solution

2.16. Consider the Quadratic forms

$$Q_1(x, y) = xy \tag{2.16.1}$$

$$Q_2(x, y) = x^2 + 2xy + y^2 \tag{2.16.2}$$

$$Q_3(x, y) = x^2 + 3xy + 2y^2 \tag{2.16.3}$$

on $\mathbb{R}^2$. Choose the correct statements from below

a) Q_1 and Q_2 are equivalent.
b) Q_1 and Q_3 are equivalent.
c) Q_2 and Q_3 are equivalent.
d) all are equivalent.

Solution: See Tables 2.16.1 2.16.2

Matrix representation	The Matrix representation of quadratic forms $$Q(x,y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{X}^T\mathbf{A}\mathbf{X} \quad (2.16.4)$$ The symmetric matrix of the quadratic form is $$\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.16.5)$$
Equivalent condition	Two quadratic forms $\mathbf{X}^T\mathbf{A}\mathbf{X}$ and $\mathbf{Y}^T\mathbf{B}\mathbf{Y}$ are called equivalent if their matrices, A and B are congruent. Two real quadratic forms are equivalent over the real field iff they have the same rank and the same index.
Rank	The rank of a quadratic form is the rank of its associated matrix.
Index	The index of the quadratic form is equal to the number of positive eigen values of the matrix of quadratic form.

TABLE 2.16.1: Definitions and results used

	Matrix	**Rank**	**Eigen Values**	**Index**
$Q_1(x, y)$	$\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_1]{R_1 \leftarrow R_2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\text{rank}(\mathbf{A}_1) = 2$	$\left\lvert \mathbf{A}_1 - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0$ $\implies \left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) = 0$ $\implies \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$	Index of $\mathbf{A}_1 = 1$
$Q_2(x, y)$	$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{A}_2) = 1$	$\left\lvert \mathbf{A}_2 - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$ $\implies (\lambda)(\lambda - 2) = 0$ $\implies \lambda_1 = 0, \lambda_2 = 2$	Index of $\mathbf{A}_2$=2
$Q_3(x, y)$	$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2} R_1} \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}$ $\text{rank}(\mathbf{A}_3) = 2$	$\left\lvert \mathbf{A}_3 - \lambda \mathbf{I} \right\rvert = 0$ $\implies \begin{vmatrix} 1-\lambda & \frac{3}{2} \\ \frac{3}{2} & 2-\lambda \end{vmatrix} = 0$ $\implies \left(\lambda - \frac{\sqrt{10}+3}{2}\right)\left(\lambda + \frac{\sqrt{10}-3}{2}\right) = 0$ $\implies \lambda_1 = \frac{3+\sqrt{10}}{2}, \lambda_2 = \frac{3-\sqrt{10}}{2}$	Index of $\mathbf{A}_3 = 1$
Conclusion	We can say that $Q_1(x, y)$ and $Q_3(x, y)$ are equivalent as the rank and index are same.			

TABLE 2.16.2: Finding which quadratic forms are equivalent

2.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array} \tag{2.17.1}$$

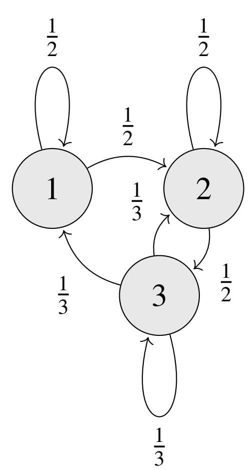
For any two states i and j, let $p_{ij}^{(n)}$ denote the n-step transition probability of going from i to j. Identify correct statements.

a) $\lim_{n\to\infty} p_{11}^{(n)} = \frac{2}{9}$
b) $\lim_{n\to\infty} p_{21}^{(n)} = 0$
c) $\lim_{n\to\infty} p_{32}^{(n)} = \frac{1}{3}$
d) $\lim_{n\to\infty} p_{13}^{(n)} = \frac{1}{3}$

Solution: See Tables 2.17.1 and 2.17.2

Irreducible Markov Chain	A Markov chain is **irreducible** if all the states communicate with each other, i.e., if there is only one communication class.
Aperiodic Markov Chain	If there is a self-transition in the chain ($p^{ii} > 0$ for some i), then the chain is called as **aperiodic**
Stationary Distribution	A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses. Typically, it is represented as a row vector π whose entries are probabilities summing to 1, and given transition matrix $\mathbf{P}$, it satisfies $$\pi = \pi\mathbf{P}$$

TABLE 2.17.1

Drawing Transition diagram	
Checking whether the chain is Irreducible and Aperiodic	Here, All the states are accessible to one another. $\implies$ They are in the same communication class. So, it is Irreducible. There exists the non- zero self-transition, which means that the chain is Aperiodic. We know that if the Markov Chain is irreducible and aperiodic then $\pi_j = \lim_{n\to\infty} P\{X_n = j\},\ j = 1, ..., N$ These are the stationary probabilities.
Finding the Stationary	Stationary Probability can be represented as

Probability Distributions	$$\pi = \pi\mathbf{P}$$ $$\implies \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\mathbf{P}$$ Equating the above equation we get $$\tfrac{1}{2}v_1 - \tfrac{1}{3}v_3 = 0$$ $$\tfrac{1}{2}v_1 - \tfrac{1}{2}v_2 + \tfrac{1}{3}v_3 = 0$$ $$\tfrac{1}{2}v_2 - \tfrac{2}{3}v_3 = 0$$ We see that summation of second and the third equation gives us the first equation only. And we know that the probability distribution will sum up to 1. $$v_1 + v_2 + v_3 = 1$$ Therefore, we get the equation form as $$\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{-1}{3} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
Solving the linear equtions	The above linear equation can be solved using Gauss-Jordan method as $$\left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 0 & \frac{-1}{3} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} & 0 \end{array}\right)$$ $$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & \frac{-1}{2} & \frac{-5}{6} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{3} & 0 \end{array}\right)$$ $$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & \frac{-1}{2} & \frac{-5}{6} & \frac{-1}{2} \\ 0 & -1 & \frac{-1}{6} & \frac{-1}{2} \end{array}\right)$$ $$\xleftrightarrow{R_2 \leftarrow \frac{-1}{2}R_2} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & -1 & \frac{-1}{6} & \frac{-1}{2} \end{array}\right)$$ $$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} \end{array}\right)$$

	$\xleftrightarrow{R_3 \leftarrow \frac{3}{2} R_3} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$ $\xleftrightarrow{R_2 \leftarrow R_2 - \frac{5}{3} R_3} \left(\begin{array}{ccc\|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$ $\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \left(\begin{array}{ccc\|c} 1 & 1 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$ $\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc\|c} 1 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$ $\therefore$, stationary probability distribution π is given by $\pi = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$
Observations	Since the given transition probability matrix **P** is irreducible and aperiodic, then $\lim_{n\to\infty} \mathbf{P}^n$ converges to a matrix with all rows identical and equal to π. We were able to find π as $\begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ $\lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ From the above matrix, we get $\lim_{n\to\infty} \mathbf{P}^n_{11} = \frac{2}{9}$ $\lim_{n\to\infty} \mathbf{P}^n_{21} = \frac{2}{9}$ $\lim_{n\to\infty} \mathbf{P}^n_{32} = \frac{4}{9}$ $\lim_{n\to\infty} \mathbf{P}^n_{13} = \frac{1}{3}$
Conclusion	From our observation we see that Options 1) and 4) are True.

TABLE 2.17.2

3 June 2018

3.1. Let $\mathbf{A}$ be a $(m \times n)$ matrix and $\mathbf{B}$ be a $(n \times m)$ matrix over real numbers with $m < n$. Then

a) $\mathbf{AB}$ is always nonsingular.
b) $\mathbf{AB}$ is always singular.
c) $\mathbf{BA}$ is always nonsingular.
d) $\mathbf{BA}$ is always singular.

Solution: See Table 3.1.1

$$rank(\mathbf{A}) \leq \min(m, n) \tag{3.1.1}$$
$$\implies \leq m, \because m < n \tag{3.1.2}$$
$$rank(\mathbf{B}) \leq \min(n, m) \tag{3.1.3}$$
$$\implies \leq m, \because m < n \tag{3.1.4}$$

We also know that $\mathbf{AB}$ will be a $m \times m$ matrix and $\mathbf{BA}$ will be a $n \times n$ matrix.

$$rank(\mathbf{AB}) \leq \min(rank(\mathbf{A}), rank(\mathbf{B})) \tag{3.1.5}$$
$$\implies \leq m \tag{3.1.6}$$
$$rank(\mathbf{BA}) \leq \min(rank(\mathbf{B}), rank(\mathbf{A})) \tag{3.1.7}$$
$$\implies \leq m \tag{3.1.8}$$

3.2. If $\mathbf{A}$ is a (2×2) matrix over $\mathbb{R}$ with $det(\mathbf{A} + \mathbf{I}) = 1 + det(\mathbf{A})$. Then we can conclude that

a) $det(\mathbf{A}) = 0$.
b) $\mathbf{A} = 0$.
c) $tr(\mathbf{A}) = 0$.
d) $\mathbf{A}$ is nonsingular.

Solution: See Table 3.2.1

Options	Explanation
AB is always nonsingular	$rank(\mathbf{AB}) \leq m$ Let,$rank(\mathbf{AB}) = k$,$k < m$. So,there are $m - k$ linearly dependent columns or rows So,**AB** will be singular Hence,incorrect
Example	$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\end{pmatrix}$,$\mathbf{B}=\begin{pmatrix}1 & 3\\2 & 6\\5 & 6\end{pmatrix}$ $\mathbf{AB}=\begin{pmatrix}20 & 33\\40 & 66\end{pmatrix}$,$rank(\mathbf{AB}) = 1$ 2^{nd} row is linearly dependent on 1^{st} row. **AB** is singular
AB is always singular	$rank(\mathbf{AB}) \leq m$ Let,$rank(\mathbf{AB}) = m$ So,there are 0 linearly dependent columns or rows So,**AB** will be nonsingular Hence,incorrect
Example	$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 5\end{pmatrix}$,$\mathbf{B}=\begin{pmatrix}1 & 3\\2 & 4\\5 & 6\end{pmatrix}$ $\mathbf{AB}=\begin{pmatrix}20 & 29\\35 & 52\end{pmatrix}$,$rank(\mathbf{AB}) = 2$ **AB** is nonsingular
BA is always nonsingular	$rank(\mathbf{BA}) \leq m$.$rank(\mathbf{BA})$ can be atmost m **BA** is $n \times n$ matrix.$n > m$. So,there are atleast $n - m$ linearly dependent columns or rows. So,**BA** will be singular always. Hence,incorrect
Example	$\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\2 & 4 & 5\end{pmatrix}$,$\mathbf{B}=\begin{pmatrix}1 & 3\\2 & 4\\5 & 6\end{pmatrix}$ $\mathbf{BA}=\begin{pmatrix}7 & 14 & 18\\10 & 20 & 26\\17 & 34 & 45\end{pmatrix}$,$rank(\mathbf{BA}) = 2$ 2^{nd} column is linearly dependent on 1^{st} column **BA** is singular
BA is always singular	$rank(\mathbf{BA}) \leq m$.$rank(\mathbf{BA})$ can be atmost m **BA** is $n \times n$ matrix.$n > m$. So,there are atleast $n - m$ linearly dependent columns or rows. So,**BA** will be singular always. Hence,correct
Example	Same example as above. **BA** is always singular.

TABLE 3.1.1: Finding Correct Option

Given	$\mathbf{A}$ be a 2×2 matrix over $\mathbb{R}$ with $$\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$
Explanation	If $\mathbf{X}$ is an eigen vector of matrix $\mathbf{A}$ corresponding to the eigen value λ i.e $$\mathbf{AX} = \lambda\mathbf{X}$$ then, $(\mathbf{I} + \mathbf{A})\,\mathbf{X} = (1 + \lambda)\,\mathbf{X}$ Thus, $\mathbf{X}$ is an eigen vector of $(\mathbf{A} + \mathbf{I})$ corresponding to the eigen value $(1 + \lambda)$. Let λ_1, λ_2 be two eigen values of $\mathbf{A}$ and $(1 + \lambda_1)\,, (1 + \lambda_2)$ be the eigen values of $(\mathbf{A} + \mathbf{I})$. $\implies$ Eigen value of $\mathbf{A} = \lambda_1, \lambda_2$ $\implies$ Eigen value of $(\mathbf{A} + \mathbf{I}) = \lambda_1 + 1, \lambda_2 + 1$ Since,
	$$\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$$ Trace of any matrix is sum of its eigen values. Determinant of matrix is product of its eigen values $$\implies (\lambda_1 + 1)\,(\lambda_2 + 1) = 1 + (\lambda_1\lambda_2)$$ $$\implies \boxed{\lambda_1 + \lambda_2 = 0}$$ $$\implies \boxed{tr(\mathbf{A}) = 0}$$
Statement 1 : $\det \mathbf{A} = 0$	False
	Let, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Here, $\det \mathbf{A} = -1$ and $\det(\mathbf{A} + \mathbf{I}) = 0$ Thus, $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$ In this case, $\det \mathbf{A} \neq 0$ but satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$

Statement 2 : $\mathbf{A} = \mathbf{0}$	False
	Let , $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Here, $\det \mathbf{A} = 0$ and $\det(\mathbf{A} + \mathbf{I}) = 1$ Thus,$1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$ In this case, $\mathbf{A} \neq \mathbf{0}$ But , satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$
Statement 3: $tr(\mathbf{A}) = 0$	True
	The given statement is true for all possible matrices. If $tr\mathbf{A} \neq 0$ then the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$ doesn't satisy. Let , $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
	Here, $\det \mathbf{A} = 0$, $\det(\mathbf{A} + \mathbf{I}) = 2, tr\mathbf{A} \neq 0$ Thus, $1 + \det(\mathbf{A}) \neq \det(\mathbf{A} + \mathbf{I})$
Statement4:$\mathbf{A}$ is non singular	False
	Non Singular Matrix: A non-singular matrix is a square one whose determinant is not zero.non-singular matrix is also a full rank matrix. Let, $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ Here, $\det \mathbf{A} = 0$ and $\det(\mathbf{A} + \mathbf{I}) = 1$ Thus,$1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$ In this case, $\mathbf{A}$ is Singular, But satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$
Conclusion	Thus, we can conclude Statement 3 is true for all possible matrices which satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$

TABLE 3.2.1: Solution Summary

3.3. The system of equations

$$x + 2x^2 + 3xy = 6 \quad (3.3.1)$$
$$x + x^2 + 3xy + y = 5 \quad (3.3.2)$$
$$x - x^2 + y = 7 \quad (3.3.3)$$

a) has solutions in rational numbers.
b) has solutions in real numbers.
c) has solutions in complex numbers.
d) has no solutions.

3.4. The trace of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \quad (3.4.1)$$

is

a) 7^{20}.
b) $2^{20} + 3^{20}$.
c) $2^{21} + 3^{20}$.
d) $2^{20} + 3^{20} + 1$.

Solution: Let,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (3.4.2)$$

To find the eigen values of $\mathbf{A}$:

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \quad (3.4.3)$$
$$\implies \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 03-\lambda & \end{vmatrix} = 0 \quad (3.4.4)$$
$$\implies (2-\lambda)(2-\lambda)(3-\lambda) = 0 \quad (3.4.5)$$
$$\implies \lambda = 2, 2, 3 \quad (3.4.6)$$

Eigen values of $\mathbf{A}$ are 2,2,3.
Hence, the eigen values of $\mathbf{A}^{20}$ are: $2^{20}, 2^{20}$ and 3^{20} respectively.
As we know that the sum of eigen values of a matrix equals the trace of the matrix, hence, the trace of $\mathbf{A}^{20}$ is:

$$tr = 2^{20} + 2^{20} + 3^{20} \quad (3.4.7)$$
$$= 2.2^{20} + 3^{20} \quad (3.4.8)$$

Therefore, option 3 is the required answer.

3.5. Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2, \quad \forall x, y \in \mathbb{R} \quad (3.5.1)$$

This implies that

a) $\lambda = -5$
b) $\lambda \geq 1$
c) $0 < \lambda < 1$
d) There is no such $\lambda \in \mathbb{R}$

Solution: Given that

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (3.5.2)$$

Arranging this in form of a matrix,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.5.3)$$

From this, we get

$$\lambda = a^2 + c^2 \quad (3.5.4)$$
$$ab + cd = 1 \quad (3.5.5)$$
$$b^2 + d^2 = 1 \quad (3.5.6)$$

Let

$$\mathbf{u} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (3.5.7)$$
$$\mathbf{v} = \begin{pmatrix} b \\ d \end{pmatrix} \quad (3.5.8)$$
$$\|\mathbf{u}\|^2 = a^2 + c^2 = \lambda \quad (3.5.9)$$
$$\|\mathbf{v}\|^2 = b^2 + d^2 = 1 \quad (3.5.10)$$

Then,

$$\mathbf{u}^T\mathbf{v} = \begin{pmatrix} a & c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = ab + cd = 1 \quad (3.5.11)$$

Using the Cauchy-Schwartz Inequality, we get

$$|\mathbf{u}^T\mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \, \|\mathbf{v}\|^2 \quad (3.5.12)$$

Now, substituing values from (3.5.9), (3.5.10), (3.5.11) above,

$$\implies 1 \leq \lambda \quad (3.5.13)$$

So from the given options, option 2) $\lambda \geq 1$ is correct.

3.6. Let $\mathbf{R}^n, n \geq 2$ be equipped with standard inner product. Let $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ be n column vectors forming an orthornormal basis of $\mathbf{R}^n$. Let $\mathbf{A}$ be a n x n matrix formed by the column vectors, $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$. Then,

a) $\mathbf{A} = \mathbf{A}^{-1}$
b) $\mathbf{A} = \mathbf{A}^T$

c) $\mathbf{A}^{-1} = \mathbf{A}^T$
d) $Det(\mathbf{A}) = 1$

Solution: Given, $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ are orthonormal and form basis.
So, when they form column vectors of matrix $\mathbf{A}$, we can say that $\mathbf{A}$ is also orthonormal.

$$\therefore \quad \mathbf{A^T A = I} \tag{3.6.1}$$
$$\implies \mathbf{A^T A A^{-1} = I A^{-1}} \tag{3.6.2}$$
$$\implies \mathbf{A^T = A^{-1}} \tag{3.6.3}$$

Clearly, option 3 is the correct answer. Let us consider an orthonormal basis for $\mathbf{R}^2$.

We can check that $\mathbf{S}= \left\{ \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right\}$ forms an orthonormal basis.
Thus the matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.6.4}$$

is the orthonormal matrix whose column vectors are the basis of $\mathbf{R}^2$. For an orthonormal matrix $\mathbf{A}$,

$$\mathbf{A^T A = I} \tag{3.6.5}$$
$$\implies \det\left(\mathbf{A^T A}\right) = \det(\mathbf{I}) \tag{3.6.6}$$
$$\implies \det\left(\mathbf{A}^T\right)\det(\mathbf{A}) = 1 \tag{3.6.7}$$
$$\implies \det(\mathbf{A})^2 = 1 \quad \because \det(\mathbf{A}) = \det\left(\mathbf{A^T}\right) \tag{3.6.8}$$
$$\implies \det(\mathbf{A}) = \pm 1 \tag{3.6.9}$$

Also, here a contradictory example:
Let,

$$\mathbf{R} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{3.6.10}$$

Clearly, $\mathbf{R}$ is an orthonormal matrix and the column vectors of it form an orthonormal basis of $\mathbf{R}^2$. But,

$$\det \mathbf{R} = \begin{vmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{vmatrix} \tag{3.6.11}$$
$$= -1 \tag{3.6.12}$$

From the above two arguments it is clear that option 4 cannot be true.

3.7. Let $\mathbb{R}, n \geq 2$, be equipped with the standard inner product. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be n column vectors forming an orthonormal basis of $\mathbb{R}^n$. Let A be the $n \times n$ matrix formed by the column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then

a) $\mathbf{A} = \mathbf{A}^{-1}$
b) $\mathbf{A} = \mathbf{A}^\top$
c) $\mathbf{A}^{-1} = \mathbf{A}^\top$
d) $det(\mathbf{A}) = 1$

3.8. Consider a Markov Chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.8.1}$$

Then,

a) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
b) $\lim_{n\to\infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$
c) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$
d) $\lim_{n\to\infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

3.9. Let V denote the vector space of all sequences $\mathbf{a} = (a_1, a_2, \ldots)$ of real numbers such that

$$\sum_n 2^n |a|_n \tag{3.9.1}$$

converges. Define

$$\|\cdot\| : V \to \mathbb{R} \tag{3.9.2}$$

by

$$\|\mathbf{a}\| = \sum_n 2^n |a|_n . \tag{3.9.3}$$

Which of the following are true?

a) V contains onlythe sequence $(0, 0, \ldots)$
b) V is finite dimensional
c) V has a countable linear basis
d) V is a complete normed space

3.10. Let V be a vector space over $\mathbb{C}$ with dimension n. Let $T : V \to V$ be a linear transformation with only1 as eigenvalue. Then which of the following must be true?

a) $T - I = 0$
b) $(T - I)^{n-1} = 0$
c) $(T - I)^n = 0$
d) $(T - I)^{2n} = 0$

3.11. If $\mathbf{A}$ is a 5×5 matrix and the dimension of the solution space of $\mathbf{Ax} = 0$ is at least two, then

a) $\text{rank}\left(\mathbf{A}^2\right) \leq 3$

Given	A $\in M_3(\mathbb{R})$ be such that $A^8 = I_{3\times3}$.
Option 1 : minimal polynomial of A can only be of degree 2	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ The Characteristic polynomial is $-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3$ Minimum polynomial is of degree 1. Hence this option is not correct
Option 2 : minimal polynomial of A can only be of degree 3	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ as given in option 1, the minimum polynomial is of degree 1. Hence this option is not correct
Option 3 : either $A = I_{3\times3}$ or $A = -I_{3\times3}$	Let $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ Here, $A^8 = I_{3\times3}$ and $A \neq I_{3\times3}$ or $A \neq -I_{3\times3}$. Hence this option is not correct

b) $\text{rank}\left(\mathbf{A}^2\right) \geq 3$
c) $\text{rank}\left(\mathbf{A}^2\right) = 3$
d) $\det\left(\mathbf{A}^2\right) = 0$

3.12. Let $\mathbf{A} \in M_3(\mathbb{R})$ be such that $\mathbf{A}^3 = \mathbf{I}_{3\times3}$. Then

a) minimal polynomial of $\mathbf{A}$ can only be of degree 2
b) minimal polynomial of $\mathbf{A}$ can only be of degree 3
c) either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A} = -\mathbf{I}$
d) there can be uncountably many $\mathbf{A}$ satisfying the above.

Solution: See Table 3.12.1.

3.13. Let $\mathbf{A}$ be an $n \times n, n > 1$ matrix satisfying

$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I} = \mathbf{0} \tag{3.13.1}$$

Then which of the following statements is true?

a) $\mathbf{A}$ is invertible
b) $t^2 - 7t + 12n = 0$ where $t = tr(\mathbf{A})$
c) $d^2 - 7d + 12 = 0$ where $d = det(\mathbf{A})$
d) $\lambda^2 - 7\lambda + 12 = 0$ where λ is an eigenvalue of $\mathbf{A}$

Solution: See Table 3.13.1

Option 4 : there are uncountably many A satisfying the above	Let A be any 3×3 involuntary matrix. **Involuntary matrix:** A matrix is said to be involuntary matrix if the matrix is its own inverse. Therefore, for an involuntary matrix, $A^2 = I$. For an involuntary matrix, A^n will be equal to A if n is odd and I if n is even. Cleary, $A^8 = I$ for all involuntary matrices. The set of involuntary matrices is uncountable. Hence there are uncountably many A which satify the above condition Hence, this option is the correct answer. Example: $$A = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$ $$A^2 = \begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$ $$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ $$\therefore A^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

TABLE 3.12.1

Given	**A** be the $n \times n$ matrix where $n > 1$ satisfying the following equation $$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I}_{n\times n} = \mathbf{0}_{n\times n} \qquad (3.13.2)$$
Explanation	The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation. Using this theorem the given equation (3.13.2) can be written as ,

	$\lambda^2 - 7\lambda + 12 = 0$ (3.13.3) $(\lambda - 4)(\lambda - 3) = 0$ (3.13.4) $\lambda_1 = 3$ (3.13.5) $\lambda_2 = 4$ (3.13.6) Here λ_1 and λ_2 were eigen values of matrix **A** We know that determinant is product of eigen values. $d = Det(\mathbf{A})$ (3.13.7) $\implies d = \lambda_1 \lambda_2$ (3.13.8) $\implies d = 12 \neq 0$ (3.13.9)
Statement 1	**A** is invertible
	From equation (3.13.9), since $d \neq 0$ the given matrix **A** is Invertible. **True Statement**
Statement 2	$t^2 - 7t + 12n = 0$ (3.13.10)
	We know that the trace is the sum of the eigen values. $t = Tr(\mathbf{A})$ (3.13.11) $\implies t = \lambda_1 + \lambda_2$ (3.13.12) $\implies t = 7$ (3.13.13) Substituting the equation (3.13.13) in (3.13.10) we get, $7^2 - 7(7) + 12n = 0$ (3.13.14) $12n = 0$ (3.13.15) Since given that $n > 1$ the equation (3.13.15) is not possible i.e $12n \neq 0$. Therefore, $t^2 - 7t + 12n = 0$ is a **False Statement**
Statement 3	$d^2 - 7d + 12 = 0$ (3.13.16)
	Substituting the equation (3.13.9) in (3.13.16), we get, $12^2 - 7(12) + 12 = 0$ (3.13.17) $72 = 0$ (3.13.18) From equation (3.13.15) it is clear that the above statement 3 is invalid. **False Statement**
Statement 4	$\lambda^2 - 7\lambda + 12 = 0$ (3.13.19)
	By Cayley Hamilton Theorem, equation (3.13.3) shows that the above statement 4 is valid. **True Statement**

TABLE 3.13.1: Explanation

3.14. Let $\mathbf{A}$ be a 6×6 matrix over $\mathbb{R}$ with characteristic polynomial

$$(x-3)^2\,(x-2)^4 \tag{3.14.1}$$

and minimal polynomial

$$(x-3)\,(x-2)^2 \tag{3.14.2}$$

Then the Jordan canonical form of $\mathbf{A}$ can be

a) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$

Solution: See Tables 3.14.1 and 3.14.1

Jordan canonical form	If **A** is a matrix of order n×n ,then the Jordan canonical form of **A** is a matrix of order n×n expressed as $$\mathbf{J} = \begin{pmatrix} \mathbf{J_1} & & \\ & \ddots & \\ & & \mathbf{J_k} \end{pmatrix} \qquad (3.14.3)$$ where $\mathbf{J_1}, ..., \mathbf{J_k}$ are the Jordan blocks.
Algebraic multiplicity A_M	Algebraic multiplicity of characteristic value λ in the characteristic polynomial determines the size of Jordan block for that eigen value $$A_M = \text{Size of Jordan block for that } \lambda \qquad (3.14.4)$$
Geometric multiplicity G_M	Geometric multiplicity determines the number of Jordan sub-blocks in a Jordan block for λ
Minimal Polynomial	The multiplicity of λ in the minimal polynomial determines the size of the largest sub-block.

TABLE 3.14.1: Definition and Properties used

Characteristic polynomial	$$p(x) = (x-3)^2 (x-2)^4 \qquad (3.14.5)$$
Algebraic Multiplicity	$$\text{For } \lambda = 3, A_M = 2 \qquad (3.14.6)$$ $$\text{For } \lambda = 2, A_M = 4 \qquad (3.14.7)$$
Minimal polynomial	$$m(x) = (x-3)(x-2)^2 \qquad (3.14.8)$$
Finding Jordan blocks for λ_1=3	For λ_1=3,We can write from table3.14.1 that $$\text{The highest order of Jordan block} = 1$$ $$\text{Size of Jordan block} = A_M = 2$$ The Jordan blocks for λ_1=3

	$\mathbf{J_1} = \begin{pmatrix}3\end{pmatrix}, \mathbf{J_2} = \begin{pmatrix}3\end{pmatrix} \qquad (3.14.9)$
Finding Jordan blocks for λ_1=2	For λ_1=2,We can write from table3.14.1 that The highest order of Jordan block = 2 Size of Jordan block = $A_M = 4$ The Jordan blocks for λ_1=3 $\mathbf{J_3} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}, \mathbf{J_4} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix} \qquad (3.14.10)$ or $\mathbf{J_3} = \begin{pmatrix}2 & 1\\0 & 2\end{pmatrix}, \mathbf{J_4} = \begin{pmatrix}2\end{pmatrix}, \mathbf{J_5} = \begin{pmatrix}2\end{pmatrix} \qquad (3.14.11)$
Jordan canonical form	Jordan canonical form of **A** is $\mathbf{J} = \begin{pmatrix}\mathbf{J_1} & & & \\ & \mathbf{J_2} & & \\ & & \mathbf{J_3} & \\ & & & \mathbf{J_4}\end{pmatrix} \text{or} \begin{pmatrix}\mathbf{J_1} & & & & \\ & \mathbf{J_2} & & & \\ & & \mathbf{J_3} & & \\ & & & \mathbf{J_4} & \\ & & & & \mathbf{J_5}\end{pmatrix} \qquad (3.14.12)$ $\begin{pmatrix}3&0&0&0&0&0\\0&3&0&0&0&0\\0&0&2&1&0&0\\0&0&0&2&0&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix} \text{or} \begin{pmatrix}3&0&0&0&0&0\\0&3&0&0&0&0\\0&0&2&1&0&0\\0&0&0&2&0&0\\0&0&0&0&2&0\\0&0&0&0&0&2\end{pmatrix} \qquad (3.14.13)$
Conclusion	From above,we can say that options 2) and 3) are correct.

TABLE 3.14.2: Finding Jordan canonical form

3.15. Let V be an inner product space and S be a subset of V. Let $\bar{S}$ denote the closure of S in V with respect to the topology induced by the metric given by the inner product. Which of the following statements is true?

a) $S = (S^{\perp})^{\perp}$

b) $\bar{S} = (S^{\perp})^{\perp}$

c) $\overline{span(S)} = (S^{\perp})^{\perp}$

d) $S^{\perp} = \left((S^{\perp})^{\perp}\right)^{\perp}$

Solution: See Tables 3.15.3, 3.15.3 and 3.15.3

Orthogonal Complement	Let $\mathcal{S}$ be a subset of an inner product space $\mathbf{V}$. The space of all vectors orthogonal to $\mathcal{S}$ is called the **orthogonal complement** of $\mathcal{S}$: $$\mathcal{S}^{\perp} = \{\mathbf{x} \in \mathbf{V} : \; < \mathbf{x}, \mathbf{y} >= 0, \quad \forall \mathbf{y} \in \mathcal{S}\}$$
Closure of subset	closure of a set $\mathcal{S}$ is the set of all limits of points from $\mathcal{S}$ Let $\mathcal{S}$ be a subset of an inner product space $\mathbf{V}$. Then closure of $\mathcal{S}$ satisfies, $$\overline{\mathcal{S}} = \{\mathbf{y} \in \mathbf{V} : \; \textit{there exist } \mathbf{x_n} \in \mathcal{S} \textit{ such that } \mathbf{x_n} \to \mathbf{y}\}$$
Projection Theorem	Let $\mathcal{S}$ be a closed subspace of a finite dimensional vector space $\mathbf{V}$, then, Every $\mathbf{x} \in \mathcal{S}$ can be expressed as, $$\mathbf{x} = \mathbf{u} + \mathbf{v}, \text{ where,}$$ $$\mathbf{u} \in \mathcal{S}, \quad \mathbf{v} \in \mathcal{S}^{\perp}$$
Theorem	If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathbf{V}$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $$\mathcal{S}_2^{\perp} \subseteq \mathcal{S}_1^{\perp} \; .$$

TABLE 3.15.1: Definitions and results used

Given	Let $\mathcal{S}$ be any set, then $\mathcal{S}^{\perp}$ is the set of all vectors that are perpendicular to all elements of $\mathcal{S}$ We will check if $\mathcal{S}^{\perp}$ is a subspace (1) Closed on Addition Let $\mathbf{u}, \mathbf{v} \in \mathcal{S}^{\perp}$, then, for $\mathbf{x} \in \mathbf{V}$, $< \mathbf{x}, \mathbf{u} + \mathbf{v} >=< \mathbf{x}, \mathbf{u} > + < \mathbf{x}, \mathbf{v} >= 0$ $\implies \mathbf{u} + \mathbf{v} \in \mathcal{S}^{\perp}$ (2) Closed on Multiplication Let $\mathbf{u} \in \mathcal{S}^{\perp}$, then, for $\mathbf{x} \in \mathbf{V}$ and scalar $\alpha \in \mathbb{F}$, $< \mathbf{x}, \alpha\mathbf{u} >= \alpha^* < \mathbf{x}, \mathbf{u} >= 0$ $\implies \alpha\mathbf{u} \in \mathcal{S}^{\perp}$ Therefore, $\mathcal{S}^{\perp}$ is a subspace Therefore, $(\mathcal{S}^{\perp})^{\perp}$ is also a subspace
	Checking the options
$\mathcal{S} = (\mathcal{S}^{\perp})^{\perp}$	We have, $$\mathcal{S}^{\perp} = \{x \in \mathbf{V} : \; < x, y >= 0, \quad \forall y \in \mathcal{S}\}$$

	$\implies \quad (\mathcal{S}^{\perp})^{\perp} = \{x \in \mathbf{V} : \ < x, y >= 0, \quad \forall y \in \mathcal{S}\}$ Let $\mathbf{s} \in \mathcal{S}$, then $< \mathbf{s}, \mathbf{v} >= 0, \quad \forall \mathbf{v} \in \mathcal{S}^{\perp}$ $\implies \mathbf{s} \in (\mathcal{S}^{\perp})^{\perp}$ Therefore, $\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp} \qquad \ldots(1)$ We have proved that $(\mathcal{S}^{\perp})^{\perp}$ is a subspace But, $\mathcal{S}$ is a subset of $\mathbf{V}$ and is not necessarily a subspace. Therefore, this option is **false**.
$\overline{\mathcal{S}} = (\mathcal{S}^{\perp})^{\perp}$	Similarly, $\overline{\mathcal{S}}$ is a subset of $\mathbf{V}$ and is not necessarily a subspace. Therefore, this option is **false**.
$\overline{span\,(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$	Let $\mathbf{v}$ is a limit of some $\mathbf{v_i}$ such that $\mathbf{v_i} \in span\,(\mathcal{S})$ $\implies \mathbf{v} \in \overline{span\,(\mathcal{S})}$ Now, Since, $\mathbf{v_i} \in span\,(\mathcal{S})$, $\implies \mathbf{v_i} = \sum \beta_j \mathbf{s_j}, \quad \mathbf{s_j} \in \mathcal{S}$ Let $\mathbf{w} \in \mathcal{S}^{\perp}$, $\implies < \mathbf{w}, \mathbf{s_j} >= 0$ Now, $< \mathbf{w}, \mathbf{v_i} >= \sum \beta_j < \mathbf{w}, \mathbf{s_j} > = 0$ Therefore, $\mathbf{w} \perp \mathbf{v_i}$, hence, $\mathbf{w} \perp \mathbf{v}$ $\implies \mathbf{v} \in (\mathcal{S}^{\perp})^{\perp}$ $\implies \overline{span\,(\mathcal{S})} \subseteq (\mathcal{S}^{\perp})^{\perp} \qquad \ldots(2)$ Therefore, this option is **false**. However, if we assume that $\mathbf{V}$ is a finite dimensional space, which implies, $\mathbf{V}$ is a hilbert space, then we have, for $\mathbf{x} \in (\mathcal{S}^{\perp})^{\perp}$, $\mathbf{x} = \mathbf{u} + \mathbf{v}, \quad \mathbf{u} \in \overline{span(\mathcal{S})}, \mathbf{v} \perp \overline{span(\mathcal{S})}$ Now, $< \mathbf{x}, \mathbf{u} >= 0$ $\implies < \mathbf{u} + \mathbf{v}, \mathbf{v} >= 0$ $\implies < \mathbf{u}, \mathbf{v} > + < \mathbf{v}, \mathbf{v} >= 0$ $\implies \\|\mathbf{v}\\| = 0$

	$\implies \mathbf{v} = 0$ $\implies \mathbf{x} = \mathbf{u} \in \overline{span(\mathcal{S})}$ $\implies (\mathcal{S}^{\perp})^{\perp} \subseteq \overline{span(\mathcal{S})} \quad \ldots(3)$ From (2) and (3), $\overline{span\,(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$ if $\mathbf{V}$ is a hilbert space.
$\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$	From (1), we have, $\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp}$ $\implies \mathcal{S}^{\perp} \subseteq \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp} \quad \ldots(4)$ We know that, $\mathcal{S}_2^{\perp} \subseteq \mathcal{S}_1^{\perp}$ Therefore, $\left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp} \subseteq \mathcal{S}^{\perp} \quad \ldots(5)$ From (4) and (5), we have, $\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$ Therefore, this option is **True**.
Example:	Let $\mathbf{V} = \mathbb{R}^2$ We want a subset $\mathcal{S}$ of $\mathbf{V}$ which is not a subspace. Let $\mathcal{S} = \left\{\begin{pmatrix} x \\ 3x+1 \end{pmatrix}\right\}, x \in \mathbb{R},$ Then, $\mathcal{S}^{\perp} = \left\{\begin{pmatrix} x \\ -\frac{1}{3}x + c \end{pmatrix}\right\} \quad \ldots(1)$ $\implies (\mathcal{S}^{\perp})^{\perp} = \left\{\begin{pmatrix} x \\ 3x + c \end{pmatrix}\right\}$ Therefore, $\mathcal{S} \subseteq (\mathcal{S}^{\perp})^{\perp}$ $\implies \boxed{\mathcal{S} \neq (\mathcal{S}^{\perp})^{\perp}}$ Similarly, $\implies \boxed{\overline{\mathcal{S}} \neq (\mathcal{S}^{\perp})^{\perp}}$ Also, $\left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp} = \left\{\begin{pmatrix} x \\ -\frac{1}{3}x + c \end{pmatrix}\right\} \quad \ldots(2)$ From (1) and (2), we get,

		$\boxed{\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}}$

TABLE 3.15.2: Solution

$\mathcal{S} = (\mathcal{S}^{\perp})^{\perp}$	**false**.
$\overline{\mathcal{S}} = (\mathcal{S}^{\perp})^{\perp}$	**false**.
$\overline{span(\mathcal{S})} = (\mathcal{S}^{\perp})^{\perp}$	**false**
$\mathcal{S}^{\perp} = \left((\mathcal{S}^{\perp})^{\perp}\right)^{\perp}$	**True**.

TABLE 3.15.3: Conclusion

3.16. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.16.1)$$

and

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \qquad (3.16.2)$$

Which of the following statements is true?

a) The matrix of second order partial derivatives of the quadratic form Q is $2\mathbf{A}$
b) The rank of the quadratic form Q is 2
c) The signature of the quadratic form Q is $++0$
d) The quadratic form Q take the value 0 for some non-zero vector $\mathbf{x}$

Solution: See Tables 3.16.1 and 3.16.2

Quadratic Form of a matrix	Let $\mathbf{V}$ be a vector space over $\mathbb{R}$. $\mathbf{A}$ be a symmetric matrix $n \times n$. Quadratic form on $\mathbf{V}$ is a real function, ($\mathbf{F} : \mathbf{V} \to \mathbb{R}$) defined as $F(x) = \mathbf{x}\mathbf{A}\mathbf{x}^T = \sum_{i,j=1}^{n} a_{ij}x_i x_j$ where $\mathbf{x} \in \mathbf{V}$
Signature of Quadratic form	The signature of quadratic form is (n_+, n_-, n_0) where n_+ is the number of positive entries, n_- is number of negative entries and n_0 is number of zero's in a_{ii}
Rank of quadratic form	Rank of quadratic form is the rank of its matrix which is maximum number of linearly independent rows/columns of a matrix

TABLE 3.16.1: Definitions

Option 1	The matrix of second order partial derivatives of the quadratic form of $\mathbf{Q}$ is $2\mathbf{A}$.
Solution	$\mathbf{Q}(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix}\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} x+2y \\ -2z \\ z \end{pmatrix} = x^2 + z^2 + 2xy - 2yz$ First order partial derivaties: $\frac{\partial \mathbf{Q}}{\partial x} = 2x + 2y \quad \frac{\partial \mathbf{Q}}{\partial y} = 2x - 2z \quad \frac{\partial \mathbf{Q}}{\partial z} = 2z - 2y$ Second order partial derivatives of: $\frac{\partial^2 \mathbf{Q}}{\partial x^2} = 2 \quad \frac{\partial^2 \mathbf{Q}}{\partial y^2} = 0 \quad \frac{\partial^2 \mathbf{Q}}{\partial z^2} = 2$ $\frac{\partial^2 \mathbf{Q}}{\partial x \partial y} = \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} = 2 \quad \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} = 0 \quad \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} = -2$ Matrix of second order partial derivatives $\mathbf{Q}$: $\begin{pmatrix} \frac{\partial^2 \mathbf{Q}}{\partial x^2} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial y^2} & \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \neq 2\mathbf{A}$ Hence, **Option 1** is not correct.
Option 2	The rank of the quadratic form of $\mathbf{Q}$ is 2
Solution	From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ Echelon form reduction: $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xleftrightarrow{R_1 = \frac{1}{2}} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$ $\xleftrightarrow{R_2 \to \frac{-1}{2} R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \xleftrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_3 \to \frac{1}{4} R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $\xleftrightarrow{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xleftrightarrow{R_2 \to R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Rank = Number of non-zero rows = $3 \neq 2$ Hence, **Option 2** is not correct.
Option 3	The signature of the quadratic form $\mathbf{Q}$ is $(+ + 0)$
Solution	From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$

	Finding eigen values: $\left\lvert\mathbf{Q}-\lambda\mathbf{I}\right\rvert=\begin{pmatrix}2-\lambda & 2 & 0\\ 2 & -\lambda & -2\\ 0 & -2 & 2-\lambda\end{pmatrix}$ $\implies (2-\lambda)\left(-2\lambda+\lambda^2+4\right)+8=0$ $\implies \lambda^3-4\lambda^2-4\lambda+16=0$ $\lambda_1=4 \quad \lambda_2=2 \quad \lambda_3=-2$ Signature $=(n_+,n_-,n_0)=(2,1,0)\neq(++0)$ Hence, **Option 3** is not correct.
Option 4	The quadratic form **Q** takes the value 0 for some non-zero vector (x,y,z)
Solution	From above we have quadratic form of $\mathbf{Q}=\begin{pmatrix}2 & 2 & 0\\ 2 & 0 & -2\\ 0 & -2 & 2\end{pmatrix}$ we can see that few elements are zero even though the vectors are non-zero. Therefore, **Option 4** is correct.

TABLE 3.16.2: Solution

3.17. Assume that a non-singular matrix

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \tag{3.17.1}$$

where $\mathbf{L}$ and $\mathbf{U}$ are lower and upper triangular matrices respectively with all diagonal entries are zero, and $\mathbf{D}$ si a diagonal matrix. Let $\mathbf{x}^*$ be the solution of $\mathbf{Ax} = \mathbf{b}$. Then the Gauss-Seidel iteration method

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k + \mathbf{c}, k = 0, 1, 2, \dots \tag{3.17.2}$$

with $\|\mathbf{H}\| < 1$ converges to $\mathbf{x}^*$ provided $\mathbf{H}$ is equal to

a) $-\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$
b) $-(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$
c) $-\mathbf{D}(\mathbf{L} + \mathbf{U})^{-1}$
d) $-(\mathbf{L} - \mathbf{D})^{-1}\mathbf{U}$

3.18. Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array} \tag{3.18.1}$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

a) $d(1) = 1$
b) $d(1) = 2$
c) $\pi_1 = \frac{1}{2}$
d) $\pi_1 = \frac{1}{3}$

Solution:

a) The period of state 1 i.e, $d(1)$ is given as:

$$d(1) = GCD\{n : P_{11}^n > 0\} \tag{3.18.2}$$

For $n = 1$,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \tag{3.18.3}$$

(3.18.4)

For $n = 2$,

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \tag{3.18.5}$$

(3.18.6)

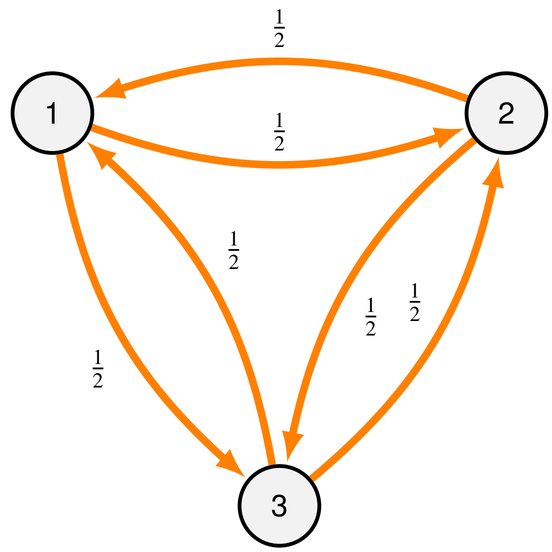


Fig. 3.18.1: State transition diagram

For $n = 3$,

$$\mathbf{P}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \tag{3.18.7}$$

(3.18.8)

For $n = 4$,

$$\mathbf{P}^4 = \begin{pmatrix} \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{8} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{pmatrix} \tag{3.18.9}$$

Thus P_{11}^n follows the sequence, that is defined as:

$$P_{11}^n = \begin{cases} 0, & \text{if } n = 1 \\ \frac{1}{2}, & \text{if } n = 2 \\ \frac{1}{2}(P_{11}^{n-1} + P_{11}^{n-2}), & \text{if } n > 2 \end{cases} \tag{3.18.10}$$

Since, for $n > 1$, $P_{11}^n > 0$

$$d(1) = GCD\{2, 3, 4, 5 \cdots\} \tag{3.18.11}$$

$$\therefore d(1) = 1 \tag{3.18.12}$$

Thus statement a is correct

b) As calucalted above in 3.18.12, $d(1) = 1$
Thus statement b is incorrect.

c) For stationary distribution,

$$\sum_{i=1}^{i=n} \pi_i = 1 \tag{3.18.13}$$

$$\implies \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = 1 \tag{3.18.14}$$

Also for a stationary distribution,

$$\pi \mathbf{P} = \pi \tag{3.18.15}$$

$$(\pi \mathbf{P})^T = \pi^T \tag{3.18.16}$$

$$\mathbf{P}^T \pi^T = \pi^T \tag{3.18.17}$$

$$\implies (\mathbf{P}^T - \mathbf{I})\pi^T = 0 \tag{3.18.18}$$

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \tag{3.18.19}$$

The given equation 3.18.14, 3.18.19 can be written as:

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.18.20}$$

We need to solve the augmented matrix to row reduced echelon form to get the solution,

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \xleftrightarrow{R_4 = R_4 + R_1} \tag{3.18.21}$$

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \xleftrightarrow{R_1 = -R_1} \tag{3.18.22}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \xleftrightarrow{R_2 = R_2 - \frac{R_1}{2}, R_3 = R_3 - \frac{R_1}{2}} \tag{3.18.23}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array}\right) \xleftrightarrow{R_3 = R_3 + R_2, R_4 = R_4 + 2R_2} \tag{3.18.24}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \xleftrightarrow{R_2 = -\frac{4}{3} R_2} \tag{3.18.25}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \xleftrightarrow{R_1 = R_1 + \frac{1}{2} R_2} \tag{3.18.26}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftrightarrow R_4} \tag{3.18.27}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftrightarrow{R_3 = \frac{R_3}{3}} \tag{3.18.28}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \xleftrightarrow{R_1 = R_1 + R_3, R_2 = R_2 + R_3} \tag{3.18.29}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{3.18.30}$$

Hence,

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3} \tag{3.18.31}$$

Thus statement c is incorrect

d) As, calculated in 3.18.31, $\pi_1 = \frac{1}{3}$
Thus statement d is correct

Hence, statements a and d are correct.

4 December 2017

4.1. Let $\mathbf{A}$ be a real symmetric matrix and $\mathbf{B} = \mathbf{I} + i\mathbf{A}$, where $i^2 = -1$. Then choose the correct option.

a) $\mathbf{B}$ is invertible if and only if $\mathbf{A}$ is invertible.
b) All Eigenvalues of $\mathbf{B}$ are necessarily real.
c) $\mathbf{B} - \mathbf{I}$ is necessarily invertible.
d) $\mathbf{B}$ is necessarily invertible.

Solution: See Table 4.1.1.

Statement 1.	**B** is invertible if and only if **A** is invertible.
False statement	Matrix **B** is invertible even if **A** is non invertible.
Example:	Consider a matrix $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.1.1}$$ a real non invertible,symmetric matrix. $$\implies \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix} \tag{4.1.2}$$ is invertible even if **A** is non invertible.
Statement 2.	All Eigenvalues of **B** are necessarily real.
False statement	Matrix **B** can have complex Eigenvalues.
Proof :	Eigen values of **B** = Eigen values of (**I**) + i (Eigen values of **A**). Clearly from (4.1.2) above Eigen values of **B** are 1 and $1 + i$ respectively. Hence **B** can also have complex Eigen value.
Statement 3.	$\mathbf{B} - \mathbf{I}$ is necessarily invertible.
False statement	$\mathbf{B} - \mathbf{I} = i\mathbf{A}$ will be invertible if **A**, is invertible.
Proof:	We have $\mathbf{B} - \mathbf{I} = i\mathbf{A}$ $\implies \mathbf{B} - \mathbf{I} = i\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}$,from (4.1.1) Hence $\mathbf{B} - \mathbf{I}$ is not invertible,unless **A** is invertible.
Statement 4.	**B** is necessarily invertible.
Correct Statement:	Matrix **B** has non zero Eigen values corresponding to Eigenvector X .
Proof:	Let X be an Eigen vector of **A** corresponding to Eigen value λ also,$\lambda \epsilon \mathbb{R}$ $\implies \mathbf{A}X = \lambda X$ $\therefore \mathbf{B}X = (\mathbf{I} + i\mathbf{A})X = \mathbf{I}X + i\mathbf{A}X = X + i\lambda X$ $\implies \mathbf{B}X = (1 + i\lambda)X$ Therefore, $1 + i\lambda$ is an Eigen value of **B**, corresponding to Eigen vector X,which are non zero. Hence, **B** is necessarily invertible.

TABLE 4.1.1: Solution summary

4.2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then the smallest positive integer n such that $\mathbf{A}^n = \mathbf{I}$ is

Solution: *Property of eigen values of A:* Let $\mathbf{A}$ be an arbitary $n \times n$ matrix of complex numbers with eigen values $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the eigen values of kth power of $\mathbf{A}$, that is the eigen values of $\mathbf{A}^k$, for any positive integer k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. Let us calculate the eigen values of $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.2.1}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{4.2.2}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \tag{4.2.3}$$

$$-\lambda(1-\lambda) + 1 = 0 \tag{4.2.4}$$

$$\lambda^2 - \lambda + 1 = 0 \tag{4.2.5}$$

$$\implies \lambda = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.2.6}$$

From the above property, the eigen values of $\mathbf{A}^n$ are λ^n. Also as it is given that $\mathbf{A}^n = \mathbf{I}$,

$$\implies \lambda^n = 1 \tag{4.2.7}$$

$$\implies \left(\frac{-1 \pm \sqrt{3}i}{2}\right)^n = 1 \tag{4.2.8}$$

Clearly $n \neq 1$. For $n = 2$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^2 = \frac{-1 \mp \sqrt{3}i}{2} \tag{4.2.9}$$

For $n = 4$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^4 = \frac{-1 \pm \sqrt{3}i}{2} \tag{4.2.10}$$

For $n = 6$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2}\right)^6 = 1 \tag{4.2.11}$$

Hence $n = 6$ is the smallest positive integer.

4.3. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$. Then the system $\mathbf{AX} = \mathbf{b}$ over the real numbers has

a) No solution when $\beta \neq 7$
b) Infinite number of solutions when $\alpha \neq 2$
c) Infinite number of solutions when $\alpha = 2$ and $\beta \neq 7$
d) A unique solution if $\alpha \neq 2$

Solution: First we derive the Row Reduced Echelon Form (RREF) of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ as follows,

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{pmatrix} \xleftrightarrow[R_3 = R_3 - 2R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \tag{4.3.1}$$

$$\xleftrightarrow{R_2 = \frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \tag{4.3.2}$$

$$\xleftrightarrow{R_1 = R_1 + R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \tag{4.3.3}$$

$$\xleftrightarrow{R_3 = R_3 - 5R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha-2 & \beta-7 \end{pmatrix} \tag{4.3.4}$$

From the RREF of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ in (4.3.4) we make the following observations for different values of α and β in Table 4.3.1. ,

Values	Observations
$\beta \neq 7$	Then the existence of solution and the number of solutions will entirely depend on value of α
$\alpha = 2$ $\beta \neq 7$	Then RREF in (4.3.4) will contain Zero Row in R_3. Moreover solvability condition will not satisfy. $\implies$ system will have Zero solutions
$\alpha \neq 2$	RREF in (4.3.4) will have all pivots $\implies$ RREF in (4.3.4) will be fullrank $\implies$ $\mathbf{AX} = \mathbf{b}$ have unique solution.

TABLE 4.3.1

Hence, if $\alpha \neq 2$ then the system $\mathbf{AX} = \mathbf{b}$ has unique solution.

4.4. Consider a Markov chain $\{X_n | n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then, $P(X_3 = 1 \mid X_0 = 1)$ equals

Solution: The three step transitional probabilities are given as,

$$P(X_3 = j \mid X_0 = i) = P(X_{n+3} = j \mid X_n = i) = (\mathbf{P}^3)_{ij}\ for\ any\ n \quad (4.4.1)$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{4} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \quad (4.4.2)$$

From (4.4.2),

$$P(X_3 = 1 \mid X_0 = 1) = (\mathbf{P}^3)_{11} = \frac{1}{4} \quad (4.4.3)$$

4.5. Let $\mathbf{A}$ be an $m \times n$ matrix with rank r. If the linear system $\mathbf{AX} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

a) $m = r$

b) the column space of $\mathbf{A}$ is a proper subspace of $\mathbf{R}^m$

c) the null space of $\mathbf{A}$ is a non-trivial subspace of $\mathbf{R}^n$ whenever $m = n$

d) $m \geq n$ implies $m = n$

Solution: *Theorem*

Theorem 4.1. *Consider the $m \times n$ system $Ax = b$, with either $b \neq 0$ or $b = 0$. We distinguish the following cases:*

a) ***Unique Solution****: If rank[A,b] = rank(A) = $n \leq m$, then and only then the system has a unique solution. In this case, indeed as many as $m - n$ equations are redundant. And the solution* $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$. *This is called as* ***Exactly Determined****.*

b) ***No Solution****: If rank[A,b] > rank(A) which necessarily implies* $\mathbf{b} \neq 0$ *and $m >$ rank(A), then and only then the system has no solution. This is called as* ***Overdetermined****.*

See Table 4.5.1 If the columns of an $m \times n$ matrix $\mathbf{A}$ span $\mathbf{R}^m$ then the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of $\mathbf{A}$ is defined to be

$$Null(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} = 0\} \quad (4.5.1)$$

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \quad (4.5.2)$$

Reduced Row Echelon form is

$$RREF(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.5.3)$$

$\therefore$ the only possible nullspace of the matrix $\mathbf{A}$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $\mathbf{B}$ be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \quad (4.5.4)$$

Reduced Row Echelon form is

$$RREF(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.5.5)$$

$\therefore$ the rank of matrix $\mathbf{B} = 3$.

4.6. Let $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right.$ and eigen values of $\mathbf{A} \in \mathbb{Q}$

a) $\mathbf{M}$ is empty

b) $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$

c) If $\mathbf{A} \in \mathbf{M}$ then the eigen values of $\mathbf{A} \in \mathbb{Z}$

d) If $\mathbf{A},\mathbf{B} \in \mathbf{M}$ such that $\mathbf{AB}=\mathbf{I}$ then $\left|\mathbf{A}\right| \in \{+1,-1\}$

Solution: See Table 4.6.1.

Options	Observations
$m = r$	The rank of any matrix **A** is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of $\mathbf{R}^m$. $\therefore$ This statement is **True**.
the column space of **A** is a proper subspace of $\mathbf{R}^m$	Any subspace of a vector space **V** other than **V** itself is considered a proper subspace of **V**. Which means that linear combination of **A** will span less than m. That will make the resultant **b** span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting. $\therefore$ This statement is **False**.
the null space of **A** is a non-trivial subsapce of $\mathbf{R}^n$ whenever $m = n$	From (4.5.2) we see that even when $m = n$ then also we are getting a trivial nullspace. $\therefore$ This statement is **False**.
$m \geq n$ implies $m = n$	It is given that the number of rows are greater than the column, and it is given that there exists a solution. If we refer to theorem (4.1) we see that the corresponding system will be **Exactly Determined** system. As an example, it will look like (4.5.4). $\therefore$ This statement is **True**.

TABLE 4.5.1: Solution

M is empty	Consider **A**=**I**=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.The elements of **A** $\in \mathbb{Z}$ and it's eigen values $1 \in \mathbb{Q}$. So, **M** is not empty.
M $=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$	Let **A**=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where elements of **A** $\in \mathbb{Z}$.The characteristic equation can be written as : $$\lambda^2 + 1 = 0 \implies \lambda = \pm i$$

	We see that $\lambda \in \mathbb{C}$ which is contradicting the main definition of $\mathbf{M}$ .So,this is not correct.
Eigen values of $\mathbf{A} \in \mathbb{Z}$	Given ,$\mathbf{A} \in \mathbf{M}$.Let λ_1,λ_2 be the eigen values of $\mathbf{A}$.The characteristic polynomial can be written as: $$\lambda^2 - tr(\mathbf{A})\lambda + \det\mathbf{A} = 0 \text{ where } tr(\mathbf{A}) = \lambda_1 + \lambda_2, \det\mathbf{A} = \lambda_1\lambda_2$$ Given the eigen values λ_1,λ_2 $\in \mathbb{Q}$,For this to be possible the discriminant of above equation should $\in \mathbb{Z}$ $$\sqrt{(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2} \in \mathbb{Z}$$ $$\implies \sqrt{(\lambda_1-\lambda_2)^2} \in \mathbb{Z}$$ $$\implies \lambda_1 - \lambda_2 \in \mathbb{Z} \text{ This is possible when both } \lambda_1,\lambda_2 \in \mathbb{Z}.$$
If **AB**=**I** then $\left\|\mathbf{A}\right\| \in$ {+1,-1}	As **A**,**B** $\in$ **M**, $\implies$ $\left\|\mathbf{A}\right\|$,$\left\|\mathbf{B}\right\| \in \mathbb{Z}$ Given **AB**=**I** $\implies$ $\left\|\mathbf{A}\right\|\left\|\mathbf{B}\right\|$=1 This is possible only when $\left\|\mathbf{A}\right\|$=$\left\|\mathbf{B}\right\|$= $\pm$ 1
Conclusion	options 3) and 4) are correct.

TABLE 4.6.1: Solution

4.7. Let A be a 3×3 matrix with real entries.Identify the correct statements.

a) A is necessarily diagonalizable over **R**
b) If A has distinct real eigen values than it is diagonalizable over**R**
c) If A has distinct eigen values than it is diagonalizable over **C**
d) If all eigen values are non zero than it is diagonalizable over **C**

Solution: See Table 4.7.1.

Statement 1.	A is necessarily diagonalizable over **R**
False statement Example:	Matrix A is diagonalizable if and only if there is a basis of $\mathbf{R}^3$consisting of eigenvectors of A. Consider a matrix $$\begin{pmatrix}1&1&0\\0&1&1\\0&0&4\end{pmatrix} \tag{4.7.1}$$ Eigen values are: $$\begin{pmatrix}1-\lambda&1&0\\0&1-\lambda&1\\0&0&4-\lambda\end{pmatrix}=0. \implies \lambda_1=1,\lambda_2=4 \tag{4.7.2}$$ $$\lambda_1 = 1 \text{ has eigen vector} \begin{pmatrix}1\\0\\0\end{pmatrix} \text{and} \lambda_2 = 4 \text{ has eigen vector} \begin{pmatrix}1\\3\\9\end{pmatrix} \tag{4.7.3}$$ We have found only two linearly independent eigenvectors for A,not diagonalisable
Statement 2.	If A has distinct real eigen values than it is diagonalizable over**R**
True statement	Distinct real eigenvalues implies linearly independent eigenvectors . and if a matrix has n linearly independent vectors than it is diagonalizable.
Proof 1: Proof 2:	**Distinct eigen values implies linearly independent vectors that spans entire space.** Consider 2 eigen vectors **v**,**w** with eigen values λ,μ respectively. such that $\lambda \neq \mu$ $$\alpha(\mathbf{v})+\beta(\mathbf{w})=0 \tag{4.7.4}$$ $$\alpha A(\mathbf{v})+\beta A(\mathbf{w})=0 \tag{4.7.5}$$ $$\alpha\lambda\mathbf{v}+\beta\mu\mathbf{w}=0 \tag{4.7.6}$$ Multiplying (4.7.4)with -λ and subtracting from (4.7.6) we have, $$\beta(\mu-\lambda)\mathbf{w}=0 \tag{4.7.7}$$ eigen values are distinct $(\mu-\lambda) \neq 0$. From equation(4.7.7) we have, $\beta = 0$ substituting $\beta = 0$ in equation (4.7.4)we have,$\alpha = 0$.As, $\mathbf{v} \neq 0$ **which proves that vectors are linearly independent**. **If a matrix has n linearly independent vectors than it is diagonalizable** If$\begin{pmatrix}\mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n}\end{pmatrix}$are n independent eigen vectors then, $A\mathbf{p_1} = \lambda\mathbf{p_1}, \cdots, A\mathbf{p_n} = \lambda\mathbf{p_n}$ $$D=\begin{pmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\\vdots&\vdots&\vdots&\vdots\\0&0&\cdots&\lambda_n\end{pmatrix} P=\begin{pmatrix}\mathbf{P_1}&\mathbf{P_2}&\cdots&\mathbf{P_n}\end{pmatrix} \tag{4.7.8}$$ Now, $A\mathbf{P_i} = \lambda_i\mathbf{P_i} \implies AP = PD$

	so,$P^{-1}AP = D$ is a diagonal matrix.
Statement 3.	If A has distinct real eigen values than it is diagonalizable over**C**
True statement . Proof: Example:	If A is an $N \times N$ complex matrix with n distinct eigenvalues, then any set of n corresponding eigenvectors form a basis for $\mathbf{C}^n$ It is sufficient to prove that the set of eigenvectors is linearly independent which is proved in statement 2. $$A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad (4.7.9)$$ Eigen values of A are: $$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6 \quad (4.7.10)$$
	Eigen vectors are: $$x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad (4.7.11)$$ Matrix A is diagonalizable because there is a basis of $\mathbf{C}^3$consisting of eigenvectors of A.
Statement 4.	If all eigen values are non zero than it is diagonalizable over **C**
False Statement:	Matrix would be diagonalizable if and only if it has linearly independent eigenvectors .
Example:	Consider a matrix $$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.7.12)$$ Eigen values are: $$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \neq 0 \quad (4.7.13)$$ $$\lambda_1 = 1 \text{ has eigen vector} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{and} \lambda_2 = 4 \text{ has eigen vector} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \quad (4.7.14)$$ We have found only two linearly independent eigenvectors for A,not diagonalisable.

TABLE 4.7.1: Solution summary

Given	V be a vector space over C of all the polynomials in a variable X of degree atmost 3 $D : P_3 \rightarrow P_3$ $D : V \rightarrow V$ be the linear operator given by differentiation wrt X $D(P(x)) \rightarrow P'(x)$ A be the matrix of D wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$

TABLE 4.8.1

4.8. Let V be a vector space over C of all the polynomials in a variable X of degree atmost 3. Let $D : V \rightarrow V$ be the linear operator given by differentiation with respect to X. Let A be the matrix of D with respect to some basis for V. Which of the following are true?

a) A is nilpotent matrix

b) A is diagonalizable matrix

c) the rank of A is 2

d) the Jordan canonical form of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: See Tables 4.8.1, 4.8.2 and 4.8.3

4.9. For every 4×4 real symmetric non-singular matrix $\mathbf{A}$ there exists a positive integer p such that

a) $p\mathbf{I} + \mathbf{A}$ is positive definite
b) $\mathbf{A}^p$ is positive definite
c) $\mathbf{A}^{-p}$ is positive definite
d) $\exp(p\mathbf{A}) - \mathbf{I}$ is positive definite

Solution: A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal,that is its eigen value decomposition is

$$\mathbf{A} = \mathbf{PDP}^T \tag{4.9.1}$$

$\mathbf{D}$ is the diagonal matrix containing the real eigen values of $\mathbf{A}$

$\mathbf{P}$ has the corresponding eigen vectors

$$\mathbf{PP}^T = \mathbf{P}^T\mathbf{P} = \mathbf{I} \tag{4.9.2}$$

A real matrix is positive definite if

$$\mathbf{x}^T\mathbf{Ax} > 0 \tag{4.9.3}$$
$$\implies \mathbf{x}^T\lambda\mathbf{x} > 0 \tag{4.9.4}$$
$$\implies \lambda\mathbf{x}^T\mathbf{x} > 0 \tag{4.9.5}$$
$$\implies \lambda > 0 \tag{4.9.6}$$

In other words, all the eigen values of A are positive See Table 4.9.1
Let $\mathbf{A}$ be

$$\mathbf{A} = \mathbf{PDP}^T \tag{4.9.7}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \tag{4.9.8}$$

From the table,the choices would be option 1,2,3

4.10. Let $\mathbf{A}$ be an $m \times n$ matrix of rank m with $n > m$. If for some non-zero real number α, we have $\mathbf{x^T A A^T x} = \alpha \mathbf{x^T x}$, for all $x \in \mathbf{R^m}$, then $\mathbf{A^T A}$ has,

a) exactly two distinct eigenvalues.
b) 0 as an eigenvalue with multiplicity $n - m$.
c) α as a non-zero eigenvalue.
d) exactly two non-zero distinct eigenvalues.

Solution: Refer Table 4.10.1.
Refer Table 4.10.2.

4.11. Consider a Markov chain with five states

$\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \qquad (4.11.1)$$

Which of the following are true?

a) 3 and 1 are in the same communicating class
b) 1 and 4 are in the same communicating class
c) 4 and 2 are in the same communicating class
d) 2 and 5 are in the same communicating class

Solution: See Tables 4.11.1 and 4.11.2

Matrix	$D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$ $D(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$ $D(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$ $D(x^2) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ $D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$ $D(x^3) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ Matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Inference	An $n \times n$ matrix with λ as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial $(A - \lambda I)^n$
Nilpotent	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ All eigen values of matrix A is 0 Thus, above matrix is nilpotent matrix Thus, above statement is true

TABLE 4.8.2

Diagonalizable	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $Rank(A) + nullity(A) =$ no of column $Rank(A) = 3$, no of column $= 4$ $nullity(A) = 4 - 3 = 1$ means there exists only one linearly independent eigen vector corresponding to 0 eigen values Thus,matrix A is not Diagonalizable. Thus, above statement is false
Rank	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Rank of matrix A is 3 Thus, above statement is false
Jordan CF	Assume characteristic polynomial of matrix A is $c_A(x)$ $c_A(x) = x^4$ Assume minimal polynomial of A is $m_A(x)$ $m_A(x)$ always divide $c_A(x)$ $m_A(x) = \{x, x^2, x^3, x^4\}$ Minimal polynomial always annihilates its matrix.Thus,we see that $m_A(A) = \{A = 0, A^2 = 0, A^3 = 0, A^4 = 0\}$ But we see that neither A is zero matrix nor A^2 and A^3 equal to zero but A^4 is equal to zero.Thus,x^4 is minimal polynomial. Algebraic Multiplicity $= a_M(\lambda = 0) = 4$ Geometric Multiplicity $= g_M(\lambda = 0) =$ $nullity(A - 0I) = nullity(A) = 1$ Hence, Jordan form of block size 4 Using Inference, $\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ $\lambda = 0$ $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which is same as given in the question. Thus,statement is true

<table>
<tr><th>OPTIONS</th><th>DERIVATIONS</th></tr>
<tr><td>Choice 1</td><td>

$$p\mathbf{I} + \mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{4.9.9}$$

$$= \mathbf{P}\mathbf{D}_1\mathbf{P}^T \tag{4.9.10}$$

$$\mathbf{D}_1 = \begin{pmatrix} \lambda_1 + p & 0 & 0 & 0 \\ 0 & \lambda_2 + p & 0 & 0 \\ 0 & 0 & \lambda_3 + p & 0 \\ 0 & 0 & 0 & \lambda_4 + p \end{pmatrix} \tag{4.9.11}$$

Some of the eigen values of $\mathbf{A}$ may be negative.
All the eigen values in $\mathbf{D}_1$ are positive only if

$$p > |\lambda_i| \ \forall i \in [1, 4] \tag{4.9.12}$$

</td></tr>
<tr><td>Choice 2</td><td>

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} \tag{4.9.13}$$

$$= (\mathbf{P}\mathbf{D}\mathbf{P}^T)(\mathbf{P}\mathbf{D}\mathbf{P}^T) \tag{4.9.14}$$

$$= \mathbf{P}\mathbf{D}^2\mathbf{P}^T \tag{4.9.15}$$

$$\text{Similarly, } \mathbf{A}^p = \mathbf{P}\mathbf{D}^p\mathbf{P}^T \tag{4.9.16}$$

$$\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \tag{4.9.17}$$

$\mathbf{A}^p$ is positive definite only if p is even.

</td></tr>
<tr><td>Choice 3</td><td>

$$\mathbf{A}^{-p} = \mathbf{P}\mathbf{D}^{-p}\mathbf{P}^T \tag{4.9.18}$$

$$\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \tag{4.9.19}$$

$\mathbf{A}^{-p}$ is positive definite only if p is even.

</td></tr>
<tr><td>Choice 4</td><td>

$$\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \tag{4.9.20}$$

$$\implies \exp(p\mathbf{A}) - \mathbf{I} = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{P}\mathbf{I}\mathbf{P}^T \tag{4.9.21}$$

$$= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \tag{4.9.22}$$

$$\exp(p\mathbf{D}) - \mathbf{I} = \begin{pmatrix} e^{\lambda_1} - 1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} - 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} - 1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4} - 1 \end{pmatrix} \tag{4.9.23}$$

$\mathbf{A}$ is non-singular

$$\implies \forall i \in [1, 4], \lambda_i \neq 0 \tag{4.9.24}$$

$$e^{\lambda_i} < 1 \tag{4.9.25}$$

So, $\exp(p\mathbf{A}) - \mathbf{I}$ is not positive definite.

</td></tr>
</table>

TABLE 4.9.1: Solution

Given	Derivation
Given	$\mathbf{A}$ is a $m \times n$ matrix of rank m with $n > m$. A non-zero real number α. To find eigenvalues of $\mathbf{A^TA}$.
Eigenvalues of $\mathbf{AA^T}$	$\mathbf{AA^T}$ is a $m \times m$ matrix and $\mathbf{A^TA}$ is a $n \times n$ matrix. Let, λ be a non-zero eigen value of $\mathbf{A^TA}$. $\mathbf{A^TAv} = \lambda\mathbf{v} \quad \mathbf{v} \in \mathbf{R^n} \qquad (4.10.1)$ $\mathbf{AA^TAv} = \lambda\mathbf{Av} \qquad (4.10.2)$ $\text{Let,} \quad \mathbf{x} = \mathbf{Av} \quad \mathbf{x} \in \mathbf{R^m} \qquad (4.10.3)$ $\mathbf{AA^Tx} = \lambda\mathbf{x} \qquad (4.10.4)$ $\mathbf{x^TAA^Tx} = \lambda\mathbf{x^Tx} \qquad (4.10.5)$ $\text{Given,} \quad \mathbf{x^TAA^Tx} = \alpha\mathbf{x^Tx} \qquad (4.10.6)$ $\implies \alpha\mathbf{x^Tx} = \lambda\mathbf{x^Tx} \qquad (4.10.7)$ From equation (4.10.7), $\lambda = \alpha$ as $\lVert\mathbf{x}\rVert \neq 0$ As rank($\mathbf{A^TA}$) = rank($\mathbf{A}$) = m and equation (4.10.7) satisfies the condition in question. Therefore the only non-zero eigen value is α $\mathbf{A^TA}$ has an eigenvalue α with multiplicity m.
Eigenvalues of $\mathbf{A^TA}$	$\mathbf{A^TA}$ is a $n \times n$ matrix. Given $n > m$, We know that, $\mathbf{A^TA}$ and $\mathbf{AA^T}$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero. 1. From above, as α is non-zero, $\mathbf{A^TA}$ has α as its eigenvalue with multiplicity m 2. $\mathbf{A^TA}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A^TA}$ are α and 0.

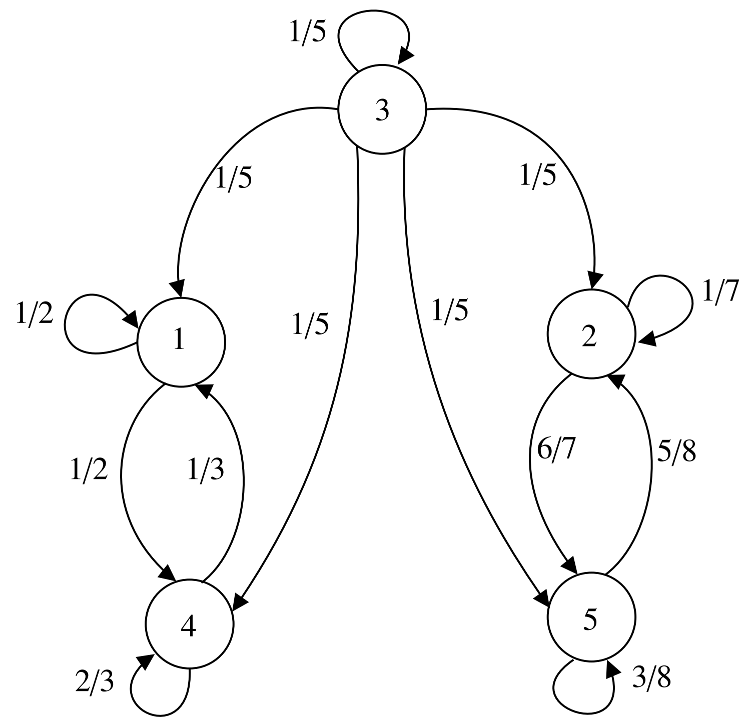
TABLE 4.10.1: Explanation

$\mathbf{A^TA}$ has exactly two distinct eigenvalues.	True statement
$\mathbf{A^TA}$ has 0 as an eigenvalue with multiplicity $n - m$	True statement
$\mathbf{A^TA}$ has α as a non-zero eigenvalue	True statement
$\mathbf{A^TA}$ has exactly two non-zero distinct eigenvalues.	False statement

TABLE 4.10.2: Solution

Accessibility of states in Markov's chain	We say that state j is accessible from state i, written as $i \to j$, if $p_{ij}^{(n)} > 0$ for some n. Every state is accessible from itself since $p_{ii}^{(0)} = 1$
Communication between states	Two states i and j are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words, $i \leftrightarrow j \text{ means } i \to j \text{ and } j \to i.$
Communicating class	For each Markov chain, there exists a unique decomposition of the state space S into a sequence of disjoint subsets $C_1, C_2, ...,$ $S = \bigcup_{i=1}^{\infty} C_i$ in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.

TABLE 4.11.1: Definition and Result used

<table>
<tr><td>Drawing Transition diagram</td><td>
</td></tr>
<tr><td>Checking whether the states 3 and 1 are in the same communicating class</td><td>Here,
State 1 is accessible from the state 3.
But, State 3 is not accessible from the state 1
i.e. $3 \to 1,\ 1 \nrightarrow 3$
$\implies \boxed{3 \nleftrightarrow 1}$

Therefore, 3 and 1 are not in the same communicating class.</td></tr>
<tr><td>Checking whether the states 1 and 4 are in the same communicating class</td><td>Here,
State 1 is accessible from the state 4.
Also, State 4 is accessible from the state 1
i.e. $3 \to 1,\ 1 \to 3$
$\implies \boxed{3 \leftrightarrow 1}$

Therefore, 1 and 4 are in the same communicating class.</td></tr>
<tr><td>Checking whether the states 4 and 2 are in the same communicating class</td><td>Here,
State 2 is not accessible from the state 4.
Also, State 4 is not accessible from the state 2
i.e. $4 \nrightarrow 2,\ 2 \nrightarrow 4$</td></tr>
</table>

	$\implies \boxed{4 \nleftrightarrow 2}$ Therefore, 4 and 2 are not in the same communicating class.
Checking whether the states 2 and 5 are in the same communicating class	Here, State 2 is accessible from the state 5. Also, State 5 is accessible from the state 2 i.e. $5 \to 2,\ 2 \to 5$ $\implies \boxed{2 \leftrightarrow 5}$ Therefore, 2 and 5 are in the same communicating class.
Conclusion	Communication classes are: $\boxed{S = \{1,4\} \cup \{3\} \cup \{2,5\}}$ Option 2) and 4) are true.

TABLE 4.11.2: Solution

5 June 2017

5.1. Let $\mathbf{A}$ be a 4×4 matrix. Suppose that the null space $N(\mathbf{A})$ of $\mathbf{A}$ is

$$\left\{(x,y,z,w)\in\mathbf{R}^4 : x+y+z=0, x+y+w=0\right\} \tag{5.1.1}$$

Then which one of the following is correct

a) dim(column space($\mathbf{A}$)) = 1
b) dim(column space($\mathbf{A}$)) = 2
c) rank($\mathbf{A}$) = 1
d) $\mathbf{S}=\{(1,1,1,0),(1,1,0,1)\}$ is a basis of $N(\mathbf{A})$

Solution: The nullspace is given by

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{5.1.2}$$

Row reducing the above matrix we get,

$$\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\xleftrightarrow[R_2\leftarrow R_2\times-1]{R_2\leftarrow R_2-R_1}\begin{pmatrix}1&1&1&0\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{5.1.3}$$

$$\xleftrightarrow{R_1\leftarrow R_1-R_2}\begin{pmatrix}1&1&0&1\\0&0&1&-1\\0&0&0&0\\0&0&0&0\end{pmatrix} \tag{5.1.4}$$

See Table 5.1.1

5.2. Let $\mathbf{A}$ and $\mathbf{B}$ be real invertible matrices such that

$$\mathbf{AB}=-\mathbf{BA}. \tag{5.2.1}$$

Then

a) trace$\mathbf{A}$ = trace($\mathbf{B}$) = 0
b) trace$\mathbf{A}$ = trace($\mathbf{B}$) = 1
c) trace$\mathbf{A}$ = 0, trace($\mathbf{B}$) = 1
d) trace($\mathbf{A}$) = 1, trace($\mathbf{B}$) = 0

Solution: See Tables 5.2.1 and 5.2.2

5.3. Let $\mathbf{A}$ be an n × n self-adjoint matrix with eigenvalues $\lambda_1,\cdots,\lambda_2$. Let,

$$\|\mathbf{X}\|_2=\sqrt{|\mathbf{X}_1^2|+\cdots+|\mathbf{X}_n^2|} \tag{5.3.1}$$

for $\mathbf{X}$=$(\mathbf{X}_1,\cdots,\mathbf{X}_n)\in\mathbb{C}^n$. If

$$p(\mathbf{A})=a_0\mathbf{I}+a_1\mathbf{A}+\cdots+a_n\mathbf{A}^n \tag{5.3.2}$$

then $sup_{\|\mathbf{X}\|_2=1}\|p(\mathbf{A})\mathbf{X}\|_2$ is equal to

Solution: We know that $\mathbf{A}$ is a self adjoint matrix and hence $\mathbf{A}=\mathbf{A}^*$ with eigen values $\lambda_1,\lambda_2,\cdots,\lambda_n$.Now as we are given,

$$p(\mathbf{A})=a_0\mathbf{I}+a_1\mathbf{A}+\cdots+a_n\mathbf{A}^n \tag{5.3.3}$$

then,

$$(p(\mathbf{A}))^*=a_0\mathbf{I}^*+a_1\mathbf{A}^*+\cdots+a_n(\mathbf{A}^*)^n \tag{5.3.4}$$

Since, $\mathbf{A}=\mathbf{A}^*$ we can state that,

$$p(\mathbf{A})(p(\mathbf{A}))^*=(p(\mathbf{A}))^*p(\mathbf{A}) \tag{5.3.5}$$

Hence p($\mathbf{A}$) is a normal matrix. Now using spectral theorem for a normal matrix,

$$\|p(\mathbf{A})\|_2=\rho(p(\mathbf{A})) \tag{5.3.6}$$

sup refers to the smallest element that is greater than or equal to every number in the set.Hence, sup of $\|p(\mathbf{A})\|_2$ will be,

$$=\max\{|\alpha| : \alpha \text{ is the eigen value of p(A)}\} \tag{5.3.7}$$

$$=\max\{|p(\lambda_j)| : j=1,2,\cdots n\} \tag{5.3.8}$$

$$=\max\{|a_0+a_1\lambda_j+\cdots+a_n\lambda_j{}^n| : j=1,2,\cdots n\} \tag{5.3.9}$$

Now, to find $sup\,\|p(\mathbf{A})\mathbf{X}\|_2$,

$$=max\{|a_0+a_1\lambda_j+\cdots+a_n\lambda_j{}^n| : j=1,2,\cdots n\}\,\|\mathbf{X}\|_2 \tag{5.3.10}$$

Since, we have to find $sup_{\|\mathbf{X}\|_2=1}$ i.e,

$$\|\mathbf{X}\|_2=\sqrt{|\mathbf{X}_1^2|+\cdots+|\mathbf{X}_n^2|}=1 \tag{5.3.11}$$

Hence the final answer will be,

$$=max\{|a_0+a_1\lambda_j+\cdots+a_n\lambda_j{}^n| : j=1,2,\cdots n\} \tag{5.3.12}$$

5.4. Let $p(x)=\alpha x^2+\beta x+\gamma$ be a polynomial, where $\alpha,\beta,\gamma\epsilon R$. Fix $X_0\epsilon R$. Let $S=\{(a,b,c)\,\epsilon R^3 : p(x)=a(x-x_0)^2+b(x-x_0)+c\}$ for all $x\epsilon R$. Find the number of elements in S is

a) 0
b) 1
c) Strictly greater than 1 but finite
d) Infinite

dim(C(**A**)) = 1	**False**. Because the number of pivot variables are 2 as obtained in (5.1.4)
dim(C(**A**)) = 2	**True**. Since the number of pivot variables are 2, the rank of **A** is 2. $\therefore dim(C(\mathbf{A})) = 2 \quad [\because dim(C(\mathbf{A})) = rank(\mathbf{A})]$
rank(**A**) = 1	**False**. Because the rank(**A**) = 2, as the number of pivot variables are 2
$\mathbf{S} = \{(1,1,1,0),(1,1,0,1)\}$ is a basis of $N(\mathbf{A})$	**False**. Let, $\mathbf{u} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \mathbf{v} = \begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ Consider, $\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\2\\0\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ Similarly, $\begin{pmatrix}1&1&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1\\1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\3\\0\\0\end{pmatrix} \neq \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ Hence, the given vectors do not form the basis.

TABLE 5.1.1

Definition	Matrix **A** is said to be similar to matrix **B** if there exists matrix **P** such that $\mathbf{A} = \mathbf{PBP}^{-1}$
Properties	Similar matrices have same eigenvalues Sum of eigenvalue of a matrix equals its trace From above two properties we can conclude that similar matrices have same trace

TABLE 5.2.1: Similar matrices and Properties

Solution:

$$p(x) = \alpha x^2 + \beta x + \gamma \tag{5.4.1}$$

$$\implies p(x) = (\alpha\beta\gamma)\left(x^2 x 1\right)^T \tag{5.4.2}$$

$$S = \{(a,b,c)\,\epsilon R^3 : p(x) = a(x-x_0)^2 + b(x-x_0) + c\},$$

$$\forall \mathbf{x} \epsilon R\,(FixX_0) \tag{5.4.3}$$

$$p(x) = (abc)\left((x-x_0)^2(x-x_0)1\right)^T \quad (5.4.4)$$
$$= a\left(x^2 - 2x_0x + x_0^2\right) + b(x-x_0) + c \quad (5.4.5)$$
$$= ax^2 + (b-2ax_0)x + \left(ax_0{}^2 - bx_0 + c\right) \quad (5.4.6)$$

Refer (5.4.2) and (5.4.6) and comparing the co-coefficients of powers of x,

$$\alpha = a, \beta = b - 2ax_0, \gamma = ax_0{}^2 - bx_0 + c \quad (5.4.7)$$
$$a = \alpha, b = \beta + 2\alpha x_0, c = \gamma - \alpha x_0{}^2 + (\beta + 2\alpha x_0)x_0 \quad (5.4.8)$$

Here α, β, γ and x_0 are the real fixed numbers. So a, b, c have unique values.
Hence S contain only 1 element.So option 2 is correct

5.5. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (5.5.1)$$

and $\mathbf{I}$ be the 3×3 identity matrix. If

$$6\mathbf{A}^{-1} = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} \quad (5.5.2)$$

for $a, b, c \in \mathbb{R}$ then (a,b,c) equals

a) (1,2,1)

b) (1,-1,2)

c) (4,1,1)

d) (1,4,1)

Solution: Finding the characteristic equation,

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} \quad (5.5.3)$$
$$\implies (1-\lambda)(-2-\lambda)(-3-\lambda) = 0 \quad (5.5.4)$$
$$\implies (\lambda^2 + \lambda - 2)(-3-\lambda) = 0 \quad (5.5.5)$$
$$\implies \lambda^3 + 4\lambda^2 + \lambda - 6 = 0 \quad (5.5.6)$$

Using Cayley-Hamilton Theorem we get,

$$\mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} - 6\mathbf{I} = 0 \quad (5.5.7)$$
$$\implies \mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} = 6\mathbf{I} \quad (5.5.8)$$
$$\implies \mathbf{A}(\mathbf{A}^2 + 4\mathbf{A} + \mathbf{I}) = 6\mathbf{I} \quad (5.5.9)$$

$\left|\mathbf{A}\right| = 6 \neq 0$ hence inverse exists. Hence (5.5.9) we get,

$$6\mathbf{A}^{-1} = \mathbf{A}^2 + 4\mathbf{A} + \mathbf{I} \quad (5.5.10)$$

Comparing (5.5.2) and (5.5.10) we get,

$$a = 1 \quad b = 4 \quad c = 1 \quad (5.5.11)$$

Hence $(a, b, c) = (1, 4, 1)$

5.6. Find the Eigenvalues of the matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 5 \\ 2 & 5 & -3 \end{pmatrix} \quad (5.6.1)$$

a) -4, 3, -3
b) 4, 3, 1
c) 4, -4$\pm\sqrt{13}$
d) 4, -2$\pm\sqrt{7}$

Solution: Using the characteristic equation of the matrix can find the Eigenvalues,

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = 0 \quad (5.6.2)$$
$$\implies \begin{vmatrix} \lambda-1 & -1 & -2 \\ -1 & \lambda+2 & -5 \\ -2 & -5 & \lambda+3 \end{vmatrix} = 0 \quad (5.6.3)$$

The expression that is obtained after expanding the determinant and simplifying it is,

$$(\lambda-1)(\lambda^2 + 5\lambda - 19) - (5\lambda + 31) = 0 \quad (5.6.4)$$

Further simplifying this we obtain the cubic equation,

$$\lambda^3 + 4\lambda^2 - 29\lambda - 12 = 0 \quad (5.6.5)$$

Solving this equation, the Eigenvalues obtained are,

$$\lambda_1 = -7.605, \ \lambda_2 = -0.394 \ and \ \lambda_3 = 4 \quad (5.6.6)$$

Therefore, the Eigenvalues of the given matrix are 4, -4 $\pm\sqrt{13}$ (Option 3)

5.7. Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers $a_0, a_1, \cdots, a_k$. For $p \in V$

$$max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} \quad (5.7.1)$$

defines a norm on V

a) only if $k < n$
b) only if $k \geq n$
c) if $k + 1 \leq n$

d) if $k \geq n+1$

Solution: Options 2 and 4 are correct as verified in the table 5.7.2
The scalar multiplication and triangle inequality properties holds true for all k.

$$max\left\{\left|\alpha p(a_j)\right|\right\} = |\alpha|\, max\left\{\left|p(a_j)\right|\right\} \quad (5.7.4)$$

$$max\left\{\left|p(a_i) + p(a_j)\right|\right\} \leq max\left\{|p(a_i)|\right\} + max\left\{\left|p(a_j)\right|\right\} \quad (5.7.5)$$

The positivity property holds true only if $k \geq n$ as more than n roots are possible when,

$$p(x) = 0 \implies \left|p(a_j)\right|_{0\leq j\leq k} = 0 \quad (5.7.6)$$

$$\implies max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} = 0 \quad (5.7.7)$$

5.8. Let **V** be the vector space of polynomials of degree at most 3 in a variable x with coefficients in $\mathbb{R}$. Let **T**=d/dx be the linear transformation of **V** to itself given by differentiation.

Which of the following are correct?

a) **T** is invertible
b) 0 is an eigenvalue of **T**
c) There is a basis with respect to which the matrix of **T** is nilpotent.
d) The matrix of **T** with respect to the basis $\left(1, 1+x, 1+x+x^2, 1+x+x^2+x^3\right)$ is diagonal.

Solution: See Tables 5.8.1 , 5.8.2 and 5.8.3.

trace($\mathbf{A}$) = 0 trace($\mathbf{B}$) = 0	From (5.2.1) we have $\mathbf{AB} = -\mathbf{BA}$ $\implies \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ So, matrix $\mathbf{A}$ and (-$\mathbf{A}$) are similar.$\therefore$ $trace(\mathbf{A}) = trace(-\mathbf{A})$ $\implies trace(\mathbf{A}) = 0$ Similarly From (5.2.1) we have $\mathbf{AB} = -\mathbf{BA}$ $\implies \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$ So, matrix $\mathbf{B}$ and (-$\mathbf{B}$) are similar.$\therefore$ $trace(\mathbf{B}) = trace(-\mathbf{B})$ $\implies trace(\mathbf{B}) = 0$ So this statement is true
trace($\mathbf{A}$) = 1 trace($\mathbf{B}$) = 1	From (5.2.1) we have $\mathbf{AB} = -\mathbf{BA}$ $\implies \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ So, matrix $\mathbf{A}$ and (-$\mathbf{A}$) are similar.$\therefore$ $trace(\mathbf{A}) = trace(-\mathbf{A})$ $\implies trace(\mathbf{A}) = 0.$ As trace($\mathbf{A}$) = 0 this statement is false
trace($\mathbf{A}$) = 0 trace($\mathbf{B}$) = 1	From (5.2.1) we have $\mathbf{AB} = -\mathbf{BA}$ $\implies \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$ So, matrix $\mathbf{B}$ and (-$\mathbf{B}$) are similar.$\therefore$ $trace(\mathbf{B}) = trace(-\mathbf{B})$ $\implies trace(\mathbf{B}) = 0.$ As trace($\mathbf{B}$) = 0 this statement is false
trace($\mathbf{A}$) = 1 trace($\mathbf{B}$) = 0	From (5.2.1) we have $\mathbf{AB} = -\mathbf{BA}$ $\implies \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ So, matrix $\mathbf{A}$ and (-$\mathbf{A}$) are similar.$\therefore$ $trace(\mathbf{A}) = trace(-\mathbf{A})$ $\implies trace(\mathbf{A}) = 0.$ As trace($\mathbf{A}$) = 0 this statement is false

TABLE 5.2.2: Calculation of trace

Properties	**Norm** $\forall x \in V$
Positivity	$\|\|x\|\| \geq 0, \|\|x\|\| = 0 \iff x = 0$
Scalar Multiplication	$\|\|\alpha x\|\| = \|\alpha\| \, \|\|x\|\|, \alpha \in F$
Triangle Inequality	$\|\|x + y\|\| \leq \|\|x\|\| + \|\|y\|\|$

TABLE 5.7.1: Properties of Norm

<table>
<tr><td colspan="2">For $p \in V$ then the norm, $max\left\{\left|p(a_j)\right| : 0 \leq j \leq k\right\} = 0 \iff \left|p(a_j)\right|_{0 \leq j \leq k} = 0$</td></tr>
<tr><td>Conditions</td><td>Explanation</td></tr>
<tr><td>only if $k < n$

Example:</td><td>A polynomial doesn't necessarily have k distinct real roots,
i.e., it may have repeated, complex roots.
let p be polynomial of degree $n = 2$ and $k = 1$ given by:-
$$p(x) = x^2 + 4x + 4 \tag{5.7.2}$$
$$\left|p(a_j)\right|_{0 \leq j \leq 1} = 0 \implies a_0 = -2, a_1 = -2 \tag{5.7.3}$$
but $a_0, a_1, \cdots, a_k$ should be distinct real numbers.
This contradicts the property of Norm. Thus condition fails.</td></tr>
<tr><td>only if $k \geq n$</td><td>p is a polynomial of degree ≤n,
it can't have more than n roots and is only possible when,
$p(x) = 0 \implies \left|p(a_j)\right|_{0 \leq j \leq k} = 0$
hence p is identically zero. Thus condition satisfies.</td></tr>
<tr><td>if $k + 1 \leq n$</td><td>Not a norm for $k < n$. Hence incorrect.</td></tr>
<tr><td>if $k \geq n + 1$</td><td>Norm for $k \geq n$. Hence correct.</td></tr>
</table>

TABLE 5.7.2: Verifying Positivity Property of Norm

Nilpotent Matrix	1. If all the eigen values of matrix is zero then it is said to nilpotent matrix 2. Determinant and trace of nilpotent matrix are always zero.
Invertible Matrix	A matrix is said to be invertible matrix if its determinant is non zero.
Diagonal matrix	diagonal matrix is a matrix in which the entries outside the main diagonal are all zero.

TABLE 5.8.1: Definition

Given	$T : P_3 \to P_3$ $T : V \to V$ be the linear operator given by differentiation wrt x $T(P(x)) \to P'(x)$ A be the matrix of T wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$

TABLE 5.8.2: Result used

Checking whether matrix of T is nilpotent	$T : V \to V$ $TP(x) = P'(x)$ Differentiating wrt x to find matrix A; $T(1) = 0 = a_1x + b_1x + c_1x^2 + d_1x^3$ $T(x) = 1 = a_2 + b_2x + c_2x^2 + d_2x^3$ $T(x^2) = 2x = a_3 + b_3x + c_3x^2 + d_3x^3$ $T(x^3) = 3x^2 = a_4 + b_4x + c_4x^2 + d_4x^3$ Representing A in matrix form ; $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ from the above matrix of T we can say it is nilpotent matrix.
Checking eigen value of matrix T	$A = \begin{pmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 2 & 0 \\ 0 & 0 & 0-\lambda & 3 \\ 0 & 0 & 0 & 0-\lambda \end{pmatrix}$ $\implies \lambda = 0$
Checking whether matrix of T is invertible	Since $\det A = 0$. Therefore matrix of T is not invertible
Checking whether Matrix of T is diagonal matrix	Let basis be $B' = \{1, 1+x, 1+x+x^2, 1+x+x^2+x^3\}$ Differentiating wrt x ;

	$T(1) = 0 = a_1x + b_1(1+x) + c_1(1+x+x^2) + d_1(1+x+x^2+x^3)$ $T(1+x) = 1 = a_2 + b_2(1+x) + c_2(1+x+x^2) + d_2(1+x+x^2x^3)$ $T(1+x+x^2) = 1+2x = a_3 + b_3(1+x) + c_3(1+x+x^2)$ $+d_3(1+x+x^2+x^3)$ $T(1+x+x^2+x^3) = 1+2x+3x^2 = a_4 + b_4(1+x) + c_4(1+x+x^2)$ $+d_4(1+x+x^2+x^3)$ $B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ above matrix is not a diagonal matrix
Conclusion	Thus we can conclude Option 2) and 3) are correct.

TABLE 5.8.3: Solution

5.9. Let m, n, r be natural numbers. Let A be an $m \times n$ matrix with real entries such that $(AA^t)^r = I$, where I is the $m \times m$ is identity matrix and A^t is the transpose of the matrix A. We can conclude that

a) $m = n$

b) AA^t is invertible

c) A^tA is invertible

d) if $m = n$, then A is invertible

Solution: Options 2) and 4) are correct. See Table 5.9.1

5.10. Let $\mathbf{A}$ be a $n \times n$ real matrix with $\mathbf{A}^2 = \mathbf{A}$. Then

a) the eigenvalues of $\mathbf{A}$ are either 0 or 1
b) $\mathbf{A}$ is a diagonal matrix with diagonal entries 0 or 1
c) $rank(\mathbf{A}) = trace(\mathbf{A})$
d) if $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A})$

Solution: See Table 5.10.1

5.11. For any $n \times n$ matrix B, let $N(B) = \{X \in \mathbb{R}^n : BX = 0\}$ be the null space of B. Let A be a 4×4 matrix with $dim(N(A - 4I)) = 2, dim(N(A - 2I)) = 1$ and $rank(A) = 3$ Which of the following are true?

a) 0,2 and 4 are eigenvalues of A
b) determinant(A)=0
c) A is not diagonalizable
d) trace(A)=8

Option	Answer
1) $m = n$	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $r = 1$ $(\mathbf{AA^T})^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ Since $m \neq n$ Option 1 is False.
2) AA^t is invertible	w.k.t $\det(A^n) = (\det(A))^n$ Since $(AA^t)^r = I$ So $\det((AA^T)^r) = \det(\mathrm{I})$ $(\det(AA^T))^r = 1$ $\implies \det(AA^T) \neq 0$ Hence AA^T is invertible Option 2 is True.
3) A^tA is invertible	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $r = 1$ $(\mathbf{A^TA})^r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ But $\det(AA^T) = 0$. $\implies AA^T$ is not invertible. Hence Option 3 is False
4) if $m = n$ then A is invertible	Since $\det(AA^T) \neq 0$ $\det(\mathrm{A}).\det(A^T) \neq 0$ $\det(\mathrm{A}).\det(\mathrm{A}) \neq 0$ $\implies A$ is invertible. Hence Option 4 is True

TABLE 5.9.1

Solution: See Table 5.11.1.

Given	A is a 4×4 matrix. $dim\,(N\,(A - 2I)) = 2$, $dim\,(N\,(A - 4I)) = 1$, and $rank\,(A) = 3$
Eigenvalues of a matrix	The number λ is an eigenvalue of a matrix A if and only if$A - \lambda I$ is singular,

i.e. $\left|A - \lambda I\right| = 0$

For $\lambda = 2$
Given, $dim\,(N\,(A - 2I)) = 2$
$\implies nullity(A - 2I) = 2$
$rank(A) + nullity(A) = n$
$\implies rank\,(A - 2I) = 4 - 2 = 2$
$\implies$ $(A - 2I)$ is not a full rank matrix
Therefore $\left|A - 2I\right| = 0$

Also,
$\implies N\,(A - 2I) = \{X \in \mathbb{R}^4 : (A - 2I)X = 0\}$
$\implies$ $(A - 2I)X = 0$ gives two eigen vectors
$\implies$ 2 is an eigenvalue of A with multiplicity 2.

Similarly, for $\lambda = 4$
Given, $dim\,(N\,(A - 4I)) = 1$
$\implies rank\,(A - 4I) = 4 - 1 = 3$
$\implies$ $(A - 4I)$ is not a full rank matrix

	Therefore $\left\|A - 4I\right\| = 0$ $\implies$ 4 is an eigenvalue of A with multiplicity 1. For $\lambda = 0$ Given that $rank\,(A) = 3$ $\implies$ A is not a full rank matrix Therefore $\left\|A\right\| = 0$ $\implies$ 0 is an eigenvalue of A with multiplicity 1.
Determinant	Given that $rank\,(A) = 3$ $\implies$ A is not a full rank matrix Therefore $\left\|A\right\| = 0$
Diagonalizability	An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen vectors. $rank\,(A) + nullity\,(A) = n$ $\implies$ for $\lambda = 0$, $nullity\,(A - \lambda I) = nullity\,(A) = 4 - 3 = 1$ $\implies$ There exists only one linearly independent eigen vector corresponding to 0 eigen value Thus, matrix A is not diagonalizable.
Trace	$Trace\,(A)$=sum of eigen values $\implies$ $Trace\,(A) = 0 + 2 + 2 + 4 = 8$
Conclusion	Option (1), (2) and (4) are correct

TABLE 5.11.1: Solution

5.12. Which of the following 3x3 matrices are diagonizable over $\mathbb{R}$?

a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: See Tables 5.12.1 and 5.12.2

Objective	Explanation
Eigenvalues of $\mathbf{A}$	Since $$\mathbf{A}^2 = \mathbf{A} \quad (5.10.1)$$ $$\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \quad (5.10.2)$$ From Cayley-Hamilton Theorem we have, $$\lambda^2 - \lambda = 0 \quad (5.10.3)$$ $$\implies \lambda(\lambda - 1) = 0 \quad (5.10.4)$$ $$\implies \lambda = 0, 1 \quad (5.10.5)$$ A matrix $\mathbf{A}$ satisfying $\mathbf{A}^2 = \mathbf{A}$ is an idempotent matrix with eigen values equal to 0 or 1.
Check if $\mathbf{A}$ is necessary diagonal	Consider $$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.6)$$ $$(5.10.7)$$ Then, $$\mathbf{A}^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.8)$$ $$= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.9)$$ $$= \mathbf{A} \quad (5.10.10)$$ Hence $\mathbf{A}$ is idempotent but not diagonal.
Relation between rank and trace of $\mathbf{A}$	Rank of matrix is defined as the number of non-zero eigenvalues. Since number of non-zero eigenvalues is 1, $$rank(\mathbf{A}) = 1 \quad (5.10.11)$$ $$trace(\mathbf{A}) = \sum_i \lambda_i = 0 + 1 = 1 \quad (5.10.12)$$ $$\implies rank(\mathbf{A}) = trace(\mathbf{A}) \quad (5.10.13)$$
Relation between rank and trace of $\mathbf{I} - \mathbf{A}$	Now for the matrix $\mathbf{I} - \mathbf{A}$ we have, $$(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \quad (5.10.14)$$ $$= \mathbf{I}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{A}^2 \quad (5.10.15)$$ $$= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \quad (5.10.16)$$ $$= \mathbf{I} - \mathbf{A} \quad (5.10.17)$$ Hence $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Therefore we conclude, $$rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \quad (5.10.18)$$
Answer	(1),(3) and (4)

TABLE 5.10.1

Test for diagonalizability	Let $\mathbf{W}_i$ be the eigenspace corresponding to eigenvalue λ_i of $\mathbf{A}$ 1)$\mathbf{A}$ is diagonalizable 2) characteristic polynomial of $\mathbf{A}$ is $f = (\mathbf{x} - \lambda_1)^{d_1}....(\mathbf{x} - \lambda_k)^{d_k}$ and $dim(\mathbf{W}_i) = d_i$ 3) $\sum_{i=1}^{k} \mathbf{W_i} = n$
Concept for diagonalization	A linear operator $\mathbf{A}$ on a n-dimensional space $\mathbb{V}$ is diagonalizable , if and only if $\mathbf{A}$ has n distinct characteristic vectors or null spaces corresponding to the characteristic values

TABLE 5.12.1: Illustration of theorem.

Option A	Given matrix is $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $\mathbf{A}$ is $det(x\mathbf{I} - \mathbf{A})$ $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-1) & -3 & -2 \\ 0 & (x-4) & -5 \\ 0 & 0 & x-6 \end{vmatrix}$ Characteristic Polynomial $= (x-1)(x-4)(x-6)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(xI - A) = 0$ which gives $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6$ Thus over $\mathbb{R}$ matrix $\mathbf{A}$ has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $dim\mathbf{W}_i = d_i$ 3) $\sum_i \mathbf{W_i} = n = 3$, which is equal to dim of $\mathbf{A}$.

Conclusion on Option A	Option A satisfy all three condition of Diagonalizability over $\mathbb{R}$.
Option B	Given matrix is $\mathbf{A}=\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $det(x\mathbf{I} - \mathbf{A})$ $det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x & -1 & 0 \\ 1 & x & 0 \\ 0 & 0 & x-1 \end{vmatrix}$ Characteristic Polynomial $= (x-1)(x+i)(x-i)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is not the product of linear factors over $\mathbb{R}$ beacuse roots of characteristic eq are complex . Thus $\mathbf{A}$ is not diagonalizable over $\mathbb{R}$.
Conclusion on Option B	Option B does not satisfy condition 1.
Option C	Given matrix is $\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $\mathbf{A}$ is $det(x\mathbf{I} - \mathbf{A})$ $det(x\mathbf{I} - \mathbf{A})= \begin{vmatrix} (x-1) & -2 & -3 \\ -2 & (x-1) & -4 \\ -3 & -4 & x-1 \end{vmatrix}$ Characteristic Polynomial $= (x+3.19)(x+0.877)(x-7.07)$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial are product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $det(x\mathbf{I} - \mathbf{A}) = 0$ which gives $\lambda_1 = -3.19, \lambda_2 = -0.887, \lambda_3 = 7.07$

	Thus over $\mathbb{R}$ matrix $\mathbf{A}$ has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $dim\mathbf{W}_i = d_i$ 3) $\sum_i \mathbf{W_i} = n = 3$, which is equal to dim of $\mathbf{A}$.
Conclusion on Option C	Option C satisfy all three condition of Diagonalizability over $\mathbb{R}$.
Option D	Given matrix is $\mathbf{A}=\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Finding Characteristics polynomial	Characteristics polynomial of the matrix $\mathbf{A}$ is $det(x\mathbf{I} - \mathbf{A})$ $det(x\mathbf{I} - \mathbf{A})= \begin{vmatrix} x & -1 & -2 \\ 0 & x & -1 \\ 0 & 0 & x \end{vmatrix}$ Characteristic Polynomial $= (x)(x)(x) = x^3$
Testing diagonalizability over $\mathbb{R}$	1) As the characteristics polynomial is product of linear factors over $\mathbb{R}$. 2) To find characteristic values of the operator $\det(x\mathbf{I} - \mathbf{A}) = 0$ $\lambda_1 = 0$ $d_1 = 3$ $\mathbf{W}_1 = \mathbf{A} - \lambda_1\mathbf{I} \implies \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $dim\mathbf{W}_1 = 2$ $dim\mathbf{W}_i \neq d_i$ Algebric Multiplicity is not equal to Geometric Multiplicity.
Conclusion on Option D	Option D does not satisfy second condition of Diagonalizability.
Answer	Option A and Option C are Diagonalizable over $\mathbb{R}$.

TABLE 5.12.2: Option Checking Table

Positive Semi Definite Matrix	A $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive semi definite if $\mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} \geq 0$ for all non-zero $\mathbf{x}$ in $\mathbb{R}^n$. Formally $\mathbf{M} \text{ is positive semi-definite} \iff \mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} \geq 0 \ \forall \ \mathbf{x} \in \mathbb{R}^n \backslash \{0\}$
Theorem	For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L}(\mathbf{V})$, following are equivalent. 1). $\mathbf{x}^{\mathbf{T}}\mathbf{M}\mathbf{x} \geq 0 \ \forall \ \mathbf{x} \in \mathbf{V}$. 2). All the eigenvalues of $\mathbf{M}$ are non-negative.

TABLE 5.13.1: Definition and Result used

Calculating eigen values of $\mathbf{A}$	Given $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Calculating, eigen values of $\mathbf{A}$, ie $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ $\implies \left\lvert\begin{pmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{pmatrix}\right\rvert = 0$ $\implies (3-\lambda)\left((2-\lambda)(1-\lambda)-9\right) - 1\left(1-\lambda-6\right) + 2\left(3-2(2-\lambda)\right) = 0$ $\implies \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$ $\implies \lambda_1 = 6, \lambda_2 = \sqrt{3} \text{ and } \lambda_3 = -\sqrt{3}$ Hence, $\mathbf{A}$ has exactly two positive eigen values.
Proving $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ using contradiction	Suppose $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Then, by theorem above in definition section, matrix $\mathbf{A}$ is positive semi definite. Hence, all the eigen values of $\mathbf{A}$ non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$. So, $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Similarly, as $\lambda_i \leq 0, \forall i$ is also not true, so $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Thus, $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$.
Correct Options	Hence, correct options are (1) and (4).

TABLE 5.13.2: Solution

5.13. Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q}(\mathbf{X}) = \mathbf{X}^{\mathbf{T}}\mathbf{A}\mathbf{X}$ for $\mathbf{X} \in \mathbb{R}^3$. Then

a) $\mathbf{A}$ has exactly two positive eigen values.
b) all the eigen values of $\mathbf{A}$ are positive.
c) $\mathbf{Q}(\mathbf{X}) \geq 0 \ \forall \ \mathbf{X} \in \mathbb{R}^3$
d) $\mathbf{Q}(\mathbf{X}) < 0$ for some $\mathbf{X} \in \mathbb{R}^3$

Solution: See Tables 5.13.1 and 5.13.2

5.14. Consider the matrix

$$A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}; x \in \mathbf{R}. \qquad (5.14.1)$$

Then,

a) A(x) has eigenvalue 0 for some $x \in \mathbf{R}$.
b) 0 is not an eigenvalue of A(x) for any $x \in \mathbf{R}$.
c) A(x) has eigenvalue 0 $\forall x \in \mathbf{R}$.
d) A(x) is invertible $\forall x \in \mathbf{R}$.

Solution: Let $\lambda = 0$ be an eigenvalue. Hence,

$$\left|A - \lambda I\right| = 0 \qquad (5.14.2)$$

$$\implies \left|A\right| = 0 \qquad (5.14.3)$$

$$\implies \left|A\right| = \begin{vmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{vmatrix} = 0 \qquad (5.14.4)$$

Performing row reduction we get,

$$\begin{vmatrix} 1+x^2 & 7 & 11 \\ 0 & \frac{2x^3-19x}{1+x^2} & \frac{4x^2-33x+4}{1+x^2} \\ 0 & 0 & \frac{26x^3-244x^2+538x-68}{2x^3-19x} \end{vmatrix} = 0 \qquad (5.14.5)$$

$$\implies 26x^3 - 244x^2 + 538x - 68 = 0 \qquad (5.14.6)$$

$$\implies x_1 = 6.01, x_2 = 3.23, x_3 = 0.13 \qquad (5.14.7)$$

See Table 5.14.1

6 December 2016

6.1. The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \qquad (6.1.1)$$

is

a) positive definite.
b) non-negative definite but not positive definite.
c) negative definite.
d) neither negative definite nor positive definite.

.

Solution:

a) For a real symmetric matrix to be positive definite the eigen values of the matrix should

OPTIONS	Explanation
Option (b)	At the Values of x given by (5.14.7), eigen value $\lambda = 0$. Hence option (b) can't be correct.
Option (c)	If one of the eigenvalue is 0 for A(x) then, $\lvert A(x)\rvert = 0 \forall x \in R$. But from (5.14.7) we have concluded that $\lvert A\rvert = 0$ only for, $x_1 = 6.01, x_2 = 3.23, x_3 = 0.13$. Hence, Option (c) is incorrect.
Option (d)	Now for the values of x given by (5.14.7), $\lvert A\rvert = 0$. Hence it is not invertible $\forall x \in \mathbf{R}$ Hence Option (d) is incorrect.
Option (a)	Now clearly from above arguments A(x) has eigenvalue 0 for some $x \in R$ Hence Option (a) is Correct.

TABLE 5.14.1

be positive.

b) For a real symmetric matrix to be negative definite the eigen values of the matrix should be negative.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

The characteristic equation of the matrix **A**is given by

$$\left|V - \lambda\mathbf{I}\right| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0 \tag{6.1.2}$$

The Eigen values of **A** are:

$$\lambda_1 = 5/2$$
$$\lambda_2 = 3/2 \tag{6.1.3}$$
$$\lambda_3 = 4$$

Since all the eigen values of matrix **A** are positive, Therefore the matrix **A** is positive definite.

6.2. Let $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x,y) = \left(x^2, y^2 + \sin x\right)$. Then the derivative of f at (x,y) is the linear transformation given by

a) $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$

b) $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$

c) $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$

d) $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

Solution: Let $f_1 = x^2$ and $f_2 = y^2 + \sin x$. Begin by finding the derivative of f(x,y)

$$Df(x,y) = \begin{pmatrix} Df_1x & Df_1y \\ Df_2x & Df_2y \end{pmatrix} \tag{6.2.1}$$
$$= \begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix} \tag{6.2.2}$$

So option 1 is correct.

Now to prove that Derivatives is a linear transformation we dwell on the definition of linear transformation that it satisfies two properties i.e additivity and homogeneity as $\mathbb{R}^n \rightarrow \mathbb{R}^m$

$$D(cf) = cD(f) \tag{6.2.3}$$
$$D(f+g) = D(f) + D(g) \tag{6.2.4}$$

Now refer (6.2.3) we proceed as

$$D(cf) = \begin{pmatrix} Dcf_1 & Dcf_1 \\ Dcf_2 & Dcf_2 \end{pmatrix} \tag{6.2.5}$$
$$= c\begin{pmatrix} Df_1 & Df_1 \\ Df_2 & Df_2 \end{pmatrix} \tag{6.2.6}$$
$$= cD(f) \tag{6.2.7}$$

Now refer (6.2.4) we proceed as

$$D(f+g) = \begin{pmatrix} D(f_1+g_1) & D(f_1+g_1) \\ D(f_2+g_2) & D(f_2+g_2) \end{pmatrix} \tag{6.2.8}$$
$$\begin{pmatrix} Df_1 & Df_1 \\ Df_2 & Df_2 \end{pmatrix} + \begin{pmatrix} Dg_1 & Dg_1 \\ Dg_2 & Dg_2 \end{pmatrix} \tag{6.2.9}$$
$$= D(f) + D(g) \tag{6.2.10}$$

Hence both properties are satisfied so we can say that it is a linear transformation

6.3. Which of the following subsets of $\mathbb{R}^4$ is a basis of $\mathbb{R}^4$?

$\mathbf{B_1}=\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1\end{pmatrix}$

$\mathbf{B_2}=\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0\\ 1 & 2 & 3 & 0\\ 1 & 2 & 3 & 4\end{pmatrix}$

$\mathbf{B_3}=\begin{pmatrix}1 & 2 & 0 & 0\\ 0 & 0 & 1 & 1\\ 2 & 1 & 0 & 0\\ -5 & 5 & 0 & 0\end{pmatrix}$

a) $\mathbf{B_1}$ and $\mathbf{B_2}$ but not $\mathbf{B_3}$.
b) $\mathbf{B_1}$,$\mathbf{B_2}$, and $\mathbf{B_3}$.
c) $\mathbf{B_1}$ and $\mathbf{B_3}$ but not $\mathbf{B_2}$.
d) Only $\mathbf{B_1}$.

Solution: See Table 6.3.1

Statement	Solution
Definition	Let $\mathbf{V}$ be a vector space. Then $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is called a basis for $\mathbf{V}$ if the following conditions hold. $$\text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} = \mathbf{V} \tag{6.3.1}$$ $$\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \text{ is linearly independent} \tag{6.3.2}$$
Given	$$\mathbf{B_1} = \begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix}, \mathbf{B_2} = \begin{pmatrix}1&1&1&1\\0&2&2&2\\0&0&3&3\\0&0&0&4\end{pmatrix}, \mathbf{B_3} = \begin{pmatrix}1&0&2&-5\\2&0&1&5\\0&1&0&0\\0&1&0&0\end{pmatrix} \tag{6.3.3}$$
Checking $\mathbf{B_1}$	$$\text{Checking for linear independence.Upon row reducing } \mathbf{B_1} \tag{6.3.4}$$ $$\begin{pmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix} \xleftrightarrow{R_1\to R_1-R_2, R_2\to R_2-R_3, R_3\to R_3-R_4} \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{6.3.5}$$ Clearly Rank of $\mathbf{B_1}$ is 4,ie full rank.Hence it forms a Basis.
Checking $\mathbf{B_2}$	$$\text{Checking for linear independence.Upon row reducing } \mathbf{B_2} \tag{6.3.6}$$ $$\begin{pmatrix}1&1&1&1\\0&2&2&2\\0&0&3&3\\0&0&0&4\end{pmatrix} \xleftrightarrow{R_2\to \frac{R_2}{2}, R_1\to R_1-R_2, R_3\to \frac{R_3}{3}, R_2\to R_2-R_3, R_4\to \frac{R_4}{4}, R_3\to R_3-R_4} \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{6.3.7}$$ Rank of $\mathbf{B_2}$ is 4, ie full rank.Hence it also forms a Basis.
Checking $\mathbf{B_3}$	$$\text{Checking for linear independence.Upon row reducing } \mathbf{B_3} \tag{6.3.8}$$ $$\begin{pmatrix}1&0&2&-5\\2&0&1&5\\0&1&0&0\\0&1&0&0\end{pmatrix} \xleftrightarrow{R_2\to R_2-2R_1, R_4\to R_4-R_2, R_3\to -\frac{R_3}{3}, R_1\to R_1-2R_3} \begin{pmatrix}1&0&0&5\\0&1&0&0\\0&0&1&-5\\0&0&0&0\end{pmatrix} \tag{6.3.9}$$ Rank of $\mathbf{B_3}$ is 3, ie not full rank.Hence it does not forms a Basis.
Conclusion	Hence option 1, ie $\mathbf{B_1}$,$\mathbf{B_2}$ and not $\mathbf{B_3}$ is the correct answer.

TABLE 6.3.1: Solution

Given	a) Matrix J of $n{\times}n$ dimension with all entries 1. b) Matrix B of $3n{\times}3n$ dimension $$B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$$
Transforming matrix B into Block diagonal matrix using transformation Matrix	$$M = \mathbf{T}(B)$$ $$M = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$$ $$M = \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}$$
Rank of Block Diagonal matrix M	It is equal to the sum of rank of individual blocks in diagonal $$r(J) = 1$$ $$\therefore r(M) = 1 + 1 + 1 = 3$$
Rank of a matrix and its transformation are same.	$\therefore$ rank of matrix B is $$r(B) = r(M) = 3$$

TABLE 6.4.1

6.4. Let J denote the matrix of order $n \times n$ with all entries 1 and let B be a $3n \times 3n$ matrix given by $B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$.
Find rank of matrix B. **Solution:** See Tables 6.4.1 and 6.4.2

6.5. Which of the following sets of functions from $\mathbb{R}e$ to $\mathbb{R}e$ is a vector space over $\mathbb{R}e$?

$$S_1 = \{f \mid \lim_{x\to 3} f(x) = 0\} \qquad (6.5.1)$$

$$S_2 = \{g \mid \lim_{x\to 3} g(x) = 1\} \qquad (6.5.2)$$

$$S_3 = \{h \mid \lim_{x\to 3} h(x)\ exists\} \qquad (6.5.3)$$

is

a) Only S_1
b) Only S_2
c) S_1 and S_3 but not S_2
d) All the three are vector spaces

Solution: Let S be a set of functions. Let $f_1, f_2 \in S$ and $\alpha, \beta \in \Re$
For a set of functions to be considered as a vector space:

a) The linear combination of f_1 and f_2 should be in S.
i.e. $\alpha f_1(x) + \beta f_2(x) \in S$
b) The $\mathbf{0}$ should belong to S
i.e. $\mathbf{0} \in S$

Case1: Test for S_1

Example	Let $n = 2$ $$J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ $$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$
Transforming matrix B into Block diagonal matrix using transformation Matrix	$$M = \mathbf{T}(B)$$ $$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$ $$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$
Rank of Block Diagonal matrix M	It is equal to the sum of rank of individual blocks in diagonal $$r(J) = 1$$ $$\therefore r(M) = 1 + 1 + 1 = 3$$
Rank of a matrix and its transformation are same.	$\therefore$ rank of matrix B is $$r(B) = r(M) = 3$$

TABLE 6.4.2

a) Let $f_1, f_2 \in S_1$ and $\alpha, \beta \in \Re$

$$\lim_{x\to 3} f_1(x) = 0$$
$$\lim_{x\to 3} f_2(x) = 0 \tag{6.5.4}$$

Then Using (6.5.4)

$$\begin{aligned} &\lim_{x\to 3}(\alpha f_1(x) + \beta f_2(x)) \\ &= \alpha\left(\lim_{x\to 3} f_1(x)\right) + \beta\left(\lim_{x\to 3} f_2(x)\right) \\ &= \alpha \times 0 + \beta \times 0 \\ &= 0 \\ &\therefore \alpha f_1(x) + \beta f_2(x) \in S_1 \end{aligned}$$

b) Let $f(x) = 0$
then

$$\lim_{x\to 3} f(x) = 0$$
$$\therefore \mathbf{0} \in S_1$$

Hence, S_1 is a vector space.

Case2: Test for S_2

a) Let $g_1, g_2 \in S_2$ and $\alpha, \beta \in \Re$

$$\begin{aligned}\lim_{x\to 3} g_1(x) &= 1 \\ \lim_{x\to 3} g_2(x) &= 1\end{aligned} \tag{6.5.5}$$

Then Using (6.5.5)

$$\begin{aligned}&\lim_{x\to 3}(\alpha g_1(x) + \beta g_2(x)) \\ &= \alpha\left(\lim_{x\to 3} g_1(x)\right) + \beta\left(\lim_{x\to 3} g_2(x)\right) \\ &= \alpha \times 1 + \beta \times 1 \\ &= \alpha + \beta\end{aligned}$$
$$\therefore \alpha g_1(x) + \beta g_2(x) \in S_1 \quad iff \quad \alpha + \beta = 1$$

b) Let $g(x) = 0$
then

$$\lim_{x\to 3} g(x) = 1$$
$$\therefore \mathbf{0} \notin S_1$$

Hence, S_2 is not a vector space.

Case3: Test for S_3

a) Let $h_1, h_2 \in S_3$ and $\alpha, \beta \in \Re$

$$\begin{aligned}&\lim_{x\to 3} h_1(x)\ exists \\ &\lim_{x\to 3} h_2(x)\ exists\end{aligned} \tag{6.5.6}$$

Then Using (6.5.6)

$$\lim_{x\to 3}(\alpha h_1(x) + \beta h_2(x))\ exists$$
$$\therefore \alpha h_1(x) + \beta h_2(x) \in S_3$$

b) Let $h(x) = 0$
then

$$\lim_{x\to 3^-} h(x) = 0 = \lim_{x\to 3^+} h(x)$$
$$\therefore \mathbf{0} \in S_1$$

Hence, S_3 is a vector space.

Therefore, Option (3) is correct.

6.6. Let $\mathbf{A}$ be an n $\times$ m matrix with each entry equal to +1,-1 or 0 such that every column has exactly one +1 and exactly one -1. We can conclude that

1. Rank $\mathbf{A} \leq n - 1$ (6.6.1)
2. Rank $\mathbf{A} = m$ (6.6.2)
3. $n \leq m$ (6.6.3)
4. $n - 1 \leq m$ (6.6.4)

Solution: See Table 6.6.1

option	Solution
1.	Let us consider **A** as follows and let s be the summation of all column entries: $$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$ $$\left\|\mathbf{A} - \lambda\mathbf{I}\right\| = \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{pmatrix} = 0$$ $$= \begin{pmatrix} a_{11} + a_{21} + \dots + an1 - \lambda & a_{11} + a_{21} + \dots + an1 - \lambda & \dots & a_{11} + a_{21} + \dots + an1 - \lambda \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{pmatrix}$$ $$\implies (s - \lambda) \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{pmatrix} = 0$$
Example	Since s=0 according to question, Therefore $\lambda = 0$ is an eigen value of **A**. Since $\lambda = 0$, Hence **A** is singular. Which means at least two rows are linearly dependent. Therefore, $$\text{Rank}(\mathbf{A}) < n$$ $$\text{Rank}(\mathbf{A}) \leq n - 1$$ Let us Consider **A** as follows,where n=4 and m=3 $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$ Calculating Row Reduced Echelon Form of **A** as follows:

	$\xleftrightarrow[R_4 \leftarrow R_2+R_4]{R_4 \leftarrow R_1+R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ $\xleftrightarrow{R_4 \leftarrow R_3+R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Since the Rank **A**=3 and n=4, Therefore the Rank **A** $\leq n-1$ statement is true.
2.	Let us Consider **A** as follows,where n=2 and m=2 $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ Applying elementary transformations on **A** as follows: $\xleftrightarrow{R_2 \leftarrow R_1+R_2} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$
Conclusion	Since the Rank **A**=1 and m=2, Therefore the Rank **A** $\neq m$, Hence the statement is false.
3.	Let us Consider **A** as follows,where n=3 and m=2 $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \quad (6.6.5)$
Conclusion	Since there exists a matrix **A** when n>m, Therefore the statement is false.
4	Let us Consider **A** as follows,where n=4 and m=2 $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.6.6)$
Conclusion	Since there exists a matrix **A** when n-1>m, Therefore the statement is false.

TABLE 6.6.1: Solution summary

Option 1	To conclude that m = n
Assumptions	**For the example:** Without loss of generality, Let m = 2, n = 3 and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
Proof	We know that $\left(\mathbf{AA^t}\right)^r = \mathbf{I}$ which is a square matrix of order m × m For any natural value of r, a square matrix ($\mathbf{I}$) of order m × m is obtained Hence, we cannot conclude that m = n because we get $\mathbf{I}$ of order m × m even if m ≠ n. To illustrate this, Consider the following example $\mathbf{AA^t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ ($\mathbf{A}$ and $\mathbf{A^t}$ from Assumptions) $\left(\mathbf{AA^t}\right)^r = \mathbf{I}$ Here m ≠ n. Therefore, **Option 1** is incorrect

TABLE 6.7.1: Option 1

Option 2	To conclude that $\mathbf{AA^t}$ is invertible
Assumptions	$\mathbf{AA^t}$ is not invertible
Proof	$\implies \left\lvert\mathbf{AA^t}\right\rvert = 0 \implies \left\lvert\left(\mathbf{AA^t}\right)^r\right\rvert = 0$ $\implies \left(\mathbf{AA^t}\right)^r \neq \mathbf{I} \left(\left\lvert\mathbf{I}\right\rvert = 1\right)$ Since, this is a contradiction to the assumption made we can conclude that $\mathbf{AA^t}$ is invertible. Therefore, **Option 2** is correct

TABLE 6.7.2: Option 2

6.7. Let m, n and r be natural numbers. Let $\mathbf{A}$ be an m × n matrix with real entries such that $\left(\mathbf{AA^t}\right)^r = \mathbf{I}$, where $\mathbf{I}$ is the m × m identity matrix and $\mathbf{A^t}$ is the transpose of the matrix $\mathbf{A}$. We can conclude that

Options:

a) m = n
b) $\mathbf{AA^t}$ is invertible
c) $\mathbf{A^tA}$ is invertible
d) if m = n, then $\mathbf{A}$ is invertible

Solution: See Tables 6.7.1, 6.7.2, 6.7.3 and 6.7.4.

6.8. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let α_n and β_n denote the two eigenvalues of $\mathbf{A}^n$ such that $|\alpha_n| \geq |\beta_n|$. Then

a) $\alpha_n \mathbb{R}ightarrow\infty$ as $n\mathbb{R}ightarrow\infty$
b) $\beta_n \mathbb{R}ightarrow 0$ as $n\mathbb{R}ightarrow\infty$
c) β_n is positive if n is even.
d) β_n is negative if n is odd.

Solution: See Table 6.8.1.

6.9. Let M_n denote the vector space of all $n \times n$ real matrices. Which of the following is a linear subspaces of M_n :-

a) $V_1 = \{A \in M_n :$ A is nonsingular$\}$
b) $V_2 = \{A \in M_n : det(A) = 0\}$
c) $V_3 = \{A \in M_n : trace(A) = 0\}$
d) $V_4 = \{BA : A \in M_n\}$, where B is some fixed matrix in M_n

Solution: See Table 6.9.1

6.10. If $\mathbf{P}$ and $\mathbf{Q}$ are invertible matrices such that $\mathbf{PQ} = -\mathbf{QP}$,then we can conclude that

a) $Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 0$
b) $Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 1$
c) $Tr(\mathbf{P}) = -Tr(\mathbf{Q})$
d) $Tr(\mathbf{P}) \neq Tr(\mathbf{Q})$

Solution: See Table 6.10.1

Option 3	To conclude that $\mathbf{A^tA}$ is invertible
Assumptions	Without loss of generality, Let m = 2, n = 3 and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\implies \mathbf{A^t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
Proof	$\implies \mathbf{A^tA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \implies \left\|\mathbf{A^tA}\right\| = 0$ This means that $\mathbf{A^tA}$ is not invertible. Therefore, **Option 3** is incorrect

TABLE 6.7.3: Option 3

Option 4	To conclude that if m = n then **A** is invertible
Assumptions	Let m = n
Proof	Since $(\mathbf{AA^t})^r = \mathbf{I} \implies \left\|(\mathbf{AA^t})^r\right\| = \left\|\mathbf{I}\right\| = 1$ $\implies \left(\left\|\mathbf{A}\right\|\left\|\mathbf{A^t}\right\|\right)^r = 1$ (**A** is a square matrix) $\implies \left(\left\|\mathbf{A}\right\|\right)^{2r} = 1$ Therefore, **Option 4** is correct

TABLE 6.7.4: Option 4

Options	**Solutions**	**True/False**
1.	Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ Now lets find the eigen values of matrix **A** $\left\|\mathbf{A} - \lambda\mathbf{I}\right\| = 0$ $\implies \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$ $\implies \lambda^2 - \lambda - 1 = 0$ On solving we get 2 eigen values $\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \beta_1 = \frac{1-\sqrt{5}}{2}$ We know that if eigenvalue of **A** is λ then eigenvalue of $\mathbf{A}^n$ is λ^n. In this problem we can say that the eigenvalues α_n and β_n of $\mathbf{A}^n$ are $\alpha_n = \alpha_1^n \quad \beta_n = \beta_1^n$ Since $\alpha_1 > 1$ we can say that $\alpha_n \to \infty$ as $n \to \infty$.	True
2.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since $-1 < \beta_1 < 0$,we can say that $\beta_n \to 0$ as $n \to \infty$.	True
3.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since β_1 is negative because $-1 < \beta_1 < 0$, if n is even then β_n is positive.	True
4.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since β_1 is negative, if n is odd then β_n is negative.	True

TABLE 6.8.1

Vector space	Is it subspace to M_n?
1)V_1: All non-singular matrices of $n \times n$	The matrices $I_{n\times n}$ and $-I_{n\times n}$ are non-singular matrices, but the sum $I_{n\times n} - I_{n\times n}$ is zero matrix and it is singular. $\therefore V_1$ does not form subspace of M_n.
2)V_2: All singular matrices of $n \times n$	The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} and \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are singular matrices, but the sum is a non-singular matrix. $\therefore V_2$ does not form subspace M_n.
3)V_3: All matrices of $n \times n$ with trace =0	Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be matrices with Trace = 0. $Tr(\mathbf{v_1} + \alpha\mathbf{v_2}) = Tr(\mathbf{v_1}) + \alpha Tr(\mathbf{v_2}) = 0$. $\therefore$ the vector space V_3 forms linear subspace of M_n.
4)V_4: F_A = BA, where B is some fixed matrix in M_n	Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be matrices in the vector space V_4. $F_{v_1+\alpha v_2} = B(\mathbf{v_1} + \alpha\mathbf{v_2})$ $=B\mathbf{v_1} + \alpha B\mathbf{v_2} =$ $F_{v_1} + \alpha F_{v_2}$. $\therefore V_4$ forms linear subspace of M_n.

TABLE 6.9.1

Given	**P** and **Q** are invertible matrices. Therefore $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ exists.

	$$\mathbf{PQ} = -\mathbf{QP} \tag{6.10.1}$$
To Prove	Tr($\mathbf{P}$)=0
Proof 1	Post multiplying equation (6.10.1) by $\mathbf{Q}^{-1}$ we get, $$\mathbf{PQQ}^{-1} = -\mathbf{QPQ}^{-1} \tag{6.10.2}$$ $$\implies \mathbf{PI} = -\mathbf{QPQ}^{-1} \tag{6.10.3}$$ $$\implies \mathbf{P} = -\mathbf{QPQ}^{-1} \tag{6.10.4}$$ Taking trace on both sides for the equation (6.10.4), $$Tr(\mathbf{P}) = Tr(-\mathbf{QPQ}^{-1}) \tag{6.10.5}$$ $$\implies Tr(\mathbf{P}) = -Tr(\mathbf{QPQ}^{-1}) \tag{6.10.6}$$ We know that Tr($\mathbf{AB}$)=Tr($\mathbf{BA}$) Let $\mathbf{A}$=$\mathbf{Q}$ and $\mathbf{B}$=$\mathbf{PQ}^{-1}$
	From the above property of trace equation (6.10.6) can be modified as $$Tr(\mathbf{P}) = -Tr(\mathbf{PQ}^{-1}\mathbf{Q}) \tag{6.10.7}$$ $$\implies Tr(\mathbf{P}) = -Tr(\mathbf{PI}) \tag{6.10.8}$$ $$\implies Tr(\mathbf{P}) = -Tr(\mathbf{P}) \tag{6.10.9}$$ $$\implies 2Tr(\mathbf{P}) = 0 \tag{6.10.10}$$ $$\implies Tr(\mathbf{P}) = 0 \tag{6.10.11}$$
To Prove	Tr($\mathbf{Q}$)=0
Proof 2	Post multiplying equation (6.10.1) by $\mathbf{P}^{-1}$ we get, $$\mathbf{PQP}^{-1} = -\mathbf{QPP}^{-1} \tag{6.10.12}$$ $$\implies \mathbf{PQP}^{-1} = -\mathbf{QI} \tag{6.10.13}$$ $$\implies \mathbf{PQP}^{-1} = -\mathbf{Q} \tag{6.10.14}$$ Taking trace on both sides for the equation (6.10.14), $$Tr(\mathbf{PQP}^{-1}) = Tr(-\mathbf{Q}) \tag{6.10.15}$$ $$\implies Tr(\mathbf{PQP}^{-1}) = -Tr(\mathbf{Q}) \tag{6.10.16}$$ We know that Tr($\mathbf{AB}$)=Tr($\mathbf{BA}$) Let $\mathbf{A}$=$\mathbf{P}$ and $\mathbf{B}$=$\mathbf{QP}^{-1}$ From the above property of trace equation (6.10.16) can be modified as $$Tr(\mathbf{QP}^{-1}\mathbf{P}) = -Tr(\mathbf{Q}) \tag{6.10.17}$$ $$\implies Tr(\mathbf{QI}) = -Tr(\mathbf{Q}) \tag{6.10.18}$$ $$\implies Tr(\mathbf{Q}) = -Tr(\mathbf{Q}) \tag{6.10.19}$$ $$\implies 2Tr(\mathbf{Q}) = 0 \tag{6.10.20}$$ $$\implies Tr(\mathbf{Q}) = 0 \tag{6.10.21}$$
Statement 1	Tr($\mathbf{P}$)=Tr($\mathbf{Q}$)=0
Explanation	From equation (6.10.11) and (6.10.21) we could say that,

	$$Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 0 \qquad (6.10.22)$$ Valid Conclusion
Statement 2	Tr($\mathbf{P}$) = $Tr(\mathbf{Q}) = 1$
Explanation	From equation (6.10.11) and (6.10.21) we could say that, $$Tr(\mathbf{P}) = Tr(\mathbf{Q}) \neq 1 \qquad (6.10.23)$$ Invalid Conclusion
Statement 3	Tr($\mathbf{P}$) = $-Tr(\mathbf{Q})$
Explanation	Substituting the conclusion 1 result equation (6.10.22) in equation (6.10.9) we get, $$Tr(\mathbf{P}) = -Tr(\mathbf{Q}) \qquad (6.10.24)$$
	Valid Conclusion
Statement 4	Tr($\mathbf{P}$) $\neq Tr(\mathbf{Q})$
Explanation	From equation (6.10.11) and (6.10.21) we could say that, $$Tr(\mathbf{P}) = Tr(\mathbf{Q}) \qquad (6.10.25)$$ Invalid Conclusion

TABLE 6.10.1: Explanation with Proofs

Let n be an odd number ≥ 7.Let,

$$\mathbf{A} = [a_{ij}] \quad (6.10.26)$$

be and $n \times n$ matrix with,

$$a_{i,i+1} = 1, \forall (i = 1, 2, ...n - 1) \quad (6.10.27)$$

and $a_{n,1} = 1$. Let $a_{ij} = 0$ for all the other pairs (i, j). Then we can conclude that,

a) $\mathbf{A}$ has 1 as an eigenvalue
b) $\mathbf{A}$ has -1 as an eigenvalue
c) $\mathbf{A}$ has at least one eigenvalue with multiplicity ≥ 2
d) $\mathbf{A}$ has no real eigenvalues

Solution: We can represent our matrix as:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & ..... & 0 & 0 \\ 0 & 0 & 1 & ..... & 0 & 0 \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & ..... & 0 & 1 \\ 1 & 0 & 0 & ..... & 0 & 0 \end{pmatrix} \quad (6.10.28)$$

$$\mathbf{A}^{\mathbf{T}} = \begin{pmatrix} 0 & 0 & 0 & ..... & 0 & 1 \\ 1 & 0 & 0 & ..... & 0 & 0 \\ 0 & 1 & 0 & ..... & 0 & 0 \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & .... & 1 & 0 \end{pmatrix} \quad (6.10.29)$$

$\mathbf{A}$ is our given matrix.We know that Characteristic Equation of $\mathbf{A}$ and $\mathbf{A}^{\mathbf{T}}$ is same.Consider the minimal polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0 \quad (6.10.30)$$

We can represent it in $n \times n$ matrix with 1's on sub-diagonals and in last column it has negative of the coefficient,and rest all 0.We represent it using $\mathbf{C}$.It is known as the companion matrix.

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & .... & 0 & -a_0 \\ 1 & 0 & 0 & .... & 0 & -a_1 \\ 0 & 1 & 0 & .... & 0 & -a_2 \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & .... & 1 & -a_{n-1} \end{pmatrix} \quad (6.10.31)$$

(6.10.30) is also the characteristic equation of $\mathbf{C}$

Comparing (6.10.29) with (6.10.31) we get:

$$a_0 = -1, a_1 = a_2 = a_3 = a_4 = ... = a_{n-1} = 0 \quad (6.10.32)$$

Substituting (6.10.32) into (6.10.30) we get:

$$x^n - 1 = 0 \quad (6.10.33)$$

By Cayley-Hamilton Theorem:

$$\lambda^n - 1 = 0 \quad (6.10.34)$$

$$(6.10.35)$$

λ=n^{th} roots of unity. See Table 6.10.2.

6.11. Let $\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ be 3 distinct subspaces of $\mathbf{R}^{10}$ such that each $\mathbf{W_i}$ has dimension of 9. Let $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$. Then we can conclude that

a) $\mathbf{W}$ may not be a subspace of $\mathbf{R}^{10}$

b) dim $\mathbf{W} \leq 8$

c) dim $\mathbf{W} \geq 7$

d) dim $\mathbf{W} \leq 3$

Solution: See Table 6.11.1

Options	Explanation
A has 1 as an eigen value	One value out of the n^{th} roots of unity is 1.So,correct
A has -1 as an eigen value	Since,n is odd.So,-1 cannot be one of the value of n^{th} roots of unity. Hence,incorrect
A has atleast one eigenvalue with multiplicity ≥ 2	All values of n^{th} roots of unity are distinct. So there is no eigenvalue with multiplicity ≥ 2. Hence,incorrect.
A has no real eigen values	One of the value is 1,which is real. Hence,incorrect.

TABLE 6.10.2: Finding Correct Option

Given	$\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ are 3 distinct subspaces of $\mathbf{R}^{10}$ Each $\mathbf{W_i}$ has dimension 9 $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$
Statement1	$\mathbf{W}$ may not be a subspace of $\mathbf{R}^{10}$
Explanation	As $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$ and $\mathbf{W_1}$, $\mathbf{W_2}$, $\mathbf{W_3}$ are subspaces of $\mathbf{W}$,then $\mathbf{W}$ must be a subspace of $\mathbf{R}^{10}$. So the first option is false.
Statement2	dim $\mathbf{W} \leq 8$
Explanation	As $\mathbf{W}$ be a subspace of a finite dimension vector space $\mathbf{R}^{10}$ and dim $\mathbf{R}^{10}$= 10, so $\mathbf{W}$ is finite dimension and dim $\mathbf{W} \leq 10$
Theorem	dim $(\mathbf{W_1} \cap \mathbf{W_2})$ $=$ dim($\mathbf{W_1}$)+dim($\mathbf{W_2}$)-dim($\mathbf{W_1} + \mathbf{W_2}$) and $\mathbf{W_1} \cap \mathbf{W_2}$ is also a subspace of $\mathbf{R}^{10}$
Proof	The minimum dimension of $\mathbf{W} = \mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}$
Explanation	Let us consider $\mathbf{V} = \mathbf{R}^{10}$ and $dim(\mathbf{V}) = 10$ and $\mathbf{U} = \mathbf{W_1} \cap \mathbf{W_2}$

	So, $dim(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = dim(\mathbf{U})$ $+dim(\mathbf{W_3})$ - $dim(\mathbf{U} + \mathbf{W_3})$ **or,** $dim(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3}) = dim(\mathbf{W_1})$ $+dim(\mathbf{W_2})+dim(\mathbf{W_3})$ - $dim(\mathbf{W_1} + \mathbf{W_1})$ $-dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3})$
	Now, $(\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3} \subseteq \mathbf{V}$ $\implies dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3}) \leq dim(\mathbf{V})$ $\implies -dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3}) \geq -dim(\mathbf{V})$ Similarly, $(\mathbf{W_1} + \mathbf{W_2}) \subseteq \mathbf{V}$ $\implies dim(\mathbf{W_1} + \mathbf{W_2}) \leq dim(\mathbf{V})$ $\implies -dim(\mathbf{W_1} + \mathbf{W_2}) \geq -dim(\mathbf{V})$
	Considering these two inequations, $-dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3})-dim(\mathbf{W_1} + \mathbf{W_2})$ $\geq -2dim(\mathbf{V})$ **or,** $dim(\mathbf{W_1}) + dim(\mathbf{W_2}) + dim(\mathbf{W_3})$ $-dim((\mathbf{W_1} \cap \mathbf{W_2}) + \mathbf{W_3})-dim(\mathbf{W_1} + \mathbf{W_2})$ $\geq dim(\mathbf{W_1}) + dim(\mathbf{W_2}) + dim(\mathbf{W_3}) - 2dim(\mathbf{V})$ **or,** $dim(\mathbf{W_1} \cap \mathbf{W_2} \cap \mathbf{W_3})$ $\geq dim(\mathbf{W_1}) + dim(\mathbf{W_2}) + dim(\mathbf{W_3}) - 2dim(\mathbf{V})$ $\implies dim(\mathbf{W}) \geq dim(\mathbf{W_1}) + dim(\mathbf{W_2})$ $+dim(\mathbf{W_3}) - 2dim(\mathbf{V})$
Statement 3	dim $\mathbf{W} \geq 7$
Explanation	As $dim(\mathbf{W}) \geq dim(\mathbf{W_1}) + dim(\mathbf{W_2})$ $+dim(\mathbf{W_3}) - 2dim(\mathbf{V})$ $\implies dim(\mathbf{W}) \geq (9+9+9)$ - (2×10) $\implies dim(\mathbf{W}) \geq 7$
Answer	$7 \leq dim(\mathbf{W}) \leq 10$

TABLE 6.11.1: **Solution summary**

Hence, we can conclude that $dim(\mathbf{W}) \geq 7$.

Theorem	Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ where $\mathbf{A}$ s an $m \times n$ matrix. a) T is **one to one** if the columns of $\mathbf{A}$ are linearly independent , which happens precisely when $\mathbf{A}$ has a pivot position in every column. b) T is **onto** if an over $\mathbb{R}$ only if the span of the columns of $\mathbf{A}$ is $\mathbb{R}^n$, which happens precisely when $\mathbf{A}$ has a pivot position in every row.
$Range(\mathbf{T})$	It is column-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v}$$ where $\mathbf{x},\mathbf{v} \in \mathbf{V}$ and We can also say that $$Range(\mathbf{T}) = C(\mathbf{A})$$ where $C(\mathbf{A})$ is column space of $\mathbf{A}$.
$rank(\mathbf{T})$	$rank(\mathbf{T}) = rank(\mathbf{A})$

TABLE 8.1.1: Definitions and Theorem

7 June 2016

8 December 2015

8.1. Let $\mathbf{V}$ be the vector space of polynomials over $\mathbb{R}$ of degree less than or equal to n. For $p(x) = a_0 + a_{n-1}x + .. + a_n x^n$ in $\mathbf{V}$, define a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ by $(\mathbf{T}p)(x) = a_n + a_{n-1}x + .. + a_0 x^n$. Then

a) $\mathbf{T}$ is one to one.
b) $\mathbf{T}$ is onto.
c) $\mathbf{T}$ is invertible.
d) $\det \mathbf{T} = \pm 1$.

Solution: See Tables 8.1.2 and 8.1.2

Given	$\mathbf{V}$ be a vector space of polynomials over $\mathbb{R}$ of degree less then n $$p(x) = a_0 + a_{n-1}x + .. + a_n x^n$$ $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ $$(\mathbf{T}p)(x) = a_n + a_{n-1}x + .. + a_0 x^n$$
Explanation	We know that Basis for a polynomial vector space $P = (p_1, p_2, .., p_n)$ is a set of vectors that spans the space, and is linearly independent . $$\text{Basis} = \left(1, x, x^2, ..., x^n\right)$$ $$\begin{aligned} &\mathbf{T}(1) = x^n = 0.1 + 0.x + .. + 0.x^{n-1} + 1.x^n \\ &\mathbf{T}(x) = x^{n-1} = 0.1 + 0.x + .. + 1.x^{n-1} + 0.x^n \\ &\vdots \\ &\mathbf{T}(x^n) = 1 = 1.1 + 0.x + .. + 0.x^{n-1} + 0.x \end{aligned}$$ Expressing $\mathbf{T}$ in matrix form $$\mathbf{T} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$
Example	For Simplicity , Let $n = 3$ $$\implies p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$ $$\implies (\mathbf{T})p(x) = a_3 + a_2 x + a_1 x^2 + a_0 x^3$$ Basis $= \left(1, x, x^2, x^3\right)$ $$\mathbf{T}(1) = 0.0 + 0.x + 0.x^2 + 1.x^3$$ $$\mathbf{T}(x) = 0.0 + 0.x + 1.x^2 + 0.x^3$$ $$\mathbf{T}(x^2) = 0.0 + 1.x + 0.x^2 + 0.x^3$$ $$\mathbf{T}(x^3) = 1.1 + 0.x + 0.x^2 + 0.x^3$$ Expressing $\mathbf{T}$ in matrix form;

	$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
Statement 1:**T** is one to one	True
.	$\mathbf{T} : \mathbf{V} \to \mathbf{V}$ be a linear transformation **T** is one-to-one if and only if the nullity of **T** is zero. According to rank-nullity theorem. $dim(\mathbf{V}) = rank(\mathbf{T}) + nullity(\mathbf{T})$ $\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ Here, $dim(\mathbf{V}) = 4$ $rank(\mathbf{T})$ = no. of linearly independent column or row = 4 $\implies nullity(\mathbf{T}) = 0$ Thus, we can conclude **T** is one to one .
Statement 2:**T** is onto	True
	A matrix transformation is onto if and only if the matrix has a pivot position in each row, if the number of pivots is equal to the number of rows. $\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ $\implies rank(\mathbf{T}) = 4$ which is equal to no of rows. Thus, we can conclude **T** is onto.
Statement 3:**T** is invertible	True
	Theorem : A linear transformation $T : V \to W$ is **invertible** if there exists another linear transformation $U : W \to V$ such that UT is the *identity* transformation on V and TU is the *identity* transformation on W , where U is called Inverse of **T**. **T** is **invertible** if and only if **T** is $one - one$ and *onto*

	$$\Longrightarrow \ \mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$ $$\mathbf{T}^{-1} = \mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{T}$$ $$\mathbf{UT} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$ Thus, we can conclude $\mathbf{T}$ is invertible.
Statement 4: $\det \mathbf{T} = \pm 1$	True
	$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$, where $\mathbf{T}$ is a permutation matrix . A permutation matrix is nonsingular matrix, and determinant is ± 1. Permutation matrix $\mathbf{A}$ satisfies $\mathbf{AA}^T = \mathbf{I}$ Here, $$\mathbf{TT}^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$ $$\mathbf{TT}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$, also an Involutory matrix . **Involutory matrix**: an involutory matrix is a matrix that is its own inverse. That is, multiplication by matrix $\mathbf{A}$ is an involution if and only if $\mathbf{A}^2 = \mathbf{I}$ and **Determinant** of an involutory matrix over any field is ± 1 Since, $\mathbf{T}^{-1} = \mathbf{T}$ and $\mathbf{T}^2 = \mathbf{I}$ We can say $\mathbf{T}$ is also an Involutory matrix. Thus, we can conclude $\det \mathbf{T} = \pm 1$

TABLE 8.1.2: Solution Summary

8.2. Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbb{R}$. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation such that $rank(\mathbf{T}^2) = rank(\mathbf{T})$. Then,

a) $Kernel(\mathbf{T}^2) = Kernel(\mathbf{T})$
b) $Range(\mathbf{T}^2) = Range(\mathbf{T})$
c) $Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$.
d) $Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$.

Solution: See Tables 8.2.1, 8.2.2, 8.2.3 and 8.2.4

$Range(\mathbf{T})$	It is column-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{A}\mathbf{x} = \mathbf{v} \quad (8.2.1)$$ where $\mathbf{x},\mathbf{v} \in \mathbf{V}$ and We can also say that $$Range(\mathbf{T}) = C(\mathbf{A}) \quad (8.2.2)$$ where $C(\mathbf{A})$ is column space of $\mathbf{A}$.
$Kernel(\mathbf{T})$	It is null-space of linear operator $\mathbf{T}$. $$\mathbf{T}(\mathbf{x}) = 0 \implies \mathbf{A}\mathbf{x} = 0 \quad (8.2.3)$$ where $\mathbf{x} \in \mathbf{V}$ and matrix $\mathbf{A}$ is same as before. We can also say that $$Kernel(\mathbf{T}) = N(\mathbf{A}) \quad (8.2.4)$$ where $N(\mathbf{A})$ is null space of $\mathbf{A}$.
$rank(\mathbf{T})$	$$rank(\mathbf{T}) = rank(\mathbf{A}) \quad (8.2.5)$$
$\mathbf{T}^2$	$$\mathbf{T}^2(\mathbf{x}) = \mathbf{A}^2\mathbf{x} \qquad \mathbf{x} \in \mathbf{V} \quad (8.2.6)$$ $$rank(\mathbf{T}^2) = rank(\mathbf{A}^2) \quad (8.2.7)$$
$\mathbf{A}$ and $\mathbf{A}^2$	The basis vectors of column-space of $\mathbf{A}$ and $\mathbf{A}^2$ are same. The basis vectors of null-space of $\mathbf{A}$ and $\mathbf{A}^2$ are same.

TABLE 8.2.1: Definitions and theorem used

Statement	Observations
Given	$\mathbf{V}$ is a finite dimensional space over $\mathbb{R}$ and $T : \mathbf{V} \to \mathbf{V}$ $$rank(\mathbf{T}) = rank(\mathbf{T}^2) \quad (8.2.8)$$ According to rank-nullity theorem. $$dim(\mathbf{V}) = rank(\mathbf{T}) + nullity(\mathbf{T}) \quad (8.2.9)$$ $$dim(\mathbf{V}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (8.2.10)$$ from (8.2.9) and (8.2.10). we get $$\implies rank(\mathbf{T}) + nullity(\mathbf{T}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (8.2.11)$$ $$\implies nullity(\mathbf{T}) = nullity(\mathbf{T}^2) \quad (8.2.12)$$

TABLE 8.2.2: Observations

Option	Solution	True/False
1	From (8.2.12), let $$nullity(\mathbf{T}) = nullity(\mathbf{T}^2) = n \quad (8.2.13)$$	

	Therefore, from table 8.2.1 and (8.2.13) we can say that both null space of linear operator $\mathbf{T}$ and null space of linear operator $\mathbf{T}^2$ will have same n number of basis. $$\implies Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \tag{8.2.14}$$	True
2	From (8.2.8), let $$rank(\mathbf{T}) = rank(\mathbf{T}^2) = r \tag{8.2.15}$$ Therefore, from table 8.2.1 and (8.2.15) we can say that both column space of linear operator $\mathbf{T}$ and column space of linear operator $\mathbf{T}^2$ will have same r number of basis. $$\implies Range(\mathbf{T}) = Range(\mathbf{T}^2) \tag{8.2.16}$$	True
3	From (8.2.13), (8.2.15) and also we can say that column space $C(\mathbf{A})$ and null space $N(\mathbf{A})$ are r-dimensional space and n-dimensional space respectively which will intersect only at origin(zero vector). And also from (8.2.2) and (8.2.4), we get $$\implies Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \tag{8.2.17}$$	True
4	From table (8.2.14), (8.2.16) and (8.2.17), we get $$\implies Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \tag{8.2.18}$$	True

TABLE 8.2.3: Solution

Statement	**Calculations and observations**
Consider vector space $\mathbf{V} = \mathbb{R}^3$ Let matrix $\mathbf{A}$ be	$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \tag{8.2.19}$$
$\mathbf{A}^2$	$$\mathbf{A}^2 = \begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \tag{8.2.20}$$
Convert both $\mathbf{A}$ and $\mathbf{A}^2$ to Row Reduced echelon form	For matrix $\mathbf{A}$, $$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xleftrightarrow[R_1 \leftarrow R_1 - 2R_2]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \tag{8.2.21}$$ $$\xleftrightarrow{R_3 \leftarrow R_3 - 5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{8.2.22}$$

	For matrix $\mathbf{A}^2$, $$\begin{pmatrix}0 & 7 & 7\\-1 & 4 & 5\\-5 & 13 & 18\end{pmatrix}\xleftrightarrow{R1\leftrightarrow R2}\begin{pmatrix}-1 & 4 & 5\\0 & 7 & 7\\-5 & 13 & 18\end{pmatrix} \quad (8.2.23)$$ $$\xleftrightarrow{R_3\leftarrow R_3-5R_1}\begin{pmatrix}-1 & 4 & 5\\0 & 7 & 7\\0 & -7 & -7\end{pmatrix}\xleftrightarrow{R_3\leftarrow R3+R_1}\begin{pmatrix}-1 & 4 & 5\\0 & 7 & 7\\0 & 0 & 0\end{pmatrix} \quad (8.2.24)$$ $$\xleftrightarrow[R_1\leftarrow -R_1]{R_2\leftarrow \frac{R_2}{7}}\begin{pmatrix}1 & -4 & -5\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}\xleftrightarrow{R1\leftarrow R_1+4R_2}\begin{pmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix} \quad (8.2.25)$$
$Range(\mathbf{T}) = Range(\mathbf{T}^2)$	Therefore, from (8.2.22) and (8.2.25) we can say that the basis vectors of $Range(\mathbf{T})$ and $Range(\mathbf{T}^2)$ are same as shown below $$\mathbf{b_1} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \mathbf{b_2} = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad (8.2.26)$$ and also we can say $$Range(\mathbf{T}) = Range(\mathbf{T}^2) \quad (8.2.27)$$
$Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2)$	Lets find the basis for null-space of linear operator $\mathbf{T}$ or $N(\mathbf{A})$. It is the solution of the equation $\mathbf{Ax} = 0$. From (8.2.22) we have, $$\mathbf{Ax} = 0 \quad (8.2.28)$$ $$\implies \begin{pmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0 \quad (8.2.29)$$ Setting the value of the free variable $x_3 = 1$ we get the solution, $$\mathbf{x} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \quad (8.2.30)$$ Hence, the basis vector of the $Kernel(\mathbf{T})$ is given by, $$\mathbf{p} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \quad (8.2.31)$$ Now, lets find the basis for null-space of linear operator $\mathbf{T}^2$ or $N(\mathbf{A}^2)$.It is the solution of the equation $\mathbf{A^2x} = 0$. From (8.2.25) we have, $$\mathbf{A}^2\mathbf{x} = 0 \quad (8.2.32)$$ $$\implies \begin{pmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 0 \quad (8.2.33)$$ Setting the value of the free variable $x_3 = 1$ we get the solution,

	$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (8.2.34)$$ Hence, from (8.2.31) and (8.2.34) we got the basis vector of $Kernel(\mathbf{T}^2)$ same as the basis vector of $Kernel(\mathbf{T})$ which is $\mathbf{p}$. Therefore, we can say that $$Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \quad (8.2.35)$$
$Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$	From (8.2.26) and (8.2.31), we got 2 basis vectors $\mathbf{b_1}$, $\mathbf{b_2}$ for $Range(\mathbf{T})$and 1 basis vector $\mathbf{p}$ for $Kernel(\mathbf{T})$. Here $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$ are linearly independent which can be proven as below. Let columns of matrix $\mathbf{M}$ are filled with vectors $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$. $$\implies \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad (8.2.36)$$ From (8.2.36), we get $rank(\mathbf{M}) = 3$.Therefore $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$ are linearly independent $Range(\mathbf{T})$ is a 2-dimensional space which is a plane in $\mathbb{R}^3$ and $Kernel(\mathbf{T})$ is a 1-dimensional space which is a line in $\mathbb{R}^3$. Since $\mathbf{b_1}$, $\mathbf{b_2}$, $\mathbf{p}$ are linearly independent then plane and line intersect at origin(zero vector). And we can say that $$Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \quad (8.2.37)$$
$Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$	From (8.2.27), (8.2.35), (8.2.37) we get $$\implies Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \quad (8.2.38)$$

TABLE 8.2.4: Example

8.3. Let **A** and **B** be $n \times n$ matrices over **C**. Then,

a) **AB** and **BA** always have the same set of eigenvalues.

b) If **AB** and **BA** have the same set of eigenvalues then **AB** = **BA**

c) If $\mathbf{A}^{-1}$ exists, then **AB** and **BA** are similar

d) The rank of **AB** is always the same as the rank of **BA**.

Solution: See Tables 8.3.1 and 8.3.2.

8.4. Let **A** be an m x n real matrix and $\mathbf{b} \in \mathbb{R}^m$ with $b \neq 0$.

a) The set of all real solutions of $\mathbf{A}x = \mathbf{b}$ is a vector space.

b) If u nd v are two solutions of $\mathbf{A}x = \mathbf{b}$ then $\lambda u + (1 - \lambda) v$ is also a solution of $\mathbf{A}x = \mathbf{b}$

c) For any two solutions u and v of $\mathbf{A}x = \mathbf{b}$, the linear combination $\lambda u + (1 - \lambda) v$ is also a solution of $\mathbf{A}x = \mathbf{b}$ only when $0 \leq \lambda \leq 1$.

d) If rank of **A** is n ,then $\mathbf{A}x = \mathbf{b}$ has at most one solution.

Solution: See Table 8.4.1

AB and **BA** always have the same set of eigenvalues.	**True**. Let λ be an eigenvalue of **AB**, and **x** be a corresponding eigenvector. Then $\mathbf{ABx}=\lambda\mathbf{x}$ Left-multiplying by **B**: $\mathbf{B}(\mathbf{AB})\mathbf{x} = \mathbf{B}(\lambda\mathbf{x})$ $(\mathbf{BA})\mathbf{Bx} = \lambda(\mathbf{Bx})$ (by associativity of multiplication) $\implies$ λ is an eigenvalue of **BA** with **Bx** as the corresponding eigenvector, assuming **Bx** is not a null vector. If **Bx** is null, then **B** is singular, so that both **AB** and **BA** are singular, and $\lambda = 0$. Since both the products are singular, 0 is an eigenvalue of both. Example: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ Then $\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$ Since **AB** and **BA** results with the same characteristic equation, $\lambda^2 + 2\lambda = 0$ they will have same set of eigenvalues that is $\lambda_1 = 0, \lambda_2 = -2$
If **AB** and **BA** have the same set of eigenvalues then **AB** = **BA**	**False**. Counter example: Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}$ $\implies$ Same eigenvalues ($\lambda_1 = 0, \lambda_2 = -2$), but $\mathbf{AB} \neq \mathbf{BA}$

TABLE 8.3.1

If $\mathbf{A}^{-1}$ exists, then $\mathbf{AB}$ and $\mathbf{BA}$ are similar	**True**. Given that $\mathbf{A}^{-1}$ exists and hence, $$\mathbf{AB} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{BA} = \mathbf{BA}.$$ Hence, $\mathbf{AB} \simeq \mathbf{BA}$ Example: Let $$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$$ then $$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} \\ &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix} \\ &= \mathbf{BA} \end{aligned}$$
The rank of $\mathbf{AB}$ is always the same as the rank of $\mathbf{BA}$.	**False**. Counter example: Let $$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ then $$\mathbf{AB} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ From the above $\mathbf{AB}$ and $\mathbf{BA}$, it is noted that the rank($\mathbf{AB}$) = 2 and rank($\mathbf{BA}$)=1. Hence the rank of $\mathbf{AB}$ need not always be same as rank of $\mathbf{BA}$.

TABLE 8.3.2

Option 1	Suppose $\mathbb{V}$ is the vector space defined as $\mathbb{V} = \{\mathbf{x} : \mathbf{Ax} = \mathbf{b}\ ,\ \mathbb{R}^n \to \mathbb{R}^m\}$ $\mathbf{v}$ and $\mathbf{u}$ are the solution to the equation $\mathbf{Ax} = \mathbf{b}$ such that $\mathbf{u}$ and $\mathbf{v} \in \mathbb{V}$ $\mathbf{Au} = \mathbf{b} \quad \mathbf{Av} = \mathbf{b}$ Checking Closure under vector addition $\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$ Which is enclosed under vector addition if and only if $\mathbf{b} = \mathbf{0}$.But here given $\mathbf{b} \neq 0$ means $\mathbf{0} \notin \mathbb{V}$ Hence does not satisfy requirements of vector space. Hence option 1 is incorrect.
Option 2	**Proof 1:** If $\mathbf{u}$ and $\mathbf{v}$ are the two solution of $\mathbf{A}x = \mathbf{b}$ $\mathbf{Au} = \mathbf{b} \quad \mathbf{Av} = \mathbf{b}$ For $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ to be a solution of $\mathbf{A}x = \mathbf{b}$,it must satisfy this equation. $\mathbf{A}(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) = \mathbf{b} \implies \mathbf{A}\lambda\mathbf{u} + \mathbf{A}(1-\lambda)\mathbf{v} = \mathbf{b} \implies \mathbf{A}\lambda\mathbf{u} + \mathbf{Av} - \mathbf{A}\lambda\mathbf{v} = \mathbf{b}$ $\mathbf{b}\lambda + \mathbf{Av} - \mathbf{b}\lambda = \mathbf{b} \implies \mathbf{Av} = \mathbf{b}$ Which satisfies the equation therefore $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is the solution of $\mathbf{A}x = \mathbf{b}$ for any λ Since the λ term cancels out therefore vaild for $\lambda \in \mathbb{R}$. **Proof 2 (Through affine Subspace with an Example)**:- Let us suppose the two solution $\mathbf{u}$ and $\mathbf{v}$ be the points on the line given by the equation $\mathbf{A}x = \mathbf{b}$ Let the Line joining these two points is given as $\mathbf{l} = \mathbf{u} - \mathbf{v}$ is line parallel to the given line $\mathbf{A}x = \mathbf{b}$ Therefore $\mathbf{v}$ belongs to solution set and is independent to other linearly independent vectors of $\mathbf{l}$ $\mathbf{x} = \mathbf{v} + \lambda\mathbf{l}$ for $\lambda \in \mathbb{R}$ on substuting $\mathbf{l}$ $\mathbf{x} = \mathbf{v} + \lambda(\mathbf{u} - \mathbf{v}) = \mathbf{v} + \lambda\mathbf{u} - \lambda\mathbf{v} = \mathbf{v}(1-\lambda) + \lambda\mathbf{u}$ Hence $\mathbf{v}(1-\lambda) + \lambda\mathbf{u}$ is also the solution of the equation $\mathbf{Ax} = \mathbf{b}$ for $\lambda \in \mathbb{R}$.

	Hence Option 2 is correct.
Option 3	Since in Option 2 we have proved that $\mathbf{v}(1-\lambda)+\lambda\mathbf{u}$ is a solution for $\mathbf{Ax}=\mathbf{b}$ for any $\lambda \in \mathbb{R}$ therefore λ can be any real value but in option 3 there is restriction on λ which is incorrect. Hence option 3 is incorrect
Option 4	$\mathbf{A}_{mxn}\mathbf{x}_{nx1} = \mathbf{b}_{mx1}$ If $\mathbf{A}$ has Full column rank($\mathbf{A}$) $= n$ then there exist one pivot in each columns and there exists no free variables thus $\mathbf{N}(\mathbf{A}) = \mathbf{0}$ so the only solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. So the solution to $\mathbf{Ax} = \mathbf{b}$ $\mathbf{x} = \mathbf{x_p}$ unique solution exists if it exist.It can be either 0 or 1. Hence at most 1 solution is possible . **Proof with example** Let $\mathbf{A} = \begin{pmatrix}1&3\\2&1\\6&1\\5&1\end{pmatrix}_{4x2} \xleftrightarrow{RREF} \begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}$ Hence $n=2$ pivot columns at both column position $\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}$ Hence no solution possible as no combination of $\mathbf{x}$ can gives the solution except $\mathbf{x} = \begin{pmatrix}0\\0\end{pmatrix}$ only if $\mathbf{b} = \mathbf{0} \implies \begin{pmatrix}1&3\\2&1\\6&1\\5&1\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ **OR** $\mathbf{x} = \begin{pmatrix}1\\1\end{pmatrix}$ only if $\mathbf{b}$ is addition of columns of $\mathbf{A} \implies \begin{pmatrix}1&3\\2&1\\6&1\\5&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\3\\7\\6\end{pmatrix}$ Hence either no solution possible or one solution possile. Therefore we say at most one solution possible. Option 4 is correct.

Answers	Option 2 and Option 4 are correct

TABLE 8.4.1: Solution

8.5. Let $\mathbf{A}$ be an $n \times n$ matrix over $\mathbb{C}$ such that every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$.Then

a) All eigen values of $\mathbf{A}$ are equal.
b) All eigen values of $\mathbf{A}$ are distinct.
c) $\mathbf{A} = \lambda\mathbf{I}$ for some $\lambda \in \mathbb{C}$, where $\mathbf{I}$ is the $n \times n$ identity matrix.
d) If $\chi_{\mathbf{A}}$ and $m_{\mathbf{A}}$ denote the characteristic polynomial and the minimal polynomial respectively, then $\chi_{\mathbf{A}} = m_{\mathbf{A}}$

Solution: See Tables 8.5.1 , 8.5.2 and 8.5.3

Given	Every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$, where $\mathbf{A}$ is an $n \times n$ matrix over $\mathbb{C}$.
Determining **A**	Since every vector is an eigen vector, the standard basis vectors are also eigen vectors $\implies \mathbf{Ae_i} = \lambda_i \mathbf{e_i} \implies \begin{pmatrix} a_1 & a_2 & . & . & . & a_n \end{pmatrix} \mathbf{e_i} = \lambda_i \mathbf{e_i} \implies a_i = \lambda_i \mathbf{e_i}$ where $\lambda_i \in \mathbb{C}$ therefore $\mathbf{A} = \begin{pmatrix} \lambda_1 \mathbf{e_1} & \lambda_2 \mathbf{e_2} & . & . & \lambda_n \mathbf{e_n} \end{pmatrix}$ Any vector $\mathbf{b}$ can be represented in the standard basis as $\mathbf{b} = b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n}$ where $b_i \in \mathbb{C}$ As every non-zero vector in $\mathbb{C}^n$ is an eigen vector $\mathbf{Ab} = \lambda \mathbf{b} \implies \mathbf{A}(b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n}) = \lambda (b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n})$ $\implies b_1\lambda_1\mathbf{e_1} + b_2\lambda_2\mathbf{e_2} + ... + b_n\lambda_n\mathbf{e_n} = \lambda (b_1\mathbf{e_1} + b_2\mathbf{e_2} + ... + b_n\mathbf{e_n})$ $\implies b_1(\lambda_1 - \lambda)\mathbf{e_1} + b_2(\lambda_2 - \lambda)\mathbf{e_2} + ... + b_n(\lambda_n - \lambda)\mathbf{e_n} = 0$ since basis are linearly independent we get $\lambda_1 = \lambda_2 = .. = \lambda_n = \lambda$ Therefore the matrix $\mathbf{A}$ is $$\mathbf{A} = \begin{pmatrix} \lambda_1 \mathbf{e_1} & \lambda_2 \mathbf{e_2} & . & . & \lambda_n \mathbf{e_n} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & . & . & \mathbf{e_n} \end{pmatrix} = \lambda \mathbf{I}_n \text{ where } \lambda \in \mathbb{C}$$

TABLE 8.5.1

option 1	Since $\mathbf{A} = \lambda \mathbf{I}_n$, all the eigen values are equal to λ. Therefore option 1 is correct as the matrix $\mathbf{A}$ is a scalar matrix.
option 2	since the matrix $\mathbf{A}$ is a scalar matrix, all the eigen values are equal. So this option is incorrect.
option 3	This option is correct. As proved in the construction the matrix $\mathbf{A} = \lambda \mathbf{I}$ for some $\lambda \in \mathbb{C}$
option 4	Since $\mathbf{A} = \lambda \mathbf{I}$ where $\lambda \in \mathbb{C}$, the characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x - \lambda)^n$ and $m_{\mathbf{A}} = (x - \lambda) \implies \chi_{\mathbf{A}} = m_{\mathbf{A}}^n$. Therefore this option is incorrect

TABLE 8.5.2: Answer

Scalar matrix	Consider a 3×3 scalar matrix $\mathbf{A} = (2 + 3i)\mathbf{I}$, for which the eigen values are $(2 + 3i), (2 + 3i), (2 + 3i)$ The eigen vectors will be the nullspace of $\mathbf{A} - \lambda \mathbf{I}$ $$\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} 2+3i & 0 & 0 \\ 0 & 2+3i & 0 \\ 0 & 0 & 2+3i \end{pmatrix} - (2+3i)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ The nullspace consists of the entire vector space so every vector is an eigen vector The characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x - (2 + 3i))^3$ and $m_{\mathbf{A}} = (x - (2 + 3i)) \implies \chi_{\mathbf{A}} = m_{\mathbf{A}}^3$ Therefore options 1 and 3 are correct.
Diagonal matrix	Consider the matrix $\mathbf{A}$ as $$\mathbf{A} = \begin{pmatrix} 2+3i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3i \end{pmatrix} \text{ The eigen values are } \lambda_1 = 2+3i,\ \lambda - 2 = 2,\ \lambda_3 = 3i$$
	The eigen vector with respect to $\lambda_1 = 2 + 3i$ will be the nullspace of $\mathbf{A} - \lambda_1 \mathbf{I}$ $$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3i & 0 \\ 0 & 0 & -2 \end{pmatrix}, \text{ so the eigen vector will be } \mathbf{e_1} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } x_1 \in \mathbb{C}$$ The eigen vector with respect to $\lambda_2 = 2$ will be the nullspace of $\mathbf{A} - \lambda_2 \mathbf{I}$ $$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} 3i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3i-2 \end{pmatrix}, \text{ so the eigen vector will be } \mathbf{e_2} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ where } x_2 \in \mathbb{C}$$

	The eigen vector with respect to $\lambda_3 = 3i$ will be the nullspace of $\mathbf{A} - \lambda_3\mathbf{I}$ $\mathbf{A} - \lambda_3\mathbf{I} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2-3i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so the eigen vector will be $\mathbf{e_3} = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ where $x_3 \in \mathbb{C}$ Consider the vector $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$ where $x_1 = x_2 = x_3 = 1$ $\mathbf{Ay} = \mathbf{Ae_1} + \mathbf{Ae_2} + \mathbf{Ae_3} = (2+3i)\,\mathbf{e_1} + 2\mathbf{e_2} + 3i\mathbf{e_3} = \begin{pmatrix} 2+3i \\ 2 \\ 3i \end{pmatrix}$ As $\mathbf{Ay}$ can not be written as $c\mathbf{y}$ where $c \in \mathbb{C}$, $\mathbf{y}$ is not an eigen vector which is a contradiction.

TABLE 8.5.3: Examples

8.6. Consider a matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} \tag{8.6.1}$$

and,

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{8.6.2}$$

Then which of following is true,

a) **A** and **B** is similar over the field of rational numbers.
b) **A** is diagonalizable over the field of rational numbers $\mathbb{Q}$.
c) **B** is the Jordan canonical form of **A**.
d) The minimal polynomial and the characteristic polynomial of **A** are the same.

Solution: Two matrix are said to be similar if their eigen values are same.
Eigen value of **A** is given as:

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{pmatrix} = 0 \tag{8.6.3}$$

$$\implies -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \tag{8.6.4}$$

$$\implies \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3. \tag{8.6.5}$$

Similarally, eigen values of **B** is givem as:

$$\begin{pmatrix} 2-\lambda & 10 & \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix} \tag{8.6.6}$$

$$\implies -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \tag{8.6.7}$$

$$\implies \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3. \tag{8.6.8}$$

Hence, matrices **A** and **B** are similar. Matrix **A** is diagonalizable if and only if there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of **A**.
From (8.6.5) , our eigenvalues for **A** are,

$$\lambda_1 = \lambda_2 = 2 \tag{8.6.9}$$

and,

$$\lambda_3 = 3. \tag{8.6.10}$$

Hence $\lambda_1 = \lambda_2$ is a repeated root with multiplicity two. Hence, We can get only two linearly independent eigenvectors for **A**, are given as :

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} and, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \tag{8.6.11}$$

But any basis for $\mathbb{R}^3$ consists of three vectors. Therefore there is no third eigenbasis for **A**, hence **A** is not diagonalizable. From (8.6.5) we have eigenvalue $\lambda_1 = 2$ with geometic multiplicity 2. Hence the Jordon canonical form of **A** can be written as :

$$\mathbf{J_A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{8.6.12}$$

Hence **B** is the Jordan canonical form of **A**. From (8.6.5), the characteristic polynomial of this matrix is:

$$f(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (\lambda-2)^2(\lambda-3) \tag{8.6.13}$$

Minimal polynomial for a matrix is a smallest polynomial for which

$$M_{\mathbf{A}}(x) = 0 \tag{8.6.14}$$

Using (8.6.14), we found minimal polynomial of **A** is :

$$M_{\mathbf{A}}(x) = (x-2)^2(x-3) \tag{8.6.15}$$

We can relate the minimal polynomial with the size of Jordan block.

Size of Jordan block = degree of minimal polynomial with geometic multiplicity of the eigen values.

From (8.6.15) we can observe that, geometric multiplicity of eigen value 2 is 2. Hence size of Jordan block is 2. which is given as:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \tag{8.6.16}$$

if geometric multiplicity of $\lambda = 2$ would be 3, then Jordan block would be:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{8.6.17}$$

In (8.6.15) geometric multiplicity of eigen value 2 is 2, and geometric multiplicity of eigen value 3 is one hence jardon block is:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{8.6.18}$$

9 June 2015

9.1. Let **A**,**B** be n × n matrices.Which of the following equals trace($\mathbf{A}^2\mathbf{B}^2$)?

a) (trace(**AB**))^2.
b) trace($\mathbf{A}\mathbf{B}^2\mathbf{A}$).
c) trace($(\mathbf{AB})^2$).
d) trace(**BABA**).

Solution: See Table 9.1.1

Statement	Solution
Definition	The trace of an n × n square matrix $\mathbf{A}$ is defined as: $$tr(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$$ where a_{ii} denotes the entry on the ith row and ith column of $\mathbf{A}$.
Properties	$$\text{The properties of the trace : } tr(c\mathbf{A}) = c\ tr(\mathbf{A}) \quad (9.1.1)$$ $$tr(\mathbf{A}^T) = tr(\mathbf{A}) \quad (9.1.2)$$ $$tr(\mathbf{A}+\mathbf{B}) = tr(\mathbf{B}+\mathbf{A}) \quad (9.1.3)$$ $$tr(\mathbf{AB}) = tr(\mathbf{BA}) \quad (9.1.4)$$ $$tr(\mathbf{A}^T\mathbf{B}) = tr(\mathbf{AB}^T) \quad (9.1.5)$$ $$tr(\mathbf{R}^{-1}\mathbf{AR}) = tr(\mathbf{R}^{-1}(\mathbf{AR})) \quad (9.1.6)$$ $$= tr((\mathbf{AR})\mathbf{R}^{-1}) = tr(\mathbf{A}) \quad (9.1.7)$$
Checking $tr(\mathbf{A}^2\mathbf{B}^2)$.	$$\text{Upon rewriting and from (9.1.4), } tr(\mathbf{A}^2\mathbf{B}^2) = tr(\mathbf{AABB}) \quad (9.1.8)$$ $$= tr(\mathbf{BAAB}) \quad (9.1.9)$$ $$= tr(\mathbf{BBAA}) \quad (9.1.10)$$ $$= tr(\mathbf{ABBA}) \quad (9.1.11)$$ $$= tr(\mathbf{AABB}) \quad (9.1.12)$$ $$= tr(\mathbf{A}^2\mathbf{B}^2) \quad (9.1.13)$$
Checking $(tr(\mathbf{AB}))^2$.	$$\text{from (9.1.4), } (tr(\mathbf{AB}))^2 = (tr(\mathbf{BA}))^2 \quad (9.1.14)$$
Checking $tr(\mathbf{AB}^2\mathbf{A})$.	$$\text{Rewriting, } tr(\mathbf{AB}^2\mathbf{A}) = tr(\mathbf{ABBA}) \quad (9.1.15)$$ $$\text{from (9.1.4), } tr(\mathbf{AB}^2\mathbf{A}) = tr(\mathbf{AABB}) = tr(\mathbf{A}^2\mathbf{B}^2) \quad (9.1.16)$$
Checking $tr(\mathbf{AB})^2$.	$$\text{from (9.1.4), } tr(\mathbf{AB})^2 = tr(\mathbf{BA})^2 \quad (9.1.17)$$
Checking $tr(\mathbf{BABA})$.	$$\text{from (9.1.4)} \quad (9.1.18)$$ $$tr(\mathbf{BABA}) = tr(\mathbf{ABAB}) \quad (9.1.19)$$ $$= tr(\mathbf{BABA}) \quad (9.1.20)$$
Conclusion	Hence, from (9.1.4), and (9.1.16) option 2, ie $tr(\mathbf{AB}^2\mathbf{A})$. is the correct answer.

TABLE 9.1.1: Solution

Options	Explanation
7 Given Rank Nullity Theorem	A:$\mathbf{R^{50}} \to \mathbf{R^{20}}$ is a linear transformation dim(row space($\mathbf{A}$))=$rank$($\mathbf{A}$)=13 $\mathbf{A}$:$\mathbf{R^{50}} \to \mathbf{R^{20}}$ is a linear transformation then, $rank(\mathbf{A})+nullity(\mathbf{A})=50$ $13 + nullity(\mathbf{A})=50$ $nullity(\mathbf{A})=37$ dim(space of solution($\mathbf{Ax} = 0$))=$nullity(\mathbf{A})=37$ Hence,incorrect
13	From above,it is obvious that it is incorrect
33	It is also incorrect.
37	From above it is correct

TABLE 9.2.1: Finding Correct Option

9.2. The row space of a 20×50 matrix $\mathbf{A}$ has dimension 13.What is the dimension of the space of solution $\mathbf{Ax} = 0$?

a) 7
b) 13
c) 33
d) 37

Solution: See Table 9.2.1

9.3. Given a 4×4 matrix $\mathbf{A}$, let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by $\mathbf{Tv} = \mathbf{Av}$, where we think of $\mathbb{R}^4$ as the set of real 4×1 matrices. For which choices of $\mathbf{A}$ given below, do Image($\mathbf{T}$) and Image($\mathbf{T}^2$) have respective dimensions 2 and 1? ($*$ denotes a nonzero entry)

a) $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$

b) $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$

c) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}$

d) $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$

Solution: We can say,

$$\mathbf{T}(\mathbf{v}) = \mathbf{Av} = \text{Image}(\mathbf{T}) = C(\mathbf{A}) \quad (9.3.1)$$

$$\mathbf{T}^2(\mathbf{v}) = \mathbf{A}^2\mathbf{v} = \text{Image}(\mathbf{T}^2) = C(\mathbf{A}^2) \quad (9.3.2)$$

where $C(\mathbf{A})$ and $C(\mathbf{A}^2)$ denote the columnspace of $\mathbf{A}$ and $\mathbf{A}^2$ respectively. Therefore,

$$\text{dimension}(\text{Image}(\mathbf{T})) = \text{dimension}(C(\mathbf{A})) = \text{rank}(\mathbf{A}) \quad (9.3.3)$$

$$\text{dimension}(\text{Image}(\mathbf{T}^2)) = \text{dimension}(C(\mathbf{A}^2)) = \text{rank}(\mathbf{A}^2) \quad (9.3.4)$$

See Table 9.3.1

1. $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}$	The number of linearly independent columns in $\mathbf{A}$ is 2
	hence, $dim(Image(\mathbf{T})) = dim(C(\mathbf{A})) = 2$ $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ The number of linearly independent columns in $\mathbf{A}^2$ is 1 hence, $dim(Image(\mathbf{T}^2)) = dim(C(\mathbf{A}^2)) = 1$ $\therefore$ This option is true.
2. $\mathbf{A} = \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$	The number of linearly independent columns in $\mathbf{A}$ is 2 hence, $\dim(\text{Image}(\mathbf{T})) = \dim(\text{C}(\mathbf{A})) = 2$ $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix}$ The number of linearly independent columns in $\mathbf{A}^2$ is 1 hence, $\dim(\text{Image}(\mathbf{T}^2)) = \dim(\text{C}(\mathbf{A}^2)) = 1$ $\therefore$ This option is true.
3. $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}$	The number of linearly independent columns in $\mathbf{A}$ is 2 hence, $dim(Image(\mathbf{T})) = dim(C(\mathbf{A})) = 2$ $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$ The number of linearly independent columns in $\mathbf{A}^2$ is 2 hence, $dim(Image(\mathbf{T}^2)) = dim(C(\mathbf{A}^2)) = 2 \neq 1$

	$\therefore$ This option is false.
4. $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$	This option is false Counter example: For some non-zero $b, c \in \mathbb{R}$, let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & b \\ 0 & 0 & c & c \end{pmatrix}$ The number of linearly independent columns in $\mathbf{A}$ is 1 hence, $dim(Image(\mathbf{T})) = dim(C(\mathbf{A})) = 1 \neq 2$

TABLE 9.3.1: Verifying with the options

9.4. Let $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle$, where $\langle , \rangle$ is the standard inner product of $\mathbb{R}^n$ and $\mathbf{A}$ is a $n \times n$ real matrix. Here D denotes the total derivative. Which of the following statements are correct?

a) $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v}) = \langle \mathbf{A}\mathbf{u}, \mathbf{y} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{v} \rangle$.

b) $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(0, 0) = 0$.

c) $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ may not exist for some $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.

d) $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ does not exist at $(\mathbf{x}, \mathbf{y}) = (0, 0)$.

Solution: See Tables 9.4.1, 9.4.2 and 9.4.3

Inner product	Inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as $$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} \quad (9.4.1)$$ Where $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$
Inner Product Property used	$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle \quad (9.4.2)$$
Total Derivative D	Total derivative is a linear transformation. For function $\mathbf{F}(\mathbf{x}, \mathbf{y})$, the total derivative is given as $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ which says that total derivative of function $\mathbf{F}$ at $(\mathbf{x}, \mathbf{y})$.

TABLE 9.4.1: Definitions and theorem used

Statement	Observations
Given	Function $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, it is given as $$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{A}^T \mathbf{y} \quad (9.4.3)$$ where $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ Using property (9.4.2), we can also get $$\implies \mathbf{F}(\mathbf{x}, \mathbf{y}) = \langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle \quad (9.4.4)$$ $$\implies \mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T \mathbf{A}\mathbf{x} \quad (9.4.5)$$
Total Derivative D	Now we will calculate $D\mathbf{F}(\mathbf{x}, \mathbf{y})$ $$D\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{x}} & \frac{\partial \mathbf{F}}{\partial \mathbf{y}} \end{pmatrix} \quad (9.4.6)$$ From (9.4.3),(9.4.5) we get $$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \mathbf{y}^T \mathbf{A} \quad (9.4.7)$$ $$\frac{\partial \mathbf{F}}{\partial \mathbf{y}} = \mathbf{x}^T \mathbf{A}^T \quad (9.4.8)$$ Substitute (9.4.7) and (9.4.8) in (9.4.6) $$D\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{y}^T \mathbf{A} & \mathbf{x}^T \mathbf{A}^T \end{pmatrix}_{1 \times n^2} \quad (9.4.9)$$

TABLE 9.4.2: Observations

Option	Solution	True/ False
1	First we calculate $(D\mathbf{F}(\mathbf{x}, \mathbf{y}))(\mathbf{u}, \mathbf{v})$ where $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$ Using (9.4.9)and block matrix multiplication we get	

	$$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \begin{pmatrix}\mathbf{y}^T\mathbf{A} & \mathbf{x}^T\mathbf{A}^T\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \quad (9.4.10)$$ $$\implies (D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \mathbf{y}^T\mathbf{A}\mathbf{u} + \mathbf{x}^T\mathbf{A}^T\mathbf{v} \quad (9.4.11)$$ $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \langle \mathbf{y}, \mathbf{A}\mathbf{u}\rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{v}\rangle \quad (9.4.12)$$ Using property (9.4.2) we get $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(\mathbf{u},\mathbf{v}) = \langle \mathbf{A}\mathbf{u}, \mathbf{y}\rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{v}\rangle \quad (9.4.13)$$	True
2.	Using (9.4.11), if $\mathbf{u} = 0$ and $\mathbf{v} = 0$ then we get $$(D\mathbf{F}(\mathbf{x},\mathbf{y}))(0,0) = 0 \quad (9.4.14)$$	True
3.	Since from (9.4.9) we can say that $D\mathbf{F}(\mathbf{x},\mathbf{y})$ will exist for any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n$.	False
4.	From (9.4.9), if $(\mathbf{x},\mathbf{y}) = (0,0)$ we get $$D\mathbf{F}(\mathbf{x},\mathbf{y})\big\|_{(0,0)} = 0 \quad (9.4.15)$$ Therefore we can say that $D\mathbf{F}(\mathbf{x},\mathbf{y})$ will exist at $(\mathbf{x},\mathbf{y}) = (0,0)$.	False

TABLE 9.4.3: Solution

9.5. An $n \times n$ complex matrix $\mathbf{A}$ satisfies $\mathbf{A}^k = \mathbf{I}_n$. the $n \times n$ identity matrix, where k is a positive integer > 1. Suppose 1 is not an eigenvalue of $\mathbf{A}$. Then which of the following statements are necessarily true?

a) $\mathbf{A}$ is diagonalizable.

b) $\mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$, the $n \times n$ zero matrix.

c) $tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = -n$

d) $\mathbf{A}^{-1} + \mathbf{A}^{-2} + ... + \mathbf{A}^{-(k-1)} = -\mathbf{I}_n$

Solution: See Tables 9.5.2 and 9.5.3

Minimal Polynomial	The minimal polynomial $\mu_{\mathbf{A}}$ of an $n \times n$ matrix $\mathbf{A}$ over a field $\mathbf{F}$ is the monic polynomial P over the field $\mathbf{F}$ of least degree such that $P(\mathbf{A}) = 0$. Any other polynomial Q with $Q(\mathbf{A}) = 0$ is polynomial multiple of $\mu_{\mathbf{A}}$.
Eigen Value and Minimal Polynomial	If λ is an eigen value of matrix $\mathbf{A}$ then λ will also be the root of the minimal polynomial $\mu_{\mathbf{A}}$.
Diagonalizability and Eigen Values	If $\mathbf{A}$ is an $n \times n$ matrix with n distinct eigenvalues, then $\mathbf{A}$ is diagonalizable
Polynomial and it's Zeros	If a polynomial is of form $x^k - 1$, it can be written as $$x^k - 1 = (x-1)(1 + x + x^2 + ... + x^{k-1})$$ The zeros to the given polynomial will be of the format $$e^{\frac{n2\pi i}{k}} \qquad \text{for } 0 \leq n < k.$$ From this we can see that all the roots of the equation $x^k - 1$ will be distinct.

Inference from the Given Data	We are given that $$\mathbf{A}^k = \mathbf{I}_n$$ This can be written as $$\mathbf{A}^k - \mathbf{I}_n = 0$$ This resembles the polynomial equation of the form $x^k - 1$, So we further write the above equation as $$\implies \mathbf{A}^k - \mathbf{I}_n = 0$$ $$\implies (\mathbf{A} - \mathbf{I}_n)(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}) = 0$$ Let $\mu_{\mathbf{A}}$ be the minimal polynomial of $\mathbf{A}$. It is given that 1 is not an eigenvalue of $\mathbf{A}$. That means $\mu_{\mathbf{A}}$ cannot divide $(\mathbf{A} - \mathbf{I}_n)$. But $\mu_{\mathbf{A}}$ will be able to divide $(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1})$ as it is a polynomial multiple of $\mathbf{A}$ i.e. $(\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1})$ is polynomial multiple of $\mu_{\mathbf{A}}$ $$\implies \mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$$

	Since we know that $1 + x + x^2 + ... + x^{k-1}$ will have distinct roots which are not equal to 1.
Option 1	We were able to find that $\implies \mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}$ is a polynomial multiple of $\mu_{\mathbf{A}}$ with $k - 1$ distinct roots. Which implies that $\mu_{\mathbf{A}}$ will also have distinct roots. Since, there are distinct roots to the minimal polynomial, it implies that **A** will be diagonalizable. $\therefore$ this statement is **True**.
Option 2	We know that $$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$$ $$\implies \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = -\mathbf{I}_n$$ $\therefore$ this statement is **False**.
Option 3	We know that $$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = 0$$ $$\implies \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1} = -\mathbf{I}_n$$ Taking *trace*() on both sides, we get $$\implies tr(\mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}) = tr(-\mathbf{I}_n)$$ $$\implies tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = tr(-\mathbf{I}_n) \qquad (\because \text{ trace() is a linear function})$$ $$\implies tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = -n$$ $\therefore$ this statement is **True**.
Option 4	We know that $$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-2} + \mathbf{A}^{k-1} = 0$$ Multiply the whole equation with $\mathbf{A}^{-(k-1)}$. We get $$\mathbf{A}^{-(k-1)} + \mathbf{A}^{1-(k-1)} + ... + \mathbf{A}^{k-2-(k-1)} + \mathbf{A}^{k-1-(k-1)} = 0$$ $$\implies \mathbf{A}^{-(k-1)} + \mathbf{A}^{1-(k-1)} + ... + \mathbf{A}^{-1} + \mathbf{I}_n = 0$$

	$\implies \mathbf{A}^{-1} + \mathbf{A}^{-2} + ... + \mathbf{A}^{-(k-1)} = -\mathbf{I}_n$ $\therefore$ this statement is **True**.
Conclusion	From our observation we see that Options 1), 3) and 4) are True.

TABLE 9.5.2

Complex Matrix Example	Let the complex matrix $\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ When $k = 4$, we get $\mathbf{A}^4 = \mathbf{I}_2$ The eigen values of the matrix $\mathbf{A}$ are $-i$ and $+i$. Since, there are two distinct eigen values for the matrix $\mathbf{A}$, $\mathbf{A}$ is diagonalizable. Now checking the equation for $\mathbf{A} + \mathbf{A}^2 + ... + \mathbf{A}^{k-1}$ $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 \qquad (\because \text{ here } k = 4)$ $\implies \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ $\implies \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}_2$ Now checking the equation for $tr(\mathbf{A}) + tr(\mathbf{A}^2) + ... + tr(\mathbf{A}^{k-1}) = -n$ $tr(\mathbf{A}) + tr(\mathbf{A}^2) + tr(\mathbf{A}^3) \qquad (\because \text{ here } k = 4)$ $\implies tr\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + tr\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + tr\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ $\implies 0 + (-2) + 0 = -2$ Now checking the equation for $\mathbf{A}^{-1} + \mathbf{A}^{-2} + ... + \mathbf{A}^{-(k-1)} = -\mathbf{I}_n$

	$\mathbf{A}^{-1} + \mathbf{A}^{-2} + \mathbf{A}^{-3} \qquad (\because \text{ here } k = 4)$ $\Longrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ $\Longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}_2$

TABLE 9.5.3

9.6. Let S be the set of 3x3 real matrices $\mathbf{A}$ with

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9.6.1)$$

Then the set contains:-

a) a Nilpotent Matrix
b) a matrix of rank one
c) a matrix of rank two
d) a non-zero skew symmetric matrix.

Solution: See Tables 9.6.1 and 9.6.2.

Proof 1 $Rank(\mathbf{A}) = Rank(\mathbf{A}^T\mathbf{A})$	Let $\mathbf{A}x$=0 and $\mathbb{N}(\mathbf{A})$ is the null space of $\mathbf{A}$ Then $\mathbf{A}^T\mathbf{A}$x=0 which means $\mathbb{N}(\mathbf{A}) \subset \mathbb{N}(\mathbf{A}^{\mathbb{T}}\mathbf{A})$ Thus if $\mathbf{A}^T\mathbf{A}$x=0 ,then $x^T\mathbf{A}^T\mathbf{A}x = 0 \implies \|\mathbf{A}x\| = 0$ Which means $\mathbf{A}x = 0$ thus $\mathbb{N}(\mathbf{A}^{\mathbb{T}}\mathbf{A}) \subset \mathbb{N}(\mathbf{A})$ From the Above two condition we can say that $N(\mathbf{A}^T\mathbf{A}) = \mathbb{N}(\mathbf{A})$ $rank(\mathbf{A}) = n - \mathbb{N}(\mathbf{A})$ $rank(\mathbf{A}) = rank(\mathbf{A}^T\mathbf{A})$ Hence Proved.
Proof 2 Rowspace($\mathbf{A}^T\mathbf{A}$)=Rowspace($\mathbf{A}$)	Suppose $\mathbf{A} = \begin{pmatrix}\mathbf{a_1} & \dots & \mathbf{a_n}\end{pmatrix}$ where $\mathbf{a_i}$ is the column vector of $\mathbf{A}$ $\mathbf{A}^T\mathbf{A} = \mathbf{A}^T\begin{pmatrix}\mathbf{a_1} & \dots & \mathbf{a_n}\end{pmatrix} = \begin{pmatrix}\mathbf{A}^T\mathbf{a_1} & \dots\mathbf{A}^T\mathbf{a_n}\end{pmatrix}$ For each column of $\mathbf{A}^T\mathbf{A}$ $\mathbf{A}^T\mathbf{a_i} = \begin{pmatrix}\mathbf{b_1} & \dots\mathbf{b_n}\end{pmatrix}\mathbf{a_i}$where $\mathbf{b_i}$ is the column vector of $\mathbf{A}^T$ and Row of $\mathbf{A}$ $= \begin{pmatrix}\mathbf{b_1} & \dots\mathbf{b_n}\end{pmatrix}\begin{pmatrix}a_{i1}\\ \vdots \\ a_{in}\end{pmatrix} = \sum_{j=1}^{n} a_{ij}b_j$ So column of $\mathbf{A}^T\mathbf{A}$ is the linear combination of rows of $\mathbf{A}$. Since rank($\mathbf{A}^T$)=rank($\mathbf{A}$) so, Row($\mathbf{A}^T\mathbf{A}$) =Column($\mathbf{A}^T\mathbf{A}$)=Row($\mathbf{A}$) Hence Proved.

TABLE 9.6.1: Proofs

Option 1	From Proof 2,Set S contained a set of matrix whose First Column is Non-zero.

Nilpotent Matrix check	$S \in \text{Set}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$ Given $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ So the only matrix $\mathbf{A}$ which satisfy $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$, $\mathbf{A}^2 = 0$ such that $\mathbf{A} \in S$ $\mathbf{A} = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} \in S$ $\mathbf{A}^T\mathbf{A} = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$ $\mathbf{A}^2 = \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$ which is a nilpotent matrix Option 1 is correct.
Option 2 matrix of rank one check	In Proof 1 we already prove that $Rank(\mathbf{A}) = Rank(\mathbf{A}^T\mathbf{A})$ Since the $Rank(\mathbf{A}^T\mathbf{A}) = 1$ so the $Rank(\mathbf{A}) = 1$ There fore Set S always contains only Rank 1 matrices. Hence Option 2 is correct.
Option 3 matrix of rank two check	Since set S contain only rank 1 matrices and none of rank 2 matrices as already proved above therefore Option 3 is incorrect.
Option 4 non-zero skew . symmetric matrix check	Proved by contradiction Assume Rank of $\mathbf{A}$ is 1 so $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{u}\mathbf{v}^T$ for any non-zero Columns vectors $\mathbf{u}$, $\mathbf{v}$ with n entries. If A is skew symmetric,we have:- $\mathbf{A}^T = -\mathbf{A}$

	$(\mathbf{uv})^T = -\mathbf{uv}^T \ \mathbf{vu}^T = -\mathbf{uv}^T$ The Column space of these matrices is same.The column space of $\mathbf{vu}^T$ is span of $\mathbf{v}$,where as the column space of $\mathbf{uv}^T$ is the span of $\mathbf{u}$, So we must have $\mathbf{v} = k\mathbf{u}$ for some $k \in \mathbb{R}$.So the equation becomes $k\mathbf{uu}^T = -k\mathbf{uu}^T$ and since $\mathbf{u} \neq 0$;We can conclude that k=0,which means $\mathbf{v} = 0$ therefore $\mathbf{A} = 0$. This Contradicts our assumption that $\mathbf{A}$has rank 1. Thus real skew symmentric matrix can never have rank=1. Hence option 4 is incorrect.
Answers	Option 1 and Option 2 are correct.

TABLE 9.6.2: Solution Table

9.7. Let $\mathbf{S} : \mathbb{R}^n \to \mathbb{R}^n$ be given by $\mathbf{S}(\mathbf{v}) = \alpha\mathbf{v}$, for a fixed $\alpha \in \mathbb{R}, \alpha \neq 0$. Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $\mathbf{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of linearly independent eigenvectors of $\mathbf{T}$. Then

a) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal
b) The matrix of $(\mathbf{T}-\mathbf{S})$ with respect to $\mathbf{B}$ is diagonal
c) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is not necessarily diagonal, but is upper triangular
d) The matrix of $\mathbf{T}$ with respect to $\mathbf{B}$ is diagonal but the matrix of $(\mathbf{T}-\mathbf{S})$ with respect to $\mathbf{B}$ is not diagonal.

Solution: Given that $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and B represents a set of linearly independent eigenvectors of $\mathbf{T}$ given as follows

$$\mathbf{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \tag{9.7.1}$$

So,

$$\mathbf{T}(\mathbf{v}_i) = \mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i \tag{9.7.2}$$

where λ_i represents the eigenvalue corresponding to $\mathbf{v}_i$. Hence, the matrix $\mathbf{T}$ with respect to $\mathbf{B}$ can be represented as

$$[\mathbf{T}]_B = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ddots & & \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix} \tag{9.7.3}$$

And,

$$(\mathbf{T}-\mathbf{S})\mathbf{v}_i = \mathbf{T}(\mathbf{v}_i) - \mathbf{S}(\mathbf{v}_i) \tag{9.7.4}$$

$$= \lambda_i\mathbf{v}_i - \alpha\mathbf{v}_i \tag{9.7.5}$$

$$= (\lambda_i - \alpha)\mathbf{v}_i \tag{9.7.6}$$

Hence, matrix of $\mathbf{T}-\mathbf{S}$ with respect to $\mathbf{B}$ can be represented as

$$[\mathbf{T}-\mathbf{S}]_B = \begin{pmatrix} \lambda_1 - \alpha & 0 & \ldots & 0 \\ 0 & \lambda_2 - \alpha & \ldots & 0 \\ \vdots & \ddots & & \\ 0 & \ldots & 0 & \lambda_n - \alpha \end{pmatrix} \tag{9.7.7}$$

1. The matrix of **T** w.r.t to **B** is diagonal	True, as seen from (9.7.3)
2. The matrix of (**T** – **S**) w.r.t **B** is diagonal	True, as seen from (9.7.7)
3. The matrix of **T** with respect to **B** is not necessarily diagonal but is upper triangular	False, as already proved $[\mathbf{T}]_B$ is diagonal
4. The matrix of **T** with respect to **B** is diagonal but the matrix of (**T** – **S**) with respect to **B** is not diagonal	False, as already proved $[\mathbf{T} - \mathbf{S}]_B$ is diagonal

TABLE 9.7.1: Verifying the given options

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\mathbf{T}(x) = \mathbf{A}\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{9.7.8}$$

Here, the eigenvalues of the above trasformation matrix are $\lambda_1 = 3, \lambda_2 = -2$. And the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Thus,

$$\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \tag{9.7.9}$$

Now,

$$\mathbf{T}(\mathbf{v}_1) = \mathbf{A}\mathbf{v}_1 \tag{9.7.10}$$
$$= \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{9.7.11}$$
$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \tag{9.7.12}$$
$$= 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{9.7.13}$$
$$= \lambda_1 \mathbf{v}_1 \tag{9.7.14}$$

And,

$$\mathbf{T}(\mathbf{v}_2) = \mathbf{A}\mathbf{v}_2 \tag{9.7.15}$$
$$= \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{9.7.16}$$
$$= \begin{pmatrix} -2 \\ -6 \end{pmatrix} \tag{9.7.17}$$
$$= -2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{9.7.18}$$
$$= \lambda_2 \mathbf{v}_2 \tag{9.7.19}$$
$$\tag{9.7.20}$$

For any vector $\mathbf{v} \in \mathbb{R}^2, \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$

$$[\mathbf{v}]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{9.7.21}$$
$$\mathbf{T}(\mathbf{v}) = \mathbf{T}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \tag{9.7.22}$$
$$= c_1\mathbf{T}(\mathbf{v}_1) + c_2\mathbf{T}(\mathbf{v}_2) \tag{9.7.23}$$
$$= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 \tag{9.7.24}$$
$$[\mathbf{T}(\mathbf{v})]_B = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} \tag{9.7.25}$$
$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{9.7.26}$$
$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} [\mathbf{v}]_B \tag{9.7.27}$$
$$= \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} [\mathbf{v}]_B \tag{9.7.28}$$
$$\mathbf{S}(\mathbf{v}) = \alpha\mathbf{v}, \alpha \neq 0 \tag{9.7.29}$$
$$= \alpha(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \tag{9.7.30}$$
$$= \alpha c_1\mathbf{v}_1 + \alpha c_2\mathbf{v}_2 \tag{9.7.31}$$

$$[\mathbf{S}(\mathbf{v})]_B = \begin{pmatrix} \alpha c_1 \\ \alpha c_2 \end{pmatrix} \tag{9.7.32}$$
$$[(\mathbf{T} - \mathbf{S})(\mathbf{v})]_B = \begin{pmatrix} \lambda_1 c_1 - \alpha c_1 \\ \lambda_2 c_2 - \alpha c_2 \end{pmatrix} \tag{9.7.33}$$
$$= \begin{pmatrix} \lambda_1 - \alpha & 0 \\ 0 & \lambda_2 - \alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{9.7.34}$$
$$= \begin{pmatrix} \lambda_1 - \alpha & 0 \\ 0 & \lambda_2 - \alpha \end{pmatrix} [\mathbf{v}]_B \tag{9.7.35}$$
$$= \begin{pmatrix} 3 - \alpha & 0 \\ 0 & -2 - \alpha \end{pmatrix} [\mathbf{v}]_B \tag{9.7.36}$$

Hence, shown from (9.7.28) and (9.7.36) that the matrix of $\mathbf{T}$ and of $\mathbf{T} - \mathbf{S}$ w.r.t to $\mathbf{B}$ is diagonal.

9.8. Let $p_n(x) = x^n$ for $x \in \mathbb{R}$ and let $\varrho = span\{p_0, p_1, p_2, ...\}$. Then
  a) ϱ is a vector space of all real valued continuous functions on $\mathbb{R}$.
  b) ϱ is a subspace of all real valued continuous functions on $\mathbb{R}$.
  c) $\{p_0, p_1, p_2, ...\}$ is a linearly independent set in the vector space of all real valued continuous functions on $\mathbb{R}$.
  d) Trigonometric functions belong to ϱ.

Solution: See Table 9.8.1

Given	$p_n(x) = x^n$ for $x \in \mathbb{R}$ and $\varrho = span\{p_0, p_1, p_2, ...\}$.
Vector space of real continuous functions on $\mathbb{R}$	The set S consisting of all real continuous functions on $\mathbb{R}$ forms a vector space. Let f and g be two real continuous functions from the set S. Since the sum of two continuous function is a continuous function. $i)$ Addition is commutative $f + g = g + f$ $ii)$ Addition is associative$f + (g + h) = (f + g) + h$ $iii)$There is unique O, zero function which maps every element to 0. $iv)$Additive inverse.For each f in S, $-f$ is a function in S. $v)$Properties of scalar multiplication.For $c, c_1, c_2 \in \mathbb{R}$, $a)$ $1f = f$ where the constant function 1 maps every element to 1. $b)$ $(c_1c_2)f = c_1(c_2f)$ $c)$ $c(f + g) = cf + cg$ $d)$ $c_1 + c_2)f = c_1f + c_2f$ Hence the set S forms a vector space.
Option 1	ϱ represents the vector space of polynomials. Polynomial functions are infintely continuously differentiable.So any function that is continuous but not differentiable can not be represented by polynomials. Example the function $\lvert x\rvert$ is continous but cannot be represented in polynomial basis.Therefore option 1 is incorrect.
Option 2	ϱ forms a subspace of all real valued continuous function on $\mathbb{R}$ Let α, β be two polynomial functions of order m and n, represented by the tuple of coefficients $(a_0, a_2, a_2..a_m)$ and $(b_0, b_1, b_2...b_n)$,then $c\alpha + \beta$ is also a polynomial function whose coefficients are $(ca_0 + b_0, ca_1 + b_1, ca_2 + b_2...)$ Therefore ϱ is a subspace of all real valued continuous functions on $\mathbb{R}$. For example consider two functions $f = \{2, 0, 4\}$ and $g = \{0, 2, 1, 5\}$,then $2f + g$ will be $2f + g = 2\left(2 + 4x^2\right) + \left(2x + x^2 + 5x^3\right) = 4 + 2x + 9x^2 + 5x^3 = \{4, 2, 9, 5\}$.
Option 3	Consider the expression $a_0p_0 + a_1p_1 + a_2p_2 + ... = O \implies a_0 = a_1 = a_2 = ... = 0$ Hence $\{p_0, p_1, p_2, ..\}$ are linearly independent set in the vector space of all real valued continuous functions on $\mathbb{R}$.
Option 4	The fundamental period of trigonometric functions is finite, where as polynomials are aperiodic. So, they cannot belong to the same class. For example $\sin x$ has a fundamental period of 2π. $\tan x$ is continuous in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, but is not defined at $k\frac{\pi}{2}$ where $k \in odd(\mathbb{N})$.

TABLE 9.8.1: Answer

9.9. Which of the following are subspaces of the vector space $\mathbb{R}^3$?

a) $(x, y, z) : x + y = 0$
b) $(x, y, z) : x - y = 0$
c) $(x, y, z) : x + y = 1$
d) $(x, y, z) : x - y = 1$

Solution: A subspace **S** of a vector space is defined as a non-empty subset that is closed under addition and scalar multiplication, i.e

a) All possible linear combinations of the vectors in **S** lie in the subspace.
b) Any vector in **S** scaled by a scalar c lies in the subspace.

We define any vector $\mathbf{V} \in \mathbf{S}$ for each of the subspaces defined in the options as:

$$\mathbf{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{9.9.1}$$

Option 1: Let $\mathbf{A} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} x_1 \\ y_1 \\ z_2 \end{pmatrix} \in \mathbf{S}$, and k_1 and k_2 be some scalars. As per definition:

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{B} = 0 \tag{9.9.2}$$

Verifying the property of the subspace by using the linear combination of **A** and **B**:

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\{k_1\mathbf{A} + k_2\mathbf{B}\} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}k_1\mathbf{A} + \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}k_2\mathbf{B} \tag{9.9.3}$$

$$\implies k_1\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{A} + k_2\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{B} = 0 \tag{9.9.4}$$

It is also evident from above that

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}c\mathbf{A} = c\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{A} = 0 \tag{9.9.5}$$

for some scalar c. Therefore, option 1 is a subspace of $\mathbb{R}^3$.

It can also be proven that option 2 is also a valid subspace of $\mathbb{R}^3$ as:

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}(c\mathbf{A}) = c\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\mathbf{A} = 0 \tag{9.9.6}$$

From the definition that $x - y = 0$

$$\implies \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\{k_1\mathbf{A} + k_2\mathbf{B}\} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}(k_1\mathbf{A}) + \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}(k_2\mathbf{B}) = 0 \in \mathbf{S} \tag{9.9.7}$$

for some scalars c, k_1 and k_2 and vectors **A** and $\mathbf{B} \in \mathbf{S}$. Option 3: Option 3 is not a valid subspace of $\mathbb{R}^3$ as it can be shown that for some scalars k_1 and k_2, **A** and $\mathbf{B} \in \mathbf{S}$ in the option:

$$\implies \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\{k_1\mathbf{A} + k_2\mathbf{B}\} = k_1\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{A} + k_2\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{B} = k_1 + k_2 \neq 1 \tag{9.9.8}$$

Because

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\mathbf{B} = 1 \tag{9.9.9}$$

from definition.

Similarly option 4 is also not a valid subspace of $\mathbb{R}^3$ as it can be be shown in similar manner that

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\{k_1\mathbf{A} + k_2\mathbf{B}\} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}(k_1\mathbf{A}) + \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}(k_2\mathbf{B}) = k_1 + k_2 \neq 1 \tag{9.9.10}$$

$$\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\mathbf{B} = 1 \tag{9.9.11}$$

Therefore, Options 1 and 2 are valid subspaces of the vector space $\mathbb{R}^3$

9.10. Let **A** be an invertible 4×4 real matrix. Which of the following are NOT true ?

a) Rank **A** =4
b) For every vector $\mathbf{b} \in \mathbb{R}$, $\mathbf{Ax} = \mathbf{b}$ has exactly one solution.
c) dim(nullspace **A**)$\geq$ 1
d) 0 is an eigenvalue of **A**

Solution: See Table 9.10.1

Given	$\mathbf{A}$ is an invertible real matrix of order 4×4
Solution	Since given $\mathbf{A}$ is an invertible matrix, $\mathbf{A}$ has full rank. $$det(\mathbf{A}) \neq 0 \quad (9.10.1)$$ $$Rank(\mathbf{A}) = 4 \quad (9.10.2)$$ Let $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the eigenvalues of matrix $\mathbf{A}$. We know that determinant of matrix $\mathbf{A}$ is the product of eigenvalues of $\mathbf{A}$. $$\lambda_1\lambda_2\lambda_3\lambda_4 \neq 0 \quad (9.10.3)$$
Statement 1	$$Rank(\mathbf{A}) = 4$$
	Since $\mathbf{A}$ is an invertible matrix,it has full rank as shown in equation (9.10.2). **True Statement**
Statement 2	For every vector $\mathbf{b} \in \mathbb{R}$, $\mathbf{Ax} = \mathbf{b}$ has exactly one solution.
	For every $\mathbf{b}$, $$\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$$ $\mathbf{x}$ will be unique solution for every $\mathbf{b}$. **True Statement**
Statement 3	dim(nullspace $\mathbf{A}$)$\geq$ 1.
	Using Rank Nullity Theorem, $$Rank(\mathbf{A}) + dim(nullspace\mathbf{A}) = n$$ $$\implies 4 + dim(nullspace\mathbf{A}) = 4$$ $$\implies dim(nullspace\mathbf{A}) = 0 \ngeq 1 \quad (9.10.4)$$
	where n is the number of columns in $\mathbf{A}$ Equation (9.10.4) proves that the given statement is **NOT True**.
Statement 4	0 is an eigenvalue of $\mathbf{A}$
	From equation (9.10.1), we could say that no eigenvalue of $\mathbf{A}$ could be 0. **NOT True Statement**

TABLE 9.10.1: Explanation

9.11. Consider non-zero vector spaces $\mathbf{V_1}, \mathbf{V_2}, \mathbf{V_3}, \mathbf{V_4}$ and linear transformations $\phi_1 : \mathbf{V_1} \rightarrow \mathbf{V_2}$, $\phi_2 : \mathbf{V_2} \rightarrow \mathbf{V_3}$, $\phi_3 : \mathbf{V_3} \rightarrow \mathbf{V_4}$ such that $Ker(\phi_1) = \{0\}$, $Range(\phi_1) = Ker(\phi_2)$, $Range(\phi_2) = Ker(\phi_3)$, $Range(\phi_3) = \mathbf{V_4}$. Then

a) $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} = 0$

b) $\sum_{i=2}^{4} (-1)^i \; dim \; \mathbf{V_i} > 0$

c) $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} < 0$

d) $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} \neq 0$

Solution: See Table 9.11.1 9.11.3

Kernel and Nullity	Given a linear transformation $L : \mathbf{V} \to \mathbf{W}$ between wo vector spaces $\mathbf{V}$ and $\mathbf{W}$, the kernel of L is the set of all vectors $\mathbf{v}$ of $\mathbf{V}$ for which $L(\mathbf{v}) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in $\mathbf{W}$. i.e. $$Ker(L) = \{\mathbf{v} \in \mathbf{V} \mid L(\mathbf{v}) = 0\}$$ Nullity of the linear transformation is the dimension of the kernel of the linear transformation i.e. $$nullity(L) = dim(Ker(L))$$
Range and Rank	Given a linear transformation $L : \mathbf{V} \to \mathbf{W}$ between wo vector spaces $\mathbf{V}$ and $\mathbf{W}$, the range of L is the set of all vectors $\mathbf{w}$ in $\mathbf{W}$ given as $$Range(L) = \{\mathbf{w} \in \mathbf{W} \mid \mathbf{w} = L(\mathbf{v}), \mathbf{v} \in \mathbf{V}\}$$ The rank of a linear transformation L is the dimension of it's range, i.e. $$rank(L) = dim(Range(L))$$
Rank-Nullity Theorem	Let $\mathbf{V}$, $\mathbf{W}$ be vector spaces, where $\mathbf{V}$ is finite dimensional. Let $L : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Then $$rank(L) + nullity(L) = dim(\mathbf{V})$$

TABLE 9.11.1

Inference from the Given Data	$Ker(\phi_1) = \{0\}$ $\implies nullity(\phi_1) = 0$ $Range(\phi_1) = Ker(\phi_2)$ $\implies rank(\phi_1) = nullity(\phi_2)$ $Range(\phi_2) = Ker(\phi_3)$ $\implies rank(\phi_2) = nullity(\phi_3)$ $Range(\phi_3) = \mathbf{V_4}$

	$\implies rank(\phi_3) = dim(\mathbf{V_4})$ Now talking about the linear transformations we can use rank-nullity theorem to determine the corresponding dimensions of the vector space. $\phi_1 : \mathbf{V_1} \to \mathbf{V_2}$ $\implies rank(\phi_1) + nullity(\phi_1) = dim(\mathbf{V_1})$ $\implies rank(\phi_1) = dim(\mathbf{V_1}) \qquad (\because nullity(\phi_1) = 0)$ $\phi_2 : \mathbf{V_2} \to \mathbf{V_3}$ $\implies rank(\phi_2) + nullity(\phi_2) = dim(\mathbf{V_2})$ $\implies rank(\phi_2) + rank(\phi_1) = dim(\mathbf{V_2}) \qquad (\because rank(\phi_1) = nullity(\phi_2))$ $\implies rank(\phi_2) + dim(\mathbf{V_1}) = dim(\mathbf{V_2}) \qquad (\because rank(\phi_1) = dim(\mathbf{V_1}))$ $\phi_3 : \mathbf{V_3} \to \mathbf{V_4}$ $\implies rank(\phi_3) + nullity(\phi_3) = dim(\mathbf{V_3})$ $\implies rank(\phi_3) + rank(\phi_2) = dim(\mathbf{V_3}) \qquad (\because rank(\phi_2) = nullity(\phi_3))$ $\implies rank(\phi_3) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) = dim(\mathbf{V_3}) \; (\because rank(\phi_2) + dim(\mathbf{V_1}) = dim(\mathbf{V_2}))$ $\implies dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) = dim(\mathbf{V_3}) \qquad (\because rank(\phi_3) = dim(\mathbf{V_4}))$ From the above equation we can infer that $dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$
Option 1	It is given that $\sum_{i=1}^{4} (-1)^i \; dim \; \mathbf{V_i} = 0$ $\implies -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) = 0$ This statement we already proved above. $\therefore$ this statement is **True**.
Option 2	It is given that $\sum_{i=2}^{4} (-1)^i \; dim \; \mathbf{V_i} > 0$ $\implies dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) > 0$

	Our original derived equation is $$\begin{aligned} & dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0 \\ \implies & dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) = dim(\mathbf{V_1}) \end{aligned}$$ It is given in the question that the vector spaces are non-zero in nature. $\implies dim(\mathbf{V_1}) > 0$ $$\therefore dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) > 0$$ $\therefore$ this statement is **True**.
Option 3	It is given that $\sum_{i=1}^{4} (-1)^i \ dim \ \mathbf{V_i} < 0$ $\implies -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) < 0$ This is contrary to our original derived equation i.e. $$dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$$ $\therefore$ this statement is **False**.
Option 4	It is given that $\sum_{i=1}^{4} (-1)^i \ dim \ \mathbf{V_i} \neq 0$ $\implies -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) \neq 0$ This is contrary to our original derived equation i.e. $$dim(\mathbf{V_4}) + dim(\mathbf{V_2}) - dim(\mathbf{V_1}) - dim(\mathbf{V_3}) = 0$$ $\therefore$ this statement is **False**.
Conclusion	From our observation we see that Options 1) and 2) are True.

Linear Transforms	Let $\phi_1 : \mathbf{R}^2 \to \mathbf{R}^3$ defined as

Example

$$\phi_1\left\{\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\}=\begin{pmatrix}x_1-x_2\\x_1+x_2\\x_2\end{pmatrix}$$

$$\implies \phi_1\left\{\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\}=\begin{pmatrix}1&-1\\1&1\\0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

For the above transformation ϕ_1 the kernel and the range are

$$Ker(\phi_1)=\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}\qquad\implies nullity(\phi_1)=0$$

$$Range(\phi_1)=\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}\qquad\implies rank(\phi_1)=2$$

We can verify the rank-nullity theorem here as

$$\begin{aligned}&nullity(\phi_1)+rank(\phi_1)\\\implies&0+2\\\implies&2=dim(\mathbf{R}^2)\end{aligned}$$

Let $\phi_2:\mathbf{R}^3\to\mathbf{R}^3$ defined as

$$\phi_2\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right\}=\begin{pmatrix}x_1-x_2+2x_3\\2x_1-2x_2+4x_3\\3x_1-3x_2+6x_3\end{pmatrix}$$

$$\implies \phi_2\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right\}=\begin{pmatrix}1&-1&2\\2&-2&4\\3&-3&6\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

For the above transformation ϕ_2 the kernel and the range are

$$Ker(\phi_2)=\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}\qquad\implies nullity(\phi_2)=2$$

$$Range(\phi_2)=\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}\qquad\implies rank(\phi_2)=1$$

We can verify the rank-nullity theorem here as

$$\begin{aligned}&nullity(\phi_2)+rank(\phi_2)\\\implies&2+1\\\implies&3=dim(\mathbf{R}^3)\end{aligned}$$

	In the above two transformations ϕ_1 and ϕ_2, we can see the following conditions being satisfied $$Ker(\phi_1) = \{0\},\ Range(\phi_1) = Ker(\phi_2)$$ Let $\phi_3 : \mathbf{R}^3 \to \mathbf{R}^2$ defined as $$\phi_3 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} x_1 + x_2 - x_3 \\ 2x_1 + \frac{1}{2}x_2 - x_3 \end{pmatrix}$$ $$\implies \phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$ For the above transformation ϕ_3 the kernel and the range are $$Ker(\phi_3) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \qquad \implies nullity(\phi_3) = 1$$ $$Range(\phi_3) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\} \qquad \implies rank(\phi_3) = 2$$ We can verify the rank-nullity theorem here as $$\begin{aligned} & nullity(\phi_3) + rank(\phi_3) \\ \implies & 1 + 2 \\ \implies & 3 = dim(\mathbf{R}^3) \end{aligned}$$ With the above ϕ_3 transformation we were able to satisfy the other conditions as well i.e. $$Range(\phi_2) = Ker(\phi_3),\ Range(\phi_3) = \mathbf{V_4}$$ Now, when we can check whether the derived equation statisfies or not. That is, $$\begin{aligned} & -dim(\mathbf{V_1}) + dim(\mathbf{V_2}) - dim(\mathbf{V_3}) + dim(\mathbf{V_4}) \\ \implies & -dim(\mathbf{R}^2) + dim(\mathbf{R}^3) - dim(\mathbf{R}^3) + dim(\mathbf{R}^2) \\ \implies & -2 + 3 - 3 + 2 = 0 \end{aligned}$$ $\therefore$ the condition is getting satisfied.

TABLE 9.11.3

9.12. Let $\mathbf{u}$ be a real n × 1 vector satisfying $\mathbf{u}^T\mathbf{u} = 1$, where $\mathbf{u}^T$ is the transpose of $\mathbf{u}$.Define
$\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ where $\mathbf{I}$ is the n^{th} order identity matrix.Which of the following statements are true?
1. $\mathbf{A}$ is singular
2. $\mathbf{A}^2 = \mathbf{A}$
3. Trace($\mathbf{A}$)=n-2
4. $\mathbf{A}^2 = \mathbf{I}$

Solution: See Table 9.12.1

Theorem 1. Let $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times k}$ be matrices such that the product $\mathbf{AB}$ is well defines. Then

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (9.12.1)$$

Proof: Matrix $\mathbf{A}$ can be treated as a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$.In that case rank of the matrix is the dimension of the image space of the transformation. If $\mathbf{T}$ is a linear transformation from $\mathbf{V}_1$ to $\mathbf{V}_2$ then clearly dim $\mathbf{T}(\mathbf{V}_1) \leq$ dim $(\mathbf{V}_1)$.Hence rank($\mathbf{AB}$) $\leq$ rank($\mathbf{B}$). Since row rank and column rank of a matrix are equal,

$$\text{Therefore rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (9.12.2)$$

Explanation

Statement	Solution
1.	$$\text{Let } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$ $$\text{Let } \mathbf{B} = \mathbf{u}\mathbf{u}^T$$ $$\therefore \mathbf{B} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$$ $$\therefore \mathbf{B} = \begin{pmatrix} u_1^2 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2^2 & \dots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_n^2 \end{pmatrix}$$ given that, $\mathbf{u}^T\mathbf{u} = 1$ $$\therefore \mathbf{u}^T\mathbf{u} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$ $$\therefore \mathbf{u}^T\mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2$$ Since $\mathbf{u}$ is non-zero vector and $\mathbf{B} = \mathbf{u}\mathbf{u}^T$. Hence $\mathbf{B}$ is a non-zero matrix. Therefore Rank of $\mathbf{B}$ is at least 1. From (9.12.2) $$\begin{aligned} \operatorname{rank}(\mathbf{B}) &\leq \min(\operatorname{rank}(\mathbf{u}), \operatorname{rank}(\mathbf{u}^T)) \\ \therefore \operatorname{rank}(\mathbf{B}) &\leq \min(1, 1) \end{aligned}$$ So Rank of $\mathbf{B}$ is at most 1. Hence Rank of $\mathbf{B}$ is equal to 1. Therefore $\mathbf{B}$ has n-1 eigenvalues equal to 0. Since the trace of a matrix is equal to the sum of its eigen values. We know that trace of $\mathbf{B} = u_1^2 + u_2^2 + \cdots + u_n^2 = 1$ $$\begin{aligned} \therefore \text{Trace of } \mathbf{B} &= \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} + \lambda_n \\ 1 &= 0 + 0 + \cdots + \lambda_n \\ \therefore \lambda_n &= 1 \end{aligned}$$ Therefore the eigen values of $\mathbf{B}$ are $\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1$ Hence the characteristic polynomial for $\mathbf{B} = x^{n-1}(x-1)$ Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ and we know the eigen values of $\mathbf{I}$ are $\lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = 1$

	and we know the eigen values of $\mathbf{uu^T}$ are $\lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_{n-1} = 0, \lambda_n = 1$ $\therefore$ The eigen values of $\mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \ldots, \lambda_{n-1} = 1, \lambda_n = -1 \quad (9.12.3)$
Example	$$\text{Let } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (9.12.4)$$ $$\text{then } \mathbf{u}^T = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (9.12.5)$$ $$\text{which satisfies } \mathbf{u}^T\mathbf{u} = 1 \quad (9.12.6)$$ $$\therefore \mathbf{uu}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9.12.7)$$ $$\text{Since } \mathbf{A} = \mathbf{I} - 2\mathbf{uu}^T \quad (9.12.8)$$ $$\therefore \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (9.12.9)$$ $$\therefore \mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (9.12.10)$$ $$\therefore \text{The eigen values of } \mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1 \quad (9.12.11)$$ $$\therefore \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (9.12.12)$$
Conclusion	From (9.12.3) Since **A** does not have 0 as an eigen value Therefore **A** is not singular. Therefore the statement is false.
2.	For $\mathbf{A}^2 = \mathbf{A}$, we know that $p(x) = x^2 - x$ $\therefore$ minimal polynomial of **A** must divide x(x-1) $\therefore$ possible eigenvalues of **A** are 0 or 1 But from (9.12.3) , we know that **A** has -1 as an eigen value Therefore $\mathbf{A}^2 = \mathbf{A}$ is false.
Conclusion	Therefore the statement is false.
3.	

	From equation (9.12.3) , Trace of $\mathbf{A} = n - 2$
Conclusion	Therefore the statement is true.
4.	Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ $\mathbf{A}^2 = (\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)(\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T$ Since $\mathbf{u}^T\mathbf{u} = 1$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T$ $\therefore \mathbf{A}^2 = \mathbf{I}$
Conclusion	Therefore the statement is true.

TABLE 9.12.1: Solution summary

10 December 2014

10.1. Let A,B be $n \times n$ matrices such that $BA + B^2 = I - BA^2$ where I is the $n \times n$ identity matrix. Which of the following is always correct

a) A is non singular
b) B is non singular
c) A+B is non singular
d) AB is non singular

Solution: See Table 10.1.1

Statement	**Solution**
Given Condition	$$BA + B^2 = I - BA^2 \tag{10.1.1}$$
Solution by Theory	We will first provide theoretical proof

Theory	As per definition of invertible matrix,A matrix 'B' in our case is defined as invertible if there exists left and right inverse of B such that BC=CB=I In that case C is called the two sided inverse of B and B is said to be invertible. Now refer(10.1.1) we get $$BA + B^2 = I - BA^2 \quad (10.1.2)$$ $$\implies BA + B^2 + BA^2 = I \quad (10.1.3)$$ $$\implies I = B\left(A + B + A^2\right) \quad (10.1.4)$$ $$(10.1.5)$$ Let C= $\left(A + B + A^2\right)$ rewrite (10.1.4) as $$I = BC \quad (10.1.6)$$ Also $$I = \left(A + B + A^2\right)B \quad (10.1.7)$$ Let D= $\left(A + B + A^2\right)$ rewrite (10.1.7) as $$I = DB \quad (10.1.8)$$ Now we can write $$D = DI \quad (10.1.9)$$ Ref (10.1.6) $$= D(BC) \quad (10.1.10)$$ $$= (DB)C \quad (10.1.11)$$ $$(10.1.12)$$ Ref (10.1.8) $$= IC \quad (10.1.13)$$ $$= C \quad (10.1.14)$$ $$\implies D = C \quad (10.1.15)$$ Hence by definition stated above we imply that Left inverse=Right inverse. So by looking at (10.1.4),we imply that B has a left and right inverse $$\implies I = BB^{-1} \quad (10.1.16)$$ $$\implies \text{B is invertible} \quad (10.1.17)$$ $\therefore$ B is non singular.Hence Option 2 is correct
Solution by examples	We will check each respective options through examples

Option 3	Let us take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.1.18}$ $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1.19}$ Take L.H.S of (10.1.1) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1.20}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{10.1.21}$ Take R.H.S of (10.1.1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.1.22}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{10.1.23}$ Our assumption satisfies (10.1.1). Now $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1.24}$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{10.1.25}$ $\therefore \left\lvert A + B \right\rvert = 0$ the respective option is Singular. Hence Option 3 is incorrect
Option 1	Now let us take $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1.26}$ Substituting(10.1.26) in (10.1.1) Take L.H.S of (10.1.1) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10.1.27}$ $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.1.28}$ Take R.H.S of (10.1.1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{10.1.29}$ $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.1.30}$ Our assumption satisfies (10.1.1) But $\left\lvert A \right\rvert = 0$ $\therefore$ the respective option is Singular. Hence Option 1 is incorrect

Option 4	Similarly $$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (10.1.31)$$ $$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (10.1.32)$$ Here also $\left\lvert AB \right\rvert = 0$ $\therefore$ the AB option is also Singular. Hence Option 4 is incorrect also
Correct Answer	So we conclude that Option 2 is correct by eliminating other options

TABLE 10.1.1: Solution

10.2. Which of the following matrices has the same row space as the matrix $\begin{pmatrix} 4 & 8 & 4 \\ 3 & 6 & 1 \\ 2 & 4 & 0 \end{pmatrix}$?

a) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

10.3. The determinant of n x n permutation matrix

$$\begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \cdot & & \\ & & \cdot & & & \\ & \cdot & & & & \\ 1 & & & & & \\ 1 & & & & & \end{pmatrix}$$

a) $(-1)^n$

b) $(-1)^{\lfloor \frac{n}{2} \rfloor}$

c) -1

d) 1

Solution: See Tables 10.3.1 and 10.3.2

Given	n x n permutation matrix $\begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & . & & \\ & & . & & & \\ & . & & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$
Proof of row exchange	The given n x n permutation matrix can be converted into identity matrix of n x n dimension by doing row exchange operations. Let $\mathbf{A} = \begin{pmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{pmatrix}$ $\begin{vmatrix} a_1 \\ . \\ . \\ a_i + a_j \\ a_i + a_j \\ . \\ . \\ a_n \end{vmatrix} = 0$ since determinant of a any matrix will be zero, if it has dependent rows. Expanding the above using linear property of determinants $\begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_i \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_i \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_j \\ . \\ . \\ a_n \end{vmatrix} = 0$

	$\implies 0 + \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_i \\ . \\ . \\ a_n \end{vmatrix} + \begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{vmatrix} + 0 = 0$ $\implies \begin{vmatrix} a_1 \\ . \\ . \\ a_j \\ a_i \\ . \\ . \\ a_n \end{vmatrix} = (-1) \begin{vmatrix} a_1 \\ . \\ . \\ a_i \\ a_j \\ . \\ . \\ a_n \end{vmatrix}$ Hence it is proved that the exchange of rows a_i *and* a_j changes the sign of the determinant. $\therefore$ for every row exchange in given permutation matrix the determinant gets multiplied by -1.
finding no of exchanges	Let $\mathbf{A} = \begin{pmatrix} a_1 & . & a_i & a_{i+1}. & a_n \end{pmatrix}$ if n is even number then the elements a_1 *to* a_i will be exchanged with a_{i+1} *to* a_n where $i = \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$. if n is odd, the center element will be a_{i+1} where $i + 1 = \lceil \frac{n}{2} \rceil$ then $i = \lfloor \frac{n}{2} \rfloor$ and the elements a_1 *to* a_i will be exchanged with a_{i+2} *to* a_n. $\therefore$ The given n x n matrix requires $\lfloor \frac{n}{2} \rfloor$ row exchanges to become identity matrix.
finding determinant	from the above results the determinant of given permutation matrix is $(-1)^{\lfloor \frac{n}{2} \rfloor} \begin{vmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & . & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & 1 & \\ & & & & & & 1 \end{vmatrix}$ we know that the determinant of identity matrix, $det(\mathbf{I}) = 1$ $\therefore$ the determinant of given n x n permutation matrix $= (-1)^{\lfloor \frac{n}{2} \rfloor}$
Conclusion	Option-2 is the right solution

TABLE 10.3.1: Solution

<table>
<tr>
<td>Example-1</td>
<td>

Let **A** is 5 x 5 permutation matrix, then

$$det(\mathbf{A}) = \begin{vmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{vmatrix}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_5} (-1) \begin{vmatrix} 1 & & & & \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ & & & & 1 \end{vmatrix}$$

$$\xleftrightarrow{R_2 \leftrightarrow R_4} (-1)(-1) \begin{vmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{vmatrix}$$

$= 1$

substituting n = 5 in the solution
$(-1)^{\lfloor \frac{5}{2} \rfloor} = 1$

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<tr>
<td>Example-2</td>
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Let **A** is 6 x 6 permutation matrix, then

$$det(\mathbf{A}) = \begin{vmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{vmatrix}$$

$$\xleftrightarrow{R_1 \leftrightarrow R_6} (-1) \begin{vmatrix} 1 & & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & & 1 \end{vmatrix}$$

$$\xleftrightarrow{R_2 \leftrightarrow R_5} (-1)(-1) \begin{vmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \end{vmatrix}$$

$$\xleftrightarrow{R_3 \leftrightarrow R_4} (-1)(-1)(-1) \begin{vmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{vmatrix}$$

$= -1$

</td>
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</table>

	substituting n = 6 in the solution $(-1)^{\lfloor \frac{6}{2} \rfloor} = -1$ Hence the proved that the solution is correct.

TABLE 10.3.2: Example

10.4. Let $\mathbf{P}$ be a 2×2 complex matrix such that

$$\mathbf{P}^{\theta}\mathbf{P} = \mathbf{I} \tag{10.4.1}$$

where $\mathbf{P}^{\theta}$ is the conjugate transpose of $\mathbf{P}$.Then the eigen values of $\mathbf{P}$ are

a) real
b) complex conjugates of each othe
c) reciprocals of each other
d) of modulus 1

Solution: See Table 10.4.1

10.5. Let $\mathbf{A}$ be a real n x n orthogonal matrix, that is, $\mathbf{A^TA} = \mathbf{AA^T} = \mathbf{I_n}$, the n x n identity matrix. which of the following statements are necessarily true?

a) $< \mathbf{Ax}, \mathbf{Ay} >=< \mathbf{x}, \mathbf{y} > \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$
b) All eigen values of $\mathbf{A}$ are either +1 or -1.
c) The rows of $\mathbf{A}$ form an orthonormal basis of $\mathbf{R}^n$.
d) $\mathbf{A}$ is diagonalizable over $\mathbf{R}$.

Solution:

$$< \mathbf{Ax}, \mathbf{Ay} > = (\mathbf{Ax})^{\mathbf{T}}\, \mathbf{Ay} \tag{10.5.1}$$
$$= \mathbf{x^TA^TAy} \tag{10.5.2}$$
$$= \mathbf{x^Ty} \quad \because \mathbf{A^TA} = \mathbf{I} \tag{10.5.3}$$
$$= < \mathbf{x}, \mathbf{y} > \tag{10.5.4}$$

Hence, option 1 is correct.

10.1 Option 2

Let λ be the eigen value and $\mathbf{v}$ be the eigen vector corresponding to it.
Then,

$$\mathbf{Av} = \lambda\mathbf{v} \tag{10.5.1}$$
$$\implies \|\mathbf{Av}\|^2 = \|\lambda\mathbf{v}\|^2 \tag{10.5.2}$$
$$\implies \|\mathbf{Av}\|^2 = |\lambda|^2\, \|\mathbf{v}\|^2 \tag{10.5.3}$$

Now,

$$\|\mathbf{Av}\|^2 = (\mathbf{Av})^T\, \mathbf{Av} \tag{10.5.4}$$
$$= \mathbf{v^TA^TAv} \tag{10.5.5}$$
$$= \mathbf{v^TIv} \tag{10.5.6}$$
$$= \mathbf{v^Tv} \tag{10.5.7}$$
$$= \|\mathbf{v}\|^2 \tag{10.5.8}$$

Comparing (10.5.3) and (10.5.8), we get,

$$|\lambda|^2 = 1 \tag{10.5.9}$$
$$\implies |\lambda| = \pm 1 \tag{10.5.10}$$

But $|\lambda|$ cannot be -1.

$$\therefore |\lambda| = 1 \tag{10.5.11}$$
$$\implies \lambda = \pm 1 \tag{10.5.12}$$

Thus, option 2 is correct.

10.2 Option 3

Let $\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_n}$ denote the row vectors of $\mathbf{A}$. Then,

$$\mathbf{AA^T} = \begin{pmatrix} \mathbf{r_1^Tr_1} & \mathbf{r_1^Tr_2} & ... & \mathbf{r_1^Tr_n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \mathbf{r_n^Tr_1} & \mathbf{r_n^Tr_2} & ... & \mathbf{r_n^Tr_n} \end{pmatrix} \tag{10.5.1}$$

But, $\mathbf{A}$ is orthogonal. So, $\mathbf{AA^T} = \mathbf{I}$. It therefore follows that

a) All diagonal elements of (10.5.1) are 1.
b) All off- diagonal elements of (10.5.1) are 0.

That is, for all $i, j = 1, 2, ..., n$,

$$\mathbf{r_i^Tr_j} = 1, \quad i = j \tag{10.5.2}$$
$$= 0, \quad i \neq j \tag{10.5.3}$$

Therefore, $\mathbf{r_1}, \mathbf{r_2}, ...\mathbf{r_n}$ are orthonormal and form a basis of $\mathbf{R}^n$.
Hence, option 3 is correct.

10.3 Option 4

Counter Example:
Let us consider a matrix in $\mathbf{R}^2$

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.5.1}$$
$$\therefore \mathbf{Q}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{10.5.2}$$

Check that $\mathbf{AA^T} = \mathbf{I}, \therefore \mathbf{Q}$ is orthogonal.
The characteristic equation is:

$$\left|\mathbf{Q} - \lambda\mathbf{I}\right| = 0 \tag{10.5.3}$$
$$\implies \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \tag{10.5.4}$$
$$\implies \lambda^2 + 1 = 0 \tag{10.5.5}$$
$$\implies \lambda = \pm i \notin \mathbf{R} \tag{10.5.6}$$

which implies $\mathbf{Q}$ is not diagonalizable over $\mathbf{R}$.

Hence, we can conclude that option 1, 2 and 3 are correct.

Options	Explanation
REAL Counter Example	$\mathbf{P} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ $\mathbf{P}^{\theta} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$ $\mathbf{P}^{\theta}\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ Eigen values of $\mathbf{P}$ are i, i which are not real Hence,incorrect.
Complex Conjugates of each other.	From above,(i, i) are not complex conjugate of each other Hence,incorrect.
Reciprocals of each other	Reciprocal of $i = \frac{1}{i} = \frac{i^4}{i} = i^3 \neq i$ Hence,incorrect.
of modulus 1 Proof	$\mathbf{PV} = \lambda\mathbf{V}$ where,$\mathbf{V}$ is eigen vector of $\mathbf{P}$ and λ is eigen value of $\mathbf{P}$ Taking conjugate transpose on both sides,we get $\mathbf{V}^{\theta}\mathbf{P}^{\theta} = \lambda^{\theta}\mathbf{V}^{\theta}$ $\mathbf{V}^{\theta}\mathbf{P}^{\theta}\mathbf{PV} = \lambda^{\theta}\mathbf{V}^{\theta}\lambda\mathbf{V} \qquad , \because \mathbf{PV} = \lambda\mathbf{V}$ $\mathbf{V}^{\theta}\mathbf{IV} = \lambda^{\theta}\lambda\mathbf{V}^{\theta}\mathbf{V} \qquad , \because \mathbf{P}^{\theta}\mathbf{P} = \mathbf{I}$ $(1 - \lambda^{\theta}\lambda)\mathbf{V}^{\theta}\mathbf{V} = 0$ Since,$\mathbf{V}$ is not zero. $(1 - \lambda^{\theta}\lambda) = 0$ $\lambda^{\theta}\lambda = 1$ $\|\lambda\|^2 = 1$ $\lambda = 1$ Hence,correct.

TABLE 10.4.1: Finding Correct Option

10.6. Which of the following matrices have Jordan canonical form equal to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}?$$

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \lvert x\mathbf{I} - \mathbf{A} \rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 10.6.1: Definitions

1. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: See Tables 10.6.1 10.6.2 and 10.6.3.

Statement	Solution
1.	Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} \neq \mathbf{0}$ Therefore, $m(x) = (x)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 0$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 0$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 1 is true.

2.	Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Since **A** is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} \neq \mathbf{0}$ Therefore, $m(x) = (x)^2$
Justification	Hence, the Jordan form of **A** is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 0$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 0$. Hence the required Jordan form of **A** is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 2 is true.

3.	Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since **A** is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} \neq \mathbf{0}$ Therefore, $m(x) = (x)^2$
Justification	Hence, the Jordan form of **A** is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 0$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 0$. Hence the required Jordan form of **A** is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 3 is true.

4.	Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Since **A** is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}^2 \neq \mathbf{0}$ Therefore, $m(x) = (x)^3$
Justification	Hence, the Jordan form of **A** is a 3×3 matrix consisting of only one block with principal diagonal values as $\lambda = 0$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of **A** is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 4 is false.

TABLE 10.6.2: Solution

For given jordan form:	$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
We have two blocks:	one block is of order 2. And one block is of order 1. And eigenvalues are all $\lambda = 0$ $\therefore$ Algebraic Multiplicity of 0 is 3. The rank of the matrix is 1.

	Geometric Multiplicity of $0 = n - \text{Rank}(\mathbf{A} - \lambda\mathbf{I})$ $= n - \text{Rank}(\mathbf{A})$ $= 2$
1.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 2. The rank of the matrix is 1. $\therefore$ The Geometric Multiplicity of 0 = 2. Therefore the matrix gives the same jordan form
2.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 2. The rank of the matrix is 1. $\therefore$ The Geometric Multiplicity of 0 = 2. Therefore the matrix gives the same jordan form
3.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 2. The rank of the matrix is 1. $\therefore$ The Geometric Multiplicity of 0 = 2. Therefore the matrix gives the same jordan form
4.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 3. The rank of the matrix is 2. $\therefore$ The Geometric Multiplicity of 0 = 1. Therefore the matrix gives different jordan form

TABLE 10.6.3: Conclusion of above Results

10.7. Let f be a non-zero symmetric bilinear form on $\mathbb{R}^3$.Suppose that there exist linear transformations $T_i : \mathbb{R}^3 \to \mathbb{R}, i = 1, 2$ such that for all $\alpha, \beta \in \mathbb{R}^3$, $f(\alpha, \beta) = T_1(\alpha) T_2(\beta)$. Then

a) rank $f = 1$

b) $\dim \{\beta \in \mathbb{R}^3 : f(\alpha, \beta) = 0 \text{ for all } \alpha \in \mathbb{R}^3\} = 2$

c) f is positive semi-definite or negative semi-definite

d) $\{\alpha : f(\alpha, \alpha) = 0\}$ is a linear subspace of dimension 2

Solution: See Tables 10.7.1, 10.7.2 and 10.7.3

Definition of bilinear form	A bilinear form on a vector space $\mathbf{V}$ is a function f, which assigns to each ordered pair of vectors α,β in $\mathbf{V}$ a scalar $f(\alpha,\beta)$ in field $\mathbf{F}$ which satisfies $i)\ f(c\alpha_1+\alpha_2,\beta)=cf(\alpha_1,\beta)+f(\alpha_2,\beta)$ $ii)\ f(\alpha,c\beta_1+\beta_2)=cf(\alpha,\beta_1)+f(\alpha,\beta_2)$
Symmetric bilinear form	A bilinear form on the vector space $\mathbf{V}$ is symmetric if $f(\alpha,\beta)=f(\beta,\alpha)$ for all vectors $\alpha,\beta\in\mathbf{V}$
Matrix of bilinear form	Let $\alpha,\beta\in\mathbb{R}^3$ be two vectors, which are represented in standard basis as $\alpha=\alpha_1\mathbf{e_1}+\alpha_2\mathbf{e_2}+\alpha_3\mathbf{e_3}$ and $\beta=\beta_1\mathbf{e_1}+\beta_2\mathbf{e_2}+\beta_3\mathbf{e_3}$,therefore $f(\alpha,\beta)$ can be represented in matrix form as $f(\alpha,\beta)=f(\alpha_1\mathbf{e_1}+\alpha_2\mathbf{e_2}+\alpha_3\mathbf{e_3},\beta_1\mathbf{e_1}+\beta_2\mathbf{e_2}+\beta_3\mathbf{e_3})$ $=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}f(\mathbf{e_1},\mathbf{e_1}) & f(\mathbf{e_1},\mathbf{e_2}) & f(\mathbf{e_1},\mathbf{e_3})\\ f(\mathbf{e_2},\mathbf{e_1}) & f(\mathbf{e_2},\mathbf{e_2}) & f(\mathbf{e_2},\mathbf{e_3})\\ f(\mathbf{e_3},\mathbf{e_1}) & f(\mathbf{e_3},\mathbf{e_2}) & f(\mathbf{e_3},\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$
Given	Given a non-zero symmetric bilinear form f such that $f(\alpha,\beta)=T_1(\alpha)T_2(\beta)$ where $\alpha,\beta\in\mathbb{R}^3$. So the symmetric bilinear form can be represented on matrix form as $f(\alpha,\beta)=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}f(\mathbf{e_1},\mathbf{e_1}) & f(\mathbf{e_1},\mathbf{e_2}) & f(\mathbf{e_1},\mathbf{e_3})\\ f(\mathbf{e_2},\mathbf{e_1}) & f(\mathbf{e_2},\mathbf{e_2}) & f(\mathbf{e_2},\mathbf{e_3})\\ f(\mathbf{e_3},\mathbf{e_1}) & f(\mathbf{e_3},\mathbf{e_2}) & f(\mathbf{e_3},\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$ $f(\alpha,\beta)=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}T_1(\mathbf{e_1})T_2(\mathbf{e_1}) & T_1(\mathbf{e_1})T_2(\mathbf{e_2}) & T_1(\mathbf{e_1})T_2(\mathbf{e_3})\\ T_1(\mathbf{e_2})T_2(\mathbf{e_1}) & T_1(\mathbf{e_2})T_2(\mathbf{e_2}) & T_1(\mathbf{e_2})T_2(\mathbf{e_3})\\ T_1(\mathbf{e_3})T_2(\mathbf{e_1}) & T_1(\mathbf{e_3})T_2(\mathbf{e_2}) & T_1(\mathbf{e_3})T_2(\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}$ $f(\alpha,\beta)=\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\end{pmatrix}\begin{pmatrix}T_1(\mathbf{e_1})\\ T_1(\mathbf{e_2})\\ T_1(\mathbf{e_3})\end{pmatrix}\begin{pmatrix}T_2(\mathbf{e_1}) & T_2(\mathbf{e_2}) & T_2(\mathbf{e_3})\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}=\alpha^T\mathbf{T_1}\mathbf{T_2}^T\beta$ where $\mathbf{T_1}=\begin{pmatrix}T_1(\mathbf{e_1})\\ T_1(\mathbf{e_2})\\ T_1(\mathbf{e_3})\end{pmatrix}$ and $\mathbf{T_2}=\begin{pmatrix}T_2(\mathbf{e_1})\\ T_2(\mathbf{e_2})\\ T_2(\mathbf{e_3})\end{pmatrix}$ are the matrix representation of the linear transformations T_1,T_2.So, the matrix representation of f is $\mathbf{T_1}\mathbf{T_2}^T$ or $\mathbf{T_2}\mathbf{T_1}^T$ since f is symmetric. *note* : Since f is non-zero symmetric bilinear form $rank(\mathbf{T_1})=rank(\mathbf{T_2})=1$

TABLE 10.7.1: Construction

Option 1	By using the property of rank of product of two matrices, we get $rank(f)=rank\left(\mathbf{T_1}\mathbf{T_2}^T\right)\le min(rank(\mathbf{T_1}),rank(\mathbf{T_2}))\le 1.$ Since f is non-zero the $rank(f)\neq 0$. Hence the $rank(f)=1$
Option 2	$\beta\in\mathbb{R}^3: f(\alpha,\beta)=0$ for all $\alpha\in\mathbb{R}^3 \implies \beta\in\mathbb{R}^3: T_2(\beta)=0$ for all $\alpha\in\mathbb{R}^3$ because $T_1(\alpha)\neq 0$ for all $\alpha\in\mathbb{R}^3$. By using rank nullity theorem
	$rank\{T_2\}+dim\{Nullspace(T_2)\}=3 \implies dim\{Nullspace(T_2)\}=2$. Similarly for T_1, we get dim{Nullspace(T_1)}=2. Therefore dim $\{\beta\in\mathbb{R}^3: f(\alpha,\beta)=0$ for all $\alpha\in\mathbb{R}^3\}=dim\{Nullspace(T_1)\}=dim\{Nullspace(T_2)\}=2$
Option 3	By using rank nullity theorem we get $rank(f)+dim\{nullspace(f)\}=3$. We know that $rank(f)=1 \implies dim\{nullspace(f)\}=2$. Therefore two eigen values of f will be 0. Since the matrix is a symmetric matrix the eigen values are real. So, the third eigen value can be either positive or negative. So, the matrix will be either positive semi-definite or negative semi-definite accordingly.This option is correct.
Option 4	$\{\alpha: f(\alpha,\alpha)=0\}$ is a linear subspace of dimension 2. Since the $dim\{nullspace(f)\}=2$, and f is diagonalizable,since it is a symmetric, the two eigen vectors corresponding to 0

	eigen values form a subspace of dimension 2.

TABLE 10.7.2: Answer

Construction	Consider the non-zero symmetric bilinear form $f(\alpha,\beta) = T_1(\alpha)T_2(\beta)$ on $\mathbb{R}^3$ where Where the matrix of linear transformations are $\mathbf{T_1} = \begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\mathbf{T_2} = \begin{pmatrix}2\\0\\2\end{pmatrix}$. The matrix of symmetric bilinear form is $f = \begin{pmatrix}2&0&2\\0&0&0\\2&0&2\end{pmatrix}$.The $rank(f) = 1$. $f(\alpha,\beta) = \alpha^T\begin{pmatrix}2&0&2\\0&0&0\\2&0&2\end{pmatrix}\beta$ The characteristic equation is $\lvert f - \lambda\mathbf{I}\rvert = \lambda^2(\lambda - 4)$. So the eigen values are $0, 0, 4$
	Therefore f is positive semi-definite. $f(\alpha,\beta) = 0$ for all $\alpha \in \mathbb{R}^3$, then $\beta = xe_1 + ye_2$ where $e_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $e_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$.Therefore $\dim\{\beta \in \mathbb{R}^3 : f(\alpha,\beta) = 0 \text{ for all } \alpha \in \mathbb{R}^3\} = 2$ $\alpha : f(\alpha,\alpha) = 0$ also has a dimension of 2 which forms the nullspace of f, where nullspace of f is the $span\{e_1, e_2\}$

TABLE 10.7.3: Example

10.8. Let **A** be 5×5 matrix and let **B** be obtained by changing one element of **A**. Let r and s be the ranks of **A** and **B** respectively. Which of the following statements is/are correct?

a) $s \leq r + 1$
b) $r - 1 \leq s$
c) $s = r - 1$
d) $s \neq r$

Solution: See Tables 10.8.1 and 10.8.2.

Theorem	If **M** and **N** are two matrices whose ranks are $rank(\mathbf{M})$ and $rank(\mathbf{N})$ respectively. Then $$rank(\mathbf{M}+\mathbf{N}) \le rank(\mathbf{M}) + rank(\mathbf{N}) \tag{10.8.1}$$

TABLE 10.8.1: Definitions and theorem used

Option	Solution	True/ False
1.	Given matrix **A** has rank r and **B** has rank s. Also given matrix **B** is obtained by changing only one element of **A**. Lets assume another matrix **P** whose addition to matrix **A** results to matrix **B** as below. $$\mathbf{A}+\mathbf{P}=\mathbf{B} \tag{10.8.2}$$ Since matrix **P** consists only single element we can say that $rank(\mathbf{P}) = 1$ From (10.8.1), (10.8.2), we get $$rank(\mathbf{A}+\mathbf{P}) \le rank(\mathbf{A}) + rank(\mathbf{P}) \tag{10.8.3}$$ $$\implies rank(\mathbf{B}) \le rank(\mathbf{A}) + rank(\mathbf{P}) \tag{10.8.4}$$ $$\implies s \le r+1 \tag{10.8.5}$$ **Example:** Let matrices **A** and **B** be as below $$\mathbf{A} = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{10.8.6}$$ $$\mathbf{B} = \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & 4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{10.8.7}$$ lets calculate rank of matrix **A**	True

$$\begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{10.8.8}$$

$$\xleftrightarrow[R_5 \leftarrow R_5 - 3R_1]{R_4 \leftarrow R_4 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \\ 0 & 0 & -6 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_5 \leftarrow R_5 - 2R_3]{R_4 \leftarrow R_4 + 3R_3} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.8.9}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_4} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.8.10}$$

$$\implies rank(\mathbf{A}) = 3 = r \tag{10.8.11}$$

Now lets calculate rank of matrix **B**

$$\begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & 4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 9 \\ 0 & 0 & -3 & 1 & -1 \\ -2 & 3 & 3 & -4 & 1 \\ 6 & -9 & 12 & 8 & 13 \end{pmatrix} \tag{10.8.12}$$

$$\xleftrightarrow[R_5 \leftarrow R_5 - 3R_1]{R_4 \leftarrow R_4 + R_1} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 9 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \\ 0 & 0 & -6 & 2 & -2 \end{pmatrix} \xleftrightarrow[R_5 \leftarrow R_5 - 2R_3]{R_4 \leftarrow R_4 + 3R_3} \begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 9 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.8.13}$$

$$\implies rank(\mathbf{B}) = 4 = s \tag{10.8.14}$$

Now matrix **P** will be

$$\mathbf{P} = \mathbf{B} - \mathbf{A} \tag{10.8.15}$$

$$\implies \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.8.16}$$

$$\implies rank(\mathbf{P}) = 1 \tag{10.8.17}$$

Now we will see equation (10.8.5) is satisfied or not

$$s \leq r + 1 \implies 4 \leq 3 + 1 \implies 4 \leq 4 \tag{10.8.18}$$

Hence satisfied

2.	From (10.8.2), If $\mathbf{P} = -\mathbf{Q}$ then we can get as below $$\mathbf{A} - \mathbf{Q} = \mathbf{B} \quad (10.8.19)$$ $$\implies \mathbf{B} + \mathbf{Q} = \mathbf{A} \quad (10.8.20)$$ Since matrix $\mathbf{Q}$ also consists only single element we can say that $rank(\mathbf{Q}) = 1$ From (10.8.1), (10.8.20), we get $$rank(\mathbf{B} + \mathbf{Q}) \leq rank(\mathbf{B}) + rank(\mathbf{Q}) \quad (10.8.21)$$ $$\implies rank(\mathbf{A}) \leq rank(\mathbf{B}) + rank(\mathbf{Q}) \quad (10.8.22)$$ $$\implies r \leq s + 1 \quad (10.8.23)$$ $$\implies r - 1 \leq s \quad (10.8.24)$$ **Example:** Let matrix $\mathbf{A}$ and $\mathbf{B}$ are considered same as in (10.8.6), (10.8.7) From (10.8.11) and (10.8.14) we got $$rank(\mathbf{A}) = r = 3 \quad (10.8.25)$$ $$rank(\mathbf{B}) = s = 4 \quad (10.8.26)$$ $$(10.8.27)$$ Here matrix $\mathbf{Q}$ will be $$\mathbf{Q} = \mathbf{A} - \mathbf{B} \quad (10.8.28)$$ $$\implies \mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \implies \mathbf{Q} = -\mathbf{P} \quad (10.8.29)$$ $$\implies rank(\mathbf{Q}) = 1 \quad (10.8.30)$$ Now we will see equation (10.8.24) is satisfied or not $$r - 1 \leq s \implies 3 - 1 \leq 4 \implies 2 \leq 4 \quad (10.8.31)$$ Hence satisfied	True
3.	Let matrix $\mathbf{A}$ be identity matrix then $rank(\mathbf{A})$ is 5 and matrix $\mathbf{B}$ can be $$\mathbf{A} = \mathbf{I}_{5\times5} \quad (10.8.32)$$ $$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (10.8.33)$$ Then $rank(\mathbf{B})$ is also 5.Therefore $s = r - 1$ is always not true.	False
4.	Similarly from (10.8.32),(10.8.33) we can say that $s \neq r$ is not true always.	False

TABLE 10.8.2: Solution

10.9. For arbitrary subspaces, U, V and W of a finite dimensional vectorspace, which of the following hold :

a) $U \cap (V + W) \subset (U \cap V) + (U \cap W)$

b) $U \cap (V + W) \supset (U \cap V) + (U \cap W)$

c) $(U \cap V) + W \subset (U + W) \cap (V + W)$

d) $(U \cap V) + W \supset (U + W) \cap (V + W)$

Solution: See Table 10.9.1

1. $U \cap (V + W) \subset (U \cap V) + (U \cap W)$	False. Counter Example: Let $\mathbf{u}_1 = (\mathbf{v}_1 + \mathbf{w}_1) \in U \cap (V + W)$ such that $(\mathbf{v}_1 + \mathbf{w}_1) \in U, \mathbf{v}_1 \in V, \mathbf{w}_1 \in W$ But since $\mathbf{w}_1 \notin V$, hence $\mathbf{v}_1 + \mathbf{w}_1 \notin V$ $\implies (\mathbf{v}_1 + \mathbf{w}_1) \notin (U \cap V)$ And since $\mathbf{v}_1 \notin W$, hence $\mathbf{v}_1 + \mathbf{w}_1 \notin W$ $\implies (\mathbf{v}_1 + \mathbf{w}_1) \notin (U \cap W)$ Therefore, $(\mathbf{v}_1 + \mathbf{w}_1) \notin (U \cap V) + (U \cap W)$ There exists an element in LHS that does not belong to RHS. $\therefore U \cap (V + W) \not\subset (U \cap V) + (U \cap W)$
2. $U \cap (V + W) \supset (U \cap V) + (U \cap W)$	Let $(\mathbf{u}_1 + \mathbf{u}_2) \in (U \cap V) + (U \cap W)$ such that $\mathbf{u}_1 \in U \cap V$ and $\mathbf{u}_2 \in U \cap W$ $\implies \mathbf{u}_1 \in U, V$ and $\mathbf{u}_2 \in U, W$ Since $\mathbf{u}_1 \in V, \mathbf{u}_2 \in W$ $\implies (\mathbf{u}_1 + \mathbf{u}_2) \in (V + W)$ And since $\mathbf{u}_1, \mathbf{u}_2 \in U$ $\implies (\mathbf{u}_1 + \mathbf{u}_2) \in U$ $\therefore (\mathbf{u}_1 + \mathbf{u}_2) \in U \cap (V + W)$ So, $(\mathbf{u}_1 + \mathbf{u}_2) \in (U \cap V) + (U \cap W) \implies (\mathbf{u}_1 + \mathbf{u}_2) \in U \cap (V + W)$ Hence, $U \cap (V + W) \supset (U \cap V) + (U \cap W)$ The given option is true.
3. $(U \cap V) + W \subset (U + W) \cap (V + W)$	Let $(\mathbf{u}_1 + \mathbf{w}_1) \in (U \cap V) + W$, such that $\mathbf{u}_1 \in (U \cap V)$ and $\mathbf{w}_1 \in W$ Since, $\mathbf{u}_1 \in (U \cap V), \implies \mathbf{u}_1 \in U, V$ Now, since $\mathbf{u}_1 \in U, \mathbf{w}_1 \in W$ $(\mathbf{u}_1 + \mathbf{w}_1) \in (U + W)$ And since, $\mathbf{u}_1 \in V, \mathbf{w}_1 \in W$ $(\mathbf{u}_1 + \mathbf{w}_1) \in (V + W)$ $\therefore (\mathbf{u}_1 + \mathbf{w}_1) \in (U + W) \cap (V + W)$ Hence, $(\mathbf{u}_1 + \mathbf{w}_1) \in (U \cap V) + W \implies (\mathbf{u}_1 + \mathbf{w}_1) \in (U + W) \cap (V + W)$ $(U \cap V) + W \subset (U + W) \cap (V + W)$ The given option is true.

4. $(U \cap V) + W \supset (U + W) \cap (V + W)$	False. Counter Example: Let $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{w}_1 \in U$ $\mathbf{v}_1 \in V, \mathbf{w}_1 \in W$ Then, since $\mathbf{v}_1 + \mathbf{w}_1 \in U \implies \mathbf{v}_1 + \mathbf{w}_1 \in U + W$ And since, $\mathbf{v}_1 \in V, \mathbf{w}_1 \in W \implies \mathbf{v}_1 + \mathbf{w}_1 \in V + W$ $\therefore \mathbf{v}_1 + \mathbf{w}_1 \in (U + W) \cap (V + W)$ Now, since $\mathbf{w}_1 \notin V \implies \mathbf{v}_1 + \mathbf{w}_1 \notin V$ $\implies \mathbf{v}_1 + \mathbf{w}_1 \notin U \cap V$ And since, $\mathbf{v}_1 \notin W \implies \mathbf{v}_1 + \mathbf{w}_1 \notin W$ $\implies \mathbf{v}_1 + \mathbf{w}_1 \notin (U \cap V) + W$ There exists an element in RHS that does not exist in LHS $\therefore (U \cap V) + W \not\supset (U + W) \cap (V + W)$

TABLE 10.9.1: Proving properties of subspaces of a vectorspace

10.10. Let **A** be a 4 x 7 real matrix and **B** be a 7 x 4 real matrix such that $\mathbf{AB} = \mathbf{I}_4$, where $\mathbf{I}_4$ is the 4 x 4 identity matrix. Which of the following is/are always true?

a) $rank(\mathbf{A}) = 4$
b) $rank(\mathbf{B}) = 7$
c) $nullity(\mathbf{B}) = 0$
d) $\mathbf{BA} = \mathbf{I}_7$, where $\mathbf{I}_7$ is the 7 x 7 identity matrix

Solution: See Tables 10.10.1 and 10.10.2

Given	**A** is 4 x 7 real matrix **B** is 7 x 4 real matrix $\mathbf{AB} = \mathbf{I}_4$
Option-1	since $\mathbf{I}_4$ is a 4 x 4 identity matrix, $rank(\mathbf{I}_4) = 4 = rank(\mathbf{AB})$ from the properties of matrices $rank(\mathbf{A}) \leq min\{\#cloumns, \#rows\}$ $rank(\mathbf{A}) \leq 4$ and $rank(\mathbf{AB}) \leq rank(\mathbf{A})$ $4 \leq rank(\mathbf{A})$ $\therefore rank(\mathbf{A}) = 4$ Hence Option-1 is True.
Option-2	Similarly from the properties of matrices $rank(\mathbf{B}) \leq min\{\#cloumns, \#rows\}$ $rank(\mathbf{B}) \leq 4$ and $rank(\mathbf{AB}) \leq rank(\mathbf{B})$ $4 \leq rank(\mathbf{B})$ $\therefore rank(\mathbf{B}) = 4$ Hence Option-2 is False.
Option-3	Since $rank(\mathbf{B}) = 4$, and **B** is a 7 x 4 matrix in finite dimensional vector space $\mathbb{V}$. the column space,$C(\mathbf{B})$ will form the basis. $\implies range(\mathbf{B}) = dim(\mathbb{V}) = 4$ from rank-nullity theorem $rank(\mathbf{B}) + nullity(\mathbf{B}) = dim(\mathbb{V})$ by substituting above values $nullity(\mathbf{B}) = 0$ Hence Option-3 is True.
Option-4	Given $\mathbf{BA} = \mathbf{I}_7$ $rank(\mathbf{I}_7) = 7 = rank(\mathbf{BA})$

	from the properties of matrices $rank(\mathbf{BA}) \leq rank(\mathbf{B})$ $7 \leq rank(\mathbf{B})$ the above conditioned can not be satisfied since we know $rank(\mathbf{B}) = 4$. Hence Option-4 is False.
Conclusion	Option-1 and 3 are True Option-2 and 4 are False

TABLE 10.10.1: Proof

Example	Proving the above results with example in lower dimensions as follows. Let $\mathbf{A}$ be a 2 x 3 matrix in vector space $\mathbb{V}$ and consider $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -4 \end{pmatrix}$ and $\mathbf{B}$ be a 3 x 2 matrix in vector space $\mathbb{V}$ and consider $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$ so that $\mathbf{AB} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a 2 x 2 matrix
Option-1	row reduced echelon form of $\mathbf{A}$ is $rref(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \end{pmatrix}$ $\implies rank(\mathbf{A}) = 2$ Hence Option-1 is True
Option-2	row reduced echelon form of $\mathbf{B}$ is $rref(\mathbf{B}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\implies rank(\mathbf{B}) = 2$ Hence Option-2 is False
Option-3	from the above rref form of $\mathbf{B}$ the $range(\mathbf{B}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$ $\implies dim(\mathbb{V}) = 2$ $nullspace(\mathbf{B}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

	$\therefore$ from rank-nullity theorem $nullity(\mathbf{B}) = 0$ Hence Option-3 is True
Option-4	$\mathbf{BA} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$ $\implies \mathbf{BA} \neq \mathbf{I}$ $rank(\mathbf{BA}) = \mathbf{I} = 2$ Hence Option-4 is False

TABLE 10.10.2: Example

10.11. Which of the following are eigen values of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}? \tag{10.11.1}$$

a) $+1$
b) -1
c) $+i$
d) $-i$

Solution: Eigen values of a real symmetric matrix are real. Proof:
Here $\mathbf{A}^T = \mathbf{A}$. Therefore matrix $\mathbf{A}$ is a symmetric matrix. Also $\mathbf{A}$ is a real matrix.
Let λ be a complex eigen value. Then the eigen vector $\mathbf{x}$ will have one or more complex elements. We have,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{10.11.2}$$

$\implies$ $\mathbf{A}\mathbf{x}$ and $\lambda\mathbf{x}$ are complex respectively.
$\implies$ their complex conjugates are also equal.
Let the conjugates of λ and $\mathbf{x}$ be $\bar{\lambda}$ and $\bar{\mathbf{x}}$ respectively.

$$\therefore \mathbf{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \tag{10.11.3}$$

$$\left[\because \bar{\mathbf{A}\mathbf{x}} = \bar{\lambda\mathbf{x}} \implies \bar{\mathbf{A}}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \implies \mathbf{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}\right] \tag{10.11.4}$$

Multiplying (10.11.2) by $\bar{\mathbf{x}}^T$ and (10.11.3) by $\mathbf{x}^T$ and subtracting,

$$\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{A}\bar{\mathbf{x}} = \left(\lambda - \bar{\lambda}\right)\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} \tag{10.11.5}$$

Each term on the LHS of (10.11.5) is scalar and $\mathbf{A}$ is symmetric

$$\therefore \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{A}\bar{\mathbf{x}} = 0 \tag{10.11.6}$$

From (10.11.5) and (10.11.6),

$$\left(\lambda - \bar{\lambda}\right)\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} = 0 \tag{10.11.7}$$

where $\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x}$ = sum of products of complex numbers times their conjugates.

$$\because \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} \neq 0 \tag{10.11.8}$$

$$\therefore \left(\lambda - \bar{\lambda}\right) = 0 \tag{10.11.9}$$

$$\implies \lambda = \bar{\lambda} \tag{10.11.10}$$

This implies λ is real.
$\therefore$ The eigen values are real. (*proved*).
Thus, we can eliminate option 3 and 4.
The sum of eigen values of a matrix is equal to the trace of the matrix.
From (10.11.1), trace of $\mathbf{A}$ = 0, which is only possible if the eigen values are +1 and -1.
Therefore, option 1 and 2 are the correct choices.

10.12. Let

$$\mathbf{A} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \tag{10.12.1}$$

where x,y $\in \mathbb{R}$ such that

$$x^2 + y^2 = 1 \tag{10.12.2}$$

Then,we must have:

a) $\mathbf{A^n} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \forall n \geq 1$
where x=cos($\frac{\theta}{n}$),y=sin($\frac{\theta}{n}$)
b) $trace(\mathbf{A}) \neq 0$
c) $\mathbf{A^T} = \mathbf{A^{-1}}$
d) $\mathbf{A}$ is similar to a diagonal matrix over $\mathbb{C}$

Solution: See Table

Options	Explanation
$\mathbf{A^n} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \forall n \geq 1$ where x=cos($\frac{\theta}{n}$),y=sin($\frac{\theta}{n}$)	$\mathbf{A} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ $\mathbf{A} = \begin{pmatrix} \cos(\frac{\theta}{n}) & \sin(\frac{\theta}{n}) \\ -\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) \end{pmatrix}$ $\mathbf{A^2} = \mathbf{A}.\mathbf{A} = \begin{pmatrix} \cos(\frac{\theta}{n}) & \sin(\frac{\theta}{n}) \\ -\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) \end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{n}) & \sin(\frac{\theta}{n}) \\ -\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) \end{pmatrix}$ $\mathbf{A^2} = \begin{pmatrix} \cos(\frac{2\theta}{n}) & \sin(\frac{2\theta}{n}) \\ -\sin(\frac{2\theta}{n}) & \cos(\frac{2\theta}{n}) \end{pmatrix}$ $\mathbf{A^3} = \mathbf{A^2}.\mathbf{A} = \begin{pmatrix} \cos(\frac{2\theta}{n}) & \sin(\frac{2\theta}{n}) \\ -\sin(\frac{2\theta}{n}) & \cos(\frac{2\theta}{n}) \end{pmatrix}\begin{pmatrix} \cos(\frac{\theta}{n}) & \sin(\frac{\theta}{n}) \\ -\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) \end{pmatrix}$ $\mathbf{A^3} = \begin{pmatrix} \cos(\frac{3\theta}{n}) & \sin(\frac{3\theta}{n}) \\ -\sin(\frac{3\theta}{n}) & \cos(\frac{3\theta}{n}) \end{pmatrix}$ .. .. .. $\mathbf{A^n} = \begin{pmatrix} \cos(\frac{n\theta}{n}) & \sin(\frac{n\theta}{n}) \\ -\sin(\frac{n\theta}{n}) & \cos(\frac{n\theta}{n}) \end{pmatrix}$ $\mathbf{A^n} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \forall n \geq 1$ Hence,correct
$trace(\mathbf{A}) \neq 0$	Let,$x = 0, y = 1$,Substitute in (10.12.1) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $trace(\mathbf{A}) = 0$ Hence,incorrect
$\mathbf{A^T} = \mathbf{A^{-1}}$ $\mathbf{AA^T}$	$\mathbf{A} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ $\mathbf{A^T} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ $\begin{pmatrix} x^2+y^2 & -xy+xy \\ -xy+xy & x^2+y^2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{A}A^T = \mathbf{I} = \mathbf{A^T}A$ $\implies \mathbf{A} = \mathbf{A^{-1}}$ $\implies \mathbf{A}$ is an orthogonal matrix. Hence,correct.

1

Options	Explanation
$\mathbf{A}$ is similar to a diagonal matrix over $\mathbb{C}$ Using Spectral Theorem	Every real orthogonal matrix is diagonalizable over $\mathbb{C}$ $\mathbf{A}$ is orthogonal from above. Since,$x, y \in \mathbb{R}$ .So,$\mathbf{A}$ is a real orthogonal matrix.
$\mathbf{A} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$	$det(\mathbf{A} - \lambda\mathbf{I})) = 0$ $(x - \lambda)^2 + y^2 = 0$ $\lambda_1 = x - iy \qquad \lambda_2 = x + iy$ For two eigen values λ_1, λ_2 let heir corresponding eigen vectors be $\mathbf{V_1}, \mathbf{V_2}$
Finding $\mathbf{V_1}$	$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{V_1} = 0$ $(\mathbf{A} - \lambda_1\mathbf{I}) = \begin{pmatrix} iy & y \\ -y & iy \end{pmatrix}$ By Elementary row operations we get, $(\mathbf{A} - \lambda_1\mathbf{I}) = \begin{pmatrix} iy & y \\ 0 & 0 \end{pmatrix}$ $\mathbf{V_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$
Finding $\mathbf{V_2}$	$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{V_2} = 0$ $(\mathbf{A} - \lambda_2\mathbf{I}) = \begin{pmatrix} -iy & y \\ -y & -iy \end{pmatrix}$ By Elementary row operations we get, $(\mathbf{A} - \lambda_2\mathbf{I}) = \begin{pmatrix} -iy & y \\ 0 & 0 \end{pmatrix}$ $\mathbf{V_2} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$
$\mathbf{A} = \mathbf{PDP^{-1}}$	$\mathbf{P}$ is a matrix containing eigen vectors of $\mathbf{A}$,$\mathbf{D}$ is the diagonal matrix where diagonals are the eigen values of $\mathbf{A}$ $\mathbf{P^{-1}} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$ $\mathbf{A} = \frac{1}{2i}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x - iy & 0 \\ 0 & x + iy \end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$ Hence,$\mathbf{A}$ is similar to a diagonal matrix over $\mathbb{C}$ Hence,correct.

TABLE : Finding Correct Option