



Linear Algebra



G V V Sharma*

CONTENTS

| | | |
|---|---------------|-----|
| 1 | June 2019 | 1 |
| 2 | December 2018 | 3 |
| 3 | June 2018 | 40 |
| 4 | December 2017 | 57 |
| 5 | June 2017 | 74 |
| 6 | December 2016 | 90 |
| 7 | June 2016 | 109 |
| 8 | December 2015 | 109 |
| 9 | June 2015 | 122 |

Abstract—This book provides solved examples on Linear Algebra.

1 JUNE 2019

1.1. Consider the vector space \mathbb{P}_n of real polynomials in x of degree $\leq n$. Define

$$T : \mathbb{P}_2 \rightarrow \mathbb{P}_3 \quad (1.1.1)$$

by

$$(Tf)(x) = \int_0^x f(t) dt + f'(x). \quad (1.1.2)$$

Then find the matrix representation of T with respect to the bases

$$\{1, x, x^2\} \text{ and } \{1, x, x^2, x^3\} \quad (1.1.3)$$

1.2. Let $P_A(x)$ denote the characteristic polynomial of a matrix A . Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \quad (1.2.1)$$

a constant?

a) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over \mathbb{R} ?

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

$$\begin{array}{ll} \text{a) } \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} & \text{c) } \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \\ \text{b) } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{d) } \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \end{array}$$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad (1.4.1)$$

1.5. Let V denote the vector space of real valued continuous functions on the close interval $[0, 1]$. Let W be the subspace of V spanned by $\{\sin x, \cos x, \tan x\}$. Find the dimension of W over \mathbb{R} .

1.6. Let V be the vector space of polynomials in the variable t of degree at most 2 over \mathbb{R} . An inner product on V is defined by

$$f^T g = \int_0^1 f(t)g(t) dt, \quad f, g \in V. \quad (1.6.1)$$

Let

$$W = \text{span}\{1 - t^2, 1 + t^2\} \quad (1.6.2)$$

and W^\perp be the orthogonal complement of W in V . Which of the following conditions is satisfied for all $h \in W^\perp$?

- a) h is an even function
- b) h is an odd function
- c) $h(t) = 0$ has a real solution
- d) $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix} (x) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (1.7.1)$$

where $m, n \in \mathbb{Z}$. With any initial vector, the scheme converges provided m, n satisfy

- a) $m + n = 3$
- b) $m > n$
- c) $m < n$
- d) $m = n$

1.8. Consider a Markov Chain with state space

$\{0, 1, 2, 3, 4\}$ and transition matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix} \quad (1.8.1)$$

Then find

$$\lim_{n \rightarrow \infty} p_{23}^{(n)} \quad (1.8.2)$$

1.9. Let $L(\mathbb{R})^n$ be the space of \mathbb{R} -linear maps from \mathbb{R}^n to \mathbb{R}^n . If $\text{Ker}(T)$ denotes the kernel of T then which of the following are true?

- a) There exists $T \in L(\mathbb{R}^5) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$
- b) There does not exist $T \in L(\mathbb{R}^5) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$
- c) There exists $T \in L(\mathbb{R}^6) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$
- d) There does not exist $T \in L(\mathbb{R}^6) \setminus \{0\}$ such that $\text{Range}(T) = \text{Ker}(T)$

1.10. Let V be a finite dimensional vector space over \mathbb{R} and $T : V \rightarrow V$ be a linear map. Can you always write $T = T_2 \circ T_1$ for some linear maps

$$T_1 : V \rightarrow W, T : W \rightarrow V, \quad (1.10.1)$$

where W is some finite dimensional vector space such that

- a) both T_1 and T_2 are onto
- b) both T_1 and T_2 are one to one
- c) T_1 is onto, T_2 is one to one
- d) T_1 is one to one, T_2 is onto

1.11. Let $A = [a_{ij}]$ be a 3×3 complex matrix. Identify the correct statements

- a) $\det \left[(-1)^{i+j} a_{ij} \right] = \det(A)$
- b) $\det \left[(-1)^{i+j} a_{ij} \right] = -\det(A)$
- c) $\det \left[(\sqrt{-1})^{i+j} a_{ij} \right] = \det(A)$
- d) $\det \left[(\sqrt{-1})^{i+j} a_{ij} \right] = -\det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad (1.12.1)$$

be a non-constant polynomial of degree $n \geq 1$.

Consider the polynomial

$$q(x) = \int_0^x p(t) dt, r(x) = \frac{d}{dx} p(x) \quad (1.12.2)$$

Let V denote the real vector space of all polynomials in x . Then which of the following are true?

- a) q and r are linearly independent in V
- b) q and r are linearly dependent in V
- c) x^n belongs to the linear span of q and r
- d) x^{n+1} belongs to the linear span of q and r .

1.13. Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over \mathbb{R} . Which of the following are true for every $n \geq 2$?

- a) there exist matrices $A, B \in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity matrix.
- b) If $A, B \in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalisable over \mathbb{R} if and only if B is diagonalisable over \mathbb{R} .
- c) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same minimal polynomial.
- d) If $A, B \in M_n(\mathbb{R})$, then AB and BA have the same eigenvalues in \mathbb{R} .

1.14. Consider a matrix

$$A = [a_{ij}], 1 \leq i, j \leq 5 \quad (1.14.1)$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \quad (1.14.2)$$

Then in which of the following cases A is a positive definite matrix?

- a) $n_i = 1 \forall i = 1, 2, 3, 4, 5$.
- b) $n_1 < n_2 < \dots < n_5$.
- c) $n_1 = n_2 = \dots = n_5$.
- d) $n_1 > n_2 > \dots > n_5$.

1.15. For a nonzero $w \in \mathbb{R}^n$, define

$$T_w : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (1.15.1)$$

by

$$T_w v = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \quad (1.15.2)$$

Which of the following are true?

- a) $\det(T_w) = 1$
- b) $T_w(v_1)_w^T(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
- c) $T_w = T_w^{-1}$
- d) $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.16.1)$$

over the field \mathbb{Q} of rationals. Which of the following matrices are of the form $P^T A P$ for suitable 2×2 invertible matrix P over \mathbb{Q} ?

- a) $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$
- b) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- c) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- d) $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 0 & \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ 1 & \frac{1}{4} & 0 & \frac{3}{4} \\ 2 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \quad (1.17.1)$$

Then which of the following are true?

- a) $\lim_{n \rightarrow \infty} p_{12}^{(n)} = 0$
- b) $\lim_{n \rightarrow \infty} p_{12}^{(n)} = \lim_{n \rightarrow \infty} p_{21}^{(n)}$
- c) $\lim_{n \rightarrow \infty} p_{22}^{(n)} = \frac{1}{8}$
- d) $\lim_{n \rightarrow \infty} p_{21}^{(n)} = \frac{1}{3}$

2 DECEMBER 2018

2.1. Consider the subspaces W_1 and W_2 of \mathbb{R}^3 given by

$$W_1 = \{\mathbf{x} \in \mathbb{R}^3 : (1 \ 1 \ 1)\mathbf{x} = 0\} \quad (2.1.1)$$

$$W_2 = \{\mathbf{x} \in \mathbb{R}^3 : (1 \ -1 \ 1)\mathbf{x} = 0\}. \quad (2.1.2)$$

If $W \subseteq \mathbb{R}^3$, such that

$$a) W \cap W_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$b) \{W \cap W_1\} \perp \{W \cap W_2\},$$

then

$$a) W = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$b) W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$c) W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

d) $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solution: Using (2.1.1),

$$\mathbf{W}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.3)$$

From (2.1.2),

$$\mathbf{W}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (2.1.4)$$

From (2.1a), we can say that, both the subspaces \mathbf{W} and \mathbf{W}_2 of \mathbf{R}^3 contains the column vector as follows: .

$$\mathbf{W} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.5)$$

$$\mathbf{W}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.6)$$

From (2.1.4) and (2.1.6),

$$\mathbf{W}_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \quad (2.1.7)$$

$$\text{Rank}(\mathbf{W}_2) = 2 \quad (2.1.8)$$

Since, $\text{rank} < 3$ and the vectors are linearly independent they span a subspace of \mathbf{R}^3 .

Consider the vector,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{W} \cap \mathbf{W}_1 \quad (2.1.9)$$

From (2.1a) and (2.1b),

The vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$\Rightarrow (x \ y \ z) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \quad (2.1.10)$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (2.1.11)$$

Since, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{W} \cap \mathbf{W}_1$:

From (2.1.3) and (2.1.11),

$$\mathbf{W}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.1.12)$$

Also from (2.1.5) and (2.1.11),

$$\mathbf{W} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.1.13)$$

Using (2.1.13),

The vectors linearly independent and $\text{rank}(\mathbf{W})=2 (< 3)$, then the vector span subspace of \mathbf{R}^3 .

Hence,

$$\mathbf{W} = \text{span}\{(0, 1, -1), (0, 1, 1)\} \Rightarrow \text{Ans : } \mathbf{1} \quad (2.1.14)$$

2.2. Let

$$C = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad (2.2.1)$$

be a basis of \mathbb{R}^2 and

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-2y \end{pmatrix}. \quad (2.2.2)$$

If $T[C]$ represents the matrix of T with respect to the basis C then which among the following is true?

a) $T[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$

b) $T[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$

c) $T[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$

d) $T[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$

Solution: See Tables 2.2.1 and 2.2.2

In above question $\mathbf{A} = \mathbf{T}, \mathbf{B} = \mathbf{T}[C], \mathbf{V} = \mathbf{C}$.

2.3. Let $W_1 = \{\mathbf{x} \in \mathbb{R}^4 : \}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.1)$$

$$\begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.2)$$

$$\begin{pmatrix} 2 & 0 & 2 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.3)$$

| | |
|-------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Linear Transformation and change of Basis | <p>If matrix \mathbf{A} represents Linear Transformation with respect to standard ordered basis and matrix \mathbf{B} represents same transformation with respect to basis \mathbf{V}, Then</p> $\mathbf{B} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ |
|-------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

TABLE 2.2.1: Linear Transformation and change of basis

$$\text{and } W_2 = \{\mathbf{x} \in \mathbb{R}^4 : \}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.4)$$

$$\begin{pmatrix} 1 & 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.5)$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} \mathbf{x} = 0. \quad (2.3.6)$$

Then which among the following is true?

- a) $\dim(W_1) = 1$
- b) $\dim(W_2) = 2$
- c) $\dim(W_1 \cap W_2) = 1$
- d) $\dim(W_1 + W_2) = 3$

2.4. Let A be an $n \times n$ complex matrix. Assume that A is self-adjoint and let B denote the inverse of $A + \mathbf{j}I$. Then all eigenvalues of $(A - \mathbf{j}I)B$ are

- a) purely imaginary
- b) of modulus one
- c) real
- d) of modulus less than one

2.5. Let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal basis of \mathbb{C}^n as column vectors. Let

$$\mathbf{M} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_k), \quad (2.5.1)$$

$$\mathbf{N} = (\mathbf{u}_{k+1} \quad \mathbf{u}_{k+2} \quad \dots \quad \mathbf{u}_n) \quad (2.5.2)$$

and \mathbf{P} be the diagonal $k \times k$ matrix with diagonal entries $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$. Then which of the following is true?

- a) $\text{rank}(\mathbf{M}\mathbf{P}\mathbf{M}^*) = k$ whenever $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$.
- b) $\text{tr}(\mathbf{M}\mathbf{P}\mathbf{M}^*) = \sum_{i=1}^k \alpha_i$
- c) $\text{rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n - k)$
- d) $\text{rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$.

Solution: See Tables 2.5.1 2.5.2 and 2.5.3

| | |
|----------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Evaluate \mathbf{T} | <p>For linear transformation \mathbf{T} we have</p> $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}$ $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow \mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ |
| Evaluate inverse of basis \mathbf{C} | <p>To find inverse of matrix \mathbf{C} we row reduce augmented matrix $\mathbf{C}\mathbf{I}$</p> $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = -\frac{1}{3}R_2]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ $\xrightarrow{R_1 = R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ $\therefore \mathbf{C}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ |
| Evaluate \mathbf{TC} | $\mathbf{TC} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$ |
| Evaluate $\mathbf{T}[\mathbf{C}] = \mathbf{C}^{-1}\mathbf{TC}$ | $\mathbf{T}[\mathbf{C}] = \mathbf{C}^{-1}\mathbf{TC}$ $= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$ $\Rightarrow \mathbf{T}[\mathbf{C}] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$ |
| Conclusion | <p>Option 3) is correct. Options 1), 2) and 4) are incorrect</p> |

TABLE 2.2.2: Calculation of $\mathbf{T}[\mathbf{C}]$

| | |
|-------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Orthonormal Basis | <p>$B = \{u_1, u_2, \dots, u_n\}$ is the Orthonormal basis for C^n if it generates every vector C^n and the inner product $\langle u_i, u_j \rangle = 0$ if $i \neq j$. That is the vectors are mutually perpendicular and $\langle u_i, u_j \rangle = 1$ otherwise.</p> |
| Trace | <p>Trace of a square matrix A, denoted by $\text{tr}(A)$ is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of A Some useful properties of Trace : $\text{tr}(AB) = \text{tr}(BA)$, where A is the $m \times n$ matrix and B is the $n \times m$ matrix</p> |
| Basis Theorem | <p>A nonempty subset of nonzero vectors in R^n is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if A matrix columns are linearly independent, then it is invertible.</p> |

TABLE 2.5.1: Definitions

| | |
|--------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>$\text{Rank}(\mathbf{MPM}^*) = \mathbf{k}$</p> | <p>Consider orthogonal vectors,</p> $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ <p>Consider $\mathbf{k} = 2$, then</p> $\mathbf{M} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ $\mathbf{MPM}^* = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p>$\Rightarrow \text{Rank}(\mathbf{MPM}^*) \leq 2$ (which is the value of k) (It depends on diagonal values α_1 and α_2) $\text{Rank}(\mathbf{MPM}^*)$ is not always k. It can be less than k if any of the entries in $\alpha_1, \alpha_2, \dots, \alpha_k$ is 0. Thus, $\text{Rank}(\mathbf{MPM}^*) \neq \mathbf{k}$ Thus, the given statement is false</p> |
| <p>$\text{Trace}(\mathbf{MPM}^*) = \sum_{i=1}^k \alpha_i$</p> | <p>Consider $\mathbf{MP} = \mathbf{A}$ and $\mathbf{M}^* = \mathbf{B}$ Using Properties, $\text{Trace}(\mathbf{AB}) = \text{Trace}(\mathbf{BA})$ We can say, $\text{Trace}(\mathbf{MPM}^*) = \text{Trace}(\mathbf{M}^*\mathbf{MP})$ $\mathbf{M} = \begin{pmatrix} u_1 & u_2 & u_3 & \dots & u_k \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \vdots \\ \bar{u}_k \end{pmatrix}$ $\mathbf{M}^*\mathbf{M} = \begin{pmatrix} \bar{u}_1 u_1 & 0 & 0 & \dots & 0 \\ 0 & \bar{u}_2 u_2 & 0 & \dots & 0 \\ 0 & 0 & \bar{u}_3 u_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \bar{u}_k u_k \end{pmatrix}$</p> |

| | |
|-------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>(Refer to Properties mentioned in Orthonormal Basis in Definition section that is $\langle u_i, u_j \rangle = 0$ if $i \neq j$)</p> $\mathbf{M}^* \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ <p>(Refer to Properties mentioned in Orthonormal Basis in Definition section that is $\langle u_i, u_j \rangle = 1$ if $i = j$)</p> <p>$\mathbf{M}^* \mathbf{M} = \mathbf{I}^k$</p> <p>$\mathbf{M}^* \mathbf{M} \mathbf{P} = \mathbf{I}^k \mathbf{P} = \mathbf{P}$</p> <p>$\text{Trace}(\mathbf{M}^* \mathbf{M} \mathbf{P}) = \text{Trace}(\mathbf{I}^k \mathbf{P}) = \text{Trace}(\mathbf{P}) = \sum_{i=1}^k \alpha_i$</p> <p>(Refer Definition section of Trace, it is sum of elements on the main diagonal)</p> <p>So, the given statement is true</p> |
| <p>$\text{Rank}(\mathbf{M}^* \mathbf{N}) = \min(k, n - k)$</p> | <p>$\mathbf{M} = \{u_1, u_2, \dots, u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, \dots, u_n\}$</p> <p>Consider orthogonal vectors,</p> $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ <p>Consider $k = 2$, then</p> $\mathbf{M} = (\mathbf{u}_1 \ \mathbf{u}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{N} = (\mathbf{u}_3 \ \mathbf{u}_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{M}^* \mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ <p>$\text{Rank}(\mathbf{M}^* \mathbf{N}) = 0$</p> <p>But, $\min(k, n - k) = (2, 2) = 2$</p> <p>And, this is clear from above that $\text{Rank}(\mathbf{M}^* \mathbf{N}) \neq \min(k, n - k)$</p> <p>Thus, above statement is false</p> |
| <p>$\text{Rank}(\mathbf{M} \mathbf{M}^* + \mathbf{N} \mathbf{N}^*) < n$</p> | <p>$\text{Rank}(\mathbf{M}) = \text{Rank}(\mathbf{M}^*)$</p> <p>$\text{Rank}(\mathbf{N}) = \text{Rank}(\mathbf{N}^*)$</p> <p>$\text{Rank}(\mathbf{M} + \mathbf{N}) \leq \text{Rank}(\mathbf{M}) + \text{Rank}(\mathbf{N})$</p> |

$\mathbf{M} = \{u_1, u_2, \dots, u_k\}$ and $\mathbf{N} = \{u_{k+1}, u_{k+2}, \dots, u_n\}$

Consider orthogonal vectors,

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Consider $k = 2$, then

$$\mathbf{M} = (u_1 \ u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Rank}(\mathbf{M}) = 2 = k$$

$$\mathbf{N} = (u_3 \ u_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Rank}(\mathbf{N}) = 2 = n - k$$

$$\text{Thus, } \text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) = \text{Rank}(\mathbf{M} + \mathbf{N}) = 4 = n$$

Thus, above statement is false

TABLE 2.5.2: Finding of True and False Statements

| | |
|--------------------------------------------------------------------|-------|
| $\text{Rank}(\mathbf{M}\mathbf{M}^*) = k$ | False |
| $\text{Trace}(\mathbf{M}\mathbf{M}^*) = \sum_{i=1}^k \alpha_i$ | True |
| $\text{Rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n - k)$ | False |
| $\text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$ | False |

TABLE 2.5.3: Conclusion of above Solutions

2.6. Let $B : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the function $B(a, b) = ab$.

Which of the following is true-

- a) B is a linear transformation
- b) B is a positive definite bilinear form
- c) B is symmetric but not positive definite
- d) B neither linear nor bilinear

Solution: Let

$$\mathbf{x} = \begin{pmatrix} x & y \end{pmatrix}^T \quad (2.6.1)$$

Then

$$B(x, y) = \mathbf{x}^T \frac{\mathbf{R}}{2} \mathbf{x} \quad (2.6.2)$$

where \mathbf{R} is the reflection matrix defined as:-

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.6.3)$$

(2.6.2) represent Quadratic form of $B(x, y)$. See Table 2.6.1

2.7. Let $B : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be the function

$$B(a, b) = ab \quad (2.7.1)$$

Which of the following is true?

- a) B is a linear transformation
- b) B is a positive definite bilinear form
- c) B is symmetric but not positive definite
- d) B is neither linear nor bilinear

2.8. Let \mathbf{A} be an invertible real $n \times n$ matrix. Define a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (2.8.1)$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T \mathbf{y} \quad (2.8.2)$$

Let $DF(\mathbf{x}, \mathbf{y})$ denote the derivate of F at (\mathbf{x}, \mathbf{y}) which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (2.8.3)$$

Then, if

- a) $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
- b) $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
- c) $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
- d) $\mathbf{x} = 0$ or $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

Solution: See Tables 2.8.1 and 2.8.2

| Options | Explanation |
|-----------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| B is a linear transformation | <p>Let the transformation be $B : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that,</p> $B(\mathbf{x}) = xy \text{ where } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ <p>Now $B(\mathbf{e}) = ab$ where $\mathbf{e} = \begin{pmatrix} a \\ b \end{pmatrix}$</p> <p>Hence, $B(c\mathbf{e}) = c^2 B(\mathbf{e})$</p> <p>Hence B is not a linear transformation.</p> <p>Hence incorrect.</p> |
| B is a positive definite bilinear form Bilinear Form Symmetric Positive Definite | <p>$f : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{F}$ where \mathbb{V} is a vector space and \mathbb{F} is a field</p> <p>f is a bilinear if the following holds true -</p> <p>If one variable is fixed then other should be linear</p> <p>Let's say x is fixed, $x=c$</p> <p>(2.6.2) becomes $B(x, y) = cy, y$ is linear</p> <p>Let's say y is fixed, $y=c$</p> <p>(2.6.2) becomes $B(x, y) = cx, x$ is linear</p> <p>Hence B is a bilinear form.</p> <p>Again a bilinear form f is symmetric if $f(\alpha, \beta) = f(\beta, \alpha)$</p> <p>Here, $B(a, b) = ab$, from (2.6.2)</p> <p>$B(b, a) = ba$, from (2.6.2)</p> <p>$ba = ab$, Hence B is symmetric.</p> <p>A symmetric bilinear f is positive definite if</p> <p>$f(\alpha, \alpha) > 0 \forall \alpha \neq 0$</p> <p>Here, $B(a, a) = a^2$ from (2.6.2)</p> <p>$a^2 > 0 \forall a \neq 0$</p> <p>Conclusion: B is symmetric and positive definite bilinear form.</p> <p>Hence Correct.</p> |
| B is symmetric but not positive definite | <p>From previous proof it is obvious that</p> <p>B is both symmetric as well as positive definite</p> <p>Hence incorrect</p> |
| B neither linear nor bilinear | <p>From previous proofs it is obvious that</p> <p>B is bilinear.</p> <p>Hence incorrect.</p> |
| Result | B is symmetric and positive definite bilinear form |

TABLE 2.6.1: Finding Correct Option

| | |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Invertible | <p>A square matrix is invertible if and only if it does not have a zero eigenvalue. So, from the definition of eigen vector we can write that</p> $\mathbf{A}\mathbf{x} \neq 0 \quad (2.8.4)$ <p>The transpose of an invertible matrix is also invertible with inverse $(\mathbf{A}^{-1})^T$.</p> $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \implies (\mathbf{A}^{-1})^T \mathbf{A}^T = \mathbf{I}^T = \mathbf{I} \quad (2.8.5)$ <p>So, similarly we can say that</p> $\mathbf{A}^T \mathbf{y} \neq 0 \quad (2.8.6)$ |
| Derivative of F | <p>Suppose $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$, the derivative of a function F is given by the Jacobian matrix</p> $\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (2.8.7)$ |
| Inner product | <p>The inner product of \mathbf{x} and \mathbf{y} is given by</p> $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} \quad (2.8.8)$ |

TABLE 2.8.1: Definition and Properties used

| | |
|----------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | $F(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle \quad (2.8.9)$ |
| using inner product definition | $F(\mathbf{x}, \mathbf{y}) = (\mathbf{A}\mathbf{x})^T \mathbf{y} = \mathbf{x}^T \mathbf{A}^T \mathbf{y} \quad (2.8.10)$ $F(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T \mathbf{A}\mathbf{x} \quad (2.8.11)$ |
| Derivative of F | <p>using (2.8.7), We can write that</p> $DF(\mathbf{x}, \mathbf{y}) = \left(\frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y} \right) = \left(\mathbf{y}^T \mathbf{A} \quad \mathbf{x}^T \mathbf{A}^T \right) \quad (2.8.12)$ |
| If $\mathbf{x} \neq 0$, then $DF(\mathbf{x}, 0) \neq 0$ | using (2.8.12), |

| | |
|---------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $DF(\mathbf{x}, 0) = \begin{pmatrix} 0 & \mathbf{x}^T \mathbf{A}^T \end{pmatrix} \quad (2.8.13)$ <p>From (2.8.4), we know that</p> $\mathbf{A}\mathbf{x} \neq 0 \quad (2.8.14)$ $\implies \mathbf{x}^T \mathbf{A}^T \neq 0 \quad (2.8.15)$ <p>So, We can say that</p> $DF(\mathbf{x}, 0) \neq 0 \quad (2.8.16)$ |
| If $\mathbf{y} \neq 0$, then $DF(0, \mathbf{y}) \neq 0$ | <p>using (2.8.12),</p> $DF(0, \mathbf{y}) = \begin{pmatrix} \mathbf{y}^T \mathbf{A} & 0 \end{pmatrix} \quad (2.8.17)$ <p>From (2.8.6), we know that</p> $\mathbf{A}^T \mathbf{y} \neq 0 \quad (2.8.18)$ $\implies \mathbf{y}^T \mathbf{A} \neq 0 \quad (2.8.19)$ <p>So, We can say that</p> $DF(0, \mathbf{y}) \neq 0 \quad (2.8.20)$ |
| If $(\mathbf{x}, \mathbf{y}) \neq 0$, then $DF(\mathbf{x}, \mathbf{y}) \neq 0$ | <p>using (2.8.12),</p> $DF(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{y}^T \mathbf{A} & \mathbf{x}^T \mathbf{A}^T \end{pmatrix} \quad (2.8.21)$ <p>As $(\mathbf{x}, \mathbf{y}) \neq 0$, $DF(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{A} = 0$</p> <p>From (2.8.4), we know that</p> $\mathbf{A} \neq 0 \quad (2.8.22)$ <p>So, We can say that</p> $DF(\mathbf{x}, \mathbf{y}) \neq 0 \quad (2.8.23)$ |
| If $\mathbf{x} = 0$ or $\mathbf{y} = 0$, then $DF(\mathbf{x}, \mathbf{y}) = 0$ | <p>From (2.8.20),</p> $DF(0, \mathbf{y}) \neq 0 \quad (2.8.24)$ <p>From (2.8.16),</p> $DF(\mathbf{x}, 0) \neq 0 \quad (2.8.25)$ <p>So, if $\mathbf{x} = 0$ or $\mathbf{y} = 0$,</p> $DF(\mathbf{x}, \mathbf{y}) \neq 0 \quad (2.8.26)$ |
| Conclusion | |

| | |
|--|----------------------------------------------------------|
| | From above,we can say that options 1),2),3) are correct. |
|--|----------------------------------------------------------|

TABLE 2.8.2: Finding derivative of linear transformation

| | |
|---------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Characteristic Polynomial | For an $n \times n$ matrix \mathbf{A} , characteristic polynomial is defined by, $p(x) = x\mathbf{I} - \mathbf{A} $ |
| Cayley-Hamilton Theorem | If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix \mathbf{A} , then, $p(\mathbf{A}) = \mathbf{0}$ |
| Minimal Polynomial | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $m(\mathbf{A}) = \mathbf{0}$ Every root of characteristic polynomial should be the root of minimal polynomial |

TABLE 2.9.1: Definitions

2.9. Let

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (2.9.1)$$

be a linear map that satisfies

$$T^2 = T - I. \quad (2.9.2)$$

Then which of the following is true?

- a) T is invertible.
- b) $T - I$ is not invertible.
- c) T has a real eigenvalue.
- d) $T^3 = -I$.

Solution: See Tables 2.9.1 and 2.9.2

| Statement | Solution |
|------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1. | <p>Given that $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ Since \mathbf{T} is a linear map from \mathbb{R}^n to \mathbb{R}^n therefore the matrix corresponding to it is of order $n \times n$.</p> <p>Since $\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$ $\therefore \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$</p> <p>$\implies p(x) = x^2 - x + 1$ will be annihilating polynomial. $\therefore p(\mathbf{T}) = \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$</p> <p>We know that minimal polynomial always divides annihilating polynomial. \therefore The roots of minimal polynomial are as follows:</p> $x = \frac{1 \pm \sqrt{3}i}{2} \quad (2.9.3)$ <p>Therefore any eigenvalue of \mathbf{T} is a root of its minimal polynomial. Since 0 is not a root of $p(x)$, Therefore 0 is not an eigen value for \mathbf{T}. Since \mathbf{T} is not invertible iff there exists an eigen value which is zero.</p> <p>$\therefore \mathbf{T}$ is invertible. $(2.9.4)$</p> |
| Conclusion | Therefore the statement is true. |
| 2. | <p>From equation (2.9.3) , Since 1 is not a root of $p(x)$, Therefore 1 is not an eigen value for \mathbf{T}. Therefore, 0 is not an eigen values of $\mathbf{T} - \mathbf{I}_n$.</p> <p>$\therefore \mathbf{T} - \mathbf{I}_n$ is invertible. $(2.9.5)$</p> |
| Conclusion | Therefore the statement is false. |

| | |
|------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 3. | <p>From equation (2.9.3) , Therefore any eigenvalue of \mathbf{T} is a root of its minimal polynomial. But the roots of minimal polynomial are not real. Therefore \mathbf{T} cant have a real eigen value.</p> |
| Conclusion | Therefore the statement is false. |
| 4. | <p>Since $\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$ (2.9.6) $\mathbf{T}^3 = \mathbf{T}(\mathbf{T} - \mathbf{I}_n)$ (2.9.7) $\therefore \mathbf{T}^3 = \mathbf{T}^2 - \mathbf{T}$ (2.9.8) $\therefore \mathbf{T}^3 = -\mathbf{I}_n$ (2.9.9)</p> |
| Conclusion | Therefore the statement is true. |

TABLE 2.9.2: Solution summary

2.10. Let

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \quad (2.10.1)$$

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix}. \quad (2.10.2)$$

Then which of the following are true?

- a) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are inconsistent.
- b) both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are consistent.
- c) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is consistent.
- d) the system $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ is inconsistent.

Solution: See Table 2.10.1

| | |
|----------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | $\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}, \mathbf{b}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix} \quad (2.10.3)$ |
| Solution | <p>Solving for $\mathbf{Mx} = \mathbf{b}_1$, Row Reducing the augmented matrix</p> $\begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 4 \end{pmatrix} \quad (2.10.4)$ $\begin{matrix} \xleftarrow{R_4 \leftarrow 2R_4 - R_1} \\ \xrightarrow{R_4 \leftarrow R_4 - 2R_2} \end{matrix} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 \\ 0 & 0 & -1 & 0 & -4 & -4 & 1 \end{pmatrix} \quad (2.10.5)$ $\xleftarrow{R_4 \leftarrow R_4 + R_3} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.10.6)$ $\Rightarrow \text{Rank}(M) = 3, \text{Rank}(\mathbf{M} \mathbf{b}_1) = 4 \quad (2.10.7)$ $\Rightarrow \text{Rank}(M) \neq \text{Rank}(\mathbf{M} \mathbf{b}_1) \quad (2.10.8)$ <p>Solving for $\mathbf{Mx} = \mathbf{b}_2$, Row Reducing the augmented matrix</p> $\begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 3 \\ 1 & 1 & 1 & 0 & 1 & 1 & 3 \end{pmatrix} \quad (2.10.9)$ $\begin{matrix} \xleftarrow{R_4 \leftarrow 2R_4 - R_1} \\ \xrightarrow{R_4 \leftarrow R_4 + 2R_2} \end{matrix} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 3 \\ 0 & 0 & -1 & 0 & -4 & -4 & -1 \end{pmatrix} \quad (2.10.10)$ $\xleftarrow{R_4 \leftarrow R_4 + R_3} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 5 \\ 0 & 1 & 0 & -1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 0 & 4 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.10.11)$ $\Rightarrow \text{Rank}(M) = 3, \text{Rank}(\mathbf{M} \mathbf{b}_2) = 4 \quad (2.10.12)$ $\Rightarrow \text{Rank}(M) \neq \text{Rank}(\mathbf{M} \mathbf{b}_2) \quad (2.10.13)$ <p>Solving for $\mathbf{Mx} = (\mathbf{b}_1 - \mathbf{b}_2)$, Row Reducing the augmented matrix</p> |

| | |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $\begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & -2 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \quad (2.10.14)$ |
| | $\begin{matrix} \xleftarrow{R_4 \leftarrow 2R_4 - R_1} \\ \xrightarrow{R_4 \leftarrow R_4 - 2R_2} \end{matrix} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & -1 & 0 & -4 & -4 & 2 \end{pmatrix} \quad (2.10.15)$ |
| | $\xleftarrow{R_4 \leftarrow R_4 + R_3} \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (2.10.16)$ |
| | $\Rightarrow \text{Rank}(M) = 3, \text{Rank}(M (\mathbf{b}_1 - \mathbf{b}_2)) = 3 \quad (2.10.17)$ |
| | $\Rightarrow \text{Rank}(M) = \text{Rank}(M (\mathbf{b}_1 - \mathbf{b}_2)) \quad (2.10.18)$ |
| Statement 1 | Both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are inconsistent |
| | $Eq.(2.10.8) \text{ and } (2.10.13) \text{ violate the condition of consistency} \quad (2.10.19)$ <p style="text-align: center;">True Statement</p> |
| Statement 2 | Both systems $\mathbf{Mx} = \mathbf{b}_1$ and $\mathbf{Mx} = \mathbf{b}_2$ are consistent |
| | $Eq.(2.10.8) \text{ and } (2.10.13) \text{ violate the condition of consistency} \quad (2.10.20)$ <p style="text-align: center;">False Statement</p> |
| Statement 3 | Systems $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ are consistent |
| | $Eq.(2.10.18) \text{ satisfy the condition of consistency} \quad (2.10.21)$ <p style="text-align: center;">True Statement</p> |
| Statement 4 | Systems $\mathbf{Mx} = \mathbf{b}_1 - \mathbf{b}_2$ are inconsistent |
| | $Eq.(2.10.18) \text{ satisfy the condition of consistency} \quad (2.10.22)$ <p style="text-align: center;">False Statement</p> |

TABLE 2.10.1: Explanation

2.11. Let

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix}. \quad (2.11.1)$$

Given that 1 is an eigenvalue of \mathbf{M} , then which among the following are correct?

- a) The minimal polynomial of \mathbf{M} is $(x - 1)(x + 4)$
- b) The minimal polynomial of \mathbf{M} is $(x - 1)^2(x + 4)$
- c) \mathbf{M} is not diagonalizable.
- d) $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I})$.

Solution: See Table 2.11.1

| | |
|--------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix} \quad (2.11.2)$ <p>One of the eigenvalue of \mathbf{M} is 1</p> |
| Solution | <p>Let the eigenvalues of matrix \mathbf{M} of order 3×3 be $\lambda_1, \lambda_2, \lambda_3$ From given , let $\lambda_1 = 1$. We know that sum of the eigenvalues of matrix is Trace of the matrix and product of eigenvalues of matrix is Determinant of the matrix. Trace of the square matrix($\text{Tr}(\mathbf{M})$) is the sum of the elements in the main diagonal of \mathbf{M}.</p> $\text{Tr}(\mathbf{M}) = 1 + 1 - 4 \quad (2.11.3)$ $\Rightarrow \text{Tr}(\mathbf{M}) = -2 \quad (2.11.4)$ $\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = -2 \quad (2.11.5)$ $\Rightarrow \lambda_2 + \lambda_3 = -3 \quad (2.11.6)$ $\Rightarrow \lambda_2 = -3 - \lambda_3 \quad (2.11.7)$ <p>By row reducing the matrix \mathbf{M}, we get ,</p> $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} \quad (2.11.8)$ |
| | $\text{Det}(\mathbf{M}) = 1 \left(3 \left(-\frac{4}{3} \right) \right) = -4 \quad (2.11.9)$ $\Rightarrow \lambda_1 \lambda_2 \lambda_3 = -4 \quad (2.11.10)$ $\Rightarrow \lambda_2 \lambda_3 = -4 \quad (2.11.11)$ <p>Solving equations (2.11.7) and (2.11.11) one of the possibilities we get,</p> $\lambda_1 = 1 \quad (2.11.12)$ $\lambda_2 = 1 \quad (2.11.13)$ $\lambda_3 = -4 \quad (2.11.14)$ |
| | <p>Using the eigenvalues the characteristic polynomial of matrix \mathbf{M} is given by,</p> $c(x) = x^3 + 2x^2 - 7x + 4 = 0 \quad (2.11.15)$ <p>The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation. Using the above theorem, the equation (2.11.15) can be written as,</p> $\mathbf{M}^3 + 2\mathbf{M}^2 - 7\mathbf{M} + 4\mathbf{I} = 0 \quad (2.11.16)$ $\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I} + 4\mathbf{M}^{-1} = 0 \quad (2.11.17)$ $\Rightarrow \mathbf{M}^{-1} = -\frac{1}{4}(\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I}) \quad (2.11.18)$ |
| Statement 1 | <p>The minimal polynomial of \mathbf{M} is $(x - 1)(x + 4)$ If $(x-1)(x+4)$ is a minimal polynomial of \mathbf{M} then,</p> |

| | |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $(\mathbf{M} - \mathbf{I})(\mathbf{M} + 4\mathbf{I}) = \mathbf{0}_{3 \times 3} \quad (2.11.19)$ <p>But,</p> $(\mathbf{M} - \mathbf{I})(\mathbf{M} + 4\mathbf{I}) = \begin{pmatrix} -4 & -4 & -4 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \neq \mathbf{0}_{3 \times 3} \quad (2.11.20)$ <p style="text-align: center;">False Statement</p> |
| Statement 2 | The minimal polynomial of \mathbf{M} is $(x - 1)^2(x + 4)$ |
| | <p>Let $m(x)$ be the minimal polynomial</p> $m(x) = (x - 1)^2(x + 4) \quad (2.11.21)$ $= x^3 + 2x^2 - 7x + 4 \quad (2.11.22)$ $= c(x)$ <p>In this case both minimal polynomial and characteristic polynomial were same. Therefore we could say that equation (2.11.21) is the minimal polynomial of \mathbf{M} as it satisfies equation (2.11.16) by Cayley Hamilton Theorem.</p> <p style="text-align: center;">True Statement</p> |
| Statement 3 | \mathbf{M} is not diagonalizable. |
| | \mathbf{M} is diagonalizable if and only if its minimal polynomial is a product of distinct monic linear factors. From equation (2.11.21) we could see that one of the factor of minimal polynomial is repeated and it is not a linear factor. Therefore, Matrix \mathbf{M} is not diagonalizable. |
| | True Statement |
| Statement 4 | $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I}) \quad (2.11.23)$ |
| | Comparing equation (2.11.18) and (2.11.23) we could say that the given statement is False Statement . |

TABLE 2.11.1: Explanation

| | |
|---------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Characteristic Polynomial | For an $n \times n$ matrix \mathbf{A} , characteristic polynomial is defined by, $p(x) = x\mathbf{I} - \mathbf{A} $ |
| Cayley-Hamilton Theorem | If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix \mathbf{A} , then, $p(\mathbf{A}) = \mathbf{0}$ |
| Minimal Polynomial | Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $m(\mathbf{A}) = \mathbf{0}$ Every root of characteristic polynomial should be the root of minimal polynomial |

TABLE 2.12.1: Definitions

2.12. Let \mathbf{A} be a real matrix with characteristic polynomial $(x - 1)^3$. Pick the correct statements from below:

- a) \mathbf{A} is necessarily diagonalizable.
- b) If the minimal polynomial of \mathbf{A} is $(x - 1)^3$, then \mathbf{A} is diagonalizable.
- c) The characteristic polynomial of \mathbf{A}^2 is $(x - 1)^3$
- d) If \mathbf{A} has exactly two Jordan blocks, then $(\mathbf{A} - \mathbf{I})^2$ is diagonalizable.

Solution: See Tables 2.12.1 and 2.12.2

| Statement | Solution |
|---------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1. | <p>Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$</p> <p>Since \mathbf{A} is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$</p> <p>Therefore, $p(x) = (x - 1)^3$</p> <p>Solving $(\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$</p> <p>Solving $(\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$</p> <p>Solving $\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$</p> <p>Since $\mathbf{A} - \mathbf{I} \neq \mathbf{0}$</p> <p>Therefore, $m(x) = (x - 1)^2$</p> |
| Justification | <p>Hence, the Jordan form of \mathbf{A} is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 1$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1.</p> <p>And one block of order 1 with $\lambda = 1$.</p> <p>Hence the required Jordan form of \mathbf{A} is,</p> $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ <p>A matrix is diagonalizable iff its jordan form is a diagonal matrix. Since \mathbf{J} is not diagonalizable therefore \mathbf{A} is not diagonalizable.</p> |
| Conclusion | Therefore the statement is false. |

| | |
|------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 2. | <p>Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$</p> <p>Since \mathbf{A} is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$</p> <p>Therefore, $p(x) = (x - 1)^3$</p> <p>Solving $(\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$</p> <p>Solving $(\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$</p> <p>Since $(\mathbf{A} - \mathbf{I})^2 \neq \mathbf{0}$</p> <p>Therefore, $m(x) = (x - 1)^3$</p> <p>Justification Hence, the Jordan form of \mathbf{A} is a 3×3 matrix consisting of only one block with principal diagonal values as $\lambda_1 = 1$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of \mathbf{A} is,</p> $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ <p>Since \mathbf{J} is not diagonalizable therefore \mathbf{A} is not diagonalizable.</p> |
| Conclusion | Therefore the statement is false. |
| 3. | <p>Give that, $p(x)$ of $\mathbf{A} = (x - 1)^3$</p> <p>Hence the eigen values of $\mathbf{A} = 1, 1, 1$</p> <p>Hence the eigen values of $\mathbf{A}^2 = 1^2, 1^2, 1^2$ or $1, 1, 1$</p> <p>Therefore $p(x)$ of $\mathbf{A}^2 = (x - 1)^3$</p> |
| Conclusion | Therefore the statement is True. |

| | |
|------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 4. | <p>We know that jordan form of a matrix is similar to the original matrix Let \mathbf{J} be the jordan form of the matrix \mathbf{A} then,</p> $\mathbf{A} = \mathbf{PJP}^{-1}$ $\mathbf{A} - \mathbf{I} = \mathbf{PJP}^{-1} - \mathbf{I}$ $\mathbf{A} - \mathbf{I} = \mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}$ $(\mathbf{A} - \mathbf{I})^2 = \mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}\mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}$ $(\mathbf{A} - \mathbf{I})^2 = \mathbf{P}(\mathbf{J} - \mathbf{I})^2\mathbf{P}^{-1}$ <p>Therefore $(\mathbf{A} - \mathbf{I})^2$ is similar to $(\mathbf{J} - \mathbf{I})^2$ Since \mathbf{A} has exactly two jordan blocks and order of \mathbf{A} is 3.</p> $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\mathbf{J} - \mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $(\mathbf{J} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ <p>Since $(\mathbf{J} - \mathbf{I})^2$ is diagonal matrix. Therefore $(\mathbf{A} - \mathbf{I})^2$ is diagonalizable.</p> |
| Conclusion | Therefore the statement is True. |

TABLE 2.12.2: Solution summary

2.13. Let P_3 be the vector space of polynomials with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \rightarrow P_3 \quad (2.13.1)$$

defined by

$$T(p(x)) = p(x-1) + p(x+1) \quad (2.13.2)$$

Which of the following properties does the matrix of T with respect to the standard basis $B = \{1, x, x^2, x^3\}$ of P_3 satisfy?

- a) $\det T = 0$.
- b) $(T - 2I)^4 = 0$ but $(T - 2I)^3 \neq 0$.
- c) $(T - 2I)^3 = 0$ but $(T - 2I)^2 \neq 0$.
- d) 2 is an eigenvalue with multiplicity 4.

Solution: Given

$$T(p(x)) = p(x+1) + p(x-1). \quad (2.13.3)$$

The matrix of T with respect to the standard basis $B = \{1, x, x^2, x^3\}$ is given by:

$$\begin{aligned} p(x) = 1 &\implies T(1) = 1 + 1 \\ &= 2 \end{aligned} \quad (2.13.4)$$

$$\begin{aligned} p(x) = x &\implies T(x) = x + 1 + x - 1 \\ &= 2x \end{aligned} \quad (2.13.5)$$

$$\begin{aligned} p(x) = x^2 &\implies T(x^2) = (x+1)^2 + (x-1)^2 \\ &= 2 + 2x^2 \end{aligned} \quad (2.13.6)$$

$$\begin{aligned} p(x) = x^3 &\implies T(x^3) = (x+1)^3 + (x-1)^3 \\ &= 6x + 2x^3 \end{aligned} \quad (2.13.7)$$

Hence, matrix of T is:

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.13.8)$$

See Table 2.13.1

2.14. Let \mathbf{M} be an $n \times n$ Hermitian matrix of rank $k, k \neq n$. If $\lambda \neq 0$ is an eigenvalue of \mathbf{M} with corresponding unit column vector \mathbf{u} , then which of the following are true?

- a) $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k - 1$.
- b) $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k$.
- c) $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k + 1$.
- d) $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*$.

Solution: See Tables 2.14.1 and 2.14.2

| | |
|------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\det(T) = 0$ | False. From (2.13.8), it is found that the determinant is not zero as the eigenvalues are nonzero. |
| $(T - 2I)^4 = 0$ but $(T - 2I)^3 \neq 0$ | False. $(T - 2I) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\implies (T - 2I)^2 = 0$ and hence $(T - 2I)^4 = 0$ and $(T - 2I)^3 = 0$ |
| $(T - 2I)^3 = 0$ but $(T - 2I)^2 \neq 0$ | False. Because $(T - 2I)^3 = 0$ and $(T - 2I)^2 = 0$ |
| 2 is an eigenvalue with multiplicity 4. | True. It is noted that the matrix of T is an upper triangular matrix having the value 2 along its principal diagonal and hence 2 is an eigenvalue with algebraic multiplicity 4. |

TABLE 2.13.1

2.15. Define a real valued function B on $\mathbb{R}^2 \times \mathbb{R}^2$ as

$$B(\mathbf{x}, \mathbf{y}) = x_1 y_1 - x_1 y_2 - x_2 y_1 + 4x_2 y_2 \quad (2.15.1)$$

Let $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$W = \{\mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0\} \quad (2.15.2)$$

Then W

- a) is not a subspace of \mathbb{R}^2 .
- b) equals $\mathbf{0}$.
- c) is the y axis
- d) is the line passing through $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution: See Tables 2.15.1, 2.15.2 and 2.15.3.

| Objective | Explanation |
|--------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Rank of $\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*$ | <p>Since</p> $\text{rank}(\mathbf{A} - \mathbf{B}) \geq \text{rank}(\mathbf{A}) - \text{rank}(\mathbf{B}) \quad (2.14.1)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) \geq \text{rank}(\mathbf{M}) - \text{rank}(\mathbf{u} \mathbf{u}^*) \quad (2.14.2)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) \geq k - \text{rank}(\mathbf{u} \mathbf{u}^*) \quad (2.14.3)$ <p>If \mathbf{A} is a non-zero column vector of order $m \times 1$ and \mathbf{B} is a non-zero row vector of order $1 \times n$ then $\text{rank}(\mathbf{AB}) = 1$. So,</p> $\text{rank}(\mathbf{u} \mathbf{u}^*) = 1 \quad (2.14.4)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) \geq k - 1 \quad (2.14.5)$ <p>Also since,</p> $\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^* = \mathbf{M} - \mathbf{M} \mathbf{u} \mathbf{u}^* = \mathbf{M}(\mathbf{I} - \mathbf{u} \mathbf{u}^*) \quad (2.14.6)$ <p>and</p> $\text{rank}(\mathbf{M}(\mathbf{I} - \mathbf{u} \mathbf{u}^*)) \leq \min(\text{rank}(\mathbf{M}), \text{rank}(\mathbf{I} - \mathbf{u} \mathbf{u}^*)) \quad (2.14.7)$ $\implies \text{rank}(\mathbf{M}(\mathbf{I} - \mathbf{u} \mathbf{u}^*)) \leq k \quad (2.14.8)$ <p>Thus we have from (2.14.5) and (2.14.8) that</p> $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k - 1 \text{ or } k \quad (2.14.9)$ <p>Consider a matrix</p> $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.14.10)$ |

TABLE 2.14.1

| Objective | Explanation |
|-------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>such that $\text{rank}(M) = 1$. The eigenvalue of \mathbf{M} is $\lambda = 1$ and the corresponding eigenvector is</p> $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.14.11)$ <p>Thus we have,</p> $\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (2.14.12)$ $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.14.13)$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.14.14)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = 0 \quad (2.14.15)$ <p>Hence if $\text{rank}(\mathbf{M}) = k$ then $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k - 1$.</p> |
| $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*$ | <p>Let the given statement be P(n): $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*$. It can be seen that P(1) is true. Assume P(n) is true for some $k \in \mathbf{N}$ such that</p> $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^k = \mathbf{M}^k - \lambda^k \mathbf{u} \mathbf{u}^* \quad (2.14.16)$ <p>Now to prove that P(k+1) is true we have</p> $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^{k+1} = (\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^k \quad (2.14.17)$ $= (\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)(\mathbf{M}^k - \lambda^k \mathbf{u} \mathbf{u}^*) \quad (2.14.18)$ $= \mathbf{M}^{k+1} - \lambda^k \mathbf{M} \mathbf{u} \mathbf{u}^* - \lambda \mathbf{M}^k \mathbf{u} \mathbf{u}^* + \lambda^{k+1} \mathbf{u} \mathbf{u}^* \mathbf{u} \mathbf{u}^* \quad (2.14.19)$ $= \mathbf{M}^{k+1} - \lambda^{k+1} \mathbf{u} \mathbf{u}^* - \lambda^{k+1} \mathbf{u} \mathbf{u}^* + \lambda^{k+1} \mathbf{u} \ \mathbf{u}\ ^2 \mathbf{u}^* \quad (2.14.20)$ $= \mathbf{M}^{k+1} - 2\lambda^{k+1} \mathbf{u} \mathbf{u}^* + \lambda^{k+1} \mathbf{u} \mathbf{u}^* \quad (2.14.21)$ $= \mathbf{M}^{k+1} - \lambda^{k+1} \mathbf{u} \mathbf{u}^* \quad (2.14.22)$ <p>Hence, by the Principle of Mathematical Induction P(n) is true for all n.</p> |
| Answer | (1) and (4) |

TABLE 2.14.2

| | |
|-----------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Subspace | A non-empty subset \mathbf{W} of \mathbf{V} is a subspace of \mathbf{V} if and only if for each pair of vectors α, β in \mathbf{W} and each scalar c in \mathbf{F} the vector $c\alpha + \beta$ is again in \mathbf{W} . |
|-----------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

TABLE 2.15.1: Definitions and theorem used

| Statement | Observations |
|-----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | $\mathbf{W} = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = 0\} \quad (2.15.3)$ |
| | $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.15.4)$ |
| | $\mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (2.15.5)$ |
| | $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.15.6)$ |
| | $\mathbf{B}(\mathbf{v}, \mathbf{w}) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \quad (2.15.7)$ |
| | we will express (2.15.7) in quadratic form. |
| | $\mathbf{B}(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{w} \quad (2.15.8)$ |
| | From (2.15.4), (2.15.6), (2.15.8) we will calculate $\mathbf{B}(\mathbf{v}_0, \mathbf{v})$ |
| | $\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = \mathbf{v}_0^T \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{v} \quad (2.15.9)$ |
| | $\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.15.10)$ |
| | $\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.15.11)$ |
| | Now we find the basis vector for \mathbf{W} , which is the basis vector of null space of $\mathbf{B}(\mathbf{v}_0, \mathbf{v})$. |
| | $\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = 0 \quad (2.15.12)$ |
| | $\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.15.13)$ |
| | $\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.15.14)$ |
| | $\Rightarrow x_1 = x_2 \quad (2.15.15)$ |
| | Therefore, the basis vector for \mathbf{W} is |
| | $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.15.16)$ |
| | Therefore |
| | $\mathbf{W} = \{k\mathbf{b} : \forall k \in \mathbb{R}\} \quad (2.15.17)$ |

TABLE 2.15.2: Observations

| Option | Solution | True/False |
|--------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| 1. | <p>Now we will see whether \mathbf{W} is a subspace or not. Let α, β be two pair of vectors in \mathbf{W} where</p> $\alpha = m\mathbf{b} \quad (2.15.18)$ $\beta = n\mathbf{b} \quad (2.15.19)$ <p>Here $m, n \in \mathbb{R}$ and now we will see whether the vector $c\alpha + \beta$ is in \mathbf{W} or not where c is a scalar value in \mathbb{R}. Here</p> $c\alpha + \beta = cm\mathbf{b} + n\mathbf{b} \quad (2.15.20)$ $\Rightarrow c\alpha + \beta = (cm + n)\mathbf{b} \quad (2.15.21)$ <p>From (2.15.21), $(cm + n) \in \mathbb{R}$ and we can say that the vector $c\alpha + \beta \in \mathbf{W}$. Therefore, \mathbf{W} is a subspace of \mathbb{R}^2</p> | |
| 2. | <p>From Table 2.15.2, we got \mathbf{W} contains the vectors which are all linear combination of basis vector \mathbf{b} as shown in (2.15.17). Therefore,</p> $\mathbf{W} \neq \{(0, 0)\} \quad (2.15.22)$ | False |
| 3. | <p>Let us consider a vector on y-axis</p> $\mathbf{p} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.15.23)$ <p>Here</p> $\mathbf{p} \neq k\mathbf{b} \quad (2.15.24)$ <p>for any $k \in \mathbb{R}$ The vector \mathbf{p} can not be written in terms of the basis vector \mathbf{b}. Then $\mathbf{p} \notin \mathbf{W}$. Therefore, the vectors in \mathbf{W} is not y-axis.</p> | False |
| 4. | <p>There is only one basis vector \mathbf{b} for \mathbf{W}. Therefore the vectors in \mathbf{W} forms a straight line in vector space \mathbb{R}^2. Since,</p> $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\mathbf{b} \quad (2.15.25)$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\mathbf{b} \quad (2.15.26)$ <p>Therefore, the line passes through (0,0) and (1,1).</p> | True |

TABLE 2.15.3: Solution

2.16. Consider the Quadratic forms

$$Q_1(x, y) = xy \quad (2.16.1)$$

$$Q_2(x, y) = x^2 + 2xy + y^2 \quad (2.16.2)$$

$$Q_3(x, y) = x^2 + 3xy + 2y^2 \quad (2.16.3)$$

on \mathbb{R}^2 . Choose the correct statements from below

- a) Q_1 and Q_2 are equivalent.
- b) Q_1 and Q_3 are equivalent.
- c) Q_2 and Q_3 are equivalent.
- d) all are equivalent.

Solution: See Tables 2.16.1 2.16.2

| | |
|-----------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Matrix representation | <p>The Matrix representation of quadratic forms</p> $Q(x, y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{X}^T \mathbf{A} \mathbf{X} \quad (2.16.4)$ <p>The symmetric matrix of the quadratic form is</p> $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.16.5)$ |
| Equivalent condition | <p>Two quadratic forms $\mathbf{X}^T \mathbf{A} \mathbf{X}$ and $\mathbf{Y}^T \mathbf{B} \mathbf{Y}$ are called equivalent if their matrices, A and B are congruent.</p> <p>Two real quadratic forms are equivalent over the real field iff they have the same rank and the same index.</p> |
| Rank | The rank of a quadratic form is the rank of its associated matrix. |
| Index | The index of the quadratic form is equal to the number of positive eigen values of the matrix of quadratic form. |

TABLE 2.16.1: Definitions and results used

| | Matrix | Rank | Eigen Values | Index |
|-------------|--------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------|
| $Q_1(x, y)$ | $\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_1]{R_1 \leftarrow R_2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\text{rank}(\mathbf{A}_1) = 2$ | $ \mathbf{A}_1 - \lambda \mathbf{I} = 0$ $\Rightarrow \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0$ $\Rightarrow (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = 0$ $\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$ | Index of $\mathbf{A}_1 = 1$ |
| $Q_2(x, y)$ | $\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{A}_2) = 1$ | $ \mathbf{A}_2 - \lambda \mathbf{I} = 0$ $\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (\lambda)(\lambda - 2) = 0$ $\Rightarrow \lambda_1 = 0, \lambda_2 = 2$ | Index of $\mathbf{A}_2 = 2$ |
| $Q_3(x, y)$ | $\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix}$ | $\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}$ $\text{rank}(\mathbf{A}_3) = 2$ | $ \mathbf{A}_3 - \lambda \mathbf{I} = 0$ $\Rightarrow \begin{vmatrix} 1-\lambda & \frac{3}{2} \\ \frac{3}{2} & 2-\lambda \end{vmatrix} = 0$ $\Rightarrow (\lambda - \frac{\sqrt{10}+3}{2})(\lambda + \frac{\sqrt{10}-3}{2}) = 0$ $\Rightarrow \lambda_1 = \frac{3+\sqrt{10}}{2}, \lambda_2 = \frac{3-\sqrt{10}}{2}$ | Index of $\mathbf{A}_3 = 1$ |
| Conclusion | We can say that $Q_1(x, y)$ and $Q_3(x, y)$ are equivalent as the rank and index are same. | | | |

TABLE 2.16.2: Finding which quadratic forms are equivalent

2.17. Consider a Markov Chain with state space $\{0, 1, 2\}$ and transition matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{matrix} \quad (2.17.1)$$

For any two states i and j , let $p_{ij}^{(n)}$ denote the n -step transition probability of going from i to j . Identify correct statements.

- a) $\lim_{n \rightarrow \infty} p_{11}^{(n)} = \frac{2}{9}$
- b) $\lim_{n \rightarrow \infty} p_{21}^{(n)} = 0$
- c) $\lim_{n \rightarrow \infty} p_{32}^{(n)} = \frac{1}{3}$
- d) $\lim_{n \rightarrow \infty} p_{13}^{(n)} = \frac{1}{3}$

Solution: See Tables 2.17.1 and 2.17.2

| | |
|--------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Irreducible Markov Chain | A Markov chain is irreducible if all the states communicate with each other, i.e., if there is only one communication class. |
| Aperiodic Markov Chain | If there is a self-transition in the chain ($p^{ii} > 0$ for some i), then the chain is called as aperiodic |
| Stationary Distribution | <p>A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses. Typically, it is represented as a row vector π whose entries are probabilities summing to 1, and given transition matrix \mathbf{P}, it satisfies</p> $\pi = \pi \mathbf{P}$ |

TABLE 2.17.1

| | |
|---------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Drawing Transition diagram | <pre> graph TD 1((1)) -- 1/2 --> 1 1 -- 1/2 --> 2((2)) 2 -- 1/2 --> 2 2 -- 1/3 --> 3((3)) 3 -- 1/3 --> 1 3 -- 1/2 --> 2 3 -- 1/3 --> 3 </pre> |
| Checking whether the chain is Irreducible and Aperiodic | <p>Here, All the states are accessible to one another. \Rightarrow They are in the same communication class. So, it is Irreducible.</p> <p>There exists the non- zero self-transition, which means that the chain is Aperiodic.</p> <p>We know that if the Markov Chain is irreducible and aperiodic then $\pi_j = \lim_{n \rightarrow \infty} P\{X_n = j\}, j = 1, \dots, N$ These are the stationary probabilities.</p> |
| Finding the Stationary | Stationary Probability can be represented as |

Probability Distributions

$$\pi = \pi \mathbf{P}$$

$$\Rightarrow \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \mathbf{P}$$

Equating the above equation we get

$$\frac{1}{2}v_1 - \frac{1}{3}v_3 = 0$$

$$\frac{1}{2}v_1 - \frac{1}{2}v_2 + \frac{1}{3}v_3 = 0$$

$$\frac{1}{2}v_2 - \frac{2}{3}v_3 = 0$$

We see that summation of second and the third equation gives us the first equation only.

And we know that the probability distribution will sum up to 1.

$$v_1 + v_2 + v_3 = 1$$

Therefore, we get the equation form as

$$\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solving the linear equations

The above linear equation can be solved using Gauss-Jordan method as

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{3} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & 0 \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -\frac{5}{6} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & 0 \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -\frac{5}{6} & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{6} & -\frac{1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & -1 & -\frac{1}{6} & -\frac{1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & \frac{11}{6} & \frac{1}{2} \end{array} \right)$$

| | |
|--------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $\xleftrightarrow{R_3 \leftarrow \frac{3}{2}R_3} \left(\begin{array}{ccc c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ $\xleftrightarrow{R_2 \leftarrow R_2 - \frac{5}{3}R_3} \left(\begin{array}{ccc c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ $\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \left(\begin{array}{ccc c} 1 & 1 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ $\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left(\begin{array}{ccc c} 1 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ <p>\therefore, stationary probability distribution π is given by</p> $\pi = \left(\frac{2}{9} \quad \frac{4}{9} \quad \frac{1}{3} \right)$ |
| Observations | <p>Since the given transition probability matrix \mathbf{P} is irreducible and aperiodic, then $\lim_{n \rightarrow \infty} \mathbf{P}^n$ converges to a matrix with all rows identical and equal to π.</p> <p>We were able to find π as $\left(\frac{2}{9} \quad \frac{4}{9} \quad \frac{1}{3} \right)$</p> $\lim_{n \rightarrow \infty} \mathbf{P}^n = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ <p>From the above matrix, we get</p> $\lim_{n \rightarrow \infty} \mathbf{P}_{11}^n = \frac{2}{9}$ $\lim_{n \rightarrow \infty} \mathbf{P}_{21}^n = \frac{2}{9}$ $\lim_{n \rightarrow \infty} \mathbf{P}_{32}^n = \frac{4}{9}$ $\lim_{n \rightarrow \infty} \mathbf{P}_{13}^n = \frac{1}{3}$ |
| Conclusion | <p>From our observation we see that</p> <p>Options 1) and 4) are True.</p> |

TABLE 2.17.2

3 JUNE 2018

3.1. Let \mathbf{A} be a $(m \times n)$ matrix and \mathbf{B} be a $(n \times m)$ matrix over real numbers with $m < n$. Then

- a) \mathbf{AB} is always nonsingular.
- b) \mathbf{AB} is always singular.
- c) \mathbf{BA} is always nonsingular.
- d) \mathbf{BA} is always singular.

Solution: See Table 3.1.1

$$\text{rank}(\mathbf{A}) \leq \min(m, n) \quad (3.1.1)$$

$$\implies \leq m, \because m < n \quad (3.1.2)$$

$$\text{rank}(\mathbf{B}) \leq \min(n, m) \quad (3.1.3)$$

$$\implies \leq m, \because m < n \quad (3.1.4)$$

We also know that \mathbf{AB} will be a $m \times m$ matrix and \mathbf{BA} will be a $n \times n$ matrix.

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (3.1.5)$$

$$\implies \leq m \quad (3.1.6)$$

$$\text{rank}(\mathbf{BA}) \leq \min(\text{rank}(\mathbf{B}), \text{rank}(\mathbf{A})) \quad (3.1.7)$$

$$\implies \leq m \quad (3.1.8)$$

3.2. If \mathbf{A} is a (2×2) matrix over \mathbb{R} with $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$. Then we can conclude that

- a) $\det(\mathbf{A}) = 0$.
- b) $\mathbf{A} = 0$.
- c) $\text{tr}(\mathbf{A}) = 0$.
- d) \mathbf{A} is nonsingular.

Solution: See Table 3.2.1

| Options | Explanation |
|---------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| AB is always nonsingular | $rank(\mathbf{AB}) \leq m$ Let, $rank(\mathbf{AB}) = k, k < m$. So, there are $m - k$ linearly dependent columns or rows So, AB will be singular Hence, incorrect Example $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 5 & 6 \end{pmatrix}$ $\mathbf{AB} = \begin{pmatrix} 20 & 33 \\ 40 & 66 \end{pmatrix}, rank(\mathbf{AB}) = 1$ 2^{nd} row is linearly dependent on 1^{st} row. AB is singular |
| AB is always singular | $rank(\mathbf{AB}) \leq m$ Let, $rank(\mathbf{AB}) = m$ So, there are 0 linearly dependent columns or rows So, AB will be nonsingular Hence, incorrect Example $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$ $\mathbf{AB} = \begin{pmatrix} 20 & 29 \\ 35 & 52 \end{pmatrix}, rank(\mathbf{AB}) = 2$ AB is nonsingular |
| BA is always nonsingular | $rank(\mathbf{BA}) \leq m, rank(\mathbf{BA})$ can be atmost m BA is $n \times n$ matrix. $n > m$. So, there are atleast $n - m$ linearly dependent columns or rows. So, BA will be singular always. Hence, incorrect Example $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} 7 & 14 & 18 \\ 10 & 20 & 26 \\ 17 & 34 & 45 \end{pmatrix}, rank(\mathbf{BA}) = 2$ 2^{nd} column is linearly dependent on 1^{st} column BA is singular |
| BA is always singular | $rank(\mathbf{BA}) \leq m, rank(\mathbf{BA})$ can be atmost m BA is $n \times n$ matrix. $n > m$. So, there are atleast $n - m$ linearly dependent columns or rows. So, BA will be singular always. Hence, correct Same example as above. BA is always singular. |

TABLE 3.1.1: Finding Correct Option

| | |
|--------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>\mathbf{A} be a 2×2 matrix over \mathbb{R} with</p> $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ |
| Explanation | <p>If \mathbf{X} is an eigen vector of matrix \mathbf{A} corresponding to the eigen value λ i.e</p> $\mathbf{AX} = \lambda\mathbf{X}$ <p>then, $(\mathbf{I} + \mathbf{A})\mathbf{X} = (1 + \lambda)\mathbf{X}$</p> <p>Thus, \mathbf{X} is an eigen vector of $(\mathbf{A} + \mathbf{I})$ corresponding to the eigen value $(1 + \lambda)$.</p> <p>Let λ_1, λ_2 be two eigen values of \mathbf{A} and $(1 + \lambda_1), (1 + \lambda_2)$ be the eigen values of $(\mathbf{A} + \mathbf{I})$.</p> <p>$\Rightarrow$ Eigen value of $\mathbf{A} = \lambda_1, \lambda_2$</p> <p>$\Rightarrow$ Eigen value of $(\mathbf{A} + \mathbf{I}) = \lambda_1 + 1, \lambda_2 + 1$</p> <p>Since,</p> |
| | $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ <p>Trace of any matrix is sum of its eigen values.</p> <p>Determinant of matrix is product of its eigen values</p> $\Rightarrow (\lambda_1 + 1)(\lambda_2 + 1) = 1 + (\lambda_1\lambda_2)$ $\Rightarrow \boxed{\lambda_1 + \lambda_2 = 0}$ $\Rightarrow \boxed{\text{tr}(\mathbf{A}) = 0}$ |
| Statement 1 : $\det \mathbf{A} = 0$ | False |
| | <p>Let, $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$</p> <p>Here, $\det \mathbf{A} = -1$ and $\det(\mathbf{A} + \mathbf{I}) = 0$</p> <p>Thus, $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> <p>In this case, $\det \mathbf{A} \neq 0$ but satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> |

| | |
|-------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Statement 2 : $\mathbf{A} = \mathbf{0}$ | False |
| | <p>Let , $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$</p> <p>Here, $\det \mathbf{A} = 0$ and $\det(\mathbf{A} + \mathbf{I}) = 1$</p> <p>Thus, $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> <p>In this case, $\mathbf{A} \neq \mathbf{0}$ But , satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> |
| Statement 3: $\text{tr}(\mathbf{A}) = 0$ | True |
| | <p>The given statement is true for all possible matrices.</p> <p>If $\text{tr} \mathbf{A} \neq 0$ then the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$ doesn't satisfy.</p> <p>Let , $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$</p> |
| | <p>Here, $\det \mathbf{A} = 0$, $\det(\mathbf{A} + \mathbf{I}) = 2$, $\text{tr} \mathbf{A} \neq 0$</p> <p>Thus, $1 + \det(\mathbf{A}) \neq \det(\mathbf{A} + \mathbf{I})$</p> |
| Statement 4: \mathbf{A} is non singular | False |
| | <p>Non Singular Matrix: A non-singular matrix is a square one whose determinant is not zero. non-singular matrix is also a full rank matrix.</p> <p>Let, $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$</p> <p>Here, $\det \mathbf{A} = 0$ and $\det(\mathbf{A} + \mathbf{I}) = 1$</p> <p>Thus, $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> <p>In this case, \mathbf{A} is Singular, But satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> |
| Conclusion | <p>Thus, we can conclude Statement 3 is true for all possible matrices which satisfy the given condition i.e $1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})$</p> |

TABLE 3.2.1: Solution Summary

3.3. The system of equations

$$x + 2x^2 + 3xy = 6 \quad (3.3.1)$$

$$x + x^2 + 3xy + y = 5 \quad (3.3.2)$$

$$x - x^2 + y = 7 \quad (3.3.3)$$

- a) has solutions in rational numbers.
- b) has solutions in real numbers.
- c) has solutions in complex numbers.
- d) has no solutions.

3.4. The trace of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \quad (3.4.1)$$

is

- a) 7^{20} .
- b) $2^{20} + 3^{20}$.
- c) $2^{21} + 3^{20}$.
- d) $2^{20} + 3^{20} + 1$.

Solution: Let,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (3.4.2)$$

To find the eigen values of \mathbf{A} :

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (3.4.3)$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \quad (3.4.4)$$

$$\Rightarrow (2-\lambda)(2-\lambda)(3-\lambda) = 0 \quad (3.4.5)$$

$$\Rightarrow \lambda = 2, 2, 3 \quad (3.4.6)$$

Eigen values of \mathbf{A} are 2,2,3.

Hence, the eigen values of \mathbf{A}^{20} are: $2^{20}, 2^{20}$ and 3^{20} respectively.

As we know that the sum of eigen values of a matrix equals the trace of the matrix, hence, the trace of \mathbf{A}^{20} is:

$$tr = 2^{20} + 2^{20} + 3^{20} \quad (3.4.7)$$

$$= 2 \cdot 2^{20} + 3^{20} \quad (3.4.8)$$

Therefore, option 3 is the required answer.

3.5. Given that there are real constants a, b, c, d such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2, \quad \forall x, y \in \mathbb{R} \quad (3.5.1)$$

This implies that

- a) $\lambda = -5$
- b) $\lambda \geq 1$
- c) $0 < \lambda < 1$
- d) There is no such $\lambda \in \mathbb{R}$

Solution: Given that

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (3.5.2)$$

Arranging this in form of a matrix,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.5.3)$$

From this, we get

$$\lambda = a^2 + c^2 \quad (3.5.4)$$

$$ab + cd = 1 \quad (3.5.5)$$

$$b^2 + d^2 = 1 \quad (3.5.6)$$

Let

$$\mathbf{u} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (3.5.7)$$

$$\mathbf{v} = \begin{pmatrix} b \\ d \end{pmatrix} \quad (3.5.8)$$

$$\|\mathbf{u}\|^2 = a^2 + c^2 = \lambda \quad (3.5.9)$$

$$\|\mathbf{v}\|^2 = b^2 + d^2 = 1 \quad (3.5.10)$$

Then,

$$\mathbf{u}^T \mathbf{v} = \begin{pmatrix} a & c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = ab + cd = 1 \quad (3.5.11)$$

Using the Cauchy-Schwartz Inequality, we get

$$|\mathbf{u}^T \mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \quad (3.5.12)$$

Now, substituting values from (3.5.9), (3.5.10), (3.5.11) above,

$$\Rightarrow 1 \leq \lambda \quad (3.5.13)$$

So from the given options, option 2) $\lambda \geq 1$ is correct.

3.6. Let $\mathbf{R}^n, n \geq 2$ be equipped with standard inner product. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n column vectors forming an orthonormal basis of \mathbf{R}^n . Let \mathbf{A} be a $n \times n$ matrix formed by the column vectors, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then,

- a) $\mathbf{A} = \mathbf{A}^{-1}$
- b) $\mathbf{A} = \mathbf{A}^T$

c) $\mathbf{A}^{-1} = \mathbf{A}^T$

d) $\det(\mathbf{A}) = 1$

Solution: Given, $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are orthonormal and form basis.

So, when they form column vectors of matrix \mathbf{A} , we can say that \mathbf{A} is also orthonormal.

$$\therefore \mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (3.6.1)$$

$$\implies \mathbf{A}^T \mathbf{A} \mathbf{A}^{-1} = \mathbf{I} \mathbf{A}^{-1} \quad (3.6.2)$$

$$\implies \mathbf{A}^T = \mathbf{A}^{-1} \quad (3.6.3)$$

Clearly, option 3 is the correct answer. Let us consider an orthonormal basis for \mathbf{R}^2 .

We can check that $\mathbf{S} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \right\}$ forms an orthonormal basis.

Thus the matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.6.4)$$

is the orthonormal matrix whose column vectors are the basis of \mathbf{R}^2 . For an orthonormal matrix \mathbf{A} ,

$$\mathbf{A}^T \mathbf{A} = \mathbf{I} \quad (3.6.5)$$

$$\implies \det(\mathbf{A}^T \mathbf{A}) = \det(\mathbf{I}) \quad (3.6.6)$$

$$\implies \det(\mathbf{A}^T) \det(\mathbf{A}) = 1 \quad (3.6.7)$$

$$\implies \det(\mathbf{A})^2 = 1 \quad \because \det(\mathbf{A}) = \det(\mathbf{A}^T) \quad (3.6.8)$$

$$\implies \det(\mathbf{A}) = \pm 1 \quad (3.6.9)$$

Also, here a contradictory example:

Let,

$$\mathbf{R} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \quad (3.6.10)$$

Clearly, \mathbf{R} is an orthonormal matrix and the column vectors of it form an orthonormal basis of \mathbf{R}^2 . But,

$$\det \mathbf{R} = \begin{vmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{vmatrix} \quad (3.6.11)$$

$$= -1 \quad (3.6.12)$$

From the above two arguments it is clear that option 4 cannot be true.

3.7. Let $\mathbb{R}, n \geq 2$, be equipped with the standard inner product. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n column

vectors forming an orthonormal basis of \mathbb{R}^n . Let \mathbf{A} be the $n \times n$ matrix formed by the column vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. Then

a) $\mathbf{A} = \mathbf{A}^{-1}$

b) $\mathbf{A} = \mathbf{A}^T$

c) $\mathbf{A}^{-1} = \mathbf{A}^T$

d) $\det(\mathbf{A}) = 1$

3.8. Consider a Markov Chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{matrix} \quad (3.8.1)$$

Then,

a) $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$

b) $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

c) $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$

d) $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

3.9. Let V denote the vector space of all sequences $\mathbf{a} = (a_1, a_2, \dots)$ of real numbers such that

$$\sum_n 2^n |a_n| \quad (3.9.1)$$

converges. Define

$$\|\cdot\| : V \rightarrow \mathbb{R} \quad (3.9.2)$$

by

$$\|\mathbf{a}\| = \sum_n 2^n |a_n|. \quad (3.9.3)$$

Which of the following are true?

a) V contains only the sequence $(0, 0, \dots)$

b) V is finite dimensional

c) V has a countable linear basis

d) V is a complete normed space

3.10. Let V be a vector space over \mathbb{C} with dimension n . Let $T : V \rightarrow V$ be a linear transformation with only 1 as eigenvalue. Then which of the following must be true?

a) $T - I = 0$

b) $(T - I)^{n-1} = 0$

c) $(T - I)^n = 0$

d) $(T - I)^{2n} = 0$

3.11. If \mathbf{A} is a 5×5 matrix and the dimension of the solution space of $\mathbf{A}\mathbf{x} = 0$ is at least two, then

a) $\text{rank}(\mathbf{A}^2) \leq 3$

- b) $\text{rank}(\mathbf{A}^2) \geq 3$
- c) $\text{rank}(\mathbf{A}^2) = 3$
- d) $\det(\mathbf{A}^2) = 0$

3.12. Let $\mathbf{A} \in M_3(\mathbb{R})$ be such that $\mathbf{A}^3 = \mathbf{I}_{3 \times 3}$. Then

- a) minimal polynomial of \mathbf{A} can only be of degree 2
- b) minimal polynomial of \mathbf{A} can only be of degree 3
- c) either $\mathbf{A} = \mathbf{I}$ or $\mathbf{A} = -\mathbf{I}$
- d) there can be uncountably many \mathbf{A} satisfying the above.

3.13. Let \mathbf{A} be an $n \times n, n > 1$ matrix satisfying

$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I} = \mathbf{0} \quad (3.13.1)$$

Then which of the following statements is true?

- a) \mathbf{A} is invertible
- b) $t^2 - 7t + 12n = 0$ where $t = \text{tr}(\mathbf{A})$
- c) $d^2 - 7d + 12 = 0$ where $d = \det(\mathbf{A})$
- d) $\lambda^2 - 7\lambda + 12 = 0$ where λ is an eigenvalue of \mathbf{A}

Solution: See Table 3.13.1

| | |
|--------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>\mathbf{A} be the $n \times n$ matrix where $n > 1$ satisfying the following equation</p> $\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I}_{n \times n} = \mathbf{0}_{n \times n} \quad (3.13.2)$ |
| Explanation | <p>The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation. Using this theorem the given equation (3.13.2) can be written as ,</p> $\lambda^2 - 7\lambda + 12 = 0 \quad (3.13.3)$ $(\lambda - 4)(\lambda - 3) = 0 \quad (3.13.4)$ $\lambda_1 = 3 \quad (3.13.5)$ $\lambda_2 = 4 \quad (3.13.6)$ <p>Here λ_1 and λ_2 were eigen values of matrix \mathbf{A} We know that determinant is product of eigen values.</p> $d = \text{Det}(\mathbf{A}) \quad (3.13.7)$ $\implies d = \lambda_1 \lambda_2 \quad (3.13.8)$ $\implies d = 12 \neq 0 \quad (3.13.9)$ |
| Statement 1 | \mathbf{A} is invertible |
| | <p>From equation (3.13.9), since $d \neq 0$ the given matrix \mathbf{A} is Invertible. True Statement</p> |
| Statement 2 | $t^2 - 7t + 12n = 0 \quad (3.13.10)$ |
| | <p>We know that the trace is the sum of the eigen values.</p> $t = \text{Tr}(\mathbf{A}) \quad (3.13.11)$ $\implies t = \lambda_1 + \lambda_2 \quad (3.13.12)$ $\implies t = 7 \quad (3.13.13)$ <p>Substituting the equation (3.13.13) in (3.13.10) we get,</p> $7^2 - 7(7) + 12n = 0 \quad (3.13.14)$ $12n = 0 \quad (3.13.15)$ <p>Since given that $n > 1$ the equation (3.13.15) is not possible i.e $12n \neq 0$. Therefore, $t^2 - 7t + 12n = 0$ is a False Statement</p> |
| Statement 3 | $d^2 - 7d + 12 = 0 \quad (3.13.16)$ |
| | <p>Substituting the equation (3.13.9) in (3.13.16), we get,</p> $12^2 - 7(12) + 12 = 0 \quad (3.13.17)$ $72 = 0 \quad (3.13.18)$ <p>From equation (3.13.15) it is clear that the above statement 3 is invalid. False Statement</p> |

| | |
|-------------|-------------------------------------------------------------------------------------------------------------------|
| Statement 4 | $\lambda^2 - 7\lambda + 12 = 0$ (3.13.19) |
| | By Cayley Hamilton Theorem, equation (3.13.3) shows that the above statement 4 is valid. True Statement |

TABLE 3.13.1: Explanation

3.14. Let \mathbf{A} be a 6×6 matrix over \mathbb{R} with characteristic polynomial

$$(x-3)^2(x-2)^4 \quad (3.14.1)$$

and minimal polynomial

$$(x-3)(x-2)^2 \quad (3.14.2)$$

Then the Jordan canonical form of \mathbf{A} can be

a)
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

b)
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

c)
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

d)
$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution: See Tables 3.14.1 and 3.14.1

| | |
|------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Jordan canonical form | <p>If \mathbf{A} is a matrix of order $n \times n$, then the Jordan canonical form of \mathbf{A} is a matrix of order $n \times n$ expressed as</p> $\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_k \end{pmatrix} \quad (3.14.3)$ <p>where $\mathbf{J}_1, \dots, \mathbf{J}_k$ are the Jordan blocks.</p> |
| Algebraic multiplicity A_M | <p>Algebraic multiplicity of characteristic value λ in the characteristic polynomial determines the size of Jordan block for that eigen value</p> $A_M = \text{Size of Jordan block for that } \lambda \quad (3.14.4)$ |
| Geometric multiplicity G_M | <p>Geometric multiplicity determines the number of Jordan sub-blocks in a Jordan block for λ</p> |
| Minimal Polynomial | <p>The multiplicity of λ in the minimal polynomial determines the size of the largest sub-block.</p> |

TABLE 3.14.1: Definition and Properties used

| | |
|-----------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Characteristic polynomial | $p(x) = (x - 3)^2 (x - 2)^4 \quad (3.14.5)$ |
| Algebraic Multiplicity | <p>For $\lambda = 3, A_M = 2$ (3.14.6) For $\lambda = 2, A_M = 4$ (3.14.7)</p> |
| Minimal polynomial | $m(x) = (x - 3)(x - 2)^2 \quad (3.14.8)$ |
| Finding Jordan blocks for $\lambda_1=3$ | <p>For $\lambda_1=3$, We can write from table 3.14.1 that</p> <p style="text-align: center;">The highest order of Jordan block = 1 Size of Jordan block = $A_M = 2$</p> <p>The Jordan blocks for $\lambda_1=3$</p> |

| | |
|-----------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $\mathbf{J}_1 = (3), \mathbf{J}_2 = (3) \quad (3.14.9)$ |
| Finding Jordan blocks for $\lambda_1=2$ | <p>For $\lambda_1=2$, We can write from table 3.14.1 that</p> <p style="text-align: center;">The highest order of Jordan block = 2 Size of Jordan block = $A_M = 4$</p> <p>The Jordan blocks for $\lambda_1=3$</p> $\mathbf{J}_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \mathbf{J}_4 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad (3.14.10)$ <p style="text-align: center;">or</p> $\mathbf{J}_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \mathbf{J}_4 = (2), \mathbf{J}_5 = (2) \quad (3.14.11)$ |
| Jordan canonical form | <p>Jordan canonical form of \mathbf{A} is</p> $\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \mathbf{J}_3 & \\ & & & \mathbf{J}_4 \end{pmatrix} \text{ or } \begin{pmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \mathbf{J}_3 & \\ & & & \mathbf{J}_4 & \\ & & & & \mathbf{J}_5 \end{pmatrix} \quad (3.14.12)$ $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (3.14.13)$ |
| Conclusion | From above, we can say that options 2) and 3) are correct. |

TABLE 3.14.2: Finding Jordan canonical form

3.15. Let V be an inner product space and S be a subset of V . Let \bar{S} denote the closure of S in V with respect to the topology induced by the metric given by the inner product. Which of the following statements is true?

- a) $S = (S^\perp)^\perp$
- b) $\bar{S} = (S^\perp)^\perp$
- c) $\text{span}(S) = (S^\perp)^\perp$
- d) $S^\perp = ((S^\perp)^\perp)^\perp$

3.16. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.16.1)$$

and

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \quad (3.16.2)$$

Which of the following statements is true?

- a) The matrix of second order partial derivatives of the quadratic form Q is $2\mathbf{A}$
- b) The rank of the quadratic form Q is 2
- c) The signature of the quadratic form Q is $++0$
- d) The quadratic form Q take the value 0 for some non-zero vector \mathbf{x}

Solution: See Tables 3.16.1 and 3.16.2

| | |
|-----------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Quadratic Form of a matrix | Let \mathbf{V} be a vector space over \mathbb{R} . \mathbf{A} be a symmetric matrix $n \times n$. Quadratic form on \mathbf{V} is a real function, $(\mathbf{F} : \mathbf{V} \rightarrow \mathbb{R})$ defined as $F(x) = \mathbf{xAx}^T = \sum_{i,j=1}^n a_{ij}x_i x_j$ where $\mathbf{x} \in \mathbf{V}$ |
| Signature of Quadratic form | The signature of quadratic form is (n_+, n_-, n_0) where n_+ is the number of positive entries, n_- is number of negative entries and n_0 is number of zero's in a_{ii} |
| Rank of quadratic form | Rank of quadratic form is the rank of its matrix which is maximum number of linearly independent rows/columns of a matrix |

TABLE 3.16.1: Definitions

| | |
|-----------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option 1 | The matrix of second order partial derivatives of the quadratic form of \mathbf{Q} is $2\mathbf{A}$. |
| Solution | $\mathbf{Q}(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x+2y \\ -2z \\ z \end{pmatrix} = x^2 + z^2 + 2xy - 2yz$ <p>First order partial derivatives: $\frac{\partial \mathbf{Q}}{\partial x} = 2x + 2y$ $\frac{\partial \mathbf{Q}}{\partial y} = 2x - 2z$ $\frac{\partial \mathbf{Q}}{\partial z} = 2z - 2y$</p> <p>Second order partial derivatives of: $\frac{\partial^2 \mathbf{Q}}{\partial x^2} = 2$ $\frac{\partial^2 \mathbf{Q}}{\partial y^2} = 0$ $\frac{\partial^2 \mathbf{Q}}{\partial z^2} = 2$</p> <p>$\frac{\partial^2 \mathbf{Q}}{\partial x \partial y} = \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} = 2$ $\frac{\partial^2 \mathbf{Q}}{\partial x \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} = 0$ $\frac{\partial^2 \mathbf{Q}}{\partial y \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} = -2$</p> <p>Matrix of second order partial derivatives \mathbf{Q}: $\begin{pmatrix} \frac{\partial^2 \mathbf{Q}}{\partial x^2} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial y^2} & \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} & \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} & \frac{\partial^2 \mathbf{Q}}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \neq 2\mathbf{A}$</p> <p>Hence, Option 1 is not correct.</p> |
| Option 2 | The rank of the quadratic form of \mathbf{Q} is 2 |
| Solution | <p>From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> <p>Echelon form reduction: $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> <p>$\xrightarrow{R_2 \rightarrow -\frac{1}{2}R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{4}R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$</p> <p>$\xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$</p> <p>Rank = Number of non-zero rows = 3 \neq 2</p> <p>Hence, Option 2 is not correct.</p> |
| Option 3 | The signature of the quadratic form \mathbf{Q} is $(+ + 0)$ |
| Solution | <p>From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> |

| | |
|-----------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>Finding eigen values: $\mathbf{Q} - \lambda \mathbf{I} = \begin{vmatrix} 2 - \lambda & 2 & 0 \\ 2 & -\lambda & -2 \\ 0 & -2 & 2 - \lambda \end{vmatrix}$</p> <p>$\Rightarrow (2 - \lambda)(-2\lambda + \lambda^2 + 4) + 8 = 0$</p> <p>$\Rightarrow \lambda^3 - 4\lambda^2 - 4\lambda + 16 = 0$</p> <p>$\lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = -2$</p> <p>Signature = $(n_+, n_-, n_0) = (2, 1, 0) \neq (+ + 0)$</p> <p>Hence, Option 3 is not correct.</p> |
| Option 4 | The quadratic form \mathbf{Q} takes the value 0 for some non-zero vector (x, y, z) |
| Solution | <p>From above we have quadratic form of $\mathbf{Q} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{pmatrix}$</p> <p>we can see that few elements are zero even though the vectors are non-zero.</p> <p>Therefore, Option 4 is correct.</p> |

TABLE 3.16.2: Solution

3.17. Assume that a non-singular matrix

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \quad (3.17.1)$$

where \mathbf{L} and \mathbf{U} are lower and upper triangular matrices respectively with all diagonal entries are zero, and \mathbf{D} is a diagonal matrix. Let \mathbf{x}^* be the solution of $\mathbf{Ax} = \mathbf{b}$. Then the Gauss-Seidel iteration method

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k + \mathbf{c}, k = 0, 1, 2, \dots \quad (3.17.2)$$

with $\|\mathbf{H}\| < 1$ converges to \mathbf{x}^* provided \mathbf{H} is equal to

- a) $-\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$
- b) $-(\mathbf{D} + \mathbf{L})^{-1} \mathbf{U}$
- c) $-\mathbf{D}(\mathbf{L} + \mathbf{U})^{-1}$
- d) $-(\mathbf{L} - \mathbf{D})^{-1} \mathbf{U}$

3.18. Consider a Markov Chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{matrix} \quad (3.18.1)$$

Let π be a stationary distribution of the Markov chain and $d(1)$ denote the period of state 1. Which of the following statements are correct?

- a) $d(1) = 1$
- b) $d(1) = 2$
- c) $\pi_1 = \frac{1}{2}$
- d) $\pi_1 = \frac{1}{3}$

Solution:

a) The period of state 1 i.e., $d(1)$ is given as:

$$d(1) = \text{GCD}\{n : P_{11}^n > 0\} \quad (3.18.2)$$

For $n = 1$,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (3.18.3)$$

$$(3.18.4)$$

For $n = 2$,

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad (3.18.5)$$

$$(3.18.6)$$

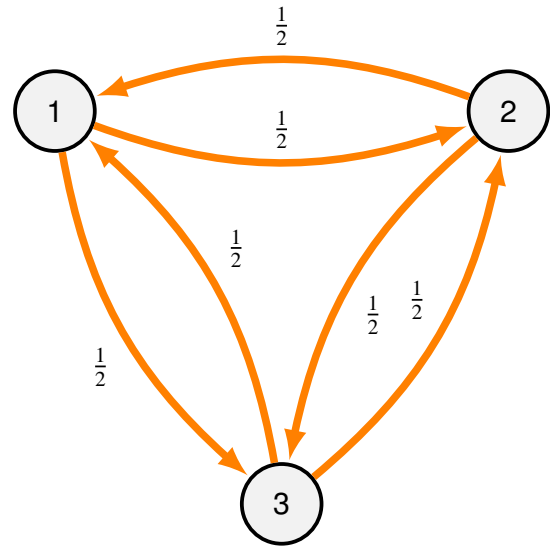


Fig. 3.18.1: State transition diagram

For $n = 3$,

$$\mathbf{P}^3 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \quad (3.18.7)$$

$$(3.18.8)$$

For $n = 4$,

$$\mathbf{P}^4 = \begin{pmatrix} \frac{3}{16} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{16} \end{pmatrix} \quad (3.18.9)$$

Thus P_{11}^n follows the sequence, that is defined as:

$$P_{11}^n = \begin{cases} 0, & \text{if } n = 1 \\ \frac{1}{2}, & \text{if } n = 2 \\ \frac{1}{2}(P_{11}^{n-1} + P_{11}^{n-2}), & \text{if } n > 2 \end{cases} \quad (3.18.10)$$

Since, for $n > 1$, $P_{11}^n > 0$

$$d(1) = \text{GCD}\{2, 3, 4, 5 \dots\} \quad (3.18.11)$$

$$\therefore d(1) = 1 \quad (3.18.12)$$

Thus statement a is correct

- b) As calculated above in 3.18.12, $d(1) = 1$
Thus statement b is incorrect.

c) For stationary distribution,

$$\sum_{i=1}^{i=n} \pi_i = 1 \quad (3.18.13)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = 1 \quad (3.18.14)$$

Also for a stationary distribution,

$$\pi \mathbf{P} = \pi \quad (3.18.15)$$

$$(\pi \mathbf{P})^T = \pi^T \quad (3.18.16)$$

$$\mathbf{P}^T \pi^T = \pi^T \quad (3.18.17)$$

$$\Rightarrow (\mathbf{P}^T - \mathbf{I}) \pi^T = 0 \quad (3.18.18)$$

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \quad (3.18.19)$$

The given equation 3.18.14, 3.18.19 can be written as:

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.18.20)$$

We need to solve the augmented matrix to row

reduced echelon form to get the solution,

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right) \xleftrightarrow{R_4=R_4+R_1} \quad (3.18.21)$$

$$\left(\begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_1=-R_1} \quad (3.18.22)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_2=R_2-\frac{R_1}{2}, R_3=R_3-\frac{R_1}{2}} \quad (3.18.23)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_3=R_3+R_2, R_4=R_4+2R_2} \quad (3.18.24)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_2=-\frac{4}{3}R_2} \quad (3.18.25)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_1=R_1+\frac{1}{2}R_2} \quad (3.18.26)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_3 \leftrightarrow R_4} \quad (3.18.27)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xleftrightarrow{R_3=\frac{R_3}{3}} \quad (3.18.28)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \xleftrightarrow{R_1=R_1+R_3, R_2=R_2+R_3} \quad (3.18.29)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (3.18.30)$$

Hence,

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3} \quad (3.18.31)$$

Thus statement c is incorrect

d) As, calculated in 3.18.31, $\pi_1 = \frac{1}{3}$

Thus statement d is correct

Hence, statements a and d are correct.

4 DECEMBER 2017

4.1. Let \mathbf{A} be a real symmetric matrix and $\mathbf{B} = \mathbf{I} + i\mathbf{A}$, where $i^2 = -1$. Then choose the correct option.

- a) \mathbf{B} is invertible if and only if \mathbf{A} is invertible.
- b) All Eigenvalues of \mathbf{B} are necessarily real.
- c) $\mathbf{B} - \mathbf{I}$ is necessarily invertible.
- d) \mathbf{B} is necessarily invertible.

Solution: See Table 4.1.1.

| | |
|--------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Statement 1. | B is invertible if and only if A is invertible. |
| False statement | Matrix B is invertible even if A is non invertible. |
| Example: | <p>Consider a matrix</p> $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.1.1)$ <p>a real non invertible,symmetric matrix.</p> $\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix} \quad (4.1.2)$ <p>is invertible even if A is non invertible.</p> |
| Statement 2. | All Eigenvalues of B are necessarily real. |
| False statement | Matrix B can have complex Eigenvalues. |
| Proof : | <p>Eigen values of B = Eigen values of (I) + i (Eigen values of A).</p> <p>Clearly from (4.1.2) above Eigen values of B are 1 and $1 + i$ respectively.</p> <p>Hence B can also have complex Eigen value.</p> |
| Statement 3. | B – I is necessarily invertible. |
| False statement | B – I = $i\mathbf{A}$ will be invertible if A , is invertible. |
| Proof: | <p>We have B – I = $i\mathbf{A}$</p> $\Rightarrow \mathbf{B} - \mathbf{I} = i\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \text{from (4.1.1)}$ <p>Hence B – I is not invertible,unless A is invertible.</p> |
| Statement 4. | B is necessarily invertible. |
| Correct Statement: | Matrix B has non zero Eigen values corresponding to Eigenvector X . |
| Proof: | <p>Let X be an Eigen vector of A corresponding to Eigen value λ</p> <p>also, $\lambda \in \mathbb{R}$</p> $\Rightarrow \mathbf{A}X = \lambda X$ $\therefore \mathbf{B}X = (\mathbf{I} + i\mathbf{A})X = \mathbf{I}X + i\mathbf{A}X = X + i\lambda X$ $\Rightarrow \mathbf{B}X = (1 + i\lambda)X$ <p>Therefore, $1 + i\lambda$ is an Eigen value of B, corresponding to Eigen vector X,which are non zero.</p> <p>Hence, B is necessarily invertible.</p> |

TABLE 4.1.1: Solution summary

4.2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Then the smallest positive integer n such that $\mathbf{A}^n = \mathbf{I}$ is

Solution: *Property of eigen values of A:* Let \mathbf{A} be an arbitrary $n \times n$ matrix of complex numbers with eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the eigen values of k^{th} power of \mathbf{A} , that is the eigen values of \mathbf{A}^k , for any positive integer k are $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$. Let us calculate the eigen values of \mathbf{A} .

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad (4.2.1)$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad (4.2.2)$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} = 0 \quad (4.2.3)$$

$$-\lambda(1 - \lambda) + 1 = 0 \quad (4.2.4)$$

$$\lambda^2 - \lambda + 1 = 0 \quad (4.2.5)$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{3}i}{2} \quad (4.2.6)$$

From the above property, the eigen values of \mathbf{A}^n are λ^n . Also as it is given that $\mathbf{A}^n = \mathbf{I}$,

$$\Rightarrow \lambda^n = 1 \quad (4.2.7)$$

$$\Rightarrow \left(\frac{-1 \pm \sqrt{3}i}{2} \right)^n = 1 \quad (4.2.8)$$

Clearly $n \neq 1$. For $n = 2$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2} \right)^2 = \frac{-1 \mp \sqrt{3}i}{2} \quad (4.2.9)$$

For $n = 4$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2} \right)^4 = \frac{-1 \pm \sqrt{3}i}{2} \quad (4.2.10)$$

For $n = 6$,

$$\left(\frac{-1 \pm \sqrt{3}i}{2} \right)^6 = 1 \quad (4.2.11)$$

Hence $n = 6$ is the smallest positive integer.

4.3. Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$. Then the system $\mathbf{AX} = \mathbf{b}$ over the real numbers has

- No solution when $\beta \neq 7$
- Infinite number of solutions when $\alpha \neq 2$
- Infinite number of solutions when $\alpha = 2$ and $\beta \neq$

7

d) A unique solution if $\alpha \neq 2$

Solution: First we derive the Row Reduced Echelon Form (RREF) of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ as follows,

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{pmatrix} \xrightarrow[R_3=R_3-2R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \quad (4.3.1)$$

$$\xrightarrow{R_2=\frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \quad (4.3.2)$$

$$\xrightarrow{R_1=R_1+R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \quad (4.3.3)$$

$$\xrightarrow{R_3=R_3-5R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha-2 & \beta-7 \end{pmatrix} \quad (4.3.4)$$

From the RREF of the augmented matrix of the system $\mathbf{AX} = \mathbf{b}$ in (4.3.4) we make the following observations for different values of α and β in Table 4.3.1. ,

| Values | Observations |
|--------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\beta \neq 7$ | Then the existence of solution and the number of solutions will entirely depend on value of α |
| $\alpha = 2$ $\beta \neq 7$ | Then RREF in (4.3.4) will contain Zero Row in R_3 . Moreover solvability condition will not satisfy. \Rightarrow system will have Zero solutions |
| $\alpha \neq 2$ | RREF in (4.3.4) will have all pivots \Rightarrow RREF in (4.3.4) will be fullrank $\Rightarrow \mathbf{AX} = \mathbf{b}$ have unique solution. |

TABLE 4.3.1

Hence, if $\alpha \neq 2$ then the system $\mathbf{AX} = \mathbf{b}$ has unique solution.

4.4. Consider a Markov chain $\{X_n | n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then, $P(X_3 = 1 | X_0 = 1)$ equals

Solution: The three step transitional probabilities are given as,

$$P(X_3 = j | X_0 = i) = P(X_{n+3} = j | X_n = i) = (\mathbf{P}^3)_{ij} \text{ for any } n \quad (4.4.1)$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \quad (4.4.2)$$

From (4.4.2),

$$P(X_3 = 1 | X_0 = 1) = (\mathbf{P}^3)_{11} = \frac{1}{4} \quad (4.4.3)$$

4.5. Let \mathbf{A} be an $m \times n$ matrix with rank r . If the linear system $\mathbf{A}\mathbf{X} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbf{R}^m$, then

- $m = r$
- the column space of \mathbf{A} is a proper subspace of \mathbf{R}^m
- the null space of \mathbf{A} is a non-trivial subspace of \mathbf{R}^n whenever $m = n$
- $m \geq n$ implies $m = n$

Solution: Theorem

Theorem 4.1. Consider the $m \times n$ system $Ax = b$, with either $b \neq 0$ or $b = 0$. We distinguish the following cases:

- Unique Solution:** If $\text{rank}[A, b] = \text{rank}(A) = n \leq m$, then and only then the system has a unique solution. In this case, indeed as many as $m - n$ equations are redundant. And the solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$. This is called as **Exactly Determined**.
- No Solution:** If $\text{rank}[A, b] > \text{rank}(A)$ which necessarily implies $\mathbf{b} \neq 0$ and $m > \text{rank}(A)$, then and only then the system has no solution. This is called as **Overdetermined**.

See Table 4.5.1 If the columns of an $m \times n$ matrix \mathbf{A} span \mathbf{R}^m then the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent for each \mathbf{b} in \mathbf{R}^m .

The **null space** of \mathbf{A} is defined to be

$$\text{Null}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = 0\} \quad (4.5.1)$$

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \quad (4.5.2)$$

Reduced Row Echelon form is

$$\text{RREF}(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.5.3)$$

\therefore the only possible nullspace of the matrix \mathbf{A} is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let \mathbf{B} be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \quad (4.5.4)$$

Reduced Row Echelon form is

$$\text{RREF}(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.5.5)$$

\therefore the rank of matrix $\mathbf{B} = 3$.

4.6. Let $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and eigen values of } \mathbf{A} \in \mathbb{Q} \right\}$

- \mathbf{M} is empty
- $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$
- If $\mathbf{A} \in \mathbf{M}$ then the eigen values of $\mathbf{A} \in \mathbb{Z}$
- If $\mathbf{A}, \mathbf{B} \in \mathbf{M}$ such that $\mathbf{AB} = \mathbf{I}$ then $|\mathbf{A}| \in \{+1, -1\}$

Solution: See Table 4.6.1.

| Options | Observations |
|---------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $m = r$ | <p>The rank of any matrix \mathbf{A} is the dimension of its column space. When the number of rows (m) is equal to the rank (r) of the matrix, then their linear combination gives us span of \mathbf{R}^m.</p> <p>\therefore This statement is True.</p> |
| the column space of \mathbf{A} is a proper subspace of \mathbf{R}^m | <p>Any subspace of a vector space \mathbf{V} other than \mathbf{V} itself is considered a proper subspace of \mathbf{V}. Which means that linear combination of \mathbf{A} will span less than m. That will make the resultant \mathbf{b} span strictly less than m. But it is given that $\mathbf{b} \in \mathbf{R}^m$, which is contradicting.</p> <p>\therefore This statement is False.</p> |
| the null space of \mathbf{A} is a non-trivial subspace of \mathbf{R}^n whenever $m = n$ | <p>From (4.5.2) we see that even when $m = n$ then also we are getting a trivial nullspace.</p> <p>\therefore This statement is False.</p> |
| $m \geq n$ implies $m = n$ | <p>It is given that the number of rows are greater than the column, and it is given that there exists a solution. If we refer to theorem (4.1) we see that the corresponding system will be Exactly Determined system.</p> <p>As an example, it will look like (4.5.4).</p> <p>\therefore This statement is True.</p> |

TABLE 4.5.1: Solution

| | |
|----------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| \mathbf{M} is empty | Consider $\mathbf{A}=\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The elements of $\mathbf{A} \in \mathbb{Z}$ and its eigen values $1 \in \mathbb{Q}$. So, \mathbf{M} is not empty. |
| $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ | <p>Let $\mathbf{A}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where elements of $\mathbf{A} \in \mathbb{Z}$. The characteristic equation can be written as :</p> $\lambda^2 + 1 = 0 \implies \lambda = \pm i$ |

| | |
|----------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | We see that $\lambda \in \mathbb{C}$ which is contradicting the main definition of \mathbf{M} . So, this is not correct. |
| Eigen values of $\mathbf{A} \in \mathbb{Z}$ | <p>Given $\mathbf{A} \in \mathbf{M}$. Let λ_1, λ_2 be the eigen values of \mathbf{A}. The characteristic polynomial can be written as:</p> $\lambda^2 - \text{tr}(\mathbf{A})\lambda + \det \mathbf{A} = 0 \text{ where } \text{tr}(\mathbf{A}) = \lambda_1 + \lambda_2, \det \mathbf{A} = \lambda_1 \lambda_2$ <p>Given the eigen values $\lambda_1, \lambda_2 \in \mathbb{Q}$, For this to be possible the discriminant of above equation should $\in \mathbb{Z}$</p> $\sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2} \in \mathbb{Z}$ $\Rightarrow \sqrt{(\lambda_1 - \lambda_2)^2} \in \mathbb{Z}$ $\Rightarrow \lambda_1 - \lambda_2 \in \mathbb{Z} \text{ This is possible when both } \lambda_1, \lambda_2 \in \mathbb{Z}.$ |
| If $\mathbf{AB}=\mathbf{I}$ then $ \mathbf{A} \in \{+1, -1\}$ | <p>As $\mathbf{A}, \mathbf{B} \in \mathbf{M} \Rightarrow \mathbf{A} , \mathbf{B} \in \mathbb{Z}$</p> <p>Given $\mathbf{AB}=\mathbf{I} \Rightarrow \mathbf{A} \mathbf{B} =1$</p> <p>This is possible only when $\mathbf{A} = \mathbf{B} = \pm 1$</p> |
| Conclusion | options 3) and 4) are correct. |

TABLE 4.6.1: Solution

4.7. Let \mathbf{A} be a 3×3 matrix with real entries. Identify the correct statements.

- a) \mathbf{A} is necessarily diagonalizable over \mathbf{R}
- b) If \mathbf{A} has distinct real eigen values then it is diagonalizable over \mathbf{R}
- c) If \mathbf{A} has distinct eigen values then it is diagonalizable over \mathbf{C}
- d) If all eigen values are non zero then it is diagonalizable over \mathbf{C}

Solution: See Table 4.7.1.

| | |
|-----------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Statement 1. | A is necessarily diagonalizable over \mathbf{R} |
| False statement Example: | <p>Matrix A is diagonalizable if and only if there is a basis of \mathbf{R}^3 consisting of eigenvectors of A. Consider a matrix</p> $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.7.1)$ <p>Eigen values are:</p> $\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \quad (4.7.2)$ <p>$\lambda_1 = 1$ has eigen vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\lambda_2 = 4$ has eigen vector $\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$ (4.7.3)</p> <p>We have found only two linearly independent eigenvectors for A, not diagonalisable</p> |
| Statement 2. | If A has distinct real eigen values than it is diagonalizable over \mathbf{R} |
| True statement | Distinct real eigenvalues implies linearly independent eigenvectors . and if a matrix has n linearly independent vectors than it is diagonalizable. |
| Proof 1: | <p>Distinct eigen values implies linearly independent vectors that spans entire space. Consider 2 eigen vectors \mathbf{v}, \mathbf{w} with eigen values λ, μ respectively. such that $\lambda \neq \mu$</p> $\alpha(\mathbf{v}) + \beta(\mathbf{w}) = 0 \quad (4.7.4)$ $\alpha A(\mathbf{v}) + \beta A(\mathbf{w}) = 0 \quad (4.7.5)$ $\alpha \lambda \mathbf{v} + \beta \mu \mathbf{w} = 0 \quad (4.7.6)$ <p>Multiplying (4.7.4) with $-\lambda$ and subtracting from (4.7.6) we have,</p> $\beta(\mu - \lambda)\mathbf{w} = 0 \quad (4.7.7)$ <p>eigen values are distinct $(\mu - \lambda) \neq 0$. From equation (4.7.7) we have, $\beta = 0$ substituting $\beta = 0$ in equation (4.7.4) we have, $\alpha = 0$. As, $\mathbf{v} \neq 0$ which proves that vectors are linearly independent.</p> <p>If a matrix has n linearly independent vectors than it is diagonalizable If $(\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n)$ are n independent eigen vectors then, $A\mathbf{p}_1 = \lambda\mathbf{p}_1, \dots, A\mathbf{p}_n = \lambda\mathbf{p}_n$</p> $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P = (\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n) \quad (4.7.8)$ <p>Now, $A\mathbf{p}_i = \lambda_i\mathbf{p}_i \implies AP = PD$</p> |
| Proof 2: | |

| | |
|------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | so, $P^{-1}AP = D$ is a diagonal matrix. |
| Statement 3. | If A has distinct real eigen values than it is diagonalizable over \mathbb{C} |
| True statement | If A is an $N \times N$ complex matrix with n distinct eigenvalues, then any set of n corresponding eigenvectors form a basis for \mathbb{C}^n |
| Proof: | It is sufficient to prove that the set of eigenvectors is linearly independent which is proved in statement 2. |
| Example: | $A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad (4.7.9)$ <p>Eigen values of A are:</p> $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6 \quad (4.7.10)$ |
| | <p>Eigen vectors are:</p> $x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad (4.7.11)$ <p>Matrix A is diagonalizable because there is a basis of \mathbb{C}^3 consisting of eigenvectors of A.</p> |
| Statement 4. | If all eigen values are non zero than it is diagonalizable over \mathbb{C} |
| False Statement: | Matrix would be diagonalizable if and only if it has linearly independent eigenvectors . |
| Example: | <p>Consider a matrix</p> $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.7.12)$ <p>Eigen values are:</p> $\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \neq 0 \quad (4.7.13)$ <p>$\lambda_1 = 1$ has eigen vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\lambda_2 = 4$ has eigen vector $\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}$ (4.7.14)</p> <p>We have found only two linearly independent eigenvectors for A, not diagonalisable.</p> |

TABLE 4.7.1: Solution summary

| | |
|-------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>V be a vector space over C of all the polynomials in a variable X of degree atmost 3</p> $D : P_3 \rightarrow P_3$ <p>$D : V \rightarrow V$ be the linear operator given by differentiation wrt X</p> $D(P(x)) \rightarrow P'(x)$ <p>A be the matrix of D wrt some basis for V</p> <p>Assume basis for V be $\{1, x, x^2, x^3\}$</p> |
|-------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

TABLE 4.8.1

4.8. Let V be a vector space over C of all the polynomials in a variable X of degree atmost 3. Let $D : V \rightarrow V$ be the linear operator given by differentiation with respect to X . Let A be the matrix of D with respect to some basis for V . Which of the following are true?

- a) A is nilpotent matrix
- b) A is diagonalizable matrix
- c) the rank of A is 2
- d) the Jordan canonical form of A is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution: See Tables 4.8.1, 4.8.2 and 4.8.3

4.9. For every 4×4 real symmetric non-singular matrix A there exists a positive integer p such that

- a) $pI + A$ is positive definite
- b) A^p is positive definite
- c) A^{-p} is positive definite
- d) $\exp(pA) - I$ is positive definite

Solution: A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal, that is its eigen value decomposition is

$$A = PDP^T \quad (4.9.1)$$

D is the diagonal matrix containing the real eigen values of A

P has the corresponding eigen vectors

$$PP^T = P^T P = I \quad (4.9.2)$$

A real matrix is positive definite if

$$\mathbf{x}^T A \mathbf{x} > 0 \quad (4.9.3)$$

$$\Rightarrow \mathbf{x}^T \lambda \mathbf{x} > 0 \quad (4.9.4)$$

$$\Rightarrow \lambda \mathbf{x}^T \mathbf{x} > 0 \quad (4.9.5)$$

$$\Rightarrow \lambda > 0 \quad (4.9.6)$$

In other words, all the eigen values of A are positive See Table 4.9.1

Let A be

$$A = PDP^T \quad (4.9.7)$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad (4.9.8)$$

From the table, the choices would be option 1,2,3

4.10. Let A be an $m \times n$ matrix of rank m with $n > m$. If for some non-zero real number α , we have $\mathbf{x}^T A A^T \mathbf{x} = \alpha \mathbf{x}^T \mathbf{x}$, for all $\mathbf{x} \in \mathbf{R}^m$, then $A^T A$ has,

- a) exactly two distinct eigenvalues.
- b) 0 as an eigenvalue with multiplicity $n - m$.
- c) α as a non-zero eigenvalue.
- d) exactly two non-zero distinct eigenvalues.

Solution: Refer Table 4.10.1.

Refer Table 4.10.2.

4.11. Consider a Markov chain with five states

$\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \quad (4.11.1)$$

Which of the following are true?

- a) 3 and 1 are in the same communicating class
- b) 1 and 4 are in the same communicating class
- c) 4 and 2 are in the same communicating class
- d) 2 and 5 are in the same communicating class

Solution: See Tables 4.11.1 and 4.11.2

| | |
|-----------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Matrix | $D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$ $D(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$ $D(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$ $D(x^2) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ $D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$ $D(x^3) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ $\text{Matrix } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ |
| Inference | <p>An $n \times n$ matrix with λ as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial $(A - \lambda I)^n$</p> |
| Nilpotent | $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p>All eigen values of matrix A is 0</p> <p>Thus, above matrix is nilpotent matrix</p> <p>Thus, above statement is true</p> |

TABLE 4.8.2

| | |
|----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Diagonalizable | $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p> $Rank(A) + nullity(A) = \text{no of column}$ $Rank(A) = 3, \text{ no of column} = 4$ $nullity(A) = 4 - 3 = 1$ means there exists only one linearly independent eigen vector corresponding to 0 eigen values Thus, matrix A is not Diagonalizable. Thus, above statement is false </p> |
| Rank | $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p> Rank of matrix A is 3 Thus, above statement is false </p> |
| Jordan CF | <p> Assume characteristic polynomial of matrix A is $c_A(x)$ $c_A(x) = x^4$ Assume minimal polynomial of A is $m_A(x)$ $m_A(x)$ always divide $c_A(x)$ $m_A(x) = \{x, x^2, x^3, x^4\}$ Minimal polynomial always annihilates its matrix. Thus, we see that $m_A(A) = \{A = 0, A^2 = 0, A^3 = 0, A^4 = 0\}$ But we see that neither A is zero matrix nor A^2 and A^3 equal to zero but A^4 is equal to zero. Thus, x^4 is minimal polynomial. Algebraic Multiplicity = $a_M(\lambda = 0) = 4$ Geometric Multiplicity = $g_M(\lambda = 0) = nullity(A - 0I) = nullity(A) = 1$ Hence, Jordan form of block size 4 Using Inference, $\mathbf{J} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ $\lambda = 0$ $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p> which is same as given in the question. Thus, statement is true </p> </p> |

| OPTIONS | DERIVATIONS |
|----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Choice 1 | $p\mathbf{I} + \mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (4.9.9)$ |
| | $= \mathbf{P}\mathbf{D}_1\mathbf{P}^T \quad (4.9.10)$ |
| | $\mathbf{D}_1 = \begin{pmatrix} \lambda_1 + p & 0 & 0 & 0 \\ 0 & \lambda_2 + p & 0 & 0 \\ 0 & 0 & \lambda_3 + p & 0 \\ 0 & 0 & 0 & \lambda_4 + p \end{pmatrix} \quad (4.9.11)$ |
| | <p>Some of the eigen values of \mathbf{A} may be negative. All the eigen values in \mathbf{D}_1 are positive only if</p> $p > \lambda_i \quad \forall i \in [1, 4] \quad (4.9.12)$ |
| Choice 2 | $\mathbf{A}^2 = \mathbf{A}\mathbf{A} \quad (4.9.13)$ |
| | $= (\mathbf{P}\mathbf{D}\mathbf{P}^T)(\mathbf{P}\mathbf{D}\mathbf{P}^T) \quad (4.9.14)$ |
| | $= \mathbf{P}\mathbf{D}^2\mathbf{P}^T \quad (4.9.15)$ |
| | <p>Similarly, $\mathbf{A}^p = \mathbf{P}\mathbf{D}^p\mathbf{P}^T \quad (4.9.16)$</p> |
| | $\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \quad (4.9.17)$ |
| | <p>\mathbf{A}^p is positive definite only if p is even.</p> |
| Choice 3 | $\mathbf{A}^{-p} = \mathbf{P}\mathbf{D}^{-p}\mathbf{P}^T \quad (4.9.18)$ |
| | $\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \quad (4.9.19)$ |
| | <p>\mathbf{A}^{-p} is positive definite only if p is even.</p> |
| Choice 4 | $\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \quad (4.9.20)$ |
| | $\Rightarrow \exp(p\mathbf{A}) - \mathbf{I} = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{P}\mathbf{I}\mathbf{P}^T \quad (4.9.21)$ |
| | $= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \quad (4.9.22)$ |
| | $\exp(p\mathbf{D}) - \mathbf{I} = \begin{pmatrix} e^{\lambda_1} - 1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} - 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} - 1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4} - 1 \end{pmatrix} \quad (4.9.23)$ |
| | <p>\mathbf{A} is non-singular</p> |
| | $\Rightarrow \forall i \in [1, 4], \lambda_i \neq 0 \quad (4.9.24)$ |
| | $e^{\lambda_i} < 1 \quad (4.9.25)$ |
| | <p>So, $\exp(p\mathbf{A}) - \mathbf{I}$ is not positive definite.</p> |

TABLE 4.9.1: Solution

| Given | Derivation |
|------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>\mathbf{A} is a $m \times n$ matrix of rank m with $n > m$. A non-zero real number α. To find eigenvalues of $\mathbf{A}^T \mathbf{A}$.</p> |
| Eigenvalues of $\mathbf{A} \mathbf{A}^T$ | <p>$\mathbf{A} \mathbf{A}^T$ is a $m \times m$ matrix and $\mathbf{A}^T \mathbf{A}$ is a $n \times n$ matrix. Let, λ be a non-zero eigen value of $\mathbf{A}^T \mathbf{A}$.</p> $\mathbf{A}^T \mathbf{A} \mathbf{v} = \lambda \mathbf{v} \quad \mathbf{v} \in \mathbf{R}^n \quad (4.10.1)$ $\mathbf{A} \mathbf{A}^T \mathbf{A} \mathbf{v} = \lambda \mathbf{A} \mathbf{v} \quad (4.10.2)$ <p>Let, $\mathbf{x} = \mathbf{A} \mathbf{v} \quad \mathbf{x} \in \mathbf{R}^m$ (4.10.3)</p> $\mathbf{A} \mathbf{A}^T \mathbf{x} = \lambda \mathbf{x} \quad (4.10.4)$ $\mathbf{x}^T \mathbf{A} \mathbf{A}^T \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} \quad (4.10.5)$ <p>Given, $\mathbf{x}^T \mathbf{A} \mathbf{A}^T \mathbf{x} = \alpha \mathbf{x}^T \mathbf{x}$ (4.10.6)</p> $\implies \alpha \mathbf{x}^T \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} \quad (4.10.7)$ <p>From equation (4.10.7), $\lambda = \alpha$ as $\ \mathbf{x}\ \neq 0$ As $\text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A}) = m$ and equation (4.10.7) satisfies the condition in question. Therefore the only non-zero eigen value is α $\mathbf{A}^T \mathbf{A}$ has an eigenvalue α with multiplicity m.</p> |
| Eigenvalues of $\mathbf{A}^T \mathbf{A}$ | <p>$\mathbf{A}^T \mathbf{A}$ is a $n \times n$ matrix. Given $n > m$,</p> <p>We know that, $\mathbf{A}^T \mathbf{A}$ and $\mathbf{A} \mathbf{A}^T$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero.</p> <ol style="list-style-type: none"> 1. From above, as α is non-zero, $\mathbf{A}^T \mathbf{A}$ has α as its eigenvalue with multiplicity m 2. $\mathbf{A}^T \mathbf{A}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A}^T \mathbf{A}$ are α and 0. |

TABLE 4.10.1: Explanation

| | |
|----------------------------------------------------------------------------|-----------------|
| $\mathbf{A}^T \mathbf{A}$ has exactly two distinct eigenvalues. | True statement |
| $\mathbf{A}^T \mathbf{A}$ has 0 as an eigenvalue with multiplicity $n - m$ | True statement |
| $\mathbf{A}^T \mathbf{A}$ has α as a non-zero eigenvalue | True statement |
| $\mathbf{A}^T \mathbf{A}$ has exactly two non-zero distinct eigenvalues. | False statement |

TABLE 4.10.2: Solution

| | |
|-------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Accessibility of states in Markov's chain | <p>We say that state j is accessible from state i, written as $i \rightarrow j$, if $p_{ij}^{(n)} > 0$ for some n. Every state is accessible from itself since $p_{ii}^{(0)} = 1$</p> |
| Communication between states | <p>Two states i and j are said to communicate, written as $i \leftrightarrow j$, if they are accessible from each other. In other words,</p> $i \leftrightarrow j \text{ means } i \rightarrow j \text{ and } j \rightarrow i.$ |
| Communicating class | <p>For each Markov chain, there exists a unique decomposition of the state space S into a sequence of disjoint subsets C_1, C_2, \dots,</p> $S = \bigcup_{i=1}^{\infty} C_i$ <p>in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.</p> |

TABLE 4.11.1: Definition and Result used

| | |
|-------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Drawing Transition diagram | |
| Checking whether the states 3 and 1 are in the same communicating class | <p>Here, State 1 is accessible from the state 3. But, State 3 is not accessible from the state 1 i.e. $3 \rightarrow 1, 1 \nrightarrow 3$ $\Rightarrow \boxed{3 \leftrightarrow 1}$</p> <p>Therefore, 3 and 1 are not in the same communicating class.</p> |
| Checking whether the states 1 and 4 are in the same communicating class | <p>Here, State 1 is accessible from the state 4. Also, State 4 is accessible from the state 1 i.e. $3 \rightarrow 1, 1 \rightarrow 3$ $\Rightarrow \boxed{3 \leftrightarrow 1}$</p> <p>Therefore, 1 and 4 are in the same communicating class.</p> |
| Checking whether the states 4 and 2 are in the same communicating class | <p>Here, State 2 is not accessible from the state 4. Also, State 4 is not accessible from the state 2 i.e. $4 \nrightarrow 2, 2 \nrightarrow 4$</p> |

| | |
|-------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $\Rightarrow \boxed{4 \leftrightarrow 2}$ <p>Therefore, 4 and 2 are not in the same communicating class.</p> |
| Checking whether the states 2 and 5 are in the same communicating class | <p>Here, State 2 is accessible from the state 5. Also, State 5 is accessible from the state 2 i.e. $5 \rightarrow 2, 2 \rightarrow 5$ $\Rightarrow \boxed{2 \leftrightarrow 5}$</p> <p>Therefore, 2 and 5 are in the same communicating class.</p> |
| Conclusion | <p>Communication classes are:</p> $\boxed{S = \{1, 4\} \cup \{3\} \cup \{2, 5\}}$ <p>Option 2) and 4) are true.</p> |

TABLE 4.11.2: Solution

5 JUNE 2017

5.1. Let \mathbf{A} be a 4×4 matrix. Suppose that the null space $N(\mathbf{A})$ of \mathbf{A} is

$$\{(x, y, z, w) \in \mathbf{R}^4 : x + y + z = 0, x + y + w = 0\} \quad (5.1.1)$$

Then which one of the following is correct

- a) $\dim(\text{column space}(\mathbf{A})) = 1$
- b) $\dim(\text{column space}(\mathbf{A})) = 2$
- c) $\text{rank}(\mathbf{A}) = 1$
- d) $\mathbf{S} = \{(1, 1, 1, 0), (1, 1, 0, 1)\}$ is a basis of $N(\mathbf{A})$

Solution: The nullspace is given by

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (5.1.2)$$

Row reducing the above matrix we get,

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_2 \leftarrow R_2 \times -1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.3)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.4)$$

See Table 5.1.1

5.2. Let \mathbf{A} and \mathbf{B} be real invertible matrices such that

$$\mathbf{AB} = -\mathbf{BA}. \quad (5.2.1)$$

Then

- a) $\text{trace} \mathbf{A} = \text{trace}(\mathbf{B}) = 0$
- b) $\text{trace} \mathbf{A} = \text{trace}(\mathbf{B}) = 1$
- c) $\text{trace} \mathbf{A} = 0, \text{trace}(\mathbf{B}) = 1$
- d) $\text{trace}(\mathbf{A}) = 1, \text{trace}(\mathbf{B}) = 0$

Solution: See Tables 5.2.1 and 5.2.2

5.3. Let \mathbf{A} be an $n \times n$ self-adjoint matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Let,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \dots + |\mathbf{X}_n^2|} \quad (5.3.1)$$

for $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n) \in \mathbb{C}^n$. If

$$p(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + \dots + a_n \mathbf{A}^n \quad (5.3.2)$$

then $\sup_{\|\mathbf{X}\|_2=1} \|p(\mathbf{A})\mathbf{X}\|_2$ is equal to

Solution: We know that \mathbf{A} is a self adjoint matrix and hence $\mathbf{A} = \mathbf{A}^*$ with eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$. Now as we are given,

$$p(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + \dots + a_n \mathbf{A}^n \quad (5.3.3)$$

then,

$$(p(\mathbf{A}))^* = a_0 \mathbf{I}^* + a_1 \mathbf{A}^* + \dots + a_n (\mathbf{A}^*)^n \quad (5.3.4)$$

Since, $\mathbf{A} = \mathbf{A}^*$ we can state that,

$$p(\mathbf{A})(p(\mathbf{A}))^* = (p(\mathbf{A}))^* p(\mathbf{A}) \quad (5.3.5)$$

Hence $p(\mathbf{A})$ is a normal matrix. Now using spectral theorem for a normal matrix,

$$\|p(\mathbf{A})\|_2 = \rho(p(\mathbf{A})) \quad (5.3.6)$$

\sup refers to the smallest element that is greater than or equal to every number in the set. Hence, \sup of $\|p(\mathbf{A})\|_2$ will be,

$$= \max \{|\alpha| : \alpha \text{ is the eigen value of } p(\mathbf{A})\} \quad (5.3.7)$$

$$= \max \{|p(\lambda_j)| : j = 1, 2, \dots, n\} \quad (5.3.8)$$

$$= \max \{|a_0 + a_1 \lambda_j + \dots + a_n \lambda_j^n| : j = 1, 2, \dots, n\} \quad (5.3.9)$$

Now, to find $\sup \|p(\mathbf{A})\mathbf{X}\|_2$,

$$= \max \{|a_0 + a_1 \lambda_j + \dots + a_n \lambda_j^n| : j = 1, 2, \dots, n\} \|\mathbf{X}\|_2 \quad (5.3.10)$$

Since, we have to find $\sup_{\|\mathbf{X}\|_2=1}$ i.e.,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \dots + |\mathbf{X}_n^2|} = 1 \quad (5.3.11)$$

Hence the final answer will be,

$$= \max \{|a_0 + a_1 \lambda_j + \dots + a_n \lambda_j^n| : j = 1, 2, \dots, n\} \quad (5.3.12)$$

5.4. Let $p(x) = \alpha x^2 + \beta x + \gamma$ be a polynomial, where $\alpha, \beta, \gamma \in \mathbf{R}$. Fix $X_0 \in \mathbf{R}$. Let $S = \{(a, b, c) \in \mathbf{R}^3 : p(x) = a(x - x_0)^2 + b(x - x_0) + c\}$ for all $x \in \mathbf{R}$. Find the number of elements in S is

- a) 0
- b) 1
- c) Strictly greater than 1 but finite
- d) Infinite

| | |
|--------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\dim(C(\mathbf{A})) = 1$ | False. Because the number of pivot variables are 2 as obtained in (5.1.4) |
| $\dim(C(\mathbf{A})) = 2$ | True. Since the number of pivot variables are 2, the rank of \mathbf{A} is 2. $\therefore \dim(C(\mathbf{A})) = 2 \quad [\because \dim(C(\mathbf{A})) = \text{rank}(\mathbf{A})]$ |
| $\text{rank}(\mathbf{A}) = 1$ | False. Because the $\text{rank}(\mathbf{A}) = 2$, as the number of pivot variables are 2 |
| $\mathbf{S} = \{(1, 1, 1, 0), (1, 1, 0, 1)\}$ is a basis of $N(\mathbf{A})$ | False. Let, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ Consider, $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Similarly, $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ Hence, the given vectors do not form the basis. |

TABLE 5.1.1

| | |
|------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Definition | Matrix \mathbf{A} is said to be similar to matrix \mathbf{B} if there exists matrix \mathbf{P} such that $\mathbf{A} = \mathbf{PBP}^{-1}$ |
| Properties | Similar matrices have same eigenvalues Sum of eigenvalue of a matrix equals its trace From above two properties we can conclude that similar matrices have same trace |

TABLE 5.2.1: Similar matrices and Properties

Solution:

$$S = \{(a, b, c) \in \mathbb{R}^3 : p(x) = a(x - x_0)^2 + b(x - x_0) + c\},$$

$$p(x) = \alpha x^2 + \beta x + \gamma \quad (5.4.1)$$

$$\Rightarrow p(x) = (\alpha\beta\gamma) \begin{pmatrix} x^2 & x & 1 \end{pmatrix}^T \quad (5.4.2)$$

$$\forall \mathbf{x} \in R(\text{Fix } X_0) \quad (5.4.3)$$

$$p(x) = (abc) \left((x - x_0)^2 (x - x_0) 1 \right)^T \quad (5.4.4)$$

$$= a(x^2 - 2x_0x + x_0^2) + b(x - x_0) + c \quad (5.4.5)$$

$$= ax^2 + (b - 2ax_0)x + (ax_0^2 - bx_0 + c) \quad (5.4.6)$$

Refer (5.4.2) and (5.4.6) and comparing the coefficients of powers of x,

$$\alpha = a, \beta = b - 2ax_0, \gamma = ax_0^2 - bx_0 + c \quad (5.4.7)$$

$$a = \alpha, b = \beta + 2\alpha x_0, c = \gamma - \alpha x_0^2 + (\beta + 2\alpha x_0)x_0 \quad (5.4.8)$$

Here α, β, γ and x_0 are the real fixed numbers. So a, b, c have unique values.

Hence S contain only 1 element. So option 2 is correct

5.5. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (5.5.1)$$

and \mathbf{I} be the 3×3 identity matrix. If

$$6\mathbf{A}^{-1} = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} \quad (5.5.2)$$

for $a, b, c \in \mathbb{R}$ then (a,b,c) equals

a) (1,2,1)

b) (1,-1,2)

c) (4,1,1)

d) (1,4,1)

Solution: Finding the characteristic equation,

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} \quad (5.5.3)$$

$$\Rightarrow (1-\lambda)(-2-\lambda)(-3-\lambda) = 0 \quad (5.5.4)$$

$$\Rightarrow (\lambda^2 + \lambda - 2)(-3-\lambda) = 0 \quad (5.5.5)$$

$$\Rightarrow \lambda^3 + 4\lambda^2 + \lambda - 6 = 0 \quad (5.5.6)$$

Using Cayley-Hamilton Theorem we get,

$$\mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} - 6\mathbf{I} = 0 \quad (5.5.7)$$

$$\Rightarrow \mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} = 6\mathbf{I} \quad (5.5.8)$$

$$\Rightarrow \mathbf{A}(\mathbf{A}^2 + 4\mathbf{A} + \mathbf{I}) = 6\mathbf{I} \quad (5.5.9)$$

$|\mathbf{A}| = 6 \neq 0$ hence inverse exists. Hence (5.5.9)

we get,

$$6\mathbf{A}^{-1} = \mathbf{A}^2 + 4\mathbf{A} + \mathbf{I} \quad (5.5.10)$$

Comparing (5.5.2) and (5.5.10) we get,

$$a = 1 \quad b = 4 \quad c = 1 \quad (5.5.11)$$

Hence $(a, b, c) = (1, 4, 1)$

5.6. Find the Eigenvalues of the matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 5 \\ 2 & 5 & -3 \end{pmatrix} \quad (5.6.1)$$

a) -4, 3, -3

b) 4, 3, 1

c) 4, $-4 \pm \sqrt{13}$

d) 4, $-2 \pm \sqrt{7}$

Solution: Using the characteristic equation of the matrix can find the Eigenvalues,

$$|\lambda\mathbf{I} - \mathbf{A}| = 0 \quad (5.6.2)$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -1 & -2 \\ -1 & \lambda+2 & -5 \\ -2 & -5 & \lambda+3 \end{vmatrix} = 0 \quad (5.6.3)$$

The expression that is obtained after expanding the determinant and simplifying it is,

$$(\lambda-1)(\lambda^2+5\lambda-19) - (5\lambda+31) = 0 \quad (5.6.4)$$

Further simplifying this we obtain the cubic equation,

$$\lambda^3 + 4\lambda^2 - 29\lambda - 12 = 0 \quad (5.6.5)$$

Solving this equation, the Eigenvalues obtained are,

$$\lambda_1 = -7.605, \lambda_2 = -0.394 \text{ and } \lambda_3 = 4 \quad (5.6.6)$$

Therefore, the Eigenvalues of the given matrix are 4, $-4 \pm \sqrt{13}$ (Option 3)

5.7. Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers a_0, a_1, \dots, a_k . For $p \in V$

$$\max \{ |p(a_j)| : 0 \leq j \leq k \} \quad (5.7.1)$$

defines a norm on V

a) only if $k < n$

b) only if $k \geq n$

c) if $k+1 \leq n$

d) if $k \geq n + 1$

Solution: Options 2 and 4 are correct as verified in the table 5.7.2

The scalar multiplication and triangle inequality properties holds true for all k .

$$\max \{ |\alpha p(a_j)| \} = |\alpha| \max \{ |p(a_j)| \} \quad (5.7.4)$$

$$\max \{ |p(a_i) + p(a_j)| \} \leq \max \{ |p(a_i)| \} + \max \{ |p(a_j)| \} \quad (5.7.5)$$

The positivity property holds true only if $k \geq n$ as more than n roots are possible when,

$$p(x) = 0 \implies |p(a_j)|_{0 \leq j \leq k} = 0 \quad (5.7.6)$$

$$\implies \max \{ |p(a_j)| : 0 \leq j \leq k \} = 0 \quad (5.7.7)$$

5.8. Let \mathbf{V} be the vector space of polynomials of degree at most 3 in a variable x with coefficients in \mathbb{R} . Let $\mathbf{T} = d/dx$ be the linear transformation of \mathbf{V} to itself given by differentiation.

Which of the following are correct?

- a) \mathbf{T} is invertible
- b) 0 is an eigenvalue of \mathbf{T}
- c) There is a basis with respect to which the matrix of \mathbf{T} is nilpotent.
- d) The matrix of \mathbf{T} with respect to the basis $(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3)$ is diagonal.

Solution: See Tables 5.8.1 , 5.8.2 and 5.8.3.

| | |
|------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\text{trace}(\mathbf{A}) = 0$ $\text{trace}(\mathbf{B}) = 0$ | <p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ <p>So, matrix \mathbf{A} and $(-\mathbf{A})$ are similar.∴</p> $\text{trace}(\mathbf{A}) = \text{trace}(-\mathbf{A})$ $\Rightarrow \text{trace}(\mathbf{A}) = 0$ <p>Similarly From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$ <p>So, matrix \mathbf{B} and $(-\mathbf{B})$ are similar.∴</p> $\text{trace}(\mathbf{B}) = \text{trace}(-\mathbf{B})$ $\Rightarrow \text{trace}(\mathbf{B}) = 0$ <p>So this statement is true</p> |
| $\text{trace}(\mathbf{A}) = 1$ $\text{trace}(\mathbf{B}) = 1$ | <p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ <p>So, matrix \mathbf{A} and $(-\mathbf{A})$ are similar.∴</p> $\text{trace}(\mathbf{A}) = \text{trace}(-\mathbf{A})$ $\Rightarrow \text{trace}(\mathbf{A}) = 0.$ <p>As $\text{trace}(\mathbf{A}) = 0$ this statement is false</p> |
| $\text{trace}(\mathbf{A}) = 0$ $\text{trace}(\mathbf{B}) = 1$ | <p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$ <p>So, matrix \mathbf{B} and $(-\mathbf{B})$ are similar.∴</p> $\text{trace}(\mathbf{B}) = \text{trace}(-\mathbf{B})$ $\Rightarrow \text{trace}(\mathbf{B}) = 0.$ <p>As $\text{trace}(\mathbf{B}) = 0$ this statement is false</p> |
| $\text{trace}(\mathbf{A}) = 1$ $\text{trace}(\mathbf{B}) = 0$ | <p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ <p>So, matrix \mathbf{A} and $(-\mathbf{A})$ are similar.∴</p> $\text{trace}(\mathbf{A}) = \text{trace}(-\mathbf{A})$ $\Rightarrow \text{trace}(\mathbf{A}) = 0.$ <p>As $\text{trace}(\mathbf{A}) = 0$ this statement is false</p> |

TABLE 5.2.2: Calculation of trace

| Properties | Norm $\forall x \in V$ |
|-----------------------|-----------------------------------------------|
| Positivity | $\ x\ \geq 0, \ x\ = 0 \iff x = 0$ |
| Scalar Multiplication | $\ \alpha x\ = \alpha \ x\ , \alpha \in F$ |
| Triangle Inequality | $\ x + y\ \leq \ x\ + \ y\ $ |

TABLE 5.7.1: Properties of Norm

| For $p \in V$ then the norm, $\max \{ p(a_j) : 0 \leq j \leq k \} = 0 \iff p(a_j) _{0 \leq j \leq k} = 0$ | |
|--------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Conditions | Explanation |
| only if $k < n$ Example: | <p>A polynomial doesn't necessarily have k distinct real roots, i.e., it may have repeated, complex roots.</p> <p>let p be polynomial of degree $n = 2$ and $k = 1$ given by:-</p> $p(x) = x^2 + 4x + 4 \quad (5.7.2)$ $ p(a_j) _{0 \leq j \leq 1} = 0 \implies a_0 = -2, a_1 = -2 \quad (5.7.3)$ <p>but a_0, a_1, \dots, a_k should be distinct real numbers.</p> <p>This contradicts the property of Norm. Thus condition fails.</p> |
| only if $k \geq n$ | <p>p is a polynomial of degree $\leq n$, it can't have more than n roots and is only possible when,</p> $p(x) = 0 \implies p(a_j) _{0 \leq j \leq k} = 0$ <p>hence p is identically zero. Thus condition satisfies.</p> |
| if $k + 1 \leq n$ | Not a norm for $k < n$. Hence incorrect. |
| if $k \geq n + 1$ | Norm for $k \geq n$. Hence correct. |

TABLE 5.7.2: Verifying Positivity Property of Norm

| | |
|-------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|
| Nilpotent Matrix | 1. If all the eigen values of matrix is zero then it is said to nilpotent matrix 2. Determinant and trace of nilpotent matrix are always zero. |
| Invertible Matrix | A matrix is said to be invertible matrix if its determinant is non zero. |
| Diagonal matrix | diagonal matrix is a matrix in which the entries outside the main diagonal are all zero. |

TABLE 5.8.1: Definition

| | |
|-------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | $T : P_3 \rightarrow P_3$ $T : V \rightarrow V$ be the linear operator given by differentiation wrt x $T(P(x)) \rightarrow P'(x)$ A be the matrix of T wrt some basis for V Assume basis for V be $\{1, x, x^2, x^3\}$ |
|-------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

TABLE 5.8.2: Result used

| | |
|---------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Checking whether matrix of T is nilpotent | $T : V \rightarrow V$ $TP(x) = P'(x)$ Differentiating wrt x to find matrix A ; $T(1) = 0 = a_1x + b_1x + c_1x^2 + d_1x^3$ $T(x) = 1 = a_2 + b_2x + c_2x^2 + d_2x^3$ $T(x^2) = 2x = a_3 + b_3x + c_3x^2 + d_3x^3$ $T(x^3) = 3x^2 = a_4 + b_4x + c_4x^2 + d_4x^3$ Representing A in matrix form ; $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ from the above matrix of T we can say it is nilpotent matrix. |
| Checking eigen value of matrix T | $A = \begin{pmatrix} 0 - \lambda & 1 & 0 & 0 \\ 0 & 0 - \lambda & 2 & 0 \\ 0 & 0 & 0 - \lambda & 3 \\ 0 & 0 & 0 & 0 - \lambda \end{pmatrix}$ $\Rightarrow \lambda = 0$ |
| Checking whether matrix of T is invertible | Since $\det A = 0$. Therefore matrix of T is not invertible |
| Checking whether Matrix of T is diagonal matrix | Let basis be $B' = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ Differentiating wrt x ; |

| | |
|------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $T(1) = 0 = a_1x + b_1(1+x) + c_1(1+x+x^2) + d_1(1+x+x^2+x^3)$ $T(1+x) = 1 = a_2 + b_2(1+x) + c_2(1+x+x^2) + d_2(1+x+x^2+x^3)$ $T(1+x+x^2) = 1+2x = a_3 + b_3(1+x) + c_3(1+x+x^2) + d_3(1+x+x^2+x^3)$ $T(1+x+x^2+x^3) = 1+2x+3x^2 = a_4 + b_4(1+x) + c_4(1+x+x^2) + d_4(1+x+x^2+x^3)$ $B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p>above matrix is not a diagonal matrix</p> |
| Conclusion | Thus we can conclude Option 2) and 3) are correct. |

TABLE 5.8.3: Solution

5.9. Let m, n, r be natural numbers. Let A be an $m \times n$ matrix with real entries such that $(AA^t)^r = I$, where I is the $m \times m$ identity matrix and A^t is the transpose of the matrix A . We can conclude that

- a) $m = n$
- b) AA^t is invertible
- c) A^tA is invertible
- d) if $m = n$, then A is invertible

Solution: Options 2) and 4) are correct. See Table 5.9.1

5.10. Let \mathbf{A} be a $n \times n$ real matrix with $\mathbf{A}^2 = \mathbf{A}$. Then

- a) the eigenvalues of \mathbf{A} are either 0 or 1
- b) \mathbf{A} is a diagonal matrix with diagonal entries 0 or 1
- c) $\text{rank}(\mathbf{A}) = \text{trace}(\mathbf{A})$
- d) if $\text{rank}(\mathbf{I} - \mathbf{A}) = \text{trace}(\mathbf{I} - \mathbf{A})$

Solution: See Table 5.10.1

5.11. For any $n \times n$ matrix B , let $N(B) = \{X \in \mathbb{R}^n : BX = 0\}$ be the null space of B . Let A be a 4×4 matrix with $\dim(N(A - 4I)) = 2$, $\dim(N(A - 2I)) = 1$ and $\text{rank}(A) = 3$ Which of the following are true?

- a) 0, 2 and 4 are eigenvalues of A
- b) $\det(A) = 0$
- c) A is not diagonalizable
- d) $\text{trace}(A) = 8$

| Option | Answer |
|--------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1) $m = n$ | <p>Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $r = 1$</p> $(\mathbf{A}\mathbf{A}^T)^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ <p>Since $m \neq n$ Option 1 is False.</p> |
| 2) AA^T is invertible | <p>w.k.t $\det(A^n) = (\det(A))^n$ Since $(AA^T)^r = I$ So $\det((AA^T)^r) = \det(I)$ $(\det(AA^T))^r = 1$ $\implies \det(AA^T) \neq 0$ Hence AA^T is invertible Option 2 is True.</p> |
| 3) $A^T A$ is invertible | <p>Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $r = 1$</p> $(\mathbf{A}^T \mathbf{A})^r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ <p>But $\det(AA^T) = 0$. $\implies AA^T$ is not invertible. Hence Option 3 is False</p> |
| 4) if $m = n$ then A is invertible | <p>Since $\det(AA^T) \neq 0$ $\det(A) \cdot \det(A^T) \neq 0$ $\det(A) \cdot \det(A) \neq 0$ $\implies A$ is invertible. Hence Option 4 is True</p> |

TABLE 5.9.1

Solution: See Table 5.11.1.

| | |
|-------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>A is a 4×4 matrix. $\dim(N(A - 2I)) = 2$, $\dim(N(A - 4I)) = 1$, and $\text{rank}(A) = 3$</p> |
| Eigenvalues of a matrix | The number λ is an eigenvalue of a matrix A if and only if $A - \lambda I$ is singular, |

i.e. $|A - \lambda I| = 0$

For $\lambda = 2$

Given, $\dim(N(A - 2I)) = 2$

$$\implies \text{nullity}(A - 2I) = 2$$

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$\implies \text{rank}(A - 2I) = 4 - 2 = 2$$

$\implies (A - 2I)$ is not a full rank matrix

Therefore $|A - 2I| = 0$

Also,

$$\implies N(A - 2I) = \{X \in \mathbb{R}^4 : (A - 2I)X = 0\}$$

$\implies (A - 2I)X = 0$ gives two eigen vectors

$\implies 2$ is an eigenvalue of A with multiplicity 2.

Similarly, for $\lambda = 4$

Given, $\dim(N(A - 4I)) = 1$

$$\implies \text{rank}(A - 4I) = 4 - 1 = 3$$

$\implies (A - 4I)$ is not a full rank matrix

| | |
|-------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>Therefore $A - 4I = 0$ $\Rightarrow 4$ is an eigenvalue of A with multiplicity 1.</p> <p>For $\lambda = 0$ Given that $\text{rank}(A) = 3$ $\Rightarrow A$ is not a full rank matrix Therefore $A = 0$ $\Rightarrow 0$ is an eigenvalue of A with multiplicity 1.</p> |
| Determinant | <p>Given that $\text{rank}(A) = 3$ $\Rightarrow A$ is not a full rank matrix Therefore $A = 0$</p> |
| Diagonalizability | <p>An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigen vectors. $\text{rank}(A) + \text{nullity}(A) = n$ \Rightarrow for $\lambda = 0$, $\text{nullity}(A - \lambda I) = \text{nullity}(A) = 4 - 3 = 1$ \Rightarrow There exists only one linearly independent eigen vector corresponding to 0 eigen value Thus, matrix A is not diagonalizable.</p> |
| Trace | <p>$\text{Trace}(A) = \text{sum of eigen values}$ $\Rightarrow \text{Trace}(A) = 0 + 2 + 2 + 4 = 8$</p> |
| Conclusion | <p>Option (1), (2) and (4) are correct</p> |

TABLE 5.11.1: Solution

5.12. Which of the following 3x3 matrices are diagonalizable over \mathbb{R} ?

- a) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$
- b) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- c) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$
- d) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution: See Tables 5.12.1 and 5.12.2

| Objective | Explanation |
|--------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Eigenvalues of \mathbf{A} | <p>Since</p> $\mathbf{A}^2 = \mathbf{A} \quad (5.10.1)$ $\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \quad (5.10.2)$ <p>From Cayley-Hamilton Theorem we have,</p> $\lambda^2 - \lambda = 0 \quad (5.10.3)$ $\implies \lambda(\lambda - 1) = 0 \quad (5.10.4)$ $\implies \lambda = 0, 1 \quad (5.10.5)$ <p>A matrix \mathbf{A} satisfying $\mathbf{A}^2 = \mathbf{A}$ is an idempotent matrix with eigen values equal to 0 or 1.</p> |
| Check if \mathbf{A} is necessary diagonal | <p>Consider</p> $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.6)$ $\quad (5.10.7)$ <p>Then,</p> $\mathbf{A}^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.8)$ $= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.9)$ $= \mathbf{A} \quad (5.10.10)$ <p>Hence \mathbf{A} is idempotent but not diagonal.</p> |
| Relation between rank and trace of \mathbf{A} | <p>Rank of matrix is defined as the number of non-zero eigenvalues. Since number of non-zero eigenvalues is 1,</p> $rank(\mathbf{A}) = 1 \quad (5.10.11)$ $trace(\mathbf{A}) = \sum_i \lambda_i = 0 + 1 = 1 \quad (5.10.12)$ $\implies rank(\mathbf{A}) = trace(\mathbf{A}) \quad (5.10.13)$ |
| Relation between rank and trace of $\mathbf{I} - \mathbf{A}$ | <p>Now for the matrix $\mathbf{I} - \mathbf{A}$ we have,</p> $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \quad (5.10.14)$ $= \mathbf{I}^2 - \mathbf{I}\mathbf{A} - \mathbf{A}\mathbf{I} + \mathbf{A}^2 \quad (5.10.15)$ $= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \quad (5.10.16)$ $= \mathbf{I} - \mathbf{A} \quad (5.10.17)$ <p>Hence $\mathbf{I} - \mathbf{A}$ is an idempotent matrix. Therefore we conclude,</p> $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \quad (5.10.18)$ |
| Answer | (1),(3) and (4) |

TABLE 5.10.1

| | |
|-----------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Test for diagonalizability | <p>Let \mathbf{W}_i be the eigenspace corresponding to eigenvalue λ_i of \mathbf{A}</p> <p>1) \mathbf{A} is diagonalizable</p> <p>2) characteristic polynomial of \mathbf{A} is</p> <p>$f = (\mathbf{x} - \lambda_1)^{d_1} \dots (\mathbf{x} - \lambda_k)^{d_k}$ and $\dim(\mathbf{W}_i) = d_i$</p> <p>3) $\sum_{i=1}^k \mathbf{W}_i = n$</p> |
| Concept for diagonalization | <p>A linear operator \mathbf{A} on a n-dimensional space \mathbb{V} is diagonalizable , if and only if \mathbf{A} has n distinct characteristic vectors or null spaces corresponding to the characteristic values</p> |

TABLE 5.12.1: Illustration of theorem.

| | |
|---------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option A | <p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ |
| Finding Characteristics polynomial | <p>Characteristics polynomial of the matrix \mathbf{A} is $\det(x\mathbf{I} - \mathbf{A})$</p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-1) & -3 & -2 \\ 0 & (x-4) & -5 \\ 0 & 0 & x-6 \end{vmatrix}$ <p>Characteristic Polynomial = $(x-1)(x-4)(x-6)$</p> |
| Testing diagonalizability over \mathbb{R} | <p>1) As the characteristics polynomial is product of linear factors over \mathbb{R} .</p> <p>2) To find characteristic values of the operator $\det(xI - A) = 0$ which gives $\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6$</p> <p>Thus over \mathbb{R} matrix \mathbf{A} has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus $\dim \mathbf{W}_i = d_i$</p> <p>3) $\sum_i \mathbf{W}_i = n = 3$, which is equal to \dim of \mathbf{A}.</p> |

| | |
|---------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Conclusion on Option A | Option A satisfy all three condition of Diagonalizability over \mathbb{R} . |
| Option B | <p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| Finding Characteristics polynomial | <p>Characteristics polynomial of the matrix $\det(x\mathbf{I} - \mathbf{A})$</p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x & -1 & 0 \\ 1 & x & 0 \\ 0 & 0 & x - 1 \end{vmatrix}$ <p>Characteristic Polynomial = $(x - 1)(x + i)(x - i)$</p> |
| Testing diagonalizability over \mathbb{R} | <p>1) As the characteristics polynomial is not the product of linear factors over \mathbb{R} beacuse roots of characteristic eq are complex . Thus \mathbf{A} is not diagonalizable over \mathbb{R}.</p> |
| Conclusion on Option B | Option B does not satisfy condition 1. |
| Option C | <p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$ |
| Finding Characteristics polynomial | <p>Characteristics polynomial of the matrix \mathbf{A} is $\det(x\mathbf{I} - \mathbf{A})$</p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x - 1) & -2 & -3 \\ -2 & (x - 1) & -4 \\ -3 & -4 & x - 1 \end{vmatrix}$ <p>Characteristic Polynomial = $(x + 3.19)(x + 0.877)(x - 7.07)$</p> |
| Testing diagonalizability over \mathbb{R} | <p>1) As the characteristics polynomial are product of linear factors over \mathbb{R} .</p> <p>2) To find characteristic values of the operator $\det(x\mathbf{I} - \mathbf{A}) = 0$ which gives $\lambda_1 = -3.19, \lambda_2 = -0.887, \lambda_3 = 7.07$</p> |

| | |
|---------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>Thus over \mathbb{R} matrix \mathbf{A} has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value .</p> <p>Thus $\dim \mathbf{W}_i = d_i$</p> <p>3) $\sum_i \mathbf{W}_i = n = 3$, which is equal to \dim of \mathbf{A}.</p> |
| Conclusion on Option C | Option C satisfy all three condition of Diagonalizability over \mathbb{R} . |
| Option D | <p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ |
| Finding Characteristics polynomial | <p>Characteristics polynomial of the matrix \mathbf{A} is $\det(x\mathbf{I} - \mathbf{A})$</p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x & -1 & -2 \\ 0 & x & -1 \\ 0 & 0 & x \end{vmatrix}$ <p>Characteristic Polynomial = $(x)(x)(x) = x^3$</p> |
| Testing diagonalizability over \mathbb{R} | <p>1) As the characteristics polynomial is product of linear factors over \mathbb{R} .</p> <p>2) To find characteristic values of the operator $\det(x\mathbf{I} - \mathbf{A}) = 0$</p> <p>$\lambda_1 = 0$</p> <p>$d_1 = 3$</p> $\mathbf{W}_1 = \mathbf{A} - \lambda_1 \mathbf{I} \Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - 0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ <p>$\dim \mathbf{W}_1 = 2$</p> <p>$\dim \mathbf{W}_i \neq d_i$</p> <p>Algebric Multiplicity is not equal to Geometric Multiplicity.</p> |
| Conclusion on Option D | Option D does not satisfy second condition of Diagonalizability. |
| Answer | Option A and Option C are Diagonalizable over \mathbb{R} . |

TABLE 5.12.2: Option Checking Table

| | |
|-------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Positive Semi Definite Matrix | A $n \times n$ symmetric real matrix \mathbf{M} is said to be positive semi definite if $\mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0$ for all non-zero \mathbf{x} in \mathbb{R}^n . Formally \mathbf{M} is positive semi-definite $\Leftrightarrow \mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ |
| Theorem | For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L}(\mathbf{V})$, following are equivalent. 1). $\mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbf{V}$. 2). All the eigenvalues of \mathbf{M} are non-negative. |

TABLE 5.13.1: Definition and Result used

| | |
|-------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Calculating eigen values of \mathbf{A} | Given $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Calculating, eigen values of \mathbf{A} , ie $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ $\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (3-\lambda)((2-\lambda)(1-\lambda)-9) - 1(1-\lambda-6) + 2(3-2(2-\lambda)) = 0$ $\Rightarrow \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$ $\Rightarrow \lambda_1 = 6, \lambda_2 = \sqrt{3} \text{ and } \lambda_3 = -\sqrt{3}$ Hence, \mathbf{A} has exactly two positive eigen values. |
| Proving $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ using contradiction | Suppose $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$. Then, by theorem above in definition section, matrix \mathbf{A} is positive semi definite. Hence, all the eigen values of \mathbf{A} non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$. So, $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Similarly, as $\lambda_2 \leq 0, \forall i$ is also not true, so $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$. Thus, $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$. |
| Correct Options | Hence, correct options are (1) and (4). |

TABLE 5.13.2: Solution

5.13. Let $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ and $\mathbf{Q}(\mathbf{X}) = \mathbf{X}^T \mathbf{A} \mathbf{X}$ for $\mathbf{X} \in \mathbb{R}^3$. Then

- \mathbf{A} has exactly two positive eigen values.
- all the eigen values of \mathbf{A} are positive.
- $\mathbf{Q}(\mathbf{X}) \geq 0 \forall \mathbf{X} \in \mathbb{R}^3$
- $\mathbf{Q}(\mathbf{X}) < 0$ for some $\mathbf{X} \in \mathbb{R}^3$

Solution: See Tables 5.13.1 and 5.13.2

5.14. Consider the matrix

$$A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}; x \in \mathbf{R}. \quad (5.14.1)$$

Then,

- $A(x)$ has eigenvalue 0 for some $x \in \mathbf{R}$.
- 0 is not an eigenvalue of $A(x)$ for any $x \in \mathbf{R}$.
- $A(x)$ has eigenvalue 0 $\forall x \in \mathbf{R}$.
- $A(x)$ is invertible $\forall x \in \mathbf{R}$.

Solution: Let $\lambda = 0$ be an eigenvalue. Hence,

$$|A - \lambda I| = 0 \quad (5.14.2)$$

$$\Rightarrow |A| = 0 \quad (5.14.3)$$

$$\Rightarrow |A| = \begin{vmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{vmatrix} = 0 \quad (5.14.4)$$

Performing row reduction we get,

$$\begin{vmatrix} 1+x^2 & 7 & 11 \\ 0 & \frac{2x^3-19x}{1+x^2} & \frac{4x^2-33x+4}{1+x^2} \\ 0 & 0 & \frac{26x^3-244x^2+538x-68}{2x^3-19x} \end{vmatrix} = 0 \quad (5.14.5)$$

$$\Rightarrow 26x^3 - 244x^2 + 538x - 68 = 0 \quad (5.14.6)$$

$$\Rightarrow x_1 = 6.01, x_2 = 3.23, x_3 = 0.13 \quad (5.14.7)$$

See Table 5.14.1

6 DECEMBER 2016

6.1. The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \quad (6.1.1)$$

is

- positive definite.
- non-negative definite but not positive definite.
- negative definite.
- neither negative definite nor positive definite.

Solution:

- For a real symmetric matrix to be positive definite the eigen values of the matrix should

| OPTIONS | Explanation |
|------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option (b) | At the Values of x given by (5.14.7), eigen value $\lambda = 0$. Hence option (b) can't be correct. |
| Option (c) | If one of the eigenvalue is 0 for $A(x)$ then, $ A(x) = 0 \forall x \in \mathbb{R}$. But from (5.14.7) we have concluded that $ A = 0$ only for, $x_1 = 6.01, x_2 = 3.23, x_3 = 0.13$. Hence, Option (c) is incorrect. |
| Option (d) | Now for the values of x given by (5.14.7), $ A = 0$. Hence it is not invertible $\forall x \in \mathbb{R}$ Hence Option (d) is incorrect. |
| Option (a) | Now clearly from above arguments $A(x)$ has eigenvalue 0 for some $x \in \mathbb{R}$ Hence Option (a) is Correct. |

TABLE 5.14.1

be positive.

- b) For a real symmetric matrix to be negative definite the eigen values of the matrix should be negative.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

The characteristic equation of the matrix \mathbf{A} is given by

$$\begin{aligned} |V - \lambda \mathbf{I}| &= \begin{vmatrix} 3 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 3 - \lambda \end{vmatrix} = 0 \\ \implies \lambda^3 - 8\lambda^2 + 19\lambda - 12 &= 0 \end{aligned} \quad (6.1.2)$$

The Eigen values of \mathbf{A} are:

$$\begin{aligned} \lambda_1 &= 5/2 \\ \lambda_2 &= 3/2 \\ \lambda_3 &= 4 \end{aligned} \quad (6.1.3)$$

Since all the eigen values of matrix \mathbf{A} are positive, Therefore the matrix \mathbf{A} is positive definite.

6.2. Let $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x, y) = (x^2, y^2 + \sin x)$. Then the derivative of f at (x, y) is the linear transformation given by

- a) $\begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix}$
b) $\begin{pmatrix} 2x & 0 \\ 2y & \cos x \end{pmatrix}$
c) $\begin{pmatrix} 2y & \cos x \\ 2x & 0 \end{pmatrix}$

d) $\begin{pmatrix} 2x & 2y \\ 0 & \cos x \end{pmatrix}$

Solution: Let $f_1 = x^2$ and $f_2 = y^2 + \sin x$.
Begin by finding the derivative of $f(x, y)$

$$Df(x, y) = \begin{pmatrix} Df_1 x & Df_1 y \\ Df_2 x & Df_2 y \end{pmatrix} \quad (6.2.1)$$

$$= \begin{pmatrix} 2x & 0 \\ \cos x & 2y \end{pmatrix} \quad (6.2.2)$$

So option 1 is correct.

Now to prove that Derivatives is a linear transformation we dwell on the definition of linear transformation that it satisfies two properties i.e additivity and homogeneity as $\mathbb{R}^n \rightarrow \mathbb{R}^m$

$$D(cf) = cD(f) \quad (6.2.3)$$

$$D(f + g) = D(f) + D(g) \quad (6.2.4)$$

Now refer (6.2.3) we proceed as

$$D(cf) = \begin{pmatrix} Dcf_1 & Dcf_1 \\ Dcf_2 & Dcf_2 \end{pmatrix} \quad (6.2.5)$$

$$= c \begin{pmatrix} Df_1 & Df_1 \\ Df_2 & Df_2 \end{pmatrix} \quad (6.2.6)$$

$$= cD(f) \quad (6.2.7)$$

Now refer (6.2.4) we proceed as

$$D(f + g) = \begin{pmatrix} D(f_1 + g_1) & D(f_1 + g_1) \\ D(f_2 + g_2) & D(f_2 + g_2) \end{pmatrix} \quad (6.2.8)$$

$$\begin{pmatrix} Df_1 & Df_1 \\ Df_2 & Df_2 \end{pmatrix} + \begin{pmatrix} Dg_1 & Dg_1 \\ Dg_2 & Dg_2 \end{pmatrix} \quad (6.2.9)$$

$$= D(f) + D(g) \quad (6.2.10)$$

Hence both properties are satisfied so we can say that it is a linear transformation

6.3. Which of the following subsets of \mathbb{R}^4 is a basis of \mathbb{R}^4 ?

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{B}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\mathbf{B}_3 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ -5 & 5 & 0 & 0 \end{pmatrix}$$

- a) \mathbf{B}_1 and \mathbf{B}_2 but not \mathbf{B}_3 .
- b) $\mathbf{B}_1, \mathbf{B}_2$, and \mathbf{B}_3 .
- c) \mathbf{B}_1 and \mathbf{B}_3 but not \mathbf{B}_2 .
- d) Only \mathbf{B}_1 .

Solution: See Table 6.3.1

| Statement | Solution |
|-------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Definition | <p>Let \mathbf{V} be a vector space. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is called a basis for \mathbf{V} if the following conditions hold.</p> $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \mathbf{V} \quad (6.3.1)$ $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \text{ is linearly independent} \quad (6.3.2)$ |
| Given | $\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{B}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \mathbf{B}_3 = \begin{pmatrix} 1 & 0 & 2 & -5 \\ 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (6.3.3)$ |
| Checking \mathbf{B}_1 | <p>Checking for linear independence. Upon row reducing \mathbf{B}_1 (6.3.4)</p> $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3, R_3 \rightarrow R_3 - R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.3.5)$ <p>Clearly Rank of \mathbf{B}_1 is 4, ie full rank. Hence it forms a Basis.</p> |
| Checking \mathbf{B}_2 | <p>Checking for linear independence. Upon row reducing \mathbf{B}_2 (6.3.6)</p> $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xleftarrow{R_2 \rightarrow \frac{R_2}{2}, R_1 \rightarrow R_1 - R_2, R_3 \rightarrow \frac{R_3}{3}, R_2 \rightarrow R_2 - R_3, R_4 \rightarrow \frac{R_4}{4}, R_3 \rightarrow R_3 - R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.3.7)$ <p>Rank of \mathbf{B}_2 is 4, ie full rank. Hence it also forms a Basis.</p> |
| Checking \mathbf{B}_3 | <p>Checking for linear independence. Upon row reducing \mathbf{B}_3 (6.3.8)</p> $\begin{pmatrix} 1 & 0 & 2 & -5 \\ 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xleftarrow{R_2 \rightarrow R_2 - 2R_1, R_4 \rightarrow R_4 - R_2, R_3 \rightarrow -\frac{R_3}{5}, R_1 \rightarrow R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.3.9)$ <p>Rank of \mathbf{B}_3 is 3, ie not full rank. Hence it does not forms a Basis.</p> |
| Conclusion | Hence option 1, ie $\mathbf{B}_1, \mathbf{B}_2$ and not \mathbf{B}_3 is the correct answer. |

TABLE 6.3.1: Solution

| | |
|--------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>a) Matrix J of $n \times n$ dimension with all entries 1.</p> <p>b) Matrix B of $3n \times 3n$ dimension</p> $B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$ |
| Transforming matrix B into Block diagonal matrix using transformation Matrix | $M = \mathbf{T}(B)$ $M = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$ $M = \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}$ |
| Rank of Block Diagonal matrix M | <p>It is equal to the sum of rank of individual blocks in diagonal</p> $r(J) = 1$ $\therefore r(M) = 1 + 1 + 1 = 3$ |
| Rank of a matrix and its transformation are same. | <p>\therefore rank of matrix B is</p> $r(B) = r(M) = 3$ |

TABLE 6.4.1

6.4. Let J denote the matrix of order $n \times n$ with all entries 1 and let B be a $3n \times 3n$ matrix given

$$\text{by } B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}.$$

Find rank of matrix B . **Solution:** See Tables 6.4.1 and 6.4.2

6.5. Which of the following sets of functions from \mathbb{R} to \mathbb{R} is a vector space over \mathbb{R} ?

$$S_1 = \{f \mid \lim_{x \rightarrow 3} f(x) = 0\} \quad (6.5.1)$$

$$S_2 = \{g \mid \lim_{x \rightarrow 3} g(x) = 1\} \quad (6.5.2)$$

$$S_3 = \{h \mid \lim_{x \rightarrow 3} h(x) \text{ exists}\} \quad (6.5.3)$$

is

a) Only S_1

b) Only S_2

c) S_1 and S_3 but not S_2

d) All the three are vector spaces

Solution: Let S be a set of functions. Let $f_1, f_2 \in S$ and $\alpha, \beta \in \mathbb{R}$

For a set of functions to be considered as a vector space:

a) The linear combination of f_1 and f_2 should be in S .

$$\text{i.e. } \alpha f_1(x) + \beta f_2(x) \in S$$

b) The $\mathbf{0}$ should belong to S

$$\text{i.e. } \mathbf{0} \in S$$

Case1: Test for S_1

| | |
|--------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Example | <p>Let $n = 2$</p> $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ |
| Transforming matrix B into Block diagonal matrix using transformation Matrix | $M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ $M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ |
| Rank of Block Diagonal matrix M | <p>It is equal to the sum of rank of individual blocks in diagonal</p> $r(J) = 1$ $\therefore r(M) = 1 + 1 + 1 = 3$ |
| Rank of a matrix and its transformation are same. | <p>\therefore rank of matrix B is</p> $r(B) = r(M) = 3$ |

TABLE 6.4.2

a) Let $f_1, f_2 \in S_1$ and $\alpha, \beta \in \mathfrak{R}$

Then Using (6.5.4)

$$\lim_{x \rightarrow 3} f_1(x) = 0$$

$$\lim_{x \rightarrow 3} f_2(x) = 0$$

(6.5.4)

$$\lim_{x \rightarrow 3} (\alpha f_1(x) + \beta f_2(x))$$

$$= \alpha \left(\lim_{x \rightarrow 3} f_1(x) \right) + \beta \left(\lim_{x \rightarrow 3} f_2(x) \right)$$

$$= \alpha \times 0 + \beta \times 0$$

$$= 0$$

$$\therefore \alpha f_1(x) + \beta f_2(x) \in S_1$$

- b) Let $f(x) = 0$
then

$$\lim_{x \rightarrow 3} f(x) = 0$$

$$\therefore \mathbf{0} \in S_1$$

Hence, S_1 is a vector space.

Case2: Test for S_2

- a) Let $g_1, g_2 \in S_2$ and $\alpha, \beta \in \mathfrak{R}$

$$\lim_{x \rightarrow 3} g_1(x) = 1$$

$$\lim_{x \rightarrow 3} g_2(x) = 1 \quad (6.5.5)$$

Then Using (6.5.5)

$$\begin{aligned} & \lim_{x \rightarrow 3} (\alpha g_1(x) + \beta g_2(x)) \\ &= \alpha \left(\lim_{x \rightarrow 3} g_1(x) \right) + \beta \left(\lim_{x \rightarrow 3} g_2(x) \right) \\ &= \alpha \times 1 + \beta \times 1 \\ &= \alpha + \beta \end{aligned}$$

$$\therefore \alpha g_1(x) + \beta g_2(x) \in S_1 \text{ if } \alpha + \beta = 1$$

- b) Let $g(x) = 0$
then

$$\lim_{x \rightarrow 3} g(x) = 1$$

$$\therefore \mathbf{0} \notin S_1$$

Hence, S_2 is not a vector space.

Case3: Test for S_3

- a) Let $h_1, h_2 \in S_3$ and $\alpha, \beta \in \mathfrak{R}$

$$\begin{aligned} & \lim_{x \rightarrow 3} h_1(x) \text{ exists} \\ & \lim_{x \rightarrow 3} h_2(x) \text{ exists} \end{aligned} \quad (6.5.6)$$

Then Using (6.5.6)

$$\begin{aligned} & \lim_{x \rightarrow 3} (\alpha h_1(x) + \beta h_2(x)) \text{ exists} \\ & \therefore \alpha h_1(x) + \beta h_2(x) \in S_3 \end{aligned}$$

- b) Let $h(x) = 0$
then

$$\lim_{x \rightarrow 3^-} h(x) = 0 = \lim_{x \rightarrow 3^+} h(x)$$

$$\therefore \mathbf{0} \in S_1$$

Hence, S_3 is a vector space.

Therefore, Option (3) is correct.

6.6. Let \mathbf{A} be an $n \times m$ matrix with each entry

equal to +1, -1 or 0 such that every column has exactly one +1 and exactly one -1. We can conclude that

$$1. \text{Rank } \mathbf{A} \leq n - 1 \quad (6.6.1)$$

$$2. \text{Rank } \mathbf{A} = m \quad (6.6.2)$$

$$3. n \leq m \quad (6.6.3)$$

$$4. n - 1 \leq m \quad (6.6.4)$$

Solution: See Table 6.6.1

| option | Solution |
|---------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1. | <p>Let us consider \mathbf{A} as follows and let s be the summation of all column entries:</p> $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$ $ \mathbf{A} - \lambda \mathbf{I} = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{vmatrix} = 0$ $= \begin{vmatrix} a_{11} + a_{21} + \dots + a_{n1} - \lambda & a_{11} + a_{21} + \dots + a_{n1} - \lambda & \dots & a_{11} + a_{21} + \dots + a_{n1} - \lambda \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{vmatrix}$ $\Rightarrow (s - \lambda) \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{vmatrix} = 0$ |
| Example | <p>Since $s=0$ according to question, Therefore $\lambda = 0$ is an eigen value of \mathbf{A}. Since $\lambda = 0$, Hence \mathbf{A} is singular. Which means at least two rows are linearly dependent. Therefore,</p> $\text{Rank}(\mathbf{A}) < n$ $\text{Rank}(\mathbf{A}) \leq n - 1$ <p>Let us Consider \mathbf{A} as follows, where $n=4$ and $m=3$</p> $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ <p>Calculating Row Reduced Echelon Form of \mathbf{A} as follows:</p> |

| | |
|------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $\begin{array}{l} \xleftrightarrow[R_4 \leftarrow R_2 + R_4]{R_4 \leftarrow R_1 + R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \\ \xleftrightarrow{R_4 \leftarrow R_3 + R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$ |
| Conclusion | Since the Rank $\mathbf{A}=3$ and $n=4$, Therefore the Rank $\mathbf{A} \leq n - 1$ statement is true. |
| 2. | <p>Let us Consider \mathbf{A} as follows,where $n=2$ and $m=2$</p> $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ <p>Applying elementary transformations on \mathbf{A} as follows:</p> $\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ |
| Conclusion | Since the Rank $\mathbf{A}=1$ and $m=2$, Therefore the Rank $\mathbf{A} \neq m$, Hence the statement is false. |
| 3. | <p>Let us Consider \mathbf{A} as follows,where $n=3$ and $m=2$</p> $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \quad (6.6.5)$ |
| Conclusion | Since there exists a matrix \mathbf{A} when $n>m$, Therefore the statement is false. |
| 4 | <p>Let us Consider \mathbf{A} as follows,where $n=4$ and $m=2$</p> $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.6.6)$ |
| Conclusion | Since there exists a matrix \mathbf{A} when $n-1>m$, Therefore the statement is false. |

TABLE 6.6.1: Solution summary

| | |
|-----------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option 1 | To conclude that $m = n$ |
| Assumptions | <p>For the example: Without loss of generality, Let $m = 2$, $n = 3$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$</p> $\Rightarrow \mathbf{A}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ |
| Proof | <p>We know that $(\mathbf{A}\mathbf{A}^t)^r = \mathbf{I}$ which is a square matrix of order $m \times m$ For any natural value of r, a square matrix (\mathbf{I}) of order $m \times m$ is obtained Hence, we cannot conclude that $m = n$ because we get \mathbf{I} of order $m \times m$ even if $m \neq n$. To illustrate this, Consider the following example</p> $\mathbf{A}\mathbf{A}^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (\mathbf{A} \text{ and } \mathbf{A}^t \text{ from Assumptions})$ $(\mathbf{A}\mathbf{A}^t)^r = \mathbf{I}$ <p>Here $m \neq n$. Therefore, Option 1 is incorrect</p> |

TABLE 6.7.1: Option 1

| | |
|-----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option 2 | To conclude that $\mathbf{A}\mathbf{A}^t$ is invertible |
| Assumptions | $\mathbf{A}\mathbf{A}^t$ is not invertible |
| Proof | $\Rightarrow \mathbf{A}\mathbf{A}^t = 0 \Rightarrow (\mathbf{A}\mathbf{A}^t)^r = 0$ $\Rightarrow (\mathbf{A}\mathbf{A}^t)^r \neq \mathbf{I} \quad (\mathbf{I} = 1)$ <p>Since, this is a contradiction to the assumption made we can conclude that $\mathbf{A}\mathbf{A}^t$ is invertible. Therefore, Option 2 is correct</p> |

TABLE 6.7.2: Option 2

6.7. Let m , n and r be natural numbers. Let \mathbf{A} be an $m \times n$ matrix with real entries such that $(\mathbf{A}\mathbf{A}^t)^r = \mathbf{I}$, where \mathbf{I} is the $m \times m$ identity matrix and \mathbf{A}^t is the transpose of the matrix \mathbf{A} . We can conclude that

Options:

- a) $m = n$
- b) $\mathbf{A}\mathbf{A}^t$ is invertible
- c) $\mathbf{A}^t\mathbf{A}$ is invertible
- d) if $m = n$, then \mathbf{A} is invertible

Solution: See Tables 6.7.1, 6.7.2, 6.7.3 and 6.7.4.

6.8. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let α_n and β_n denote the two eigenvalues of \mathbf{A}^n such that $|\alpha_n| \geq |\beta_n|$. Then

- a) $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$
- b) $\beta_n \rightarrow 0$ as $n \rightarrow \infty$
- c) β_n is positive if n is even.
- d) β_n is negative if n is odd.

Solution: See Table 6.8.1.

6.9. Let M_n denote the vector space of all $n \times n$ real

matrices. Which of the following is a linear subspaces of M_n :-

- a) $V_1 = \{A \in M_n : A \text{ is nonsingular}\}$
- b) $V_2 = \{A \in M_n : \det(A) = 0\}$
- c) $V_3 = \{A \in M_n : \text{trace}(A) = 0\}$
- d) $V_4 = \{BA : A \in M_n\}$, where B is some fixed matrix in M_n

Solution: See Table 6.9.1

6.10. If \mathbf{P} and \mathbf{Q} are invertible matrices such that $\mathbf{P}\mathbf{Q} = -\mathbf{Q}\mathbf{P}$, then we can conclude that

- a) $\text{Tr}(\mathbf{P}) = \text{Tr}(\mathbf{Q}) = 0$
- b) $\text{Tr}(\mathbf{P}) = \text{Tr}(\mathbf{Q}) = 1$
- c) $\text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{Q})$
- d) $\text{Tr}(\mathbf{P}) \neq \text{Tr}(\mathbf{Q})$

Solution: See Table 6.10.1

| | |
|-----------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option 3 | To conclude that $\mathbf{A}^t \mathbf{A}$ is invertible |
| Assumptions | Without loss of generality, Let $m = 2$, $n = 3$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\Rightarrow \mathbf{A}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ |
| Proof | $\Rightarrow \mathbf{A}^t \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{A}^t \mathbf{A} = 0$ This means that $\mathbf{A}^t \mathbf{A}$ is not invertible. Therefore, Option 3 is incorrect |

TABLE 6.7.3: Option 3

| | |
|-----------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option 4 | To conclude that if $m = n$ then \mathbf{A} is invertible |
| Assumptions | Let $m = n$ |
| Proof | Since $(\mathbf{A} \mathbf{A}^t)^r = \mathbf{I} \Rightarrow (\mathbf{A} \mathbf{A}^t)^r = \mathbf{I} = 1$ $\Rightarrow (\mathbf{A} \mathbf{A}^t)^r = 1$ (\mathbf{A} is a square matrix) $\Rightarrow (\mathbf{A})^{2r} = 1$ Therefore, Option 4 is correct |

TABLE 6.7.4: Option 4

| Options | Solutions | True/False |
|---------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------|
| 1. | Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ Now lets find the eigen values of matrix \mathbf{A} $ \mathbf{A} - \lambda \mathbf{I} = 0$ $\Rightarrow \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$ $\Rightarrow \lambda^2 - \lambda - 1 = 0$ On solving we get 2 eigen values $\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \beta_1 = \frac{1-\sqrt{5}}{2}$ We know that if eigenvalue of \mathbf{A} is λ then eigenvalue of \mathbf{A}^n is λ^n . In this problem we can say that the eigenvalues α_n and β_n of \mathbf{A}^n are $\alpha_n = \alpha_1^n \quad \beta_n = \beta_1^n$ Since $\alpha_1 > 1$ we can say that $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. | True |
| 2. | We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since $-1 < \beta_1 < 0$, we can say that $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. | True |
| 3. | We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since β_1 is negative because $-1 < \beta_1 < 0$, if n is even then β_n is positive. | True |
| 4. | We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$. Since β_1 is negative, if n is odd then β_n is negative. | True |

TABLE 6.8.1

| Vector space | Is it subspace to M_n ? |
|---------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1) V_1 : All non-singular matrices of $n \times n$ | The matrices $I_{n \times n}$ and $-I_{n \times n}$ are non-singular matrices, but the sum $I_{n \times n} - I_{n \times n}$ is zero matrix and it is singular. $\therefore V_1$ does not form subspace of M_n . |
| 2) V_2 : All singular matrices of $n \times n$ | The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are singular matrices, but the sum is a non-singular matrix. $\therefore V_2$ does not form subspace M_n . |
| 3) V_3 : All matrices of $n \times n$ with trace =0 | Let \mathbf{v}_1 and \mathbf{v}_2 be matrices with Trace = 0. $Tr(\mathbf{v}_1 + \alpha \mathbf{v}_2) = Tr(\mathbf{v}_1) + \alpha Tr(\mathbf{v}_2) = 0$. \therefore the vector space V_3 forms linear subspace of M_n . |
| 4) V_4 : $F_A = BA$, where B is some fixed matrix in M_n | Let \mathbf{v}_1 and \mathbf{v}_2 be matrices in the vector space V_4 . $F_{\mathbf{v}_1 + \alpha \mathbf{v}_2} = B(\mathbf{v}_1 + \alpha \mathbf{v}_2)$ $= B\mathbf{v}_1 + \alpha B\mathbf{v}_2 =$ $F_{\mathbf{v}_1} + \alpha F_{\mathbf{v}_2}$. $\therefore V_4$ forms linear subspace of M_n . |

TABLE 6.9.1

| | |
|-------|---------------------------------------------------------------------------------------------------------------------|
| Given | \mathbf{P} and \mathbf{Q} are invertible matrices. Therefore \mathbf{P}^{-1} and \mathbf{Q}^{-1} exists. |
|-------|---------------------------------------------------------------------------------------------------------------------|

| | |
|--------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $PQ = -QP$ (6.10.1) |
| To Prove | $\text{Tr}(\mathbf{P})=0$ |
| Proof 1 | <p>Post multiplying equation (6.10.1) by \mathbf{Q}^{-1} we get,</p> $PQQ^{-1} = -QPQ^{-1} \quad (6.10.2)$ $\Rightarrow \mathbf{PI} = -QPQ^{-1} \quad (6.10.3)$ $\Rightarrow \mathbf{P} = -QPQ^{-1} \quad (6.10.4)$ <p>Taking trace on both sides for the equation (6.10.4),</p> $\text{Tr}(\mathbf{P}) = \text{Tr}(-QPQ^{-1}) \quad (6.10.5)$ $\Rightarrow \text{Tr}(\mathbf{P}) = -\text{Tr}(QPQ^{-1}) \quad (6.10.6)$ <p>We know that $\text{Tr}(\mathbf{AB})=\text{Tr}(\mathbf{BA})$ Let $\mathbf{A}=\mathbf{Q}$ and $\mathbf{B}=\mathbf{PQ}^{-1}$</p> <p>From the above property of trace equation (6.10.6) can be modified as</p> $\text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{PQ}^{-1}\mathbf{Q}) \quad (6.10.7)$ $\Rightarrow \text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{PI}) \quad (6.10.8)$ $\Rightarrow \text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{P}) \quad (6.10.9)$ $\Rightarrow 2\text{Tr}(\mathbf{P}) = 0 \quad (6.10.10)$ $\Rightarrow \text{Tr}(\mathbf{P}) = 0 \quad (6.10.11)$ |
| To Prove | $\text{Tr}(\mathbf{Q})=0$ |
| Proof 2 | <p>Post multiplying equation (6.10.1) by \mathbf{P}^{-1} we get,</p> $PQP^{-1} = -QPP^{-1} \quad (6.10.12)$ $\Rightarrow PQP^{-1} = -\mathbf{QI} \quad (6.10.13)$ $\Rightarrow PQP^{-1} = -\mathbf{Q} \quad (6.10.14)$ <p>Taking trace on both sides for the equation (6.10.14),</p> $\text{Tr}(PQP^{-1}) = \text{Tr}(-\mathbf{Q}) \quad (6.10.15)$ $\Rightarrow \text{Tr}(PQP^{-1}) = -\text{Tr}(\mathbf{Q}) \quad (6.10.16)$ <p>We know that $\text{Tr}(\mathbf{AB})=\text{Tr}(\mathbf{BA})$ Let $\mathbf{A}=\mathbf{P}$ and $\mathbf{B}=\mathbf{QP}^{-1}$</p> <p>From the above property of trace equation (6.10.16) can be modified as</p> $\text{Tr}(\mathbf{QP}^{-1}\mathbf{P}) = -\text{Tr}(\mathbf{Q}) \quad (6.10.17)$ $\Rightarrow \text{Tr}(\mathbf{QI}) = -\text{Tr}(\mathbf{Q}) \quad (6.10.18)$ $\Rightarrow \text{Tr}(\mathbf{Q}) = -\text{Tr}(\mathbf{Q}) \quad (6.10.19)$ $\Rightarrow 2\text{Tr}(\mathbf{Q}) = 0 \quad (6.10.20)$ $\Rightarrow \text{Tr}(\mathbf{Q}) = 0 \quad (6.10.21)$ |
| Statement 1 | $\text{Tr}(\mathbf{P})=\text{Tr}(\mathbf{Q})=0$ |
| Explanation | From equation (6.10.11) and (6.10.21) we could say that, |

| | |
|--------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 0 \quad (6.10.22)$ <p>Valid Conclusion</p> |
| Statement 2 | $Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 1$ |
| Explanation | <p>From equation (6.10.11) and (6.10.21) we could say that,</p> $Tr(\mathbf{P}) = Tr(\mathbf{Q}) \neq 1 \quad (6.10.23)$ <p>Invalid Conclusion</p> |
| Statement 3 | $Tr(\mathbf{P}) = -Tr(\mathbf{Q})$ |
| Explanation | <p>Substituting the conclusion 1 result equation (6.10.22) in equation (6.10.9) we get,</p> $Tr(\mathbf{P}) = -Tr(\mathbf{Q}) \quad (6.10.24)$ <p>Valid Conclusion</p> |
| Statement 4 | $Tr(\mathbf{P}) \neq Tr(\mathbf{Q})$ |
| Explanation | <p>From equation (6.10.11) and (6.10.21) we could say that,</p> $Tr(\mathbf{P}) = Tr(\mathbf{Q}) \quad (6.10.25)$ <p>Invalid Conclusion</p> |

TABLE 6.10.1: Explanation with Proofs

Let n be an odd number ≥ 7 . Let,

$$\mathbf{A} = [a_{ij}] \quad (6.10.26)$$

be an $n \times n$ matrix with,

$$a_{i,i+1} = 1, \forall (i = 1, 2, \dots, n-1) \quad (6.10.27)$$

and $a_{n,1} = 1$. Let $a_{ij} = 0$ for all the other pairs (i, j) . Then we can conclude that,

- \mathbf{A} has 1 as an eigenvalue
- \mathbf{A} has -1 as an eigenvalue
- \mathbf{A} has at least one eigenvalue with multiplicity ≥ 2
- \mathbf{A} has no real eigenvalues

Solution: We can represent our matrix as:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (6.10.28)$$

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (6.10.29)$$

\mathbf{A} is our given matrix. We know that Characteristic Equation of \mathbf{A} and \mathbf{A}^T is same. Consider the minimal polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 \quad (6.10.30)$$

We can represent it in $n \times n$ matrix with 1's on sub-diagonals and in last column it has negative of the coefficient, and rest all 0. We represent it using \mathbf{C} . It is known as the companion matrix.

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix} \quad (6.10.31)$$

(6.10.30) is also the characteristic equation of \mathbf{C}

Comparing (6.10.29) with (6.10.31) we get:

$$a_0 = -1, a_1 = a_2 = a_3 = a_4 = \dots = a_{n-1} = 0 \quad (6.10.32)$$

Substituting (6.10.32) into (6.10.30) we get:

$$x^n - 1 = 0 \quad (6.10.33)$$

By Cayley-Hamilton Theorem:

$$\lambda^n - 1 = 0 \quad (6.10.34)$$

$$(6.10.35)$$

$\lambda = n^{\text{th}}$ roots of unity. See Table 6.10.2.

- 6.11. Let $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ be 3 distinct subspaces of \mathbf{R}^{10} such that each \mathbf{W}_i has dimension of 9. Let $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$. Then we can conclude that

a) \mathbf{W} may not be a subspace of \mathbf{R}^{10}

b) $\dim \mathbf{W} \leq 8$

c) $\dim \mathbf{W} \geq 7$

d) $\dim \mathbf{W} \leq 3$

Solution: See Table 6.11.1

| Options | Explanation |
|---------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------|
| A has 1 as an eigen value | One value out of the n^{th} roots of unity is 1. So, correct |
| A has -1 as an eigen value | Since, n is odd. So, -1 cannot be one of the value of n^{th} roots of unity. Hence, incorrect |
| A has atleast one eigenvalue with multiplicity ≥ 2 | All values of n^{th} roots of unity are distinct. So there is no eigenvalue with multiplicity ≥ 2 . Hence, incorrect. |
| A has no real eigen values | One of the value is 1, which is real. Hence, incorrect. |

TABLE 6.10.2: Finding Correct Option

| | |
|-------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ are 3 distinct subspaces of \mathbf{R}^{10} Each \mathbf{W}_i has dimension 9 $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$ |
| Statement1 | \mathbf{W} may not be a subspace of \mathbf{R}^{10} |
| Explanation | As $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$ and $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ are subspaces of \mathbf{W} , then \mathbf{W} must be a subspace of \mathbf{R}^{10} . So the first option is false. |
| Statement2 | $\dim \mathbf{W} \leq 8$ |
| Explanation | As \mathbf{W} be a subspace of a finite dimension vector space \mathbf{R}^{10} and $\dim \mathbf{R}^{10} = 10$, so \mathbf{W} is finite dimension and $\dim \mathbf{W} \leq 10$ |
| Theorem | $\dim (\mathbf{W}_1 \cap \mathbf{W}_2)$ $= \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) - \dim(\mathbf{W}_1 + \mathbf{W}_2)$ and $\mathbf{W}_1 \cap \mathbf{W}_2$ is also a subspace of \mathbf{R}^{10} |
| Proof | The minimum dimension of $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$ |
| Explanation | Let us consider $\mathbf{V} = \mathbf{R}^{10}$ and $\dim(\mathbf{V}) = 10$ and $\mathbf{U} = \mathbf{W}_1 \cap \mathbf{W}_2$ |

| | |
|--------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>So, $\dim(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) = \dim(\mathbf{U}) + \dim(\mathbf{W}_3) - \dim(\mathbf{U} + \mathbf{W}_3)$</p> <p>or, $\dim(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) = \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - \dim(\mathbf{W}_1 + \mathbf{W}_2) - \dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3)$</p> |
| | <p>Now, $(\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3 \subseteq \mathbf{V}$ $\Rightarrow \dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) \leq \dim(\mathbf{V})$ $\Rightarrow -\dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) \geq -\dim(\mathbf{V})$</p> <p>Similarly, $(\mathbf{W}_1 + \mathbf{W}_2) \subseteq \mathbf{V}$ $\Rightarrow \dim(\mathbf{W}_1 + \mathbf{W}_2) \leq \dim(\mathbf{V})$ $\Rightarrow -\dim(\mathbf{W}_1 + \mathbf{W}_2) \geq -\dim(\mathbf{V})$</p> |
| | <p>Considering these two inequations, $-\dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) - \dim(\mathbf{W}_1 + \mathbf{W}_2) \geq -2\dim(\mathbf{V})$</p> <p>or, $\dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - \dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) - \dim(\mathbf{W}_1 + \mathbf{W}_2) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})$</p> <p>or, $\dim(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})$</p> <p>$\Rightarrow \dim(\mathbf{W}) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})$</p> |
| Statement 3 | $\dim \mathbf{W} \geq 7$ |
| Explanation | <p>As $\dim(\mathbf{W}) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})$ $\Rightarrow \dim(\mathbf{W}) \geq (9+9+9) - (2 \times 10)$ $\Rightarrow \dim(\mathbf{W}) \geq 7$</p> |
| Answer | $7 \leq \dim(\mathbf{W}) \leq 10$ |

TABLE 6.11.1: Solution summary

Hence, we can conclude that $\dim(\mathbf{W}) \geq 7$.

7 JUNE 2016

8 DECEMBER 2015

8.1. Let \mathbf{V} be a finite dimensional vector space over \mathbb{R} . Let $T : \mathbf{V} \rightarrow \mathbf{V}$ be a linear transformation such that $\text{rank}(\mathbf{T}^2) = \text{rank}(\mathbf{T})$. Then,

- a) $\text{Kernel}(\mathbf{T}^2) = \text{Kernel}(\mathbf{T})$
- b) $\text{Range}(\mathbf{T}^2) = \text{Range}(\mathbf{T})$
- c) $\text{Kernel}(\mathbf{T}) \cap \text{Range}(\mathbf{T}) = \{0\}$.
- d) $\text{Kernel}(\mathbf{T}^2) \cap \text{Range}(\mathbf{T}^2) = \{0\}$.

Solution: See Tables 8.1.1, 8.1.2, 8.1.3 and 8.1.4

| | |
|---------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $Range(\mathbf{T})$ | <p>It is column-space of linear operator \mathbf{T}.</p> $\mathbf{T}(\mathbf{x}) = \mathbf{v} \implies \mathbf{Ax} = \mathbf{v} \quad (8.1.1)$ <p>where $\mathbf{x}, \mathbf{v} \in \mathbf{V}$ and We can also say that</p> $Range(\mathbf{T}) = C(\mathbf{A}) \quad (8.1.2)$ <p>where $C(\mathbf{A})$ is column space of \mathbf{A}.</p> |
| $Kernel(\mathbf{T})$ | <p>It is null-space of linear operator \mathbf{T}.</p> $\mathbf{T}(\mathbf{x}) = 0 \implies \mathbf{Ax} = 0 \quad (8.1.3)$ <p>where $\mathbf{x} \in \mathbf{V}$ and matrix \mathbf{A} is same as before. We can also say that</p> $Kernel(\mathbf{T}) = N(\mathbf{A}) \quad (8.1.4)$ <p>where $N(\mathbf{A})$ is null space of \mathbf{A}.</p> |
| $rank(\mathbf{T})$ | $rank(\mathbf{T}) = rank(\mathbf{A}) \quad (8.1.5)$ |
| \mathbf{T}^2 | $\mathbf{T}^2(\mathbf{x}) = \mathbf{A}^2\mathbf{x} \quad \mathbf{x} \in \mathbf{V} \quad (8.1.6)$ $rank(\mathbf{T}^2) = rank(\mathbf{A}^2) \quad (8.1.7)$ |
| \mathbf{A} and \mathbf{A}^2 | <p>The basis vectors of column-space of \mathbf{A} and \mathbf{A}^2 are same. The basis vectors of null-space of \mathbf{A} and \mathbf{A}^2 are same.</p> |

TABLE 8.1.1: Definitions and theorem used

| Statement | Observations |
|-----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | <p>\mathbf{V} is a finite dimensional space over \mathbb{R} and $T : \mathbf{V} \rightarrow \mathbf{V}$</p> $rank(\mathbf{T}) = rank(\mathbf{T}^2) \quad (8.1.8)$ <p>According to rank-nullity theorem.</p> $dim(\mathbf{V}) = rank(\mathbf{T}) + nullity(\mathbf{T}) \quad (8.1.9)$ $dim(\mathbf{V}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (8.1.10)$ <p>from (8.1.9) and (8.1.10). we get</p> $\implies rank(\mathbf{T}) + nullity(\mathbf{T}) = rank(\mathbf{T}^2) + nullity(\mathbf{T}^2) \quad (8.1.11)$ $\implies nullity(\mathbf{T}) = nullity(\mathbf{T}^2) \quad (8.1.12)$ |

TABLE 8.1.2: Observations

| Option | Solution | True/False |
|--------|--------------------------------------------------------------------------------------------|------------|
| 1 | <p>From (8.1.12), let</p> $nullity(\mathbf{T}) = nullity(\mathbf{T}^2) = n \quad (8.1.13)$ | |

| | | |
|---|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------|
| | <p>Therefore, from table 8.1.1 and (8.1.13) we can say that both null space of linear operator \mathbf{T} and null space of linear operator \mathbf{T}^2 will have same n number of basis.</p> $\implies \text{Kernel}(\mathbf{T}) = \text{Kernel}(\mathbf{T}^2) \quad (8.1.14)$ | True |
| 2 | <p>From (8.1.8), let</p> $\text{rank}(\mathbf{T}) = \text{rank}(\mathbf{T}^2) = r \quad (8.1.15)$ <p>Therefore, from table 8.1.1 and (8.1.15) we can say that both column space of linear operator \mathbf{T} and column space of linear operator \mathbf{T}^2 will have same r number of basis.</p> $\implies \text{Range}(\mathbf{T}) = \text{Range}(\mathbf{T}^2) \quad (8.1.16)$ | True |
| 3 | <p>From (8.1.13), (8.1.15) and also we can say that column space $C(\mathbf{A})$ and null space $N(\mathbf{A})$ are r-dimensional space and n-dimensional space respectively which will intersect only at origin(zero vector). And also from (8.1.2) and (8.1.4), we get</p> $\implies \text{Kernel}(\mathbf{T}) \cap \text{Range}(\mathbf{T}) = \{0\} \quad (8.1.17)$ | True |
| 4 | <p>From table (8.1.14), (8.1.16) and (8.1.17), we get</p> $\implies \text{Kernel}(\mathbf{T}^2) \cap \text{Range}(\mathbf{T}^2) = \{0\} \quad (8.1.18)$ | True |

TABLE 8.1.3: Solution

| Statement | Calculations and observations |
|------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>Consider vector space $\mathbf{V} = \mathbb{R}^3$</p> <p>Let matrix \mathbf{A} be</p> | $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \quad (8.1.19)$ |
| \mathbf{A}^2 | $\mathbf{A}^2 = \begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \quad (8.1.20)$ |
| Convert both \mathbf{A} and \mathbf{A}^2 to Row Reduced echelon form | <p>For matrix \mathbf{A},</p> $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix} \xrightarrow[R_1 \leftarrow R_1 - 2R_2]{R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix} \quad (8.1.21)$ $\xrightarrow{R_3 \leftarrow R_3 - 5R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (8.1.22)$ |

| | |
|---------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>For matrix \mathbf{A}^2,</p> $\begin{pmatrix} 0 & 7 & 7 \\ -1 & 4 & 5 \\ -5 & 13 & 18 \end{pmatrix} \xleftrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ -5 & 13 & 18 \end{pmatrix} \quad (8.1.23)$ $\xleftrightarrow{R3 \leftarrow R3 - 5R1} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ 0 & -7 & -7 \end{pmatrix} \xleftrightarrow{R3 \leftarrow R3 + R1} \begin{pmatrix} -1 & 4 & 5 \\ 0 & 7 & 7 \\ 0 & 0 & 0 \end{pmatrix} \quad (8.1.24)$ $\xleftrightarrow{\substack{R2 \leftarrow \frac{R2}{7} \\ R1 \leftarrow -R1}} \begin{pmatrix} 1 & -4 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xleftrightarrow{R1 \leftarrow R1 + 4R2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (8.1.25)$ |
| $Range(\mathbf{T}) = Range(\mathbf{T}^2)$ | <p>Therefore, from (8.1.22) and (8.1.25) we can say that the basis vectors of $Range(\mathbf{T})$ and $Range(\mathbf{T}^2)$ are same as shown below</p> $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (8.1.26)$ <p>and also we can say</p> $Range(\mathbf{T}) = Range(\mathbf{T}^2) \quad (8.1.27)$ |
| $Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2)$ | <p>Lets find the basis for null-space of linear operator \mathbf{T} or $N(\mathbf{A})$. It is the solution of the equation $\mathbf{A}\mathbf{x} = 0$. From (8.1.22) we have,</p> $\mathbf{A}\mathbf{x} = 0 \quad (8.1.28)$ $\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (8.1.29)$ <p>Setting the value of the free variable $x_3 = 1$ we get the solution,</p> $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (8.1.30)$ <p>Hence, the basis vector of the $Kernel(\mathbf{T})$ is given by,</p> $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (8.1.31)$ <p>Now, lets find the basis for null-space of linear operator \mathbf{T}^2 or $N(\mathbf{A}^2)$. It is the solution of the equation $\mathbf{A}^2\mathbf{x} = 0$. From (8.1.25) we have,</p> $\mathbf{A}^2\mathbf{x} = 0 \quad (8.1.32)$ $\Rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad (8.1.33)$ <p>Setting the value of the free variable $x_3 = 1$ we get the solution,</p> |

| | |
|---------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (8.1.34)$ <p>Hence, from (8.1.31) and (8.1.34) we got the basis vector of $Kernel(\mathbf{T}^2)$ same as the basis vector of $Kernel(\mathbf{T})$ which is \mathbf{p}. Therefore, we can say that</p> $Kernel(\mathbf{T}) = Kernel(\mathbf{T}^2) \quad (8.1.35)$ |
| $Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\}$ | <p>From (8.1.26) and (8.1.31), we got 2 basis vectors $\mathbf{b}_1, \mathbf{b}_2$ for $Range(\mathbf{T})$ and 1 basis vector \mathbf{p} for $Kernel(\mathbf{T})$. Here $\mathbf{b}_1, \mathbf{b}_2, \mathbf{p}$ are linearly independent which can be proven as below. Let columns of matrix \mathbf{M} are filled with vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{p}$.</p> $\Rightarrow \mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad (8.1.36)$ <p>From (8.1.36), we get $rank(\mathbf{M}) = 3$. Therefore $\mathbf{b}_1, \mathbf{b}_2, \mathbf{p}$ are linearly independent $Range(\mathbf{T})$ is a 2-dimensional space which is a plane in \mathbb{R}^3 and $Kernel(\mathbf{T})$ is a 1-dimensional space which is a line in \mathbb{R}^3. Since $\mathbf{b}_1, \mathbf{b}_2, \mathbf{p}$ are linearly independent then plane and line intersect at origin (zero vector). And we can say that</p> $Kernel(\mathbf{T}) \cap Range(\mathbf{T}) = \{0\} \quad (8.1.37)$ |
| $Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\}$ | <p>From (8.1.27), (8.1.35), (8.1.37) we get</p> $\Rightarrow Kernel(\mathbf{T}^2) \cap Range(\mathbf{T}^2) = \{0\} \quad (8.1.38)$ |

TABLE 8.1.4: Example

8.2. Let \mathbf{A} be an $m \times n$ real matrix and $\mathbf{b} \in \mathbb{R}^m$ with $b \neq 0$.

- a) The set of all real solutions of $\mathbf{Ax} = \mathbf{b}$ is a vector space.
- b) If u and v are two solutions of $\mathbf{Ax} = \mathbf{b}$ then $\lambda u + (1 - \lambda)v$ is also a solution of $\mathbf{Ax} = \mathbf{b}$
- c) For any two solutions u and v of $\mathbf{Ax} = \mathbf{b}$, the linear combination $\lambda u + (1 - \lambda)v$ is also a solution of $\mathbf{Ax} = \mathbf{b}$ only when $0 \leq \lambda \leq 1$.
- d) If rank of \mathbf{A} is n , then $\mathbf{Ax} = \mathbf{b}$ has at most one solution.

Solution: See Table 8.2.1

| | |
|----------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Option 1 | <p>Suppose \mathbb{V} is the vector space defined as $\mathbb{V} = \{\mathbf{x} : \mathbf{Ax} = \mathbf{b}, \mathbb{R}^n \rightarrow \mathbb{R}^m\}$</p> <p>$\mathbf{v}$ and \mathbf{u} are the solution to the equation $\mathbf{Ax} = \mathbf{b}$ such that \mathbf{u} and $\mathbf{v} \in \mathbb{V}$</p> <p>$\mathbf{Au} = \mathbf{b} \quad \mathbf{Av} = \mathbf{b}$</p> <p>Checking Closure under vector addition</p> <p>$\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$</p> <p>Which is enclosed under vector addition if and only if $\mathbf{b} = \mathbf{0}$. But here given $\mathbf{b} \neq \mathbf{0}$ means $\mathbf{0} \notin \mathbb{V}$</p> <p>Hence does not satisfy requirements of vector space.</p> <p>Hence option 1 is incorrect.</p> |
| Option 2 | <p>Proof 1:</p> <p>If \mathbf{u} and \mathbf{v} are the two solution of $\mathbf{Ax} = \mathbf{b}$</p> <p>$\mathbf{Au} = \mathbf{b} \quad \mathbf{Av} = \mathbf{b}$</p> <p>For $\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$ to be a solution of $\mathbf{Ax} = \mathbf{b}$, it must satisfy this equation.</p> <p>$\mathbf{A}(\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}) = \mathbf{b} \implies \mathbf{A}\lambda\mathbf{u} + \mathbf{A}(1 - \lambda)\mathbf{v} = \mathbf{b} \implies \mathbf{A}\lambda\mathbf{u} + \mathbf{Av} - \mathbf{A}\lambda\mathbf{v} = \mathbf{b}$</p> <p>$\mathbf{b}\lambda + \mathbf{Av} - \mathbf{b}\lambda = \mathbf{b} \implies \mathbf{Av} = \mathbf{b}$</p> <p>Which satisfies the equation therefore $\lambda\mathbf{u} + (1 - \lambda)\mathbf{v}$ is the solution of $\mathbf{Ax} = \mathbf{b}$ for any λ</p> <p>Since the λ term cancels out therefore valid for $\lambda \in \mathbb{R}$.</p> <p>Proof 2 (Through affine Subspace with an Example):-</p> <p>Let us suppose the two solution \mathbf{u} and \mathbf{v} be the points on the line given by the equation $\mathbf{Ax} = \mathbf{b}$</p> <p>Let the Line joining these two points is given as</p> <p>$\mathbf{l} = \mathbf{u} - \mathbf{v}$ is line parallel to the given line $\mathbf{Ax} = \mathbf{b}$</p> <p>Therefore \mathbf{v} belongs to solution set and is independent to other linearly independent vectors of \mathbf{l}</p> <p>$\mathbf{x} = \mathbf{v} + \lambda\mathbf{l}$ for $\lambda \in \mathbb{R}$ on substituting \mathbf{l}</p> <p>$\mathbf{x} = \mathbf{v} + \lambda(\mathbf{u} - \mathbf{v}) = \mathbf{v} + \lambda\mathbf{u} - \lambda\mathbf{v} = \mathbf{v}(1 - \lambda) + \lambda\mathbf{u}$</p> <p>Hence $\mathbf{v}(1 - \lambda) + \lambda\mathbf{u}$ is also the solution of the equation $\mathbf{Ax} = \mathbf{b}$ for $\lambda \in \mathbb{R}$.</p> |

| | |
|----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | Hence Option 2 is correct. |
| Option 3 | <p>Since in Option 2 we have proved that $\mathbf{v}(1 - \lambda) + \lambda\mathbf{u}$ is a solution for $\mathbf{Ax} = \mathbf{b}$ for any $\lambda \in \mathbb{R}$ therefore λ can be any real value but in option 3 there is restriction on λ which is incorrect.</p> <p>Hence option 3 is incorrect</p> |
| Option 4 | <p>$\mathbf{A}_{m \times n} \mathbf{x}_{n \times 1} = \mathbf{b}_{m \times 1}$</p> <p>If \mathbf{A} has Full column rank(\mathbf{A}) = n then there exist one pivot in each columns and there exists no free variables thus $\mathbf{N}(\mathbf{A}) = \mathbf{0}$ so the only solution to $\mathbf{Ax} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.</p> <p>So the solution to $\mathbf{Ax} = \mathbf{b}$</p> <p>$\mathbf{x} = \mathbf{x}_p$ unique solution exists if it exist. It can be either 0 or 1.</p> <p>Hence at most 1 solution is possible .</p> <p>Proof with example</p> <p>Let $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{pmatrix}_{4 \times 2} \xleftrightarrow{RREF} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ Hence $n = 2$ pivot columns at both column position</p> <p>$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ Hence no solution possible as no combination of \mathbf{x} can gives the solution except</p> <p>$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only if $\mathbf{b} = \mathbf{0} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ OR</p> <p>$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ only if \mathbf{b} is addition of columns of $\mathbf{A} \Rightarrow \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 7 \\ 6 \end{pmatrix}$</p> <p>Hence either no solution possible or one solution possible. Therefore we say at most one solution possible.</p> <p>Option 4 is correct.</p> |

| | |
|---------|-----------------------------------|
| Answers | Option 2 and Option 4 are correct |
|---------|-----------------------------------|

TABLE 8.2.1: Solution

- 8.3. Let \mathbf{A} be an $n \times n$ matrix over \mathbb{C} such that every non-zero vector \mathbb{C}^n is an eigen vector of \mathbf{A} . Then
- a) All eigen values of \mathbf{A} are equal.
 - b) All eigen values of \mathbf{A} are distinct.
 - c) $\mathbf{A} = \lambda \mathbf{I}$ for some $\lambda \in \mathbb{C}$, where \mathbf{I} is the $n \times n$ identity matrix.
 - d) If $\chi_{\mathbf{A}}$ and $m_{\mathbf{A}}$ denote the characteristic polynomial and the minimal polynomial respectively, then $\chi_{\mathbf{A}} = m_{\mathbf{A}}$

Solution: See Tables 8.3.1 , 8.3.2 and 8.3.3

| | |
|--------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | Every non-zero vector \mathbb{C}^n is an eigen vector of \mathbf{A} , where \mathbf{A} is an $n \times n$ matrix over \mathbb{C} . |
| Determining \mathbf{A} | <p>Since every vector is an eigen vector, the standard basis vectors are also eigen vectors</p> $\Rightarrow \mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i \Rightarrow \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \mathbf{e}_i = \lambda_i \mathbf{e}_i \Rightarrow a_i = \lambda_i \mathbf{e}_i \text{ where } \lambda_i \in \mathbb{C}$ <p>therefore $\mathbf{A} = (\lambda_1 \mathbf{e}_1 \quad \lambda_2 \mathbf{e}_2 \quad \dots \quad \lambda_n \mathbf{e}_n)$</p> <p>Any vector \mathbf{b} can be represented in the standard basis as</p> $\mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n \text{ where } b_i \in \mathbb{C}$ <p>As every non-zero vector in \mathbb{C}^n is an eigen vector</p> $\mathbf{A}\mathbf{b} = \lambda \mathbf{b} \Rightarrow \mathbf{A}(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n) = \lambda(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n)$ $\Rightarrow b_1 \lambda_1 \mathbf{e}_1 + b_2 \lambda_2 \mathbf{e}_2 + \dots + b_n \lambda_n \mathbf{e}_n = \lambda(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n)$ $\Rightarrow b_1(\lambda_1 - \lambda) \mathbf{e}_1 + b_2(\lambda_2 - \lambda) \mathbf{e}_2 + \dots + b_n(\lambda_n - \lambda) \mathbf{e}_n = 0$ <p>since basis are linearly independent we get $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$</p> <p>Therefore the matrix \mathbf{A} is</p> $\mathbf{A} = (\lambda_1 \mathbf{e}_1 \quad \lambda_2 \mathbf{e}_2 \quad \dots \quad \lambda_n \mathbf{e}_n) = \lambda(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \dots \quad \mathbf{e}_n) = \lambda \mathbf{I}_n \text{ where } \lambda \in \mathbb{C}$ |

TABLE 8.3.1

| | |
|----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| option 1 | Since $\mathbf{A} = \lambda \mathbf{I}_n$, all the eigen values are equal to λ . Therefore option 1 is correct as the matrix \mathbf{A} is a scalar matrix. |
| option 2 | since the matrix \mathbf{A} is a scalar matrix, all the eigen values are equal. So this option is incorrect. |
| option 3 | This option is correct. As proved in the construction the matrix $\mathbf{A} = \lambda \mathbf{I}$ for some $\lambda \in \mathbb{C}$ |
| option 4 | Since $\mathbf{A} = \lambda \mathbf{I}$ where $\lambda \in \mathbb{C}$, the characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x - \lambda)^n$ and $m_{\mathbf{A}} = (x - \lambda) \Rightarrow \chi_{\mathbf{A}} = m_{\mathbf{A}}^n$. Therefore this option is incorrect |

TABLE 8.3.2: Answer

| | |
|-----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Scalar matrix | <p>Consider a 3×3 scalar matrix $\mathbf{A} = (2 + 3i) \mathbf{I}$, for which the eigen values are $(2 + 3i), (2 + 3i), (2 + 3i)$</p> <p>The eigen vectors will be the nullspace of $\mathbf{A} - \lambda \mathbf{I}$</p> $\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} 2 + 3i & 0 & 0 \\ 0 & 2 + 3i & 0 \\ 0 & 0 & 2 + 3i \end{pmatrix} - (2 + 3i) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ <p>The nullspace consists of the entire vector space so every vector is an eigen vector</p> <p>The characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x - (2 + 3i))^3$ and $m_{\mathbf{A}} = (x - (2 + 3i)) \Rightarrow \chi_{\mathbf{A}} = m_{\mathbf{A}}^3$</p> <p>Therefore options 1 and 3 are correct.</p> |
| Diagonal matrix | <p>Consider the matrix \mathbf{A} as</p> $\mathbf{A} = \begin{pmatrix} 2 + 3i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3i \end{pmatrix}$ <p>The eigen values are $\lambda_1 = 2 + 3i, \lambda_2 = 2, \lambda_3 = 3i$</p> |
| | <p>The eigen vector with respect to $\lambda_1 = 2 + 3i$ will be the nullspace of $\mathbf{A} - \lambda_1 \mathbf{I}$</p> $\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3i & 0 \\ 0 & 0 & -2 \end{pmatrix}, \text{ so the eigen vector will be } \mathbf{e}_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } x_1 \in \mathbb{C}$ <p>The eigen vector with respect to $\lambda_2 = 2$ will be the nullspace of $\mathbf{A} - \lambda_2 \mathbf{I}$</p> $\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} 3i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3i - 2 \end{pmatrix}, \text{ so the eigen vector will be } \mathbf{e}_2 = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ where } x_2 \in \mathbb{C}$ |

The eigen vector with respect to $\lambda_3 = 3i$ will be the nullspace of $\mathbf{A} - \lambda_3 \mathbf{I}$

$\mathbf{A} - \lambda_3 \mathbf{I} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 3i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so the eigen vector will be $\mathbf{e}_3 = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ where $x_3 \in \mathbb{C}$

Consider the vector $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ where $x_1 = x_2 = x_3 = 1$

$$\mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{e}_1 + \mathbf{A}\mathbf{e}_2 + \mathbf{A}\mathbf{e}_3 = (2 + 3i)\mathbf{e}_1 + 2\mathbf{e}_2 + 3i\mathbf{e}_3 = \begin{pmatrix} 2 + 3i \\ 2 \\ 3i \end{pmatrix}$$

As $\mathbf{A}\mathbf{y}$ can not be written as $c\mathbf{y}$ where $c \in \mathbb{C}$, \mathbf{y} is not an eigen vector which is a contradiction.

TABLE 8.3.3: Examples

8.4. Consider a matrix,

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} \quad (8.4.1)$$

and,

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (8.4.2)$$

Then which of following is true,

- \mathbf{A} and \mathbf{B} is similar over the field of rational numbers.
- \mathbf{A} is diagonalizable over the field of rational numbers \mathbb{Q} .
- \mathbf{B} is the Jordan canonical form of \mathbf{A} .
- The minimal polynomial and the characteristic polynomial of \mathbf{A} are the same.

Solution: Two matrix are said to be similar if their eigen values are same.

Eigen value of \mathbf{A} is given as:

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \quad (8.4.3)$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \quad (8.4.4)$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3. \quad (8.4.5)$$

Similarly, eigen values of \mathbf{B} is given as:

$$\begin{vmatrix} 2-\lambda & 10 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} \quad (8.4.6)$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \quad (8.4.7)$$

$$\Rightarrow \lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 3. \quad (8.4.8)$$

Hence, matrices \mathbf{A} and \mathbf{B} are similar. Matrix \mathbf{A} is diagonalizable if and only if there is a basis of \mathbb{R}^3 consisting of eigenvectors of \mathbf{A} .

From (8.4.5), our eigenvalues for \mathbf{A} are,

$$\lambda_1 = \lambda_2 = 2 \quad (8.4.9)$$

and,

$$\lambda_3 = 3. \quad (8.4.10)$$

Hence $\lambda_1 = \lambda_2$ is a repeated root with multiplicity two. Hence, We can get only two linearly

independent eigenvectors for \mathbf{A} , are given as :

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad (8.4.11)$$

But any basis for \mathbb{R}^3 consists of three vectors. Therefore there is no third eigenbasis for \mathbf{A} , hence \mathbf{A} is not diagonalizable. From (8.4.5) we have eigenvalue $\lambda_1 = 2$ with geometric multiplicity 2. Hence the Jordan canonical form of \mathbf{A} can be written as :

$$\mathbf{J}_\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (8.4.12)$$

Hence \mathbf{B} is the Jordan canonical form of \mathbf{A} . From (8.4.5), the characteristic polynomial of this matrix is:

$$f(\lambda) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (\lambda - 2)^2(\lambda - 3) \quad (8.4.13)$$

Minimal polynomial for a matrix is a smallest polynomial for which

$$M_\mathbf{A}(x) = 0 \quad (8.4.14)$$

Using (8.4.14), we found minimal polynomial of \mathbf{A} is :

$$M_\mathbf{A}(x) = (x - 2)^2(x - 3) \quad (8.4.15)$$

We can relate the minimal polynomial with the size of Jordan block.

Size of Jordan block = degree of minimal polynomial with geometric multiplicity of the eigen values.

From (8.4.15) we can observe that, geometric multiplicity of eigen value 2 is 2. Hence size of Jordan block is 2. which is given as:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad (8.4.16)$$

if geometric multiplicity of $\lambda = 2$ would be 3, then Jordan block would be:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad (8.4.17)$$

In (8.4.15) geometric multiplicity of eigen

value 2 is 2, and geometric multiplicity of eigen value 3 is one hence jardon block is:

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (8.4.18)$$

Hence, matrix of $\mathbf{T} - \mathbf{S}$ with respect to \mathbf{B} can be represented as

$$[\mathbf{T} - \mathbf{S}]_B = \begin{pmatrix} \lambda_1 - \alpha & 0 & \dots & 0 \\ 0 & \lambda_2 - \alpha & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \lambda_n - \alpha \end{pmatrix} \quad (9.1.7)$$

9 JUNE 2015

9.1. Let $\mathbf{S} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be given by $\mathbf{S}(\mathbf{v}) = \alpha\mathbf{v}$, for a fixed $\alpha \in \mathbb{R}, \alpha \neq 0$. Let $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation such that $\mathbf{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of linearly independent eigenvectors of \mathbf{T} . Then

- The matrix of \mathbf{T} with respect to \mathbf{B} is diagonal
- The matrix of $(\mathbf{T} - \mathbf{S})$ with respect to \mathbf{B} is diagonal
- The matrix of \mathbf{T} with respect to \mathbf{B} is not necessarily diagonal, but is upper triangular
- The matrix of \mathbf{T} with respect to \mathbf{B} is diagonal but the matrix of $(\mathbf{T} - \mathbf{S})$ with respect to \mathbf{B} is not diagonal.

Solution: Given that $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and \mathbf{B} represents a set of linearly independent eigenvectors of \mathbf{T} given as follows

$$\mathbf{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \quad (9.1.1)$$

So,

$$\mathbf{T}(\mathbf{v}_i) = \mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad (9.1.2)$$

where λ_i represents the eigenvalue corresponding to \mathbf{v}_i . Hence, the matrix \mathbf{T} with respect to \mathbf{B} can be represented as

$$[\mathbf{T}]_B = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \quad (9.1.3)$$

And,

$$(\mathbf{T} - \mathbf{S})\mathbf{v}_i = \mathbf{T}(\mathbf{v}_i) - \mathbf{S}(\mathbf{v}_i) \quad (9.1.4)$$

$$= \lambda_i\mathbf{v}_i - \alpha\mathbf{v}_i \quad (9.1.5)$$

$$= (\lambda_i - \alpha)\mathbf{v}_i \quad (9.1.6)$$

| | |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------|
| 1. The matrix of \mathbf{T} w.r.t to \mathbf{B} is diagonal | True, as seen from (9.1.3) |
| 2. The matrix of $(\mathbf{T} - \mathbf{S})$ w.r.t \mathbf{B} is diagonal | True, as seen from (9.1.7) |
| 3. The matrix of \mathbf{T} with respect to \mathbf{B} is not necessarily diagonal but is upper triangular | False, as already proved $[\mathbf{T}]_B$ is diagonal |
| 4. The matrix of \mathbf{T} with respect to \mathbf{B} is diagonal but the matrix of $(\mathbf{T} - \mathbf{S})$ with respect to \mathbf{B} is not diagonal | False, as already proved $[\mathbf{T} - \mathbf{S}]_B$ is diagonal |

TABLE 9.1.1: Verifying the given options

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where

$$\mathbf{T}(x) = \mathbf{A}\mathbf{x} = \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (9.1.8)$$

Here, the eigenvalues of the above transformation matrix are $\lambda_1 = 3, \lambda_2 = -2$. And the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Thus,

$$\mathbf{B} = \{\mathbf{v}_1, \mathbf{v}_2\} \quad (9.1.9)$$

Now,

$$\mathbf{T}(\mathbf{v}_1) = \mathbf{A}\mathbf{v}_1 \quad (9.1.10)$$

$$= \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (9.1.11)$$

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \quad (9.1.12)$$

$$= 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (9.1.13)$$

$$= \lambda_1 \mathbf{v}_1 \quad (9.1.14)$$

And,

$$\mathbf{T}(\mathbf{v}_2) = \mathbf{A}\mathbf{v}_2 \quad (9.1.15)$$

$$= \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (9.1.16)$$

$$= \begin{pmatrix} -2 \\ -6 \end{pmatrix} \quad (9.1.17)$$

$$= -2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad (9.1.18)$$

$$= \lambda_2 \mathbf{v}_2 \quad (9.1.19)$$

$$(9.1.20)$$

For any vector $\mathbf{v} \in \mathbb{R}^2, \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$

$$[\mathbf{v}]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (9.1.21)$$

$$\mathbf{T}(\mathbf{v}) = \mathbf{T}(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) \quad (9.1.22)$$

$$= c_1 \mathbf{T}(\mathbf{v}_1) + c_2 \mathbf{T}(\mathbf{v}_2) \quad (9.1.23)$$

$$= c_1 \lambda_1 \mathbf{v}_1 + c_2 \lambda_2 \mathbf{v}_2 \quad (9.1.24)$$

$$[\mathbf{T}(\mathbf{v})]_B = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \end{pmatrix} \quad (9.1.25)$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (9.1.26)$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} [\mathbf{v}]_B \quad (9.1.27)$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} [\mathbf{v}]_B \quad (9.1.28)$$

$$\mathbf{S}(\mathbf{v}) = \alpha \mathbf{v}, \alpha \neq 0 \quad (9.1.29)$$

$$= \alpha(c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2) \quad (9.1.30)$$

$$= \alpha c_1 \mathbf{v}_1 + \alpha c_2 \mathbf{v}_2 \quad (9.1.31)$$

$$[\mathbf{S}(\mathbf{v})]_B = \begin{pmatrix} \alpha c_1 \\ \alpha c_2 \end{pmatrix} \quad (9.1.32)$$

$$[(\mathbf{T} - \mathbf{S})(\mathbf{v})]_B = \begin{pmatrix} \lambda_1 c_1 - \alpha c_1 \\ \lambda_2 c_2 - \alpha c_2 \end{pmatrix} \quad (9.1.33)$$

$$= \begin{pmatrix} \lambda_1 - \alpha & 0 \\ 0 & \lambda_2 - \alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad (9.1.34)$$

$$= \begin{pmatrix} \lambda_1 - \alpha & 0 \\ 0 & \lambda_2 - \alpha \end{pmatrix} [\mathbf{v}]_B \quad (9.1.35)$$

$$= \begin{pmatrix} 3 - \alpha & 0 \\ 0 & -2 - \alpha \end{pmatrix} [\mathbf{v}]_B \quad (9.1.36)$$

Hence, shown from (9.1.28) and (9.1.36) that the matrix of \mathbf{T} and of $\mathbf{T} - \mathbf{S}$ w.r.t to \mathbf{B} is diagonal.

9.2. Let \mathbf{A} be an invertible 4×4 real matrix. Which of the following are NOT true ?

- a) Rank $\mathbf{A} = 4$
- b) For every vector $\mathbf{b} \in \mathbb{R}$, $\mathbf{A}\mathbf{x} = \mathbf{b}$ has exactly one solution.
- c) $\dim(\text{nullspace } \mathbf{A}) \geq 1$
- d) 0 is an eigenvalue of \mathbf{A}

Solution: See Table 9.2.1

| | |
|-------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Given | A is an invertible real matrix of order 4×4 |
| Solution | <p>Since given A is an invertible matrix, A has full rank.</p> $\det(\mathbf{A}) \neq 0 \quad (9.2.1)$ $\text{Rank}(\mathbf{A}) = 4 \quad (9.2.2)$ <p>Let $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the eigenvalues of matrix A. We know that determinant of matrix A is the product of eigenvalues of A.</p> $\lambda_1 \lambda_2 \lambda_3 \lambda_4 \neq 0 \quad (9.2.3)$ |
| Statement 1 | $\text{Rank}(\mathbf{A}) = 4$ |
| | <p>Since A is an invertible matrix, it has full rank as shown in equation (9.2.2). True Statement</p> |
| Statement 2 | For every vector $\mathbf{b} \in \mathbb{R}$, $\mathbf{Ax} = \mathbf{b}$ has exactly one solution. |
| | <p>For every \mathbf{b},</p> $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ <p>\mathbf{x} will be unique solution for every \mathbf{b}. True Statement</p> |
| Statement 3 | $\dim(\text{nullspace } \mathbf{A}) \geq 1$. |
| | <p>Using Rank Nullity Theorem,</p> $\begin{aligned} \text{Rank}(\mathbf{A}) + \dim(\text{nullspace } \mathbf{A}) &= n \\ \implies 4 + \dim(\text{nullspace } \mathbf{A}) &= 4 \\ \implies \dim(\text{nullspace } \mathbf{A}) &= 0 \not\geq 1 \end{aligned} \quad (9.2.4)$ |
| | <p>where n is the number of columns in A Equation (9.2.4) proves that the given statement is NOT True.</p> |
| Statement 4 | 0 is an eigenvalue of A |
| | <p>From equation (9.2.1), we could say that no eigenvalue of A could be 0. NOT True Statement</p> |

TABLE 9.2.1: Explanation

9.3. Consider non-zero vector spaces $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$ and linear transformations $\phi_1 : \mathbf{V}_1 \rightarrow \mathbf{V}_2$, $\phi_2 : \mathbf{V}_2 \rightarrow \mathbf{V}_3$, $\phi_3 : \mathbf{V}_3 \rightarrow \mathbf{V}_4$ such that $\text{Ker}(\phi_1) = \{0\}$, $\text{Range}(\phi_1) = \text{Ker}(\phi_2)$, $\text{Range}(\phi_2) = \text{Ker}(\phi_3)$, $\text{Range}(\phi_3) = \mathbf{V}_4$. Then

a) $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i = 0$

b) $\sum_{i=2}^4 (-1)^i \dim \mathbf{V}_i > 0$

c) $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i < 0$

d) $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i \neq 0$

Solution: See Table 9.3.1 9.3.3

| | |
|----------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Kernel and Nullity | <p>Given a linear transformation $L : \mathbf{V} \rightarrow \mathbf{W}$ between two vector spaces \mathbf{V} and \mathbf{W}, the kernel of L is the set of all vectors \mathbf{v} of \mathbf{V} for which $L(\mathbf{v}) = \mathbf{0}$, where $\mathbf{0}$ denotes the zero vector in \mathbf{W}. i.e.</p> $\text{Ker}(L) = \{\mathbf{v} \in \mathbf{V} \mid L(\mathbf{v}) = \mathbf{0}\}$ <p>Nullity of the linear transformation is the dimension of the kernel of the linear transformation i.e.</p> $\text{nullity}(L) = \dim(\text{Ker}(L))$ |
| Range and Rank | <p>Given a linear transformation $L : \mathbf{V} \rightarrow \mathbf{W}$ between two vector spaces \mathbf{V} and \mathbf{W}, the range of L is the set of all vectors \mathbf{w} in \mathbf{W} given as</p> $\text{Range}(L) = \{\mathbf{w} \in \mathbf{W} \mid \mathbf{w} = L(\mathbf{v}), \mathbf{v} \in \mathbf{V}\}$ <p>The rank of a linear transformation L is the dimension of its range, i.e.</p> $\text{rank}(L) = \dim(\text{Range}(L))$ |
| Rank-Nullity Theorem | <p>Let \mathbf{V}, \mathbf{W} be vector spaces, where \mathbf{V} is finite dimensional. Let $L : \mathbf{V} \rightarrow \mathbf{W}$ be a linear transformation. Then</p> $\text{rank}(L) + \text{nullity}(L) = \dim(\mathbf{V})$ |

TABLE 9.3.1

| | |
|-------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Inference from the Given Data | $\text{Ker}(\phi_1) = \{0\}$ $\implies \text{nullity}(\phi_1) = 0$ $\text{Range}(\phi_1) = \text{Ker}(\phi_2)$ $\implies \text{rank}(\phi_1) = \text{nullity}(\phi_2)$ $\text{Range}(\phi_2) = \text{Ker}(\phi_3)$ $\implies \text{rank}(\phi_2) = \text{nullity}(\phi_3)$ $\text{Range}(\phi_3) = \mathbf{V}_4$ |
|-------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

$$\implies \text{rank}(\phi_3) = \dim(\mathbf{V}_4)$$

Now talking about the linear transformations we can use rank-nullity theorem to determine the corresponding dimensions of the vector space.

$$\phi_1 : \mathbf{V}_1 \rightarrow \mathbf{V}_2$$

$$\implies \text{rank}(\phi_1) + \text{nullity}(\phi_1) = \dim(\mathbf{V}_1)$$

$$\implies \text{rank}(\phi_1) = \dim(\mathbf{V}_1) \quad (\because \text{nullity}(\phi_1) = 0)$$

$$\phi_2 : \mathbf{V}_2 \rightarrow \mathbf{V}_3$$

$$\implies \text{rank}(\phi_2) + \text{nullity}(\phi_2) = \dim(\mathbf{V}_2)$$

$$\implies \text{rank}(\phi_2) + \text{rank}(\phi_1) = \dim(\mathbf{V}_2) \quad (\because \text{rank}(\phi_1) = \text{nullity}(\phi_2))$$

$$\implies \text{rank}(\phi_2) + \dim(\mathbf{V}_1) = \dim(\mathbf{V}_2) \quad (\because \text{rank}(\phi_1) = \dim(\mathbf{V}_1))$$

$$\phi_3 : \mathbf{V}_3 \rightarrow \mathbf{V}_4$$

$$\implies \text{rank}(\phi_3) + \text{nullity}(\phi_3) = \dim(\mathbf{V}_3)$$

$$\implies \text{rank}(\phi_3) + \text{rank}(\phi_2) = \dim(\mathbf{V}_3) \quad (\because \text{rank}(\phi_2) = \text{nullity}(\phi_3))$$

$$\implies \text{rank}(\phi_3) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) = \dim(\mathbf{V}_3) \quad (\because \text{rank}(\phi_2) + \dim(\mathbf{V}_1) = \dim(\mathbf{V}_2))$$

$$\implies \dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) = \dim(\mathbf{V}_3) \quad (\because \text{rank}(\phi_3) = \dim(\mathbf{V}_4))$$

From the above equation we can infer that

$$\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$$

Option 1

It is given that

$$\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i = 0$$

$$\implies -\dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) = 0$$

This statement we already proved above.

\therefore this statement is **True**.

Option 2

It is given that

$$\sum_{i=2}^4 (-1)^i \dim \mathbf{V}_i > 0$$

$$\implies \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) > 0$$

| | |
|------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>Our original derived equation is</p> $\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$ $\implies \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) = \dim(\mathbf{V}_1)$ <p>It is given in the question that the vector spaces are non-zero in nature.</p> $\implies \dim(\mathbf{V}_1) > 0$ $\therefore \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) > 0$ <p>\therefore this statement is True.</p> |
| Option 3 | <p>It is given that</p> $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i < 0$ $\implies -\dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) < 0$ <p>This is contrary to our original derived equation i.e.</p> $\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$ <p>\therefore this statement is False.</p> |
| Option 4 | <p>It is given that</p> $\sum_{i=1}^4 (-1)^i \dim \mathbf{V}_i \neq 0$ $\implies -\dim(\mathbf{V}_1) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_3) + \dim(\mathbf{V}_4) \neq 0$ <p>This is contrary to our original derived equation i.e.</p> $\dim(\mathbf{V}_4) + \dim(\mathbf{V}_2) - \dim(\mathbf{V}_1) - \dim(\mathbf{V}_3) = 0$ <p>\therefore this statement is False.</p> |
| Conclusion | <p>From our observation we see that</p> <p>Options 1) and 2) are True.</p> |

Example

$$\phi_1 \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \phi_1 \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

For the above transformation ϕ_1 the kernel and the range are

$$Ker(\phi_1) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \Rightarrow nullity(\phi_1) = 0$$

$$Range(\phi_1) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \Rightarrow rank(\phi_1) = 2$$

We can verify the rank-nullity theorem here as

$$\begin{aligned} & nullity(\phi_1) + rank(\phi_1) \\ \Rightarrow & 0 + 2 \\ \Rightarrow & 2 = dim(\mathbf{R}^2) \end{aligned}$$

Let $\phi_2 : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ defined as

$$\phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 - 2x_2 + 4x_3 \\ 3x_1 - 3x_2 + 6x_3 \end{pmatrix}$$

$$\Rightarrow \phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For the above transformation ϕ_2 the kernel and the range are

$$Ker(\phi_2) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} \Rightarrow nullity(\phi_2) = 2$$

$$Range(\phi_2) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \Rightarrow rank(\phi_2) = 1$$

We can verify the rank-nullity theorem here as

$$\begin{aligned} & nullity(\phi_2) + rank(\phi_2) \\ \Rightarrow & 2 + 1 \\ \Rightarrow & 3 = dim(\mathbf{R}^3) \end{aligned}$$

In the above two transformations ϕ_1 and ϕ_2 , we can see the following conditions being satisfied

$$Ker(\phi_1) = \{0\}, Range(\phi_1) = Ker(\phi_2)$$

Let $\phi_3 : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ defined as

$$\phi_3 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} x_1 + x_2 - x_3 \\ 2x_1 + \frac{1}{2}x_2 - x_3 \end{pmatrix}$$

$$\Rightarrow \phi_2 \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For the above transformation ϕ_3 the kernel and the range are

$$Ker(\phi_3) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \Rightarrow nullity(\phi_3) = 1$$

$$Range(\phi_3) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \right\} \Rightarrow rank(\phi_3) = 2$$

We can verify the rank-nullity theorem here as

$$\begin{aligned} & nullity(\phi_3) + rank(\phi_3) \\ \Rightarrow & 1 + 2 \\ \Rightarrow & 3 = dim(\mathbf{R}^3) \end{aligned}$$

With the above ϕ_3 transformation we were able to satisfy the other conditions as well i.e.

$$Range(\phi_2) = Ker(\phi_3), Range(\phi_3) = \mathbf{V}_4$$

Now, when we can check whether the derived equation statisfies or not. That is,

$$\begin{aligned} & -dim(\mathbf{V}_1) + dim(\mathbf{V}_2) - dim(\mathbf{V}_3) + dim(\mathbf{V}_4) \\ \Rightarrow & -dim(\mathbf{R}^2) + dim(\mathbf{R}^3) - dim(\mathbf{R}^3) + dim(\mathbf{R}^2) \\ \Rightarrow & -2 + 3 - 3 + 2 = 0 \end{aligned}$$

\therefore the condition is getting satisfied.

TABLE 9.3.3

9.4. Let \mathbf{u} be a real $n \times 1$ vector satisfying $\mathbf{u}^T \mathbf{u} = 1$, where \mathbf{u}^T is the transpose of \mathbf{u} . Define $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ where \mathbf{I} is the n^{th} order identity matrix. Which of the following statements are true?

1. \mathbf{A} is singular
2. $\mathbf{A}^2 = \mathbf{A}$
3. $\text{Trace}(\mathbf{A}) = n-2$
4. $\mathbf{A}^2 = \mathbf{I}$

Solution: See Table 9.4.1

Theorem 1. Let $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{n \times k}$ be matrices such that the product \mathbf{AB} is well defines. Then

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (9.4.1)$$

Proof: Matrix \mathbf{A} can be treated as a linear transformation from \mathbb{F}^n to \mathbb{F}^m . In that case rank of the matrix is the dimension of the image space of the transformation. If \mathbf{T} is a linear transformation from \mathbf{V}_1 to \mathbf{V}_2 then clearly $\dim \mathbf{T}(\mathbf{V}_1) \leq \dim (\mathbf{V}_1)$. Hence $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$. Since row rank and column rank of a matrix are equal,

$$\text{Therefore } \text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (9.4.2)$$

Explanation

| Statement | Solution |
|-----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1. | $\text{Let } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ $\text{Let } \mathbf{B} = \mathbf{u}\mathbf{u}^T$ $\therefore \mathbf{B} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$ $\therefore \mathbf{B} = \begin{pmatrix} u_1^2 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2^2 & \dots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_n^2 \end{pmatrix}$ <p>given that, $\mathbf{u}^T\mathbf{u} = 1$</p> $\therefore \mathbf{u}^T\mathbf{u} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ $\therefore \mathbf{u}^T\mathbf{u} = u_1^2 + u_2^2 + \dots + u_n^2$ <p>Since \mathbf{u} is non-zero vector and $\mathbf{B} = \mathbf{u}\mathbf{u}^T$. Hence \mathbf{B} is a non-zero matrix. Therefore Rank of \mathbf{B} is at least 1. From (9.4.2)</p> $\text{rank}(\mathbf{B}) \leq \min(\text{rank}(\mathbf{u}), \text{rank}(\mathbf{u}^T))$ $\therefore \text{rank}(\mathbf{B}) \leq \min(1, 1)$ <p>So Rank of \mathbf{B} is at most 1. Hence Rank of \mathbf{B} is equal to 1. Therefore \mathbf{B} has n-1 eigenvalues equal to 0. Since the trace of a matrix is equal to the sum of its eigen values. We know that trace of $\mathbf{B} = u_1^2 + u_2^2 + \dots + u_n^2 = 1$</p> $\therefore \text{Trace of } \mathbf{B} = \lambda_1 + \lambda_2 + \dots + \lambda_{n-1} + \lambda_n$ $1 = 0 + 0 + \dots + \lambda_n$ $\therefore \lambda_n = 1$ <p>Therefore the eigen values of \mathbf{B} are $\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1$ Hence the characteristic polynomial for $\mathbf{B} = x^{n-1}(x - 1)$ Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ and we know the eigen values of \mathbf{I} are $\lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = 1$</p> |

| | |
|------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | <p>and we know the eigen values of \mathbf{uu}^T are $\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1$</p> <p>$\therefore$ The eigen values of $\mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = -1$ (9.4.3)</p> |
| Example | <p>Let $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (9.4.4)</p> <p>then $\mathbf{u}^T = (1 \ 0 \ 0)$ (9.4.5)</p> <p>which satisfies $\mathbf{u}^T \mathbf{u} = 1$ (9.4.6)</p> <p>$\therefore \mathbf{uu}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (9.4.7)</p> <p>Since $\mathbf{A} = \mathbf{I} - 2\mathbf{uu}^T$ (9.4.8)</p> <p>$\therefore \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (9.4.9)</p> <p>$\therefore \mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (9.4.10)</p> <p>\therefore The eigen values of $\mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$ (9.4.11)</p> <p>$\therefore \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (9.4.12)</p> |
| Conclusion | <p>From (9.4.3)</p> <p>Since \mathbf{A} does not have 0 as an eigen value</p> <p>Therefore \mathbf{A} is not singular.</p> <p>Therefore the statement is false.</p> |
| 2. | <p>For $\mathbf{A}^2 = \mathbf{A}$,</p> <p>we know that $p(x) = x^2 - x$</p> <p>\therefore minimal polynomial of \mathbf{A} must divide $x(x-1)$</p> <p>\therefore possible eigenvalues of \mathbf{A} are 0 or 1</p> <p>But from (9.4.3) , we know that \mathbf{A} has -1 as an eigen value</p> <p>Therefore $\mathbf{A}^2 = \mathbf{A}$ is false.</p> |
| Conclusion | <p>Therefore the statement is false.</p> |
| 3. | |

| | |
|------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| | From equation (9.4.3) , Trace of $\mathbf{A} = n - 2$ |
| Conclusion | Therefore the statement is true. |
| 4. | <p>Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$</p> $\mathbf{A}^2 = (\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)(\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T$ <p>Since $\mathbf{u}^T\mathbf{u} = 1$</p> $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T$ $\therefore \mathbf{A}^2 = \mathbf{I}$ |
| Conclusion | Therefore the statement is true. |

TABLE 9.4.1: Solution summary