



# Linear Algebra



G V V Sharma\*

## CONTENTS

1	June 2019	1
2	December 2018	3
3	June 2018	36
4	December 2017	52
5	June 2017	69
6	December 2016	85
7	June 2016	103
8	December 2015	103
9	June 2015	106

**Abstract**—This book provides solved examples on Linear Algebra.

1 JUNE 2019

1.1. Consider the vector space  $\mathbb{P}_n$  of real polynomials in  $x$  of degree  $\leq n$ . Define

$$T : \mathbb{P}_2 \rightarrow \mathbb{P}_3 \quad (1.1.1)$$

\*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in. All content in this manual is released under GNU GPL. Free and open source.

by

$$(Tf)(x) = \int_0^x f(t) dt + f'(x). \quad (1.1.2)$$

Then find the matrix representation of  $T$  with respect to the bases

$$\{1, x, x^2\} \text{ and } \{1, x, x^2, x^3\} \quad (1.1.3)$$

1.2. Let  $P_A(x)$  denote the characteristic polynomial of a matrix  $A$ . Then for which of the following matrices is

$$P_A(x) - P_{A^{-1}}(x) \quad (1.2.1)$$

a constant?

a)  $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$

b)  $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

c)  $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$

d)  $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

1.3. Which of the following matrices is not diagonalizable over  $\mathbb{R}$ ?

a)  $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

b)  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

c)  $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

d)  $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

1.4. What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad (1.4.1)$$

1.5. Let  $V$  denote the vector space of real valued continuous functions on the close interval  $[0, 1]$ . Let  $W$  be the subspace of  $V$  spanned by  $\{\sin x, \cos x, \tan x\}$ . Find the dimension of  $W$  over  $\mathbb{R}$ .

1.6. Let  $V$  be the vector space of polynomials in the variable  $t$  of degree at most 2 over  $\mathbb{R}$ . An inner product on  $V$  is defined by

$$f^T g = \int_0^1 f(t)g(t) dt, \quad f, g \in V. \quad (1.6.1)$$

Let

$$W = \text{span}\{1 - t^2, 1 + t^2\} \quad (1.6.2)$$

and  $W^\perp$  be the orthogonal complement of  $W$  in  $V$ . Which of the following conditions is satisfied for all  $h \in W^\perp$ ?

- a)  $h$  is an even function
- b)  $h$  is an odd function
- c)  $h(t) = 0$  has a real solution
- d)  $h(0) = 0$

1.7. Consider solving the following system by Jacobi iteration scheme

$$\begin{pmatrix} 1 & 2m & -2m \\ n & 1 & n \\ 2m & 2m & 1 \end{pmatrix} (x) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad (1.7.1)$$

where  $m, n \in \mathbb{Z}$ . With any initial vector, the scheme converges provided  $m, n$  satisfy

- a)  $m + n = 3$
- b)  $m > n$
- c)  $m < n$
- d)  $m = n$

1.8. Consider a Markov Chain with state space  $\{0, 1, 2, 3, 4\}$  and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 2 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 3 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 4 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (1.8.1)$$

Then find

$$\lim_{n \rightarrow \infty} p_{23}^{(n)} \quad (1.8.2)$$

1.9. Let  $L(\mathbb{R})^n$  be the space of  $\mathbb{R}$ -linear maps from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . If  $\text{Ker}(T)$  denotes the kernel of  $T$  then which of the following are true?

- a) There exists  $T \in L(\mathbb{R}^5) \setminus \{0\}$  such that  $\text{Range}(T) = \text{Ker}(T)$
- b) There does not exist  $T \in L(\mathbb{R}^5) \setminus \{0\}$  such that  $\text{Range}(T) = \text{Ker}(T)$
- c) There exists  $T \in L(\mathbb{R}^6) \setminus \{0\}$  such that  $\text{Range}(T) = \text{Ker}(T)$
- d) There does not exist  $T \in L(\mathbb{R}^6) \setminus \{0\}$  such that  $\text{Range}(T) = \text{Ker}(T)$

1.10. Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$  and  $T : V \rightarrow V$  be a linear map. Can you always write  $T = T_2 \circ T_1$  for some linear maps

$$T_1 : V \rightarrow W, T : W \rightarrow V, \quad (1.10.1)$$

where  $W$  is some finite dimensional vector space such that

- a) both  $T_1$  and  $T_2$  are onto
- b) both  $T_1$  and  $T_2$  are one to one
- c)  $T_1$  is onto,  $T_2$  is one to one
- d)  $T_1$  is one to one,  $T_2$  is onto

1.11. Let  $A = [a_{ij}]$  be a  $3 \times 3$  complex matrix. Identify the correct statements

- a)  $\det \begin{bmatrix} (-1)^{i+j} a_{ij} \end{bmatrix} = \det(A)$
- b)  $\det \begin{bmatrix} (-1)^{i+j} a_{ij} \end{bmatrix} = -\det(A)$
- c)  $\det \begin{bmatrix} (\sqrt{-1})^{i+j} a_{ij} \end{bmatrix} = \det(A)$
- d)  $\det \begin{bmatrix} (\sqrt{-1})^{i+j} a_{ij} \end{bmatrix} = -\det(A)$

1.12. Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \quad (1.12.1)$$

be a non-constant polynomial of degree  $n \geq 1$ . Consider the polynomial

$$q(x) = \int_0^x p(t) dt, r(x) = \frac{d}{dx} p(x) \quad (1.12.2)$$

Let  $V$  denote the real vector space of all polynomials in  $x$ . Then which of the following are true?

- a)  $q$  and  $r$  are linearly independent in  $V$
- b)  $q$  and  $r$  are linearly dependent in  $V$
- c)  $x^n$  belongs to the linear span of  $q$  and  $r$
- d)  $x^{n+1}$  belongs to the linear span of  $q$  and  $r$ .

1.13. Let  $M_n(\mathbb{R})$  be the ring of  $n \times n$  matrices over  $\mathbb{R}$ . Which of the following are true for every  $n \geq 2$ ?

- a) there exist matrices  $A, B \in M_n(\mathbb{R})$  such that  $AB - BA = I_n$ , where  $I_n$  denotes the identity matrix.
- b) If  $A, B \in M_n(\mathbb{R})$  and  $AB = BA$ , then  $A$  is diagonalisable over  $\mathbb{R}$  if and only if  $B$  is diagonalisable over  $\mathbb{R}$ .
- c) If  $A, B \in M_n(\mathbb{R})$ , then  $AB$  and  $BA$  have the same minimal polynomial.
- d) If  $A, B \in M_n(\mathbb{R})$ , then  $AB$  and  $BA$  have the same eigenvalues in  $\mathbb{R}$ .

1.14. Consider a matrix

$$A = [a_{ij}], 1 \leq i, j \leq 5 \quad (1.14.1)$$

such that

$$a_{ij} = \frac{1}{n_i + n_j + 1}, \quad n_i, n_j \in \mathbb{N} \quad (1.14.2)$$

Then in which of the following cases  $A$  is a positive definite matrix?

- a)  $n_i = 1 \forall i = 1, 2, 3, 4, 5$ .
- b)  $n_1 < n_2 < \dots < n_5$ .
- c)  $n_1 = n_2 = \dots = n_5$ .
- d)  $n_1 > n_2 > \dots > n_5$ .

1.15. For a nonzero  $w \in \mathbb{R}^n$ , define

$$T_w : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (1.15.1)$$

by

$$T_w v = v - \frac{2v^T w}{w^T w} w, \quad v \in \mathbb{R}^n \quad (1.15.2)$$

Which of the following are true?

- a)  $\det(T_w) = 1$
- b)  $T_w(v_1)^T_w(v_2) = v_1^T v_2 \forall v_1, v_2 \in \mathbb{R}^n$
- c)  $T_w = T_w^{-1}$
- d)  $T_{2w} = 2T_w$

1.16. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1.16.1)$$

over the field  $\mathbb{Q}$  of rationals. Which of the following matrices are of the form  $P^T A P$  for suitable  $2 \times 2$  invertible matrix  $P$  over  $\mathbb{Q}$ ?

- a)  $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$
- b)  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- c)  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- d)  $\begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$

1.17. Consider a Markov Chain with state space  $\{0, 1, 2\}$  and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 0 & \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ 1 & \frac{1}{4} & 0 & \frac{3}{4} \\ 2 & \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \quad (1.17.1)$$

Then which of the following are true?

- a)  $\lim_{n \rightarrow \infty} p_{12}^{(n)} = 0$
- b)  $\lim_{n \rightarrow \infty} p_{12}^{(n)} = \lim_{n \rightarrow \infty} p_{21}^{(n)}$
- c)  $\lim_{n \rightarrow \infty} p_{22}^{(n)} = \frac{1}{8}$
- d)  $\lim_{n \rightarrow \infty} p_{21}^{(n)} = \frac{1}{3}$

2 DECEMBER 2018

2.1. Consider the subspaces  $W_1$  and  $W_2$  of  $\mathbb{R}^3$  given by

$$W_1 = \{x \in \mathbb{R}^3 : (1 \ 1 \ 1)x = 0\} \quad (2.1.1)$$

$$W_2 = \{x \in \mathbb{R}^3 : (1 \ -1 \ 1)x = 0\}. \quad (2.1.2)$$

If  $W \subseteq \mathbb{R}^3$ , such that

- a)  $W \cap W_2 = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
- b)  $\{W \cap W_1\} \perp \{W \cap W_2\}$ ,  
then

- a)  $W = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
- b)  $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$
- c)  $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$
- d)  $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

**Solution:** Using (2.1.1),

$$W_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.3)$$

From (2.1.2),

$$W_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad (2.1.4)$$

From (2.1a), we can say that, both the subspaces  $\mathbf{W}$  and  $\mathbf{W}_2$  of  $\mathbf{R}^3$  contains the column vector as follows: .

$$\mathbf{W} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.5)$$

$$\mathbf{W}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad (2.1.6)$$

From (2.1.4) and (2.1.6),

$$\mathbf{W}_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \quad (2.1.7)$$

$$\text{Rank}(\mathbf{W}_2) = 2 \quad (2.1.8)$$

Since,  $\text{rank} < 3$  and the vectors are linearly independent they span a subspace of  $\mathbf{R}^3$ .

Consider the vector,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{W} \cap \mathbf{W}_1 \quad (2.1.9) \quad 2.2. \text{ Let}$$

From (2.1a) and (2.1b),

The vector  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$  is orthogonal to  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ .

$$\Rightarrow (x \ y \ z) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 \quad (2.1.10)$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (2.1.11)$$

Since,  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{W} \cap \mathbf{W}_1$ :

From (2.1.3) and (2.1.11),

$$\mathbf{W}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.1.12)$$

Also from (2.1.5) and (2.1.11),

$$\mathbf{W} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (2.1.13)$$

Using (2.1.13),

The vectors linearly independent and  $\text{rank}(\mathbf{W})=2 (< 3)$ , then the vector span subspace of  $\mathbf{R}^3$ .

Hence,

$$\mathbf{W} = \text{span}\{(0, 1, -1), (0, 1, 1)\} \Rightarrow \text{Ans : 1} \quad (2.1.14)$$

$$C = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \quad (2.2.1)$$

be a basis of  $\mathbb{R}^2$  and

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-2y \end{pmatrix}. \quad (2.2.2)$$

If  $T[C]$  represents the matrix of  $T$  with respect to the basis  $C$  then which among the following is true?

$$\text{a) } T[C] = \begin{pmatrix} -3 & -2 \\ 3 & 1 \end{pmatrix}$$

$$\text{b) } T[C] = \begin{pmatrix} 3 & -2 \\ -3 & 1 \end{pmatrix}$$

$$\text{c) } T[C] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$$

$$\text{d) } T[C] = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}$$

**Solution:** See Tables 2.2.1 and 2.2.2

In above question  $\mathbf{A} = \mathbf{T}, \mathbf{B} = \mathbf{T}[\mathbf{C}], \mathbf{V} = \mathbf{C}$ .

2.3. Let  $W_1 = \{\mathbf{x} \in \mathbb{R}^4 : \}$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.1)$$

$$\begin{pmatrix} 0 & 2 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.2)$$

$$\begin{pmatrix} 2 & 0 & 2 & -1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.3)$$

Linear Transformation and change of Basis	<p>If matrix <math>\mathbf{A}</math> represents Linear Transformation with respect to standard ordered basis and matrix <math>\mathbf{B}</math> represents same transformation with respect to basis <math>\mathbf{V}</math>, Then</p> $\mathbf{B} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$
---	---

TABLE 2.2.1: Linear Transformation and change of basis

$$\text{and } W_2 = \{\mathbf{x} \in \mathbb{R}^4 : \}$$

$$\begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.4)$$

$$\begin{pmatrix} 1 & 0 & 1 & -2 \end{pmatrix} \mathbf{x} = 0 \quad (2.3.5)$$

$$\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} \mathbf{x} = 0. \quad (2.3.6)$$

Then which among the following is true?

- a)  $\dim(W_1) = 1$
- b)  $\dim(W_2) = 2$
- c)  $\dim(W_1 \cap W_2) = 1$
- d)  $\dim(W_1 + W_2) = 3$

2.4. Let  $A$  be an  $n \times n$  complex matrix. Assume that  $A$  is self-adjoint and let  $B$  denote the inverse of  $A + \mathbf{j}I$ . Then all eigenvalues of  $(A - \mathbf{j}I)B$  are

- a) purely imaginary
- b) of modulus one
- c) real
- d) of modulus less than one

2.5. Let  $\{u_1, u_2, \dots, u_n\}$  be an orthonormal basis of  $\mathbb{C}^n$  as column vectors. Let

$$\mathbf{M} = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \dots \quad \mathbf{u}_k), \quad (2.5.1)$$

$$\mathbf{N} = (\mathbf{u}_{k+1} \quad \mathbf{u}_{k+2} \quad \dots \quad \mathbf{u}_n) \quad (2.5.2)$$

and  $\mathbf{P}$  be the diagonal  $k \times k$  matrix with diagonal entries  $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ . Then which of the following is true?

- a)  $\text{rank}(\mathbf{M}\mathbf{P}\mathbf{M}^*) = k$  whenever  $\alpha_i \neq \alpha_j, 1 \leq i, j \leq k$ .
- b)  $\text{tr}(\mathbf{M}\mathbf{P}\mathbf{M}^*) = \sum_{i=1}^k \alpha_i$
- c)  $\text{rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n - k)$
- d)  $\text{rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$ .

**Solution:** See Tables 2.5.1 2.5.2 and 2.5.3

Evaluate $\mathbf{T}$	<p>For linear transformation <math>\mathbf{T}</math> we have</p> $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - 2y \end{pmatrix}$ $\mathbf{T} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\Rightarrow \mathbf{T} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$
Evaluate inverse of basis $\mathbf{C}$	<p>To find inverse of matrix <math>\mathbf{C}</math> we row reduce augmented matrix <math>\mathbf{C}\mathbf{I}</math></p> $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 = -\frac{1}{3}R_2]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ $\xrightarrow{R_1 = R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$ $\therefore \mathbf{C}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$
Evaluate $\mathbf{TC}$	$\mathbf{TC} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $= \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$
Evaluate $\mathbf{T}[\mathbf{C}] = \mathbf{C}^{-1}\mathbf{TC}$	$\mathbf{T}[\mathbf{C}] = \mathbf{C}^{-1}\mathbf{TC}$ $= \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix}$ $\Rightarrow \mathbf{T}[\mathbf{C}] = \begin{pmatrix} -3 & -1 \\ 3 & 2 \end{pmatrix}$
Conclusion	<p>Option 3) is correct. Options 1), 2) and 4) are incorrect</p>

TABLE 2.2.2: Calculation of  $\mathbf{T}[\mathbf{C}]$

Orthonormal Basis	<p><math>B = \{u_1, u_2, \dots, u_n\}</math> is the Orthonormal basis for <math>C^n</math> if it generates every vector <math>C^n</math> and the inner product <math>\langle u_i, u_j \rangle = 0</math> if <math>i \neq j</math>. That is the vectors are mutually perpendicular and <math>\langle u_i, u_j \rangle = 1</math> otherwise.</p>
Trace	<p>Trace of a square matrix <math>A</math>, denoted by <math>\text{tr}(A)</math> is defined to be the sum of elements on the main diagonal(from the upper left to lower right) of <math>A</math>  Some useful properties of Trace :  <math>\text{tr}(AB) = \text{tr}(BA)</math>, where <math>A</math> is the <math>m \times n</math> matrix and <math>B</math> is the <math>n \times m</math> matrix</p>
Basis Theorem	<p>A nonempty subset of nonzero vectors in <math>R^n</math> is called an orthogonal set if every pair of distinct vectors in the set is orthogonal. Any Orthogonal sets of vectors are automatically linearly independent and if <math>A</math> matrix columns are linearly independent, then it is invertible.</p>

TABLE 2.5.1: Definitions

$\text{Rank}(\mathbf{MPM}^*) = \mathbf{k}$	<p>Consider orthogonal vectors,</p> $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ <p>Consider <math>\mathbf{k} = 2</math>, then</p> $\mathbf{M} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{P} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ $\mathbf{MPM}^* = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p><math>\Rightarrow \text{Rank}(\mathbf{MPM}^*) \leq 2</math> (which is the value of <math>k</math>)          (It depends on diagonal values <math>\alpha_1</math> and <math>\alpha_2</math>)  <math>\text{Rank}(\mathbf{MPM}^*)</math> is not always <math>k</math>.          It can be less than <math>k</math> if any of the entries in <math>\alpha_1, \alpha_2, \dots, \alpha_k</math> is 0.          Thus, <math>\text{Rank}(\mathbf{MPM}^*) \neq \mathbf{k}</math>          Thus, the given statement is false</p>
$\text{Trace}(\mathbf{MPM}^*) = \sum_{i=1}^k \alpha_i$	<p>Consider <math>\mathbf{MP} = \mathbf{A}</math> and <math>\mathbf{M}^* = \mathbf{B}</math>          Using Properties, <math>\text{Trace}(\mathbf{AB}) = \text{Trace}(\mathbf{BA})</math>          We can say, <math>\text{Trace}(\mathbf{MPM}^*) = \text{Trace}(\mathbf{M}^*\mathbf{MP})</math></p> $\mathbf{M} = \begin{pmatrix} u_1 & u_2 & u_3 & \dots & u_k \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \vdots \\ \bar{u}_k \end{pmatrix}$ $\mathbf{M}^*\mathbf{M} = \begin{pmatrix} \bar{u}_1 u_1 & 0 & 0 & \dots & 0 \\ 0 & \bar{u}_2 u_2 & 0 & \dots & 0 \\ 0 & 0 & \bar{u}_3 u_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \bar{u}_k u_k \end{pmatrix}$

	<p>(Refer to Properties mentioned in Orthonormal Basis in Definition section that is <math>\langle u_i, u_j \rangle = 0</math> if <math>i \neq j</math>)</p> $\mathbf{M}^* \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$ <p>(Refer to Properties mentioned in Orthonormal Basis in Definition section that is <math>\langle u_i, u_j \rangle = 1</math> if <math>i = j</math>)</p> <p><math>\mathbf{M}^* \mathbf{M} = \mathbf{I}^k</math></p> <p><math>\mathbf{M}^* \mathbf{M} \mathbf{P} = \mathbf{I}^k \mathbf{P} = \mathbf{P}</math></p> <p><math>\text{Trace}(\mathbf{M}^* \mathbf{M} \mathbf{P}) = \text{Trace}(\mathbf{I}^k \mathbf{P}) = \text{Trace}(\mathbf{P}) = \sum_{i=1}^k \alpha_i</math></p> <p>(Refer Definition section of Trace, it is sum of elements on the main diagonal)</p> <p>So, the given statement is true</p>
<p><math>\text{Rank}(\mathbf{M}^* \mathbf{N}) = \min(k, n - k)</math></p>	<p><math>\mathbf{M} = \{u_1, u_2, \dots, u_k\}</math> and <math>\mathbf{N} = \{u_{k+1}, u_{k+2}, \dots, u_n\}</math></p> <p>Consider orthogonal vectors,</p> $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ <p>Consider <math>k = 2</math>, then</p> $\mathbf{M} = (\mathbf{u}_1 \ \mathbf{u}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\mathbf{M}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\mathbf{N} = (\mathbf{u}_3 \ \mathbf{u}_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mathbf{M}^* \mathbf{N} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ <p><math>\text{Rank}(\mathbf{M}^* \mathbf{N}) = 0</math></p> <p>But, <math>\min(k, n - k) = (2, 2) = 2</math></p> <p>And, this is clear from above that <math>\text{Rank}(\mathbf{M}^* \mathbf{N}) \neq \min(k, n - k)</math></p> <p>Thus, above statement is false</p>
<p><math>\text{Rank}(\mathbf{M} \mathbf{M}^* + \mathbf{N} \mathbf{N}^*) &lt; n</math></p>	<p><math>\text{Rank}(\mathbf{M}) = \text{Rank}(\mathbf{M}^*)</math></p> <p><math>\text{Rank}(\mathbf{N}) = \text{Rank}(\mathbf{N}^*)</math></p> <p><math>\text{Rank}(\mathbf{M} + \mathbf{N}) \leq \text{Rank}(\mathbf{M}) + \text{Rank}(\mathbf{N})</math></p>



$\mathbf{M} = \{u_1, u_2, \dots, u_k\}$  and  $\mathbf{N} = \{u_{k+1}, u_{k+2}, \dots, u_n\}$

Consider orthogonal vectors,

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \mathbf{u}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Consider  $k = 2$ , then

$$\mathbf{M} = (u_1 \ u_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{Rank}(\mathbf{M}) = 2 = k$$

$$\mathbf{N} = (u_3 \ u_4) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Rank}(\mathbf{N}) = 2 = n - k$$

$$\text{Thus, } \text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) = \text{Rank}(\mathbf{M} + \mathbf{N}) = 4 = n$$

Thus, above statement is false

TABLE 2.5.2: Finding of True and False Statements

$\text{Rank}(\mathbf{M}\mathbf{M}^*) = k$	False
$\text{Trace}(\mathbf{M}\mathbf{M}^*) = \sum_{i=1}^k \alpha_i$	True
$\text{Rank}(\mathbf{M}^*\mathbf{N}) = \min(k, n - k)$	False
$\text{Rank}(\mathbf{M}\mathbf{M}^* + \mathbf{N}\mathbf{N}^*) < n$	False

TABLE 2.5.3: Conclusion of above Solutions

2.6. Let  $B : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the function

$$B(a, b) = ab \quad (2.6.1)$$

Which of the following is true?

- a)  $B$  is a linear transformation
- b)  $B$  is a positive definite bilinear form
- c)  $B$  is symmetric but not positive definite
- d)  $B$  is neither linear nor bilinear

2.7. Let  $\mathbf{A}$  be an invertible real  $n \times n$  matrix. Define a function

$$F : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (2.7.1)$$

by

$$F(\mathbf{x}, \mathbf{y}) = (F\mathbf{x})^T \mathbf{y} \quad (2.7.2)$$

Let  $DF(\mathbf{x}, \mathbf{y})$  denote the derivate of  $F$  at  $(\mathbf{x}, \mathbf{y})$  which is a linear transformation from

$$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \quad (2.7.3)$$

Then, if

- a)  $\mathbf{x} \neq 0, DF(\mathbf{x}, \mathbf{0}) \neq 0$
- b)  $\mathbf{y} \neq 0, DF(\mathbf{0}, \mathbf{y}) \neq 0$
- c)  $(\mathbf{x}, \mathbf{y}) \neq (\mathbf{0}, \mathbf{0}), DF(\mathbf{x}, \mathbf{0}) \neq 0$
- d)  $\mathbf{x} = 0$  or  $\mathbf{y} = 0, DF(\mathbf{x}, \mathbf{y}) = 0$

**Solution:** See Tables 2.7.1 and 2.7.2

Invertible	<p>A square matrix is invertible if and only if it does not have a zero eigenvalue. So, from the definition of eigen vector we can write that</p> $\mathbf{A}\mathbf{x} \neq 0 \quad (2.7.4)$ <p>The transpose of an invertible matrix is also invertible with inverse <math>(\mathbf{A}^{-1})^T</math>.</p> $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \implies (\mathbf{A}^{-1})^T \mathbf{A}^T = \mathbf{I}^T = \mathbf{I} \quad (2.7.5)$ <p>So, similarly we can say that</p> $\mathbf{A}^T \mathbf{y} \neq 0 \quad (2.7.6)$
Derivative of F	<p>Suppose <math>F: \mathbb{R}^n \rightarrow \mathbb{R}^m</math>, the derivative of a function F is given by the Jacobian matrix</p> $\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (2.7.7)$
Inner product	<p>The inner product of <math>\mathbf{x}</math> and <math>\mathbf{y}</math> is given by</p> $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} \quad (2.7.8)$

TABLE 2.7.1: Definition and Properties used

Given	$F(\mathbf{x}, \mathbf{y}) = \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle \quad (2.7.9)$
using inner product definition	$F(\mathbf{x}, \mathbf{y}) = (\mathbf{A}\mathbf{x})^T \mathbf{y} = \mathbf{x}^T \mathbf{A}^T \mathbf{y} \quad (2.7.10)$ $F(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T \mathbf{A}\mathbf{x} \quad (2.7.11)$
Derivative of F	<p>using (2.7.7), We can write that</p> $DF(\mathbf{x}, \mathbf{y}) = \left( \frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y} \right) = \left( \mathbf{y}^T \mathbf{A} \quad \mathbf{x}^T \mathbf{A}^T \right) \quad (2.7.12)$
If $\mathbf{x} \neq 0$ , then $DF(\mathbf{x}, 0) \neq 0$	using (2.7.12),

	$DF(\mathbf{x}, 0) = \begin{pmatrix} 0 & \mathbf{x}^T \mathbf{A}^T \end{pmatrix} \quad (2.7.13)$ <p>From (2.7.4), we know that</p> $\mathbf{A}\mathbf{x} \neq 0 \quad (2.7.14)$ $\implies \mathbf{x}^T \mathbf{A}^T \neq 0 \quad (2.7.15)$ <p>So, We can say that</p> $DF(\mathbf{x}, 0) \neq 0 \quad (2.7.16)$
If $\mathbf{y} \neq 0$ , then $DF(0, \mathbf{y}) \neq 0$	<p>using (2.7.12),</p> $DF(0, \mathbf{y}) = \begin{pmatrix} \mathbf{y}^T \mathbf{A} & 0 \end{pmatrix} \quad (2.7.17)$ <p>From (2.7.6), we know that</p> $\mathbf{A}^T \mathbf{y} \neq 0 \quad (2.7.18)$ $\implies \mathbf{y}^T \mathbf{A} \neq 0 \quad (2.7.19)$ <p>So, We can say that</p> $DF(0, \mathbf{y}) \neq 0 \quad (2.7.20)$
If $(\mathbf{x}, \mathbf{y}) \neq 0$ , then $DF(\mathbf{x}, \mathbf{y}) \neq 0$	<p>using (2.7.12),</p> $DF(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{y}^T \mathbf{A} & \mathbf{x}^T \mathbf{A}^T \end{pmatrix} \quad (2.7.21)$ <p>As <math>(\mathbf{x}, \mathbf{y}) \neq 0</math>, <math>DF(\mathbf{x}, \mathbf{y}) = 0</math> iff <math>\mathbf{A} = 0</math></p> <p>From (2.7.4), we know that</p> $\mathbf{A} \neq 0 \quad (2.7.22)$ <p>So, We can say that</p> $DF(\mathbf{x}, \mathbf{y}) \neq 0 \quad (2.7.23)$
If $\mathbf{x} = 0$ or $\mathbf{y} = 0$ , then $DF(\mathbf{x}, \mathbf{y}) = 0$	<p>From (2.7.20),</p> $DF(0, \mathbf{y}) \neq 0 \quad (2.7.24)$ <p>From (2.7.16),</p> $DF(\mathbf{x}, 0) \neq 0 \quad (2.7.25)$ <p>So, if <math>\mathbf{x} = 0</math> or <math>\mathbf{y} = 0</math>,</p> $DF(\mathbf{x}, \mathbf{y}) \neq 0 \quad (2.7.26)$
Conclusion	

	From above,we can say that options 1),2),3) are correct.
--	--

TABLE 2.7.2: Finding derivative of linear transformation

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by, $p(x) =  x\mathbf{I} - \mathbf{A} $
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ , then, $p(\mathbf{A}) = \mathbf{0}$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $m(\mathbf{A}) = \mathbf{0}$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 2.8.1: Definitions

2.8. Let

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (2.8.1)$$

be a linear map that satisfies

$$T^2 = T - I. \quad (2.8.2)$$

Then which of the following is true?

- a)  $T$  is invertible.
- b)  $T - I$  is not invertible.
- c)  $T$  has a real eigenvalue.
- d)  $T^3 = -I$ .

**Solution:** See Tables 2.8.1 and 2.8.2

Statement	Solution
1.	<p>Given that <math>\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n</math>  Since <math>\mathbf{T}</math> is a linear map from <math>\mathbb{R}^n</math> to <math>\mathbb{R}^n</math> therefore the matrix corresponding to it is of order <math>n \times n</math>.</p> <p>Since <math>\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n</math>  <math>\therefore \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}</math></p> <p><math>\implies p(x) = x^2 - x + 1</math> will be annihilating polynomial.  <math>\therefore p(\mathbf{T}) = \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}</math></p> <p>We know that minimal polynomial always divides annihilating polynomial.  <math>\therefore</math> The roots of minimal polynomial are as follows:</p> $x = \frac{1 \pm \sqrt{3}i}{2} \quad (2.8.3)$ <p>Therefore any eigenvalue of <math>\mathbf{T}</math> is a root of its minimal polynomial.  Since 0 is not a root of <math>p(x)</math>, Therefore 0 is not an eigen value for <math>\mathbf{T}</math>.  Since <math>\mathbf{T}</math> is not invertible iff there exists an eigen value which is zero.</p> <p><math>\therefore \mathbf{T}</math> is invertible. <math>(2.8.4)</math></p>
Conclusion	Therefore the statement is true.
2.	<p>From equation (2.8.3) ,  Since 1 is not a root of <math>p(x)</math>, Therefore 1 is not an eigen value for <math>\mathbf{T}</math>.  Therefore, 0 is not an eigen values of <math>\mathbf{T} - \mathbf{I}_n</math>.</p> <p><math>\therefore \mathbf{T} - \mathbf{I}_n</math> is invertible. <math>(2.8.5)</math></p>
Conclusion	Therefore the statement is false.

3.	<p>From equation (2.8.3) ,  Therefore any eigenvalue of <math>\mathbf{T}</math> is a root of its minimal polynomial.  But the roots of minimal polynomial are not real.  Therefore <math>\mathbf{T}</math> cant have a real eigen value.</p>
Conclusion	Therefore the statement is false.
4.	<p>Since <math>\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n</math>      (2.8.6)  <math>\mathbf{T}^3 = \mathbf{T}(\mathbf{T} - \mathbf{I}_n)</math>      (2.8.7)  <math>\therefore \mathbf{T}^3 = \mathbf{T}^2 - \mathbf{T}</math>      (2.8.8)  <math>\therefore \mathbf{T}^3 = -\mathbf{I}_n</math>      (2.8.9)</p>
Conclusion	Therefore the statement is true.

TABLE 2.8.2: Solution summary



2.9. Let

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & 3 & 2 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 0 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} \quad (2.9.1)$$

$$\mathbf{b}_1 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 5 \\ 1 \\ 3 \\ 3 \end{pmatrix}. \quad (2.9.2)$$

Then which of the following are true?

- a) both systems  $\mathbf{M}\mathbf{x} = \mathbf{b}_1$  and  $\mathbf{M}\mathbf{x} = \mathbf{b}_2$  are inconsistent.
- b) both systems  $\mathbf{M}\mathbf{x} = \mathbf{b}_1$  and  $\mathbf{M}\mathbf{x} = \mathbf{b}_2$  are consistent.
- c) the system  $\mathbf{M}\mathbf{x} = \mathbf{b}_1 - \mathbf{b}_2$  is consistent.
- d) the system  $\mathbf{M}\mathbf{x} = \mathbf{b}_1 - \mathbf{b}_2$  is inconsistent.

2.10. Let

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix}. \quad (2.10.1)$$

Given that 1 is an eigenvalue of  $\mathbf{M}$ , then which among the following are correct?

- a) The minimal polynomial of  $\mathbf{M}$  is  $(x - 1)(x + 4)$
- b) The minimal polynomial of  $\mathbf{M}$  is  $(x - 1)^2(x + 4)$
- c)  $\mathbf{M}$  is not diagonalizable.
- d)  $\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I})$ .

**Solution:** See Table 2.10.1

Given	$\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 4 \\ -2 & 1 & -4 \end{pmatrix} \quad (2.10.2)$ <p>One of the eigenvalue of <math>\mathbf{M}</math> is 1</p>
Solution	<p>Let the eigenvalues of matrix <math>\mathbf{M}</math> of order <math>3 \times 3</math> be <math>\lambda_1, \lambda_2, \lambda_3</math>  From given , let <math>\lambda_1 = 1</math>.  We know that sum of the eigenvalues of matrix is Trace of the matrix and product of eigenvalues of matrix is Determinant of the matrix.  Trace of the square matrix(<math>\text{Tr}(\mathbf{M})</math>) is the sum of the elements in the main diagonal of <math>\mathbf{M}</math>.</p> $\text{Tr}(\mathbf{M}) = 1 + 1 - 4 \quad (2.10.3)$ $\Rightarrow \text{Tr}(\mathbf{M}) = -2 \quad (2.10.4)$ $\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = -2 \quad (2.10.5)$ $\Rightarrow \lambda_2 + \lambda_3 = -3 \quad (2.10.6)$ $\Rightarrow \lambda_2 = -3 - \lambda_3 \quad (2.10.7)$ <p>By row reducing the matrix <math>\mathbf{M}</math>, we get ,</p> $\mathbf{M} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} \quad (2.10.8)$
	$\text{Det}(\mathbf{M}) = 1 \left( 3 \left( -\frac{4}{3} \right) \right) = -4 \quad (2.10.9)$ $\Rightarrow \lambda_1 \lambda_2 \lambda_3 = -4 \quad (2.10.10)$ $\Rightarrow \lambda_2 \lambda_3 = -4 \quad (2.10.11)$ <p>Solving equations (2.10.7) and (2.10.11) one of the possibilities we get,</p> $\lambda_1 = 1 \quad (2.10.12)$ $\lambda_2 = 1 \quad (2.10.13)$ $\lambda_3 = -4 \quad (2.10.14)$
	<p>Using the eigenvalues the characteristic polynomial of matrix <math>\mathbf{M}</math> is given by,</p> $c(x) = x^3 + 2x^2 - 7x + 4 = 0 \quad (2.10.15)$ <p>The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation.  Using the above theorem, the equation (2.10.15) can be written as,</p> $\mathbf{M}^3 + 2\mathbf{M}^2 - 7\mathbf{M} + 4\mathbf{I} = 0 \quad (2.10.16)$ $\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I} + 4\mathbf{M}^{-1} = 0 \quad (2.10.17)$ $\Rightarrow \mathbf{M}^{-1} = -\frac{1}{4}(\mathbf{M}^2 + 2\mathbf{M} - 7\mathbf{I}) \quad (2.10.18)$
<b>Statement 1</b>	<p>The minimal polynomial of <math>\mathbf{M}</math> is <math>(x - 1)(x + 4)</math>  If <math>(x-1)(x+4)</math> is a minimal polynomial of <math>\mathbf{M}</math> then,</p>

	$(\mathbf{M} - \mathbf{I})(\mathbf{M} + 4\mathbf{I}) = \mathbf{0}_{3 \times 3} \quad (2.10.19)$ <p>But,</p> $(\mathbf{M} - \mathbf{I})(\mathbf{M} + 4\mathbf{I}) = \begin{pmatrix} -4 & -4 & -4 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \neq \mathbf{0}_{3 \times 3} \quad (2.10.20)$ <p style="text-align: center;"><b>False Statement</b></p>
<b>Statement 2</b>	The minimal polynomial of $\mathbf{M}$ is $(x - 1)^2(x + 4)$
	<p>Let <math>m(x)</math> be the minimal polynomial</p> $m(x) = (x - 1)^2(x + 4) \quad (2.10.21)$ $= x^3 + 2x^2 - 7x + 4 \quad (2.10.22)$ $= c(x)$ <p>In this case both minimal polynomial and characteristic polynomial were same. Therefore we could say that equation (2.10.21) is the minimal polynomial of <math>\mathbf{M}</math> as it satisfies equation (2.10.16) by Cayley Hamilton Theorem.</p> <p style="text-align: center;"><b>True Statement</b></p>
<b>Statement 3</b>	$\mathbf{M}$ is not diagonalizable.
	$\mathbf{M}$ is diagonalizable if and only if its minimal polynomial is a product of distinct monic linear factors. From equation (2.10.21) we could see that one of the factor of minimal polynomial is repeated and it is not a linear factor. Therefore, Matrix $\mathbf{M}$ is not diagonalizable.
	<b>True Statement</b>
<b>Statement 4</b>	$\mathbf{M}^{-1} = \frac{1}{4}(\mathbf{M} + 3\mathbf{I}) \quad (2.10.23)$
	Comparing equation (2.10.18) and (2.10.23) we could say that the given statement is <b>False Statement</b> .

TABLE 2.10.1: Explanation

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$ , characteristic polynomial is defined by, $p(x) =  x\mathbf{I} - \mathbf{A} $
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$ , then, $p(\mathbf{A}) = \mathbf{0}$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $m(\mathbf{A}) = \mathbf{0}$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 2.11.1: Definitions

2.11. Let  $\mathbf{A}$  be a real matrix with characteristic polynomial  $(x - 1)^3$ . Pick the correct statements from below:

- a)  $\mathbf{A}$  is necessarily diagonalizable.
- b) If the minimal polynomial of  $\mathbf{A}$  is  $(x - 1)^3$ , then  $\mathbf{A}$  is diagonalizable.
- c) The characteristic polynomial of  $\mathbf{A}^2$  is  $(x - 1)^3$
- d) If  $\mathbf{A}$  has exactly two Jordan blocks, then  $(\mathbf{A} - \mathbf{I})^2$  is diagonalizable.

**Solution:** See Tables 2.11.1 and 2.11.2

Statement	Solution
1.	<p>Let <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 2 \\ 0 &amp; 0 &amp; 1 \end{pmatrix}</math></p> <p>Since <math>\mathbf{A}</math> is upper triangular matrix, <math>\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1</math></p> <p>Therefore, <math>p(x) = (x - 1)^3</math></p> <p>Solving <math>(\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \end{pmatrix}</math></p> <p>Solving <math>(\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \end{pmatrix}</math></p> <p>Solving <math>\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 2 \\ 0 &amp; 0 &amp; 0 \end{pmatrix}</math></p> <p>Since <math>\mathbf{A} - \mathbf{I} \neq \mathbf{0}</math></p> <p>Therefore, <math>m(x) = (x - 1)^2</math></p>
Justification	<p>Hence, the Jordan form of <math>\mathbf{A}</math> is a <math>3 \times 3</math> matrix consisting of two block: one block of order 2 with principal diagonal value as <math>\lambda = 1</math> and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1.</p> <p>And one block of order 1 with <math>\lambda = 1</math>.</p> <p>Hence the required Jordan form of <math>\mathbf{A}</math> is,</p> $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ <p>A matrix is diagonalizable iff its jordan form is a diagonal matrix. Since <math>\mathbf{J}</math> is not diagonalizable therefore <math>\mathbf{A}</math> is not diagonalizable.</p>
Conclusion	Therefore the statement is false.

2.	<p style="text-align: center;"> <math display="block">\text{Let } \mathbf{A} = \begin{pmatrix} 1 &amp; 2 &amp; 0 \\ 0 &amp; 1 &amp; 2 \\ 0 &amp; 0 &amp; 1 \end{pmatrix}</math> </p> <p>Since <math>\mathbf{A}</math> is upper triangular matrix, <math>\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1</math></p> <p>Therefore, <math>p(x) = (x - 1)^3</math></p> <p>Solving <math>(\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \end{pmatrix}</math></p> <p>Solving <math>(\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 &amp; 0 &amp; 4 \\ 0 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 0 \end{pmatrix}</math></p> <p>Since <math>(\mathbf{A} - \mathbf{I})^2 \neq \mathbf{0}</math></p> <p>Therefore, <math>m(x) = (x - 1)^3</math></p> <p>Justification Hence, the Jordan form of <math>\mathbf{A}</math> is a <math>3 \times 3</math> matrix consisting of only one block with principal diagonal values as <math>\lambda_1 = 1</math> and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of <math>\mathbf{A}</math> is,</p> <p style="text-align: center;"> <math display="block">\therefore \mathbf{J} = \begin{pmatrix} 1 &amp; 1 &amp; 0 \\ 0 &amp; 1 &amp; 1 \\ 0 &amp; 0 &amp; 1 \end{pmatrix}</math> </p> <p>Since <math>\mathbf{J}</math> is not diagonalizable therefore <math>\mathbf{A}</math> is not diagonalizable.</p>
Conclusion	Therefore the statement is false.
3.	<p style="text-align: center;">Give that, <math>p(x)</math> of <math>\mathbf{A} = (x - 1)^3</math></p> <p style="text-align: center;">Hence the eigen values of <math>\mathbf{A} = 1, 1, 1</math></p> <p style="text-align: center;">Hence the eigen values of <math>\mathbf{A}^2 = 1^2, 1^2, 1^2</math> or <math>1, 1, 1</math></p> <p style="text-align: center;">Therefore <math>p(x)</math> of <math>\mathbf{A}^2 = (x - 1)^3</math></p>
Conclusion	Therefore the statement is True.

4.	<p>We know that jordan form of a matrix is similar to the original matrix Let <math>\mathbf{J}</math> be the jordan form of the matrix <math>\mathbf{A}</math> then,</p> $\mathbf{A} = \mathbf{PJP}^{-1}$ $\mathbf{A} - \mathbf{I} = \mathbf{PJP}^{-1} - \mathbf{I}$ $\mathbf{A} - \mathbf{I} = \mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}$ $(\mathbf{A} - \mathbf{I})^2 = \mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}\mathbf{P}(\mathbf{J} - \mathbf{I})\mathbf{P}^{-1}$ $(\mathbf{A} - \mathbf{I})^2 = \mathbf{P}(\mathbf{J} - \mathbf{I})^2\mathbf{P}^{-1}$ <p>Therefore <math>(\mathbf{A} - \mathbf{I})^2</math> is similar to <math>(\mathbf{J} - \mathbf{I})^2</math> Since <math>\mathbf{A}</math> has exactly two jordan blocks and order of <math>\mathbf{A}</math> is 3.</p> $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\mathbf{J} - \mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $(\mathbf{J} - \mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ <p>Since <math>(\mathbf{J} - \mathbf{I})^2</math> is diagonal matrix. Therefore <math>(\mathbf{A} - \mathbf{I})^2</math> is diagonalizable.</p>
Conclusion	Therefore the statement is True.

TABLE 2.11.2: Solution summary

2.12. Let  $P_3$  be the vector space of polynomials with real coefficients and of degree at most 3. Consider the linear map

$$T : P_3 \rightarrow P_3 \quad (2.12.1)$$

defined by

$$T(p(x)) = p(x-1) + p(x+1) \quad (2.12.2)$$

Which of the following properties does the matrix of  $T$  with respect to the standard basis  $B = \{1, x, x^2, x^3\}$  of  $P_3$  satisfy?

- a)  $\det T = 0$ .
- b)  $(T - 2I)^4 = 0$  but  $(T - 2I)^3 \neq 0$ .
- c)  $(T - 2I)^3 = 0$  but  $(T - 2I)^2 \neq 0$ .
- d) 2 is an eigenvalue with multiplicity 4.

**Solution:** Given

$$T(p(x)) = p(x+1) + p(x-1). \quad (2.12.3)$$

The matrix of  $T$  with respect to the standard basis  $B = \{1, x, x^2, x^3\}$  is given by:

$$\begin{aligned} p(x) = 1 &\implies T(1) = 1 + 1 \\ &= 2 \end{aligned} \quad (2.12.4)$$

$$\begin{aligned} p(x) = x &\implies T(x) = x + 1 + x - 1 \\ &= 2x \end{aligned} \quad (2.12.5)$$

$$\begin{aligned} p(x) = x^2 &\implies T(x^2) = (x+1)^2 + (x-1)^2 \\ &= 2 + 2x^2 \end{aligned} \quad (2.12.6)$$

$$\begin{aligned} p(x) = x^3 &\implies T(x^3) = (x+1)^3 + (x-1)^3 \\ &= 6x + 2x^3 \end{aligned} \quad (2.12.7)$$

Hence, matrix of  $T$  is:

$$\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (2.12.8)$$

See Table 2.12.1

2.13. Let  $\mathbf{M}$  be an  $n \times n$  Hermitian matrix of rank  $k, k \neq n$ . If  $\lambda \neq 0$  is an eigenvalue of  $\mathbf{M}$  with corresponding unit column vector  $\mathbf{u}$ , then which of the following are true?

- a)  $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k - 1$ .
- b)  $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k$ .
- c)  $\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k + 1$ .
- d)  $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*$ .

**Solution:** See Tables 2.13.1 and 2.13.2

$\det(T) = 0$	<b>False.</b> From (2.12.8), it is found that the determinant is not zero as the eigenvalues are nonzero.
$(T - 2I)^4 = 0$ but $(T - 2I)^3 \neq 0$	<b>False.</b> $(T - 2I) = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\implies (T - 2I)^2 = 0$ and hence $(T - 2I)^4 = 0$ and $(T - 2I)^3 = 0$
$(T - 2I)^3 = 0$ but $(T - 2I)^2 \neq 0$	<b>False.</b> Because $(T - 2I)^3 = 0$ and $(T - 2I)^2 = 0$
2 is an eigenvalue with multiplicity 4.	<b>True.</b> It is noted that the matrix of $T$ is an upper triangular matrix having the value 2 along its principal diagonal and hence 2 is an eigenvalue with algebraic multiplicity 4.

TABLE 2.12.1

2.14. Define a real valued function  $B$  on  $\mathbb{R}^2 \times \mathbb{R}^2$  as

$$B(\mathbf{x}, \mathbf{y}) = x_1 y_1 - x_1 y_2 - x_2 y_1 + 4x_2 y_2 \quad (2.14.1)$$

Let  $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and

$$W = \{\mathbf{v} \in \mathbb{R}^2 : B(\mathbf{v}_0, \mathbf{v}) = 0\} \quad (2.14.2)$$

Then  $W$

- a) is not a subspace of  $\mathbb{R}^2$ .
- b) equals  $\mathbf{0}$ .
- c) is the  $y$  axis
- d) is the line passing through  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

**Solution:** See Tables 2.14.1, 2.14.2 and 2.14.3.



Objective	Explanation
Rank of $\mathbf{M} - \lambda \mathbf{u}\mathbf{u}^*$	<p>Since</p> $\text{rank}(\mathbf{A} - \mathbf{B}) \geq \text{rank}(\mathbf{A}) - \text{rank}(\mathbf{B}) \quad (2.13.1)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u}\mathbf{u}^*) \geq \text{rank}(\mathbf{M}) - \text{rank}(\mathbf{u}\mathbf{u}^*) \quad (2.13.2)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u}\mathbf{u}^*) \geq k - \text{rank}(\mathbf{u}\mathbf{u}^*) \quad (2.13.3)$ <p>If <math>\mathbf{A}</math> is a non-zero column vector of order <math>m \times 1</math> and <math>\mathbf{B}</math> is a non-zero row vector of order <math>1 \times n</math> then <math>\text{rank}(\mathbf{AB}) = 1</math>. So,</p> $\text{rank}(\mathbf{u}\mathbf{u}^*) = 1 \quad (2.13.4)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u}\mathbf{u}^*) \geq k - 1 \quad (2.13.5)$ <p>Also since,</p> $\mathbf{M} - \lambda \mathbf{u}\mathbf{u}^* = \mathbf{M} - \mathbf{M}\mathbf{u}\mathbf{u}^* = \mathbf{M}(\mathbf{I} - \mathbf{u}\mathbf{u}^*) \quad (2.13.6)$ <p>and</p> $\text{rank}(\mathbf{M}(\mathbf{I} - \mathbf{u}\mathbf{u}^*)) \leq \min(\text{rank}(\mathbf{M}), \text{rank}(\mathbf{I} - \mathbf{u}\mathbf{u}^*)) \quad (2.13.7)$ $\implies \text{rank}(\mathbf{M}(\mathbf{I} - \mathbf{u}\mathbf{u}^*)) \leq k \quad (2.13.8)$ <p>Thus we have from (2.13.5) and (2.13.8) that</p> $\text{rank}(\mathbf{M} - \lambda \mathbf{u}\mathbf{u}^*) = k - 1 \text{ or } k \quad (2.13.9)$ <p>Consider a matrix</p> $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.13.10)$

TABLE 2.13.1

Objective	Explanation
	<p>such that <math>\text{rank}(M) = 1</math>. The eigenvalue of <math>\mathbf{M}</math> is <math>\lambda = 1</math> and the corresponding eigenvector is</p> $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.13.11)$ <p>Thus we have,</p> $\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (2.13.12)$ $= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.13.13)$ $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.13.14)$ $\implies \text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = 0 \quad (2.13.15)$ <p>Hence if <math>\text{rank}(\mathbf{M}) = k</math> then <math>\text{rank}(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*) = k - 1</math>.</p>
$(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*$	<p>Let the given statement be P(n): <math>(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^n = \mathbf{M}^n - \lambda^n \mathbf{u} \mathbf{u}^*</math>. It can be seen that P(1) is true. Assume P(n) is true for some <math>k \in \mathbf{N}</math> such that</p> $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^k = \mathbf{M}^k - \lambda^k \mathbf{u} \mathbf{u}^* \quad (2.13.16)$ <p>Now to prove that P(k+1) is true we have</p> $(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^{k+1} = (\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)(\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)^k \quad (2.13.17)$ $= (\mathbf{M} - \lambda \mathbf{u} \mathbf{u}^*)(\mathbf{M}^k - \lambda^k \mathbf{u} \mathbf{u}^*) \quad (2.13.18)$ $= \mathbf{M}^{k+1} - \lambda^k \mathbf{M} \mathbf{u} \mathbf{u}^* - \lambda \mathbf{M}^k \mathbf{u} \mathbf{u}^* + \lambda^{k+1} \mathbf{u} \mathbf{u}^* \mathbf{u} \mathbf{u}^* \quad (2.13.19)$ $= \mathbf{M}^{k+1} - \lambda^{k+1} \mathbf{u} \mathbf{u}^* - \lambda^{k+1} \mathbf{u} \mathbf{u}^* + \lambda^{k+1} \mathbf{u} \ \mathbf{u}\ ^2 \mathbf{u}^* \quad (2.13.20)$ $= \mathbf{M}^{k+1} - 2\lambda^{k+1} \mathbf{u} \mathbf{u}^* + \lambda^{k+1} \mathbf{u} \mathbf{u}^* \quad (2.13.21)$ $= \mathbf{M}^{k+1} - \lambda^{k+1} \mathbf{u} \mathbf{u}^* \quad (2.13.22)$ <p>Hence, by the Principle of Mathematical Induction P(n) is true for all <math>n</math>.</p>
Answer	(1) and (4)

TABLE 2.13.2

<b>Subspace</b>	A non-empty subset $\mathbf{W}$ of $\mathbf{V}$ is a subspace of $\mathbf{V}$ if and only if for each pair of vectors $\alpha, \beta$ in $\mathbf{W}$ and each scalar $c$ in $\mathbf{F}$ the vector $c\alpha + \beta$ is again in $\mathbf{W}$ .
-----------------	---

TABLE 2.14.1: Definitions and theorem used

Statement	Observations
Given	$\mathbf{W} = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = 0\} \quad (2.14.3)$
	$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.14.4)$
	$\mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (2.14.5)$
	$\mathbf{v}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.14.6)$
	$\mathbf{B}(\mathbf{v}, \mathbf{w}) = x_1y_1 - x_1y_2 - x_2y_1 + 4x_2y_2 \quad (2.14.7)$
	we will express (2.14.7) in quadratic form.
	$\mathbf{B}(\mathbf{v}, \mathbf{w}) = \mathbf{v}^T \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{w} \quad (2.14.8)$
	From (2.14.4), (2.14.6), (2.14.8) we will calculate $\mathbf{B}(\mathbf{v}_0, \mathbf{v})$
	$\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = \mathbf{v}_0^T \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{v} \quad (2.14.9)$
	$\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.14.10)$
	$\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.14.11)$
	Now we find the basis vector for $\mathbf{W}$ , which is the basis vector of null space of $\mathbf{B}(\mathbf{v}_0, \mathbf{v})$ .
	$\Rightarrow \mathbf{B}(\mathbf{v}_0, \mathbf{v}) = 0 \quad (2.14.12)$
	$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.14.13)$
	$\Rightarrow \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad (2.14.14)$
	$\Rightarrow x_1 = x_2 \quad (2.14.15)$
	Therefore, the basis vector for $\mathbf{W}$ is
	$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.14.16)$
	Therefore
	$\mathbf{W} = \{k\mathbf{b} : \forall k \in \mathbb{R}\} \quad (2.14.17)$

TABLE 2.14.2: Observations

Option	Solution	True/False
1.	<p>Now we will see whether <math>\mathbf{W}</math> is a subspace or not.  Let <math>\alpha, \beta</math> be two pair of vectors in <math>\mathbf{W}</math> where</p> $\alpha = m\mathbf{b} \quad (2.14.18)$ $\beta = n\mathbf{b} \quad (2.14.19)$ <p>Here <math>m, n \in \mathbb{R}</math> and now we will see whether the vector <math>c\alpha + \beta</math> is in <math>\mathbf{W}</math> or not where <math>c</math> is a scalar value in <math>\mathbb{R}</math>.  Here</p> $c\alpha + \beta = cm\mathbf{b} + n\mathbf{b} \quad (2.14.20)$ $\Rightarrow c\alpha + \beta = (cm + n)\mathbf{b} \quad (2.14.21)$ <p>From (2.14.21), <math>(cm + n) \in \mathbb{R}</math> and we can say that the vector <math>c\alpha + \beta \in \mathbf{W}</math>.  Therefore, <math>\mathbf{W}</math> is a subspace of <math>\mathbb{R}^2</math></p>	
2.	<p>From Table 2.14.2, we got <math>\mathbf{W}</math> contains the vectors which are all linear combination of basis vector <math>\mathbf{b}</math> as shown in (2.14.17).  Therefore,</p> $\mathbf{W} \neq \{(0, 0)\} \quad (2.14.22)$	False
3.	<p>Let us consider a vector on y-axis</p> $\mathbf{p} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (2.14.23)$ <p>Here</p> $\mathbf{p} \neq k\mathbf{b} \quad (2.14.24)$ <p>for any <math>k \in \mathbb{R}</math>  The vector <math>\mathbf{p}</math> can not be written in terms of the basis vector <math>\mathbf{b}</math>. Then <math>\mathbf{p} \notin \mathbf{W}</math>.  Therefore, the vectors in <math>\mathbf{W}</math> is not y-axis.</p>	False
4.	<p>There is only one basis vector <math>\mathbf{b}</math> for <math>\mathbf{W}</math>. Therefore the vectors in <math>\mathbf{W}</math> forms a straight line in vector space <math>\mathbb{R}^2</math>.  Since,</p> $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\mathbf{b} \quad (2.14.25)$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\mathbf{b} \quad (2.14.26)$ <p>Therefore, the line passes through (0,0) and (1,1).</p>	True

TABLE 2.14.3: Solution

2.15. Consider the Quadratic forms

$$Q_1(x, y) = xy \quad (2.15.1)$$

$$Q_2(x, y) = x^2 + 2xy + y^2 \quad (2.15.2)$$

$$Q_3(x, y) = x^2 + 3xy + 2y^2 \quad (2.15.3)$$

on  $\mathbb{R}^2$ . Choose the correct statements from below

- a)  $Q_1$  and  $Q_2$  are equivalent.
- b)  $Q_1$  and  $Q_3$  are equivalent.
- c)  $Q_2$  and  $Q_3$  are equivalent.
- d) all are equivalent.

**Solution:** See Tables 2.15.1 2.15.2

Matrix representation	<p>The Matrix representation of quadratic forms</p> $Q(x, y) = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{X}^T \mathbf{A} \mathbf{X} \quad (2.15.4)$ <p>The symmetric matrix of the quadratic form is</p> $\mathbf{A} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.15.5)$
Equivalent condition	<p>Two quadratic forms <math>\mathbf{X}^T \mathbf{A} \mathbf{X}</math> and <math>\mathbf{Y}^T \mathbf{B} \mathbf{Y}</math> are called equivalent if their matrices, A and B are congruent.</p> <p>Two real quadratic forms are equivalent over the real field iff they have the same rank and the same index.</p>
Rank	The rank of a quadratic form is the rank of its associated matrix.
Index	The index of the quadratic form is equal to the number of positive eigen values of the matrix of quadratic form.

TABLE 2.15.1: Definitions and results used

	Matrix	Rank	Eigen Values	Index
$Q_1(x, y)$	$\mathbf{A}_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_1]{R_1 \leftarrow R_2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\text{rank}(\mathbf{A}_1) = 2$	$ \mathbf{A}_1 - \lambda \mathbf{I}  = 0$ $\Rightarrow \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0$ $\Rightarrow (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = 0$ $\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$	Index of $\mathbf{A}_1 = 1$
$Q_2(x, y)$	$\mathbf{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\text{rank}(\mathbf{A}_2) = 1$	$ \mathbf{A}_2 - \lambda \mathbf{I}  = 0$ $\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (\lambda)(\lambda - 2) = 0$ $\Rightarrow \lambda_1 = 0, \lambda_2 = 2$	Index of $\mathbf{A}_2 = 2$
$Q_3(x, y)$	$\mathbf{A}_3 = \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 2 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{1}{4} \end{pmatrix}$ $\text{rank}(\mathbf{A}_3) = 2$	$ \mathbf{A}_3 - \lambda \mathbf{I}  = 0$ $\Rightarrow \begin{vmatrix} 1-\lambda & \frac{3}{2} \\ \frac{3}{2} & 2-\lambda \end{vmatrix} = 0$ $\Rightarrow (\lambda - \frac{\sqrt{10}+3}{2})(\lambda + \frac{\sqrt{10}-3}{2}) = 0$ $\Rightarrow \lambda_1 = \frac{3+\sqrt{10}}{2}, \lambda_2 = \frac{3-\sqrt{10}}{2}$	Index of $\mathbf{A}_3 = 1$
Conclusion	We can say that $Q_1(x, y)$ and $Q_3(x, y)$ are equivalent as the rank and index are same.			

TABLE 2.15.2: Finding which quadratic forms are equivalent

2.16. Consider a Markov Chain with state space  $\{0, 1, 2\}$  and transition matrix

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{matrix} \quad (2.16.1)$$

For any two states  $i$  and  $j$ , let  $p_{ij}^{(n)}$  denote the  $n$ -step transition probability of going from  $i$  to  $j$ . Identify correct statements.

- a)  $\lim_{n \rightarrow \infty} p_{11}^{(n)} = \frac{2}{9}$
- b)  $\lim_{n \rightarrow \infty} p_{21}^{(n)} = 0$
- c)  $\lim_{n \rightarrow \infty} p_{32}^{(n)} = \frac{1}{3}$
- d)  $\lim_{n \rightarrow \infty} p_{13}^{(n)} = \frac{1}{3}$

**Solution:** See Tables 2.16.1 and 2.16.2



Irreducible Markov Chain	A Markov chain is <b>irreducible</b> if all the states communicate with each other, i.e., if there is only one communication class.
Aperiodic Markov Chain	If there is a self-transition in the chain ( $p^{ii} > 0$ for some $i$ ), then the chain is called as <b>aperiodic</b>
Stationary Distribution	<p>A stationary distribution of a Markov chain is a probability distribution that remains unchanged in the Markov chain as time progresses. Typically, it is represented as a row vector <math>\pi</math> whose entries are probabilities summing to 1, and given transition matrix <math>\mathbf{P}</math>, it satisfies</p> $\pi = \pi \mathbf{P}$

TABLE 2.16.1

Drawing Transition diagram	<pre> graph TD     1((1)) -- 1/2 --&gt; 1     1 -- 1/2 --&gt; 2((2))     2 -- 1/2 --&gt; 2     2 -- 1/3 --&gt; 3((3))     3 -- 1/3 --&gt; 1     3 -- 1/2 --&gt; 2     3 -- 1/3 --&gt; 3 </pre>
Checking whether the chain is Irreducible and Aperiodic	<p>Here, All the states are accessible to one another. <math>\Rightarrow</math> They are in the same communication class. So, it is Irreducible.</p> <p>There exists the non- zero self-transition, which means that the chain is Aperiodic.</p> <p>We know that if the Markov Chain is irreducible and aperiodic then  <math display="block">\pi_j = \lim_{n \rightarrow \infty} P\{X_n = j\}, j = 1, \dots, N</math> These are the stationary probabilities.</p>
Finding the Stationary	Stationary Probability can be represented as

## Probability Distributions

$$\pi = \pi \mathbf{P}$$

$$\Rightarrow \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \mathbf{P}$$

Equating the above equation we get

$$\frac{1}{2}v_1 - \frac{1}{3}v_3 = 0$$

$$\frac{1}{2}v_1 - \frac{1}{2}v_2 + \frac{1}{3}v_3 = 0$$

$$\frac{1}{2}v_2 - \frac{2}{3}v_3 = 0$$

We see that summation of second and the third equation gives us the first equation only.

And we know that the probability distribution will sum up to 1.

$$v_1 + v_2 + v_3 = 1$$

Therefore, we get the equation form as

$$\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

## Solving the linear equations

The above linear equation can be solved using Gauss-Jordan method as

$$\left( \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{3} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & 0 \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -\frac{5}{6} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{3} & 0 \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - \frac{1}{2}R_1} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -\frac{5}{6} & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{6} & -\frac{1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & -1 & -\frac{1}{6} & -\frac{1}{2} \end{array} \right)$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + R_2} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & \frac{2}{3} & \frac{1}{2} \end{array} \right)$$

	$\xleftrightarrow{R_3 \leftarrow \frac{3}{2}R_3} \left( \begin{array}{ccc c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{5}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ $\xleftrightarrow{R_2 \leftarrow R_2 - \frac{5}{3}R_3} \left( \begin{array}{ccc c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ $\xleftrightarrow{R_1 \leftarrow R_1 - R_3} \left( \begin{array}{ccc c} 1 & 1 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ $\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \left( \begin{array}{ccc c} 1 & 0 & 0 & \frac{2}{9} \\ 0 & 1 & 0 & \frac{4}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array} \right)$ <p><math>\therefore</math>, stationary probability distribution <math>\pi</math> is given by</p> $\pi = \left( \frac{2}{9} \quad \frac{4}{9} \quad \frac{1}{3} \right)$
Observations	<p>Since the given transition probability matrix <math>\mathbf{P}</math> is irreducible and aperiodic, then <math>\lim_{n \rightarrow \infty} \mathbf{P}^n</math> converges to a matrix with all rows identical and equal to <math>\pi</math>.</p> <p>We were able to find <math>\pi</math> as <math>\left( \frac{2}{9} \quad \frac{4}{9} \quad \frac{1}{3} \right)</math></p> $\lim_{n \rightarrow \infty} \mathbf{P}^n = \begin{pmatrix} \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \\ \frac{2}{9} & \frac{4}{9} & \frac{1}{3} \end{pmatrix}$ <p>From the above matrix, we get</p> $\lim_{n \rightarrow \infty} \mathbf{P}_{11}^n = \frac{2}{9}$ $\lim_{n \rightarrow \infty} \mathbf{P}_{21}^n = \frac{2}{9}$ $\lim_{n \rightarrow \infty} \mathbf{P}_{32}^n = \frac{4}{9}$ $\lim_{n \rightarrow \infty} \mathbf{P}_{13}^n = \frac{1}{3}$
Conclusion	<p>From our observation we see that</p> <p>Options 1) and 4) are True.</p>

TABLE 2.16.2

3 JUNE 2018

3.1. Let  $\mathbf{A}$  be a  $(m \times n)$  matrix and  $\mathbf{B}$  be a  $(n \times m)$  matrix over real numbers with  $m < n$ . Then

- a)  $\mathbf{AB}$  is always nonsingular.
- b)  $\mathbf{AB}$  is always singular.
- c)  $\mathbf{BA}$  is always nonsingular.
- d)  $\mathbf{BA}$  is always singular.

**Solution:** See Table 3.1.1

$$\text{rank}(\mathbf{A}) \leq \min(m, n) \quad (3.1.1)$$

$$\implies \leq m, \because m < n \quad (3.1.2)$$

$$\text{rank}(\mathbf{B}) \leq \min(n, m) \quad (3.1.3)$$

$$\implies \leq m, \because m < n \quad (3.1.4)$$

We also know that  $\mathbf{AB}$  will be a  $m \times m$  matrix and  $\mathbf{BA}$  will be a  $n \times n$  matrix.

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (3.1.5)$$

$$\implies \leq m \quad (3.1.6)$$

$$\text{rank}(\mathbf{BA}) \leq \min(\text{rank}(\mathbf{B}), \text{rank}(\mathbf{A})) \quad (3.1.7)$$

$$\implies \leq m \quad (3.1.8)$$

3.2. If  $\mathbf{A}$  is a  $(2 \times 2)$  matrix over  $\mathbb{R}$  with  $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ . Then we can conclude that

- a)  $\det(\mathbf{A}) = 0$ .
- b)  $\mathbf{A} = 0$ .
- c)  $\text{tr}(\mathbf{A}) = 0$ .
- d)  $\mathbf{A}$  is nonsingular.

**Solution:** See Table 3.2.1

Options	Explanation
<b>AB</b> is always nonsingular	$rank(\mathbf{AB}) \leq m$ Let, $rank(\mathbf{AB}) = k, k < m$ . So, there are $m - k$ linearly dependent columns or rows So, <b>AB</b> will be singular Hence, incorrect  Example $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 5 & 6 \end{pmatrix}$ $\mathbf{AB} = \begin{pmatrix} 20 & 33 \\ 40 & 66 \end{pmatrix}, rank(\mathbf{AB}) = 1$ $2^{nd}$ row is linearly dependent on $1^{st}$ row. <b>AB</b> is singular
<b>AB</b> is always singular	$rank(\mathbf{AB}) \leq m$ Let, $rank(\mathbf{AB}) = m$ So, there are 0 linearly dependent columns or rows So, <b>AB</b> will be nonsingular Hence, incorrect  Example $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$ $\mathbf{AB} = \begin{pmatrix} 20 & 29 \\ 35 & 52 \end{pmatrix}, rank(\mathbf{AB}) = 2$ <b>AB</b> is nonsingular
<b>BA</b> is always nonsingular	$rank(\mathbf{BA}) \leq m, rank(\mathbf{BA})$ can be atmost $m$ <b>BA</b> is $n \times n$ matrix. $n > m$ . So, there are atleast $n - m$ linearly dependent columns or rows. So, <b>BA</b> will be singular always. Hence, incorrect  Example $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} 7 & 14 & 18 \\ 10 & 20 & 26 \\ 17 & 34 & 45 \end{pmatrix}, rank(\mathbf{BA}) = 2$ $2^{nd}$ column is linearly dependent on $1^{st}$ column <b>BA</b> is singular
<b>BA</b> is always singular	$rank(\mathbf{BA}) \leq m, rank(\mathbf{BA})$ can be atmost $m$ <b>BA</b> is $n \times n$ matrix. $n > m$ . So, there are atleast $n - m$ linearly dependent columns or rows. So, <b>BA</b> will be singular always. Hence, correct Same example as above. <b>BA</b> is always singular.

TABLE 3.1.1: Finding Correct Option

Given	<p><math>\mathbf{A}</math> be a <math>2 \times 2</math> matrix over <math>\mathbb{R}</math> with</p> $\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$
Explanation	<p>If <math>\mathbf{X}</math> is an eigen vector of matrix <math>\mathbf{A}</math> corresponding to the eigen value <math>\lambda</math> i.e</p> $\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$ <p>then, <math>(\mathbf{I} + \mathbf{A})\mathbf{X} = (1 + \lambda)\mathbf{X}</math></p> <p>Thus, <math>\mathbf{X}</math> is an eigen vector of <math>(\mathbf{A} + \mathbf{I})</math> corresponding to the eigen value <math>(1 + \lambda)</math>.</p> <p>Let <math>\lambda_1, \lambda_2</math> be two eigen values of <math>\mathbf{A}</math> and <math>(1 + \lambda_1), (1 + \lambda_2)</math> be the eigen values of <math>(\mathbf{A} + \mathbf{I})</math>.</p> <p><math>\Rightarrow</math> Eigen value of <math>\mathbf{A} = \lambda_1, \lambda_2</math></p> <p><math>\Rightarrow</math> Eigen value of <math>(\mathbf{A} + \mathbf{I}) = \lambda_1 + 1, \lambda_2 + 1</math></p> <p>Since,</p>
	$\det(\mathbf{A} + \mathbf{I}) = 1 + \det(\mathbf{A})$ <p>Trace of any matrix is sum of its eigen values.</p> <p>Determinant of matrix is product of its eigen values</p> $\Rightarrow (\lambda_1 + 1)(\lambda_2 + 1) = 1 + (\lambda_1\lambda_2)$ $\Rightarrow \boxed{\lambda_1 + \lambda_2 = 0}$ $\Rightarrow \boxed{tr(\mathbf{A}) = 0}$
<b>Statement 1</b> : $\det \mathbf{A} = 0$	False
	<p>Let, <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 \\ 0 &amp; -1 \end{pmatrix}</math></p> <p>Here, <math>\det \mathbf{A} = -1</math> and <math>\det(\mathbf{A} + \mathbf{I}) = 0</math></p> <p>Thus, <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p> <p>In this case,  <math>\det \mathbf{A} \neq 0</math>  but satisfy the given condition i.e <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p>

<b>Statement 2 : <math>\mathbf{A} = \mathbf{0}</math></b>	False
	<p>Let , <math>\mathbf{A} = \begin{pmatrix} 0 &amp; 1 \\ 0 &amp; 0 \end{pmatrix}</math></p> <p>Here, <math>\det \mathbf{A} = 0</math> and <math>\det(\mathbf{A} + \mathbf{I}) = 1</math></p> <p>Thus, <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p> <p>In this case,  <math>\mathbf{A} \neq \mathbf{0}</math>          But , satisfy the given condition i.e <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p>
<b>Statement 3: <math>\text{tr}(\mathbf{A}) = 0</math></b>	True
	<p>The given statement is true for all possible matrices.</p> <p>If <math>\text{tr} \mathbf{A} \neq 0</math> then the given condition i.e <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math> doesn't satisfy.</p> <p>Let , <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 0 \end{pmatrix}</math></p>
	<p>Here, <math>\det \mathbf{A} = 0</math> , <math>\det(\mathbf{A} + \mathbf{I}) = 2</math>, <math>\text{tr} \mathbf{A} \neq 0</math></p> <p>Thus, <math>1 + \det(\mathbf{A}) \neq \det(\mathbf{A} + \mathbf{I})</math></p>
<b>Statement 4: <math>\mathbf{A}</math> is non singular</b>	False
	<p>Non Singular Matrix: A non-singular matrix is a square one whose determinant is not zero. non-singular matrix is also a full rank matrix.</p> <p>Let, <math>\mathbf{A} = \begin{pmatrix} 0 &amp; 0 \\ 0 &amp; 0 \end{pmatrix}</math></p> <p>Here, <math>\det \mathbf{A} = 0</math> and <math>\det(\mathbf{A} + \mathbf{I}) = 1</math></p> <p>Thus, <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p> <p>In this case,  <math>\mathbf{A}</math> is Singular,          But satisfy the given condition i.e <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p>
<b>Conclusion</b>	<p>Thus, we can conclude Statement 3 is true for all possible matrices which satisfy the given condition i.e <math>1 + \det(\mathbf{A}) = \det(\mathbf{A} + \mathbf{I})</math></p>

TABLE 3.2.1: Solution Summary

## 3.3. The system of equations

$$x + 2x^2 + 3xy = 6 \quad (3.3.1)$$

$$x + x^2 + 3xy + y = 5 \quad (3.3.2)$$

$$x - x^2 + y = 7 \quad (3.3.3)$$

- a) has solutions in rational numbers.
- b) has solutions in real numbers.
- c) has solutions in complex numbers.
- d) has no solutions.

## 3.4. The trace of the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20} \quad (3.4.1)$$

is

- a)  $7^{20}$ .
- b)  $2^{20} + 3^{20}$ .
- c)  $2^{21} + 3^{20}$ .
- d)  $2^{20} + 3^{20} + 1$ .

**Solution:** Let,

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad (3.4.2)$$

To find the eigen values of  $\mathbf{A}$ :

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad (3.4.3)$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \quad (3.4.4)$$

$$\Rightarrow (2-\lambda)(2-\lambda)(3-\lambda) = 0 \quad (3.4.5)$$

$$\Rightarrow \lambda = 2, 2, 3 \quad (3.4.6)$$

Eigen values of  $\mathbf{A}$  are 2,2,3.

Hence, the eigen values of  $\mathbf{A}^{20}$  are:  $2^{20}, 2^{20}$  and  $3^{20}$  respectively.

As we know that the sum of eigen values of a matrix equals the trace of the matrix, hence, the trace of  $\mathbf{A}^{20}$  is:

$$tr = 2^{20} + 2^{20} + 3^{20} \quad (3.4.7)$$

$$= 2 \cdot 2^{20} + 3^{20} \quad (3.4.8)$$

Therefore, option 3 is the required answer.

3.5. Given that there are real constants  $a, b, c, d$  such that the identity

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2, \quad \forall x, y \in \mathbb{R} \quad (3.5.1)$$

This implies that

- a)  $\lambda = -5$
- b)  $\lambda \geq 1$
- c)  $0 < \lambda < 1$
- d) There is no such  $\lambda \in \mathbb{R}$

**Solution:** Given that

$$\lambda x^2 + 2xy + y^2 = (ax + by)^2 + (cx + dy)^2 \quad (3.5.2)$$

Arranging this in form of a matrix,

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.5.3)$$

From this, we get

$$\lambda = a^2 + c^2 \quad (3.5.4)$$

$$ab + cd = 1 \quad (3.5.5)$$

$$b^2 + d^2 = 1 \quad (3.5.6)$$

Let

$$\mathbf{u} = \begin{pmatrix} a \\ c \end{pmatrix} \quad (3.5.7)$$

$$\mathbf{v} = \begin{pmatrix} b \\ d \end{pmatrix} \quad (3.5.8)$$

$$\|\mathbf{u}\|^2 = a^2 + c^2 = \lambda \quad (3.5.9)$$

$$\|\mathbf{v}\|^2 = b^2 + d^2 = 1 \quad (3.5.10)$$

Then,

$$\mathbf{u}^T \mathbf{v} = \begin{pmatrix} a & c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} = ab + cd = 1 \quad (3.5.11)$$

Using the Cauchy-Schwartz Inequality, we get

$$|\mathbf{u}^T \mathbf{v}|^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \quad (3.5.12)$$

Now, substituting values from (3.5.9), (3.5.10), (3.5.11) above,

$$\Rightarrow 1 \leq \lambda \quad (3.5.13)$$

So from the given options, option 2)  $\lambda \geq 1$  is correct.

3.6. Let  $\mathbb{R}^n, n \geq 2$ , be equipped with the standard inner product. Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be  $n$  column vectors forming an orthonormal basis of  $\mathbb{R}^n$ . Let  $A$  be the  $n \times n$  matrix formed by the column vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . Then



- a)  $\mathbf{A} = \mathbf{A}^{-1}$       c)  $\mathbf{A}^{-1} = \mathbf{A}^\top$   
 b)  $\mathbf{A} = \mathbf{A}^\top$       d)  $\det(\mathbf{A}) = 1$

3.7. Consider a Markov Chain with state space  $\{1, 2, 3, 4\}$  and transition matrix

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Then,

- a)  $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$   
 b)  $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 0, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$   
 c)  $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} = \infty$   
 d)  $\lim_{n \rightarrow \infty} p_{22}^{(n)} = 1, \sum_{n=0}^{\infty} p_{22}^{(n)} < \infty$

3.8. Let  $V$  denote the vector space of all sequences  $\mathbf{a} = (a_1, a_2, \dots)$  of real numbers such that

$$\sum_n 2^n |a_n| \quad (3.8.1)$$

converges. Define

$$\|\cdot\| : V \rightarrow \mathbb{R} \quad (3.8.2)$$

by

$$\|\mathbf{a}\| = \sum_n 2^n |a_n|. \quad (3.8.3)$$

Which of the following are true?

- a)  $V$  contains only the sequence  $(0, 0, \dots)$   
 b)  $V$  is finite dimensional  
 c)  $V$  has a countable linear basis  
 d)  $V$  is a complete normed space

3.9. Let  $V$  be a vector space over  $\mathbb{C}$  with dimension  $n$ . Let  $T : V \rightarrow V$  be a linear transformation with only 1 as eigenvalue. Then which of the following must be true?

- a)  $T - I = 0$   
 b)  $(T - I)^{n-1} = 0$   
 c)  $(T - I)^n = 0$   
 d)  $(T - I)^{2n} = 0$

3.10. If  $\mathbf{A}$  is a  $5 \times 5$  matrix and the dimension of the solution space of  $\mathbf{A}\mathbf{x} = 0$  is at least two, then

- a)  $\text{rank}(\mathbf{A}^2) \leq 3$   
 b)  $\text{rank}(\mathbf{A}^2) \geq 3$   
 c)  $\text{rank}(\mathbf{A}^2) = 3$   
 d)  $\det(\mathbf{A}^2) = 0$

3.11. Let  $\mathbf{A} \in M_3(\mathbb{R})$  be such that  $\mathbf{A}^3 = \mathbf{I}_{3 \times 3}$ . Then

- a) minimal polynomial of  $\mathbf{A}$  can only be of degree 2  
 b) minimal polynomial of  $\mathbf{A}$  can only be of degree 3  
 c) either  $\mathbf{A} = \mathbf{I}$  or  $\mathbf{A} = -\mathbf{I}$   
 d) there can be uncountably many  $\mathbf{A}$  satisfying the above.

(3.7.1) 3.12. Let  $\mathbf{A}$  be an  $n \times n, n > 1$  matrix satisfying

$$\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I} = \mathbf{0} \quad (3.12.1)$$

Then which of the following statements is true?

- a)  $\mathbf{A}$  is invertible  
 b)  $t^2 - 7t + 12n = 0$  where  $t = \text{tr}(\mathbf{A})$   
 c)  $d^2 - 7d + 12 = 0$  where  $d = \det(\mathbf{A})$   
 d)  $\lambda^2 - 7\lambda + 12 = 0$  where  $\lambda$  is an eigenvalue of  $\mathbf{A}$

**Solution:** See Table 3.12.1

Given	<p><math>\mathbf{A}</math> be the <math>n \times n</math> matrix where <math>n &gt; 1</math> satisfying the following equation</p> $\mathbf{A}^2 - 7\mathbf{A} + 12\mathbf{I}_{n \times n} = \mathbf{0}_{n \times n} \quad (3.12.2)$
Explanation	<p>The Cayley Hamilton Theorem states that every square matrix satisfies its own characteristic equation. Using this theorem the given equation (3.12.2) can be written as ,</p> $\lambda^2 - 7\lambda + 12 = 0 \quad (3.12.3)$ $(\lambda - 4)(\lambda - 3) = 0 \quad (3.12.4)$ $\lambda_1 = 3 \quad (3.12.5)$ $\lambda_2 = 4 \quad (3.12.6)$ <p>Here <math>\lambda_1</math> and <math>\lambda_2</math> were eigen values of matrix <math>\mathbf{A}</math> We know that determinant is product of eigen values.</p> $d = \text{Det}(\mathbf{A}) \quad (3.12.7)$ $\implies d = \lambda_1 \lambda_2 \quad (3.12.8)$ $\implies d = 12 \neq 0 \quad (3.12.9)$
<b>Statement 1</b>	$\mathbf{A}$ is invertible
	<p>From equation (3.12.9), since <math>d \neq 0</math> the given matrix <math>\mathbf{A}</math> is Invertible. <b>True Statement</b></p>
<b>Statement 2</b>	$t^2 - 7t + 12n = 0 \quad (3.12.10)$
	<p>We know that the trace is the sum of the eigen values.</p> $t = \text{Tr}(\mathbf{A}) \quad (3.12.11)$ $\implies t = \lambda_1 + \lambda_2 \quad (3.12.12)$ $\implies t = 7 \quad (3.12.13)$ <p>Substituting the equation (3.12.13) in (3.12.10) we get,</p> $7^2 - 7(7) + 12n = 0 \quad (3.12.14)$ $12n = 0 \quad (3.12.15)$ <p>Since given that <math>n &gt; 1</math> the equation (3.12.15) is not possible i.e <math>12n \neq 0</math>. Therefore, <math>t^2 - 7t + 12n = 0</math> is a <b>False Statement</b></p>
<b>Statement 3</b>	$d^2 - 7d + 12 = 0 \quad (3.12.16)$
	<p>Substituting the equation (3.12.9) in (3.12.16), we get,</p> $12^2 - 7(12) + 12 = 0 \quad (3.12.17)$ $72 = 0 \quad (3.12.18)$ <p>From equation (3.12.15) it is clear that the above statement 3 is invalid. <b>False Statement</b></p>

Statement 4	$\lambda^2 - 7\lambda + 12 = 0$ (3.12.19)
	By Cayley Hamilton Theorem, equation (3.12.3) shows that the above statement 4 is valid. <b>True Statement</b>

TABLE 3.12.1: Explanation

3.13. Let  $\mathbf{A}$  be a  $6 \times 6$  matrix over  $\mathbb{R}$  with characteristic polynomial

$$(x-3)^2(x-2)^4 \quad (3.13.1)$$

and minimal polynomial

$$(x-3)(x-2)^2 \quad (3.13.2)$$

Then the Jordan canonical form of  $\mathbf{A}$  can be

a) 
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

b) 
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

c) 
$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

d) 
$$\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

**Solution:** See Tables 3.13.1 and 3.13.1

Jordan canonical form	<p>If <math>\mathbf{A}</math> is a matrix of order <math>n \times n</math>, then the Jordan canonical form of <math>\mathbf{A}</math> is a matrix of order <math>n \times n</math> expressed as</p> $\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & & \\ & \ddots & \\ & & \mathbf{J}_k \end{pmatrix} \quad (3.13.3)$ <p>where <math>\mathbf{J}_1, \dots, \mathbf{J}_k</math> are the Jordan blocks.</p>
Algebraic multiplicity $A_M$	<p>Algebraic multiplicity of characteristic value <math>\lambda</math> in the characteristic polynomial determines the size of Jordan block for that eigen value</p> $A_M = \text{Size of Jordan block for that } \lambda \quad (3.13.4)$
Geometric multiplicity $G_M$	<p>Geometric multiplicity determines the number of Jordan sub-blocks in a Jordan block for <math>\lambda</math></p>
Minimal Polynomial	<p>The multiplicity of <math>\lambda</math> in the minimal polynomial determines the size of the largest sub-block.</p>

TABLE 3.13.1: Definition and Properties used

Characteristic polynomial	$p(x) = (x - 3)^2 (x - 2)^4 \quad (3.13.5)$
Algebraic Multiplicity	<p>For <math>\lambda = 3, A_M = 2</math> <span style="float: right;">(3.13.6)</span>  For <math>\lambda = 2, A_M = 4</math> <span style="float: right;">(3.13.7)</span></p>
Minimal polynomial	$m(x) = (x - 3)(x - 2)^2 \quad (3.13.8)$
Finding Jordan blocks for $\lambda_1=3$	<p>For <math>\lambda_1=3</math>, We can write from table 3.13.1 that</p> <p style="text-align: center;">The highest order of Jordan block = 1  Size of Jordan block = <math>A_M = 2</math></p> <p>The Jordan blocks for <math>\lambda_1=3</math></p>

	$\mathbf{J}_1 = (3), \mathbf{J}_2 = (3) \quad (3.13.9)$
Finding Jordan blocks for $\lambda_1=2$	<p>For <math>\lambda_1=2</math>, We can write from table 3.13.1 that</p> <p style="text-align: center;">The highest order of Jordan block = 2 Size of Jordan block = <math>A_M = 4</math></p> <p>The Jordan blocks for <math>\lambda_1=3</math></p> $\mathbf{J}_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \mathbf{J}_4 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad (3.13.10)$ <p style="text-align: center;">or</p> $\mathbf{J}_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \mathbf{J}_4 = (2), \mathbf{J}_5 = (2) \quad (3.13.11)$
Jordan canonical form	<p>Jordan canonical form of <math>\mathbf{A}</math> is</p> $\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \mathbf{J}_3 & \\ & & & \mathbf{J}_4 \end{pmatrix} \text{ or } \begin{pmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \mathbf{J}_3 & \\ & & & \mathbf{J}_4 & \\ & & & & \mathbf{J}_5 \end{pmatrix} \quad (3.13.12)$ $\begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad (3.13.13)$
Conclusion	From above, we can say that options 2) and 3) are correct.

TABLE 3.13.2: Finding Jordan canonical form

3.14. Let  $V$  be an inner product space and  $S$  be a subset of  $V$ . Let  $\bar{S}$  denote the closure of  $S$  in  $V$  with respect to the topology induced by the metric given by the inner product. Which of the following statements is true?

- a)  $S = (S^\perp)^\perp$
- b)  $\bar{S} = (S^\perp)^\perp$
- c)  $\text{span}(S) = (S^\perp)^\perp$
- d)  $S^\perp = ((S^\perp)^\perp)^\perp$

3.15. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.15.1)$$

and

$$Q(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \quad (3.15.2)$$

Which of the following statements is true?

- a) The matrix of second order partial derivatives of the quadratic form  $Q$  is  $2\mathbf{A}$
- b) The rank of the quadratic form  $Q$  is 2
- c) The signature of the quadratic form  $Q$  is  $++0$
- d) The quadratic form  $Q$  take the value 0 for some non-zero vector  $\mathbf{x}$

**Solution:** See Tables 3.15.1 and 3.15.2

Quadratic Form of a matrix	Let $\mathbf{V}$ be a vector space over $\mathbb{R}$ . $\mathbf{A}$ be a symmetric matrix $n \times n$ . Quadratic form on $\mathbf{V}$ is a real function, $(\mathbf{F} : \mathbf{V} \rightarrow \mathbb{R})$ defined as $F(x) = \mathbf{xAx}^T = \sum_{i,j=1}^n a_{ij}x_i x_j$ where $\mathbf{x} \in \mathbf{V}$
Signature of Quadratic form	The signature of quadratic form is $(n_+, n_-, n_0)$ where $n_+$ is the number of positive entries, $n_-$ is number of negative entries and $n_0$ is number of zero's in $a_{ii}$
Rank of quadratic form	Rank of quadratic form is the rank of its matrix which is maximum number of linearly independent rows/columns of a matrix

TABLE 3.15.1: Definitions

<b>Option 1</b>	The matrix of second order partial derivatives of the quadratic form of $\mathbf{Q}$ is $2\mathbf{A}$ .
Solution	$\mathbf{Q}(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} x+2y \\ -2z \\ z \end{pmatrix} = x^2 + z^2 + 2xy - 2yz$ <p>First order partial derivatives: <math>\frac{\partial \mathbf{Q}}{\partial x} = 2x + 2y</math> <math>\frac{\partial \mathbf{Q}}{\partial y} = 2x - 2z</math> <math>\frac{\partial \mathbf{Q}}{\partial z} = 2z - 2y</math></p> <p>Second order partial derivatives of: <math>\frac{\partial^2 \mathbf{Q}}{\partial x^2} = 2</math> <math>\frac{\partial^2 \mathbf{Q}}{\partial y^2} = 0</math> <math>\frac{\partial^2 \mathbf{Q}}{\partial z^2} = 2</math></p> <p><math>\frac{\partial^2 \mathbf{Q}}{\partial x \partial y} = \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} = 2</math> <math>\frac{\partial^2 \mathbf{Q}}{\partial x \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} = 0</math> <math>\frac{\partial^2 \mathbf{Q}}{\partial y \partial z} = \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} = -2</math></p> <p>Matrix of second order partial derivatives <math>\mathbf{Q}</math>: <math>\begin{pmatrix} \frac{\partial^2 \mathbf{Q}}{\partial x^2} &amp; \frac{\partial^2 \mathbf{Q}}{\partial x \partial y} &amp; \frac{\partial^2 \mathbf{Q}}{\partial x \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial y \partial x} &amp; \frac{\partial^2 \mathbf{Q}}{\partial y^2} &amp; \frac{\partial^2 \mathbf{Q}}{\partial y \partial z} \\ \frac{\partial^2 \mathbf{Q}}{\partial z \partial x} &amp; \frac{\partial^2 \mathbf{Q}}{\partial z \partial y} &amp; \frac{\partial^2 \mathbf{Q}}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 &amp; 2 &amp; 0 \\ 2 &amp; 0 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix} \neq 2\mathbf{A}</math></p> <p>Hence, <b>Option 1</b> is not correct.</p>
<b>Option 2</b>	The rank of the quadratic form of $\mathbf{Q}$ is 2
Solution	<p>From above we have quadratic form of <math>\mathbf{Q} = \begin{pmatrix} 2 &amp; 2 &amp; 0 \\ 2 &amp; 0 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix}</math></p> <p>Echelon form reduction: <math>\begin{pmatrix} 2 &amp; 2 &amp; 0 \\ 2 &amp; 0 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix} \xrightarrow{R_1 = \frac{1}{2}R_1} \begin{pmatrix} 1 &amp; 1 &amp; 0 \\ 2 &amp; 0 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 &amp; 1 &amp; 0 \\ 0 &amp; -2 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix}</math></p> <p><math>\xrightarrow{R_2 \rightarrow -\frac{1}{2}R_2} \begin{pmatrix} 1 &amp; 1 &amp; 0 \\ 0 &amp; 1 &amp; 1 \\ 0 &amp; -2 &amp; 2 \end{pmatrix} \xrightarrow{R_3 \rightarrow R_3 + 2R_2} \begin{pmatrix} 1 &amp; 1 &amp; 0 \\ 0 &amp; 1 &amp; 1 \\ 0 &amp; 0 &amp; 4 \end{pmatrix} \xrightarrow{R_3 \rightarrow \frac{1}{4}R_3} \begin{pmatrix} 1 &amp; 1 &amp; 0 \\ 0 &amp; 1 &amp; 1 \\ 0 &amp; 0 &amp; 1 \end{pmatrix}</math></p> <p><math>\xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 1 \\ 0 &amp; 0 &amp; 1 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - R_3} \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 0 \\ 0 &amp; 0 &amp; 1 \end{pmatrix}</math></p> <p>Rank = Number of non-zero rows = 3 <math>\neq</math> 2</p> <p>Hence, <b>Option 2</b> is not correct.</p>
<b>Option 3</b>	The signature of the quadratic form $\mathbf{Q}$ is $(+ + 0)$
Solution	<p>From above we have quadratic form of <math>\mathbf{Q} = \begin{pmatrix} 2 &amp; 2 &amp; 0 \\ 2 &amp; 0 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix}</math></p>



	<p>Finding eigen values: <math> \mathbf{Q} - \lambda \mathbf{I}  = \begin{vmatrix} 2 - \lambda &amp; 2 &amp; 0 \\ 2 &amp; -\lambda &amp; -2 \\ 0 &amp; -2 &amp; 2 - \lambda \end{vmatrix}</math></p> <p><math>\Rightarrow (2 - \lambda)(-2\lambda + \lambda^2 + 4) + 8 = 0</math></p> <p><math>\Rightarrow \lambda^3 - 4\lambda^2 - 4\lambda + 16 = 0</math></p> <p><math>\lambda_1 = 4 \quad \lambda_2 = 2 \quad \lambda_3 = -2</math></p> <p>Signature = <math>(n_+, n_-, n_0) = (2, 1, 0) \neq (+ + 0)</math></p> <p>Hence, <b>Option 3</b> is not correct.</p>
<b>Option 4</b>	The quadratic form $\mathbf{Q}$ takes the value 0 for some non-zero vector $(x, y, z)$
Solution	<p>From above we have quadratic form of <math>\mathbf{Q} = \begin{pmatrix} 2 &amp; 2 &amp; 0 \\ 2 &amp; 0 &amp; -2 \\ 0 &amp; -2 &amp; 2 \end{pmatrix}</math></p> <p>we can see that few elements are zero even though the vectors are non-zero.</p> <p>Therefore, <b>Option 4</b> is correct.</p>

TABLE 3.15.2: Solution

3.16. Assume that a non-singular matrix

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U} \quad (3.16.1)$$

where  $\mathbf{L}$  and  $\mathbf{U}$  are lower and upper triangular matrices respectively with all diagonal entries are zero, and  $\mathbf{D}$  is a diagonal matrix. Let  $\mathbf{x}^*$  be the solution of  $\mathbf{Ax} = \mathbf{b}$ . Then the Gauss-Seidel iteration method

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k + \mathbf{c}, k = 0, 1, 2, \dots \quad (3.16.2)$$

with  $\|\mathbf{H}\| < 1$  converges to  $\mathbf{x}^*$  provided  $\mathbf{H}$  is equal to

- a)  $-\mathbf{D}^{-1}(\mathbf{L} + \mathbf{U})$
- b)  $-(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}$
- c)  $-\mathbf{D}(\mathbf{L} + \mathbf{U})^{-1}$
- d)  $-(\mathbf{L} - \mathbf{D})^{-1}\mathbf{U}$

3.17. Consider a Markov Chain with state space  $S = \{1, 2, 3\}$  and transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{matrix} \quad (3.17.1)$$

Let  $\pi$  be a stationary distribution of the Markov chain and  $d(1)$  denote the period of state 1. Which of the following statements are correct?

- a)  $d(1) = 1$
- b)  $d(1) = 2$
- c)  $\pi_1 = \frac{1}{2}$
- d)  $\pi_1 = \frac{1}{3}$

**Solution:**

a) The period of state 1 i.e.,  $d(1)$  is given as:

$$d(1) = \text{GCD}\{n : P_{11}^n > 0\} \quad (3.17.2)$$

For  $n = 1$ ,

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (3.17.3)$$

$$(3.17.4)$$

For  $n = 2$ ,

$$\mathbf{P}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad (3.17.5)$$

$$(3.17.6)$$

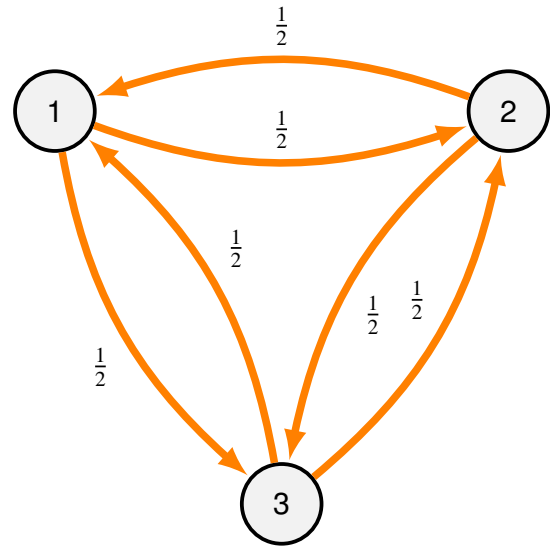


Fig. 3.17.1: State transition diagram

For  $n = 3$ ,

$$\mathbf{P}^3 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \end{pmatrix} \quad (3.17.7)$$

$$(3.17.8)$$

For  $n = 4$ ,

$$\mathbf{P}^4 = \begin{pmatrix} \frac{3}{16} & \frac{5}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{3}{16} & \frac{5}{16} \\ \frac{5}{16} & \frac{5}{16} & \frac{3}{8} \end{pmatrix} \quad (3.17.9)$$

Thus  $P_{11}^n$  follows the sequence, that is defined as:

$$P_{11}^n = \begin{cases} 0, & \text{if } n = 1 \\ \frac{1}{2}, & \text{if } n = 2 \\ \frac{1}{2}(P_{11}^{n-1} + P_{11}^{n-2}), & \text{if } n > 2 \end{cases} \quad (3.17.10)$$

Since, for  $n > 1$ ,  $P_{11}^n > 0$

$$d(1) = \text{GCD}\{2, 3, 4, 5, \dots\} \quad (3.17.11)$$

$$\therefore d(1) = 1 \quad (3.17.12)$$

Thus statement a is correct

- b) As calculated above in 3.17.12,  $d(1) = 1$   
Thus statement b is incorrect.

c) For stationary distribution,

$$\sum_{i=1}^{i=n} \pi_i = 1 \quad (3.17.13)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = 1 \quad (3.17.14)$$

Also for a stationary distribution,

$$\pi \mathbf{P} = \pi \quad (3.17.15)$$

$$(\pi \mathbf{P})^T = \pi^T \quad (3.17.16)$$

$$\mathbf{P}^T \pi^T = \pi^T \quad (3.17.17)$$

$$\Rightarrow (\mathbf{P}^T - \mathbf{I}) \pi^T = 0 \quad (3.17.18)$$

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \quad (3.17.19)$$

The given equation 3.17.14, 3.17.19 can be written as:

$$\begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.17.20)$$

We need to solve the augmented matrix to row

reduced echelon form to get the solution,

$$\left( \begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right) \xleftrightarrow{R_4=R_4+R_1} \quad (3.17.21)$$

$$\left( \begin{array}{ccc|c} -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_1=-R_1} \quad (3.17.22)$$

$$\left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_2=R_2-\frac{R_1}{2}, R_3=R_3-\frac{R_1}{2}} \quad (3.17.23)$$

$$\left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{array} \right) \xleftrightarrow{R_3=R_3+R_2, R_4=R_4+2R_2} \quad (3.17.24)$$

$$\left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_2=-\frac{4}{3}R_2} \quad (3.17.25)$$

$$\left( \begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_1=R_1+\frac{1}{2}R_2} \quad (3.17.26)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{array} \right) \xleftrightarrow{R_3 \leftrightarrow R_4} \quad (3.17.27)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \xleftrightarrow{R_3=\frac{R_3}{3}} \quad (3.17.28)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \xleftrightarrow{R_1=R_1+R_3, R_2=R_2+R_3} \quad (3.17.29)$$

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (3.17.30)$$

Hence,

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3} \quad (3.17.31)$$

Thus statement c is incorrect

d) As, calculated in 3.17.31,  $\pi_1 = \frac{1}{3}$

Thus statement d is correct

Hence, statements a and d are correct.

4 DECEMBER 2017

4.1. Let  $\mathbf{A}$  be a real symmetric matrix and  $\mathbf{B} = \mathbf{I} + i\mathbf{A}$ , where  $i^2 = -1$ . Then choose the correct option.

- a)  $\mathbf{B}$  is invertible if and only if  $\mathbf{A}$  is invertible.
- b) All Eigenvalues of  $\mathbf{B}$  are necessarily real.
- c)  $\mathbf{B} - \mathbf{I}$  is necessarily invertible.
- d)  $\mathbf{B}$  is necessarily invertible.

**Solution:** See Table 4.1.1.

Statement 1.	<b>B</b> is invertible if and only if <b>A</b> is invertible.
False statement	Matrix <b>B</b> is invertible even if <b>A</b> is non invertible.
Example:	<p>Consider a matrix</p> $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.1.1)$ <p>a real non invertible,symmetric matrix.</p> $\Rightarrow \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix} \quad (4.1.2)$ <p>is invertible even if <b>A</b> is non invertible.</p>
Statement 2.	All Eigenvalues of <b>B</b> are necessarily real.
False statement	Matrix <b>B</b> can have complex Eigenvalues.
Proof :	<p>Eigen values of <b>B</b> = Eigen values of (<b>I</b>) + i (Eigen values of <b>A</b>).</p> <p>Clearly from (4.1.2) above Eigen values of <b>B</b> are 1 and <math>1 + i</math> respectively.</p> <p>Hence <b>B</b> can also have complex Eigen value.</p>
Statement 3.	<b>B – I</b> is necessarily invertible.
False statement	<b>B – I</b> = $i\mathbf{A}$ will be invertible if <b>A</b> , is invertible.
Proof:	<p>We have <b>B – I</b> = <math>i\mathbf{A}</math></p> $\Rightarrow \mathbf{B} - \mathbf{I} = i\mathbf{A} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \text{from (4.1.1)}$ <p>Hence <b>B – I</b> is not invertible,unless <b>A</b> is invertible.</p>
Statement 4.	<b>B</b> is necessarily invertible.
Correct Statement:	Matrix <b>B</b> has non zero Eigen values corresponding to Eigenvector $X$ .
Proof:	<p>Let <math>X</math> be an Eigen vector of <b>A</b> corresponding to Eigen value <math>\lambda</math></p> <p>also, <math>\lambda \in \mathbb{R}</math></p> $\Rightarrow \mathbf{A}X = \lambda X$ $\therefore \mathbf{B}X = (\mathbf{I} + i\mathbf{A})X = \mathbf{I}X + i\mathbf{A}X = X + i\lambda X$ $\Rightarrow \mathbf{B}X = (1 + i\lambda)X$ <p>Therefore, <math>1 + i\lambda</math> is an Eigen value of <b>B</b>, corresponding to Eigen vector <math>X</math>,which are non zero.</p> <p>Hence, <b>B</b> is necessarily invertible.</p>

TABLE 4.1.1: Solution summary

4.2. Let  $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ . Then the smallest positive integer  $n$  such that  $\mathbf{A}^n = \mathbf{I}$  is

**Solution:** *Property of eigen values of A:* Let  $\mathbf{A}$  be an arbitrary  $n \times n$  matrix of complex numbers with eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then the eigen values of  $k^{\text{th}}$  power of  $\mathbf{A}$ , that is the eigen values of  $\mathbf{A}^k$ , for any positive integer  $k$  are  $\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k$ . Let us calculate the eigen values of  $\mathbf{A}$ .

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad (4.2.1)$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad (4.2.2)$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} = 0 \quad (4.2.3)$$

$$-\lambda(1 - \lambda) + 1 = 0 \quad (4.2.4)$$

$$\lambda^2 - \lambda + 1 = 0 \quad (4.2.5)$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{3}i}{2} \quad (4.2.6)$$

From the above property, the eigen values of  $\mathbf{A}^n$  are  $\lambda^n$ . Also as it is given that  $\mathbf{A}^n = \mathbf{I}$ ,

$$\Rightarrow \lambda^n = 1 \quad (4.2.7)$$

$$\Rightarrow \left( \frac{-1 \pm \sqrt{3}i}{2} \right)^n = 1 \quad (4.2.8)$$

Clearly  $n \neq 1$ . For  $n = 2$ ,

$$\left( \frac{-1 \pm \sqrt{3}i}{2} \right)^2 = \frac{-1 \mp \sqrt{3}i}{2} \quad (4.2.9)$$

For  $n = 4$ ,

$$\left( \frac{-1 \pm \sqrt{3}i}{2} \right)^4 = \frac{-1 \pm \sqrt{3}i}{2} \quad (4.2.10)$$

For  $n = 6$ ,

$$\left( \frac{-1 \pm \sqrt{3}i}{2} \right)^6 = 1 \quad (4.2.11)$$

Hence  $n = 6$  is the smallest positive integer.

4.3. Let  $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 3 & \alpha \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ \beta \end{pmatrix}$ . Then the system  $\mathbf{AX} = \mathbf{b}$  over the real numbers has

- No solution when  $\beta \neq 7$
- Infinite number of solutions when  $\alpha \neq 2$
- Infinite number of solutions when  $\alpha = 2$  and  $\beta \neq$

7

d) A unique solution if  $\alpha \neq 2$

**Solution:** First we derive the Row Reduced Echelon Form (RREF) of the augmented matrix of the system  $\mathbf{AX} = \mathbf{b}$  as follows,

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & \alpha & \beta \end{pmatrix} \xrightarrow[R_3=R_3-2R_1]{R_2=R_2-R_1} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \quad (4.3.1)$$

$$\xrightarrow{R_2=\frac{1}{2}R_2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \quad (4.3.2)$$

$$\xrightarrow{R_1=R_1+R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 5 & \alpha-2 & \beta-2 \end{pmatrix} \quad (4.3.3)$$

$$\xrightarrow{R_3=R_3-5R_2} \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \alpha-2 & \beta-7 \end{pmatrix} \quad (4.3.4)$$

From the RREF of the augmented matrix of the system  $\mathbf{AX} = \mathbf{b}$  in (4.3.4) we make the following observations for different values of  $\alpha$  and  $\beta$  in Table 4.3.1. ,

Values	Observations
$\beta \neq 7$	Then the existence of solution and the number of solutions will entirely depend on value of $\alpha$
$\alpha = 2$ $\beta \neq 7$	Then RREF in (4.3.4) will contain Zero Row in $R_3$ . Moreover solvability condition will not satisfy. $\Rightarrow$ system will have Zero solutions
$\alpha \neq 2$	RREF in (4.3.4) will have all pivots $\Rightarrow$ RREF in (4.3.4) will be fullrank $\Rightarrow \mathbf{AX} = \mathbf{b}$ have unique solution.

TABLE 4.3.1

Hence, if  $\alpha \neq 2$  then the system  $\mathbf{AX} = \mathbf{b}$  has unique solution.

4.4. Consider a Markov chain  $\{X_n | n \geq 0\}$  with state space  $\{1, 2, 3\}$  and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Then,  $P(X_3 = 1 | X_0 = 1)$  equals

**Solution:** The three step transitional probabilities are given as,

$$P(X_3 = j | X_0 = i) = P(X_{n+3} = j | X_n = i) = (\mathbf{P}^3)_{ij} \text{ for any } n \quad (4.4.1)$$

$$\mathbf{P}^3 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}^3 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \quad (4.4.2)$$

From (4.4.2),

$$P(X_3 = 1 | X_0 = 1) = (\mathbf{P}^3)_{11} = \frac{1}{4} \quad (4.4.3)$$

4.5. Let  $\mathbf{A}$  be an  $m \times n$  matrix with rank  $r$ . If the linear system  $\mathbf{A}\mathbf{X} = \mathbf{b}$  has a solution for each  $\mathbf{b} \in \mathbf{R}^m$ , then

- $m = r$
- the column space of  $\mathbf{A}$  is a proper subspace of  $\mathbf{R}^m$
- the null space of  $\mathbf{A}$  is a non-trivial subspace of  $\mathbf{R}^n$  whenever  $m = n$
- $m \geq n$  implies  $m = n$

**Solution:** Theorem

**Theorem 4.1.** Consider the  $m \times n$  system  $Ax = b$ , with either  $b \neq 0$  or  $b = 0$ . We distinguish the following cases:

- Unique Solution:** If  $\text{rank}[A, b] = \text{rank}(A) = n \leq m$ , then and only then the system has a unique solution. In this case, indeed as many as  $m - n$  equations are redundant. And the solution  $\mathbf{X} = \mathbf{A}^{-1}\mathbf{b}$ . This is called as **Exactly Determined**.
- No Solution:** If  $\text{rank}[A, b] > \text{rank}(A)$  which necessarily implies  $\mathbf{b} \neq 0$  and  $m > \text{rank}(A)$ , then and only then the system has no solution. This is called as **Overdetermined**.

See Table 4.5.1 If the columns of an  $m \times n$  matrix  $\mathbf{A}$  span  $\mathbf{R}^m$  then the equation  $\mathbf{A}\mathbf{x} = \mathbf{b}$  is consistent for each  $\mathbf{b}$  in  $\mathbf{R}^m$ .

The **null space** of  $\mathbf{A}$  is defined to be

$$\text{Null}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = 0\} \quad (4.5.1)$$

$$\mathbf{A} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \end{pmatrix} \quad (4.5.2)$$

Reduced Row Echelon form is

$$\text{RREF}(\mathbf{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.5.3)$$

$\therefore$  the only possible nullspace of the matrix  $\mathbf{A}$  is  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ .

Let  $\mathbf{B}$  be given as

$$\mathbf{B} = \begin{pmatrix} -3 & -2 & 4 \\ 14 & 8 & -18 \\ 4 & 2 & -4 \\ 28 & 16 & -36 \\ 8 & 4 & -8 \end{pmatrix} \quad (4.5.4)$$

Reduced Row Echelon form is

$$\text{RREF}(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.5.5)$$

$\therefore$  the rank of matrix  $\mathbf{B} = 3$ .

4.6. Let  $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and eigen values of } \mathbf{A} \in \mathbb{Q} \right\}$

- $\mathbf{M}$  is empty
- $\mathbf{M} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$
- If  $\mathbf{A} \in \mathbf{M}$  then the eigen values of  $\mathbf{A} \in \mathbb{Z}$
- If  $\mathbf{A}, \mathbf{B} \in \mathbf{M}$  such that  $\mathbf{AB} = \mathbf{I}$  then  $|\mathbf{A}| \in \{+1, -1\}$

**Solution:** See Table 4.6.1.

Options	Observations
$m = r$	<p>The rank of any matrix <math>\mathbf{A}</math> is the dimension of its column space. When the number of rows (<math>m</math>) is equal to the rank (<math>r</math>) of the matrix, then their linear combination gives us span of <math>\mathbf{R}^m</math>.</p> <p><math>\therefore</math> This statement is <b>True</b>.</p>
the column space of $\mathbf{A}$ is a proper subspace of $\mathbf{R}^m$	<p>Any subspace of a vector space <math>\mathbf{V}</math> other than <math>\mathbf{V}</math> itself is considered a proper subspace of <math>\mathbf{V}</math>. Which means that linear combination of <math>\mathbf{A}</math> will span less than <math>m</math>. That will make the resultant <math>\mathbf{b}</math> span strictly less than <math>m</math>. But it is given that <math>\mathbf{b} \in \mathbf{R}^m</math>, which is contradicting.</p> <p><math>\therefore</math> This statement is <b>False</b>.</p>
the null space of $\mathbf{A}$ is a non-trivial subspace of $\mathbf{R}^n$ whenever $m = n$	<p>From (4.5.2) we see that even when <math>m = n</math> then also we are getting a trivial nullspace.</p> <p><math>\therefore</math> This statement is <b>False</b>.</p>
$m \geq n$ implies $m = n$	<p>It is given that the number of rows are greater than the column, and it is given that there exists a solution. If we refer to theorem (4.1) we see that the corresponding system will be <b>Exactly Determined</b> system.</p> <p>As an example, it will look like (4.5.4).</p> <p><math>\therefore</math> This statement is <b>True</b>.</p>

TABLE 4.5.1: Solution



<p><math>\mathbf{M}</math> is empty</p>	<p>Consider <math>\mathbf{A}=\mathbf{I}=\begin{pmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{pmatrix}</math>. The elements of <math>\mathbf{A} \in \mathbb{Z}</math> and its eigen values <math>1 \in \mathbb{Q}</math>. So, <math>\mathbf{M}</math> is not empty.</p>
<p><math>\mathbf{M} = \left\{ \begin{pmatrix} a &amp; b \\ c &amp; d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}</math></p>	<p>Let <math>\mathbf{A}=\begin{pmatrix} 0 &amp; -1 \\ 1 &amp; 0 \end{pmatrix}</math> where elements of <math>\mathbf{A} \in \mathbb{Z}</math>. The characteristic equation can be written as :</p> $\lambda^2 + 1 = 0 \implies \lambda = \pm i$

	We see that $\lambda \in \mathbb{C}$ which is contradicting the main definition of $\mathbf{M}$ . So, this is not correct.
Eigen values of $\mathbf{A} \in \mathbb{Z}$	<p>Given <math>\mathbf{A} \in \mathbf{M}</math>. Let <math>\lambda_1, \lambda_2</math> be the eigen values of <math>\mathbf{A}</math>. The characteristic polynomial can be written as:</p> $\lambda^2 - \text{tr}(\mathbf{A})\lambda + \det \mathbf{A} = 0 \text{ where } \text{tr}(\mathbf{A}) = \lambda_1 + \lambda_2, \det \mathbf{A} = \lambda_1 \lambda_2$ <p>Given the eigen values <math>\lambda_1, \lambda_2 \in \mathbb{Q}</math>, For this to be possible the discriminant of above equation should <math>\in \mathbb{Z}</math></p> $\sqrt{(\lambda_1 + \lambda_2)^2 - 4\lambda_1 \lambda_2} \in \mathbb{Z}$ $\Rightarrow \sqrt{(\lambda_1 - \lambda_2)^2} \in \mathbb{Z}$ $\Rightarrow \lambda_1 - \lambda_2 \in \mathbb{Z} \text{ This is possible when both } \lambda_1, \lambda_2 \in \mathbb{Z}.$
If $\mathbf{AB}=\mathbf{I}$ then $ \mathbf{A}  \in \{+1, -1\}$	<p>As <math>\mathbf{A}, \mathbf{B} \in \mathbf{M} \Rightarrow  \mathbf{A} ,  \mathbf{B}  \in \mathbb{Z}</math></p> <p>Given <math>\mathbf{AB}=\mathbf{I} \Rightarrow  \mathbf{A}  \mathbf{B} =1</math></p> <p>This is possible only when <math> \mathbf{A} = \mathbf{B} = \pm 1</math></p>
Conclusion	options 3) and 4) are correct.

TABLE 4.6.1: Solution

4.7. Let  $\mathbf{A}$  be a  $3 \times 3$  matrix with real entries. Identify the correct statements.

- a)  $\mathbf{A}$  is necessarily diagonalizable over  $\mathbf{R}$
- b) If  $\mathbf{A}$  has distinct real eigen values then it is diagonalizable over  $\mathbf{R}$
- c) If  $\mathbf{A}$  has distinct eigen values then it is diagonalizable over  $\mathbf{C}$
- d) If all eigen values are non zero then it is diagonalizable over  $\mathbf{C}$

**Solution:** See Table 4.7.1.

Statement 1.	A is necessarily diagonalizable over $\mathbf{R}$
False statement Example:	<p>Matrix A is diagonalizable if and only if there is a basis of <math>\mathbf{R}^3</math> consisting of eigenvectors of A. Consider a matrix</p> $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.7.1)$ <p>Eigen values are:</p> $\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \quad (4.7.2)$ <p><math>\lambda_1 = 1</math> has eigen vector <math>\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}</math> and <math>\lambda_2 = 4</math> has eigen vector <math>\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}</math> (4.7.3)</p> <p>We have found only two linearly independent eigenvectors for A, not diagonalisable</p>
Statement 2.	If A has distinct real eigen values than it is diagonalizable over $\mathbf{R}$
True statement	Distinct real eigenvalues implies linearly independent eigenvectors . and if a matrix has n linearly independent vectors than it is diagonalizable.
Proof 1:	<p><b>Distinct eigen values implies linearly independent vectors that spans entire space.</b> Consider 2 eigen vectors <math>\mathbf{v}, \mathbf{w}</math> with eigen values <math>\lambda, \mu</math> respectively. such that <math>\lambda \neq \mu</math></p> $\alpha(\mathbf{v}) + \beta(\mathbf{w}) = 0 \quad (4.7.4)$ $\alpha A(\mathbf{v}) + \beta A(\mathbf{w}) = 0 \quad (4.7.5)$ $\alpha \lambda \mathbf{v} + \beta \mu \mathbf{w} = 0 \quad (4.7.6)$ <p>Multiplying (4.7.4) with <math>-\lambda</math> and subtracting from (4.7.6) we have,</p> $\beta(\mu - \lambda)\mathbf{w} = 0 \quad (4.7.7)$ <p>eigen values are distinct <math>(\mu - \lambda) \neq 0</math> . From equation (4.7.7) we have, <math>\beta = 0</math> substituting <math>\beta = 0</math> in equation (4.7.4) we have, <math>\alpha = 0</math>. As, <math>\mathbf{v} \neq 0</math> <b>which proves that vectors are linearly independent.</b></p> <p><b>If a matrix has n linearly independent vectors than it is diagonalizable</b> If <math>(\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n)</math> are n independent eigen vectors then, <math>A\mathbf{p}_1 = \lambda\mathbf{p}_1, \dots, A\mathbf{p}_n = \lambda\mathbf{p}_n</math></p> $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} P = (\mathbf{p}_1 \ \mathbf{p}_2 \ \cdots \ \mathbf{p}_n) \quad (4.7.8)$ <p>Now, <math>A\mathbf{p}_i = \lambda_i\mathbf{p}_i \implies AP = PD</math></p>
Proof 2:	

	so, $P^{-1}AP = D$ is a diagonal matrix.
Statement 3.	If A has distinct real eigen values than it is diagonalizable over $\mathbb{C}$
True statement	If A is an $N \times N$ complex matrix with n distinct eigenvalues, then any set of n corresponding eigenvectors form a basis for $\mathbb{C}^n$
Proof:	It is sufficient to prove that the set of eigenvectors is linearly independent which is proved in statement 2.
Example:	$A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad (4.7.9)$ <p>Eigen values of A are:</p> $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6 \quad (4.7.10)$
	<p>Eigen vectors are:</p> $x_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \quad (4.7.11)$ <p>Matrix A is diagonalizable because there is a basis of <math>\mathbb{C}^3</math> consisting of eigenvectors of A.</p>
Statement 4.	If all eigen values are non zero than it is diagonalizable over $\mathbb{C}$
False Statement:	Matrix would be diagonalizable if and only if it has linearly independent eigenvectors .
Example:	<p>Consider a matrix</p> $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \quad (4.7.12)$ <p>Eigen values are:</p> $\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{pmatrix} = 0. \implies \lambda_1 = 1, \lambda_2 = 4 \neq 0 \quad (4.7.13)$ <p><math>\lambda_1 = 1</math> has eigen vector <math>\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}</math> and <math>\lambda_2 = 4</math> has eigen vector <math>\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix}</math> (4.7.14)</p> <p>We have found only two linearly independent eigenvectors for A, not diagonalisable.</p>

TABLE 4.7.1: Solution summary

Given	<p><math>V</math> be a vector space over <math>C</math> of all the polynomials in a variable <math>X</math> of degree atmost 3</p> $D : P_3 \rightarrow P_3$ <p><math>D : V \rightarrow V</math> be the linear operator given by differentiation wrt <math>X</math></p> $D(P(x)) \rightarrow P'(x)$ <p><math>A</math> be the matrix of <math>D</math> wrt some basis for <math>V</math></p> <p>Assume basis for <math>V</math> be <math>\{1, x, x^2, x^3\}</math></p>
-------	---

TABLE 4.8.1

4.8. Let  $V$  be a vector space over  $C$  of all the polynomials in a variable  $X$  of degree atmost 3. Let  $D : V \rightarrow V$  be the linear operator given by differentiation with respect to  $X$ . Let  $A$  be the matrix of  $D$  with respect to some basis for  $V$ . Which of the following are true?

- a)  $A$  is nilpotent matrix
- b)  $A$  is diagonalizable matrix
- c) the rank of  $A$  is 2
- d) the Jordan canonical form of  $A$  is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Solution:** See Tables 4.8.1, 4.8.2 and 4.8.3

4.9. For every  $4 \times 4$  real symmetric non-singular matrix  $A$  there exists a positive integer  $p$  such that

- a)  $pI + A$  is positive definite
- b)  $A^p$  is positive definite
- c)  $A^{-p}$  is positive definite
- d)  $\exp(pA) - I$  is positive definite

**Solution:** A matrix is real symmetric implies its eigen values are real and eigen vectors are orthogonal, that is its eigen value decomposition is

$$A = PDP^T \quad (4.9.1)$$

$D$  is the diagonal matrix containing the real eigen values of  $A$

$P$  has the corresponding eigen vectors

$$PP^T = P^T P = I \quad (4.9.2)$$

A real matrix is positive definite if

$$\mathbf{x}^T A \mathbf{x} > 0 \quad (4.9.3)$$

$$\implies \mathbf{x}^T \lambda \mathbf{x} > 0 \quad (4.9.4)$$

$$\implies \lambda \mathbf{x}^T \mathbf{x} > 0 \quad (4.9.5)$$

$$\implies \lambda > 0 \quad (4.9.6)$$

In other words, all the eigen values of  $A$  are positive See Table 4.9.1

Let  $A$  be

$$A = PDP^T \quad (4.9.7)$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \quad (4.9.8)$$

From the table, the choices would be option 1,2,3

4.10. Let  $A$  be an  $m \times n$  matrix of rank  $m$  with  $n > m$ . If for some non-zero real number  $\alpha$ , we have  $\mathbf{x}^T A A^T \mathbf{x} = \alpha \mathbf{x}^T \mathbf{x}$ , for all  $\mathbf{x} \in \mathbf{R}^m$ , then  $A^T A$  has,

- a) exactly two distinct eigenvalues.
- b) 0 as an eigenvalue with multiplicity  $n - m$ .
- c)  $\alpha$  as a non-zero eigenvalue.
- d) exactly two non-zero distinct eigenvalues.

**Solution:** Refer Table 4.10.1.

Refer Table 4.10.2.

4.11. Consider a Markov chain with five states

$\{1, 2, 3, 4, 5\}$  and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{7} & 0 & 0 & \frac{6}{7} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{5}{8} & 0 & 0 & \frac{3}{8} \end{pmatrix} \quad (4.11.1)$$

Which of the following are true?

- a) 3 and 1 are in the same communicating class
- b) 1 and 4 are in the same communicating class
- c) 4 and 2 are in the same communicating class
- d) 2 and 5 are in the same communicating class

**Solution:** See Tables 4.11.1 and 4.11.2

Matrix	$D(1) = 0 = 0.1 + 0.x + 0.x^2 + 0.x^3$ $D(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x) = 1 = 1.1 + 0.x + 0.x^2 + 0.x^3$ $D(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $D(x^2) = 2x = 0.1 + 2.x + 0.x^2 + 0.x^3$ $D(x^2) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ $D(x^3) = 3x^2 = 0.1 + 0.x + 3.x^2 + 0.x^3$ $D(x^3) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ $\text{Matrix } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
Inference	<p>An <math>n \times n</math> matrix with <math>\lambda</math> as diagonal elements, ones on the super diagonal and zeroes in all other entries is nilpotent with minimal polynomial <math>(A - \lambda I)^n</math></p>
Nilpotent	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p>All eigen values of matrix <math>A</math> is 0  Thus, above matrix is nilpotent matrix  Thus, above statement is true</p>

TABLE 4.8.2

Diagonalizable	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p> <math>Rank(A) + nullity(A) = \text{no of column}</math>  <math>Rank(A) = 3, \text{ no of column} = 4</math>  <math>nullity(A) = 4 - 3 = 1</math>  means there exists only one linearly independent eigen vector corresponding to 0 eigen values  Thus, matrix <math>A</math> is not Diagonalizable.  Thus, above statement is false </p>
Rank	$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p> Rank of matrix <math>A</math> is 3  Thus, above statement is false </p>
Jordan CF	<p> Assume characteristic polynomial of matrix <math>A</math> is <math>c_A(x)</math>  <math>c_A(x) = x^4</math>  Assume minimal polynomial of <math>A</math> is <math>m_A(x)</math>  <math>m_A(x)</math> always divide <math>c_A(x)</math>  <math>m_A(x) = \{x, x^2, x^3, x^4\}</math>  Minimal polynomial always annihilates its matrix. Thus, we see that  <math>m_A(A) = \{A = 0, A^2 = 0, A^3 = 0, A^4 = 0\}</math>  But we see that neither <math>A</math> is zero matrix nor <math>A^2</math> and <math>A^3</math> equal to zero but <math>A^4</math> is equal to zero. Thus, <math>x^4</math> is minimal polynomial.  Algebraic Multiplicity = <math>a_M(\lambda = 0) = 4</math>  Geometric Multiplicity = <math>g_M(\lambda = 0) = nullity(A - 0I) = nullity(A) = 1</math>  Hence, Jordan form of block size 4  Using Inference, <math>\mathbf{J} = \begin{pmatrix} \lambda &amp; 1 &amp; 0 &amp; 0 \\ 0 &amp; \lambda &amp; 1 &amp; 0 \\ 0 &amp; 0 &amp; \lambda &amp; 1 \\ 0 &amp; 0 &amp; 0 &amp; \lambda \end{pmatrix}</math>  <math>\lambda = 0</math>  <math display="block">\begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \\ 0 &amp; 0 &amp; 1 &amp; 0 \\ 0 &amp; 0 &amp; 0 &amp; 1 \\ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix}</math> <p> which is same as given in the question. Thus, statement is true </p> </p>



OPTIONS	DERIVATIONS
Choice 1	$p\mathbf{I} + \mathbf{A} = \mathbf{P}(p\mathbf{I})\mathbf{P}^T + \mathbf{P}\mathbf{D}\mathbf{P}^T \quad (4.9.9)$ $= \mathbf{P}\mathbf{D}_1\mathbf{P}^T \quad (4.9.10)$ $\mathbf{D}_1 = \begin{pmatrix} \lambda_1 + p & 0 & 0 & 0 \\ 0 & \lambda_2 + p & 0 & 0 \\ 0 & 0 & \lambda_3 + p & 0 \\ 0 & 0 & 0 & \lambda_4 + p \end{pmatrix} \quad (4.9.11)$ <p>Some of the eigen values of <math>\mathbf{A}</math> may be negative. All the eigen values in <math>\mathbf{D}_1</math> are positive only if</p> $p >  \lambda_i  \quad \forall i \in [1, 4] \quad (4.9.12)$
Choice 2	$\mathbf{A}^2 = \mathbf{A}\mathbf{A} \quad (4.9.13)$ $= (\mathbf{P}\mathbf{D}\mathbf{P}^T)(\mathbf{P}\mathbf{D}\mathbf{P}^T) \quad (4.9.14)$ $= \mathbf{P}\mathbf{D}^2\mathbf{P}^T \quad (4.9.15)$ <p>Similarly, <math>\mathbf{A}^p = \mathbf{P}\mathbf{D}^p\mathbf{P}^T \quad (4.9.16)</math></p> $\mathbf{D}^p = \begin{pmatrix} \lambda_1^p & 0 & 0 & 0 \\ 0 & \lambda_2^p & 0 & 0 \\ 0 & 0 & \lambda_3^p & 0 \\ 0 & 0 & 0 & \lambda_4^p \end{pmatrix} \quad (4.9.17)$ <p><math>\mathbf{A}^p</math> is positive definite only if <math>p</math> is even.</p>
Choice 3	$\mathbf{A}^{-p} = \mathbf{P}\mathbf{D}^{-p}\mathbf{P}^T \quad (4.9.18)$ $\mathbf{D}^{-p} = \begin{pmatrix} \lambda_1^{-p} & 0 & 0 & 0 \\ 0 & \lambda_2^{-p} & 0 & 0 \\ 0 & 0 & \lambda_3^{-p} & 0 \\ 0 & 0 & 0 & \lambda_4^{-p} \end{pmatrix} \quad (4.9.19)$ <p><math>\mathbf{A}^{-p}</math> is positive definite only if <math>p</math> is even.</p>
Choice 4	$\exp(p\mathbf{A}) = \sum_{k=0}^{\infty} \frac{(p\mathbf{A})^k}{k!} \quad (4.9.20)$ $\Rightarrow \exp(p\mathbf{A}) - \mathbf{I} = \mathbf{P}\exp(p\mathbf{D})\mathbf{P}^T - \mathbf{P}\mathbf{I}\mathbf{P}^T \quad (4.9.21)$ $= \mathbf{P}(\exp(p\mathbf{D}) - \mathbf{I})\mathbf{P}^T \quad (4.9.22)$ $\exp(p\mathbf{D}) - \mathbf{I} = \begin{pmatrix} e^{\lambda_1} - 1 & 0 & 0 & 0 \\ 0 & e^{\lambda_2} - 1 & 0 & 0 \\ 0 & 0 & e^{\lambda_3} - 1 & 0 \\ 0 & 0 & 0 & e^{\lambda_4} - 1 \end{pmatrix} \quad (4.9.23)$ <p><math>\mathbf{A}</math> is non-singular</p> $\Rightarrow \forall i \in [1, 4], \lambda_i \neq 0 \quad (4.9.24)$ $e^{\lambda_i} < 1 \quad (4.9.25)$ <p>So, <math>\exp(p\mathbf{A}) - \mathbf{I}</math> is not positive definite.</p>

TABLE 4.9.1: Solution

Given	Derivation
Given	$\mathbf{A}$ is a $m \times n$ matrix of rank $m$ with $n > m$ . A non-zero real number $\alpha$ . To find eigenvalues of $\mathbf{A}^T \mathbf{A}$ .
Eigenvalues of $\mathbf{A} \mathbf{A}^T$	$\mathbf{A} \mathbf{A}^T$ is a $m \times m$ matrix and $\mathbf{A}^T \mathbf{A}$ is a $n \times n$ matrix. Let, $\lambda$ be a non-zero eigen value of $\mathbf{A}^T \mathbf{A}$ . $\mathbf{A}^T \mathbf{A} \mathbf{v} = \lambda \mathbf{v} \quad \mathbf{v} \in \mathbf{R}^n \quad (4.10.1)$ $\mathbf{A} \mathbf{A}^T \mathbf{A} \mathbf{v} = \lambda \mathbf{A} \mathbf{v} \quad (4.10.2)$ Let, $\mathbf{x} = \mathbf{A} \mathbf{v} \quad \mathbf{x} \in \mathbf{R}^m \quad (4.10.3)$ $\mathbf{A} \mathbf{A}^T \mathbf{x} = \lambda \mathbf{x} \quad (4.10.4)$ $\mathbf{x}^T \mathbf{A} \mathbf{A}^T \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} \quad (4.10.5)$ Given, $\mathbf{x}^T \mathbf{A} \mathbf{A}^T \mathbf{x} = \alpha \mathbf{x}^T \mathbf{x} \quad (4.10.6)$ $\implies \alpha \mathbf{x}^T \mathbf{x} = \lambda \mathbf{x}^T \mathbf{x} \quad (4.10.7)$ <p>From equation (4.10.7), <math>\lambda = \alpha</math> as <math>\ \mathbf{x}\  \neq 0</math>  As <math>\text{rank}(\mathbf{A}^T \mathbf{A}) = \text{rank}(\mathbf{A}) = m</math> and equation (4.10.7) satisfies the condition in question.  Therefore the only non-zero eigen value is <math>\alpha</math>  <math>\mathbf{A}^T \mathbf{A}</math> has an eigenvalue <math>\alpha</math> with multiplicity <math>m</math>.</p>
Eigenvalues of $\mathbf{A}^T \mathbf{A}$	$\mathbf{A}^T \mathbf{A}$ is a $n \times n$ matrix. Given $n > m$ , We know that, $\mathbf{A}^T \mathbf{A}$ and $\mathbf{A} \mathbf{A}^T$ have same number of non-zero eigenvalues and if one of them has more number of eigenvalues than the other then these eigenvalues are zero. 1. From above, as $\alpha$ is non-zero, $\mathbf{A}^T \mathbf{A}$ has $\alpha$ as its eigenvalue with multiplicity $m$ 2. $\mathbf{A}^T \mathbf{A}$ has 0 as its eigenvalue with multiplicity $n - m$ 3. Therefore, the two distinct eigenvalues of $\mathbf{A}^T \mathbf{A}$ are $\alpha$ and 0.

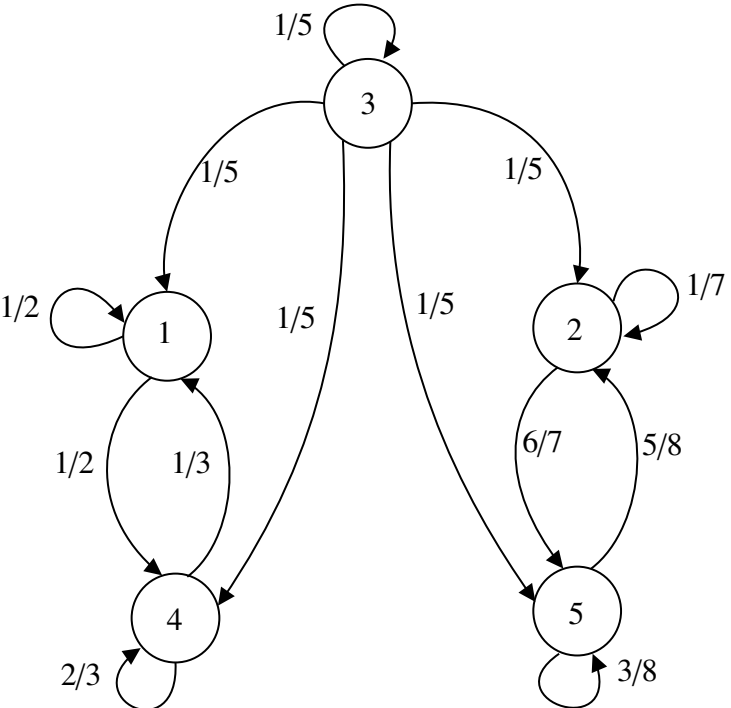
TABLE 4.10.1: Explanation

$\mathbf{A}^T \mathbf{A}$ has exactly two distinct eigenvalues.	True statement
$\mathbf{A}^T \mathbf{A}$ has 0 as an eigenvalue with multiplicity $n - m$	True statement
$\mathbf{A}^T \mathbf{A}$ has $\alpha$ as a non-zero eigenvalue	True statement
$\mathbf{A}^T \mathbf{A}$ has exactly two non-zero distinct eigenvalues.	False statement

TABLE 4.10.2: Solution

Accessibility of states in Markov's chain	We say that state $j$ is accessible from state $i$ , written as $i \rightarrow j$ , if $p_{ij}^{(n)} > 0$ for some $n$ . Every state is accessible from itself since $p_{ii}^{(0)} = 1$
Communication between states	Two states $i$ and $j$ are said to communicate, written as $i \leftrightarrow j$ , if they are accessible from each other. In other words, $i \leftrightarrow j \text{ means } i \rightarrow j \text{ and } j \rightarrow i.$
Communicating class	For each Markov chain, there exists a unique decomposition of the state space $S$ into a sequence of disjoint subsets $C_1, C_2, \dots$ , $S = \bigcup_{i=1}^{\infty} C_i$ in which each subset has the property that all states within it communicate. Each such subset is called a communication class of the Markov chain.

TABLE 4.11.1: Definition and Result used

Drawing Transition diagram	
Checking whether the states 3 and 1 are in the same communicating class	<p>Here,        State 1 is accessible from the state 3.        But, State 3 is not accessible from the state 1        i.e. <math>3 \rightarrow 1, 1 \nrightarrow 3</math>  <math>\Rightarrow \boxed{3 \leftrightarrow 1}</math></p> <p>Therefore, 3 and 1 are not in the same communicating class.</p>
Checking whether the states 1 and 4 are in the same communicating class	<p>Here,        State 1 is accessible from the state 4.        Also, State 4 is accessible from the state 1        i.e. <math>3 \rightarrow 1, 1 \rightarrow 3</math>  <math>\Rightarrow \boxed{3 \leftrightarrow 1}</math></p> <p>Therefore, 1 and 4 are in the same communicating class.</p>
Checking whether the states 4 and 2 are in the same communicating class	<p>Here,        State 2 is not accessible from the state 4.        Also, State 4 is not accessible from the state 2        i.e. <math>4 \nrightarrow 2, 2 \nrightarrow 4</math></p>

	$\Rightarrow \boxed{4 \leftrightarrow 2}$ <p>Therefore, 4 and 2 are not in the same communicating class.</p>
Checking whether the states 2 and 5 are in the same communicating class	<p>Here,          State 2 is accessible from the state 5.          Also, State 5 is accessible from the state 2          i.e. <math>5 \rightarrow 2, 2 \rightarrow 5</math>  <math display="block">\Rightarrow \boxed{2 \leftrightarrow 5}</math></p> <p>Therefore, 2 and 5 are in the same communicating class.</p>
Conclusion	<p>Communication classes are:</p> $\boxed{S = \{1, 4\} \cup \{3\} \cup \{2, 5\}}$ <p>Option 2) and 4) are true.</p>

TABLE 4.11.2: Solution

5 JUNE 2017

5.1. Let  $\mathbf{A}$  be a  $4 \times 4$  matrix. Suppose that the null space  $N(\mathbf{A})$  of  $\mathbf{A}$  is

$$\{(x, y, z, w) \in \mathbf{R}^4 : x + y + z = 0, x + y + w = 0\} \quad (5.1.1)$$

Then which one of the following is correct

- a)  $\dim(\text{column space}(\mathbf{A})) = 1$
- b)  $\dim(\text{column space}(\mathbf{A})) = 2$
- c)  $\text{rank}(\mathbf{A}) = 1$
- d)  $\mathbf{S} = \{(1, 1, 1, 0), (1, 1, 0, 1)\}$  is a basis of  $N(\mathbf{A})$

**Solution:** The nullspace is given by

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (5.1.2)$$

Row reducing the above matrix we get,

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_2 \leftarrow R_2 \times -1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.3)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - R_2} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.1.4)$$

See Table 5.1.1

5.2. Let  $\mathbf{A}$  and  $\mathbf{B}$  be real invertible matrices such that

$$\mathbf{AB} = -\mathbf{BA}. \quad (5.2.1)$$

Then

- a)  $\text{trace} \mathbf{A} = \text{trace}(\mathbf{B}) = 0$
- b)  $\text{trace} \mathbf{A} = \text{trace}(\mathbf{B}) = 1$
- c)  $\text{trace} \mathbf{A} = 0, \text{trace}(\mathbf{B}) = 1$
- d)  $\text{trace}(\mathbf{A}) = 1, \text{trace}(\mathbf{B}) = 0$

**Solution:** See Tables 5.2.1 and 5.2.2

5.3. Let  $\mathbf{A}$  be an  $n \times n$  self-adjoint matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$ . Let,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \dots + |\mathbf{X}_n^2|} \quad (5.3.1)$$

for  $\mathbf{X} = (\mathbf{X}_1, \dots, \mathbf{X}_n) \in \mathbb{C}^n$ . If

$$p(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + \dots + a_n \mathbf{A}^n \quad (5.3.2)$$

then  $\sup_{\|\mathbf{X}\|_2=1} \|p(\mathbf{A})\mathbf{X}\|_2$  is equal to

**Solution:** We know that  $\mathbf{A}$  is a self adjoint matrix and hence  $\mathbf{A} = \mathbf{A}^*$  with eigen values  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Now as we are given,

$$p(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + \dots + a_n \mathbf{A}^n \quad (5.3.3)$$

then,

$$(p(\mathbf{A}))^* = a_0 \mathbf{I}^* + a_1 \mathbf{A}^* + \dots + a_n (\mathbf{A}^*)^n \quad (5.3.4)$$

Since,  $\mathbf{A} = \mathbf{A}^*$  we can state that,

$$p(\mathbf{A})(p(\mathbf{A}))^* = (p(\mathbf{A}))^* p(\mathbf{A}) \quad (5.3.5)$$

Hence  $p(\mathbf{A})$  is a normal matrix. Now using spectral theorem for a normal matrix,

$$\|p(\mathbf{A})\|_2 = \rho(p(\mathbf{A})) \quad (5.3.6)$$

$\sup$  refers to the smallest element that is greater than or equal to every number in the set. Hence,  $\sup$  of  $\|p(\mathbf{A})\|_2$  will be,

$$= \max \{|\alpha| : \alpha \text{ is the eigen value of } p(\mathbf{A})\} \quad (5.3.7)$$

$$= \max \{|p(\lambda_j)| : j = 1, 2, \dots, n\} \quad (5.3.8)$$

$$= \max \{|a_0 + a_1 \lambda_j + \dots + a_n \lambda_j^n| : j = 1, 2, \dots, n\} \quad (5.3.9)$$

Now, to find  $\sup \|p(\mathbf{A})\mathbf{X}\|_2$ ,

$$= \max \{|a_0 + a_1 \lambda_j + \dots + a_n \lambda_j^n| : j = 1, 2, \dots, n\} \|\mathbf{X}\|_2 \quad (5.3.10)$$

Since, we have to find  $\sup_{\|\mathbf{X}\|_2=1}$  i.e.,

$$\|\mathbf{X}\|_2 = \sqrt{|\mathbf{X}_1^2| + \dots + |\mathbf{X}_n^2|} = 1 \quad (5.3.11)$$

Hence the final answer will be,

$$= \max \{|a_0 + a_1 \lambda_j + \dots + a_n \lambda_j^n| : j = 1, 2, \dots, n\} \quad (5.3.12)$$

5.4. Let  $p(x) = \alpha x^2 + \beta x + \gamma$  be a polynomial, where  $\alpha, \beta, \gamma \in \mathbf{R}$ . Fix  $X_0 \in \mathbf{R}$ . Let  $S = \{(a, b, c) \in \mathbf{R}^3 : p(x) = a(x - x_0)^2 + b(x - x_0) + c\}$  for all  $x \in \mathbf{R}$ . Find the number of elements in  $S$  is

- a) 0
- b) 1
- c) Strictly greater than 1 but finite
- d) Infinite

$\dim(C(\mathbf{A})) = 1$	<b>False.</b> Because the number of pivot variables are 2 as obtained in (5.1.4)
$\dim(C(\mathbf{A})) = 2$	<b>True.</b> Since the number of pivot variables are 2, the rank of $\mathbf{A}$ is 2. $\therefore \dim(C(\mathbf{A})) = 2 \quad [\because \dim(C(\mathbf{A})) = \text{rank}(\mathbf{A})]$
$\text{rank}(\mathbf{A}) = 1$	<b>False.</b> Because the $\text{rank}(\mathbf{A}) = 2$ , as the number of pivot variables are 2
$\mathbf{S} = \{(1, 1, 1, 0), (1, 1, 0, 1)\}$ is a basis of $N(\mathbf{A})$	<p><b>False.</b></p> <p>Let,</p> $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ <p>Consider,</p> $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ <p>Similarly,</p> $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ <p>Hence, the given vectors do not form the basis.</p>

TABLE 5.1.1

Definition	Matrix $\mathbf{A}$ is said to be similar to matrix $\mathbf{B}$ if there exists matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PBP}^{-1}$
Properties	<p>Similar matrices have same eigenvalues</p> <p>Sum of eigenvalue of a matrix equals its trace</p> <p>From above two properties we can conclude that similar matrices have same trace</p>

TABLE 5.2.1: Similar matrices and Properties

**Solution:**

$$S = \{(a, b, c) \in \mathbb{R}^3 : p(x) = a(x - x_0)^2 + b(x - x_0) + c\},$$

$$p(x) = \alpha x^2 + \beta x + \gamma \quad (5.4.1)$$

$$\Rightarrow p(x) = (\alpha\beta\gamma) \begin{pmatrix} x^2 & x & 1 \end{pmatrix}^T \quad (5.4.2)$$

$$\forall \mathbf{x} \in R(\text{Fix } X_0) \quad (5.4.3)$$

$$p(x) = (abc) \left( (x - x_0)^2 (x - x_0) 1 \right)^T \quad (5.4.4)$$

$$= a(x^2 - 2x_0x + x_0^2) + b(x - x_0) + c \quad (5.4.5)$$

$$= ax^2 + (b - 2ax_0)x + (ax_0^2 - bx_0 + c) \quad (5.4.6)$$

Refer (5.4.2) and (5.4.6) and comparing the coefficients of powers of x,

$$\alpha = a, \beta = b - 2ax_0, \gamma = ax_0^2 - bx_0 + c \quad (5.4.7)$$

$$a = \alpha, b = \beta + 2\alpha x_0, c = \gamma - \alpha x_0^2 + (\beta + 2\alpha x_0)x_0 \quad (5.4.8)$$

Here  $\alpha, \beta, \gamma$  and  $x_0$  are the real fixed numbers. So  $a, b, c$  have unique values.

Hence S contain only 1 element. So option 2 is correct

5.5. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad (5.5.1)$$

and  $\mathbf{I}$  be the  $3 \times 3$  identity matrix. If

$$6\mathbf{A}^{-1} = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} \quad (5.5.2)$$

for  $a, b, c \in \mathbb{R}$  then (a,b,c) equals

a) (1,2,1)

b) (1,-1,2)

c) (4,1,1)

d) (1,4,1)

**Solution:** Finding the characteristic equation,

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & -3-\lambda \end{vmatrix} \quad (5.5.3)$$

$$\Rightarrow (1-\lambda)(-2-\lambda)(-3-\lambda) = 0 \quad (5.5.4)$$

$$\Rightarrow (\lambda^2 + \lambda - 2)(-3-\lambda) = 0 \quad (5.5.5)$$

$$\Rightarrow \lambda^3 + 4\lambda^2 + \lambda - 6 = 0 \quad (5.5.6)$$

Using Cayley-Hamilton Theorem we get,

$$\mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} - 6\mathbf{I} = 0 \quad (5.5.7)$$

$$\Rightarrow \mathbf{A}^3 + 4\mathbf{A}^2 + \mathbf{A} = 6\mathbf{I} \quad (5.5.8)$$

$$\Rightarrow \mathbf{A}(\mathbf{A}^2 + 4\mathbf{A} + \mathbf{I}) = 6\mathbf{I} \quad (5.5.9)$$

$|\mathbf{A}| = 6 \neq 0$  hence inverse exists. Hence (5.5.9)

we get,

$$6\mathbf{A}^{-1} = \mathbf{A}^2 + 4\mathbf{A} + \mathbf{I} \quad (5.5.10)$$

Comparing (5.5.2) and (5.5.10) we get,

$$a = 1 \quad b = 4 \quad c = 1 \quad (5.5.11)$$

Hence  $(a, b, c) = (1, 4, 1)$

5.6. Find the Eigenvalues of the matrix,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 5 \\ 2 & 5 & -3 \end{pmatrix} \quad (5.6.1)$$

a) -4, 3, -3

b) 4, 3, 1

c) 4,  $-4 \pm \sqrt{13}$

d) 4,  $-2 \pm \sqrt{7}$

**Solution:** Using the characteristic equation of the matrix can find the Eigenvalues,

$$|\lambda\mathbf{I} - \mathbf{A}| = 0 \quad (5.6.2)$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & -1 & -2 \\ -1 & \lambda+2 & -5 \\ -2 & -5 & \lambda+3 \end{vmatrix} = 0 \quad (5.6.3)$$

The expression that is obtained after expanding the determinant and simplifying it is,

$$(\lambda-1)(\lambda^2+5\lambda-19) - (5\lambda+31) = 0 \quad (5.6.4)$$

Further simplifying this we obtain the cubic equation,

$$\lambda^3 + 4\lambda^2 - 29\lambda - 12 = 0 \quad (5.6.5)$$

Solving this equation, the Eigenvalues obtained are,

$$\lambda_1 = -7.605, \lambda_2 = -0.394 \text{ and } \lambda_3 = 4 \quad (5.6.6)$$

Therefore, the Eigenvalues of the given matrix are 4,  $-4 \pm \sqrt{13}$  (Option 3)

5.7. Consider the vector space V of real polynomials of degree less than or equal to n. Fix distinct real numbers  $a_0, a_1, \dots, a_k$ . For  $p \in V$

$$\max \{ |p(a_j)| : 0 \leq j \leq k \} \quad (5.7.1)$$

defines a norm on V

a) only if  $k < n$

b) only if  $k \geq n$

c) if  $k+1 \leq n$

d) if  $k \geq n + 1$

**Solution:** Options 2 and 4 are correct as verified in the table 5.7.2

The scalar multiplication and triangle inequality properties holds true for all  $k$ .

$$\max \{ |\alpha p(a_j)| \} = |\alpha| \max \{ |p(a_j)| \} \quad (5.7.4)$$

$$\max \{ |p(a_i) + p(a_j)| \} \leq \max \{ |p(a_i)| \} + \max \{ |p(a_j)| \} \quad (5.7.5)$$

The positivity property holds true only if  $k \geq n$  as more than  $n$  roots are possible when,

$$p(x) = 0 \implies |p(a_j)|_{0 \leq j \leq k} = 0 \quad (5.7.6)$$

$$\implies \max \{ |p(a_j)| : 0 \leq j \leq k \} = 0 \quad (5.7.7)$$

5.8. Let  $\mathbf{V}$  be the vector space of polynomials of degree at most 3 in a variable  $x$  with coefficients in  $\mathbb{R}$ . Let  $\mathbf{T} = d/dx$  be the linear transformation of  $\mathbf{V}$  to itself given by differentiation.

Which of the following are correct?

- a)  $\mathbf{T}$  is invertible
- b) 0 is an eigenvalue of  $\mathbf{T}$
- c) There is a basis with respect to which the matrix of  $\mathbf{T}$  is nilpotent.
- d) The matrix of  $\mathbf{T}$  with respect to the basis  $(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3)$  is diagonal.

**Solution:** See Tables 5.8.1 , 5.8.2 and 5.8.3.



$\text{trace}(\mathbf{A}) = 0$ $\text{trace}(\mathbf{B}) = 0$	<p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ <p>So, matrix <math>\mathbf{A}</math> and <math>(-\mathbf{A})</math> are similar.∴</p> $\text{trace}(\mathbf{A}) = \text{trace}(-\mathbf{A})$ $\Rightarrow \text{trace}(\mathbf{A}) = 0$ <p>Similarly From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$ <p>So, matrix <math>\mathbf{B}</math> and <math>(-\mathbf{B})</math> are similar.∴</p> $\text{trace}(\mathbf{B}) = \text{trace}(-\mathbf{B})$ $\Rightarrow \text{trace}(\mathbf{B}) = 0$ <p>So this statement is true</p>
$\text{trace}(\mathbf{A}) = 1$ $\text{trace}(\mathbf{B}) = 1$	<p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ <p>So, matrix <math>\mathbf{A}</math> and <math>(-\mathbf{A})</math> are similar.∴</p> $\text{trace}(\mathbf{A}) = \text{trace}(-\mathbf{A})$ $\Rightarrow \text{trace}(\mathbf{A}) = 0.$ <p>As <math>\text{trace}(\mathbf{A}) = 0</math> this statement is false</p>
$\text{trace}(\mathbf{A}) = 0$ $\text{trace}(\mathbf{B}) = 1$	<p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{B} = \mathbf{A}^{-1}(-\mathbf{B})\mathbf{A}$ <p>So, matrix <math>\mathbf{B}</math> and <math>(-\mathbf{B})</math> are similar.∴</p> $\text{trace}(\mathbf{B}) = \text{trace}(-\mathbf{B})$ $\Rightarrow \text{trace}(\mathbf{B}) = 0.$ <p>As <math>\text{trace}(\mathbf{B}) = 0</math> this statement is false</p>
$\text{trace}(\mathbf{A}) = 1$ $\text{trace}(\mathbf{B}) = 0$	<p>From (5.2.1) we have</p> $\mathbf{AB} = -\mathbf{BA}$ $\Rightarrow \mathbf{A} = \mathbf{B}(-\mathbf{A})\mathbf{B}^{-1}$ <p>So, matrix <math>\mathbf{A}</math> and <math>(-\mathbf{A})</math> are similar.∴</p> $\text{trace}(\mathbf{A}) = \text{trace}(-\mathbf{A})$ $\Rightarrow \text{trace}(\mathbf{A}) = 0.$ <p>As <math>\text{trace}(\mathbf{A}) = 0</math> this statement is false</p>

TABLE 5.2.2: Calculation of trace

Properties	Norm $\forall x \in V$
Positivity	$\ x\  \geq 0, \ x\  = 0 \iff x = 0$
Scalar Multiplication	$\ \alpha x\  =  \alpha  \ x\ , \alpha \in F$
Triangle Inequality	$\ x + y\  \leq \ x\  + \ y\ $

TABLE 5.7.1: Properties of Norm

For $p \in V$ then the norm, $\max \{  p(a_j)  : 0 \leq j \leq k \} = 0 \iff  p(a_j) _{0 \leq j \leq k} = 0$	
Conditions	Explanation
only if $k < n$  Example:	<p>A polynomial doesn't necessarily have <math>k</math> distinct real roots, i.e., it may have repeated, complex roots.</p> <p>let <math>p</math> be polynomial of degree <math>n = 2</math> and <math>k = 1</math> given by:-</p> $p(x) = x^2 + 4x + 4 \quad (5.7.2)$ $ p(a_j) _{0 \leq j \leq 1} = 0 \implies a_0 = -2, a_1 = -2 \quad (5.7.3)$ <p>but <math>a_0, a_1, \dots, a_k</math> should be distinct real numbers.</p> <p>This contradicts the property of Norm. Thus condition fails.</p>
only if $k \geq n$	<p><math>p</math> is a polynomial of degree <math>\leq n</math>, it can't have more than <math>n</math> roots and is only possible when,</p> $p(x) = 0 \implies  p(a_j) _{0 \leq j \leq k} = 0$ <p>hence <math>p</math> is identically zero. Thus condition satisfies.</p>
if $k + 1 \leq n$	Not a norm for $k < n$ . Hence incorrect.
if $k \geq n + 1$	Norm for $k \geq n$ . Hence correct.

TABLE 5.7.2: Verifying Positivity Property of Norm

Nilpotent Matrix	1. If all the eigen values of matrix is zero then it is said to nilpotent matrix 2. Determinant and trace of nilpotent matrix are always zero.
Invertible Matrix	A matrix is said to be invertible matrix if its determinant is non zero.
Diagonal matrix	diagonal matrix is a matrix in which the entries outside the main diagonal are all zero.

TABLE 5.8.1: Definition

Given	$T : P_3 \rightarrow P_3$  $T : V \rightarrow V$ be the linear operator given by differentiation wrt $x$ $T(P(x)) \rightarrow P'(x)$  $A$ be the matrix of $T$ wrt some basis for $V$ Assume basis for $V$ be $\{1, x, x^2, x^3\}$
-------	--

TABLE 5.8.2: Result used

Checking whether matrix of $T$ is nilpotent	$T : V \rightarrow V$ $TP(x) = P'(x)$ Differentiating wrt $x$ to find matrix $A$ ; $T(1) = 0 = a_1x + b_1x + c_1x^2 + d_1x^3$ $T(x) = 1 = a_2 + b_2x + c_2x^2 + d_2x^3$ $T(x^2) = 2x = a_3 + b_3x + c_3x^2 + d_3x^3$ $T(x^3) = 3x^2 = a_4 + b_4x + c_4x^2 + d_4x^3$ Representing $A$ in matrix form ; $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ from the above matrix of $T$ we can say it is nilpotent matrix.
Checking eigen value of matrix $T$	$A = \begin{pmatrix} 0 - \lambda & 1 & 0 & 0 \\ 0 & 0 - \lambda & 2 & 0 \\ 0 & 0 & 0 - \lambda & 3 \\ 0 & 0 & 0 & 0 - \lambda \end{pmatrix}$ $\Rightarrow \lambda = 0$
Checking whether matrix of $T$ is invertible	Since $\det A = 0$ . Therefore matrix of $T$ is not invertible
Checking whether Matrix of $T$ is diagonal matrix	Let basis be $B' = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3\}$ Differentiating wrt $x$ ;

	$T(1) = 0 = a_1x + b_1(1+x) + c_1(1+x+x^2) + d_1(1+x+x^2+x^3)$ $T(1+x) = 1 = a_2 + b_2(1+x) + c_2(1+x+x^2) + d_2(1+x+x^2+x^3)$ $T(1+x+x^2) = 1+2x = a_3 + b_3(1+x) + c_3(1+x+x^2) + d_3(1+x+x^2+x^3)$ $T(1+x+x^2+x^3) = 1+2x+3x^2 = a_4 + b_4(1+x) + c_4(1+x+x^2) + d_4(1+x+x^2+x^3)$ $B = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ <p>above matrix is not a diagonal matrix</p>
Conclusion	Thus we can conclude Option 2) and 3) are correct.

TABLE 5.8.3: Solution

5.9. Let  $m, n, r$  be natural numbers. Let  $A$  be an  $m \times n$  matrix with real entries such that  $(AA^t)^r = I$ , where  $I$  is the  $m \times m$  identity matrix and  $A^t$  is the transpose of the matrix  $A$ . We can conclude that

- a)  $m = n$
- b)  $AA^t$  is invertible
- c)  $A^tA$  is invertible
- d) if  $m = n$ , then  $A$  is invertible

**Solution:** Options 2) and 4) are correct. See Table 5.9.1

5.10. Let  $\mathbf{A}$  be a  $n \times n$  real matrix with  $\mathbf{A}^2 = \mathbf{A}$ . Then

- a) the eigenvalues of  $\mathbf{A}$  are either 0 or 1
- b)  $\mathbf{A}$  is a diagonal matrix with diagonal entries 0 or 1
- c)  $\text{rank}(\mathbf{A}) = \text{trace}(\mathbf{A})$
- d) if  $\text{rank}(\mathbf{I} - \mathbf{A}) = \text{trace}(\mathbf{I} - \mathbf{A})$

**Solution:** See Table 5.10.1

5.11. For any  $n \times n$  matrix  $B$ , let  $N(B) = \{X \in \mathbb{R}^n : BX = 0\}$  be the null space of  $B$ . Let  $A$  be a  $4 \times 4$  matrix with  $\dim(N(A - 4I)) = 2$ ,  $\dim(N(A - 2I)) = 1$  and  $\text{rank}(A) = 3$  Which of the following are true?

- a) 0, 2 and 4 are eigenvalues of  $A$
- b)  $\det(A) = 0$
- c)  $A$  is not diagonalizable
- d)  $\text{trace}(A) = 8$

Option	Answer
1) $m = n$	<p>Let <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 0 \end{pmatrix}</math> and <math>r = 1</math></p> $(\mathbf{A}\mathbf{A}^T)^r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ <p>Since <math>m \neq n</math> Option 1 is False.</p>
2) $AA^T$ is invertible	<p>w.k.t <math>\det(A^n) = (\det(A))^n</math>          Since <math>(AA^T)^r = I</math>          So <math>\det((AA^T)^r) = \det(I)</math>  <math>(\det(AA^T))^r = 1</math>  <math>\implies \det(AA^T) \neq 0</math>          Hence <math>AA^T</math> is invertible          Option 2 is True.</p>
3) $A^T A$ is invertible	<p>Let <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 0 \end{pmatrix}</math> and <math>r = 1</math></p> $(\mathbf{A}^T \mathbf{A})^r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ <p>But <math>\det(AA^T) = 0</math>.  <math>\implies AA^T</math> is not invertible.          Hence Option 3 is False</p>
4) if $m = n$ then $A$ is invertible	<p>Since <math>\det(AA^T) \neq 0</math>  <math>\det(A) \cdot \det(A^T) \neq 0</math>  <math>\det(A) \cdot \det(A) \neq 0</math>  <math>\implies A</math> is invertible.          Hence Option 4 is True</p>

TABLE 5.9.1

**Solution:** See Table 5.11.1.

Given	<p><math>A</math> is a <math>4 \times 4</math> matrix.  <math>\dim(N(A - 2I)) = 2</math>,  <math>\dim(N(A - 4I)) = 1</math>, and  <math>\text{rank}(A) = 3</math></p>
Eigenvalues of a matrix	The number $\lambda$ is an eigenvalue of a matrix $A$ if and only if $A - \lambda I$ is singular,

i.e.  $|A - \lambda I| = 0$

For  $\lambda = 2$

Given,  $\dim(N(A - 2I)) = 2$

$$\implies \text{nullity}(A - 2I) = 2$$

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$\implies \text{rank}(A - 2I) = 4 - 2 = 2$$

$\implies (A - 2I)$  is not a full rank matrix

Therefore  $|A - 2I| = 0$

Also,

$$\implies N(A - 2I) = \{X \in \mathbb{R}^4 : (A - 2I)X = 0\}$$

$$\implies (A - 2I)X = 0 \text{ gives two eigen vectors}$$

$$\implies 2 \text{ is an eigenvalue of } A \text{ with multiplicity } 2.$$

Similarly, for  $\lambda = 4$

Given,  $\dim(N(A - 4I)) = 1$

$$\implies \text{rank}(A - 4I) = 4 - 1 = 3$$

$\implies (A - 4I)$  is not a full rank matrix

	<p>Therefore <math> A - 4I  = 0</math>  <math>\Rightarrow 4</math> is an eigenvalue of <math>A</math> with multiplicity 1.</p> <p>For <math>\lambda = 0</math>  Given that <math>\text{rank}(A) = 3</math>  <math>\Rightarrow A</math> is not a full rank matrix  Therefore <math> A  = 0</math>  <math>\Rightarrow 0</math> is an eigenvalue of <math>A</math> with multiplicity 1.</p>
Determinant	<p>Given that <math>\text{rank}(A) = 3</math>  <math>\Rightarrow A</math> is not a full rank matrix  Therefore <math> A  = 0</math></p>
Diagonalizability	<p>An <math>n \times n</math> matrix <math>A</math> is diagonalizable if and only if <math>A</math> has <math>n</math> linearly independent eigen vectors.  <math>\text{rank}(A) + \text{nullity}(A) = n</math>  <math>\Rightarrow</math> for <math>\lambda = 0</math>,  <math>\text{nullity}(A - \lambda I) = \text{nullity}(A) = 4 - 3 = 1</math>  <math>\Rightarrow</math> There exists only one linearly independent eigen vector corresponding to 0 eigen value  Thus, matrix <math>A</math> is not diagonalizable.</p>
Trace	<p><math>\text{Trace}(A) = \text{sum of eigen values}</math>  <math>\Rightarrow \text{Trace}(A) = 0 + 2 + 2 + 4 = 8</math></p>
Conclusion	<p>Option (1), (2) and (4) are correct</p>

TABLE 5.11.1: Solution

5.12. Which of the following 3x3 matrices are diagonalizable over  $\mathbb{R}$ ?

- a)  $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$
- b)  $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- c)  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$
- d)  $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

**Solution:** See Tables 5.12.1 and 5.12.2



Objective	Explanation
Eigenvalues of $\mathbf{A}$	<p>Since</p> $\mathbf{A}^2 = \mathbf{A} \quad (5.10.1)$ $\implies \mathbf{A}^2 - \mathbf{A} = \mathbf{O} \quad (5.10.2)$ <p>From Cayley-Hamilton Theorem we have,</p> $\lambda^2 - \lambda = 0 \quad (5.10.3)$ $\implies \lambda(\lambda - 1) = 0 \quad (5.10.4)$ $\implies \lambda = 0, 1 \quad (5.10.5)$ <p>A matrix <math>\mathbf{A}</math> satisfying <math>\mathbf{A}^2 = \mathbf{A}</math> is an idempotent matrix with eigen values equal to 0 or 1.</p>
Check if $\mathbf{A}$ is necessary diagonal	<p>Consider</p> $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.6)$ $\quad (5.10.7)$ <p>Then,</p> $\mathbf{A}^2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.8)$ $= \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad (5.10.9)$ $= \mathbf{A} \quad (5.10.10)$ <p>Hence <math>\mathbf{A}</math> is idempotent but not diagonal.</p>
Relation between rank and trace of $\mathbf{A}$	<p>Rank of matrix is defined as the number of non-zero eigenvalues. Since number of non-zero eigenvalues is 1,</p> $rank(\mathbf{A}) = 1 \quad (5.10.11)$ $trace(\mathbf{A}) = \sum_i \lambda_i = 0 + 1 = 1 \quad (5.10.12)$ $\implies rank(\mathbf{A}) = trace(\mathbf{A}) \quad (5.10.13)$
Relation between rank and trace of $\mathbf{I} - \mathbf{A}$	<p>Now for the matrix <math>\mathbf{I} - \mathbf{A}</math> we have,</p> $(\mathbf{I} - \mathbf{A})^2 = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \quad (5.10.14)$ $= \mathbf{I}^2 - \mathbf{I}\mathbf{A} - \mathbf{A}\mathbf{I} + \mathbf{A}^2 \quad (5.10.15)$ $= \mathbf{I} - \mathbf{A} - \mathbf{A} + \mathbf{A} \quad (5.10.16)$ $= \mathbf{I} - \mathbf{A} \quad (5.10.17)$ <p>Hence <math>\mathbf{I} - \mathbf{A}</math> is an idempotent matrix. Therefore we conclude,</p> $rank(\mathbf{I} - \mathbf{A}) = trace(\mathbf{I} - \mathbf{A}) \quad (5.10.18)$
Answer	(1),(3) and (4)

TABLE 5.10.1

Test for diagonalizability	<p>Let <math>\mathbf{W}_i</math> be the eigenspace corresponding to eigenvalue <math>\lambda_i</math> of <math>\mathbf{A}</math></p> <p>1) <math>\mathbf{A}</math> is diagonalizable</p> <p>2) characteristic polynomial of <math>\mathbf{A}</math> is</p> <p><math>f = (\mathbf{x} - \lambda_1)^{d_1} \dots (\mathbf{x} - \lambda_k)^{d_k}</math> and <math>\dim(\mathbf{W}_i) = d_i</math></p> <p>3) <math>\sum_{i=1}^k \mathbf{W}_i = n</math></p>
Concept for diagonalization	<p>A linear operator <math>\mathbf{A}</math> on a <math>n</math>-dimensional space <math>\mathbb{V}</math> is diagonalizable , if and only if <math>\mathbf{A}</math> has <math>n</math> distinct characteristic vectors or null spaces corresponding to the characteristic values</p>

TABLE 5.12.1: Illustration of theorem.

Option A	<p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$
Finding Characteristics polynomial	<p>Characteristics polynomial of the matrix <math>\mathbf{A}</math> is <math>\det(x\mathbf{I} - \mathbf{A})</math></p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x-1) & -3 & -2 \\ 0 & (x-4) & -5 \\ 0 & 0 & x-6 \end{vmatrix}$ <p>Characteristic Polynomial = <math>(x-1)(x-4)(x-6)</math></p>
Testing diagonalizability over $\mathbb{R}$	<p>1) As the characteristics polynomial is product of linear factors over <math>\mathbb{R}</math> .</p> <p>2) To find characteristic values of the operator <math>\det(xI - A) = 0</math> which gives <math>\lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 6</math></p> <p>Thus over <math>\mathbb{R}</math> matrix <math>\mathbf{A}</math> has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value . Thus <math>\dim \mathbf{W}_i = d_i</math></p> <p>3) <math>\sum_i \mathbf{W}_i = n = 3</math> , which is equal to <math>\dim</math> of <math>\mathbf{A}</math>.</p>

Conclusion on Option A	Option A satisfy all three condition of Diagonalizability over $\mathbb{R}$ .
Option B	<p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Finding Characteristics polynomial	<p>Characteristics polynomial of the matrix <math>\det(x\mathbf{I} - \mathbf{A})</math></p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x & -1 & 0 \\ 1 & x & 0 \\ 0 & 0 & x - 1 \end{vmatrix}$ <p>Characteristic Polynomial = <math>(x - 1)(x + i)(x - i)</math></p>
Testing diagonalizability over $\mathbb{R}$	<p>1) As the characteristics polynomial is not the product of linear factors over <math>\mathbb{R}</math> beacuse roots of characteristic eq are complex . Thus <math>\mathbf{A}</math> is not diagonalizable over <math>\mathbb{R}</math>.</p>
Conclusion on Option B	Option B does not satisfy condition 1.
Option C	<p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$
Finding Characteristics polynomial	<p>Characteristics polynomial of the matrix <math>\mathbf{A}</math> is <math>\det(x\mathbf{I} - \mathbf{A})</math></p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} (x - 1) & -2 & -3 \\ -2 & (x - 1) & -4 \\ -3 & -4 & x - 1 \end{vmatrix}$ <p>Characteristic Polynomial = <math>(x + 3.19)(x + 0.877)(x - 7.07)</math></p>
Testing diagonalizability over $\mathbb{R}$	<p>1) As the characteristics polynomial are product of linear factors over <math>\mathbb{R}</math> .</p> <p>2) To find characteristic values of the operator <math>\det(x\mathbf{I} - \mathbf{A}) = 0</math> which gives <math>\lambda_1 = -3.19, \lambda_2 = -0.887, \lambda_3 = 7.07</math></p>

	<p>Thus over <math>\mathbb{R}</math> matrix <math>\mathbf{A}</math> has three distinct characteristic values. There will be atleast one characteristics vector i.e., one dimension with each characteristics value .</p> <p>Thus <math>\dim \mathbf{W}_i = d_i</math></p> <p>3) <math>\sum_i \mathbf{W}_i = n = 3</math> , which is equal to <math>\dim</math> of <math>\mathbf{A}</math>.</p>
Conclusion on Option C	Option C satisfy all three condition of Diagonalizability over $\mathbb{R}$ .
Option D	<p>Given matrix is</p> $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Finding Characteristics polynomial	<p>Characteristics polynomial of the matrix <math>\mathbf{A}</math> is <math>\det(x\mathbf{I} - \mathbf{A})</math></p> $\det(x\mathbf{I} - \mathbf{A}) = \begin{vmatrix} x & -1 & -2 \\ 0 & x & -1 \\ 0 & 0 & x \end{vmatrix}$ <p>Characteristic Polynomial = <math>(x)(x)(x) = x^3</math></p>
Testing diagonalizability over $\mathbb{R}$	<p>1) As the characteristics polynomial is product of linear factors over <math>\mathbb{R}</math> .</p> <p>2) To find characteristic values of the operator <math>\det(x\mathbf{I} - \mathbf{A}) = 0</math></p> <p><math>\lambda_1 = 0</math></p> <p><math>d_1 = 3</math></p> $\mathbf{W}_1 = \mathbf{A} - \lambda_1 \mathbf{I} \Rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - 0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ <p><math>\dim \mathbf{W}_1 = 2</math></p> <p><math>\dim \mathbf{W}_i \neq d_i</math></p> <p>Algebric Multiplicity is not equal to Geometric Multiplicity.</p>
Conclusion on Option D	Option D does not satisfy second condition of Diagonalizability.
Answer	Option A and Option C are Diagonalizable over $\mathbb{R}$ .

TABLE 5.12.2: Option Checking Table

Positive Semi Definite Matrix	A $n \times n$ symmetric real matrix $\mathbf{M}$ is said to be positive semi definite if $\mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0$ for all non-zero $\mathbf{x}$ in $\mathbb{R}^n$ . Formally $\mathbf{M}$ is positive semi-definite $\Leftrightarrow \mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbb{R}^n \setminus \{0\}$
Theorem	For a symmetric $n \times n$ matrix $\mathbf{M} \in \mathbf{L}(\mathbf{V})$ , following are equivalent. 1). $\mathbf{x}^T \mathbf{M} \mathbf{x} \geq 0 \forall \mathbf{x} \in \mathbf{V}$ . 2). All the eigenvalues of $\mathbf{M}$ are non-negative.

TABLE 5.13.1: Definition and Result used

Calculating eigen values of $\mathbf{A}$	Given $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ Calculating, eigen values of $\mathbf{A}$ , ie $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ $\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$ $\Rightarrow (3-\lambda)((2-\lambda)(1-\lambda)-9) - 1(1-\lambda-6) + 2(3-2(2-\lambda)) = 0$ $\Rightarrow \lambda^3 - 6\lambda^2 - 3\lambda + 18 = 0$ $\Rightarrow \lambda_1 = 6, \lambda_2 = \sqrt{3} \text{ and } \lambda_3 = -\sqrt{3}$ Hence, $\mathbf{A}$ has exactly two positive eigen values.
Proving $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ using contradiction	Suppose $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^3$ . Then, by theorem above in definition section, matrix $\mathbf{A}$ is positive semi definite. Hence, all the eigen values of $\mathbf{A}$ non-negative, but this is not the case as one of eigen value is $\lambda_3 = -\sqrt{3}$ . So, $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$ . Similarly, as $\lambda_2 \leq 0, \forall i$ is also not true, so $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 0$ is not true for all $\mathbf{x} \in \mathbb{R}^3$ . Thus, $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for some $\mathbf{x} \in \mathbb{R}^3$ .
Correct Options	Hence, correct options are (1) and (4).

TABLE 5.13.2: Solution

5.13. Let  $\mathbf{A} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$  and  $\mathbf{Q}(\mathbf{X}) = \mathbf{X}^T \mathbf{A} \mathbf{X}$  for  $\mathbf{X} \in \mathbb{R}^3$ . Then

- $\mathbf{A}$  has exactly two positive eigen values.
- all the eigen values of  $\mathbf{A}$  are positive.
- $\mathbf{Q}(\mathbf{X}) \geq 0 \forall \mathbf{X} \in \mathbb{R}^3$
- $\mathbf{Q}(\mathbf{X}) < 0$  for some  $\mathbf{X} \in \mathbb{R}^3$

**Solution:** See Tables 5.13.1 and 5.13.2

5.14. Consider the matrix

$$A(x) = \begin{pmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{pmatrix}; x \in \mathbf{R}. \quad (5.14.1)$$

Then,

- $A(x)$  has eigenvalue 0 for some  $x \in \mathbf{R}$ .
- 0 is not an eigenvalue of  $A(x)$  for any  $x \in \mathbf{R}$ .
- $A(x)$  has eigenvalue 0  $\forall x \in \mathbf{R}$ .
- $A(x)$  is invertible  $\forall x \in \mathbf{R}$ .

**Solution:** Let  $\lambda = 0$  be an eigenvalue. Hence,

$$|A - \lambda I| = 0 \quad (5.14.2)$$

$$\Rightarrow |A| = 0 \quad (5.14.3)$$

$$\Rightarrow |A| = \begin{vmatrix} 1+x^2 & 7 & 11 \\ 3x & 2x & 4 \\ 8x & 17 & 13 \end{vmatrix} = 0 \quad (5.14.4)$$

Performing row reduction we get,

$$\begin{vmatrix} 1+x^2 & 7 & 11 \\ 0 & \frac{2x^3-19x}{1+x^2} & \frac{4x^2-33x+4}{1+x^2} \\ 0 & 0 & \frac{26x^3-244x^2+538x-68}{2x^3-19x} \end{vmatrix} = 0 \quad (5.14.5)$$

$$\Rightarrow 26x^3 - 244x^2 + 538x - 68 = 0 \quad (5.14.6)$$

$$\Rightarrow x_1 = 6.01, x_2 = 3.23, x_3 = 0.13 \quad (5.14.7)$$

See Table 5.14.1

6 DECEMBER 2016

6.1. The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} \quad (6.1.1)$$

is

- positive definite.
- non-negative definite but not positive definite.
- negative definite.
- neither negative definite nor positive definite.

**Solution:**

- For a real symmetric matrix to be positive definite the eigen values of the matrix should

OPTIONS	Explanation
Option (b)	At the Values of x given by (5.14.7), eigen value $\lambda = 0$ . Hence option (b) can't be correct.
Option (c)	If one of the eigenvalue is 0 for $A(x)$ then, $ A(x)  = 0 \forall x \in R$ . But from (5.14.7) we have concluded that $ A  = 0$ only for, $x_1 = 6.01, x_2 = 3.23, x_3 = 0.13$ . Hence, Option (c) is incorrect.
Option (d)	Now for the values of x given by (5.14.7), $ A  = 0$ . Hence it is not invertible $\forall x \in R$ Hence Option (d) is incorrect.
Option (a)	Now clearly from above arguments $A(x)$ has eigenvalue 0 for some $x \in R$ Hence Option (a) is Correct.

TABLE 5.14.1

be positive.

- b) For a real symmetric matrix to be negative definite the eigen values of the matrix should be negative.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

The characteristic equation of the matrix  $\mathbf{A}$  is given by

$$|V - \lambda \mathbf{I}| = \begin{vmatrix} 3 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 3 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$$

(6.1.2)

The Eigen values of  $\mathbf{A}$  are:

$$\begin{aligned} \lambda_1 &= 5/2 \\ \lambda_2 &= 3/2 \\ \lambda_3 &= 4 \end{aligned}$$

(6.1.3)

Since all the eigen values of matrix  $\mathbf{A}$  are positive, Therefore the matrix  $\mathbf{A}$  is positive definite.

- 6.2. Which of the following subsets of  $\mathbb{R}^4$  is a basis of  $\mathbb{R}^4$  ?

$$\mathbf{B}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{B}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\mathbf{B}_3 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ -5 & 5 & 0 & 0 \end{pmatrix}$$

- a)  $\mathbf{B}_1$  and  $\mathbf{B}_2$  but not  $\mathbf{B}_3$ .  
b)  $\mathbf{B}_1, \mathbf{B}_2$ , and  $\mathbf{B}_3$ .  
c)  $\mathbf{B}_1$  and  $\mathbf{B}_3$  but not  $\mathbf{B}_2$ .  
d) Only  $\mathbf{B}_1$ .

**Solution:** See Table 6.2.1

Statement	Solution
Definition	<p>Let <math>\mathbf{V}</math> be a vector space. Then <math>\{\mathbf{v}_1, \dots, \mathbf{v}_n\}</math> is called a basis for <math>\mathbf{V}</math> if the following conditions hold.</p> $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \mathbf{V} \quad (6.2.1)$ $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \text{ is linearly independent} \quad (6.2.2)$
Given	$\mathbf{B}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{B}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \mathbf{B}_3 = \begin{pmatrix} 1 & 0 & 2 & -5 \\ 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (6.2.3)$
Checking $\mathbf{B}_1$	<p>Checking for linear independence. Upon row reducing <math>\mathbf{B}_1</math> (6.2.4)</p> $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xleftarrow{R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3, R_3 \rightarrow R_3 - R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.2.5)$ <p>Clearly Rank of <math>\mathbf{B}_1</math> is 4, ie full rank. Hence it forms a Basis.</p>
Checking $\mathbf{B}_2$	<p>Checking for linear independence. Upon row reducing <math>\mathbf{B}_2</math> (6.2.6)</p> $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \xleftarrow{R_2 \rightarrow \frac{R_2}{2}, R_1 \rightarrow R_1 - R_2, R_3 \rightarrow \frac{R_3}{3}, R_2 \rightarrow R_2 - R_3, R_4 \rightarrow \frac{R_4}{4}, R_3 \rightarrow R_3 - R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.2.7)$ <p>Rank of <math>\mathbf{B}_2</math> is 4, ie full rank. Hence it also forms a Basis.</p>
Checking $\mathbf{B}_3$	<p>Checking for linear independence. Upon row reducing <math>\mathbf{B}_3</math> (6.2.8)</p> $\begin{pmatrix} 1 & 0 & 2 & -5 \\ 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xleftarrow{R_2 \rightarrow R_2 - 2R_1, R_4 \rightarrow R_4 - R_2, R_3 \rightarrow -\frac{R_3}{5}, R_1 \rightarrow R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.2.9)$ <p>Rank of <math>\mathbf{B}_3</math> is 3, ie not full rank. Hence it does not forms a Basis.</p>
Conclusion	Hence option 1, ie $\mathbf{B}_1, \mathbf{B}_2$ and not $\mathbf{B}_3$ is the correct answer.

TABLE 6.2.1: Solution

Given	<p>a) Matrix <math>J</math> of <math>n \times n</math> dimension with all entries 1.</p> <p>b) Matrix <math>B</math> of <math>3n \times 3n</math> dimension</p> $B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$
Transforming matrix $B$ into Block diagonal matrix using transformation Matrix	$M = \mathbf{T}(B)$ $M = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}$ $M = \begin{pmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}$
Rank of Block Diagonal matrix $M$	<p>It is equal to the sum of rank of individual blocks in diagonal</p> $r(J) = 1$ $\therefore r(M) = 1 + 1 + 1 = 3$
Rank of a matrix and its transformation are same.	<p><math>\therefore</math> rank of matrix <math>B</math> is</p> $r(B) = r(M) = 3$

TABLE 6.3.1

6.3. Let  $J$  denote the matrix of order  $n \times n$  with all entries 1 and let  $B$  be a  $3n \times 3n$  matrix given

$$\text{by } B = \begin{pmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{pmatrix}.$$

Find rank of matrix  $B$ . **Solution:** See Tables 6.3.1 and 6.3.2

6.4. Which of the following sets of functions from  $\mathfrak{R}$  to  $\mathfrak{R}$  is a vector space over  $\mathfrak{R}$ ?

$$S_1 = \{f \mid \lim_{x \rightarrow 3} f(x) = 0\} \quad (6.4.1)$$

$$S_2 = \{g \mid \lim_{x \rightarrow 3} g(x) = 1\} \quad (6.4.2)$$

$$S_3 = \{h \mid \lim_{x \rightarrow 3} h(x) \text{ exists}\} \quad (6.4.3)$$

is

a) Only  $S_1$

b) Only  $S_2$

c)  $S_1$  and  $S_3$  but not  $S_2$

d) All the three are vector spaces

**Solution:** Let  $S$  be a set of functions. Let  $f_1, f_2 \in S$  and  $\alpha, \beta \in \mathfrak{R}$

For a set of functions to be considered as a vector space:

a) The linear combination of  $f_1$  and  $f_2$  should be in  $S$ .

$$\text{i.e. } \alpha f_1(x) + \beta f_2(x) \in S$$

b) The  $\mathbf{0}$  should belong to  $S$

$$\text{i.e. } \mathbf{0} \in S$$

Case1: Test for  $S_1$



Example	<p>Let <math>n = 2</math></p> $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
Transforming matrix $B$ into Block diagonal matrix using transformation Matrix	$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ $M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
Rank of Block Diagonal matrix $M$	<p>It is equal to the sum of rank of individual blocks in diagonal</p> $r(J) = 1$ $\therefore r(M) = 1 + 1 + 1 = 3$
Rank of a matrix and its transformation are same.	<p><math>\therefore</math> rank of matrix <math>B</math> is</p> $r(B) = r(M) = 3$

TABLE 6.3.2

a) Let  $f_1, f_2 \in S_1$  and  $\alpha, \beta \in \mathfrak{R}$

Then Using (6.4.4)

$$\lim_{x \rightarrow 3} f_1(x) = 0$$

$$\lim_{x \rightarrow 3} f_2(x) = 0$$

(6.4.4)

$$\lim_{x \rightarrow 3} (\alpha f_1(x) + \beta f_2(x))$$

$$= \alpha \left( \lim_{x \rightarrow 3} f_1(x) \right) + \beta \left( \lim_{x \rightarrow 3} f_2(x) \right)$$

$$= \alpha \times 0 + \beta \times 0$$

$$= 0$$

$$\therefore \alpha f_1(x) + \beta f_2(x) \in S_1$$

- b) Let  $f(x) = 0$   
then

$$\lim_{x \rightarrow 3} f(x) = 0$$

$$\therefore \mathbf{0} \in S_1$$

Hence,  $S_1$  is a vector space.

Case2: Test for  $S_2$

- a) Let  $g_1, g_2 \in S_2$  and  $\alpha, \beta \in \mathfrak{R}$

$$\lim_{x \rightarrow 3} g_1(x) = 1$$

$$\lim_{x \rightarrow 3} g_2(x) = 1 \quad (6.4.5)$$

Then Using (6.4.5)

$$\begin{aligned} & \lim_{x \rightarrow 3} (\alpha g_1(x) + \beta g_2(x)) \\ &= \alpha \left( \lim_{x \rightarrow 3} g_1(x) \right) + \beta \left( \lim_{x \rightarrow 3} g_2(x) \right) \\ &= \alpha \times 1 + \beta \times 1 \\ &= \alpha + \beta \end{aligned}$$

$$\therefore \alpha g_1(x) + \beta g_2(x) \in S_1 \text{ if } \alpha + \beta = 1$$

- b) Let  $g(x) = 0$   
then

$$\lim_{x \rightarrow 3} g(x) = 1$$

$$\therefore \mathbf{0} \notin S_1$$

Hence,  $S_2$  is not a vector space.

Case3: Test for  $S_3$

- a) Let  $h_1, h_2 \in S_3$  and  $\alpha, \beta \in \mathfrak{R}$

$$\lim_{x \rightarrow 3} h_1(x) \text{ exists}$$

$$\lim_{x \rightarrow 3} h_2(x) \text{ exists} \quad (6.4.6)$$

Then Using (6.4.6)

$$\begin{aligned} & \lim_{x \rightarrow 3} (\alpha h_1(x) + \beta h_2(x)) \text{ exists} \\ & \therefore \alpha h_1(x) + \beta h_2(x) \in S_3 \end{aligned}$$

- b) Let  $h(x) = 0$   
then

$$\lim_{x \rightarrow 3^-} h(x) = 0 = \lim_{x \rightarrow 3^+} h(x)$$

$$\therefore \mathbf{0} \in S_1$$

Hence,  $S_3$  is a vector space.

Therefore, Option (3) is correct.

6.5. Let  $\mathbf{A}$  be an  $n \times m$  matrix with each entry

equal to +1, -1 or 0 such that every column has exactly one +1 and exactly one -1. We can conclude that

$$1. \text{Rank } \mathbf{A} \leq n - 1 \quad (6.5.1)$$

$$2. \text{Rank } \mathbf{A} = m \quad (6.5.2)$$

$$3. n \leq m \quad (6.5.3)$$

$$4. n - 1 \leq m \quad (6.5.4)$$

**Solution:** See Table 6.5.1

option	Solution
1.	<p>Let us consider <math>\mathbf{A}</math> as follows and let <math>s</math> be the summation of all column entries:</p> $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$ $ \mathbf{A} - \lambda \mathbf{I}  = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{vmatrix} = 0$ $= \begin{vmatrix} a_{11} + a_{21} + \dots + a_{n1} - \lambda & a_{11} + a_{21} + \dots + a_{n1} - \lambda & \dots & a_{11} + a_{21} + \dots + a_{n1} - \lambda \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{vmatrix}$ $\Rightarrow (s - \lambda) \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} - \lambda \end{vmatrix} = 0$
Example	<p>Since <math>s=0</math> according to question,  Therefore <math>\lambda = 0</math> is an eigen value of <math>\mathbf{A}</math>.  Since <math>\lambda = 0</math>, Hence <math>\mathbf{A}</math> is singular.  Which means at least two rows are linearly dependent.  Therefore,</p> $\text{Rank}(\mathbf{A}) < n$ $\text{Rank}(\mathbf{A}) \leq n - 1$ <p>Let us Consider <math>\mathbf{A}</math> as follows, where <math>n=4</math> and <math>m=3</math></p> $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ <p>Calculating Row Reduced Echelon Form of <math>\mathbf{A}</math> as follows:</p>

	$\begin{array}{l} \xleftrightarrow[R_4 \leftarrow R_2 + R_4]{R_4 \leftarrow R_1 + R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \\ \xleftrightarrow{R_4 \leftarrow R_3 + R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$
Conclusion	Since the Rank $\mathbf{A}=3$ and $n=4$ , Therefore the Rank $\mathbf{A} \leq n - 1$ statement is true.
2.	<p>Let us Consider <math>\mathbf{A}</math> as follows,where <math>n=2</math> and <math>m=2</math></p> $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ <p>Applying elementary transformations on <math>\mathbf{A}</math> as follows:</p> $\xleftrightarrow{R_2 \leftarrow R_1 + R_2} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$
Conclusion	Since the Rank $\mathbf{A}=1$ and $m=2$ , Therefore the Rank $\mathbf{A} \neq m$ , Hence the statement is false.
3.	<p>Let us Consider <math>\mathbf{A}</math> as follows,where <math>n=3</math> and <math>m=2</math></p> $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \quad (6.5.5)$
Conclusion	Since there exists a matrix $\mathbf{A}$ when $n>m$ , Therefore the statement is false.
4	<p>Let us Consider <math>\mathbf{A}</math> as follows,where <math>n=4</math> and <math>m=2</math></p> $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (6.5.6)$
Conclusion	Since there exists a matrix $\mathbf{A}$ when $n-1>m$ , Therefore the statement is false.

TABLE 6.5.1: Solution summary

<b>Option 1</b>	To conclude that $m = n$
Assumptions	<p><b>For the example:</b> Without loss of generality, Let <math>m = 2</math>, <math>n = 3</math> and <math>\mathbf{A} = \begin{pmatrix} 1 &amp; 0 &amp; 0 \\ 0 &amp; 1 &amp; 0 \end{pmatrix}</math></p> $\Rightarrow \mathbf{A}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
Proof	<p>We know that <math>(\mathbf{A}\mathbf{A}^t)^r = \mathbf{I}</math> which is a square matrix of order <math>m \times m</math>  For any natural value of <math>r</math>, a square matrix (<math>\mathbf{I}</math>) of order <math>m \times m</math> is obtained  Hence, we cannot conclude that <math>m = n</math> because we get <math>\mathbf{I}</math> of order <math>m \times m</math>  even if <math>m \neq n</math>. To illustrate this, Consider the following example</p> $\mathbf{A}\mathbf{A}^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \quad (\mathbf{A} \text{ and } \mathbf{A}^t \text{ from Assumptions})$ $(\mathbf{A}\mathbf{A}^t)^r = \mathbf{I}$ <p>Here <math>m \neq n</math>. Therefore, <b>Option 1</b> is incorrect</p>

TABLE 6.6.1: Option 1

<b>Option 2</b>	To conclude that $\mathbf{A}\mathbf{A}^t$ is invertible
Assumptions	$\mathbf{A}\mathbf{A}^t$ is not invertible
Proof	$\Rightarrow  \mathbf{A}\mathbf{A}^t  = 0 \Rightarrow  (\mathbf{A}\mathbf{A}^t)^r  = 0$ $\Rightarrow (\mathbf{A}\mathbf{A}^t)^r \neq \mathbf{I} \quad ( \mathbf{I}  = 1)$ <p>Since, this is a contradiction to the assumption made we can conclude that <math>\mathbf{A}\mathbf{A}^t</math> is invertible. Therefore, <b>Option 2</b> is correct</p>

TABLE 6.6.2: Option 2

6.6. Let  $m$ ,  $n$  and  $r$  be natural numbers. Let  $\mathbf{A}$  be an  $m \times n$  matrix with real entries such that  $(\mathbf{A}\mathbf{A}^t)^r = \mathbf{I}$ , where  $\mathbf{I}$  is the  $m \times m$  identity matrix and  $\mathbf{A}^t$  is the transpose of the matrix  $\mathbf{A}$ . We can conclude that

**Options:**

- a)  $m = n$
- b)  $\mathbf{A}\mathbf{A}^t$  is invertible
- c)  $\mathbf{A}^t\mathbf{A}$  is invertible
- d) if  $m = n$ , then  $\mathbf{A}$  is invertible

**Solution:** See Tables 6.6.1, 6.6.2, 6.6.3 and 6.6.4.

6.7. Let  $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$  and let  $\alpha_n$  and  $\beta_n$  denote the two eigenvalues of  $\mathbf{A}^n$  such that  $|\alpha_n| \geq |\beta_n|$ . Then

- a)  $\alpha_n \rightarrow \infty$  as  $n \rightarrow \infty$
- b)  $\beta_n \rightarrow 0$  as  $n \rightarrow \infty$
- c)  $\beta_n$  is positive if  $n$  is even.
- d)  $\beta_n$  is negative if  $n$  is odd.

**Solution:** See Table 6.7.1.

6.8. Let  $M_n$  denote the vector space of all  $n \times n$  real

matrices. Which of the following is a linear subspaces of  $M_n$  :-

- a)  $V_1 = \{A \in M_n : A \text{ is nonsingular}\}$
- b)  $V_2 = \{A \in M_n : \det(A) = 0\}$
- c)  $V_3 = \{A \in M_n : \text{trace}(A) = 0\}$
- d)  $V_4 = \{BA : A \in M_n\}$ , where  $B$  is some fixed matrix in  $M_n$

**Solution:** See Table 6.8.1

6.9. If  $\mathbf{P}$  and  $\mathbf{Q}$  are invertible matrices such that  $\mathbf{P}\mathbf{Q} = -\mathbf{Q}\mathbf{P}$ , then we can conclude that

- a)  $\text{Tr}(\mathbf{P}) = \text{Tr}(\mathbf{Q}) = 0$
- b)  $\text{Tr}(\mathbf{P}) = \text{Tr}(\mathbf{Q}) = 1$
- c)  $\text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{Q})$
- d)  $\text{Tr}(\mathbf{P}) \neq \text{Tr}(\mathbf{Q})$

**Solution:** See Table 6.9.1

<b>Option 3</b>	To conclude that $\mathbf{A}^t \mathbf{A}$ is invertible
Assumptions	Without loss of generality, Let $m = 2$ , $n = 3$ and $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\Rightarrow \mathbf{A}^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$
Proof	$\Rightarrow \mathbf{A}^t \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow  \mathbf{A}^t \mathbf{A}  = 0$ This means that $\mathbf{A}^t \mathbf{A}$ is not invertible. Therefore, <b>Option 3</b> is incorrect

TABLE 6.6.3: Option 3

<b>Option 4</b>	To conclude that if $m = n$ then $\mathbf{A}$ is invertible
Assumptions	Let $m = n$
Proof	Since $(\mathbf{A} \mathbf{A}^t)^r = \mathbf{I} \Rightarrow  (\mathbf{A} \mathbf{A}^t)^r  =  \mathbf{I}  = 1$ $\Rightarrow ( \mathbf{A}   \mathbf{A}^t )^r = 1$ ( $\mathbf{A}$ is a square matrix) $\Rightarrow ( \mathbf{A} )^{2r} = 1$ Therefore, <b>Option 4</b> is correct

TABLE 6.6.4: Option 4

Options	Solutions	True/False
1.	Given $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ Now lets find the eigen values of matrix $\mathbf{A}$ $ \mathbf{A} - \lambda \mathbf{I}  = 0$ $\Rightarrow \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$ $\Rightarrow \lambda^2 - \lambda - 1 = 0$ On solving we get 2 eigen values $\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \beta_1 = \frac{1-\sqrt{5}}{2}$ We know that if eigenvalue of $\mathbf{A}$ is $\lambda$ then eigenvalue of $\mathbf{A}^n$ is $\lambda^n$ . In this problem we can say that the eigenvalues $\alpha_n$ and $\beta_n$ of $\mathbf{A}^n$ are $\alpha_n = \alpha_1^n \quad \beta_n = \beta_1^n$ Since $\alpha_1 > 1$ we can say that $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$ .	True
2.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$ . Since $-1 < \beta_1 < 0$ , we can say that $\beta_n \rightarrow 0$ as $n \rightarrow \infty$ .	True
3.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$ . Since $\beta_1$ is negative because $-1 < \beta_1 < 0$ , if $n$ is even then $\beta_n$ is positive.	True
4.	We got $\beta_1 = \frac{1-\sqrt{5}}{2}$ and $\beta_n = \beta_1^n$ . Since $\beta_1$ is negative, if $n$ is odd then $\beta_n$ is negative.	True

TABLE 6.7.1

Vector space	Is it subspace to $M_n$ ?
1) $V_1$ : All non-singular matrices of $n \times n$	The matrices $I_{n \times n}$ and $-I_{n \times n}$ are non-singular matrices, but the sum $I_{n \times n} - I_{n \times n}$ is zero matrix and it is singular. $\therefore V_1$ does not form subspace of $M_n$ .
2) $V_2$ : All singular matrices of $n \times n$	The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are singular matrices, but the sum is a non-singular matrix. $\therefore V_2$ does not form subspace $M_n$ .
3) $V_3$ : All matrices of $n \times n$ with trace =0	Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be matrices with Trace = 0. $Tr(\mathbf{v}_1 + \alpha \mathbf{v}_2) = Tr(\mathbf{v}_1) + \alpha Tr(\mathbf{v}_2) = 0$ . $\therefore$ the vector space $V_3$ forms linear subspace of $M_n$ .
4) $V_4$ : $F_A = BA$ , where B is some fixed matrix in $M_n$	Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be matrices in the vector space $V_4$ . $F_{\mathbf{v}_1 + \alpha \mathbf{v}_2} = B(\mathbf{v}_1 + \alpha \mathbf{v}_2)$ $= B\mathbf{v}_1 + \alpha B\mathbf{v}_2 =$ $F_{\mathbf{v}_1} + \alpha F_{\mathbf{v}_2}$ . $\therefore V_4$ forms linear subspace of $M_n$ .

TABLE 6.8.1

Given	$\mathbf{P}$ and $\mathbf{Q}$ are invertible matrices. Therefore $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ exists.
-------	---



	$\mathbf{PQ} = -\mathbf{QP} \quad (6.9.1)$
To Prove	$\text{Tr}(\mathbf{P})=0$
Proof 1	<p>Post multiplying equation (6.9.1) by <math>\mathbf{Q}^{-1}</math> we get,</p> $\mathbf{PQQ}^{-1} = -\mathbf{QPQ}^{-1} \quad (6.9.2)$ $\Rightarrow \mathbf{PI} = -\mathbf{QPQ}^{-1} \quad (6.9.3)$ $\Rightarrow \mathbf{P} = -\mathbf{QPQ}^{-1} \quad (6.9.4)$ <p>Taking trace on both sides for the equation (6.9.4),</p> $\text{Tr}(\mathbf{P}) = \text{Tr}(-\mathbf{QPQ}^{-1}) \quad (6.9.5)$ $\Rightarrow \text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{QPQ}^{-1}) \quad (6.9.6)$ <p>We know that <math>\text{Tr}(\mathbf{AB})=\text{Tr}(\mathbf{BA})</math>  Let <math>\mathbf{A}=\mathbf{Q}</math> and <math>\mathbf{B}=\mathbf{PQ}^{-1}</math></p>
	<p>From the above property of trace equation (6.9.6) can be modified as</p> $\text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{PQ}^{-1}\mathbf{Q}) \quad (6.9.7)$ $\Rightarrow \text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{PI}) \quad (6.9.8)$ $\Rightarrow \text{Tr}(\mathbf{P}) = -\text{Tr}(\mathbf{P}) \quad (6.9.9)$ $\Rightarrow 2\text{Tr}(\mathbf{P}) = 0 \quad (6.9.10)$ $\Rightarrow \text{Tr}(\mathbf{P}) = 0 \quad (6.9.11)$
To Prove	$\text{Tr}(\mathbf{Q})=0$
Proof 2	<p>Post multiplying equation (6.9.1) by <math>\mathbf{P}^{-1}</math> we get,</p> $\mathbf{PQP}^{-1} = -\mathbf{QPP}^{-1} \quad (6.9.12)$ $\Rightarrow \mathbf{PQP}^{-1} = -\mathbf{QI} \quad (6.9.13)$ $\Rightarrow \mathbf{PQP}^{-1} = -\mathbf{Q} \quad (6.9.14)$ <p>Taking trace on both sides for the equation (6.9.14),</p> $\text{Tr}(\mathbf{PQP}^{-1}) = \text{Tr}(-\mathbf{Q}) \quad (6.9.15)$ $\Rightarrow \text{Tr}(\mathbf{PQP}^{-1}) = -\text{Tr}(\mathbf{Q}) \quad (6.9.16)$ <p>We know that <math>\text{Tr}(\mathbf{AB})=\text{Tr}(\mathbf{BA})</math>  Let <math>\mathbf{A}=\mathbf{P}</math> and <math>\mathbf{B}=\mathbf{QP}^{-1}</math></p> <p>From the above property of trace equation (6.9.16) can be modified as</p> $\text{Tr}(\mathbf{QP}^{-1}\mathbf{P}) = -\text{Tr}(\mathbf{Q}) \quad (6.9.17)$ $\Rightarrow \text{Tr}(\mathbf{QI}) = -\text{Tr}(\mathbf{Q}) \quad (6.9.18)$ $\Rightarrow \text{Tr}(\mathbf{Q}) = -\text{Tr}(\mathbf{Q}) \quad (6.9.19)$ $\Rightarrow 2\text{Tr}(\mathbf{Q}) = 0 \quad (6.9.20)$ $\Rightarrow \text{Tr}(\mathbf{Q}) = 0 \quad (6.9.21)$
<b>Statement 1</b>	$\text{Tr}(\mathbf{P})=\text{Tr}(\mathbf{Q})=0$
Explanation	From equation (6.9.11) and (6.9.21) we could say that,

	$Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 0 \quad (6.9.22)$ <p>Valid Conclusion</p>
<b>Statement 2</b>	$Tr(\mathbf{P}) = Tr(\mathbf{Q}) = 1$
Explanation	<p>From equation (6.9.11) and (6.9.21) we could say that,</p> $Tr(\mathbf{P}) = Tr(\mathbf{Q}) \neq 1 \quad (6.9.23)$ <p>Invalid Conclusion</p>
<b>Statement 3</b>	$Tr(\mathbf{P}) = -Tr(\mathbf{Q})$
Explanation	<p>Substituting the conclusion 1 result equation (6.9.22) in equation (6.9.9) we get,</p> $Tr(\mathbf{P}) = -Tr(\mathbf{Q}) \quad (6.9.24)$ <p>Valid Conclusion</p>
<b>Statement 4</b>	$Tr(\mathbf{P}) \neq Tr(\mathbf{Q})$
Explanation	<p>From equation (6.9.11) and (6.9.21) we could say that,</p> $Tr(\mathbf{P}) = Tr(\mathbf{Q}) \quad (6.9.25)$ <p>Invalid Conclusion</p>

TABLE 6.9.1: Explanation with Proofs

Let  $n$  be an odd number  $\geq 7$ . Let,

$$\mathbf{A} = [a_{ij}] \quad (6.9.26)$$

be an  $n \times n$  matrix with,

$$a_{i,i+1} = 1, \forall (i = 1, 2, \dots, n-1) \quad (6.9.27)$$

and  $a_{n,1} = 1$ . Let  $a_{ij} = 0$  for all the other pairs  $(i, j)$ . Then we can conclude that,

- a)  $\mathbf{A}$  has 1 as an eigenvalue
- b)  $\mathbf{A}$  has -1 as an eigenvalue
- c)  $\mathbf{A}$  has at least one eigenvalue with multiplicity  $\geq 2$
- d)  $\mathbf{A}$  has no real eigenvalues

**Solution:** We can represent our matrix as:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (6.9.28)$$

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (6.9.29)$$

$\mathbf{A}$  is our given matrix. We know that Characteristic Equation of  $\mathbf{A}$  and  $\mathbf{A}^T$  is same. Consider the minimal polynomial

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0 \quad (6.9.30)$$

We can represent it in  $n \times n$  matrix with 1's on sub-diagonals and in last column it has negative of the coefficient, and rest all 0. We represent it using  $\mathbf{C}$ . It is known as the companion matrix.

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix} \quad (6.9.31)$$

(6.9.30) is also the characteristic equation of  $\mathbf{C}$ . Comparing (6.9.29) with (6.9.31) we get:

$$a_0 = -1, a_1 = a_2 = a_3 = a_4 = \dots = a_{n-1} = 0 \quad (6.9.32)$$

Substituting (6.9.32) into (6.9.30) we get:

$$x^n - 1 = 0 \quad (6.9.33)$$

By Cayley-Hamilton Theorem:

$$\lambda^n - 1 = 0 \quad (6.9.34)$$

$$(6.9.35)$$

$\lambda = n^{\text{th}}$  roots of unity. See Table 6.9.2.

6.10. Let  $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$  be 3 distinct subspaces of  $\mathbf{R}^{10}$  such that each  $\mathbf{W}_i$  has dimension of 9. Let  $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$ . Then we can conclude that

a)  $\mathbf{W}$  may not be a subspace of  $\mathbf{R}^{10}$

b)  $\dim \mathbf{W} \leq 8$

c)  $\dim \mathbf{W} \geq 7$

d)  $\dim \mathbf{W} \leq 3$

**Solution:** See Table 6.10.1

Options	Explanation
A has 1 as an eigen value	One value out of the $n^{\text{th}}$ roots of unity is 1. So, correct
A has -1 as an eigen value	Since, $n$ is odd. So, -1 cannot be one of the value of $n^{\text{th}}$ roots of unity. Hence, incorrect
A has atleast one eigenvalue with multiplicity $\geq 2$	All values of $n^{\text{th}}$ roots of unity are distinct. So there is no eigenvalue with multiplicity $\geq 2$ . Hence, incorrect.
A has no real eigen values	One of the value is 1, which is real. Hence, incorrect.

TABLE 6.9.2: Finding Correct Option

<b>Given</b>	$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ are 3 distinct subspaces of $\mathbf{R}^{10}$  Each $\mathbf{W}_i$ has dimension 9  $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$
<b>Statement1</b>	$\mathbf{W}$ may not be a subspace of $\mathbf{R}^{10}$
Explanation	As $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$ and $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ are subspaces of $\mathbf{W}$ , then $\mathbf{W}$ must be a subspace of $\mathbf{R}^{10}$ . So the first option is false.
<b>Statement2</b>	$\dim \mathbf{W} \leq 8$
Explanation	As $\mathbf{W}$ be a subspace of a finite dimension vector space $\mathbf{R}^{10}$ and $\dim \mathbf{R}^{10} = 10$ , so $\mathbf{W}$ is finite dimension and $\dim \mathbf{W} \leq 10$
<b>Theorem</b>	$\dim (\mathbf{W}_1 \cap \mathbf{W}_2)$ $= \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) - \dim(\mathbf{W}_1 + \mathbf{W}_2)$ and $\mathbf{W}_1 \cap \mathbf{W}_2$ is also a subspace of $\mathbf{R}^{10}$
<b>Proof</b>	The minimum dimension of $\mathbf{W} = \mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3$
Explanation	Let us consider $\mathbf{V} = \mathbf{R}^{10}$ and $\dim(\mathbf{V}) = 10$ and $\mathbf{U} = \mathbf{W}_1 \cap \mathbf{W}_2$

	<p>So, <math>\dim(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) = \dim(\mathbf{U}) + \dim(\mathbf{W}_3) - \dim(\mathbf{U} + \mathbf{W}_3)</math></p> <p>or, <math>\dim(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) = \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - \dim(\mathbf{W}_1 + \mathbf{W}_2) - \dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3)</math></p>
	<p>Now, <math>(\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3 \subseteq \mathbf{V}</math>  <math>\Rightarrow \dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) \leq \dim(\mathbf{V})</math>  <math>\Rightarrow -\dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) \geq -\dim(\mathbf{V})</math></p> <p>Similarly, <math>(\mathbf{W}_1 + \mathbf{W}_2) \subseteq \mathbf{V}</math>  <math>\Rightarrow \dim(\mathbf{W}_1 + \mathbf{W}_2) \leq \dim(\mathbf{V})</math>  <math>\Rightarrow -\dim(\mathbf{W}_1 + \mathbf{W}_2) \geq -\dim(\mathbf{V})</math></p>
	<p>Considering these two inequations,  <math>-\dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) - \dim(\mathbf{W}_1 + \mathbf{W}_2) \geq -2\dim(\mathbf{V})</math></p> <p>or, <math>\dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - \dim((\mathbf{W}_1 \cap \mathbf{W}_2) + \mathbf{W}_3) - \dim(\mathbf{W}_1 + \mathbf{W}_2) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})</math></p> <p>or, <math>\dim(\mathbf{W}_1 \cap \mathbf{W}_2 \cap \mathbf{W}_3) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})</math></p> <p><math>\Rightarrow \dim(\mathbf{W}) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})</math></p>
<b>Statement 3</b>	$\dim \mathbf{W} \geq 7$
Explanation	<p>As <math>\dim(\mathbf{W}) \geq \dim(\mathbf{W}_1) + \dim(\mathbf{W}_2) + \dim(\mathbf{W}_3) - 2\dim(\mathbf{V})</math>  <math>\Rightarrow \dim(\mathbf{W}) \geq (9+9+9) - (2 \times 10)</math>  <math>\Rightarrow \dim(\mathbf{W}) \geq 7</math></p>
<b>Answer</b>	$7 \leq \dim(\mathbf{W}) \leq 10$

TABLE 6.10.1: Solution summary

Hence, we can conclude that  $\dim(\mathbf{W}) \geq 7$ .

7 JUNE 2016

8 DECEMBER 2015

8.1. Let  $\mathbf{A}$  be an  $n \times n$  matrix over  $\mathbb{C}$  such that every non-zero vector  $\mathbb{C}^n$  is an eigen vector of  $\mathbf{A}$ . Then

- a) All eigen values of  $\mathbf{A}$  are equal.
- b) All eigen values of  $\mathbf{A}$  are distinct.
- c)  $\mathbf{A} = \lambda \mathbf{I}$  for some  $\lambda \in \mathbb{C}$ , where  $\mathbf{I}$  is the  $n \times n$  identity matrix.
- d) If  $\chi_{\mathbf{A}}$  and  $m_{\mathbf{A}}$  denote the characteristic polynomial and the minimal polynomial respectively, then  $\chi_{\mathbf{A}} = m_{\mathbf{A}}$

**Solution:** See Tables 8.1.1 , 8.1.2 and 8.1.3

Given	Every non-zero vector $\mathbb{C}^n$ is an eigen vector of $\mathbf{A}$ , where $\mathbf{A}$ is an $n \times n$ matrix over $\mathbb{C}$ .
Determining $\mathbf{A}$	<p>Since every vector is an eigen vector, the standard basis vectors are also eigen vectors</p> $\Rightarrow \mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i \Rightarrow \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix} \mathbf{e}_i = \lambda_i \mathbf{e}_i \Rightarrow a_i = \lambda_i \mathbf{e}_i \text{ where } \lambda_i \in \mathbb{C}$ <p>therefore <math>\mathbf{A} = \begin{pmatrix} \lambda_1 \mathbf{e}_1 &amp; \lambda_2 \mathbf{e}_2 &amp; \dots &amp; \lambda_n \mathbf{e}_n \end{pmatrix}</math></p> <p>Any vector <math>\mathbf{b}</math> can be represented in the standard basis as</p> $\mathbf{b} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n \text{ where } b_i \in \mathbb{C}$ <p>As every non-zero vector in <math>\mathbb{C}^n</math> is an eigen vector</p> $\mathbf{A}\mathbf{b} = \lambda \mathbf{b} \Rightarrow \mathbf{A}(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n) = \lambda(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n)$ $\Rightarrow b_1 \lambda_1 \mathbf{e}_1 + b_2 \lambda_2 \mathbf{e}_2 + \dots + b_n \lambda_n \mathbf{e}_n = \lambda(b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + \dots + b_n \mathbf{e}_n)$ $\Rightarrow b_1(\lambda_1 - \lambda) \mathbf{e}_1 + b_2(\lambda_2 - \lambda) \mathbf{e}_2 + \dots + b_n(\lambda_n - \lambda) \mathbf{e}_n = 0$ <p>since basis are linearly independent we get <math>\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda</math></p> <p>Therefore the matrix <math>\mathbf{A}</math> is</p> $\mathbf{A} = \begin{pmatrix} \lambda_1 \mathbf{e}_1 & \lambda_2 \mathbf{e}_2 & \dots & \lambda_n \mathbf{e}_n \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \end{pmatrix} = \lambda \mathbf{I}_n \text{ where } \lambda \in \mathbb{C}$

TABLE 8.1.1

option 1	Since $\mathbf{A} = \lambda \mathbf{I}_n$ , all the eigen values are equal to $\lambda$ . Therefore option 1 is correct as the matrix $\mathbf{A}$ is a scalar matrix.
option 2	since the matrix $\mathbf{A}$ is a scalar matrix, all the eigen values are equal. So this option is incorrect.
option 3	This option is correct. As proved in the construction the matrix $\mathbf{A} = \lambda \mathbf{I}$ for some $\lambda \in \mathbb{C}$
option 4	Since $\mathbf{A} = \lambda \mathbf{I}$ where $\lambda \in \mathbb{C}$ , the characteristic polynomial and the minimal polynomial are $\chi_{\mathbf{A}} = (x - \lambda)^n$ and $m_{\mathbf{A}} = (x - \lambda) \Rightarrow \chi_{\mathbf{A}} = m_{\mathbf{A}}^n$ . Therefore this option is incorrect

TABLE 8.1.2: Answer

Scalar matrix	<p>Consider a <math>3 \times 3</math> scalar matrix <math>\mathbf{A} = (2 + 3i) \mathbf{I}</math>, for which the eigen values are <math>(2 + 3i), (2 + 3i), (2 + 3i)</math></p> <p>The eigen vectors will be the nullspace of <math>\mathbf{A} - \lambda \mathbf{I}</math></p> $\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} 2 + 3i & 0 & 0 \\ 0 & 2 + 3i & 0 \\ 0 & 0 & 2 + 3i \end{pmatrix} - (2 + 3i) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ <p>The nullspace consists of the entire vector space so every vector is an eigen vector</p> <p>The characteristic polynomial and the minimal polynomial are <math>\chi_{\mathbf{A}} = (x - (2 + 3i))^3</math> and <math>m_{\mathbf{A}} = (x - (2 + 3i)) \Rightarrow \chi_{\mathbf{A}} = m_{\mathbf{A}}^3</math></p> <p>Therefore options 1 and 3 are correct.</p>
Diagonal matrix	<p>Consider the matrix <math>\mathbf{A}</math> as</p> $\mathbf{A} = \begin{pmatrix} 2 + 3i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3i \end{pmatrix}$ <p>The eigen values are <math>\lambda_1 = 2 + 3i, \lambda_2 = 2, \lambda_3 = 3i</math></p>
	<p>The eigen vector with respect to <math>\lambda_1 = 2 + 3i</math> will be the nullspace of <math>\mathbf{A} - \lambda_1 \mathbf{I}</math></p> $\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3i & 0 \\ 0 & 0 & -2 \end{pmatrix}, \text{ so the eigen vector will be } \mathbf{e}_1 = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } x_1 \in \mathbb{C}$ <p>The eigen vector with respect to <math>\lambda_2 = 2</math> will be the nullspace of <math>\mathbf{A} - \lambda_2 \mathbf{I}</math></p> $\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} 3i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3i - 2 \end{pmatrix}, \text{ so the eigen vector will be } \mathbf{e}_2 = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ where } x_2 \in \mathbb{C}$



The eigen vector with respect to  $\lambda_3 = 3i$  will be the nullspace of  $\mathbf{A} - \lambda_3 \mathbf{I}$

$\mathbf{A} - \lambda_3 \mathbf{I} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 3i & 0 \\ 0 & 0 & 0 \end{pmatrix}$ , so the eigen vector will be  $\mathbf{e}_3 = x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$  where  $x_3 \in \mathbb{C}$

Consider the vector  $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$  where  $x_1 = x_2 = x_3 = 1$

$$\mathbf{A}\mathbf{y} = \mathbf{A}\mathbf{e}_1 + \mathbf{A}\mathbf{e}_2 + \mathbf{A}\mathbf{e}_3 = (2 + 3i)\mathbf{e}_1 + 2\mathbf{e}_2 + 3i\mathbf{e}_3 = \begin{pmatrix} 2 + 3i \\ 2 \\ 3i \end{pmatrix}$$

As  $\mathbf{A}\mathbf{y}$  can not be written as  $c\mathbf{y}$  where  $c \in \mathbb{C}$ ,  $\mathbf{y}$  is not an eigen vector which is a contradiction.

TABLE 8.1.3: Examples

9 JUNE 2015

9.1. Let  $\mathbf{A}$  be an invertible  $4 \times 4$  real matrix. Which of the following are NOT true ?

- a) Rank  $\mathbf{A} = 4$
- b) For every vector  $\mathbf{b} \in \mathbb{R}$ ,  $\mathbf{Ax} = \mathbf{b}$  has exactly one solution.
- c)  $\dim(\text{nullspace } \mathbf{A}) \geq 1$
- d) 0 is an eigenvalue of  $\mathbf{A}$

**Solution:** See Table 9.1.1

Given	<b>A</b> is an invertible real matrix of order $4 \times 4$
Solution	<p>Since given <b>A</b> is an invertible matrix, <b>A</b> has full rank.</p> $\det(\mathbf{A}) \neq 0 \quad (9.1.1)$ $\text{Rank}(\mathbf{A}) = 4 \quad (9.1.2)$ <p>Let <math>\lambda_1, \lambda_2, \lambda_3</math> and <math>\lambda_4</math> be the eigenvalues of matrix <b>A</b>. We know that determinant of matrix <b>A</b> is the product of eigenvalues of <b>A</b>.</p> $\lambda_1 \lambda_2 \lambda_3 \lambda_4 \neq 0 \quad (9.1.3)$
Statement 1	$\text{Rank}(\mathbf{A}) = 4$
	<p>Since <b>A</b> is an invertible matrix, it has full rank as shown in equation (9.1.2). <b>True Statement</b></p>
Statement 2	For every vector $\mathbf{b} \in \mathbb{R}$ , $\mathbf{Ax} = \mathbf{b}$ has exactly one solution.
	<p>For every <math>\mathbf{b}</math>,</p> $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ <p><math>\mathbf{x}</math> will be unique solution for every <math>\mathbf{b}</math>. <b>True Statement</b></p>
Statement 3	$\dim(\text{nullspace } \mathbf{A}) \geq 1$ .
	<p>Using Rank Nullity Theorem,</p> $\begin{aligned} \text{Rank}(\mathbf{A}) + \dim(\text{nullspace } \mathbf{A}) &= n \\ \implies 4 + \dim(\text{nullspace } \mathbf{A}) &= 4 \\ \implies \dim(\text{nullspace } \mathbf{A}) &= 0 \not\geq 1 \end{aligned} \quad (9.1.4)$
	<p>where <math>n</math> is the number of columns in <b>A</b> Equation (9.1.4) proves that the given statement is <b>NOT True</b>.</p>
Statement 4	0 is an eigenvalue of <b>A</b>
	<p>From equation (9.1.1), we could say that no eigenvalue of <b>A</b> could be 0. <b>NOT True Statement</b></p>

TABLE 9.1.1: Explanation