Linear Algebra Manual

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CONTENTS

***Abstract*—This book provides an introduction to college level Linear Algebra based solved exercises from the UGC-NET exam.**

1 Spectral Decomposition

1.1 Characteristic Polynomial

1.1.1. *Cayley-Hamilton Theorem:* Let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{1.1.1.1}$$

and $\mathbf{I}$ be the 3×3 identity matrix. If

$$6\mathbf{A}^{-1} = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} \tag{1.1.1.2}$$

for $a, b, c \in \mathbb{R}$ then find (a,b,c)

Solution: (1.1.1.2) can be expressed as

$$a\mathbf{A}^3 + b\mathbf{A}^2 + c\mathbf{A} - 6\mathbf{I} = 0 \tag{1.1.1.3}$$

The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{vmatrix} 1-x & 0 & 2 \\ 1 & -2-x & 0 \\ 0 & 0 & -3-x \end{vmatrix} = 0 \tag{1.1.1.4}$$

$$\implies x^3 + 4x^2 + x - 6 = 0 \tag{1.1.1.5}$$

Comparing (1.1.1.5) and (1.1.1.3),

$$(a, b, c) = (1, 4, 1) \tag{1.1.1.6}$$

Consider the following matrices for the following examples

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.7}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{pmatrix} \mathbf{D} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.8}$$

1.1.2. The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{vmatrix} 1-x & 2 & 3 \\ 0 & 4-x & 5 \\ 0 & 0 & 6-x \end{vmatrix} = 0 \tag{1.1.2.1}$$

$$\implies (x-1)(x-4)(x-6) = 0 \tag{1.1.2.2}$$

The eigenvalues are $x = 1, 4, 6$.

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1.1.3. The characteristic polynomial of $\mathbf{B}$ is

$$\begin{vmatrix} x & -1 & 0 \\ 1 & x & 0 \\ 0 & 0 & x-1 \end{vmatrix} = 0 \tag{1.1.3.1}$$

$$\implies (x-1)\left(x^2+1\right) = 0 \tag{1.1.3.2}$$

The eigenvalues are $x = 1, \pm\jmath$.

1.1.4. The characteristic polynomial of $\mathbf{C}$ is

$$\begin{vmatrix} 1-x & 2 & 3 \\ 2 & 1-x & 4 \\ 3 & 4 & 1-x \end{vmatrix} \tag{1.1.4.1}$$

$$\implies x^3 - 3x^2 - 26x - 20 = 0 \tag{1.1.4.2}$$

The eigenvalues are $x = 7.07467358, -0.88679099, -3.1878826$.

1.1.5. The characteristic polynomial of $\mathbf{D}$ is

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \tag{1.1.5.1}$$

$$\implies x^3 = 0 \tag{1.1.5.2}$$

The matrix has a single eigenvalue at $x = 0$.

1.2 Diagonalizability

1.2.1. $\mathbf{A}$ and $\mathbf{C}$ have distinct real eigenvalues. Hence they are diagonalizable.

Proof. Suppose λ_1, λ_2 are eigenvalues of $\mathbf{A}$ with the same eigenvector $\mathbf{v}$. Then

$$\mathbf{Av} = \lambda_1 \mathbf{v} \tag{1.2.1.1}$$

$$\mathbf{Av} = \lambda_2 \mathbf{v} \tag{1.2.1.2}$$

$$\implies (\lambda_1 - \lambda_2)\mathbf{v} = 0, \text{or}, \lambda_1 \quad = \lambda_2 \tag{1.2.1.3}$$

1.2.2. $\mathbf{B}$ has distinct eigenvalues, but some are complex. Hence, $\mathbf{B}$ is diagonalizable over $\mathbb{C}$, but not $\mathbb{R}$.

1.2.3. $\mathbf{D}$ has only one eigenvalue 0, whose multiplicity is 3. This is defined as the *algebraic multiplicty* of the eigenvalue. The corresponding eigenvectors are obtained from

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v} = 0 \tag{1.2.3.1}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{v} = 0 \tag{1.2.3.2}$$

$$\implies \mathbf{v} = k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{1.2.3.3}$$

Thus, $\mathbf{D}$ has only one eigenvector and is not diagonalizable.

1.2.4. Thus,

$$\sum_{i=1}^{3} \text{nullity}\left(\mathbf{A} - \lambda_i \mathbf{I}\right) = 3 \tag{1.2.4.1}$$

$$\sum_{i=1}^{1} \text{nullity}\left(\mathbf{D} - \lambda_i \mathbf{I}\right) = 1 < 3 \tag{1.2.4.2}$$

1.2.5. If the characteristic polynomial of an $n \times n$ matrix $\mathbf{X}$ is

$$\prod_{i=1}^{k} (x - \lambda_i)^{a_i} = 0 \tag{1.2.5.1}$$

where

$$\sum_{i=1}^{k} a_i = n \tag{1.2.5.2}$$

define the *geometric multiplicity* of λ_i as

$$g_i = \text{nullity}\left(\mathbf{X} - \lambda_i \mathbf{I}\right) \tag{1.2.5.3}$$

Then $\mathbf{X}$ is diagonalizable iff

$$g_i = a_i \tag{1.2.5.4}$$

$$\sum_{i=1}^{k} g_i = n \tag{1.2.5.5}$$

1.3 Example

For any $n \times n$ matrix B, let $N(\mathbf{B}) = \{X \in \mathbb{R}^n : \mathbf{Bx} = 0\}$ be the null space of $\mathbf{B}$. Let $\mathbf{B}$ be a 4×4 matrix with $\dim(N(\mathbf{A} - 4\mathbf{I})) = 2, \dim(N(\mathbf{A} - 2\mathbf{I})) = 1$ and $\text{rank}(\mathbf{A}) = 3$.

1.3.1. $\because$ rank($\mathbf{A}$) = 3, using the rank-nullity theorem,

$$\text{nullity}\left(\mathbf{A}\right) = 4 - 3 = 1 \tag{1.3.1.1}$$

$$\implies \exists \mathbf{x} \ni \mathbf{Ax} = 0 \tag{1.3.1.2}$$

Thus, 0 is an eigenvalue of $\mathbf{A}$ and the eigenvalues of $\mathbf{A}$ are $0, 2, 4$.

1.3.2. From the given information, the geometric multiplicities of 0, 2, 4 are respectively,

$$g_1 = 1, g_2 = 1, g_3 = 2 \tag{1.3.2.1}$$

and substituting in 1.2.5.5,

$$\sum_{i=1}^{3} g_i = 4 \tag{1.3.2.2}$$

$\therefore$ $\mathbf{A}$ is diagonalizable.

2 Jordan Canonical Form

2.1 Motivation

2.1. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix} \tag{2.1.1}$$

The matrix $\mathbf{A}$ in (2.1.1) has the characteristic polynomial

$$\begin{aligned} p(x) &= \det(x\mathbf{I} - \mathbf{A}) && (2.1.2) \\ &= x^4 - 11x^3 + 42x^2 - 64x + 32 && (2.1.3) \\ &= (x-1)(x-2)(x-4)^2. && (2.1.4) \end{aligned}$$

Thus, the eigenvalues are given by $x = 1, 2, 4$.

2.2. The eigenvectors corresponding to the above eigenvalues are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \tag{2.2.1}$$

Clearly, the dimension of the eigenspace is < 4. Hence, the matrix $\mathbf{A}$ is not diagonalizable.

2.2 Jordan Form

2.1. The Jordan decomposition of $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P} \tag{2.1.1}$$

where

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{2.1.2}$$

2.2. The matrix $\mathbf{P}$ in is given by

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} \tag{2.2.1}$$

where

$$\mathbf{A}\mathbf{v}_4 = \mathbf{v}_3 \tag{2.2.2}$$

$\mathbf{v}_4$ is defined to be the *generalized* eigenvector of $\mathbf{A}$.