CP 2 - Solution of 1D-1G Diffusion Equation
NPRE 445

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04/06/2020

Contents

Project was written in Latex and programming was done in Python.

1 The Question

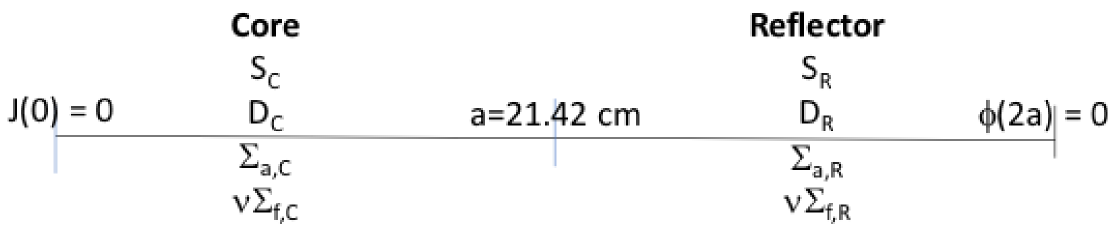

Fig. 1: System Parameters

We were asked to solve for the flux profile and plot it for the above set-up problem. It involves a core and reflector region. The solution was carried out analytically and numerically for two scenarios - with a fixed source and with a fission source.

2 Fixed Source Problem

Table 1: List of Constants

Symbol	**Variable**	**Value**
D_C	diffusion coefficient for the Core region	$0.90\ cm$
D_R	diffusion coefficient for the Reflector region	$0.18\ cm$
S_C	source term for the Core region	$2.5\ \frac{n}{cm^3}$
S_R	source term for the Reflector region	$0\frac{n}{cm^3}$
$\Sigma_{a,C}$	absorption cross section for the Core region	$0.066\ cm^{-1}$
$\Sigma_{a,R}$	absorption cross section for the Reflector region	$0.02\ cm^{-1}$

2.1 Analytical Solution

We will start with the main diffusion equation.

$$-D\frac{d^2}{dx^2}\phi(x) + \Sigma_a\phi(x) = S$$

For the fixed-source problem, we can see that since the source terms have different values for the two regions, the resulting diffusion equations will also be different. Equations (1) and (2) are for the Core and Reflector region respectively.

$$\frac{d^2}{dx^2}\phi_C(x) - \frac{1}{L_C^2}\phi_C(x) = \frac{-S_C}{D_C} \tag{1}$$

$$\frac{d^2}{dx^2}\phi_R(x) - \frac{1}{L_R^2}\phi_R(x) = 0 \tag{2}$$

We already know the form of the solution for these second order differential equations, which happens to be the neutron flux,

$$\phi_C(x) = C_1 \cosh\left(\frac{x}{L_C}\right) + C_2 \sinh\left(\frac{x}{L_C}\right) + \frac{s_C}{\Sigma_{a,C}} \tag{3}$$

$$\phi_R(x) = C_3 \cosh\left(\frac{x}{L_R}\right) + C_4 \sinh\left(\frac{x}{L_R}\right) \tag{4}$$

This problem will have three important boundary conditions:
1. $J(x=0)=0$ [symmetry boundary condition]
2. $\phi(x=2a)=0$ [extrapolated boundary condition]
3. $\phi_R(x=a)=\phi_C(x=a)$ and $J_R(x=a)=J_C(x=a)$ [continuity of current]

These help simplify (3) and (4) to the following

$$\phi_C(x) = C_1 \cosh\left(\frac{x}{L_C}\right) + \frac{s_C}{\Sigma_{a,C}} \tag{5}$$

$$\phi_R(x) = C_4 \sinh\left(\frac{2a-x}{L_R}\right) \tag{6}$$

The constants were calculated separately to be $C_1 = \frac{-S_C}{\Sigma_{a,C}\left(\cosh\left(\frac{a}{L_C}\right)+\frac{L_R D_C}{L_C D_R}\sinh\left(\frac{a}{L_C}\right)\tanh\left(\frac{a}{L_R}\right)\right)}$ and $C_4 = -C_1 \frac{L_R D_C}{L_C D_R}\frac{\sinh\left(\frac{a}{L_C}\right)}{\cosh\left(\frac{a}{L_R}\right)}$

2.2 Pseudo Algorithm of Numerical Solution

The width of the problem region, that is, the reactor and core regions, were divided into meshes of size 10, 20, 40, 60, 80, 160 and 320. The mesh number was increased for better granularity and a max mesh of 10,000 was reached. Equation (7) is the element with which we diagonally populate A matrix, which is originally a [mesh x mesh] zeros matrix.

$$-D^i \frac{d^2}{dx^2}\phi^i + \Sigma_a^i \phi^i = S^i \tag{7}$$

The ϕ and S matrix are single columned zero matrices with the mesh number as their length. The S matrix was then populated with the given source terms for the core and reflector regions corresponding to the correct indexes. Next, the linear algebra solver in numpy was used to solve for flux according to (8).

$$\bar{\bar{A}}\bar{\phi} = \bar{S} \tag{8}$$

2.3 Plots

Three solutions were plotted according to Question 4 - the analytical, and numerical for 10 and 80 meshes.

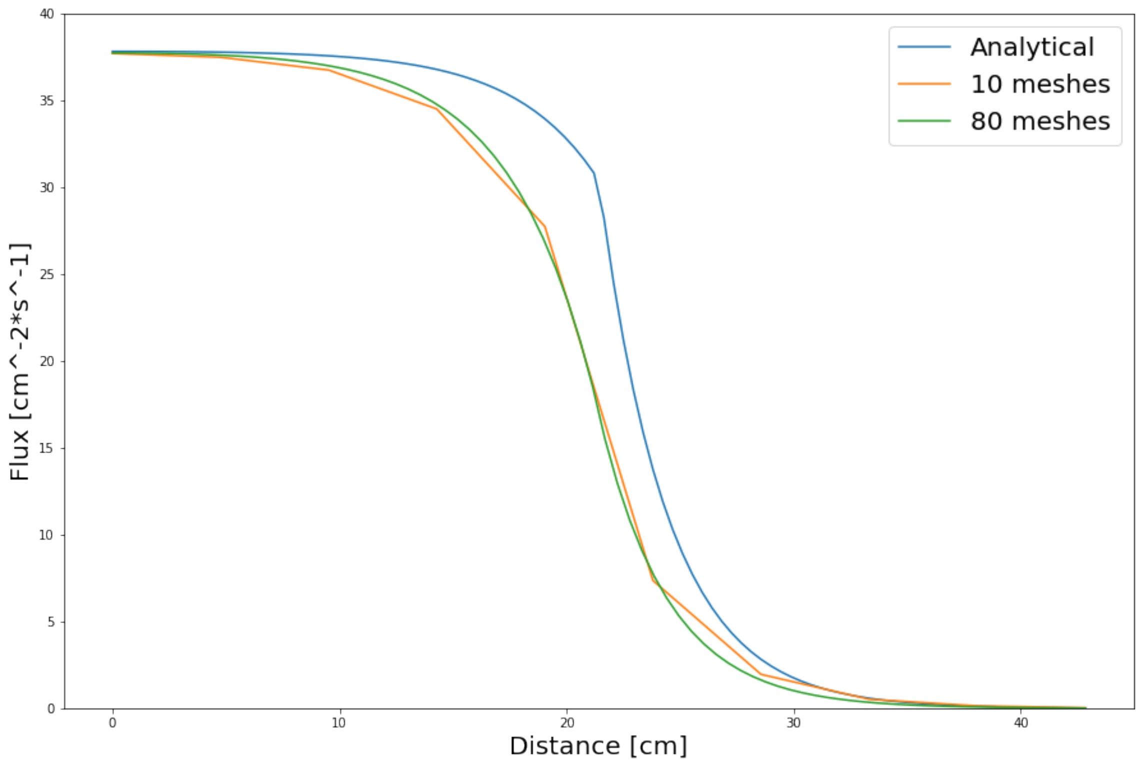

Fig. 2: Flux vs Distance (Question 4)

Two solutions were plotted according to Question 5 - the analytical, and numerical for max meshes.

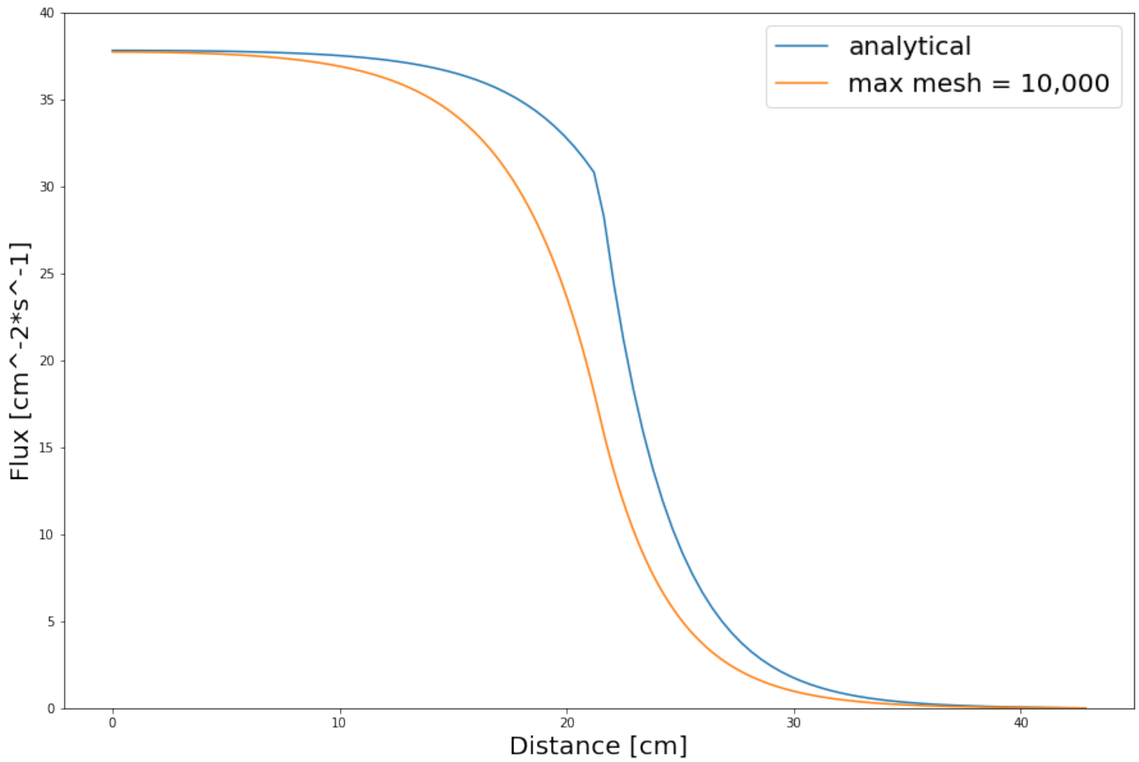

Fig. 3: Flux vs Distance (Question 5)

2.4 Comparison of Solutions

Table 2: Percent Error in the Numerical Solutions

Number of Meshes	Percent Error (between numerical and analytical solution) [%]
10	52.4924
20	50.2884
40	50.4392
80	48.8948
160	46.1512
320	45.7184

As we can see in Table 2, the percent error between the numerical and analytic solution is decreasing as mesh number increases.

3 Criticality Problem

Table 3: List of Constants

Symbol	**Variable**	**Value**
D_C	diffusion coefficient for the Core region	0.90 cm
D_R	diffusion coefficient for the Reflector region	0.18 cm
$\nu\Sigma_{f,C}$	fission term for the Core region	0.075 cm^{-1}
$\nu\Sigma_{f,R}$	fission term for the Reflector region	0 cm^{-1}
$\Sigma_{a,C}$	absorption cross section for the Core region	0.066 cm^{-1}
$\Sigma_{a,R}$	absorption cross section for the Reflector region	0.02 cm^{-1}

3.1 Analytical Solution

For the criticality problem, the fixed source in the Core is replaced by a fission source. The new diffusion equations for the Core and Reflector region are shown below.

$$\frac{d^2}{dx^2}\phi_C(x) + B_m^2\phi_C(x) = 0 \tag{9}$$

$$\frac{d^2}{dx^2}\phi_R(x) - \frac{1}{L_R^2}\phi_R(x) = 0 \tag{10}$$

We know the form of the solutions to these second order differential equations as shown in (11) and (12) respectively.

$$\phi_C(x) = C_1\cos(B_m x) + C_2\sin(B_m x) \tag{11}$$

$$\phi_R(x) = C_3\cosh\left(\frac{x}{L_R}\right) + C_4\sinh\left(\frac{x}{L_R}\right) \tag{12}$$

Using the boundary conditions, these simplify to (13) and (14).

$$\phi_C(x) = C_1\cos(B_m x) \tag{13}$$

$$\phi_R(x) = C_4 \sinh\left(\frac{2a - x}{L_R}\right) \tag{14}$$

Normalising the flux at $x = 0$ makes $C_1 = 1$. From here we can solve for C_4 shown in equation (15). We can then obtain the criticality condition as shown in (16).

$$C_4 = \frac{\cos\left(B_m a\right)}{\sinh\left(\frac{a}{L_R}\right)} \tag{15}$$

$$B_g = \frac{D_R}{L_R D_C} \frac{\cot\left(B_m a\right)}{\tanh\left(\frac{a}{L_R}\right)} \tag{16}$$

3.2 Pseudo Algorithm of Numerical Solution

The algorithm here is very similar to that in the previous problem. (17) shows the primary element used to populate the A matrix diagonally. F matrix is populated with $\nu\Sigma$.

$$-D^i \frac{d^2}{dx^2}\phi^i + \Sigma_a^i \phi^i = \frac{1}{k}\nu\Sigma_f^i \phi^i \tag{17}$$

$$\bar{\bar{A}}\bar{\phi} = \frac{1}{k}\bar{\bar{F}}\bar{\phi} \tag{18}$$

Equation (18) is solved numerically by iterating equation (19) until the condition in (20) and (21) is satisfied. This will give us the flux distribution and K_{eff}

$$k^{n+1} = k^n \frac{\Sigma_i \nu \Sigma_f^i \bar{\bar{\phi}}_{n+1}}{\Sigma_i \nu \Sigma_f^i \bar{\phi}_n} \tag{19}$$

$$\left| \frac{k^n - k^{n-1}}{k^n} \right| < 10^{-5} \tag{20}$$

$$\left| \frac{\max\left(s^n - s^{n-1}\right)}{\max\left(s^n\right)} \right| < 10^{-5} \tag{21}$$

3.3 Plots

Three solutions were plotted according to Question 4 - the analytical, and numerical for 10 and 80 meshes.

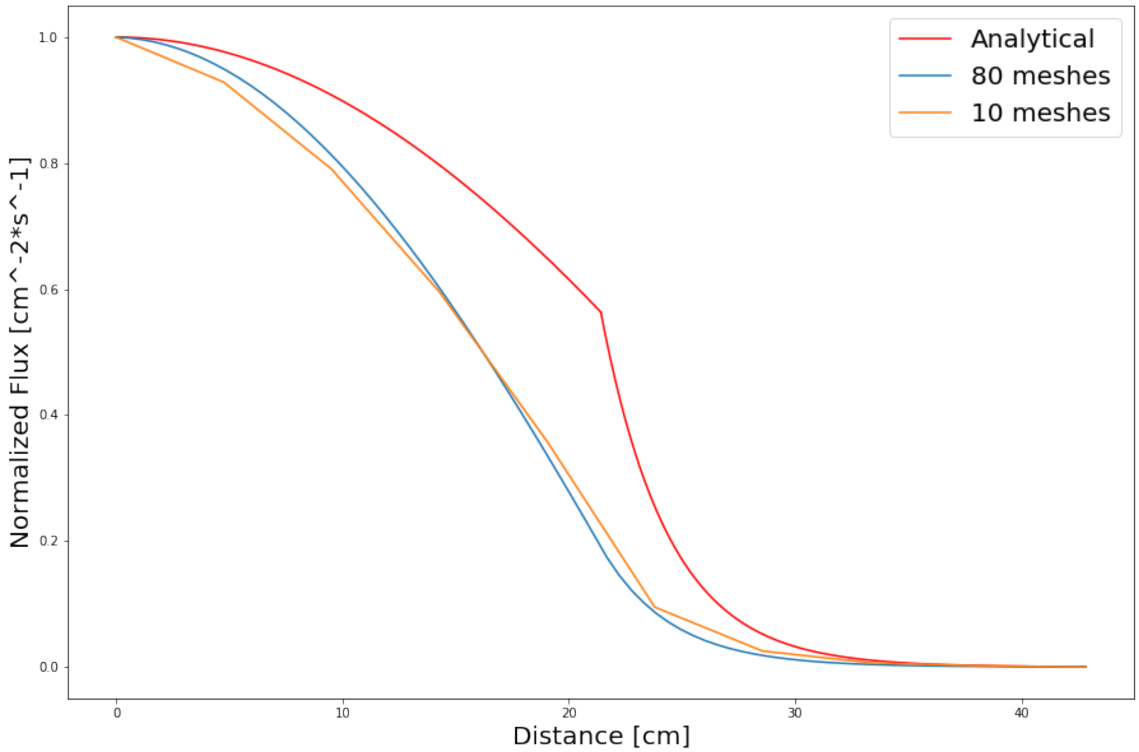

Fig. 4: Flux vs Distance (Question 4)

Two solutions were plotted according to Question 5 - the analytical, and numerical for max meshes.

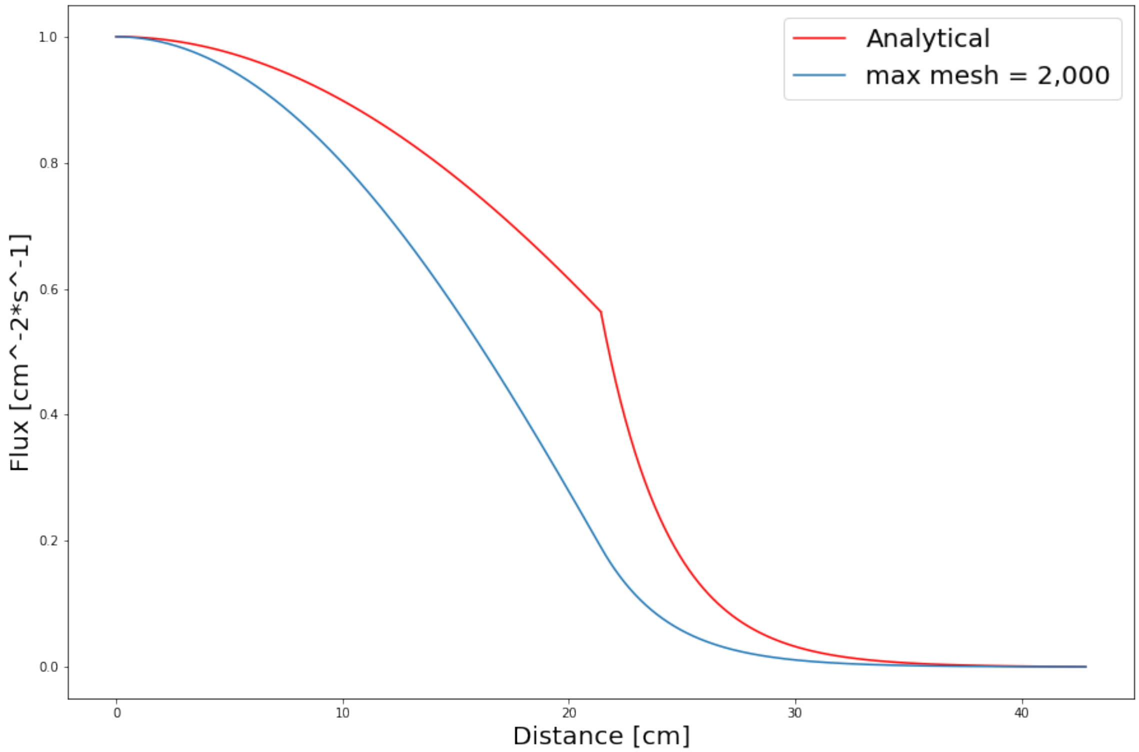

Fig. 5: Flux vs Distance (Question 5)

3.4 Comparison of Solutions

As mentioned in the caption, Table 4 shows the percent error between the numerical and analytic solution for the different mesh settings.

As we can see, the percent errors show a steady decrease as the mesh numbers increase. Next, the K effective are tabulated in Table 5 according to the mesh number.

Table 4: Percent Error in the Numerical Solutions

Number of Meshes	Percent Error (between numerical and analytical solution) [%]
10	65.5351
20	64.7613
40	63.1711
80	59.8267
160	52.5548
320	36.4114

Table 5: k effective for various mesh numbers

Number of Meshes	k_{eff}
analytic	11.05297345
10	10.75484146
20	10.75495663
40	10.75541619
80	10.75723686
160	10.76425463
320	10.78898987

4 References

1. www.sharelatex.com
2. www.stackoverflow.com
3. Nuclear Reactor Analysis by Duderstadt and Hamilton