Lecture 22 (March 6)

Solve the IVP using Laplace Transform

$$y'' + 2y' + 10y = 0$$
$$y(0) = 1, \quad y'(0) = 2$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = 0$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 0$$

$$[s^2 Y(s) - s\,y(0) - y'(0)] + 2[sY(s) - y(0)] + 10\,Y(s) = 0$$

$$s^2 Y(s) - s - 2 + 2(sY(s) - 1) + 10\,Y(s) = 0$$

$$(s^2 + 2s + 10)Y(s) - s - 4 = 0$$

$$Y(s) = \frac{s+4}{s^2 + 2s + 10}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+2s+10}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+1)+3}{(s+1)^2+9}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$= e^{-t}\cos 3t + e^{-t}\sin 3t$$

Example. $y'' + y = \sin 2t$

$y(0) = 2, \quad y'(0) = 1$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\sin 2t\}$$

$$s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

Partial fraction decomposition for $\frac{2}{(s^2+1)(s^2+4)}$

$$\left[\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}\right] \times (s^2+1)(s^2+4)$$

$$2 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$= As^3+4As+Bs^2+4B$$

$$+Cs^3+Ds^2+Cs+D$$

$$= (A+C)s^3+(B+D)s^2+(4A+C)s+4B+D$$

$$\boxed{\begin{array}{l} A+C=0, \\ 4A+C=0 \end{array}} \quad \boxed{\begin{array}{l} B+D=0 \\ 4B+D=2 \end{array}}$$

$$A=0,\ C=0 \qquad B=\frac{2}{3},\ D=-\frac{2}{3}$$

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{2}{3}\cdot\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

$$Y(s) = \frac{2}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$= \frac{5}{3}\frac{1}{s^2+1} - \frac{1}{3}\frac{2}{s^2+4} + 2\cdot\frac{s}{s^2+1}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + 2\cdot\cos t$$

Example: $y'' + y = e^{-2t}$

$y(0) = 0, \quad y'(0) = 0$

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{e^{-2t}\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \frac{1}{s+2}$$

$$s^2 Y(s) + Y(s) = \frac{1}{s+2}$$

$$(s^2+1)\,Y(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{1}{(s+2)(s^2+1)}$$

Need to find $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)(s^2+1)}\right\}$

– partial fraction decomposition

$$\left[\frac{1}{(s+2)(s^2+1)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+1} \right] (s+2)(s^2+1)$$

$$1 = A(s^2+1) + (Bs+C)(s+2)$$
$$= As^2 + A + Bs^2 + Cs + 2Bs + 2C$$
$$= (A+B)s^2 + (C+2B)s + A + 2C$$

$$\begin{cases} A+B=0 & ① \\ C+2B=0 & ② \\ A+2C=1 & ③ \end{cases}$$

Use ① & ③ to eliminate A (③ − ①)

$$2C - B = 1$$

together with $C+2B=0$ ② $\Rightarrow C = -2B$

$$-4B - B = 1$$
$$B = -\frac{1}{5}$$

$$A = -B = \frac{1}{5}, \quad C = -2B = \frac{2}{5}$$

$$\frac{1}{(s+2)(s^2+1)} = \frac{1/5}{s+2} + \frac{-\frac{1}{5}s + \frac{2}{5}}{s^2+1}$$

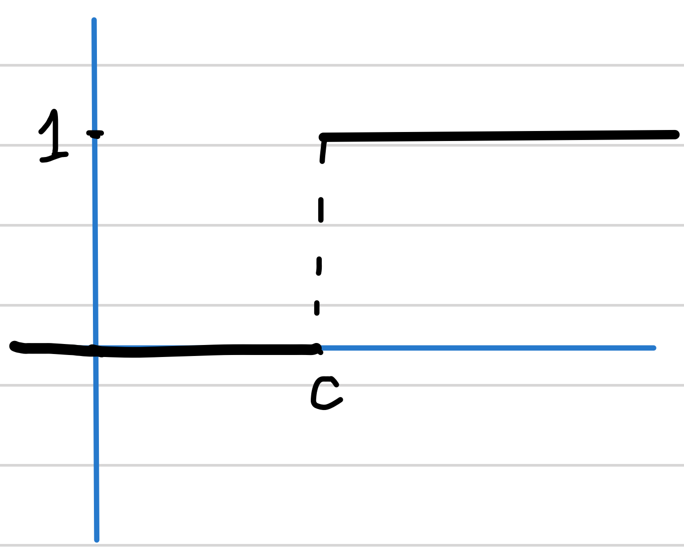
$$= \frac{1}{5}\cdot\frac{1}{s+2} - \frac{1}{5}\,\frac{s}{s^2+1} + \frac{2}{5}\,\frac{1}{s^2+1}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$y(t) = \underbrace{\frac{1}{5}e^{-2t}}_{\text{special soln}} \underbrace{-\frac{1}{5}\cos t + \frac{2}{5}\sin t}_{\text{homogeneous solution}}$$

We can solve inhomogeneous eqn with piecewise continuous right hand side $f(t)$ using Laplace Transform

Heaviside function (unit step function)



$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t > c \end{cases}$$

$$\mathcal{L}\{u_c(t)\} = \int_0^{+\infty} e^{-st} u_c(t)\,dt$$

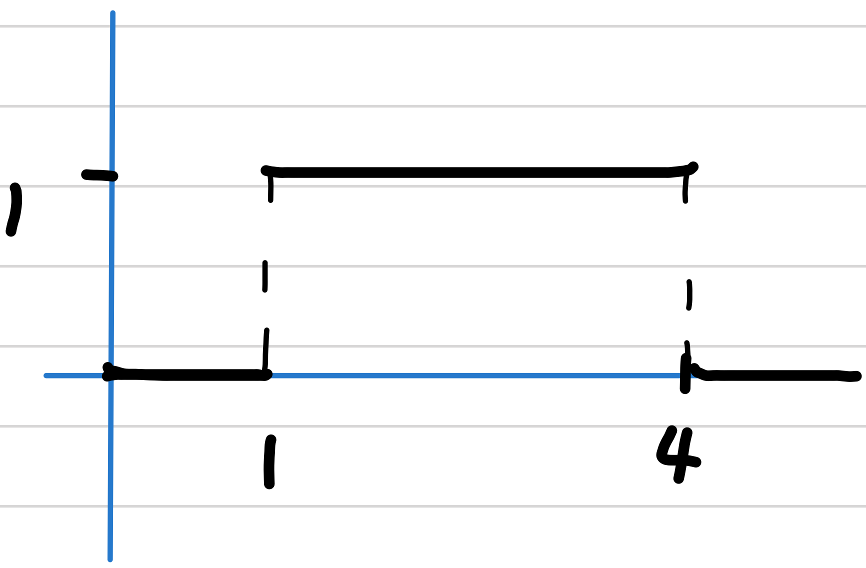
$$= \int_c^{+\infty} e^{-st}\cdot 1\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_c^{+\infty}$$

$$= 0 - \left(-\frac{1}{s}e^{-s\cdot c}\right) \qquad s > 0$$

$$= \frac{1}{s}e^{-cs}$$

Example: $h(t) = u_1(t) - u_4(t)$

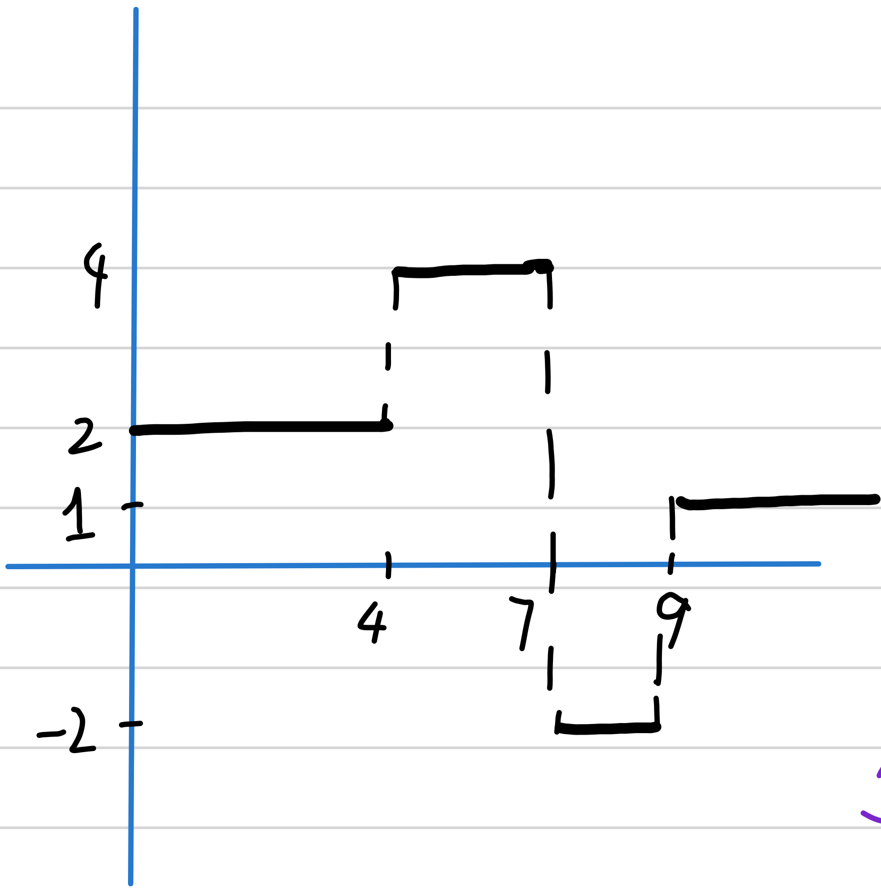


$$\mathcal{L}\{h(t)\}$$

$$= \mathcal{L}\{u_1(t)\} - \mathcal{L}\{u_4(t)\}$$

$$= \frac{e^{-s}}{s} - \frac{e^{-4s}}{s}$$

Example. $f(t) = \begin{cases} 2 & 0 \le t < 4 \\ 4 & 4 < t < 7 \\ -2 & 7 < t < 9 \\ 1 & t > 9 \end{cases}$



start with

$f_1(t) = 2$

jump of 2 at

$t = 4$

$f_2(t) = f_1(t) + 2u_4(t)$

$= 2 + 2u_4(t)$

jump of -6 at

$t = 7$

$f_3(t) = f_2(t) - 6u_7(t)$

$= 2 + 2u_4(t) - 6u_7(t)$

jump of 3 at

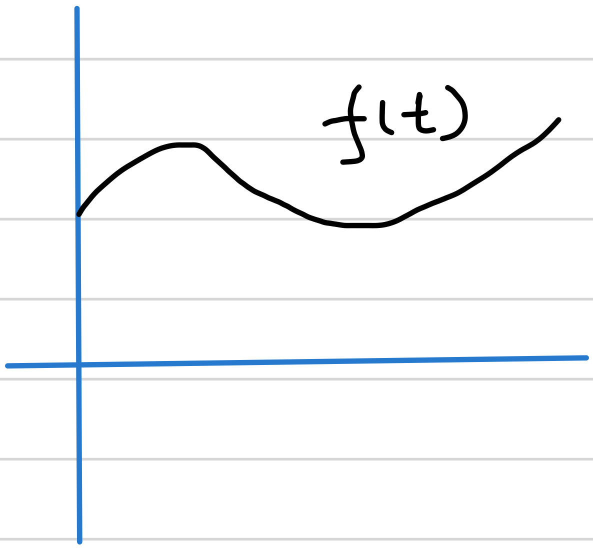
$t = 9$

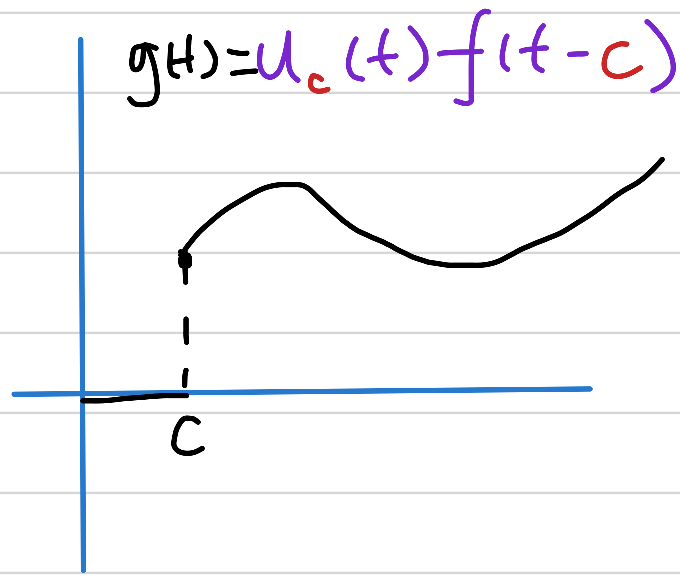
$f_4(t) = f_3(t) + 3u_9(t)$

$= 2 + 2u_4(t) - 6u_7(t) + 3u_9(t)$

$f(t) = 2 + 2u_4(t) - 6u_7(t) + 3u_9(t)$

$f(t)$ delayed by c





$$g(t) = \begin{cases} 0 & t < c \\ f(t-c) & t \geq c \end{cases}$$

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^{+\infty} e^{-st} u_c(t) f(t-c)\, dt$$

$$= \int_c^{\infty} e^{-st} f(t-c)\, dt$$

$(t - c = \xi)$ $$= \int_0^{+\infty} e^{-s(\xi + c)} f(\xi)\, d\xi$$

$$= e^{-sc} \int_0^{+\infty} e^{-s\xi} f(\xi)\, d\xi$$

$$= e^{-cs} \mathcal{L}\{f(t)\}$$