

Ficha de trabalho - Valores e Vetores Próprios

1.

a)

$$|A - \lambda I_3| = 0$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 4 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 & 2 \\ 0 & -1-\lambda & 4 \\ -1 & 0 & -2-\lambda \end{bmatrix}$$

$$|A - \lambda I_3| = (1-\lambda)(-1)\det \begin{bmatrix} -1-\lambda & 4 \\ 0 & -2-\lambda \end{bmatrix} + (-1)(-1)\det \begin{bmatrix} -1 & 2 \\ -1-\lambda & 4 \end{bmatrix}$$

$$= (1-\lambda)\det \begin{bmatrix} -1-\lambda & 4 \\ 0 & -2-\lambda \end{bmatrix} - \det \begin{bmatrix} -1 & 2 \\ -1-\lambda & 4 \end{bmatrix}$$

$$= (1-\lambda)((-1-\lambda)(-2-\lambda) - 0 \times 4) - (-1 \times 4 - (-1-\lambda) \times (2))$$

$$= (1-\lambda)(2 + 2\lambda + \lambda + \lambda^2) - (-4 + 2 + 2\lambda)$$

$$= (1-\lambda)(2 + 3\lambda + \lambda^2) - (-2 + 2\lambda)$$

$$= (1-\lambda)(2 + 3\lambda + \lambda^2) + 2(1-\lambda)$$

$$= (1-\lambda)(2 + 3\lambda + \lambda^2 + 2)$$

$$= (1-\lambda)(\lambda^2 + 3\lambda + 4)$$

$$\begin{aligned} 1-\lambda &= 0 \quad \vee \quad \lambda^2 + 3\lambda + 4 = 0 \quad \Rightarrow \\ \Rightarrow \lambda &= 1 \quad \vee \quad \lambda = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 4}}{2 \times 1} \end{aligned}$$

$$\Rightarrow \lambda = 1 \quad \vee \quad \text{Imp}$$

A só tem um valor próprio real 1 que é multiplicação algébrica de 1.

b)

$$(A - I_3)X = 0 \quad (\Rightarrow)$$

$$A - I_3 = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 4 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & -2 & 4 \\ -1 & 0 & -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & -1 & 2 & 0 \\ 0 & -2 & 4 & 0 \\ -1 & 0 & -3 & 0 \end{array} \right] \xrightarrow{l_3 \leftrightarrow l_1} \left[\begin{array}{ccc|c} -1 & 0 & -3 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array} \right] \xrightarrow{l_3 \rightarrow 2l_3 - l_2} \left[\begin{array}{ccc|c} -1 & 0 & -3 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{cases} -x - 3z = 0 \\ -2y + 4z = 0 \end{cases} \Rightarrow \begin{cases} x = -3z \\ y = 2z \end{cases} \quad z \in \mathbb{R}$$

C.S. = $\{ (-3z, 2z, z) : z \in \mathbb{R} \} = \{ z(-3, 2, 1) : z \in \mathbb{R} \}$
vetores próprios associados ao valor próprio 1 são os elementos do conjunto:

$$z \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} : z \in \mathbb{R} \setminus \{0\} = \left\langle \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle \setminus \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

2.

a)

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -2 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{bmatrix}$$

$$(1-\lambda)(-1)^2 \det \begin{bmatrix} -1-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix} = (1-\lambda) ((-1-\lambda)(1-\lambda) - (2 \times 0))$$

$$= (1-\lambda) (-1+\lambda-\lambda+\lambda^2)$$

$$= (1-\lambda) (-1+\lambda^2)$$

$$|A - \lambda I_3| = 0 \Rightarrow (1-\lambda) = 0 \vee -1+\lambda^2 = 0 \Leftrightarrow$$

$$\Rightarrow \lambda = 1 \vee \lambda = 1 \vee \lambda = -1$$

Valores propios: 1, -1

$$A - 1I_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \rightarrow x_2 = 0 \quad \left\{ (x_1, 0, x_3) : x_1, x_3 \in \mathbb{R} \right\} =$$

$$= \left\{ x_1 (1, 0, 0) + x_3 (0, 0, 1) \in \mathbb{R}^3 \right\}$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle \mid \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$b) A + 1I_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \xrightarrow{l_2 \leftrightarrow l_3} \begin{bmatrix} 2 & -2 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2x_1 - 2x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow x_1 = x_3$$

$$2x_2 + 2x_3 = 0 \Rightarrow x_2 = -x_3$$

$$\left\{ (x_1, -x_3, x_3) : x_3 \in \mathbb{R} \right\} = \left\{ x_3 (1, -1, 1) : x_3 \in \mathbb{R} \right\}$$

$$= \left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle \mid \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

b)

$$|A - \lambda I_3| \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -1-\lambda & 0 \\ 0 & 2 & -\lambda \end{bmatrix}$$

$$|A - \lambda I_3| = (1-\lambda)(-1)^2 \begin{bmatrix} -1-\lambda & 0 \\ 2 & -\lambda \end{bmatrix}$$

$$= (1-\lambda) \{ (-1-\lambda) \times (-\lambda) \} = (1-\lambda) (\lambda + \lambda^2)$$

$$= -\lambda + \lambda^2 - \lambda^2 + \lambda^3$$

$$= \lambda^3 + \lambda$$

$$\lambda^3 + \lambda = 0 \Rightarrow \lambda(\lambda+1) = 0 \rightarrow \lambda = 0 \vee \lambda + 1 = 0 \Rightarrow$$

$$\lambda = 0 \vee \lambda = -1$$