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On Translation-like Actions, Amenability and Homology on Discrete Spaces

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Preface

The aim of my work is to analyse actions of finitely generated groups on Cayley graphs. From a coarse geometric viewpoint, we analyse the properties of both amenable and nonamenable groups. For this purpose it is central to consider functions among discrete metric spaces and homologies on Cayley graphs.

I wish to thank M. Jonas Deré for finding me this wonderful topic, selecting well written literature and helping me out whenever I was in trouble. We often had fruitful conversations where neither of us hesitated to contradict the other whenever necessary. I also wish to thank him for a recommendation letter and congratulate him with his son.

I will also send my acknowledgements to Rainier Van Es. He was there to discuss together some of the doubts I had and guide me to some sources, specially the ones from Jean-Pierre Serre.

Beside the academical help, this work would never have been possible without my close friends and my family. They were always very supportive and their presence was very important to get through this hard year. My family was also there to help me through the difficult process of applying for a second master.

Last but not least, I wish to thank Younes Ben Messaoud for always offering both academic advice and a sympathetic ear. Talking with him helped me clear my doubts and his tips made hard decisions easy and clear.

Summary

In the previous century, several conjectures where stated about whether a group contains a subgroup or not. The following 2 are particularly famous:

1. **Burnside's Problem:** Does every finitely generated infinite group contain $\mathbb{Z}$ as a subgroup? (i.e. does it have a non-torsion element)

2. **The Geometric Von Neumann conjecture:** Does every nonamenable finitely generated group contain the free group $\mathbb{F}_2$?

The above conjectures were answered negatively. They are proven wrong by among others the existence of Tarski groups by Olshanskii in 1979. Kevin Whyte [16] suggested to reformulate the fact of having a subgroup as having a translation-like action by a subgroup. This seems to be the perfect adjustment to turn these two statements into *if and only if*-statements.

Theorem 6.1 (Geometric Burnside Problem)**.** *A finitely generated group G is infinite if and only if it admits a translation-like action by $\mathbb{Z}$.*

Theorem 5.1 (Geometric Von Neumann theorem)**.** *A finitely generated group G is non-amenable if and only if it admits a translation-like action by the free group $\mathbb{F}_2$.*

We prove these two theorems, originally proven in resp. [15] and [16].

As in the paper of Brandon Steward [15], we then make two improvements on these theorems. First, we ask for transitive actions. This is possible for the geometric Von Neumann theorem, for the Geometric Burnside Problem this works if and only if the group has at most two ends.[1] The second improvement, is to restate translation-like actions as *having a subgraph* Φ. After applying these adaptations, both results can then be merged in an elegant way.

Corollary 6.13 (Regular spanning trees of Cayley graphs)**.** *Let G be a finitely generated infinite group, then there exists a Cayley graph* $\mathrm{Cay}(G, S)$ *of G allowing a regular spanning tree. (Possibly 2-regular)*

The second main purpose of this thesis, is to study the 0-dimensional uniformly finite homology H_0^{uf} on Cayley graphs and UDBG spaces. This leads to numerous applications. It turns out a group is nonamenable if and only if H_0^{uf} vanishes. This homology also tells one whether a quasi-isometry can be "upgraded" to a bilipschitz equivalence (a bijective quasi-isometry).

[1] In particular, this works for amenable groups.

Sources and chronological overview

In this thesis, we elaborate in further detail the results of the papers [16] and [15]. To give a gentle preliminary and define all necessary concepts, we also used [10]. This very useful source also contains most of the necessary definitions and many examples.

The preliminaries offer a good way to get familiar with the concepts. Thanks to many examples, basic lemmas and characterisations, one obtains a toolkit to solve not only the problems coming later, but a wide variety of problems in geometric group theory.

As homology groups are central to this thesis, they are given their own chapter. However, chapter 2 can be seen as an extension of the preliminaries, rather than results, it contains examples and intuitive ways to think about homologies.

In chapter 3, we see that the uniformly finite homology $H_0^{uf}(X)$ is linked to the bilipschitz structure $S_{\text{bilip}}(X)$ by following theorem. The bilipschitz structure is a quotient of quasi-isometries over bilipschitz equivalences.

Theorem 3.3. *We have the following isomorphism of abelian semigroups,*

$$S_{bilip}(X) \cong H_0^{uf^+}(X).$$

In practice, previous isomorphism expresses itself by following theorem.

Theorem 3.4. *Let $f : X \to Y$ be a quasi-isometry with*

$$f_*[X] = [Y],$$

then there is a bilipschitz equivalence at bounded distance from f.

An important application of this is that nonamenable quasi-isometric spaces are bilipschitz equivalent. This fact is stated for trees in chapter 4 and in general in chapter 7.

In Chapter 4, we study the bilipschitz equivalence of trees.[2] This result is omnipresent in the proofs in chapters 5 and 6. Furthermore, this chapter gives a setting to restate geometric properties in a graph theoretic way.

Chapters 5 and 6 can be seen as the core of the thesis. Using the results of chapters 3 and 4, they tell the main story brought in the summary. Chapter 5 proves the transitive Geometric Von Neumann theorem 5.9. Chapter 6 mainly focuses on following theorem and further proves corollary 6.13.

Theorem 6.4 (The transitive Geometric Burnside problem)**.** *Let G be a finitely generated group with at most two ends, then there exists a Cayley graph* $\mathrm{Cay}(G, S)$ *of G allowing a bi-infinite Hamiltonian path.*

Chapter 7 concludes with quasi-isometry classes of free products of surface groups.[3] This is an application of the theorem proven at the beginning of the chapter, which states a group is amenable if and only if its uniformly finite homology does not vanish.

[2]Trees with uniformly bounded valencies larger than 3.

[3]Surface groups are fundamental groups of n-holed tori.

List of Symbols

G	Finitely generated discrete infinite group
X, Y, Z	UDBG space
x, y, z	Elements in (often resp.) X, Y, Z
$B_r(x)$	Ball of radius r with center x
$B_r(U)$	All elements of X at distance r from some $u \in U$
D_r	Maximum size of a ball of radius r in a graph
$G \curvearrowright X$	A left action by G on X
$g \cdot x$	The left action of $g \in G$ on $x \in X$
$x \cdot g$	The right action of $g \in G$ on $x \in X$
$\mathrm{diam}(S)$	Diameter of a (finite) subset $S \subset X$ i.e. largest distance between two elements.
$\lvert S \rvert$	Number of elements in a set or subset $S \subset X$
Γ, Φ, Λ	Graph
Φ, Ψ	Usually a subgraph of Γ
$\{x, y\}$	Undirected edge
(x, y)	Directed edge
$\deg_\Gamma(x)$	Valency or degree of x in Γ
Γ_r	Graph made from a UDBG space X. (or a graph Γ)
E_r	Set of edges of Γ_r defined by $E_r := \{\{x, y\} \mid d(x, y) \leq r\}$.
$V(\Gamma)$	Vertex set of a graph Γ
$E(\Gamma)$	Edge set of a graph Γ
f, g, h	Functions, mostly quasi-isometries and h a bilipschitz equivalence.
$X \sim_{QI} Y$	Two spaces are quasi-isometric
$X \sim_{\text{Bilip}} Y$	two spaces are bilipschitz equivalent
$f \approx g$	Two functions at bounded distance of each other i.e. $d(f, g) := \sup_{x \in X} d(f(x), g(x)) < \infty$
$S_{\text{bilip}}(X)$	Bilipschitz structure of X
$\mathbb{F}_p$	Free group with p generators. Embedded in the word metric (like all groups).
$\mathbb{Z}_k$	The cyclic group of order k
$\mathrm{Cay}(G, S)$	Cayley graph of G with respect to the finite generating set S

$R_r(X)$	r-Rips complex
∂_k	Boundary operator on k-simplices
S_k	k-simplices
$S_\bullet$	Simplicial complex
C_k	$\mathbb{Z}$-Vector space (or $\mathbb{R}$-Vector space) generated by k-simplices
H_k	k-dimensional homology
$H_\bullet$	Homology
C_k^{uf}	Uniformly finite k-dimensional cycles
H_k^{uf}	Uniformly finite k-dimensional homology
$[x]$	0-simplex
$[x, y]$	1-simplex
c	cycle mostly in $C_0^{uf}(X)$
b	(often 1-dimensional)
$[c]$	Class in $H_0^{uf}(X)$
f_*	Induced function by f on the (0-dimensional) homology
$C_0^{uf^+}(X)$	Positive cycles
$H_0^{uf^+}(X)$	Positive classes
$\sqcup$	Disjoint union of two spaces.
$\vee$	Wedge sum of two surfaces
$G * H$	Free product of two groups
$\coprod_{i\in I} G_i$	Free product of an arbitrary number of groups
$[X]$	Uniformly finite class
S_g	The g-holed thorus
Σ_g	Surface group of S_g

Contents

Chapter 1

Preliminaries

The first objects we will consider are discrete metric spaces. We add the extra condition that there is a minimal distance between two points and that for $r > 0$ the cardinality of all balls $\{B_r(x) \quad | \quad x \in X\}$ is uniformly bounded. We call such spaces UDBG spaces i.e. uniformly discrete spaces of bounded geometry. These UDBG spaces are a generalization of Cayley graphs of finitely generated groups. Our goal is to prove things about these groups using their geometry. Furthermore, we cannot speak of geometry without invariants, and for this we need functions. The most important type of function here is a quasi-isometry. This gives us the large scale structure of a group. It can even link these discrete groups with continuous spaces. A bijective quasi-isometry is called a bilipschitz equivalence. We proceed with important sets of functions, with a particular focus on the bilipschitz structure $S_{\text{bilip}}(X)$. Lastly, taking a well behaved map from a group G to a function space $X \to X$, the notion of action arises. By far, the most important action is the left translation action on the Cayley graph. But the more abstract notion of an action becomes important when we want to show that a graph Γ or UDBG space X, looks like a group G or a subgroup.

1.1 UDBG spaces

We start with the necessary definitions to define uniformly discrete spaces of bounded geometry (UDBG spaces) and give some examples.

Definition 1.1 (Uniformly discrete)**.** *Let X be a metric space, then X is uniformly discrete if there is an $\epsilon \in \mathbb{R}^+$ such that for all different $x, y \in X$:*

$$d(x, y) > \epsilon.$$

Let $x \in X$ and let $U \subset X$. Denote $d(U, x) = \inf_{u \in U} d(u, x)$. Let us now denote $B_r(x)$ for the ball of radius r around x

$$B_r(x) := \{y \quad | \quad d(x, y) \leq r\}.$$

Similarly, denote $B_r(U)$ for the ball of radius r around U

$$B_r(U) = \{y \quad | \quad d(U, y) \leq r\} = \bigcup_{u \in U} B_r(u).$$

Next, we want to avoid situations where balls can have an infinite cardinality like in the trivial metric. Indeed, if we define the trivial metric on X as

$$d_{triv}(x, y) = \delta_{x,y} \quad \forall x, y \in X,$$

where $\delta_{x,y}$ is the Kronecker delta, then for all $x \in X$ we have that

$$B_1(x) = X.$$

This has an infinite cardinality for infinite sets X. We wish to avoid this infinite balls, and we also wish to have a uniform bound on the size of these balls $|B_r(x)|$.

Definition 1.2 (Bounded geometry)**.** *A metric space X is said to be of bounded geometry if for all $r > 0$:*

$$D_r = \sup_{x \in X} |B_r(x)| < \infty.$$

It is crucial that the bound D_r only depends on the radius r and not on the center x.

As we will see later, this is certainly the case in Cayley graphs of groups. As by the very nature of groups, these balls will always have the same size.

We give some examples and counterexamples.

- Uniformly discrete subsets of $\mathbb{R}^n$, considered with the euclidean metric, are UDBG spaces. Some examples of UDBG spaces are $\mathbb{Z}^n$, the set of squares $\{n^2 \mid n \in \mathbb{N}\} \subset \mathbb{R}$, and the set of primes in $\mathbb{N} \subset \mathbb{R}$.

- Subsets of UDBG spaces are UDBG spaces. Subsets are uniformly discrete with the same constant ϵ and of bounded geometry as balls can only contain at most the same number of elements.

- Sets with an accumulation point, like $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ are not UDBG spaces as they are not uniformly discrete.

- Another counterexample is given by the set

  $$\{(n, 1/n) \mid n \in \mathbb{N}\} \cup \{(n, -1/n) \mid n \in \mathbb{N}\} \subset \mathbb{R}^2.$$

  This is not uniformly discrete as the points $(n, 1/n)$ and $(n, -1/n)$ get arbitrarily close as n grows.

- Take $X = \{(n, m) \mid n > m, n, m \in \mathbb{N}\}$ with the metric

  $$d((n_1, m_1), (n_2, m_2)) = d_{Eucl}(n_1, n_2) + d_{triv}(m_1, m_2),$$ [1]

  as shown in figure 1.1. Then we see that $|B_1(n, m)| \geq |\{(n, m') \mid 0 \leq m' < n\}| \geq n$ so that $\{|B_1(n, m)| \mid (n, m) \in X\}$ is not uniformly bounded. Hence, this is not of bounded geometry.

[1] This set could be seen as a union of sets with trivial metric $X_n = \{(n, m) \mid n > m, m \in \mathbb{N}\}$ of n elements. Similar constructions can be made with our first counterexample taking $X_n = \{1/m \mid m < n\}$.

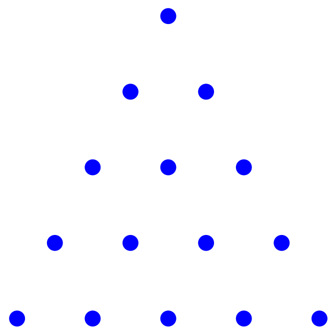


Figure 1.1: The set X on the last example. One should consider all horizontal rows to be sets with diameter one.

1.2 Cayley graphs

Cayley graphs offer a way to work with groups in a geometric way. They offer a natural way to represent a finitely generated group as a graph.

Definition 1.3 (Cayley graph)**.** *Let G be a finitely generated group and let $S \subset G$ be a generating set. Then we define the Cayley graph of G as*

$$\mathrm{Cay}(G, S) := (G, E),$$

where

$$\{x, y\} \in E \iff x = sy$$

for some $s \in S \cup S^{-1}$. [2]

We now define an intuitive metric on such a graph.

Definition 1.4 (Geodesic distance, word metric)**.** *The geodesic distance on* Cay(G, S) *is defined to be the number of edges in shortest path from x to y. This is equivalent with the word metric on G, given as*

$$d(x, y) = n$$

wheren $\in \mathbb{N}$ is the smallest number such that

$$xy^{-1} = s_1 ... s_n$$

for some $s_1, ..., s_n \in S \cup S^{-1}$.

The geodesic distance can be defined similarly on any connected graph Γ. Graphs will be seen as metric spaces using this distance throughout the paper.

Examples

Examples of such graphs are the n-dimensional grids

$$\mathrm{Cay}(\mathbb{Z}^n, \{e_1, ..., e_n\}),$$

as on figure 1.2 or also the free group shown in Figure 1.3,

$$\mathrm{Cay}(\mathbb{F}_2, \{a, b\}).$$

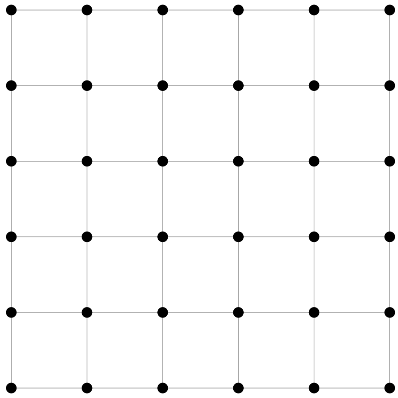


Figure 1.2: A picture of (a part of) $\mathrm{Cay}(\mathbb{Z}^2, \{(1,0),(0,1)\})$.

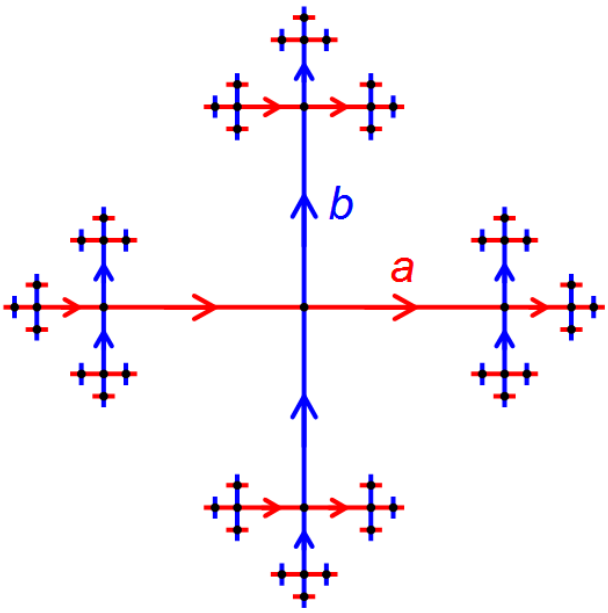


Figure 1.3: The Cayley graph of $\mathbb{F}_2$.[1]

Unfortunately, these graphs depend on the generating set S. Here is an example from [10, page 126, example 5.2.12]. Take the infinite dihedral group

$$D_\infty = \langle a, b \quad | \quad b^2 = abab = 1 \rangle.$$

Every element in this group is of the form a^n or ba^n with $n \in \mathbb{Z}$. We see that the most intuitive graph, $\mathrm{Cay}(D_\infty, \{a, b\})$, looks like a double line.

$$\begin{tikzcd}
\dots & ba^{-2} \arrow[l, dotted] & ba^{-1} \arrow[l, "a"'] & b \arrow[l, "a"'] & ba \arrow[l, "a"'] & ba^2 \arrow[l, "a"'] \arrow[r, dotted] & \dots \\
\dots & a^{-2} \arrow[l, dotted] \arrow[u, "b"] \arrow[r, "a"] & a^{-1} \arrow[u, "b"] \arrow[r, "a"] & 1 \arrow[u, "b"] \arrow[r, "a"] & a \arrow[u, "b"] \arrow[r, "a"] & a^2 \arrow[u, "b"] \arrow[r, dotted] & \dots
\end{tikzcd}$$

However in $\mathrm{Cay}(D_\infty, \{ab, b\})$, we have that ab and b are its own inverses. This implies $|S \cup S^{-1}| = 2$. So each element has only 2 adjacent edges (one says the valence or degree is 2, see below). Such a connected graph must be a line.

$$\begin{tikzcd}
\dots & bab = a^{-1} \arrow[l, dotted] \arrow[r, "b"] & ab \arrow[r, "ab"] & 1 \arrow[r, "b"] & b \arrow[r, "ab"] & a \arrow[r, dotted] & \dots
\end{tikzcd}$$

To show that these representations are not isomorphic, we recall the following definition.

Definition 1.5 (Isomorphism of graphs)**.** *Let $\Gamma = (X, E)$ and $\Gamma' = (X', E')$ be two graphs, then an isomorphism $f : X \to X'$ is a bijection such that*

$$E' = \big\{ \{f(x), f(y)\} \quad | \quad \{x, y\} \in E \big\}.$$

If such a function exists, the two graphs are said to be isomorphic. We denote this as

$$\Gamma \cong \Gamma'.$$

One sees that

$$\mathrm{Cay}(D_\infty, \{ab, b\}) \cong \mathrm{Cay}(\mathbb{Z}, 1).$$

However, looking at valencies, it is clear that

$$\mathrm{Cay}(D_\infty, \{a, b\}) \ncong \mathrm{Cay}(\mathbb{Z}, 1).$$

We conclude that a Cayley graph depends on S.

This dependence on S can be bypassed if we only consider the large scale geometry. As we will see, both of these graphs are bilipschitz equivalent. For this reason, when looking at large scale geometry, the generating set can be omitted from notation. So we can just write $\mathrm{Cay}(G)$.

Recall following definition.

Definition 1.6 (Valency/degree)**.** *Let Γ be a graph and $x \in V(\Gamma)$ a vertex. Then we define the valency or degree of x in Γ as the number of edges in $E(\Gamma)$ containing x. We denote the valency as*

$$\deg_\Gamma(x).$$

[2] Here $S^{-1} = \{s^{-1} \quad | \quad s \in S\}$.

We now define uniformly bounded valencies on a graph. This is an analogue of the bounded geometry condition on metric spaces.

Definition 1.7 (Uniformly bounded valencies). *Let Γ be a graph, then Γ is said to have uniformly bounded valencies if and only if*

$$r = \sup_{g \in \Gamma} \deg g < \infty.$$

Remark 1.8 (Groups and graphs as UDBG spaces). It is crucial to this thesis to hold in mind that the groups and graphs we study,[3] can be seen as UDBG spaces in a natural way. For a graph we define the *geodesic distance* (or *path length metric*) as the number of minimal edges one needs to traverse to get from one vertex to another. As an example, in the graph

$$\mathrm{Cay}(D_\infty, \{a, b\}),$$

we see that the elements a^{-1} and ba^2 are at distance 4. A group can then be seen as a UDBG space by taking a Cayley graph and then using the geodesic distance. This is often called the *word metric* on a group G with respect to its generating set S. Similarly, we obtain that using the word metric in the dihedral infinite group $d(a^{-1}, ba^2) = 4$.

1.3 Maps between UDBG spaces

In this section we define functions between discrete spaces. We start with a Lipschitz function. The reader shall remark that in discrete spaces, to be a Lipschitz function is a rather weak notion. That is why we will examine stronger notions as well. The strongest notion one considers in coarse geometry is the notion of a bilipschitz equivalence.

Definition 1.9 (Lipschitz, bilipschitz and bilipschitz equivalence). *A function, $f : X \to Y$ is C-Lipschitz if and only if*

$$\forall x_1, x_2 \in X : \quad d(f(x_1), f(x_2)) \leq C d(x_1, x_2).$$

If f is C-Lipschitz for some $C \geq 0$, we say f is Lipschitz.

A function, $f : X \to Y$ is C-bilipschitz if and only if

$$\forall x_1, x_2 \in X : \quad C^{-1} d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq C d(x_1, x_2).$$

If f is (C-)bilipschitz and there exists a (C-)bilipschitz function $g : Y \to X$ such that

$$f \circ g = \mathrm{Id}_Y \quad \text{and} \quad g \circ f = \mathrm{Id}_X,$$

then f is said to be a bilipschitz equivalence. The spaces X and Y are said to be (C-)bilipschitz equivalent. We denote this as

$$X \sim_{Bilip} Y.$$

Of these three, we will mostly treat bilipschitz equivalences. To work with them, we use a working definition.[9] Later on, we will mainly use lemma 1.19.

[3] This means finitely generated discrete infinite groups and graphs with uniformly bounded valencies.

Characterisation 1.10. *A function is a bilipschitz equivalence if and only if it is a bilipschitz bijection.*

Proof. $\boxed{\Rightarrow}$ This is trivial, if the bilipschitz equivalence would not be a bijection, one could never find an inverse g.
$\boxed{\Leftarrow}$ Define $g = f^{-1}$ and note that

$$C^{-1}d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq Cd(x_1, x_2) \quad \forall x_1, x_2 \in X,$$

implies

$$C^{-1}d(f(x_1), f(x_2)) \leq d(x_1, x_2) \leq Cd(f(x_1), f(x_2)) \quad \forall x_1, x_2 \in X.$$

Using the bijectivity of f, this is equivalent to

$$C^{-1}d(g(y_1), g(y_2)) \leq d(y_1, y_2) \leq Cd(y_1, y_2) \quad \forall y_1, y_2 \in Y.$$

Therefore g is also a bilipschitz equivalence. □

Definition 1.11 (Distance of 2 functions)**.** *Let $f : X \to Y$ and $h : X \to Y$ then we define their distance as*

$$d(f, h) := \sup_{x \in X} d(f(x), h(x)).$$

We often write $f \approx g$ to say they are at bounded distance from each other, meaning that $d(f, g) < \infty$.

We now define a weaker notion of a Lipschitz function. The difference is that we add an additive constant b to the equation.

Definition 1.12 (Coarse Lipschitz function)**.** *A function $f : X \to Y$ is a (b, C)-coarse Lipschitz function if and only if*

$$\forall x_1, x_2 \in X : d(f(x_1), f(x_2)) \leq Cd(x_1, x_2) + b.$$

We say f is coarse Lipschitz if it is (b, C)-coarse Lipschitz for some $b, C \geq 0$.

Similar to the notion of a coarse Lipschitz function is the notion of a quasi-isometric embedding. A quasi-isometric embedding is what one could call a "coarse bilipschitz function". This notion is preferred in [10], whereas [16] uses mainly coarse Lipschitz functions. In this thesis, preference is given to quasi-isometric embeddings.

Definition 1.13 (Quasi-isometric embedding)**.** *A function $f : X \to Y$ is a (b, C)-quasi-isometric embedding if and only if*

$$\forall x_1, x_2 \in X : C^{-1}d(x_1, x_2) - b \leq d(f(x_1), f(x_2)) \leq Cd(x_1, x_2) + b. \tag{1.1}$$

Both Lipschitz functions and quasi-isometric embeddings allow us to define quasi-isometries. This is the most famous concept in coarse geometry. This thesis will mostly work with quasi-isometries and bilipschitz equivalences.

Definition 1.14 (Quasi-isometry)**.** *Let $f : X \to Y$ be a (b, C)-quasi-isometric embedding. If there exists a (b, C)-quasi-isometric embedding $g : Y \to X$ such that $f \circ g \approx \mathrm{Id}_Y$ and $g \circ f \approx \mathrm{Id}_X$, then f is a (b, C)-quasi-isometry. We then call the spaces X and Y (b, C)-quasi-isometric, and write this as*

$$X \sim_{QI} Y.$$

Furthermore, g is called a quasi-inverse of f.

In coarse geometry, functions on a finite distance from each other can essentially be seen as the same function. Following lemma motivates this.

Lemma 1.15. *If f is a quasi-isometry and $f \approx g$ then g is a quasi-isometry.*

The same statement holds for among others coarse Lipschitz functions and quasi-dense functions (see definition 1.24). It does not work for bilipschitz functions as injectivity and surjectivity are not invariant i.e. if we take f to be injective, g will not necessarily be injective.

proof of lemma 1.15. Let f satisfy equation (1.1) for some $b, C \geq 0$, and let $d(f,g) = M$. Then we have $\forall x_1, x_2 \in X$:

$$d(g(x_1), g(x_2)) \leq d(f(x_1), f(x_2)) + 2M \leq Cd(x_1, x_2) + b + 2M$$

and

$$C^{-1}d(x_1, x_2) - b - 2M \leq d(f(x_1), f(x_2)) - 2M \leq d(g(x_1), g(x_2)).$$

Therefore, g satisfies equation (1.1) with constants $(b + 2M, C)$. □

We can also define quasi-isometries using Lipschitz functions.

Remark 1.16. If there are Lipschitz functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \approx \mathrm{Id}_Y$ and $g \circ f \approx \mathrm{Id}_X$ then f is a quasi-isometry.

In particular, we have obviously that quasi-isometric embeddings are coarse Lipschitz.

Proposition 1.17. *A quasi-isometric embedding is coarse Lipschitz.*

The converse however is not true. For instance, a constant function $\mathbb{Z} \to \mathbb{Z} : n \mapsto 0$ is coarse Lipschitz but not a quasi-isometric embedding.

Following equivalent definition of a quasi-isometric embedding is given in [16].

Characterisation 1.18 (Quasi-isometric embedding)**.** *A function $f : X \to Y$ is a quasi-isometric embedding if and only if it is a quasi-isometry to its image $f : X \to f(X)$.*

Next lemma gives the link between quasi-isometries and bilipschitz equivalences. We will use it often to prove a function is a bilipschitz equivalence or upgrade a quasi-isometries into a bilipschitz equivalence.

Lemma 1.19. *Let X, Y be UDBG spaces. A function $f : X \to Y$ is a bilipschitz equivalence if and only if it is bijective and a quasi isometry.*

As we will see, the clue in this proof is to use the uniform discreteness to turn the additive constant b into a multiplicative one b/ϵ.

Proof. We know that for some $b, C \geq 1$ we have

$$\forall x_1, x_2 \in X : \quad d(f(x_1), f(x_2)) \leq Cd(x_1, x_2) + b.$$

Let ϵ be, like in the definition of uniformly discrete, the minimal distance between two elements but now in both X and Y i.e.

$$\epsilon \leq \min\{d(x_1, x_2), d(y_1, y_2) \quad | \quad x_1, x_2 \in X, y_1, y_2 \in Y\}.$$

Then defining

$$C' := C + \frac{b}{\epsilon},$$

we find that

$$\forall x_1, x_2 \in X : \quad d(f(x_1), f(x_2)) \leq Cd(x_1, x_2) + b\frac{d(x_1, x_2)}{\epsilon} \leq C'd(x_1, x_2).$$

On the other hand, we know that for $b, C \geq 1$ we have

$$\forall x_1, x_2 \in X : \quad C^{-1}d(x_1, x_2) - b \leq d(f(x_1), f(x_2))$$

or equivalently,

$$\forall x_1, x_2 \in X : \quad d(x_1, x_2) \leq C\Big(d(f(x_1), f(x_2)) + b\Big)$$

Then defining

$$C'' := C\Big(1 + \frac{b}{\epsilon}\Big),$$

we find that

$$\begin{aligned} d(x_1, x_2) &\leq C\Big(d(f(x_1), f(x_2)) + b\Big) \\ &\leq C\Big(d(f(x_1), f(x_2)) + b\frac{d(f(x_1), f(x_2))}{\epsilon}\Big) \\ &\leq C''d(f(x_1), f(x_2)). \end{aligned}$$

Note that $C'' \geq C'$ as $C \geq 1$. Hence, we conclude that

$$\forall x_1, x_2 \in X : \quad C''^{-1}d(x_1, x_2) \leq d(f(x_1), f(x_2)) \leq C''d(x_1, x_2).$$

This concludes the proof. □

The main goal of defining quasi-isometries is that up to quasi-isometry (and bilipschitz equivalence), a Cayley graph $\mathrm{Cay}(G, S)$ does not depend on its generating set S, but only on the group G.

Theorem 1.20 (Quasi-isometry invariance of the generating set)**.** *Let G be a group and S and S' two finite generating sets of G, then* $\mathrm{Cay}(G, S)$ *and* $\mathrm{Cay}(G, S')$ *are bilipschitz equivalent (hence also quasi-isometric).* [4]

Furthermore, the identity function on G

$$\mathrm{Id} : \mathrm{Cay}(G, S') \to \mathrm{Cay}(G, S) : g \mapsto g,$$

is a bilipschitz equivalence.

Proof. As S generates G, we can write every $s' \in S' \cup S'^{-1}$ as a word $s_1 ... s_n$ with $s_1, ..., s_n \in S \cup S^{-1}$. This means there is a function

$$\sigma : S' \cup S'^{-1} \to \{ \text{ words in } S \cup S^{-1}\}$$

[4] By this we mean the metric spaces (G, d_S) and $(G, d_{S'})$ are bilipschitz equivalent. See remark 1.8.

sending every s' to some word as above. Let M be the maximal length of $\sigma(s')$ for $s' \in S' \cup S'^{-1}$. Now for any $g = g_1 ... g_k \in G$ with word length k in S' and where $g_1, ..., g_k \in S' \cup S'^{-1}$, we can write

$$g = \sigma(g_1)...\sigma(g_k).$$

Here every $\sigma(g_i)$ contains at most M letters by our choice of M. Hence $\sigma(g_1)...\sigma(g_k)$ contains at most kM letters.

So the identity map

$$\mathrm{Id} : \mathrm{Cay}(G, S') \to \mathrm{Cay}(G, S) : g \mapsto g.$$

satisfies

$$d_{S'}(e, g) \leq M d_S(e, g).$$

By right translation invariance of the word metric, this implies

$$d_{S'}(h, g) \leq M d_S(h, g),$$

for all $g, h \in G$. We conclude that Id is a M-Lipschitz function. Switching S and S' in the reasoning, the inverse function

$$\mathrm{Id}^{-1} : \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S') : g \mapsto g,$$

will also be Lipschitz (for some M'). We conclude that Id is a bilipschitz equivalence and hence also a quasi-isometry. □

As an example of this, we recall our discussion of D_∞ at page 5. The identity function between the two representations of D_∞ is indeed a 2-bilipschitz equivalence.

But let us give another argument to show that both representations of D_∞ are quasi-isometric. The double line $\mathrm{Cay}(\{0,1\} \times \mathbb{Z}, \{(1,0),(0,1)\})$ is quasi-isometric to $\mathbb{Z}$ by the quasi-isometry

$$\pi_2 : \{0,1\} \times \mathbb{Z} \to \mathbb{Z} : (n, m) \mapsto m,$$

and its quasi-inverse

$$i : \mathbb{Z} \to \{0,1\} \times \mathbb{Z} : m \mapsto (0, m).$$

Check that,

$$\pi_2 \circ i : \mathbb{Z} \to \mathbb{Z}$$

is the identity. And that

$$i \circ \pi_2 : \{0,1\} \times \mathbb{Z} \to \{0,1\} \times \mathbb{Z} : (n, m) \mapsto (0, m)$$

is at distance 1 from the identity.

Last example can be viewed in a much wider range of examples. This is given in following lemma.

Lemma 1.21. *Let X be a UDBG space and F a finite metric space (necessarily UDBG). Define the cartesian products of the two spaces as $(F \times X, d_F + d_X)$.*

Then the projection,

$$\pi_2 : F \times X \to X : (a, x) \mapsto x,$$

is a quasi-isometry with quasi-inverse

$$i_{a_0} : X \to F \times X : x \mapsto (a_0, x).$$

Here we fixed any $a_0 \in F$.

Still, the most important examples of quasi-isometric embeddings are given by the following theorem.

Theorem 1.22. *Let G be a group and H be a subgroup of finite index, then the natural embedding*

$$i : H \hookrightarrow G$$

is a quasi-isometry. (The metrics on H and G are given by the word metric.)

For the proof I refer to [12]. We will now see the concept of a quasi-dense function. This can be seen as a coarse notion of surjectivity of functions.

Definition 1.23 (Quasi-dense subspace)**.** *Let X be a UDBG space. A subset $S \subset X$ is called C-dense for $C \geq 0$ if and only if*

$$B_C(S) = Y$$

or equivalently, for all $y \in Y$:

$$d(y, S) \leq C.$$

A subset $S \subset X$ is quasi-dense if and only if it is C-dense for some $C \geq 0$.

Definition 1.24 (Quasi-dense function)**.** *A function $f : X \to Y$ is called C-dense for $C \geq 0$ if and only if*

$$B_C(f(X)) = Y$$

or equivalently, for all $y \in Y$:

$$d(y, f(X)) \leq C.$$

A function is quasi-dense if and only if it is C-dense for some $C \geq 0$.

To prove a function is a quasi-isometry, we will often use following lemma.

Lemma 1.25. *A function $f : X \to Y$ is a quasi-isometry if and only if it is a quasi-isometric embedding and quasi-dense.*

Proof. $\boxed{\Rightarrow}$Let $f : X \to Y$ be a quasi-isometry, then by definition it is a quasi-isometric embedding. Let g be its quasi-inverse, then $f \circ g \approx \mathrm{Id}$. So there is a $C \geq 0$ such that for all $y \in Y$

$$d(y, f \circ g(y)) \leq C.$$

In particular, this implies

$$d(y, f \circ g(Y)) \leq C.$$

So $f \circ g(Y)$ is quasi-dense in Y. Since $f(X) \supset f \circ g(Y)$, it follows that f is quasi-dense in Y.

$\boxed{\Leftarrow}$ Let f be C-dense and let it be a (b', C')-quasi-isometric embedding. Define the quasi-inverse g as follows (see also Figure 1.4):

- Send all points $y \in f(X)$ to any $x \in f^{-1}(y)$.
- For $y' \notin f(X)$, take any $y \in f(X)$ at distance C from y'. So $d(y, y') \leq C$. Send y' to $g(y)$.

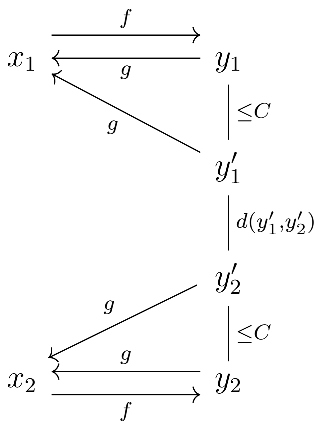


Figure 1.4: The construction of g visualised.

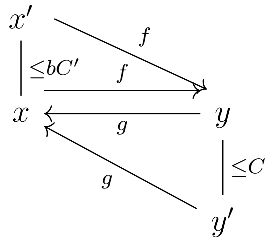


Figure 1.5: Here we see that $f \circ g$ is at distance C from the identity as y is chosen such that $d(y, y') \leq C$. On the other hand, $g \circ f$ is at distance bC' from the identity as it holds that $C'^{-1}d(x, x') - b \leq d(y, y) = 0$.

We now show that g is a quasi-isometry using remark 1.16. Take any $y_1', y_2' \in Y$. Denote x_1, x_2, y_1, y_2 as on figure 1.4. We then have that $d(y_1, y_1') \leq C$ and $d(y_2, y_2') \leq C$. Since f is a quasi-isometry we have that

$$C'^{-1}d(x_1, x_2) - b' \leq d(y_1, y_2) \leq C'd(x_1, x_2) + b'.$$

The triangle inequality, ensures that

$$d(y_1, y_2) - 2C \leq d(y_1', y_2') \leq d(y_1, y_2) + 2C.$$

So we get that

$$C'^{-1}d(x_1, x_2) - b' - 2C \leq d(y_1', y_2') \leq C'd(x_1, x_2) + b' + 2C.$$

One can rewrite this to see that g is a $(C'(b' + 2C), C')$-quasi-isometric embedding. We can see in Figure 1.5 that $f \circ g$ is at distance bC'^{-1} from Id_X and $g \circ f$ is at distance C from the identity on Y. This concludes the proof.

□

To end this section, we make a construction that will often be used throughout the thesis. It plays a particularly important role in section 4.2 to move from geometric concepts to graph theory.

Example 1.26. Let $\Gamma = (X, E)$ be a connected graph and X the corresponding metric space using the geodesic distance. For $r \in \mathbb{Z}_{\geq 1}$, we define the graph

$$\Gamma_r := (X, E_r),$$

where

$$E_r := \{\{x, y\} \quad | \quad d(x, y) \leq r\}.$$

This is seen to be r-bilipschitz equivalent by the identity function on X

$$\mathrm{Id} : \Gamma \to \Gamma_r,$$

as it divides distances by r and rounds up.

Similarly, Γ_r can also be defined starting from any UDBG space X. However, this construction has not the same strong properties. Here Γ_r and X may be quite different.

Often UDBG spaces of the type $\{n^2 \quad | \quad n \in \mathbb{N}\} \subset \mathbb{R}$ will generate annoying counterexamples. For this purpose, we define *coarsely connected* UDBG spaces. Most theorems are thought for this type of UDBG spaces. This is similar to asking that the graphs we use are connected.

Definition 1.27. *A UDBG space X is said to be coarsely connected if and only if there is a $r \geq 0$ such that Γ_r as defined above, is connected.*

1.4 Spaces of functions

We now define the bilipschitz structure $S_{\text{bilip}}(X)$ of a UDBG space X as done in [16, section 3]. Proves where added for the examples from that same source. In brief, $S_{\text{bilip}}(X)$ is the quotient of all quasi-isometries over the ones that are at bounded distance from each other after a bilipschitz equivalence.

Definition 1.28. *Let X be a UDBG space, then we define the space of all quasi-isometries to X as*

$$QIF(X) = \{(Y, f) \quad | \quad f : Y \to X \text{ is a quasi-isometry}\}.$$

Definition 1.29. *Let X be a UDBG space, then we define the space of all quasi-isometric functions on X as*

$$QI(X) = \{f | f : X \to X \text{ is a quasi-isometry}\}.$$

We can now define $S_{\text{bilip}}(X)$.

Definition 1.30 (Bilipschitz structure)**.** *Let X be a UDBG space, then we define the bilipschitz structures on X as*

$$S_{bilip}(X) = \frac{QIF(X)}{\sim}.$$

Here we say that $(Y_1, f_1) \sim (Y_2, f_2)$ if and only if for some bilipschitz equivalence h, the following diagram commutes up to finite distance.

$$\begin{tikzcd} Y_1 \arrow[r, "h"] \arrow[dr, "f_1"'] & Y_2 \arrow[d, "f_2"] \\ & X \end{tikzcd}$$

i.e. $f_1 \approx f_2 \circ h$ *or in more details*

$$\exists D \in \mathbb{R}^+ : \forall y \in Y_1 : d_X(f_1(y), f_2 \circ h(y)) \leq D.$$

Remark 1.31. $S_{\text{bilip}}(X)$ can be defined as a semigroup by defining addition as follows. Take $(Y, f), (Z, g) \in S_{\text{bilip}}(X)$ then

$$(Y, f) + (Z, g) = (Y \sqcup Z, f \sqcup g)$$

where $(f \sqcup g)(y) = f(y)$ for $y \in Y$ and $(f \sqcup g)(z) = g(z)$ for $z \in Z$.

Examples

Let us look at the bilipschitz structure of a singleton $S_{\text{bilip}}(\{x\})$.

Proposition 1.32. *We have the isomorphism of semigroups*

$$S_{bilip}(\{x\}) \cong \mathbb{Z}_{\geq 0},$$

using the bijection between both sets

$$\varphi : S_{bilip}(\{x\}) \to \mathbb{Z}_{\geq 0} : (X, f) \mapsto |X|.$$

Proof. We first show that $X \sim_{QI} \{x\}$ if and only if X is finite. Let X be finite then any function between X and $\{x\}$ will be a $(\text{diam}(X), 0)$-quasi-isometry. Let X be infinite, then by our bounded geometry condition $\text{diam}(X) = \infty$. Hence any function $\{x\} \to X$ will not be quasi-dense, which is a necessary condition by lemma 1.25.

Next, we show that cardinality is a complete invariant. Take $(X, f), (Y, g) \in S_{\text{bilip}}(\{x\})$. First let $|X| = |Y|$, take any bijection $\phi : X \to Y$. Any function on finite sets is a C-quasi-isometry, where C is obtained by dividing the maximal possible distance by the smallest distance,

$$C := \max \left\{ \frac{\text{diam}(Y)}{\min_{x_1, x_2 \in X} d(x_1, x_2)}, \frac{\text{diam}(X)}{\min_{y_1, y_2 \in Y} d(y_1, y_2)} \right\}.$$

Now lemma 1.19 guaranties that ϕ is a bilipschitz equivalence. The diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & \{x\} \end{tikzcd}$$

will clearly commute since the destination is always the same x.

Secondly, if $|X| \neq |Y|$, then there is no bijection between them.

□

Remark 1.33. Actually, $S_{\text{bilip}}(X)$ is the same for all finite sets X, this is because $X \sim_{QI} \{x\}$ by lemma 1.21 and $X \cong X \times \{x\}$ is finite. Also commuting up to bounded distance is always the case in a finite (hence bounded) image.

Let us look at a second example, that of the integers.

Proposition 1.34. *The bilipschitz structure $S_{bilip}(\mathbb{Z})$ is uncountable.*

The idea is that $(\mathbb{Z}, f_r)$ where $f_r(n) = \lfloor rn \rfloor$ give rise to different elements in $S_{\text{bilip}}(\mathbb{Z})$ for all $r \in \mathbb{R}$.

Proof. Let $(\mathbb{Z}, f_r), (\mathbb{Z}, f_s) \in S_{\text{bilip}}(\mathbb{Z})$ with $s > r > 0$. Let, for the sake of contraposition, h be a bijection, such that following diagram commutes up to finite distance C.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "h"] \arrow[dr, "f_r"'] & \mathbb{Z} \arrow[d, "f_s"] \\ & \mathbb{Z} \end{tikzcd}$$

We show that for large n, we have $h(n) < n$. We know by the commutativity of the diagram that

$$\begin{aligned} & \lfloor sh(n) \rfloor \approx \lfloor rn \rfloor \\ \Rightarrow \quad & \lfloor sh(n) \rfloor - \lfloor rn \rfloor \leq C \end{aligned}$$

As the floor function is at distance 1 from the identity on $\mathbb{R}$, we can just omit them replacing C by $C + 2$. ($C + 1$ works as well)

$$\begin{aligned} & sh(n) - rn \leq C \\ \iff \quad & s(h(n) - n) + sn - rn \leq C \\ \iff \quad & sn - rn - C \leq s(n - h(n)) \\ \iff \quad & (s - r)n - C \leq s(n - h(n)). \end{aligned}$$

As n grows, so will the left side of the equation, this will be strictly positive for all $n > n_0$ for some $n_0 > 0$. Hence, the right hand side must be strictly positive as well. This implies $(n - h(n)) > 0$ i.e. $n > h(n)$.

For negative n we can start from the equation

$$\lfloor sh(n) \rfloor - \lfloor rn \rfloor \geq -C.$$

Similarly, we obtain

$$(s - r)n + C \geq s(n - h(n)).$$

So that for large negative n, say $n < n_1$ for some $n_1 < 0$, we have $h(n) > n$. Joining both conclusions we obtain

$$h([n_1, n_0]) \subset (n_1, n_0).$$

We conclude that

$$|h([n_1, n_0])| \leq |(n_1, n_0)| < |[n_1, n_0]|.$$

This implies that h cannot be a bijection. We conclude that $(\mathbb{Z}, f_r)$ and $(\mathbb{Z}, f_s)$ are two different elements in $S_{\text{bilip}}(\mathbb{Z})$.

So the family $\{(\mathbb{Z}, f_r) \mid r \in \mathbb{R}\}$ gives rise to an uncountable amount of distinct elements. This finishes the proof. $\square$

It is clear that the bilipschitz structure of a UDBG space can be much larger than the UDBG space itself. However, for nonamenable spaces, we will see that this structure is trivial, see theorems 3.3 and 7.1.

1.5 Actions

A function from a group G to bijections on a set X satisfying some compatibility conditions, is called an action. On UDBG spaces, one requires these bijections to have more restrictions concerning distances. This will then define translation-like actions. The most natural example of such an action is the translation action on a Cayley graph. At the end of the section we define amenable discrete metric spaces using Følner sets. One of the main goals of this thesis is to better understand amenability using translation-like actions.

Definition 1.35 (Action on a UDBG space)**.** *Let G be a group and X a set. A left group action $G \curvearrowright X$ is a function*

$$G \to \{f : X \to X \quad | \quad f \textit{ is a permutation}\} : g \mapsto (g\cdot)$$

satisfying

$$\begin{aligned}(e\cdot) &= \mathrm{Id}_X,\\ (g\cdot)\circ(h\cdot) &= (gh\cdot).\end{aligned}$$

From the definition it follows that $(g\cdot)$ is a bijection. Similarly, we define an action on a (connected) graph Γ by seeing the graph as a metric space with the geodesic distance.

A right group action is defined similarly as

$$G \to \operatorname{Aut} X : g \mapsto (\cdot g)$$

satisfying

$$\begin{aligned}(\cdot e) &= \mathrm{Id}_X\\ (x\cdot g)\cdot h &= x \cdot gh.\end{aligned}$$

The most important examples are left translation actions. A group G acts naturally on $\mathrm{Cay}(G)$ by

$$G \curvearrowright \mathrm{Cay}(G) : g \cdot g' = gg'$$

for all $g, g' \in G$. Let H be a subgroup of G, then the left translation action by H on $\mathrm{Cay}(G)$ is defined as,

$$H \curvearrowright \mathrm{Cay}(G) : h \cdot g = hg.$$

for all $h \in H, g \in G$.

We now introduce special kinds of actions.

Definition 1.36 (Free action)**.** *Let G be a group and X a set. A group action is called free if it satisfies,*

$$g \cdot x \neq x$$

for all $g \in G\backslash\{e\}$ and $x \in X$.

The translation action is an example of a free action.

Theorem 1.37. *The translation action on a group G (or equivalently $\mathrm{Cay}(G,S)$) with the word metric d_S, is a free action.*

We now introduce one of the most important concepts of this thesis.

Definition 1.38 (Translation-like action)**.** *An action by G on X is translation-like if it is free, and satisfies for all g,*

$$(g\cdot) \approx \mathrm{Id}_X .$$

An equivalent formulation for the condition is that $\{d(g \cdot x, x) \quad | \quad x \in X\}$ is bounded for all g.

Remark 1.39. Next to the left translation action one also has a right translation action. Both can be more suitable depending on ones purposes. For this thesis we will exclusively use left actions. The right translation action acts by graph isomorphisms (1-bilipschitz equivalences). Indeed, we see that an edge $\{g, sg\}$ is moved to $\{g, sg\} \cdot h = \{gh, sgh\}$ which is still an edge.

The left translation action does not share this property. Here an edge $\{g, sg\} \in E(\mathrm{Cay}(G))$ is moved to $h \cdot \{g, sg\}\{hg, hsg\}$. These two elements are at distance bounded by $2|h|+1$. As this bound only depends on $h \in H$ we have that the left translation action is still a translation-like action. However, the reason we use left (translation) actions is that we can embed a Cayley graph $\mathrm{Cay}(H,U)$ of H in $\mathrm{Cay}(G,S)$ in a natural way, if we have that $U \subset S$. [5] Moreover, the graph

$$\Phi = (G, \{\{g, u \cdot g\} \quad | \quad g \in G, u \in U\})$$

satisfies that its connected components are isomorphic to $\mathrm{Cay}(H,U)$. We write this as

$$\Phi \cong \bigsqcup \mathrm{Cay}(H,U).$$

By this we mean that Φ is an arbitrary union of disconnected copies of $\mathrm{Cay}(H,U)$. This gives a way to translate *having a translation-like action* to *having a subgraph* $\Phi \cong \bigsqcup \mathrm{Cay}(H,U)$. For more information see theorem 4.6.

The main purpose of our thesis is to link translation-like actions to amenability. Let us therefore define amenability on metric space. By viewing groups and graphs as metric spaces using respectively the word metric or the geodesic distance, we can extend this definition to groups and graphs as well.

Definition 1.40 (Amenable)**.** *A Følner sequence $(F_n)_{n\in\mathbb{N}}$ is a sequence of finite sets F_n such that for any $R > 0$:*

$$\lim \frac{|\partial_R F_n|}{|F_n|} = 0.$$

A UDBG space X is amenable if and only if it admits a Følner sequence.

A graph Γ is amenable if and only if its vertex set $V(\Gamma)$ embedded in the geodesic distance, is amenable as a metric space.

A group G is amenable if and only if there is a Cayley graph $\mathrm{Cay}(G,S)$ that is amenable as a graph.

[5] By applying the embedding $\iota : H \to G$ to graphs $\mathrm{Cay}(H,U) \to \mathrm{Cay}(G,S)$. See sections 4.2 and 6.3.

Note that whether a Cayley graph $\mathrm{Cay}(G, S)$ that is amenable does not depend on the chosen S. Intuitively, it suffices in many cases to check whether growing balls $B_n(e)$ of growing radius n, form a Følner sequence.

- If balls have polynomial growth, then the space is amenable. This is the case in $\mathbb{Z}^n$.
- If balls have exponential growth, then the space is nonamenable. This is the case in free groups $\mathbb{F}_k$.

Chapter 2

Homology theory

We now introduce homology on UDBG. Crucial will be the construction of r-Rips complexes. This defines a simplicial complex on a UDBG space X. Section 3.1 takes the limit of r to infinity to define $H_\bullet(X)$. This way we get rid of the dependency of some constant r. In coarse geometry, it is important to avoid such dependencies. In section 3.2, we start with theorem 2.12, stating that $H_\bullet(X)$ always gives the same boring object. That is why several norm closures of simplicial complexes are analysed. First, we describe the p-homology $H_\bullet^{(p)}(X)$. Secondly, we analyse the most important homology for this thesis, the uniformly finite homology $H_\bullet^{uf}(X)$. Intuition and examples are added with a particular focus on the 0 dimensional homology.

Definition 2.1. *Let X be a set, a k-simplex $[x_0, ..., x_k]$ is an ordered set of $k+1$ different elements of X. A simplicial complex $S_\bullet$ or $S_\bullet(X)$ of X is a sequence of sets of k-simplices $(S_k)_{k\in\mathbb{N}}$ satisfying that if $l \leq k$ and $[y_0, ..., y_l]$ is a subset*[1] *of $[x_0, ..., x_k] \in S_k$ then $[y_0, ..., y_l] \in S_l$.*

Let us now denote with C_k the free $\mathbb{Z}$-module (or $\mathbb{R}$-module) over S_k and $C_\bullet = (C_k)_{k\in\mathbb{Z}_{\geq 0}}$. An element $c \in C_k$, called a chain, can be seen as a finitely supported function from k-simplices to $\mathbb{Z}$ or equivalently as a finite sum of $s \in S_k$. We denote c_x for the image of x by c.

Notation 2.2. When speaking of chains, we use both the notation via functions $c : S_k \to \mathbb{Z}$ as the sum notation $c = \sum_{i=0}^{n} c_{s_i} s_i$ where $c_{s_i} \in \mathbb{Z}$ and $s_i \in S_k$. Functions are more used when describing sets of chains like in example 2.3 below. On the other hand sums are more used when describing specific chains. Examples are

$$C_0 = \{c : S_0 \to \mathbb{Z} \quad | \quad \mathrm{supp}(c) \text{ is finite}\}.$$
$$C_1 = \{b : S_1 \to \mathbb{Z} \quad | \quad \mathrm{supp}(b) \text{ is finite}\}.$$

We often use c for a 0-chain and b for a 1-chain. Often we will be speaking about chains/homology on a graph. If this is the case $c \in C_0$ is a sum of vertices and $b \in C_1$ is a sum of (directed) edges.

Example 2.3. Take X the infinite cyclic group $\langle t \rangle$, then elements c of C_0 are

$$5[t^3] - 3[t^{-1}], -2[t], 3[e]...$$

[1] By this we mean

$$\{y_0, ..., y_l\} \subset \{x_0, ..., x_k\}.$$

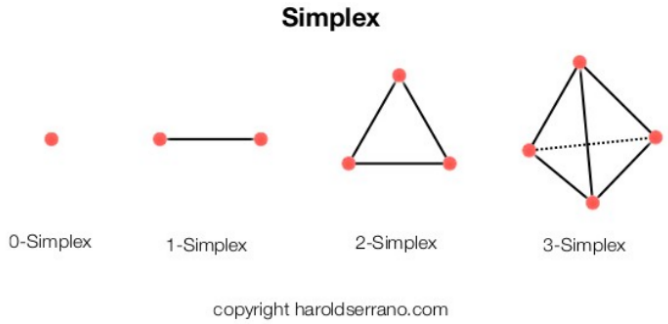


Figure 2.1: We see examples of k-simplices. In the last picture the outer loop is a 1-boundary as it can be triangulated. This figure was published in [5].

Let us get more specific in our context. For UDBG spaces, one can associate Rips complexes. For $r \in \mathbb{R}$ an r-Rips complex $R_r(X)$ can be represented as a graph $\Gamma_r = (X, E_r)$ where $\{x, y\} \in E_r$ if and only if $d(x, y) \leq r$ as in example 1.26. For any k, the k-simplices S_k are all complete subgraphs of $k+1$ vertices.

Definition 2.4 (The r-Rips complex $R_r(X)$)**.** *Let X be a UDBG space, we define the r-Rips complex $R_r(X)$ to be the simplicial complex with $[x_0, ..., x_k] \in S_k(R_r)$ if and only if:*

$$d(x_i, x_j) \leq r$$

for all $i, j \leq k$.

Remark that a r-Rips complex forms indeed a simplicial complex. As if all elements in $[x_0, ..., x_k]$ are on distance smaller than r, then the same will hold for any subset of $[x_0, ..., x_k]$.

Remark 2.5. Note that by this definition $S_0(R_r)$ and $S_1(R_r)$ can easily be seen as respectively the vertices and the edges on the graph Γ_r.

Furthermore, the sets $C_0(R_r)$ and $C_1(R_r)$ can be seen as sums of vertices or directed edges. The notation $[x, y]$ is then replaced by the notation for directed edges (x, y). Changing the direction of an edge $(x, y) \in C_1(R_r)$ gives you the opposite element $(y, x) \in C_1(R_r)$.

We next need to define a chain on such complexes.

Definition 2.6 (Chain complex)**.** *Let $R_r = (C_k)_k$ be a Rips complex. then we define a chain by functions $\partial_k : C_k \to C_{k-1}$ such that $\partial_{k-1} \circ \partial_k = 0$ or equivalently* $\ker \partial_{k-1} \supset \operatorname{Im} \partial_k$.

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots$$

Unless mentioned otherwise, ∂_k is defined by

$$\partial_0 = 0,$$

and

$$\partial_k : C_k(R_r) \to C_{k-1}(R_r) : [x_0, ..., x_k] \mapsto \sum_{i=0}^{k} (-1)^i [x_0, ..., \widehat{x_i}, ..., x_k].$$

We now define the homology as follows.

Definition 2.7 (Homology of a chain complex)**.** *For a chain we define*

$$Z_k = \ker \partial_k$$

as the k-cycles, and

$$B_k = \operatorname{Im} \partial_{k+1}$$

as the k-boundaries. Next we define the k-homology as,

$$H_k = \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}} = \frac{Z_k}{B_k}.$$

and $H_\bullet = (H_k)_{k \in \mathbb{Z}_{\geq 0}}$.

Remark 2.8. Note that a cycle $c \in Z_k$ is not identical to its class $[c] \in H_k$. However, in the remainder of this thesis, we will not always write this distinction to simplify notation, for example in chapter 3.2, we write $f[X]$ for both the cycle $c = \sum_{x \in X}[f(x)]$ as the class $[c]$.

Note that in the 0-dimensional complex, cycles and chains are the same since

$$H_0^{uf} = \frac{C_0^{uf}}{B_0^{uf}}.$$

I will refer to these objects as cycles.

Analogously to the definition of chains, we can define a cochain as a sequence

$$\ldots \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} \ldots$$

where here as well $\partial^{n+1} \circ \partial^n = 0$ for all $n \in \mathbb{N}$. We can just as well define,

$$H^k = \frac{\ker \partial^k}{\operatorname{Im} \partial^{k-1}}.$$

This is called a cohomology. Here we call $\ker \partial^k$ the k-cocycles, and $\operatorname{Im} \partial^{k-1}$ the k-coboundaries.

2.1 Homology on UDBG spaces

Let us have a Rips complex $R_r = R_r(X)$ on some UDBG space X and the "corresponding" graph $\Gamma_r = (X, E_r)$ where $\{x, y\} \in E_r$ if and only if $d(x, y) \leq r$. Recall that we defined next linear functions

$$\partial_k : C_k(R_r) \to C_{k-1}(R_r) : [x_0, ..., x_k] \mapsto \sum_{i=0}^{k} (-1)^i [x_0, ..., \widehat{x}_i, ..., x_k].$$

This gives rise to a very intuitive homology. The intuition can be seen in many sources about topological data analysis like [5]. In H_0 for instance, we see $x - y$ is a boundary if and only if there is a path b from x to y. The dimension of H_0 as a vector space hence equals the number of connected components. In H_1, we find that some cycle b consists of sums of closed paths. Such a closed path is a boundary if and only if it can be triangulated (see Figure 2.1). Let $a \in C_2$ be the sum of these triangles then ∂a gives b. This hence measures the one dimensional holes, like the one in a torus.

As we are interested in large scale geometry, the dependence on r is an inconvenience. Ideally, we would like homologies to be the same for quasi-isometric spaces. For this purpose we look at what happens when r grows to infinity. We herefore define a direct limit of groups. We briefly mention this abstract definition before applying it to Rips complexes. [17]

Definition 2.9 (Direct limit of groups). *Let I be an ordered set and let $G_* = (G_i)_{i \in I}$ where G_i is a group. Further, let $f_{ij} : G_i \to G_j$ be morphisms $\forall i \leq j \in I$ satisfying two conditions*

- *f_{ii} is the identity.*
- *Composition: $f_{jk} \circ f_{ij} = f_{ik}$ for $i \leq j \leq k$.*

Then $\langle G_*, f_{ij} \rangle$ is called a direct system. We define its direct limit as

$$\lim_{\to} G_* = \sqcup_i G_i / \sim$$

Here, two elements $g_i \in G_i$ and $g_j \in G_j$ are equivalent if and only if there exists a $k \in I$ such that $f_{ik}(g_i) = f_{jk}(g_j)$.

Also, one will notice that often G_i is just an ascending sequence and that f_{ij} is just the natural inclusion $G_i \hookrightarrow G_j$. In this case we simply have that $\lim_{\to} G_* = \bigcup_{i \in \mathbb{R}} G_i$. This will be the case for chain sequences $C_\bullet$.

Let us now apply this on homologies. Set $I = \mathbb{R}^+$ and $G_r = H_\bullet(R_r(X))$. We define the functions $f_{rr'}$ as natural injections.

Definition 2.10 (Homology of a UDBG space). *Let X be a UDBG space. Let for all $0 \leq r \leq r'$ the function $f_{rr'}$ be defined as the natural injection $C_\bullet(R_r) \hookrightarrow C_\bullet(R_{r'})$. We then define the simplicial complex of X as*

$$C_\bullet(X) = \lim_{\to} C_\bullet(R_r(X))$$

and the (normal) homology as

$$H_\bullet(X) = \lim_{\to} H_\bullet(R_r(X)),$$

where $\overline{f_{rr'}} : H_\bullet(R_r(X)) \hookrightarrow H_\bullet(R_{r'}(X))$ is the induced map of $f_{rr'}$.[2]

Remark 2.11. It is important to remember that we took a union to start with. This means that every element sits in G_i for some i. So for any cycle $c \in C_k(X)$, one finds r such that $c \in C_k(R_r(X))$.

[2]Note that this are not injective anymore. A nontrivial element c in $H_k(R_r(X))$ might become a boundary for larger r. However the functions $\overline{f_{rr'}}$ still form group morphisms and satisfy the conditions of the definition 2.9.

2.2 Examples of homologies

This section handles different norm closures of $H_\bullet(X)$ and gives an intuition of the (mostly 0 dimensional) homology. Firstly, we look at the normal homology, here the 0 dimensional homology consists of finitely supported functions from $X \to \mathbb{Z}$, up to boundaries.[3] One can change $\mathbb{Z}$ to $\mathbb{R}$ to obtain a different homology. The second paragraph looks at closures in the p-norm. Here we must take $\mathbb{R}$-modules to obtain something meaningful. The third paragraph, handles the case $p = \infty$, the uniformly bounded functions. Taking functions to $\mathbb{Z}$, this is called the uniformly finite homology $H_0^{uf}(X)$. The notation $H^{(\infty)}(X)$ is reserved for functions to $\mathbb{R}$, this homology is called the infinite homology.

Throughout the further chapters of this thesis, we focus on the uniformly finite homology. It will be used throughout chapter 3 and to give a characterisation of amenability, theorem 7.1.

Normal homology $H_\bullet(X)$

Following discussion and theorem I assume to be commonly known.

Let us analyze $H_\bullet(X)$ for a finite set X. We can make

$$r > \operatorname{diam}(X) := \sup_{x,y \in X} d(x, y)$$

so that the graph of X is complete. Then the homology is the same as the homology of a single point. This means that $H_0(X) \cong \mathbb{Z}$, and for $k > 0$, this means that $H_k = 0$ as all k-cycles are k-boundaries. We can prove the first isomorphism using the augmentation function

$$\mathcal{E} : C_0 \to \mathbb{Z} : c \mapsto \sum_{x \in X} c(x).$$

Indeed, fix x_0, and let $b \in H_1(X)$ be the sum of paths from x to x_0 taken c_x times. Then this maps $\sum_{x \in X} c_x$ to $[c] = [c + \partial b] = \mathcal{E}(c)[x_0]$. We see that $\ker \mathcal{E} \subset B_0$.

Conversely, take $\partial b \in B_0$ for any $b \in C_1$, then every edge adds one at its arriving point and subtract one at its starting point. Hence it adds up to zero. We conclude that $\ker \mathcal{E} \supset B_0$.

Let now X be an infinite set. Taking any $c \in C_k$, we can look at the finite support of c and neglect the rest. Therefore, this situation is analogous to the case where X is finite. Again we find $H_0 \cong \mathbb{Z}$ and $H_k = 0$ for $k > 0$.

Let us finally note that we supposed $\operatorname{Im} c \subset \mathbb{Z}$. One can also look at the group with $\operatorname{Im} c \subset \mathbb{R}$. The discussion of this is analogous. We find $H_0 \cong \mathbb{R}$ and $H_k = 0$ for $k > 0$. We have now proven the following theorem.

Theorem 2.12. *Let X be a UDBG space, then the homology $H_\bullet(X)$ as $\mathbb{Z}$-module is given by*

$$\begin{aligned} &H_0(X) \cong \mathbb{Z}; \\ &H_k(X) = 0 \quad \forall k > 0. \end{aligned}$$

[3]Most often an element is denoted as a finite sum of elements x of X. The two representations are possible.

And as $\mathbb{R}$*-module, it is given by*

$$\begin{aligned} H_0(X) &\cong \mathbb{R}; \\ H_k(X) &= 0 \quad \forall k > 0. \end{aligned}$$

p-homology $H_\bullet^{(p)}(X)$

Let us take now take $\mathbb{R}$-modules and the closure of $C_\bullet(X)$ with respect to the norm

$$\|c\|_p = \Big(\sum_{x \in S_k(X)} |c_x|^p \Big)^{p^{-1}},$$

where $c \in C_k$. This c can now have infinite support. Still, remark 2.11 gives an important condition on the elements. For any c, its support must be in $C_k(R_r(X))$ for some r.

Let $c \in C_0^{(p)}(X)$. We now encounter 3 problems in previous reasoning. Firstly, it can happen that we cannot connect $\operatorname{supp} c$. Look for instance at $\operatorname{supp} c = X = \{n^2 \mid n \in \mathbb{Z}\}$. Then $\forall r$ if $n > r$ then we can see that $(n-1)^2$ and n^2 are not connected in R_r. [4] Secondly, for $p > 1$, the functions c do not need to be absolutely summable, this will prevent us from shrinking c to a point. Thirdly, as we have now more boundaries, more elements vanish. We illustrate this below.

Remark 2.13. Take $p > 1$ and $c = [e]$ in $C_0^{(p)}(\mathbb{F}_2)$.[5] We show $[c] = 0 \in H_0^{(p)}(\mathbb{F}_2)$ or $c + \partial b = 0$ for some $b \in C_1^{(p)}(\mathbb{F}_2)$. Take for b an infinite binary probability tree. In the figure below, I show the first 3 steps of how such a 1-chain looks like.

$$\begin{tikzcd}
 & & & xxx & xx \arrow[l, "1/8"'] \arrow[r, "1/8"] & yxx \\
 & e \arrow[rrr, "1/2"] \arrow[dd, "1/2"] & & & x \arrow[u, "1/4"] \arrow[d, "1/4"] & \\
xxy & & xyy & xyx & yx \arrow[l, "1/8"'] \arrow[r, "1/8"] & yyx \\
xy \arrow[u, "1/8"] \arrow[d, "1/8"] & y \arrow[l, "1/4"'] \arrow[r, "1/4"] & yy \arrow[u, "1/8"] \arrow[d, "1/8"] & & & \\
yxy & & yyy & & &
\end{tikzcd}$$

Calculate the norm of b

$$\|b\|_p^p = \sum_{i=1}^{\infty} 2^i \frac{1}{(2^i)^p} = \sum_{i=1}^{\infty} (2^i)^{1-p}.$$

As this serie decays exponentially, it is finite.

[4] This types of possible counterexamples are often regarded as pathological. When we want to exclude them we talk of *coarsely connected* UDBG spaces. This means there is a r such that $R_r(X)$ is connected.

[5] Where $\mathbb{F}_2$ has the defining word metric.

For amenable groups it is not possible to embed a tree in its Cayley graph, like we just did for $\mathbb{F}_2$. However, we still can shrink any finite supported part c' of c to a point, just like in the normal homology. Here $c - c'$ can be made arbitrarily small in the p norm and the infinity norm.

The infinite homology $H_\bullet^{(\infty)}(X)$ and the uniformly finite homology $H_\bullet^{uf}(X)$

The main studied homology in this thesis will be $H_0^{uf}(X)$. This consists of the uniformly bounded functions i.e. functions $c : X \to \mathbb{Z}$ with finite infinite norm

$$\|c\|_\infty = \sup |c_x|.$$

Here we can use both $\mathbb{Z}$ and $\mathbb{R}$-modules. To distinguish both we call the $\mathbb{R}$-modules

$$H_n^{(\infty)} := \frac{\{c : S_n \to \mathbb{R} \quad | \quad \partial c = 0, \quad \|c\|_\infty < \infty\}}{\{\partial b \quad | \quad b \in C_{n+1}\}},$$

the infinite homology and the $\mathbb{Z}$-modules

$$H_n^{uf} := \frac{\{c : S_n \to \mathbb{Z} \quad | \quad \partial c = 0, \quad \|c\|_\infty < \infty\}}{\{\partial b \quad | \quad b \in C_{n+1}\}},$$ [6]

the uniformly finite homology.

Last but not least, just like in the p-homology, we have that for any $x \in X$ we have that $[x] = 0$ in $H_0^{uf}(X)$. But unlike in the p-homology, it is not necessary in the uniformly finite homology to have a nonamenable space. It will suffice to be in an infinite coarsely connected UDBG space $|X| = \infty$ as shown by example 2.14 below. Remark that this homology characterizes amenable spaces, as it will be possible to vanish the uniformly finite class

$$[X] := \sum_{x \in X} [x]$$

if and only if the space X is nonamenable (see theorem 7.1).

Example 2.14. Take for instance $X = \langle t \rangle$ to be the infinite cyclic group then $[e]$ is a boundary. Take an infinite path going to e

$$b = \sum_{n \in \mathbb{N}} [t^{n+1}, t^n].$$

Then $[e] + \partial b = 0$. Here b is illustrated

$$e \longrightarrow t \longrightarrow t^2 \longrightarrow t^3 \longrightarrow t^4 \dashrightarrow \ldots$$

Following theorem gives a generating set of $C_0^{uf}(X)$.

[6] One could also denote this set as

$$H_n^{uf} := \frac{\{c = \sum_{s_i \in S_n} m_i s_i \quad | \quad m_i \in \mathbb{Z}, s_i \in S_n, m \in \mathbb{N}\}}{\{\partial b \quad | \quad b \in C_{n+1}\}}.$$

See notation 2.2

Theorem 2.15. *Let $S \subset X$, then we define the cycle (or class)*

$$[S] := \sum_{s \in S} [s].$$

The set $\{[S] | S \subset X\}$ is a generating set of $C_0^{uf}(X)$ (and of $H_0^{uf}(X)$).

Proof. Take any $c \in C_0^{uf}(X)$, then c can be represented as a function $c : X \to \mathbb{Z} : x \mapsto c_x$. Look at the sets $c^{-1}(n) \subset X$ for an integer n. Note that $c_x \leq M := \|c\|_\infty$. Then we have the equality,

$$c = \sum_{n=-M}^{M} n[c^{-1}(n)],$$

where one sees that $c^{-1}(n) \subset X$. This concludes the proof for cycles. The theorem for classes is an easy consequence as

$$H_0^{uf}(X) = \frac{C_0^{uf}(X)}{B_0^{uf}(X)}.$$

□

Note that the class $[X]$ called the uniformly finite class is of particular importance to us. Let us now see a concrete example. [16, section 7]

Proposition 2.16. *The uniformly finite homology of the integers can be represented by following isomorphism*

$$H_0^{uf}(\mathbb{Z}) \cong \frac{\{b : \{[n, n+1] \quad | \quad n \in \mathbb{Z}\} \to \mathbb{Z} \quad | \quad \|\partial b\|_\infty < \infty\}}{\{b : \{[n, n+1] \quad | \quad n \in \mathbb{Z}\} \to \mathbb{Z} \quad | \quad \|b\|_\infty < \infty\}}.$$

Proof. Look at the function

$$f : \frac{\{b : \{[n, n+1] \quad | \quad n \in \mathbb{Z}\} \to \mathbb{Z} \quad | \quad \|\partial b\|_\infty < \infty\}}{\{b \quad | \quad \|b\|_\infty < \infty\}} \to H_0^{uf}(\mathbb{Z}) : \overline{b} \mapsto \overline{\partial b}.$$

Note that f is the same as ∂_1. Seeing b as a finite sum, ∂_1 is determined by following formula

$$\partial_1(\{n, n+1\}) = [n+1] - [n].$$

More concretely, f sends an element

$$\overline{b} = \sum_{n=-\infty}^{\infty} a_n[n, n+1] + C_1$$

to the element

$$f(\overline{b}) = \overline{\partial b} = \sum_{n=-\infty}^{\infty} (a_{n-1} - a_n)[n] + B_0.$$

We see indeed that, by definition, the denominator is mapped to the 0-boundaries.

Furthermore, the condition $\|\partial b\|_\infty < \infty$, ensures that this function maps into $C^{uf}(\mathbb{Z})$. Finally, this is shown to be onto. Define the inverse function g with

$$g(c)_{[n,n+1]} = \begin{cases} \sum_{i=1}^{n} c_i & \text{if } n \geq 0 \\ \sum_{i=n+1}^{0} c_i & \text{if } n < 0 \end{cases}$$

Note that what I did is simply adding infinite paths. For $n > 0$ the paths go to the right, for $n \leq 0$ they move to the left.

□

Remark 2.17 (Cohomologies on UDBG spaces). As explained in [3], we can also define cohomologies on UDBG spaces, using following ∂ functions

$$\partial : \overline{C_n}(X) \to \overline{C_{n+1}}(X)$$

where

$$\overline{C_n} = \hom_{\mathbb{Z}}(C_n, \mathbb{Z}).$$

These functions are defined by the equations

$$\partial f([x_0, ..., x_{n+1}]) = f \circ \partial([x_0, ..., x_{n+1}]) = \sum_{i=0}^{n+1} (-1)^i f([x_0, ..., \widehat{x}_i ..., x_{n+1}]).$$

These operators ∂ are a discrete version of a differential. For example for $n = 1$, we obtain the formula

$$\partial f([x_0, x_1]) = f(x_1) - f(x_0).$$

This matches our intuition of a differential along an edge.

Chapter 3

The bilipschitz structure and homology

This section is based on [16, sections 3 and 4]. However, the proof of lemma 3.11 has been clarified, modified and corrected where needed.

Definition 3.1. *A 0-cycle $c \in C_0^{uf}(X)$ is positive if and only if $c_x \geq 0$ for all $x \in X$ and* $\operatorname{supp} c$ *is quasi-dense in X. We denote the set of positive cycles as ${C_0^{uf}}^+(X)$.*

A class $[c] \in H_0^{uf}(X)$ is positive if and only if it can be represented by a positive cycle $c' \in [c]$. We denote the set of positive cycles as ${H_0^{uf}}^+(X)$.

A function $f : X \to Y$ induces a function on the uniformly finite 0-dimensional homology.

Definition 3.2. *Let $f : X \to Y$ be a quasi-isometric embedding, then f induces the function*

$$f_* : H_0^{uf}(X) \to H_0^{uf}(Y) : c = \sum_{x \in X} c_x[x] \mapsto \sum_{x \in X} c_x[f(x)].$$

This section is about the proof of next theorem.

Theorem 3.3. *We have the following isomorphism of abelian semigroups,*

$$S_{bilip}(X) \cong {H_0^{uf}}^+(X).$$

This is given by the isomorphism

$$\chi : S_{bilip}(X) \to {H_0^{uf}}^+(X) : (Y, f) \mapsto f_*[Y] = \sum_{y \in Y} [f(y)].$$

Here addition in $H_0^{uf}(X)$ is defined by simply adding two cycles and taking the class. Addition in $S_{\text{bilip}}(X)$ is defined as in remark 1.31.

In the first section, we give a proof of this theorem using the following one.

Theorem 3.4. *Let $f : X \to Y$ be a quasi-isometry with*

$$f_*[X] = [Y],$$

then there is a bilipschitz equivalence at bounded distance from f.

The second section consists of the proof of theorem 3.4.

Before we start, let us briefly illustrate this theorem on finite X. The bilipschitz structure $S_{\text{bilip}}(X)$ equals that of a singleton $S_{\text{bilip}}(\{x\})$ by remark 1.33, and in 1.32 we concluded $S_{\text{bilip}}(\{x\}) \cong \mathbb{N}$. As X is a finite metric space, the normal homology is equal to the uniformly finite homology. From theorem 2.12, we know that $H_0^{uf^+}(X) = H_0^+(X) \cong \mathbb{N}$. For a function

$$f : Y \to \{x\} : y \mapsto x,$$

we can also calculate that

$$\chi(Y, f) = f_*[Y] = \sum_{y \in Y} [f(y)] = |Y|[x].$$

It is easy to check that addition behaves well.

3.1 Proof of theorem 3.3

Let us define following function

$$\chi : S_{\text{bilip}}(X) \to H_0^{uf^+}(X) : (Y, f) \mapsto f_*[Y]. \tag{3.1}$$

To show this is an isomorphism we need three conditions to hold.

1. χ is well defined. If $z_1 = (Y_1, f_1) \sim (Y_2, f_2) = z_2$, then $\chi(z_1) = \chi(z_2)$.
2. $\operatorname{Im} \chi = H_0^{uf^+}(X)$.
3. χ is injective. (i.e. a complete invariant)

1. For the first point, let us assume we have following diagram

$$\begin{tikzcd} Y_1 \arrow[r, "h"] \arrow[dr, "f_1"'] & Y_2 \arrow[d, "f_2"] \\ & X \end{tikzcd}$$

with $f_1 \approx f_2 \circ h$, where $\approx$ denotes that they are at bounded distance from each other. By the bijectivity of h we have

$$h_*[Y_1] = [Y_2].$$

Apply f_{2*} to get

$$(f_2 \circ h)_*[Y_1] = f_{2*}[Y_2].$$

Since $d := d(f_1, f_2 \circ h) < \infty$, all edges of the form

$$(f_1(y_1), f_2 \circ h(y_1))$$

for all $y_1 \in Y_1$, have length smaller or equal to d. Hence, we can define $b \in C_1^{uf}(R_d(X))$ to be the sum of these edges

$$b = \sum_{y_1 \in Y_1} \big(f_1(y_1), (f_2 \circ h)(y_1)\big)$$

Then we obtain a boundary

$$f_{2*}[Y_2] - f_{1*}[Y_1] = \partial b.$$

Let now f_1 be a (b, C)-quasi-isometry. The 1-cycle b is uniformly bounded since for any edge (x_1, x_2) we bound the number of paths going through it by $D_C M'$, where

$$|B_d(\{x_1, x_2\})| \leq (d(x_1, x_2) + b)C =: D_C,$$
$$|f_1^{-1}(x)| \leq |B_b(x)| \leq M' \quad \forall x \in B_d(\{x_1, x_2\}).$$

Here the first bound is obtained from the bounded geometry and since f_1 is a (b, C)-quasi-isometry. The second bound holds since f_1 is a (b, C)-quasi-isometry.

2. We now show that $\operatorname{Im} \chi = H_0^{uf^+}(X)$ by splitting up in 2 inclusions.

Theorem 3.5. *Let X be a UDBG space and let χ be defined as in equation* (3.1)*, then*

$$\operatorname{Im} \chi = H_0^{uf^+}(X).$$

Proof. $\boxed{\subset}$ We know that $f_*[Y]$ is positive since $f(Y)$ is quasi-dense in X. Thus $\operatorname{Im} \chi \subset H_0^{uf^+}(X)$.
$\boxed{\supset}$ Take $c \in C_0^{uf^+}(X)$ and let $\|c\|_\infty = M$. Define

$$Y_c = \{(x, n) \quad | \quad \forall x \in X, 1 \leq n \leq c_x\} \subset X \times \{1, ..., M\},$$

and

$$\pi : Y_c \to X : (x, n) \mapsto x.$$

Then precisely c_x points of Y_c are mapped to x. Conclude that

$$c = \pi_*[Y_c] = \chi(Y_c, \pi).$$

We see that the projection of $X \times \{1, ..., M\} \to X$ is a quasi-isometry by lemma 1.21. By that same reasoning, the projection $\pi : Y_c \to X$ is a quasi-isometry. $\square$

3. We will next show that χ is injective. This mainly uses theorem 3.4, which is proven in next section. Take $z_1 = (Y_1, f_1)$ and $z_2 = (Y_2, f_2)$ in $S_{\text{bilip}}(X)$ and assume $\chi(z_1) = \chi(z_2)$. We want to prove that there is a a bilipschitz equivalence h making the diagram commute up to bounded distance.

$$\begin{tikzcd} Y_1 \arrow[r, "h"] \arrow[dr, "f_1"'] & Y_2 \arrow[d, "f_2"] \\ & X \end{tikzcd}$$

Let g_2 be the coarse inverse of f_2. Then $g_2 \circ f_1 : Y_1 \to Y_2$ is a quasi-isometry.

$$\begin{tikzcd} Y_1 \arrow[r, "h"] \arrow[dr, "f_1"'] & Y_2 \\ & X \arrow[u, "g_2"'] \end{tikzcd}$$

Further, we find

$$\begin{aligned} & \chi(z_1) = \chi(z_2) \\ \iff & f_{1*}[Y_1] = f_{2*}[Y_2]. \end{aligned}$$

Using that $(g_2 \circ f_2) \approx \mathrm{Id}_{Y_2}$, we have that [1]

$$(g_2 \circ f_1)_*[Y_1] = (g_2 \circ f_2)_*[Y_2] = [Y_2].$$

So $f = g_2 \circ f_2$ is a quasi-isometry with $f_*[Y_1] = [Y_2]$. Applying theorem 3.4, we finish the proof of theorem 3.3.

3.2 Bilipschitz equivalence on finite distance

To finish the proof of 3.3 we only need the following theorem to hold. In this section we will give a proof, similar to [16, section 4].

Theorem 3.6. *Let $f : X \to Y$ be a quasi-isometry with*

$$f_*[X] = [Y],$$

then there is a bilipschitz equivalence h at bounded distance from f.

Remember that by lemma 1.19, what we need is a bijection h at finite distance of f. The hard part here is to find an injection. Let us first briefly discuss how to make a bijection given we have an injection in each direction. Let us have 2 quasi-isometric injections,

$$h_1 : X \to Y,$$
$$h_2 : Y \to X,$$

close to resp. f and a quasi-inverse g of f (h_2 is found similarly using that we have $g[Y] = [X]$). Then we find, by the Cantor-Bernstein-Schröder theorem below, a bijection $h : X \to Y$ at bounded distance from h_1 and $h_2^{-1} : h_2(Y) \to Y$ (which are at bounded distance from each other on the domain $h_2(Y)$).

Theorem 3.7. *[Cantor-Bernstein-Schröder theorem] Let X, Y be two sets, and let*

$$\begin{aligned} h_1 &: X \to Y \\ h_2 &: Y \to X, \end{aligned}$$

be two injections, then there exists a bijection

$$h : X \to Y$$

[1] The last equality under the footnote is shown by taking

$$b = \sum_{y \in Y_2} \left((g_2 \circ f_2)(y), y\right)$$

We then obtain that

$$(g_2 \circ f_2)_*[Y_2] + \partial b = [Y_2].$$

This is similar to what we did in page 30.

such that if

$$h(x) = y,$$

then

$$h_1(x) = y \text{ or } h_2(y) = x.$$

I added the last condition on the bijection h, as this will be crucial in our discussion. It is clear from the proof in [8] that this condition holds.

Let us now show that this extra condition ensures that $f \approx h$. If $h_1(x) = y$, then $d(h_1(x), f(x))$ is bounded by $d(h_1, f)$. If $h_2(y) = x$ then similarly $d(h_2(y), g(y))$ is bounded by $d(h_2, g)$. So $g(y)$ and x are close. To show that y and $f(x)$ are close, we argue as follows. As $g(y)$ and x are close,

$$d((f \circ g)(y), f(x)) \leq Cd(h_2, g) + b,$$

as f is a (b, C)-quasi-isometry. We also have that $h(x) = y$ is close to $(f \circ g)(y)$ as $(f \circ g) \approx \mathrm{Id}_{Y_2}$. Together, this gives a uniform bound for $d(y, f(x))$

$$d(h(x), f(x)) \leq \max\{d((f \circ g), \mathrm{Id}_{Y_2}) + Cd(h_2, g) + b, d(h_1, f)\}.$$

Let us now focus on the injection. We need the following.

Lemma 3.8. *There exists an injective map at bounded distance from f.*

The problem consists to find an $r \in \mathbb{R}$ so we can insert every $x \in X$ into $B_r(f(x))$ in an injective way. For this kind of problems, we always refer to Hall's selection theorem. Intuitively, this theorem states that if you can insert every finite subset $F \subset X$ as desired, then we can insert X as desired.[2]

Theorem 3.9 (Hall's selection theorem [6])**.** *Let X, Y be countable infinite sets. Let $\alpha : X \to \mathrm{Fin}(Y)$, be a function from X to finite subsets of Y.*

Then there exists an injective $h : X \to Y$ with

$$h(x) \in \alpha(x) \quad \forall x \in X$$

if and only if for all for all finite $S \subset X$ we have

$$|S| \leq |\alpha(S)|\,, \tag{3.2}$$

where $\alpha(S) := \bigcup_{s \in S} \alpha(s)$.

A proof of this theorem is given in [6]. We use this theorem with $\alpha = B_r \circ f$. To link equation (3.2) with our condition $f_*[X] = [Y]$, we need to replace S by $f^{-1}(f(S)) \supset S$. This way equation (3.2) becomes following stronger condition

$$|f^{-1}(f(S))| \leq |B_r(f(S))|.$$

To simplify this, we will use a finite subset $S \subset Y$ instead of $f(S)$. So we derive the following consequence of the Hall theorem 3.9.

[2]Hall's selection theorem is easily seen to be equivalent with Hall's marriage theorem.

Lemma 3.10. *Let for all finite $S \subset Y$*

$$\left|f^{-1}(S)\right| \leq |B_r(S)|, \tag{3.3}$$

then there is an injection h at distance r of f.

To show that an injection h exists, it suffices to proof equation (3.3) for all finite $S \subset Y$. Let us use that $f_*[X] = [Y]$. We have

$$\forall y \in Y : f_*[X]_y = \left|f^{-1}(y)\right|,$$

where $f_*[X]_y$ is the image of y if we see $f_*[X]$ as a function $Y \to \mathbb{Z}$. Let us use following notation as in [16],

$$\sum_S c := \sum_{s \in S} c_s,$$

then we have that

$$\left|f^{-1}(S)\right| = \sum_S f_*[X].$$

So condition 3.3 reduces to

$$\sum_S f_*[X] \leq |B_r(S)|,$$

or equivalently subtracting $|S|$ on both sides,

$$\sum_S f_*[X] - \sum_S [Y] = \sum_S (f_*[X] - [Y]) \leq |B_r(S)| - |S| = |\partial_r(S)|.$$

We know that $f_*[X]$ is positive, hence $(f_*[X] - [Y])_y \geq -1$ and nonnegative on $\operatorname{Im} f$ which is C-dense in Y, by 1.25. As $f_*[X] = [Y]$ in the homology, we know that as cycles $f_*[X] - [Y] = \partial b$ for some $b \in C_1^{uf}(X)$. This brings us to next lemma.

Lemma 3.11. *Let X be a UDBG space, and $c \in C_0^{uf}(X)$ such that $c_x \geq -1$ and non-negative on a C-dense set of X. If c is exact, i.e. $c = \partial b$ for some $b \in C_1^{uf}(X)$, then we have for all finite $S \subset X$:*

$$\sum_S c \leq |\partial_r(S)|. \tag{3.4}$$

Before we prove this, let us describe these objects. The fact that $b \in C_1^{uf}(X)$ implies, by remark 2.11, that $b \in C_1^{uf}(R_\alpha(X))$ for some $\alpha > 0$. We can then, by remark 2.5, denote b as a multiset of directed edges in

$$E(\Gamma_\alpha) = E_\alpha = \{\{x, y\} \quad | \quad d(x, y) \leq \alpha\}$$

defined in example 1.26.[3] [4] This means that all edges in b have length smaller than α. Since X is of bounded geometry, the valency $\deg_{\Gamma_\alpha}(x)$ of any vertex $x \in X$ is bounded by $D_\alpha = \sup_{x \in X} B_\alpha(x)$. Furthermore, as b is the uniformly finite homology, $b_{(x,y)}$ is uniformly bounded for $\{x, y\} \in E_\alpha$ by some $M' = \|c\|_\infty$. Hence the total number of edges (counting the number of appearances) at a point is uniformly bounded by $V = D_\alpha M'$.

$$
\begin{tikzcd}
-1 \arrow[d] & -1 \arrow[d] & \\
-1 \arrow[r] \arrow[d] & -4 \arrow[r, "5"] \arrow[d] & 5 \\
1 & 1 &
\end{tikzcd}
$$

Figure 3.1: An example of c and b. The vertices form c and the edges b. The numbers stand for the multiplicities (or weight) of the vertex or edge.

We can see $c = \partial b$ graphically, we have that $c_x = \mathbf{in} - \mathbf{out}$ i.e. c_x equals the incoming edges minus the outgoing edges, as in figure 3.1. In particular, the condition $c_x \geq -1$ means $\mathbf{in} \geq \mathbf{out} - 1$. And $\sum_S c$ is the number of edges going to S minus the outgoing.

Lastly, we also remove all loops of b as this does not change ∂b. We also assume it contains no opposite pair of edges (see previous footnote).

Proof. Fix a finite subset $S \subset X$. We will make a set P of paths going to S. Set $c' = c$ and $b' = b$ and apply following procedure to c' and b' (always keeping the equality $c' = \partial b'$).

While there is an $s \in S$ with $c'_s > 0$:

- Take s.
- Take an incoming edge of s and follow it backwards.
- Keep following edges backward to make a backward path. This process either continues and gets an infinite path, or it gets stuck at some starting point x' with no incoming edges. As the last edge of our path was an outgoing edge of x', we have $c'_x = -1$.
- Add this path to P and subtract this path from b' (and hence from $c' = \partial b'$).

Example 3.12. For example let S be the right vertex with a 5 of figure 3.1. Then subtracting the path

$$
\begin{tikzcd}
-1 \arrow[d] & & \\
0 \arrow[r] & 0 \arrow[r] & 1
\end{tikzcd}
$$

We obtain:

$$
\begin{tikzcd}
0 & -1 \arrow[d] & \\
-1 \arrow[d] & -4 \arrow[r, "4"] \arrow[d] & 4 \\
1 & 1 &
\end{tikzcd}
$$

[3]A multiset (bag or mset) is a set allowing repetition.

[4]Note that the edges taken a negative amount of times, can be reversed. The reversed edge will be taken a positive amount of times. Then we can add them that positive number of times to the multiset.

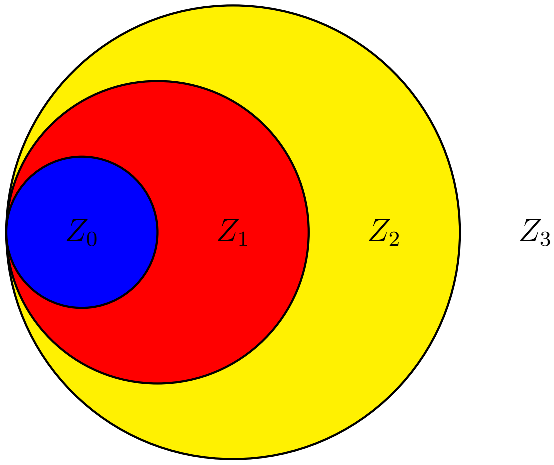


Figure 3.2: The zones Z_i where the paths of P_i start.

If we keep subtracting paths towards S like in the procedure of the proof, we end up with,

$$\begin{array}{ccc} 0 & 0 & \\ & & \\ -1 & -1 & 0 \\ \downarrow & \downarrow & \\ 1 & 1 & \end{array}$$

Take now $r > C$. We divide P into 4 sets depending on the starting point of a path p as illustrated in figure 3.2.

- P_0 contains paths starting in $Z_0 = S$.
- P_1 contains paths starting in $Z_1 = \partial_C S$.
- P_2 contains paths starting in $Z_2 = \partial_r S - Z_1$.
- P_3 contains paths starting outside $\partial_r S$ (in Z_3) and infinite paths.

Looking at the construction of P we see that

$$\sum_S c \leq |P_1| + |P_2| + |P_3|\,. \tag{3.5}$$

This equation holds because every path $p \in P_0$ has no net contribution to $\sum_S c$ and every other path $p \in P_1 \cup P_2 \cup P_3$ adds 1. Furthermore, at the end of the procedure we have $\sum_S c' \leq 0$.

We will now bound the 3 terms of the right hand side of equation (3.5) separately.

Bound P_1. First we bound P_1. Note that Z_1 has finite size, and there can only be one path starting at every $x \in Z_1$ with $c_x = -1$. It follows that

$$|P_1| \leq |Z_1^-|, \quad \text{where } Z_1^- := \{x \in Z_1 | c_x = -1\}.$$

Denote $Z_1^+ = Z_1 - Z_1^-$ the set of all elements with nonnegative image and similarly S^- and S^+ for all $S \subset X$.

Bound P_2. We now proceed with P_2. This is bounded by following lemma.

Lemma 3.13. *There exists an $A > 1$ such that*

$$A\,|P_2| \leq \max(\left|\partial_{r+C}(S)^+\right|, \left|Z_2^-\right|).$$

Proof. Let the injective function

$$\sigma : P_2 \to Z_2^-,$$

assign a path p to its starting point x. Let

$$g : Z_2^- \to B_C(Z_2)^+,$$

assign a starting point to a nonnegative element on a distance less than C. This is possible since X^+ is C-dense. Note this function is not injective. But $|g^{-1}(x)|$ is bounded by $B_C(x) \leq \sup_{y\in X} B_C(y) = D_C$, due to the fact that X has bounded geometry. Composing the two functions gives

$$g \circ \sigma : P_2 \to B_C(Z_2)^+.$$

We hence get the following inequalities,

$$\begin{aligned} |P_2| &\leq \left|Z_2^-\right|, \\ |P_2|\,/D_C &\leq \left|B_C(Z_2)^+\right| \leq \left|\partial_{r+C}(S)^+\right|. \end{aligned}$$

Sum these two, and fix $A = 1 + 1/D_C$. We get,

$$A\,|P_2| \leq \max(\left|\partial_{r+C}(S)^+\right|, \left|Z_2^-\right|).$$

□

Bound P_3. And finally, $|P_3|$ is bounded by next lemma.

Lemma 3.14. *For all $r \geq C + \alpha$, we have*

$$|P_3| \leq \frac{V\,|Z_2|}{\left\lfloor \frac{r-C}{\alpha} \right\rfloor}.$$

Here V is the maximum number of (weighted) edges used by b containing a specific vertex.

Proof. As edges have length at most α, every path in P_3 passes by at least $\left\lfloor \frac{r-C}{\alpha} \right\rfloor$ vertices in Z_2. The number of (weighted) edges at a vertex is bounded by V. So the number of paths is bounded as follows,

$$\left\lfloor \frac{r-C}{\alpha} \right\rfloor |P_3| \leq V\,|Z_2|.$$

□

We now end the proof of 3.11. Set r large enough so that,

$$\left\lfloor \frac{r-C}{\alpha} \right\rfloor \geq \frac{AV}{A-1}, \tag{3.6}$$

and $r' = r + C$. Fix $S \in X$. We now look at two distinct cases.

If $|P_2| + |P_3| \leq A\,|P_2|$, then we have,

$$|P_1| \leq \left|Z_1^-\right|,$$

and by 3.13,

$$|P_2| + |P_3| \leq A\,|P_2| \leq \max(\left|\partial_{r+C}(S)^+\right|, \left|Z_2^-\right|).$$

Hence,

$$\sum_S c \leq \left|Z_1^-\right| + \max(\left|\partial_{r+C}(S)^+\right|, \left|Z_2^-\right|).$$

As $\partial_{r+C}(S)$ contains the two disjoint unions $Z_1^- \sqcup \partial_{r+C}(S)^+$ and $Z_1^- \sqcup Z_2^-$. We can confidently add both bounds and conclude that,

$$\sum_S c \leq |\partial_{r+C}(S)| = |\partial_{r'}(S)|\,.$$

Let now $|P_2| + |P_3| \geq A\,|P_2|$. We make following calculations to bound the left hand side $|P_2| + |P_3|$ by $|P_3|$.

$$\begin{aligned} & |P_3| \geq (A-1)\,|P_2| \\ \iff & |P_2| \leq \frac{|P_3|}{A-1} \\ \iff & |P_2| + |P_3| \leq |P_3| \left(\frac{1}{A-1} + 1\right) = \frac{A\,|P_3|}{A-1}. \end{aligned}$$

By lemma 3.14 and (3.6),

$$\frac{A\,|P_3|}{A-1} \leq \frac{AV}{A-1} \frac{|Z_2|}{\left\lfloor \frac{r-C}{\alpha} \right\rfloor} \leq |Z_2|\,.$$

Conclude that,

$$\begin{aligned} \sum_S c &\leq |P_1| + |P_2| + |P_3| \\ &\leq |Z_1| + |Z_2| = |\partial_r(S)| \\ &\leq |\partial_{r'}(S)|\,. \end{aligned}$$

In the two cases, r' satisfies equation (3.4). $\square$

We can thus find an injection as stated by lemma 3.8. By the argument at the beginning of the section, one finds a bijection at bounded distance of f. This bijective quasi-isometry will be a bilipschitz equivalence, by lemma 1.19. The whole of this chapter proves theorem 3.3.

Chapter 4

Bilipschitz equivalences among trees

In this short chapter, based on [15, pages 194-200], we will see that all trees with uniformly bounded valencies larger than 3, are bilipschitz equivalent.

Theorem 4.1 (Bilipschitz equivalence of nonamenable trees)**.** *Let T, T' be two trees with uniformly bounded valencies.*[1] *Let all valencies be equal or larger than 3, then $T \sim_{Bilip} T'$.*

This will be very useful throughout the next two chapters. For most important theorems, we first construct a spanning tree of a Cayley graph. Theorem 4.1 allows us to state that this spanning tree is bilipschitz equivalent to a regular spanning tree T_k for any desired k.

Another important theorem for the next two chapters is theorem 4.6. This can be seen as a dictionary between geometry and graph theory. It translates *There is a translation-like action $H \curvearrowright G$* to *There exists a Cayley graph* $\mathrm{Cay}(G)$ *allowing a subgraph* $\Phi \cong \bigsqcup \mathrm{Cay}(H)$. We also see how to deal with bilipschitz equivalent graphs in this context using the construction of Γ_r from example 1.26.

For many reasonings, we will speak of *the* root v_0 of a tree. This root can be chosen arbitrarily, however when speaking of $\mathbb{F}_k$, we assume the root to be the identity element e. Just as throughout the thesis, we always represent $\mathbb{F}_k$ as a $2k$-regular tree by using its defining generating set.

Definition 4.2 (Regular trees)**.** *Let $k \in \mathbb{N}$, we define the k-regular tree T_k as the tree where each vertex has valency k.*

Next we recall the definition of uniformly bounded valencies on a graph. This is the analogous of the bounded geometry in UDBG spaces. It is also a property present in any Cayley graph $\mathrm{Cay}(G, S)$ as all vertices here have valency equal to $|S \cup S^{-1}|$.

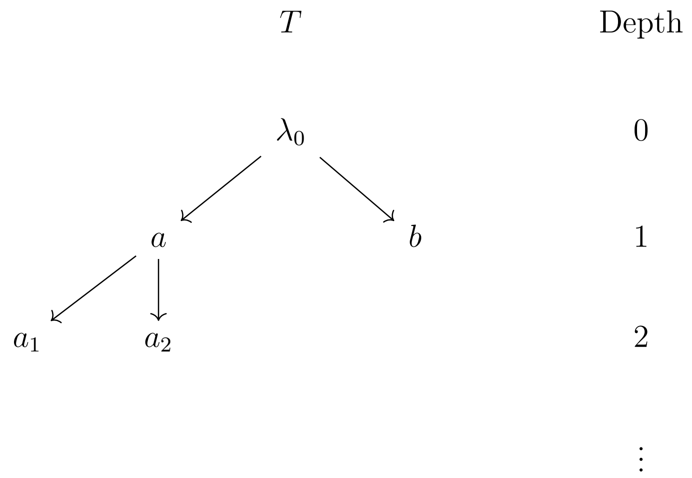
Definition 1.7 (uniformly bounded valencies)**.** *Let Γ be a graph, then Γ is said to have uniformly bounded valencies if and only if*

$$r = \sup_{g \in \Gamma} \deg g < \infty.$$

[1] Remember valencies and degrees are synonyms.

4.1 Trees

There are two main ways to represent a tree. The first one, represents a tree like one would represent a graph. Often the root is then placed at the center. The second one will place the root at the top, and have all edges pointing down like in a decision tree. We will use the second representation. A little example is shown below.[2]



Here, the depth of a vertex is equal to its distance from the root. Therefore, it is easy to identify the balls $B_T(\lambda_0, n)$ as the $n+1$ top layers of the tree (zero to n). We abbreviate notation by writing

$$B_T(n) := B_T(\lambda_0, n)$$

Similarly, we will use the notation

$$S_T(n) := B_T(n) - B_T(n-1)$$

to speak of the n-th layer.

Let us introduce some concepts involving trees from [13].[3] Although, there are a lot of them, most become intuitive when considering a tree as a family tree. Every vertex (or node) can have a certain number of *children* or no children at all (for the trees we consider, every vertex has at least two children). Every vertex also has a unique *parent*, except the root, which has none. Furthermore, two nodes that share the same parent are *siblings*.

Note that this author defines the degree of a vertex as follows: "The *degree* of a node is the number children it has." This is inconsistent with the rest of our thesis. Herefore, we define the number of children of a vertex a as $|\operatorname{Children} a|$ and the total number of neighbours as $\deg a$. We then have

$$|\operatorname{Children} \lambda_0| = \deg \lambda_0,$$

[2]It is important to remark that we handle infinite trees. In these simplified drawings, vertices that have no children on the drawing should be assumed to have children (and even an infinite number of descendants). However, if children are drawn for some vertex, then they are all drawn.

[3]I would like to notice that this source is rather hard to find. Luckily, any course notes on (decision) trees will satisfy.

for the root λ_0 of a tree, and

$$|\operatorname{Children} a| = \deg a - 1,$$

for all other vertices a. For example in previous tree T we have $|\operatorname{Children} a| = 2$ and $\deg a = 3$.

Next, we define the following partial order $\preceq$ on $V(T)$. For $u, v \in V(t)$ we say $u \preceq v$ if the unique shortest path from v_0 to v traverses u. [4] If $u \preceq v$ we say v is a *descendant* of u and u is an *ancestor* of v. Note that $u \preceq u$ for all $u \in V(T)$

A last important concept is that of a rooted subtree.

Definition 4.3 (Rooted subtree)**.** *Let T be a tree and $a \in V(T)$ be any vertex. Then we define the rooted subtree T_a of T in a as the descendants of a and a, and all edges in T between two such vertices.*

In previous example, we would obtain following subtree T_a rooted in a:

$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & \\
a_1 & & a_2
\end{tikzcd}$$

In a rooted subtree every vertex still maintains the same children.

We already state a first step for the proof of 4.1.

Theorem 4.4. *Let T be a tree with $\deg t \geq 3$ for all $t \in T$, then $H_0^{uf}(T) = 0$.*[5]

Sketch of proof. Look at a binary tree [6] and move all elements in $V(T)$ downwards using an edge only once as follows. For every vertex take first the edge to the left, and then always the right edge as in figure 4.1. We now prove by induction that all edges are used at most once.

Let $n \in \mathbb{N}$. Assume all edges in $E(B_T(n))$ [7] are taken by at most one vertex. Let $v \in S_T(n)$. We then see that v can only have (at most) one incoming vertex v' coming from its parent. The vertex v' will be send to the right whereas v will be send to the left. This way, every edge departing from v will only be used at most once. By analogy, this will hold for any $v \in S_T(n)$. Hence, we have proven that this holds for any edge in $E(B_T(n+1))$. By induction, we conclude that this holds for any edge in T.

Let now $S \subset V(T)$ be any subset. Denote the sum of the infinitely many edges used by all $v \in S$ as $b \in C_1^{uf}(T)$. We then see $[S] + \partial b = 0$ so that $[S] = 0$ in the homology $H_0^{uf}(T)$. As the cycles of the form $[S]$ for $S \subset V(T)$ generate $C_0^{uf}(T)$, this concludes the proof when T is a binary tree.

One can check that the same procedure works for many trees with vertices having possibly more than two children. Instead of "the" right and left edges of a vertex, one can take any two different downward edges.

$\square$

[4] or equivalently if all paths from v_0 to v traverse u.

[5] Note one could equivalently state $[T] = 0$.

[6] This is almost the 3-valent tree, only the root has valency 2.

[7] By this I mean all edges between vertices of depth smaller than n.

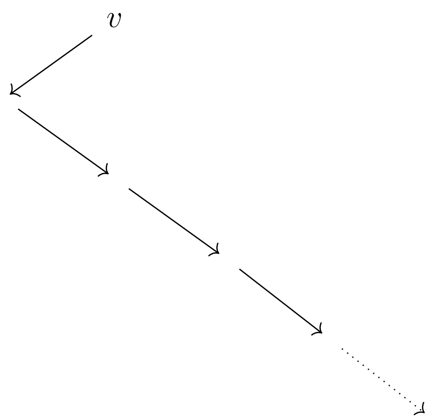


Figure 4.1: The infinite path a vertex v must take in a binary tree.

This theorem, together with theorem 3.4, tells us that being quasi-isometric and bilipschitz equivalent is the same for trees. To prove theorem 4.1, we still need to prove all trees are quasi-isometric.

Theorem 4.5. *Let $r \in \mathbb{N}$. Let T, T' be two trees with valencies between 3 and r, then*

$$T \sim_{QI} T' \iff T \sim_{Bilip} T'.$$

Proof. It is trivial that bilipschitz equivalent spaces are quasi-isometric. We prove the converse. We just have shown that $H_0^{uf}(T) = H_0^{uf}(T') = 0$, or equivalently $[T] = [T'] = 0$. If we suppose that there is a quasi-isometry $f : T \to T'$ then theorem 3.4, tells us there is a bilipschitz equivalence $h : T \to T'$ at bounded distance from f. $\square$

This theorem will be extended to nonamenable graphs in theorem 7.1.

4.2 Geometry $\to$ graph theory

We will now highlight one of the key insights of Brandon Stewards [15]. As mentioned at the beginning of the section, the goal is to translate a theorem asking for a translation-like action, to one asking for a subgraph.

Theorem 4.6. *Let H, G be finitely generated infinite groups and $H \curvearrowright G$ be a translation-like action. Let $\mathrm{Cay}(H, U)$ be any Cayley graph of H, then there exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that $\mathrm{Cay}(G, W)$ contains a spanning subgraph Φ whose connected components are isomorphic to $\mathrm{Cay}(H, U)$.*

More precisely, one defines $E(\Phi)$ as

$$\{h \cdot g, h' \cdot g | \{h, h'\} \in E(\mathrm{Cay}(H, U)), g \in G\} \in E(\Phi) \subset E(\mathrm{Cay}(G, W)) \quad \forall g \in G.$$

In particular, if the action is transitive, then $\Phi \cong \mathrm{Cay}(H, U)$.

Conversely, if there exists a Cayley graph $\mathrm{Cay}(G, W)$ containing a spanning subgraph Φ' whose connected components are isomorphic to some Cayley graph $\mathrm{Cay}(H, U)$ of H. Then there exists a translation-like action $H \curvearrowright G$.

Proof. Let $\mathrm{Cay}(G,V)$ and $\mathrm{Cay}(H,U)$ be any Cayley graphs of G and H with corresponding metrics d_V, d_U. Take the edges $\{\{e, u\cdot e\} | u \in U\}$. Call $c_u = d_V(e, u\cdot e) \in \mathbb{N}$ and $c = \max c_u$.[8] Define now

$$W = V \cup V^2 \cup ... \cup V^c.$$

Here we denote $V^2 := \{vv' | v, v' \in V\}$. I claim that $\mathrm{Cay}(G,W)$ satisfies the condition of the theorem.

One can see that $\{e, u\cdot e\} \in V^{c_u} \subset W$. It follows that

$$\{\{e, u\cdot e\} | u \in U\} \subset E(\mathrm{Cay}(G,W)).$$

By right invariance, it also holds that

$$\{\{h, u\cdot h\} | h \in H, u \in U\} = E(\mathrm{Cay}(H,U)) \subset E(\mathrm{Cay}(G,W)).$$

This proves the theorem.

For the "Furthermore" part, we take the subgraph:

$$(G, \{\{g, u\cdot g\} | g \in G, u \in U\}).$$

Let us now suppose there are a Cayley graphs $\mathrm{Cay}(G,W)$ and $\mathrm{Cay}(H,U)$ such that there is a spanning subgraph $\Phi \subset \mathrm{Cay}(G,W)$ whose connected components are isomorphic to $\mathrm{Cay}(H,U)$. Let

$$\varphi : \bigsqcup \mathrm{Cay}(H,U) \to \Phi,$$

be a graph isomorphism. Then for every edge $\{h_i, uh_i\}$ in $E(\bigsqcup \mathrm{Cay}(H,U))$, define the action by $u \in U$ as

$$u \cdot \varphi(h_i) = \varphi(uh_i).$$

As φ is a bijection, this is well defined. Remark now that the actions $\{(u\cdot) | u \in U\}$ fix the actions $\{(h\cdot) | h \in H\}$. Let $h = u_1...u_n$ then

$$h \cdot g = u_1 \cdot (u_2 ... (u_n \cdot g)).$$

Also as

$$d_W(u\cdot g, g) = 1$$

we have that

$$d_W(h\cdot g, g) \leq n.$$

In general we have that $d_W(h\cdot g, g) \leq |h|$. This completes the proof. □

We now add two theorems that deal with bilipschitz equivalences and subgraphs. First, when we have found a spanning subgraph $\Phi \subset \Gamma$, but we need to modify Φ to obtain a bilipschitz equivalent graph Φ' then we still have that $\Phi' \subset \Gamma'$ for some $\Gamma' \sim_{\mathrm{Bilip}} \Gamma$. Secondly, we can turn a Lipschitz bijection into a subgraph of some larger graph. These two theorems have a huge consequence on the rest of the paper as we do not need to speak of translation-like actions anymore.

[8]Here d_V stands for $d_{\mathrm{Cay}(G,V)}$, the number of edges necessary to get from one vertex to the other on the graph $\mathrm{Cay}(G,V)$.

Theorem 4.7. *Let Γ be a graph containing a spanning subgraph $\Phi \subset \Gamma$. Let T be a graph C-bilipschitz equivalent to Φ. Then there exists a graph Γ' such that $\Gamma' \sim_{Bilip} \Gamma$ and a spanning subgraph $\Phi' \subset \Gamma'$ such that*

$$T \cong \Phi'.$$

Proof. Let

$$f : T \to \Phi$$

be a C-bilipschitz equivalence. We can define

$$\Phi' = \Big(V(\Gamma), \{\{f(t), f(t')\} \quad | \quad \{t, t'\} \in E(T)\}\Big).$$

and

$$\Gamma' := \Gamma_C = \Big(V(\Gamma), \{\{v, v'\} \quad | \quad d_\Gamma(v, v') \leq C\}\Big).$$

These graphs satisfy what we wanted to prove. □

Theorem 4.8. *Let Γ be a graph and let Φ a graph such that, $V(\Gamma) = V(\Phi) =: X$ and the identity function on the vertex sets*

$$i : \Phi \to \Gamma : x \in X \to x,$$

is a C-Lipschitz bijection.

Then $\Phi \subset \Gamma'$ where

$$\Gamma' := \Gamma_C = (V(\Gamma), \{\{v, v'\} \quad | \quad d_\Gamma(v, v') \leq C\}.$$

Proof. It is easy to see that every edge $\{v, v'\}$ in Φ satisfies that $d_\Gamma(v, v') \leq C$. Therefore the edge will be in Γ'. □

4.3 Bilipschitz equivalence of trees

The aim of this section is to prove theorem 4.1. We will start with the case of free groups (which are isomorphic to T_{2k}) as a motivation for theorem 4.1. This case is given by theorem 4.10. Next, we give the proof of theorem 4.1.

We start with an auxiliary theorem for 4.10.

Theorem 4.9. *Let $\mathbb{F}_p, \mathbb{F}_q$ be free groups with $p, q \geq 2$ then $\mathbb{F}_p$ and $\mathbb{F}_q$ are commensurable.*

An easy consequence is theorem 4.1 for the special case of free groups.

Theorem 4.10. *Let $\mathbb{F}_p, \mathbb{F}_q$ be finitely generated free groups with $p, q \geq 2$ then $\mathbb{F}_p$ and $\mathbb{F}_q$ are bilipschitz equivalent.*

Proof. As free groups are commensurable, they are quasi-isometric. We know the graphs of these groups are the regular trees T_{2p} and T_{2q}. So their uniformly finite homology is zero. By theorem 3.4, we conclude that we can upgrade a quisi-isometry between these groups to a bilipschitz equivalence. This concludes the proof. □

To have the necessary building blocks for the proof. We start with the concept of a perimeter. This is the discreet counterpart of a surrounding surface.[9] Next, we state a technical lemma that will be used in an inductive way throughout the proof of theorem 4.1. Last, we state the proof of theorem 4.1.

Definition 4.11 (Perimeter)**.** *Let T be a tree, a perimeter $\mathcal{P} \subset V(T)$ is a finite set satisfying the following. There exists an $R \geq 0$ such that for all v for which $d(v_0, v) \geq R$, there exists a unique $p \in \mathcal{P}$ such that $p \preceq v$.*

One defines the radius of the perimeter $\mathcal{P}$ as

$$R := \max\{d(v_0, p) \quad | \quad p \in \mathcal{P}\}.$$

The easiest example of a perimeter consists of some set $S(n)$ for $n \in \mathbb{Z}_{\geq 1}$. For every vertex v with $d(v, v_0) \geq n$, the $(n+1)$-th element of the path from v_0 to v is in $S(n)$.

For the proof of theorem 4.1, perimeters will allow us to construct a quasi-isometry F inductively, "layer by layer" or perimeter by perimeter. Sending perimeters $S(n)$ for $n \in \mathbb{N}$ to growing perimeters is important to obtain a quasi-dense image. Each inductive step in the proof repeatedly uses following lemma.

Lemma 4.12. *Let T be a tree, $r \geq 3$ and the root v_0 be such that $2 \leq \deg(v_0) \leq r - 1$. Let the other vertices $v \in V(T)$ satisfy $3 \leq \deg(v) \leq r - 1$. Then there exists a perimeter $\mathcal{P}$ of radius at most r, and a function $d : \mathcal{P} \to \mathbb{N}$ such that:*

1. $1 \leq d(p) \leq |\operatorname{Children}(p)|$ *for all* $p \in \mathcal{P}$.
2. $\sum_{p \in \mathcal{P}} d(p) = r$.

Proof. We prove this by induction on r. Let $r = 3$ then $\deg(v_0) = 2$. This case is depicted below,

$$\begin{tikzcd}
 & v_0 \arrow[dl] \arrow[dr] & \\
a & & b
\end{tikzcd}$$

Set $\mathcal{P} = \{a, b\}$ and $d(a) = 2, d(b) = 1$ then this case is proven.

Let now $r \geq 4$. By Cauchy's theorem, we can write r as

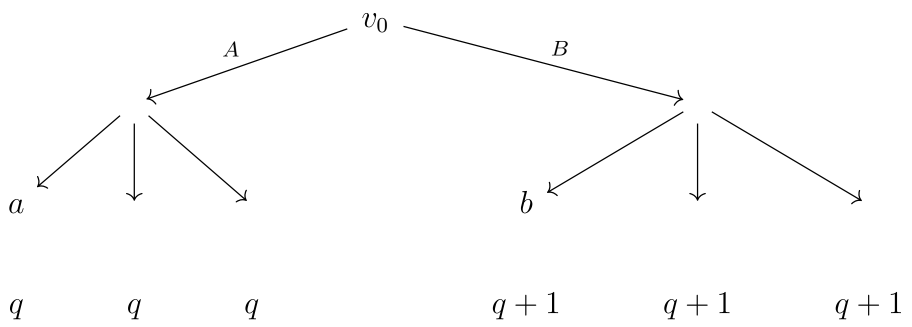
$$r = q \deg(v_0) + s,$$

with $q, s \in \mathbb{N}$ and $s < \deg(v_0)$. We divide the children of v_0 in two groups as in the picture below, B containing s children and A containing all other children.

The idea is as follows. For any vertex $a \in A$, have d assign a total of q to some perimeter $\mathcal{P}_a$ of the rooted subtree T_a of a. Similarly, for any vertex $b \in B$, have d assign a total of $q + 1$ to some perimeter $\mathcal{P}_b$ of the rooted subtree T_b of b. We illustrate this below.[10]

[9] Surrounding the origin of a space. Similarly, a perimeter should "surround" the root of the graph.

[10] In this tree a and b are children of v_0. The extra arrows with superscript A and B are drawn in order to show what the sets A and B are.



The total will then be

$$|A|q + |B|(q+1) = q(|A| + |B|) + s = q \deg v_0 + s = r.$$

Take now any $a \in A$ (or $b \in B$, as we will do the same for them). We will denote $\mathcal{P}_a$ and d_a for the parts of $\mathcal{P}$ and d lying in the descendants of a (including a itself). If $q \leq |\operatorname{Children} a|$ then we can simply assign $\mathcal{P}_a = \{a\}$ and $d_a(a) = q$.

Let now $q > |\operatorname{Children} a|$. We will then be able to apply the induction hypothesis on $q < r$ for the tree T_a rooted in a. By this hypothesis, we can find a perimeter $\mathcal{P}_a$ of T_a with radius at most q. We also find a function d_a on $\mathcal{P}_a$ such that the sum of the images is q.

We apply the same reasoning for every $a \in A$ and then for every $b \in B$ replacing q by $q+1$. We can now set

$$\mathcal{P} := \bigcup_{c \in A \cup B} \mathcal{P}_c,$$
$$d := \bigcup_{c \in A \cup B} d_c,$$

where all unions are disjoint. We already argued that the sum of the values of d is indeed r. The last thing we need to prove is that the radius of $\mathcal{P}$ is less than r.

We see by the induction hypothesis, that for every b, $\mathcal{P}_b$ has radius less or equal to $q+1$ and $\mathcal{P}_a$ has radius less or equal to q. Hence $\mathcal{P}$ is contained in the ball $B(v_0, q+2)$, this is equivalent to say that it has radius at most $q+2$. Lastly, we notice that

$$q \leq \frac{r}{\deg v_0} \leq \frac{r}{2},$$

and as $r \geq 4$,

$$q + 2 \leq \frac{r}{2} + 2 \leq r.$$

So we conclude that the radius of $\mathcal{P}$ is indeed at most r.

$\square$

We can now proceed with the proof of theorem 4.1 that all trees with valencies between 3 and r for any $r \in \mathbb{N}$,[11] are bilipschitz equivalent.

Proof of Theorem 4.1. By uniform boundedness of valencies, we can set $r = \sup_{t \in T \cup T'} \deg t < \infty$. We show that T is bilipschitz equivalent to the regular tree T_M, where $M = r + 1$.

[11] In other words, with uniformly bounded valency.

By analogy the same will follow for T' and by composition of bilipschitz equivalences, we obtain that T and T' are bilipschitz equivalent.

The proof starts with the construction of a function $F : T_M \to T$. Next, we will show that F is a quasi-isometry. Lastly, we use theorem 3.4 to show that both trees are bilipschitz equivalent.

Construction of F

First set $F(t_0) = \lambda_0$. Then, we define F on $S_{T_M}(n)$ by induction on $n \in \mathbb{Z}_{\geq 1}$. Assume F is defined on $B_{T_M}(n-1)$. Take any $y \in F(S_{T_M}(n-1))$ and assume the following conditions

I. $|F^{-1}(y)| \leq |\,\mathrm{Children}\, y|$.

II. $F^{-1}(y) = \{v_1 ... v_k\}$ consists of siblings.

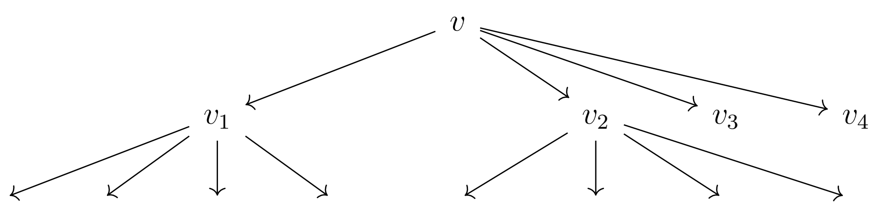
III. $F(S_{T_M}(n-1))$ is a perimeter.

By the first condition, it is possible to partition Children(y) into nonempty sets A_{v_i}. We next assign Children(v_i) to the descendants of A_{v_i}. Doing this for all i and then for all other $y \in F(S_{T_M}(n-1))$, will define F on $S_{T_M}(n)$.

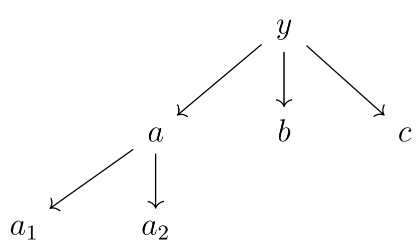
Example of the construction

Let us illustrate (an inductive step of) the construction where $M = 4$. Below we sketch (parts of) the trees of our example $T_M = T_4$ and T.

T_M



T



Let us assume $F^{-1}(y) = \{v_1, v_2\}$, $A_{v_1} = \{a, b\}$ and $A_{v_2} = \{c\}$. To determine the image of Children(v_1), take the subtree $T_{A_{v_1}}$:[12]

[12]For the image of Children(v_2), omit y and directly take the subtree rooted in c. Basically, add y iff $|A_{v_i}| = 1$.

$$
\begin{tikzcd}
 & & y \arrow[dl] \arrow[d] \\
 & a \arrow[dl] \arrow[d] & b \\
a_1 & a_2 &
\end{tikzcd}
$$

We can define a perimeter $\mathcal{P}$ and a function d as in lemma 4.12. For example, we can take $\mathcal{P} = \{a_1, a_2, b\}$ and $d(a_1) = 1$, $d(a_2) = 2$ and $d(b) = 1$.

$$
\begin{tikzcd}
 & & y \arrow[dl] \arrow[d] \\
 & a \arrow[dl] \arrow[d] & 1 \\
1 & 2 &
\end{tikzcd}
$$

We now simply assign two children of v_1 to a_2, one to a_1 and one to b. [13]

If we now apply the same strategy everywhere, we define F on the n-th layer.

Formal construction

Let

$$
\begin{aligned}
F^{-1}(y) &= \{v_1, ..., v_k\}, \\
\text{Children}(y) &= \{\alpha_1, ..., \alpha_l\},
\end{aligned}
$$

and let $\{A_{v_1}, ..., A_{v_k}\}$ be any partition of Children(y) with A_{v_1} nonempty for all i.

We determine the image of Children(v_1). The others are determined analogously. For the image of Children(v_1), take the subtree $T_{A_{v_1}}$. If $|A_{v_1}| = 1$, say $A_{v_1} = \{\alpha_1\}$, then $T_{A_{v_1}} = T_{\alpha_1}$ i.e. the rooted subtree in α_1. Otherwise, $V(T_{A_{v_1}})$ consists of the vertices y, A_{v_1} and the descendants of A_{v_1}.

We can now find a perimeter $\mathcal{P} = \{c_1, ..., c_m\} \subset V(T_{A_{v_1}})$ and a function $d : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{m} d(c_i) = M$. We now simply assign $d(c_i)$ children of v_1 to c_i for all $1 \leq i \leq m$. This way, we assign all children of v_1.

We now apply the same procedure for all other $v_i \in F^{-1}(y)$. This whole procedure is then applied to all $y' \in F(S(n-1))$. This way we define F on the n-th layer. We do this inductively on $n \in \mathbb{Z}_{\geq 1}$ to define F completely.

We only need to check that the three conditions above are satisfied, where y is now replaced by any child v_i of y. By analogy it will then hold for all the elements v in the perimeter $S_{T_M}(n)$. The first condition follows immediately from the way we chose our perimeter using lemma 4.12. The second one holds as only children of v_i are mapped to $\mathcal{P} \subset T_{A_{v_i}}$. For the third one, we remark that if we had a perimeter containing y, then replacing y by a perimeters in T_y, we still get a perimeter.

F is a quasi-isometry

Using 1.25, we prove that F is a quasi-isometry by proving it is a quasi-isometric embedding that is quasi-dense. We need to show that $d_{T_M} = d(u, v)$ and $d_T = d(F(u), F(v))$ can

[13] The algorithm of the proof of 4.12 would give $\mathcal{P} = \{a, b\}$ and $d(a) = d(b) = 2$. Anyway, both solutions for $\mathcal{P}$ and d have desired properties.

be bounded by a linear function of each other, independent of $u, v \in V(T_M)$. To bound d_T, I claim that

$$d_T \leq M d_{T_M}. \tag{4.1}$$

This follows from the condition on the radius of the perimeters. In the example, we can see $d(w, v_1) = 1$ and $d(F(w), F(v_1)) = d(a_1, y)$. This last distance equals two in the example, but by the condition on the radius of the perimeter in lemma 4.12, we know that, in general, a_1 must lie in $B(y, r)$. It follows that $d(a_1, y) \leq r$. For the descendants of elements like c that are alone in the sets A_{v_i}, we can only tell the perimeter $\mathcal{P}_c$ is in $B(c, r) \subset B(y, r+1) = B(y, M)$. Hence we have proven equation (4.1) if $d(u, v) = 1$. Note that it is sufficient to show relation (4.1) for all u and v being neighbours, as it then easily follows for all u, v by the triangle inequality.

Let us now proof following bound

$$d_{T_M} \leq d_T + M + 1, \tag{4.2}$$

by dividing into three situations depending on u, v. We can further assume without loss of generality that u is closer to the root than v or just as close as v.

1. By the construction of F, one can see that if $u \preceq v$,

$$d_{T_M} \leq d_T.$$

To see this, look at the following illustration, we see that an edge on the left hand side corresponds to between 1 and M edges on the right hand side.

$$\begin{tikzcd}
u \arrow[d] & & F(u) \arrow[dd, dotted] \\
u_1 \arrow[d, dotted] & & \\
v & & F(u_1) \arrow[d, dotted] \\
& & F(v)
\end{tikzcd}$$

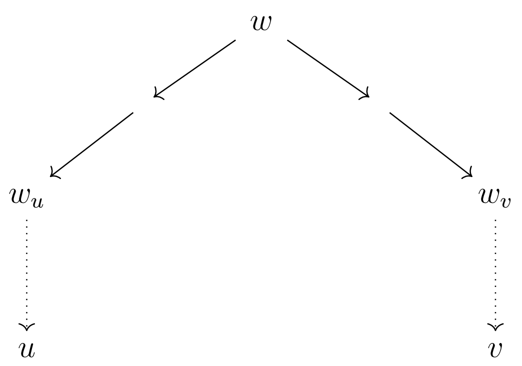
2. Let us now assume we have the following situation, where the first common ancestor w of u, v is the parent of u.

$$\begin{tikzcd}
& w \arrow[dl] \arrow[ddr, dotted] & \\
u & & \\
& & v
\end{tikzcd}$$

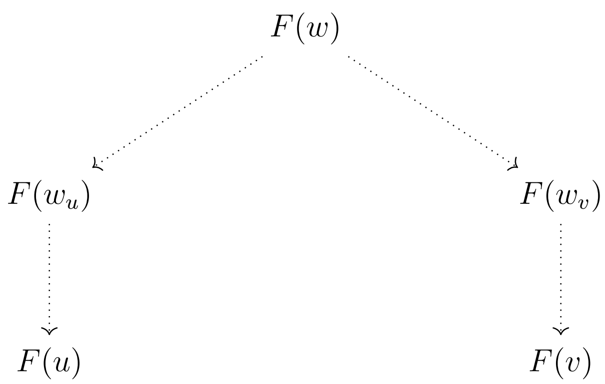
Here we calculate that

$$\begin{aligned}
d(u,v) &= 1 + d(w,v) \\
&\leq 1 + d(F(w),F(v)) \\
&\leq 1 + d(F(u),F(v)) + d(F(w),F(u)) \\
&\leq 1 + d(F(u),F(v)) + M \\
&= d(F(u),F(v)) + M + 1.
\end{aligned}$$

3. If none of the previous cases can be applied to u, v then we know we have following drawing.[14]



In this case, $F(w_u)$ and $F(w_v)$ are different as w_u, w_v are not siblings. We hence get the following diagram.



Here the two upper dotted lines represent at least one edge. We can now bound all parts and calculate that

$$\begin{aligned}
d(u,v) &= d(u,w_u) + d(v,w_v) + d(w_u,w_v) \\
&= d(u,w_u) + d(v,w_v) + 4 \\
&\leq d(F(u),F(w_u)) + d(F(v),F(w_v)) + 4 \\
&\leq \big(d(F(u),F(w_u)) + d(F(v),F(w_v)) + 2\big) + 2 \\
&\leq d(F(u),F(v)) + 2 \\
&\leq d(F(u),F(v)) + M + 1.
\end{aligned}$$

[14] Here we can have dotted lines of length 0, i.e. one could have $w_u = u$ or $w_v = v$.

So we have proven 2 bounds, given by equations (4.1) and (4.2). It follows that F is a quasi-isometric embedding.

To prove that F is a quasi-isometry, we prove that F is $(M+1)$-quasi-dense. Let $y \in T$, we need to prove that there is a $u \in T_M$ such that $F(u) \in B_T(y, M+1)$. We first argue that there is a v such that $y \preceq F(v)$. We know that $F(S_{T_M}(n))$ is a perimeter for every $n \in \mathbb{N}$. This means the perimeter $F(S_{T_M}(n))$ contains either several descendants of y (in which case we are done), or an ancestor. As y has only finitely many ancestors, this second case can only happen a finite number of times.

Now we know there is a descendant $F(v)$ of y, look at the path $(v_0 = t_0, v_1, v_2, ..., v_k = v)$. The elements y and all $F(v_i)$ must lie on the path between λ_0 and $F(v)$. Let n be such that $F(v_{n-1}) \preceq y \preceq F(v_n)$. As $d_T(F(v_{n-1}), F(v_n)) \leq M$, they both must lie inside $B_T(y, M+1)$. We conclude that F is quasi-dense and hence a quasi-isometry.

Conclusion

We have now that T and T_M are quasi-isometric. We next use theorem 4.4 to show their uniformly finite homology is zero. Theorem 3.4 then tells us that the trees are bilipschitz equivalent. This concludes the proof.

□

I end with elaborating on why theorem 4.1 fails for trees T without uniformly bounded valencies. Suppose that for all M, we can find elements t in T such that $\deg t \geq M$. Let us try to insert T into a tree T' with uniformly bounded valencies using an injective C-Lipschitz function f for some C. We can then take an element $t \in T$ such that $|B_1(t)| > D_C = \sup_{x \in T'} |B_C(x)|$. As $|B_1(t)| > |B_C(f(t))|$, it is clear that some element s of $B_1(t)$ must lie outside of $B_C(f(t))$. Therefore,

$$Cd(t, s) = C \cdot 1 < d(f(t), f(s)).$$

Therefore f cannot exist.

Chapter 5

The Von Neumann conjecture

This chapter is about theorems based upon the Von Neumann conjecture. Note that what I call the (transitive) geometric Von Neumann theorem has nothing to do with the well-known Von Neumann theorem of operator theory. In the first section, we prove the geometric Von Neumann theorem as in [16, pages 106-107]. In a second section, we elaborate on two contributions of Brandon Steward [15, page 200]. First, we restate *having a translation-like action by* $\mathbb{F}_2$ as *having a Cayley graph containing a subgraph isomorphic to (a number of copies of)* $\mathbb{F}_2$ using theorem 4.6. Secondly, we show that if there exists a translation-like action, then there exists a transitive translation-like action.[1] This we call the transitive geometric Von Neumann theorem 5.9.

A key ingredient in both the geometric Von Neumann theorem and the transitive geometric Von Neumann theorem is the bilipschitz equivalence of large trees, theorem 4.1. This theorem furthermore ensures that we can replace $\mathbb{F}_2$ by T_k or any tree with valencies uniformly bounded and strictly larger than two.

5.1 The Von Neumann conjecture

This section is dedicated to the proof of the geometric Von Neumann theorem. We make a little generalisation by proving it for a coarsely connected UDBG space X instead of a group G.

Theorem 5.1 (Geometric Von Neumann theorem). *A coarsely connected UDBG space X is nonamenable if and only if it admits a translation-like action by the free group $\mathbb{F}_2$.*

Note that such an action would partition X into its orbits $[x]$, as being in the same orbit is an equivalence relation. As the action is translation-like $(a\cdot)$ and $(b\cdot)$ are B-Lipschitz functions (as are their inverses). We can hence make a graph $\Gamma = (X, E)$ where

$$E = \left\{ \{x, s \cdot x\} \quad | \quad x \in X, s \in \{a, b, a^{-1}, b^{-1}\} \right\}.$$

The length of the edges is uniformly bounded by B. So having a translation like action by $\mathbb{F}_2$, is equivalent with having uniformly Lipschitz embedded copies of $\mathbb{F}_2$.

[1] This means one can remove "a number of copies of" and simply state there is a spanning tree isomorphic to $\mathbb{F}_2$.

Theorem 5.2. *A coarsely connected UDBG space X is nonamenable if and only if, for some $B > 0$, it admits a partition whose pieces are B-Lipschitz embedded copies of the 4-valent tree $\mathbb{F}_2$.*

The left implication $\boxed{\Leftarrow}$ is easy. As $\mathbb{F}_2$ can be embedded in X it is clear that X is nonamenable.

Let us focus on the right implication $\boxed{\Rightarrow}$. I start with a sketch of the proof. The rest of this section will then elaborate the argument. First, we construct a bilipschitz equivalence

$$h : X \times \{0, 1, 2\} \to X.$$

Then we make a directed graph with edges $(x, h(x, i))$ for $i \in \{0, 1, 2\}$. This will give a graph where every edge has 3 outgoing edges, and at most one incoming edge. The connected components will almost be 4-valence trees, the only possible difference is that they can contain at most one vertex with no incoming edge, or at most one loop. Using theorem 4.1, we show that such a connected component is bilipschitz equivalent to $\mathbb{F}_2$.

Let us now deepen our argument by proving some lemmas we used. We start constructing h.

Lemma 5.3. *There exists a bilipschitz equivalence*

$$h : X \times \{0, 1, 2\} \to X.$$

satisfying $h(x, i) \in B_C(x)\backslash\{x\} = \partial_C(x)$ for all $x \in X$ and $i \in \{0, 1, 2\}$ and for some $C > 0$.

Proof. To show that h exists, we proceed similarly as in the proof of theorem 3.4. By Hall's selection theorem, it suffices to find a $C > 0$ such that for any finite $S \subset X$,

$$|\partial_C(S)| \geq |S \times \{0, 1, 2\}| = 3|S|.$$

This is possible as X is nonamenable.

This gives an injection $h_1 : X \times \{0, 1, 2\} \to X$ at a distance C of the natural projection, satisfying

$$h_1(x, i) \neq x \quad \forall i \in \{0, 1, 2\}.$$

Similarly as,

$$|\partial_C(S \times \{0, 1, 2\}) \geq |S|,$$

one finds an injection $h_2 : X \to X \times \{0, 1, 2\}$ at distance C of the natural injection $x \to (x, 0)$, satisfying

$$h_2(x) \neq (x, i) \quad \forall i \in \{0, 1, 2\}.$$

Now using the Cantor-Bernstein-Schröder theorem gives a bilipschitz bijection h. □

We can now define the directed graph $\Gamma = (X, E)$ with $E = \{(x, h(x, i)) \mid \forall x \in X, i \in \{0, 1, 2\}\}$, and similarly, the induced subgraph $\Gamma_x = ([x], E_x)$ of every connected component $[x] \subset X$ of Γ. Define now the parent function of the graph Γ_x and its backward (or ancestral) path, similarly to how we defined parents in trees.

$$
\begin{tikzcd}
& z \arrow[dl, dashed] \arrow[dr, dashed] & \\
x & & y
\end{tikzcd}
\qquad
\begin{tikzcd}
& y \arrow[dl, dashed] \\
x &
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[dr, dashed] & \\
& y
\end{tikzcd}
$$

Figure 5.1: Possible paths between x and y.

Definition 5.4 (Parent function)**.** *Let $\Gamma = (X, E)$ be a directed graph with at most one incoming edge at every vertex. For all $x \in X$ with an incoming edge (y, x), we define the parent function as*

$$p(x) = y.$$

We say y is the father of x. If x has no incoming edge, then $p(x)$ is not defined and we call x a root of the graph.

Definition 5.5 (Backward path)**.** *Let Γ be a graph as before, then for any $x \in X$ we define the backward path from x as*

$$P(x) = (x, p(x), p^2(x), ...),$$

where $p^n = p \circ ... \circ p$ means we apply the function p n times. This is either infinite or ends at some root $p^n(x)$.

In following lemma, we exploit these concepts. This will enable us to prove our next important result, lemma 5.7.

Lemma 5.6. *We have*

$$[x] = [y]$$

if and only if their backward paths have a common vertex i.e. if for some $n, m \in \mathbb{N}$ we have

$$p^n(x) = p^m(y).$$

Note that by unicity of the backward path, these paths will be equal from this point on.

Proof. The left implication $\boxed{\Leftarrow}$ is trivial. If $p^n(x) = p^m(y)$ then both vertices x and y are connected to the common vertex $p^n(x)$. We now show that if x and y are connected then it must be one of the paths shown in figure 5.1. This is the easy to see as one can never have encounter the following in a path.

$$
\begin{tikzcd}
{} \arrow[dr, dashed] & & {} \arrow[dl, dashed] \\
& z &
\end{tikzcd}
$$

This is not possible because in this case z has two incoming vertices, which would mean it has two inverse images in h. We see that in the 3 cases of figure 5.1, the upper vertex is the common ancestor. This finishes the proof. $\square$

Lemma 5.7. *A connected component $\Gamma_x \subset \Gamma$ is isomorphic with one of the following 3 graphs.*

1. *The 4-valent tree $\mathbb{F}_2$.*

2. *A tree, with 4-valent vertices, except one which is 3-valent.*

3. *A graph with exactly one loop and 4-valent vertices.*

Proof. Look at the backward path $P(x)$. This can end up in 3 ways. It can stop at a root (case 2), end up periodically[2] (case 3), or it can continue infinitely and injectively (case 1). $P(y)$ must end the same way for all $y \in [x]$, by previous lemma.

Let $[x]$ have a root r then $P(r) = (r)$, so for all $y \in [x]$ we have $P(y) = (y, ..., r)$.

Note that a loop must be orientable in Γ_x as there is at most one incoming edge at every vertex. By orientable I mean of the form below.

$$\begin{tikzcd} z_1 \arrow[r] & z_0 \arrow[d] \\ z_2 \arrow[u] & z_n \arrow[l, dashed] \end{tikzcd}$$

Let Γ_x have a loop $(z_0, ..., z_n)$, then $P(z_0) = (z_0, ..., z_n, z_0, ...)$. Hence, all other $P(y)$ must end repeating the same loop.

The two above implications tell us that Γ_x cannot have any other loops or roots than the one by which the backward path ends. Conclude that for any of the 3 ending possibilities, we end up with one of the 3 cases. □

Lemma 5.8. *For every $x \in X$, we have that Γ_x and $\mathbb{F}_2$ are bilipschitz equivalent as graphs.*

Proof. In case there is a loop, we can remove one edge from the loop.[3]

One now has 2 3-valent vertices: one root, and one edge with two outgoing edges. So in any of the cases of lemma 5.7 Γ_x can be seen as a tree with only $0, 1$ or 2 3-valent edges. By theorem 4.1 these trees are bilipschitz equivalent to $\mathbb{F}_2$. This concludes the proof. □

We have concluded the proof of theorem 5.2.

5.2 A transitive version

After the work of Kevin Whyte in [16], Brandon Steward [15] made a stronger version of this theorem in two different ways. On the one hand, he added that one could require the translation-like action to be transitive. On the other hand, he also saw that by taking a larger generating set W to make a Cayley graph of, one could request a spanning subgraph $\Phi \cong \mathbb{F}_2$ instead of a transitive translation-like action. This section is dedicated to his work.

The main result we will prove is the transitive geometric Von Neumann theorem.

Theorem 5.9 (Transitive geometric Von Neumann theorem)**.** *Let G be a finitely generated infinite group and $k \in \mathbb{Z}_{\geq 3}$, then the following statements are equivalent.*

[2] As soon as we find two equal elements in $P(x)$, it will repeat the loop gone through between these two equal elements.

[3] Note that this only increases distances by the length of the loop minus two, as any path that was using the removed edge can use the all other edges instead. Hence removing the loop still gives a metric space that is bilipschitz equivalent in a natural way.

1. *G is nonamenable.*

2. *G admits a translation-like action by T_k.*

3. *G admits a transitive translation-like action by T_k.*

Similarly to the start of this chapter, we can state this in a more elegant graph theoretic way. The respective statements are equivalent by theorem 4.6.

Theorem 5.10 (Transitive geometric Von Neumann theorem)**.** *Let G be a finitely generated infinite group and $k \in \mathbb{Z}_{\geq 3}$, then the following statements are equivalent.*

1. *G is nonamenable.*

2. *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a subgraph $\Phi \subset \mathrm{Cay}(G, W)$ such that*
   - $V(\Phi) = V(\mathrm{Cay}(G, W))$.
   - $E(\Phi) \subset E(\mathrm{Cay}(G, W))$.
   - $\Phi \cong \bigsqcup T_k$.[4]

3. *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a subgraph $\Phi' \subset \mathrm{Cay}(G, W)$ such that*
   - $V(\Phi') = V(\mathrm{Cay}(G, W))$.
   - $E(\Phi') \subset E(\mathrm{Cay}(G, W))$.
   - $\Phi' \cong T_k$.

A more elegant way to state previous theorem is the following.

Theorem 5.11 (Transitive geometric Von Neumann theorem)**.** *Let G be a finitely generated infinite group and $k \in \mathbb{Z}_{\geq 3}$, then the following statements are equivalent.*

1. *G is nonamenable.*

2. *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a spanning subgraph $\Phi \subset \mathrm{Cay}(G, W)$ such that $\Phi \cong \bigsqcup T_k$.*

3. *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a k-regular spanning tree $\Phi' \subset \mathrm{Cay}(G, W)$.*

We already have proven most of theorem 5.11 (or equivalently 5.9). The first two statements are equivalent due to the geometric Von Neumann theorem 5.2, the implication from the third to the second is trivial. The only missing part is from the second to the third.

Lemma 5.12. *Let G be a finitely generated infinite group and $k \in \mathbb{Z}_{\geq 3}$, if we have that:*

- *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a spanning subgraph $\Phi \subset \mathrm{Cay}(G, W)$ such that $\Phi \cong \bigsqcup T_k$.*

[4] By this I mean an arbitrary number of disjoint, disconnected copies of T_k.

Then it holds that:

- *There exists a Cayley graph* $\mathrm{Cay}(G,W)$ *of* G *such that there exists a* k*-regular spanning tree* $\Phi' \subset \mathrm{Cay}(G,W)$.

Proof. Assume $\mathrm{Cay}(G,W)$ and Φ such as stated in the theorem. The idea of the proof is to add edges to Φ to turn it into a spanning tree T. Next, we use theorem 4.1 to find that T is bilipschitz equivalent with a 4-valent tree $\Phi' \cong \mathbb{F}_2$.

As $\Phi \subset \mathrm{Cay}(G,W)$ is a spanning subgraph containing no loops and $\mathrm{Cay}(G,W)$ is connected, we can add edges of $\mathrm{Cay}(G,W)$ to Φ to obtain a spanning tree T. This is done by adding one by one edges that connect 2 disconnected components. We then use the axiom of choice to add an arbitrary number of edges until there is only one connected component left in T.

Now T is a spanning tree of $\mathrm{Cay}(G,W)$. As the valencies in $\mathrm{Cay}(G,W)$ are all equal to $|W \cup W^{-1}|$, those in T are uniformly bounded by $|W \cup W^{-1}|$. As any $g \in G$ has valency 4 in Φ, its valency is at least 4 in T.

We can therefore apply theorem 4.1, and conclude that

$$T \sim_{\mathrm{Bilip}} \mathbb{F}_2.$$

As illustrated in theorem 4.7, we can find a $\Phi' \cong \mathbb{F}_2$ with $V(\Phi') = G$ and $T \sim_{\mathrm{Bilip}} \Phi'$ where the natural map

$$\Phi' \to \mathrm{Cay}(G,W) : g \mapsto g,$$

is a C-Lipschitz bijection for some $C \in \mathbb{N}$.

We can again ensure that $\Phi' \subset \mathrm{Cay}(G,W')$ where

$$W' := W \cup W^2 \cup ... \cup W^C.$$

This concludes the proof. $\square$

Chapter 6

Burnside's problem

The main purpose of this chapter will be to show that a finitely generated infinite group allows a translation-like action by $\mathbb{Z}$. As we have seen in previous chapter, in graph terminology we say that we can find a Cayley graph with a subgraph who's connected components are isomorphic to (a number of copies of) $\mathbb{Z}$. As a reference I used [15, pages 187-192].

Theorem 6.1 (The Geometric Burnside problem). *Let G be a finitely generated infinite group, then there exists a Cayley graph $\mathrm{Cay}(G, S)$ of G having a subgraph Ψ such that*

- $\mathrm{Cay}(G, S)$ *and Ψ have the same vertex set. i.e. $G = V(\Psi)$.*
- *The connected components of Ψ are isomorphic to $\mathbb{Z}$.*

When G has only finitely many ends, it can have at most 2 ends due to following theorem of [10, page 264, theorem 8.2.11].

Theorem 6.2 (Number of ends of a group). *Let G be a finitely generated group with finitely many ends, then G (or any Cayley graph of G) has at most two ends.*

If this is the case, we will even go further and look for a transitive translation-like action. In graph theoretic language, we will find a subgraph Φ isomorphic to $\mathbb{Z}$. Such a subgraph (or spanning tree) is called a (bi-infinite) Hamiltonian path.

Theorem 6.3 (The transitive Geometric Burnside problem). *Let Γ be a connected infinite graph with uniformly bounded valencies $\deg(v) \leq D$, then Γ is bilipschitz equivalent to a graph Γ' admitting a bi-infinite Hamiltonian path if and only if Γ has at most two ends.*

Or, less generally, stated for Cayley graphs, we obtain the following version. In this chapter we proof the more general theorem 6.3.

Theorem 6.4 (The transitive Geometric Burnside problem for groups). *Let G be a finitely generated group with at most two ends, then there exists a Cayley graph $\mathrm{Cay}(G, S)$ of G allowing a bi-infinite Hamiltonian path.*

6.1 Hamiltonian paths in Cayley graphs

We start defining Hamiltonian and Eulerian paths. Our goal will then be to show theorem 6.4. For this purpose, we will use theorem 6.9 as a black box theorem. The largest part of the proof of theorem 6.4 will then consist in using theorem 6.9 to construct a path P that passes at least 1 and at most D times at each vertex. Lastly, Hall's selection theorem will help us refine this path to a Hamiltonian path P'.

We denote a path P in a graph Γ as a function $I \subset \mathbb{Z} \to V(\Gamma)$, where I is an interval. However we also see P as a subgraph of Γ. Therefore, we use the notation $V(P)$ to denote the vertices used by the path P (the image of P) and $E(P)$ to denote the edges used by P. We say $\{u, v\} \in E(\Gamma)$ is used by P if and only if $\{u, v\} = \{P(n), P(n+1)\}$ for some n such that $n, n+1$ are in the domain of P.

Definition 6.5 (Hamiltonian path). *Let Γ be a graph, a Hamiltonian path is a bijective path $P : I \subset \mathbb{Z} \to V(\Gamma)$, where I is a (possibly infinite) interval. In other words, it is a path P passing through each vertex exactly once.*

Note that we will only consider bi-infinite Hamiltonian paths. In this case $I = \mathbb{Z}$.

Definition 6.6 (Eulerian path). *Let Γ be a graph, an Eulerian path is a path P passing through each edge exactly once.*

We now state following theorem about the existence of Eulerian paths. For the proof we refer to [11, Section I.3, theorem 1.3].

Theorem 6.7 (Existence of Eulerian paths). *A countably infinite connected multigraph*[1] *Γ admits an Eulerian path if and only if the following conditions are satisfied:*

I. The degree of every vertex is even. (or infinite)

II. Γ has at most 2 ends.

III. Removing a finite subgraph $\Phi \subset \Gamma$ with even valencies, we have that $\Gamma - E(\Phi)$ has one infinite connected component.[2]

We can now prove theorem 6.4. For this purpose, rewrite it in graph theoretic version.

Theorem 6.8 (The transitive Geometric Burnside problem). *Let Γ be a connected infinite graph with uniformly bounded valencies $\deg(v) \leq D$, then Γ is bilipschitz equivalent to a graph Γ' admitting a bi-infinite Hamiltonian path if and only if Γ has at most two ends.*

Proof of theorem 6.8. $\boxed{\Rightarrow}$ Suppose there is a Hamiltonian path P' in a bilipschitz equivalent graph Γ'. If Γ has at least three ends, then so does Γ'. So we can find 3 ends A, B, C of Γ' separated by a finite set $F \subset V(\Gamma')$. We obtain a contradiction due to the following observation.

P' only passes a finite number of times in F ($|F|$ times to be precise). Hence before the first passage in F, P' can only appear in one of the three ends, say A. After the last passage in F, P' can once again, only appear in one of the three ends this could be either

[1] This is a graph allowing loops and duplicated edges.

[2] Brandon Steward [15] speaks here of an even subgraph. This term is often used to refer to another concept. Therefore, I will speak of a (sub)graph with even valencies.

again A or another, say B. In any case, P' will only have the possibility to be in C for a finite number of vertices. It is then impossible for P' to pass by the infinite number of vertices in C.

$\boxed{\Leftarrow}$ We now prove the harder implication.

Summary

As Γ is infinite,[3] Γ cannot have 0 ends, hence it has 1 or 2 ends.

We first construct a path P, traversing every vertex at least once and edge at most twice. For this we use theorem 6.7. Due to the uniformly bounded degrees, P traverses every vertex at most D times. We then use Hall's selection theorem 3.9 to find a Hamiltonian path P' such that $P'(n)$ and $P'(n+1)$ are at most $M := D+1$ edges away for any $n \in \mathbb{Z}$. Similarly, to what we do on Cayley graphs, we then define the bilipschitz equivalent graph $\Gamma' = \Gamma_{2M-1}$ as

$$(V(\Gamma), \big\{\{v, v'\} | d_\Gamma(v, v') \leq 2M-1\big\}).$$

This graph will then contain the path P'.

Find an Eulerian path P

Suppose Γ has one end. Let Λ' be the multigraph obtained from doubling every edge in Γ. Then Λ' satisfies the three conditions of 6.7. We hence find an Eulerian path P as desired. This path then uses every edge of Γ exactly two times.

Let now Γ have two ends. We want to remove a finite number of edges $E \subset E(\Gamma)$ such that $\Gamma - E$ has exactly two connected components that are infinite. Start taking a finite set E_1 such that $\Gamma - E_1$ has two infinite components C_1', C_2' (and possibly some finite components). This is possible as Γ has two ends. Next we wish to refine E_1 to connect all finite components to infinite components without connecting the two infinite components. For this we proceed in two steps.

- Define E_2 as the edges from E_1 that have an endpoint in C_1'. This way E_2 only sits around C_1'. Define C_1'', C_2'' as the infinite connected components of $\Gamma - E_2$. Note that $C_1' = C_1''$.

- Next define E as the edges of E_2 that have the other endpoint in C_2''. This way we connect all remaining finite components to C_1''.

Now E has the desired properties. Define the new infinite components of $\Gamma - E$ as C_1, C_2. (again note that $C_2 = C_2''$)

Take a vertex $w \in C_1$ adjacent to C_2 in Γ. We define $E_w := E - \{(w, v) \quad | \quad v \in V(\Gamma)\}$. We now utilise the graph $\Lambda = (X, E(\Gamma) - E_w)$. Let then $\Lambda_1, \Lambda_2 \subset \Lambda$ be the induced graphs by $C_1, C_2 \cup \{w\}$, respectively. These partition the edges of Λ.

As $\Lambda_1, \Lambda_2 \subset \Lambda$ are infinite connected subgraphs, we can now find injective paths $P_i : \mathbb{N} \to V(\Lambda_i)$ for $i \in \{1, 2\}$, such that $P_i(0) = w$.

Just as before we double edges. More specifically, we define the graph Λ' as follows.

- $V(\Lambda') = V(\Gamma)$.

- If $\{u, v\} \notin E(\Lambda)$ then $\{u, v\} \notin E(\Lambda')$.

- If $\{u, v\} \in E(\Lambda)$ and $\{u, v\} \in E(P_i)$, for some i,[4] then $\{u, v\}$ appears once in $E(\Lambda')$.

[3] and connected and locally finite

[4] Recall $\{u, v\} \in E(P_i)$ if and only if $\{u, v\} = \{P_i(n), P_i(n+1)\}$ for some natural number n.

- If $\{u,v\} \in E(\Lambda)$ and $\{u,v\} \notin E(P_i)$, for any i, then $\{u,v\}$ appears twice in $E(\Lambda')$.

As we did for Λ, we also define Λ_1' and Λ_2' as the induced subgraphs of Λ' by respectively C_1 and $C_2 \cup \{w\}$.

We now claim that Λ' satisfies the 3 conditions in 6.7. The second one is trivial. For the first one, notice that it suffices to count the edges of a vertex that are not doubled. If $v \neq w$ is not in $V(P_i)$ (the image of P_i) then it has no single edges. If $v \neq w$ is in $V(P_i)$, then it has exactly 2 single edges: one incoming and one outgoing. For w itself, we see that the only single edges it has are one in P_1 and one edge in P_2. Therefore, it has even degree as well.

For the third condition, take any finite graph $\Phi \subset \Lambda'$ with even valencies. Define the following subgraphs Φ_i of Φ
$$\Phi_i = \Phi \cap \Lambda_i'.$$
As w is the only vertex joining Λ_1' and Λ_2', this partitions $E(\Phi)$ and maintains the degrees of all vertices except w. Hence Φ_i has even valencies at all vertices except possibly w. We show that $\deg_{\Phi_i} w$ is even by noting that the sum of the valencies of a finite graph is even (two times the number of edges). So w cannot be the only vertex with odd degree in Φ_i and, therefore, w has even degree in Φ_i. As $E(\Phi_i)$ has even valency at every vertex, $\Lambda_i' - E(\Phi_i)$ has, just as Λ_i', exactly one vertex with odd valency: w. It follows, by the same argument about even valencies, that w cannot be in any finite component of $\Lambda_i' - E(\Phi_i)$. So w must lie in the infinite component[5] of $\Lambda_i' - E(\Phi_i)$.

We conclude that w must lie in both the infinite component of $\Lambda_1' - E(\Phi_1)$ and the infinite component of $\Lambda_2' - E(\Phi_2)$. We conclude that the two ends in $\Lambda' - E(\Phi)$ are connected. Hence we have only 1 infinite component. This concludes the third condition.

We can now apply theorem 6.7, to obtain an Eulerian path P on Λ'. We hence have found a path P on Γ passing by every vertex at least once and every edge at most twice.

Regardless of whether Γ has one or two ends, we have found a path P with the desired property.

Construct a Hamiltonian path P'

In short, we make a restriction
$$P|_S : S \subset \mathbb{Z} \to \Gamma$$
of P that reaches every vertex in Γ exactly once, using Hall's selection theorem. We can then define $P' : \mathbb{Z} \to \Gamma$ to be the path passing by the same vertices as $P|_S$ in the same order. For P' to be in a graph Γ' bilipschitz equivalent to Γ, we request that for all $k \in \mathbb{Z}$
$$S \cap [kM, (k+1)M) \neq \emptyset.^6 \tag{6.1}$$

This will imply that two consecutive numbers n_1, n_2 in S differ at most $2M-1$. So $P|_S(n_1)$ and $P|_S(n_2)$ lie at most $2M-1$ edges away (following the edges used by P)
$$|n_1 - n_2| \leq 2M - 1.$$
Define now $\Gamma' = \Gamma_{2M-1}$ as $V(\Gamma') = V(\Gamma)$ and
$$\{u,v\} \in E(\Gamma') \iff d_\Gamma(u,v) \leq 2M-1.$$

[5]The unicity of this component follows from the fact that Λ_i' has exactly one end. So removing a finite part $E(\Phi_i) \subset E(\Lambda_i')$ one still has one connected component.

[6]Here $[kM, (k+1)M) \subset \mathbb{R}$ denotes an interval.

Now Γ' defines a bilipschitz equivalent graph containing P'.

Recall that we had uniformly bounded valencies in Γ,

$$D := \sup_{v \in V(\Gamma)} \deg_\Gamma(v) < \infty.$$

So $\deg_\Lambda(v) \leq 2D$. Anytime P passes through v it uses two edges adjacent to v. It follows that P can pass at most D times at every vertex v. Define $M = D + 1$. Let us now make an injection $f : M\mathbb{Z} \to V(\Gamma)$ such that

$$f(kM) = P(kM + l_k), \quad 0 \leq l_k < M.$$

We show this is possible using Hall's selection theorem. For every finite subset $F \in M\mathbb{Z}$, we have a total of $|F|M$ possibilities for the integers $kM + l_k$. Out of all these options, we cannot have more than D possibilities equal. Hence we have at least

$$\frac{|F|M}{D} \geq |F|$$

choices to insert the numbers in $F \in M\mathbb{Z}$. We hence have satisfied the condition of Hall's selection theorem 3.9 for any finite subset $F \in M\mathbb{Z}$. The theorem now ensures the injection f exists.

This f clearly has the same image as the injection $P|_{S_1}$ with

$$S_1 = \{kM + l_k | k \in \mathbb{Z}\}.$$

We also clearly have that

$$S_1 \cap [kM, (k+1)M) = \{kM + l_k\} \neq \emptyset.$$

We next simply extend S_1 by adding a number $n_v \in P^{-1}(v)$ for each $v \in V(\Gamma)$ not yet in $V(P|_{S_1})$ (any choice for n_v will do). Call the new obtained set S. We have now ensured that $P|_S$ is a bijection and S satisfies equation (6.1) as S_1 does.

It should be clear that we now can find an order preserving bijection $\varphi : \mathbb{Z} \to S$ and this way define the path $P' = P|_S \circ \varphi$.

We then define the $(2M-1)$-bilipschitz equivalent graph $\Gamma' = \Gamma_{2M-1}$ as

$$(V(\Gamma), \big\{\{v, v'\} | d_\Gamma(v, v') \leq 2M - 1\big\}).$$

As 2 consequent vertices of P' are joined by at most $2M - 1$ edges along P, the graph Γ' will contain the path P'. We have hence found a Hamiltonian path P' in a graph Γ' bilipschitz equivalent to Γ. $\square$

6.2 The Geometric Burnside problem

We have already handled the groups with finitely many ends. It remains to show the geometric Burnside problem for groups with infinitely many ends. For this, the crucial part will be to show that such a group G always has an element h of infinite order. Once such an element $h \in G$ is found, it suffices to take the graph Φ with

$$E(\Phi) := \{\{g, hg\} \quad | \quad g \in G\}.$$

One will then see that every connected component $(\langle h \rangle g, \{\{h^n g, h^{n+1} g\} \quad | \quad n \in \mathbb{Z}\})$ of Φ will be isomorphic to $\mathbb{Z}$.

Theorem 6.9 (Existence of nontorsion elements)**.** *Let G be a finitely generated group with infinitely many ends then G has a nontorsion element h.*

For the proof of this theorem, we use a universal way to write down these groups. Following theorem proven in [4, Section I.8, Theorem 8.32, clause (5)] will be used as a black box theorem.

Theorem 6.10. *Let G be a finitely generated group with infinitely many ends then G is*

1. *an amalgated product $G = A *_C B$. where $C \neq A$ and $C \neq B$. Furthermore, $[A : C] \geq 3$ or,*
2. *a HNN extension $G = A*_C = \langle A, t \quad | \quad \forall c \in C : tct^{-1} = \phi(c)\rangle$ with C finite.*

Proof of theorem 6.9. We analise both cases:

1. Let $G = A *_C B$ and a, b be elements from respectively $A\backslash C$ and $B\backslash C$. Then ab a nontorsion element in G. One can see this because in the natural morphism
$$\phi : A *_C B \to A/C * B/C,$$
the elements $(aCbC)^n$ are nonzero in the free group $A/C * B/C$.
2. Let $C, C' \subset A$ be two isomorphic subgroups and let $\phi : C \to C'$ be an injective morphism. Let now $G = A*_C = \langle A, t \quad | \quad \forall c \in C : tct^{-1} = \phi(c)\rangle$ be a HNN extension. Then t is a nontorsion element. One can indeed see that we can make a morphism
$$\psi : A*_C \to \mathbb{Z}$$
fixed by the equations
$$\begin{aligned} \psi(a) &= 0 \quad \forall a \in A, \\ \psi(t) &= 1. \end{aligned}$$
This is indeed well defined as all relations are mapped to zero
$$\psi(tct^{-1}\phi(c)^{-1}) = 1 + 0 - 1 + 0 = 0.$$

This concludes the proof.

□

We can now proof the Burnside problem for groups with infinitely many ends.

Theorem 6.11. *Let G be a finitely generated infinite group with infinitely many ends. Then there exists a Cayley graph* $\mathrm{Cay}(G, S)$ *of G having a subgraph Φ such that*

- $\mathrm{Cay}(G, S)$ *and Φ have the same vertex set. i.e. $G = V(\Phi)$.*
- *The connected components of Φ are isomorphic to $\mathbb{Z}$.*

Proof. By theorem 6.9, we find a nontorsion element $h \in G$. Take now any generating set S containing h. We define following graph $\Phi \subset \mathrm{Cay}(G, S)$ as $V(\Phi) = G$ and

$$E(\Phi) = \{\{g, hg\} \quad | \quad g \in G\}.$$

The connected components of Φ will then be $(\langle h \rangle g, \{\{h^n g, h^{n+1} g\} \quad | \quad n \in \mathbb{Z}\})$ and will be isomorphic to $\mathbb{Z}$ as graphs.[7] □

Now the geometric Burnside problem 6.1 easily follows from the transitive geometric Burnside problem 6.8 and previous theorem.

6.3 Regular trees in Cayley graphs

The following two sections are based upon the work in [15, pages 201-203].

Here we summarize our work of chapters 5 and 6. Together the transitive geometric Von Neumann theorem 5.11 and the transitive geometric Burnside problem 6.4 can be summarized by following theorem.

Theorem 6.12. *Let G be a finitely generated group and $k \in \mathbb{Z}_{\geq 3}$, then*

1. *G is nonamenable if and only if there exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a regular spanning tree $\Phi' \subset \mathrm{Cay}(G, W)$*

2. *G has finitely many ends if and only if there exists a Cayley graph $\mathrm{Cay}(G, W)$ of G allowing a Hamiltonian path.*

An elegant way to summarize this, is given by following corollary.

Corollary 6.13 (Regular spanning trees of Cayley graphs)**.** *Let G be a finitely generated infinite group, then there exists a Cayley graph $\mathrm{Cay}(G, S)$ of G allowing a regular spanning tree. (Possibly 2-regular)*

Proof. By previous theorem, we know that this holds true for nonamenable groups and groups with finitely many ends. It hence suffices to show that groups with infinitely many ends are nonamenable. We prove this below. □

Theorem 6.14. *Let G be a finitely generated group with infinitely many ends. Then G is nonamenable.*

Sketch of proof. This proof can be found in [15, page 202], it furthermore extensively uses [4]. Let G have infinitely many ends. We show that $\mathbb{F}_2 \subset G$. This clearly implies that G is nonamenable.

Recall from theorem 6.10 that G can be of 2 possible forms. We analyse in separate cases.

1. **Let $G = A * B$.** By [4, proposition 4], we know that the kernel of the natural homomorphism
$$A * B \to A \times B$$

[7] Where the graph $\mathbb{Z}$ means $\mathrm{Cay}(\mathbb{Z}, \{1\})$.

is a free group with free basis $\{aba^{-1}b^{-1} \quad | \quad a \in A, b \in B\}$. Hence it suffices to find two independent basis elements to prove that $\mathbb{F}_2 \subset G$. As $|A| \geq 3$, we can find two nontrivial different elements $a_1, a_2 \in A$. we now have that

$$\mathbb{F}_2 \cong \langle a_1ba_1^{-1}b^{-1}, a_2ba_2^{-1}b^{-1}\rangle \subset G.$$

2. **Let $G = A*_C B$, with C nontrivial.** By [4, Section III.Γ.6, Lemma 6.4], one can see that if we have elements $a_1 \in A, a_2..., a_n \in A\backslash C$ and $b_1, b_2, ..., b_{n-1} \in B\backslash C, b_n \in B$ then
$$a_1b_1...a_nb_n \neq e,$$
where e is the identity element in G. Using this property, one can find two elements generating a free group in G. For more detail here, see [15, page 202].

3. **Let $G = A*_C = \langle A, t \quad | \quad t^{-1}ct = \phi(c)\rangle$,** where $\phi : C \to A$ is an isomorphism and $[A : C] \geq 3$. As
$$|A| \geq 3|C| > 2|C| \geq |C \cup C'|,$$
we can find an element $a \in A\backslash C \cup C'$. Then (again using the lemma [4, Section III.Γ.6, Lemma 6.4]) we see that
$$\mathbb{F}_2 \cong \langle tat, t^2at^2\rangle \subset G.$$

We conclude that in all cases G contains a free nonabelian subgroup. □

It is not yet known to which extend the existence of regular spanning trees holds for **all** Cayley graphs.

For nonamenable groups, it is not always the case that Cayley graphs allow a regular spanning tree. Following example illustrates this. [15, page 203, Proposition 5.9]

Example 6.15. Let $G = \mathbb{Z} * \mathbb{Z}_3$. We denote the generators of G as t, u where $\mathbb{Z} = \langle t\rangle$ and $\mathbb{Z}_3 = \langle u\rangle$. We now look at the graph $\Gamma = \mathrm{Cay}(G, \{t, u, t^{-1}, u^{-1}\})$ shown in figure 6.1. This graph looks like the 6-regular tree T_6 where the vertices are converted to triangles.

Suppose we have a regular spanning tree Λ of Γ. For Λ to attain all vertices of one triangle, it needs to use exactly 2 of the edges of the triangle. For Λ to attain all triangles, it needs to use **all** edges between the triangles. One can easily see that if this is the case, one vertex of the triangle has degree 4 and the other two have degree 3. It follows that Λ is not a regular tree. Hence this Cayley graph cannot have a regular spanning tree.

Problems

We end the section with a few problems. The following two come from [15].

1. Does every infinite Cayley graph with finitely many ends allow a Hamiltonian path?

2. Does every infinite nonamenable Cayley graph with finitely many ends allow a Hamiltonian path?

In this thesis, we add following questions related to the bilipschitz equivalences of trees, theorem 4.1.

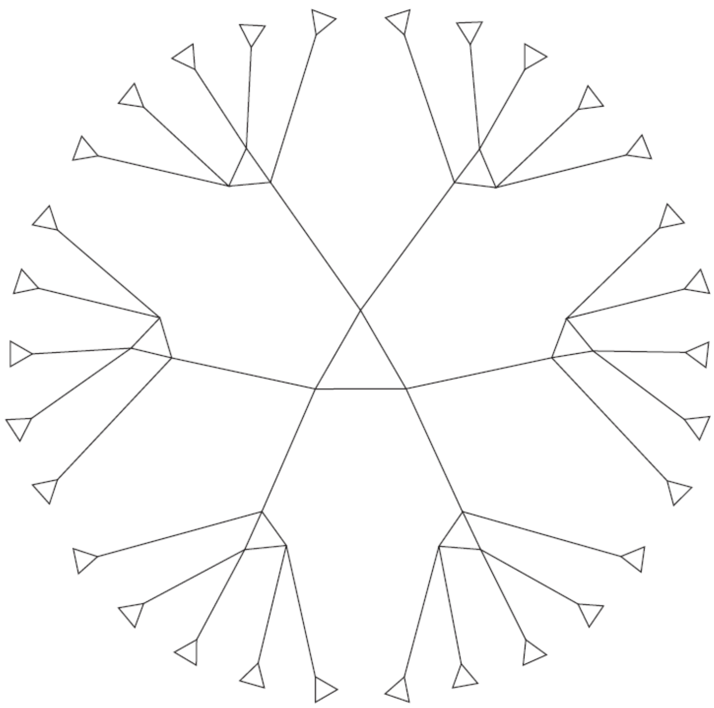


Figure 6.1: A finite portion of the natural Cayley graph $\mathrm{Cay}(\mathbb{Z} * \mathbb{Z}_3, S)$. This image is taken from [15, page 204].

1. A tree with valencies strictly larger than 2 is nonamenable. It is clear that having a finite number of 2-valent vertices we still obtain a nonamenable tree. Is there a way to refine the condition on the valencies of vertices for a tree to be (non)amenable?

2. Can theorem 4.1 be extended to all nonamenable trees? If not, to which can it be extended?

For this last question, I would suggest as a counter example any tree containing an infinite sequence of 2-valent vertices, as in figure 6.2.

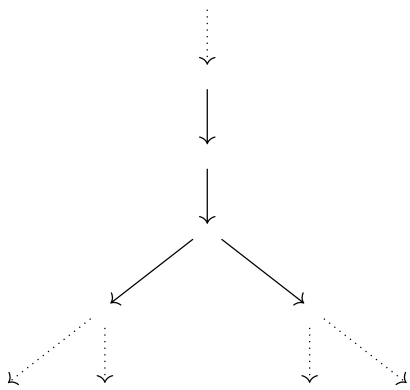


Figure 6.2: A tree consisting of a binary tree and an upper part isomorphic to $\mathbb{N}$ i.e. containing an infinite sequence of 2-valent vertices.

Chapter 7

Free products of surface groups

Like in chapters 2 and 3 at the beginning of the thesis, this last chapter focuses more on metric spaces and the uniformly finite homology rather than on graph theory. The chapter contains two parts. First, we show that a group is amenable if and only if its uniformly finite homology H_0^{uf} is nontrivial. Secondly, we apply this characterization of amenability to classify free products of surface groups into 5 quasi-isometry classes. Herefore, we use multiple covering maps of finite index.

7.1 Amenability and the uniformly finite homology

We will now discuss next theorem. In the context of a graph, it intuitively tells you that a graph $\Gamma = (X, E)$ is nonamenable if you can send every vertex x to infinity. This means that for every $x \in X$ we have a path p_x with no starting vertex and ending at x, so that every edge is only used by a uniformly bounded number of paths p_x.

Theorem 7.1. *Let X be a coarsely connected UDBG space. Then X is nonamenable if and only if $H_0^{uf}(X) = 0$.*

A proof of this theorem can be found in [2], or [16, pages 108-111]. Yet the proof in this section is a proof of my own. Before I start with a proof, let us make following claim.

Theorem 7.2. *Let X be a coarsely connected UDBG space, and let $[X] = 0$ in $H_0^{uf}(X)$. Then one can find*

- *some $b \in H_1^{uf}(R_r(X))$ such that*

$$[X] + \partial b = 0. \tag{7.1}$$

- *for every $x \in X$ a path p_x such that*

$$b = \sum_{x \in X} p_x.$$

Proof. Note that we can see b as a sum (or multiset) of edges in Γ_r as defined in example 1.26 on the space X. As X is coarsely connected, we can also suppose r to be large enough for Γ_r to be connected.

We start defining $b' = b$ and will then inductively remove paths p_x from b' similarly to the procedure at the end of chapter 3. During the procedure we use two claims.

- Claim 1: Every time a path p_x reaches a vertex x', that vertex has an outgoing edge.
- Claim 2: At every moment we have $\partial b'_x = 0$ if the path p_x is not yet removed from b' and $\partial b'_x = -1$ if the path p_x is already removed from b'.

The procedure is as follows. For $x \in X$, we start having $\partial b'_x = -1$ by claim 2. We also know $\partial b'_x = \mathbf{in} - \mathbf{out}$. Hence, we can find an edge starting in x. Take the edge (x, x'). Take now an edge starting in x' and continue forever to form the path p_x (using claim 1).

Next, we remove p_x from b'. As $\partial p_x = -[x]$, the only thing that changes for b' is that now $\partial b'_x = 0$ where it was -1.

Using the axiom of choice, we can remove all paths $\{p_x \quad | \quad x \in X\}$ from b'. At that point, we have that by claim 2

$$\partial b'_x = 0 \quad \forall x \in X,$$

or

$$\partial b' = 0.$$

It is not necessarily the case that $b' = 0$. We clearly have that, at the end, $b - b'$ equals the paths we removed during the procedure. We conclude that

$$b - b' = \sum_{x \in X} p_x.$$

If we redefine b as $b - b'$. Then this satisfies the conditions of the theorem.

We finish with proofs of the claims.

Proof of claim 1: Let us suppose that the path p_x arrives at a vertex x' with no outgoing edges i.e. $\mathbf{out} = 0$. As the path p_x has reached the vertex x', it must have used some incoming edge. We hence have $\mathbf{in} \geq 1$. However, this would imply that

$$\partial b'_x = \mathbf{in} - \mathbf{out} \geq 1.$$

This is not possible by claim 2.

Proof of claim 2: Recall that at the beginning we have

$$\partial b'_x = -1 \quad \forall x \in X$$

by equation (7.1). The proof of claim 2 can then be derived from the sentence: "As $\partial p_x = -[x]$, the only thing that changes for b' is that now $\partial b'_x = 0$ where it was -1."

□

Remark 7.3. From the proof one sees that $[X] = 0$ will imply $[S] = 0$. This is because if we can have a sum of uniformly bounded paths

$$\sum_{x \in X} p_x,$$

then the sum of paths

$$\sum_{x \in S} p_x,$$

will definitely be uniformly bounded. Hence, we have the following equivalence

$$[X] = 0 \iff H_0^{uf}(X) = 0.$$

Let us come back to theorem 7.1. I will first prove the implication $\boxed{\Rightarrow}$ using contraposition.

Lemma 7.4. *Let X be amenable then $H_0^{uf}(X) \neq 0$.*

We know that $H_0^{uf}(X)$ is generated by $\{[S] | S \subset X\}$. Hence for this lemma, it is equivalent to prove that $[S] \neq 0$ for some $S \subset X$. We will prove that $[X] \neq 0$.

Proof. Suppose $[X] + \partial b = 0$ for some $b \in C_1^{uf}(X)$. We find a contradiction by proving that b cannot be uniformly bounded, it will follow that $[X]$ is not a 0-boundary in the uniformly finite homology $H_0^{uf}(X)$, and so $[X] \neq 0$. Let r be such that $b \in C_1^{uf}(R_r(X))$, further denote the graph $\Gamma_r = (X, E_r)$ where

$$\{x, y\} \in E_r \iff d_X(x, y) \leq r.$$

Recall that we can see $b \in C_1^{uf}(R_r(X))$ as a multiset of directed edges in E_r as on page 34.

By theorem 7.2, b must have a path p_x in Γ_r starting in x and having no ending point, for each $x \in X$.[1] It is clear that the infinite path p_x must leave any finite set at some point. One can assume, without loss of generality, that two paths do not pass the same edge in opposite direction, see remark 7.6 below.

Recall from definition 1.40 that as X is amenable, it contains a Følner sequence, i.e. there is a sequence $(F_n)_{n \in \mathbb{N}}$ where $F_n \subset X$ such that for all r (so in particular our r)

$$\lim_{n \to \infty} \frac{|\partial_r F_n|}{|F_n|} = 0. \tag{7.2}$$

For every $x \in F_n$, p_x leaves F_n at least once, and goes to some point $y \in \partial_r F_n$. Since p_x does that for every vertex $x \in F_n$, b contains at least $|F_n|$ edges from $|F_n|$ to $|\partial_r F_n|$.[2] Then the average number of times a path uses a vertex $y \in \partial_r F_n$ is at least $|F_n|/|\partial_r F_n|$. By equation (7.2), this grows to infinity as n grows. Hence the maximal number of times a vertex $y \in \partial_r F_n$ is used diverges to infinity as $n \to \infty$.

We conclude that b is not uniform bounded as follows. By the bounded geometry of X, we can bound the number of unweighted edges from F_n to y by

$$B_y(r) < D_r$$

where D_r is independent of y. So on average an edge is taken more than

$$\frac{|F_n|}{|\partial_r F_n| \, D_r}$$

times. This diverges to infinite as $n \to \infty$. As this is an average, there must be some edge e_n taken at least this number of times. So b cannot be uniformly finite. This concludes the proof. $\square$

[1] See the remark 7.5 below for an alternative proof without this claim.

[2] Counting multiplicity (and sign).

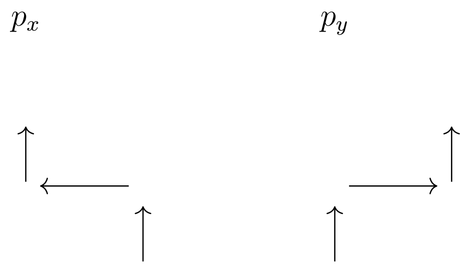
Remark 7.5. It is possible to avoid the claim that $b = \sum_{x \in X} p_x$. For this note that for $[X] + \partial b = 0$ to hold b must have $|F_n|$ edges more going from F_n to $\partial_r F_n$ than there are from $\partial_r F_n$ to F_n (counting the number of times an edge is used). Indeed one must have that

$$\sum_{F_n} ([X] + \partial b) = 0.$$

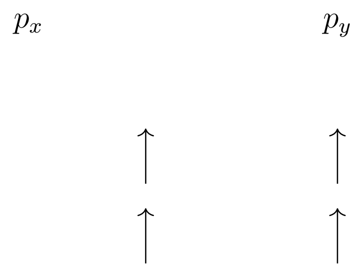
Furthermore, we know that $\sum_{F_n} [X] = |F_n|$ and that $\sum_{F_n} \partial b$ equals the incoming edges minus the outgoing edges of b. We hence arrive at the same conclusion that there are at least $|F_n|$ edges from F_n to $\partial_r F_n$.

Remark 7.6. If two paths p_x, p_y use the same edge in opposit direction. Delete the use of that edge and switch the the remainder of the paths between p_x and p_y.

For example, if p_x and p_y look like



Then we switch them to



We now prove the other implication of theorem 7.1.

Lemma 7.7. *Let X be nonamenable then $H_0^{uf}(X) = 0$.*

Proof. By the transitive Von Neumann theorem 5.11, we have a C-Lipschitz bijection

$$\phi : \mathbb{F}_2 \to X.$$

By theorem 4.4, we have that $[\mathbb{F}_2] = 0$ and $H_0^{uf}(\mathbb{F}_2) = 0$. Let $[\mathbb{F}_2] + \partial b = 0$ as cycles. Using

$$b' = \sum_{(g,h) \in \operatorname{supp} b} b_{(g,h)} \big(\phi(g), \phi(h)\big),$$

we have that b' uses edges of length at most C and that

$$[X] + \partial b' = 0$$

$\square$

As we will see, theorem 7.1 combined with theorem 3.4, has some important implications. Firstly, we can see that being bilipschitz equivalent and being quasi-isometric means the same thing for nonamenable spaces.

Theorem 7.8. *Let X, Y be two nonamenable coarsely connected UDBG spaces, then*

$$X \sim_{QI} Y \iff X \sim_{Bilip} Y.$$

Proof. It is trivial that bilipschitz equivalent spaces are quasi-isometric. We prove the converse. We just have shown that $H_0^{uf}(X) = H_0^{uf}(Y) = 0$, or equivalently $[X] = [Y] = 0$. If we suppose that there is a quasi-isometry $\phi : X \to Y$ then theorem 3.4, tells us there is a bilipschitz equivalence $h : X \to Y$ at bounded distance from f. □

Theorem 7.9. *Let G_1, G_2 be nonamenable finitely generated groups and let $G_1 \sim_{QI} G_2$. Then for any finitely generated group G, we have*

$$G_1 * G \sim_{Bilip} G_2 * G.$$

Proof. Let $\phi : G_1 \to G_2$ be a quasi-isometry. By previous theorem, we obtain a C-bilipschitz equivalence

$$h : G_1 \to G_2.$$

Remember that being a bilipschitz equivalence is the same as being quasi-isometric and bijective. We will now construct a bilipschitz equivalence

$$f : G_1 * G \to G_2 * G.$$

Fix f by the following properties

$$\begin{cases} f(g) & = g & \forall g \in G \\ f(\gamma) & = h(\gamma) & \forall \gamma \in G_1 \\ f(g\gamma) & = f(g)f(\gamma) & \forall g \in G, \gamma \in G_1 \\ f(\gamma g) & = f(\gamma)f(g) & \forall g \in G, \gamma \in G_1 \end{cases}$$

To prove that f is indeed a bilipschitz equivalence, see that the bijectivity follows from the bijectivity of h. Furthermore, showing that f is a Lipschitz (or quasi-isometric embedding) is sufficient, since f^{-1} will be a quasi-isometric embedding by analogy. To finish the proof we will need following lemma.

Lemma 7.10. *Let $a = g_1\gamma_1 \dots g_n\gamma_n$ be an element of $G_1 * G$ with $g_i \neq e \neq \gamma_i$ for all i (except possibly g_1 and γ_n). Let $\| \cdot \|$ denote the word metric on $G_1 * G$ made composing the word metrics of G and G. Then*

$$\|a\| = \sum_{i=0}^{n} \|g_i\| + \|\gamma_i\|.$$

Take now any $a = g_1\gamma_1 \dots g_n\gamma_n$ as above. Then

$$\begin{aligned} \|a\| &= \sum_{i=0}^{n} \|g_i\| + \|\gamma_i\| \\ \|f(a)\| &= \sum_{i=0}^{n} \|g_i\| + \|h(\gamma_i)\| \\ &\leq \sum_{i=0}^{n} \|g_i\| + C\|\gamma_i\| \\ &\leq \max(1, C)(\sum_{i=0}^{n} \|g_i\| + \|\gamma_i\|) \\ &\leq \max(1, C)\|a\|. \end{aligned}$$

So f is a $\max(1, C)$-Lipschitz function.

□

To see the power of this theorem, remark that in the amenable case this is far from true. For instance,

$$\mathbb{Z}_2 * \mathbb{Z}_2 \sim_{QI} D_\infty \not\sim_{QI} e * \mathbb{Z}_2 \sim_{QI} \mathbb{Z}_2$$

with $\mathbb{Z}_2$ denoting the group of order two and not the 2-adic group. And

$$\mathbb{Z}_2 * \mathbb{Z}_2 \sim_{QI} D_\infty \not\sim_{QI} \mathbb{Z}_3 * \mathbb{Z}_2$$

are not quasi-isometric either as

$$F_2 \cong \langle (1, 0)(0, 1), (2, 0)(0, 1) \rangle \subset \mathbb{Z}_3 * \mathbb{Z}_2.$$

So $\mathbb{Z}_3 * \mathbb{Z}_2$ grows exponentially where D_∞ only has linear growth.

7.2 Classification of free products of surface groups

Following corollary 7.16 is one of the many surprising classifications made possible by 7.9. Denote with Σ_g the fundamental group of a surface S_g of genus g, this is isomorphic to [10, appendix, page 323]

$$\left\langle a_1, b_1 \dots, a_g, b_g \middle| \prod_{i \in \{1, \dots, g\}} [a_i, b_i] \right\rangle.$$

In particular, we have that $\Sigma_1 = \mathbb{Z}^2$.

We will now talk about Euler characteristics on groups. First, we recall the Euler characteristics on manifolds. For a surface M with a triangulation with V vertices, E edges and F faces, we have the formula

$$\chi(M) = V - E + F.$$

This notion can be generalized by following definition. Note that a triangulation is in particular a simplicial d-complex with d the dimension of the triangulated manifold.

Definition 7.11 (Euler characteristic)**.** *Let M be a compact manifold of dimension d. Let $\mathcal{K}$ be a triangulation and let k_n denote the number of n-simplices. Then the Euler characteristics of M is defined as*

$$\chi(M) = k_0 - k_1 + k_2 ... + (-1)^d k_d.$$

For the Euler characteristic on groups, we give no formal definition, but just assume that $\chi(\pi_1(M)) = \chi(M)$. This implies that $\chi(\Sigma_g) = 2 - 2g$ and $\chi(\mathbb{Z}^2) = 0$. For a formal and general definition, using groupoids, I refer to a paper by Jean-Pierre Serre, [14]. For our purpose it will suffice to look at properties with free products and finite coverings.

For free products of groups, we make the following objection. Let M and N be two manifolds then we know that

$$\pi_1(M \vee N) = \pi_1(M) * \pi_1(N).$$

On the other hand, taking triangulations of M and N, we see that

$$\chi(M \vee N) = \chi(M) + \chi(N).$$

Here $\vee$ denotes the wedge sum. It means that we identify one point of the manifolds M and N to connect the manifolds. This equation gives a useful calculation tool.

Lemma 7.12 (Euler characteristic of free products)**.** *Let $G = G_1 * G_2$ with $G_1 = \pi_1(M)$ and $G_2 = \pi_1(N)$ then,*

$$\chi(G) = \chi(G_1) + \chi(G_2).$$

Proof. This follows from equations 7.2 and 7.2. We make following observations

$$\begin{aligned}
\chi(G) &= \chi(G_1 * G_2) \\
&= \chi(\pi_1(M) * \pi_1(M)) \\
&= \chi(\pi_1(M \vee N)) \\
&= \chi(M \vee N) \\
&= \chi(M) + \chi(N) \\
&= \chi(G_1) + \chi(G_2).
\end{aligned}$$

□

Let us now look at finite index coverings. Let $\pi : M \to N$ be of index n, then any triangulation on N can be pulled back to M. We see that the number of vertices are multiplied by n, the same happens for edges and so on. We obtain,

$$\chi(M) = n\chi(N).$$

This gives a second calculation tool.

Lemma 7.13 (Euler characteristic of finite coverings)**.** *Let $\pi : M \to N$ be a finite covering of index n, then*

$$\chi(M) = n\chi(N).$$

Finite coverings also tell that fundamental groups are of finite index. This gives a third and most important tool.

Lemma 7.14 (Fundamental groups of finite coverings)**.** *Let $\pi : M \to N$ be a finite covering of index n, then $\pi_1(M)$ is of index n in $\pi_1(N)$.*

Proof. Take a loop γ in N starting at y. Then γ has n liftings, each starting at a point of $\pi^{-1}(y)$. □

We now prove following properties, these will then classify all free products of surface groups up to quasi isometry.

Proposition 7.15. *Let $g \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 2}$ and $a_i \in \mathbb{Z}_{\geq 1}$ for $i \leq n$. Now,*

$$\coprod_{i=1}^{n} \Sigma_{a_i}$$

defines a free product of surface groups. The following holds

1. *Σ_g is of finite index in Σ_2,*
2. *Let $a_i > 1$ for all i, then $\coprod_{i=1}^{n} \Sigma_{a_i} \sim_{QI} \Sigma_2 * \Sigma_2$,*[3]
3. *If all $a_i = 1$ then $\coprod_{i=1}^{n} \mathbb{Z}^2 \sim_{QI} \mathbb{Z}^2 * \mathbb{Z}^2$.*
4. *Let some $a_i = 1$ and some $a_i > 1$, then $\coprod_{i=1}^{n} \Sigma_{a_i} \sim_{QI} \mathbb{Z}^2 * \Sigma_2$,*

Proof. $\boxed{1.}$ On figure 7.1, we see a finite covering $\pi : S_g \to S_2$. By theorem 7.14, the groups Σ_g and Σ_2 are of finite index of each other.

$\boxed{2.}$ With finite coverings, we see following groups are quasi-isometric.

- $\coprod_{i=1}^{n} \Sigma_{a_i} \sim_{QI} \Sigma_{n+1} * \coprod_{i=1}^{n-1} \Sigma_2$: By applying n times the first property and theorem 7.9.
- $\Sigma_n * \coprod_{i=1}^{n-1} \Sigma_2 \sim_{QI} \Sigma_2 * \Sigma_2$: By defining a covering map
$$p : S_n \vee \bigvee_{i=1}^{n-1} S_2 \to S_2 \vee S_2$$
of index $n-1$ as follows. We project S_n to S_2 with index $n-1$ as on figure 7.1. You should then imagine that at every outer handle of S_n, a double torus S_2 is attached. These attached surfaces $\bigvee_{i=1}^{n-1} S_2$ are mapped to the second surface S_2 with index $n-1$. Note that the $n-1$ points where two surfaces join in the domain of p, should be send to the intersection point in the two surfaces in the image of p.

This gives the desired property.

$\boxed{3.}$ One can see that by passing through finite subgroups that

$$\coprod_{i=1}^{n} \mathbb{Z}^2 \sim_{QI} \mathbb{Z}^2 * \mathbb{Z}^2.$$

This is shown again by making finite index covering maps $\bigvee_{i=1}^{n} S_1 \to S_1 \vee S_1$. To make the construction one must notice that there exists a $n-1$ index covering map $S_1 \to S_1$. If we identify S_1 with $\mathbb{R}^2/\mathbb{Z}^2$ this map is given by $(x, y) \mapsto ((n-1)x, y)$. It now suffices to paste the $n-1$ remaining copies of S_1 at the right place.

$\boxed{4.}$ This is similar to the proof of the previous two points. □

[3] $\coprod$ is here used to denote the free product of groups.

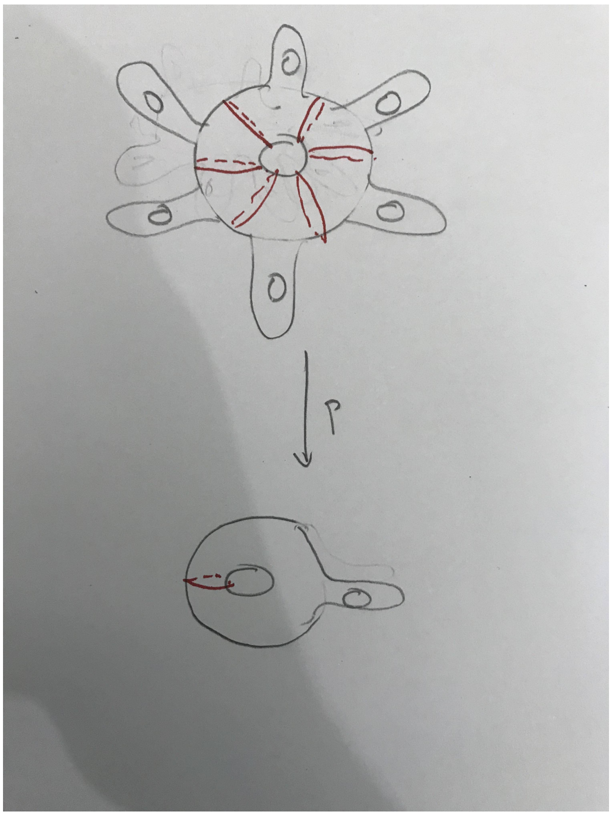


Figure 7.1: This shows a finite cover from S_g to the double torus S_2 [7].

From these properties, we easily deduce following classification theorem.

Corollary 7.16 (Classification of products of surface groups)**.** *Let* $G = \Sigma_{g_1} * \ldots \Sigma_{g_n} = \coprod_{i=1}^{n} \Sigma_{g_i}$*, then* G *is quasi-isometric to one of following groups:* $\mathbb{Z}^2, \Sigma_2, \mathbb{Z}^2 * \mathbb{Z}^2, \mathbb{Z}^2 * \Sigma_2, \Sigma_2 * \Sigma_2$*.*

Proof. Let $G = \coprod_{i=1}^{n} \Sigma_{g_i}$, by our first property of 7.15, this is of finite index in

$$\coprod_{i=1}^{l} \mathbb{Z}^2 * \coprod_{j=1}^{k} \Sigma_2$$

where $n = k + l$. If n equals 1 or 2, then this already coincides with one of the five groups. Let now $n \geq 3$.

If $l = 0$, then the second property of 7.15 assures that

$$G \sim_{QI} \Sigma_2 * \Sigma_2.$$

If $k = 0$, then the fourth property of 7.15 assures that

$$G \sim_{QI} \mathbb{Z}^2 * \mathbb{Z}^2.$$

Otherwise, the second property of 7.15 assures that

$$G \sim_{QI} \mathbb{Z}^2 * \Sigma_2.$$

$\square$

Even if all these products of surface groups can be classified in only five classes, commensurability is another issue. Following theorem and its proof can be found in [16].

Proposition 7.17. *Let $G_1 = \Sigma_{g_1} * \dots \Sigma_{g_m}$ and $G_2 = \Sigma_{h_1} * \dots \Sigma_{h_n}$, then G_1 is commensurable with G_2 if and only if*

$$\frac{\chi(G_1)}{m-1} = \frac{\chi(G_2)}{n-1}.$$

The main reason we cannot apply previous proof in the same way, is that we do not have that $\Sigma_2 * G$ and $\Sigma_g * G$ are commensurable.

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