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Regular Spanning Trees in Cayley Graphs

Geometric versions of the Von Neumann conjecture and Burnside's problem

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Main theorem of the day

Theorem: Regular trees of Cayley graphs

Let G be a finitely generated infinite group, then there exists a Cayley graph $\mathsf{Cay}(G, S)$ of G allowing a regular spanning tree.

2 failed conjectures

1. **Burnside's Problem:** Does every finitely generated infinite group contain $\mathbb{Z}$ as a subgroup?
2. **The Von Neumann conjecture:** Does every nonamenable finitely generated group contain the free group $\mathbb{F}_2$?

The 2 parts of the theorem

- If G amenable, 2-regular spanning tree (= bi-infinite Hamiltonian path)
  Burnside problem
- If G nonamenable, k-regular spanning tree for all $k \geq 3$.
  Von Neumann conjecture

Sources

[7] Clara Löh. *Geometric group theory*. Springer, 2017.

[8] Brandon Seward. Burnside's problem, spanning trees and tilings. *Geometry & Topology*, 18(1):179–210, 2014.

[9] Kevin Whyte. Amenability, bilipschitz equivalence, and the von neumann conjecture. *arXiv preprint math/9704202*, 2008.

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Introduction

Early questions?

History and results of my thesis

Proof of the bilipschitz equivalences of trees

Proof of the transitive geometric Von Neumann theorem

Cayley graphs

Definition (Cayley graph)

Let G be a finitely generated group and let $S \subset G$ be a generating set. Then we define the Cayley graph of G as

$$\mathsf{Cay}(G, S) := (G, E),$$

where the edges E link the elements $x, y \in G$ for which

$$x = sy$$

for some $s \in S \cup S^{-1}$.

Cayley graphs Examples

- the n-dimensional grids $\mathrm{Cay}(\mathbb{Z}^n, \{e_1, ..., e_n\})$

Cayley graphs Examples

- the n-dimensional grids $\mathrm{Cay}(\mathbb{Z}^n, \{e_1, ..., e_n\})$
- $\mathrm{Cay}(\mathbb{F}_2, \{a, b\})$

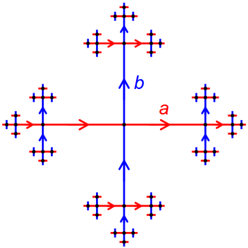


Figure: The Cayley graph of $\mathbb{F}_2$.

- The **degree** or **valency** of a vertex is the number of adjacent edges.

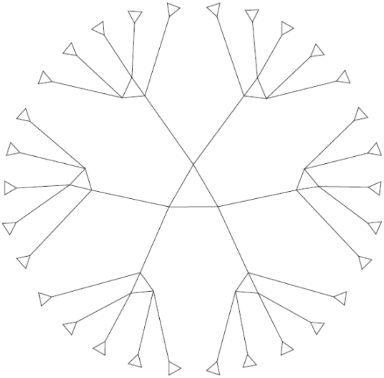


Figure: Representation of $G = \mathbb{Z} * \mathbb{Z}_3$.

Cayley graphs $D_\infty = \langle a, b | b^2, abab \rangle$

$$\begin{tikzcd}
\dots & ba^{-2} \arrow[l, dotted] & ba^{-1} \arrow[l, "a"] & b \arrow[l, "a"] & ba \arrow[l, "a"] & ba^2 \arrow[l, "a"] \arrow[r, dotted] & \dots \\
\dots & a^{-2} \arrow[l, dotted] \arrow[u, "b"] \arrow[r, "a"] & a^{-1} \arrow[u, "b"] \arrow[r, "a"] & 1 \arrow[u, "b"] \arrow[r, "a"] & a \arrow[u, "b"] \arrow[r, "a"] & a^2 \arrow[u, "b"] \arrow[r, dotted] & \dots
\end{tikzcd}$$

Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

$$\begin{tikzcd}
\dots & bab = a^{-1} \arrow[l, dotted] \arrow[r, "b"] & ab \arrow[r, "ab"] & 1 \arrow[r, "b"] & b \arrow[r, "ab"] & a \arrow[r, dotted] & \dots
\end{tikzcd}$$

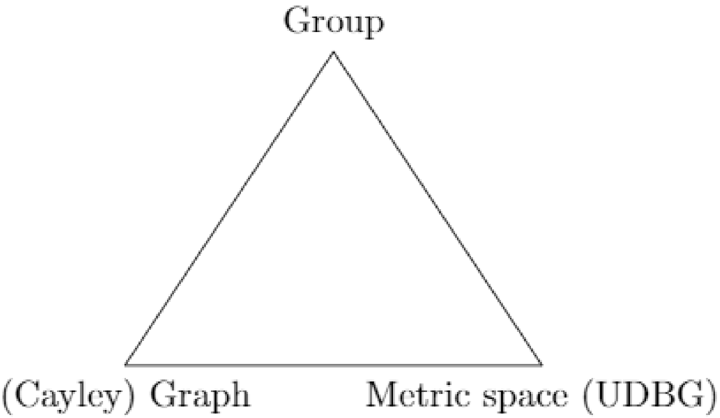
Figure: The graph $\mathrm{Cay}(D_\infty, \{ab, b\})$.

A connected graph is a metric space

$$
\begin{tikzcd}
\dots & ba^{-2} \arrow[l, dotted] & ba^{-1} \arrow[l, "a"'] & b \arrow[l, "a"'] & ba \arrow[l, "a"'] & ba^2 \arrow[l, "a"'] \arrow[r, dotted] & \dots \\
\dots & a^{-2} \arrow[l, dotted] \arrow[u, "b"] \arrow[r, "a"] & a^{-1} \arrow[u, "b"] \arrow[r, "a"] & 1 \arrow[u, "b"] \arrow[r, "a"] & a \arrow[u, "b"] \arrow[r, "a"] & a^2 \arrow[u, "b"] \arrow[r, dotted] & \dots
\end{tikzcd}
$$

Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

Holy Trinity (Drie-eenheid)



Definition Regular spanning trees

Definition (Regular trees)

Let $k \in \mathbb{N}$, we define the k-regular tree T_k as the tree where each vertex has valency k.

Definition (Spanning subgraph)

Let Γ be a graph. A spanning subgraph $\Phi \subset \Gamma$ satisfies

$$V(\Gamma) = V(\Phi).$$

Hamiltonian path

Definition (Hamiltonian path)

Let Γ be a graph. A Hamiltonian path P on Γ is a path hitting **every vertex once.**

Examples

$$\begin{tikzcd}
\ldots & \arrow[l, dashed] ba^{-2} & \arrow[l, "a"'] ba^{-1} & \arrow[l, "a"'] b & \arrow[l, "a"'] ba & \arrow[l, "a"'] ba^2 \arrow[r, dashed] & \ldots \\
\ldots & \arrow[l, dashed] a^{-2} \arrow[u, "b"] \arrow[r, "a"] & a^{-1} \arrow[u, "b"] \arrow[r, "a"] & 1 \arrow[u, "b"] \arrow[r, "a"] & a \arrow[u, "b"] \arrow[r, "a"] & a^2 \arrow[u, "b"] \arrow[r, dashed] & \ldots
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Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

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Figure: The graph $\mathrm{Cay}(D_\infty, \{ab, b\})$.

Functions

A function $f : X \to Y$ is

- a quasi-isometric embedding if and only if

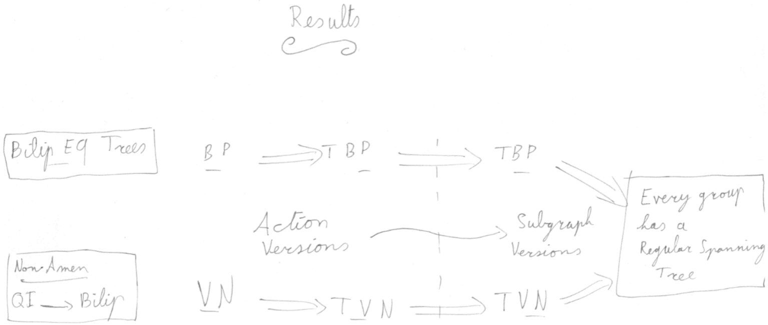
$$\forall x_1, x_2 \in X : C^{-1} d(x_1, x_2) - b < d(f(x_1), f(x_2)) < Cd(x_1, x_2) + b.$$

- a quasi-isometry if it also is C-dense ie $f(X)$ is C-dense in Y.

$$B_C(f(X)) = Y$$

- A bilipschitz equivalence if it also is bijective.

A quasi-isometry: just a bad printer...

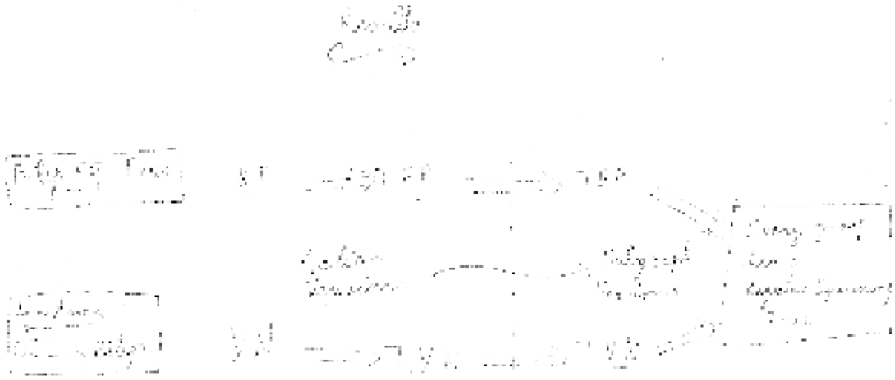


A quasi-isometry: just a bad printer...

Bilip Eq Trees

B P

A quasi-isometry: just a bad printer...



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Definition amenable

Definition (Amenable)

A discrete metric space X is amenable if and only if it contains a Følner sequence. A Følner sequence consists of finite sets F_n (growing balls) such that for any $R > 0$:

$$\lim \frac{|\partial_R F_n|}{|F_n|} = 0.$$

(Volume $>>$ boundary)

Amenability

- Examples are $\mathbb{Z}^n$ and $\mathbb{R}^n$
- Volume ball of size $R \approx R^n$.
- Surface/Boundary ball of size $R \approx R^{n-1}$.

Amenability

- Examples are $\mathbb{Z}^n$ and $\mathbb{R}^n$
- Volume ball of size $R \approx R^n$.
- Surface/Boundary ball of size $R \approx R^{n-1}$.
- Why is Usain Bolt quicker than Blake.

Amenability

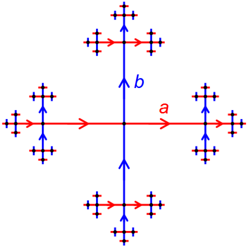

Figure: The Cayley graph of $\mathbb{F}_2$.

Early questions?

Everything clear about:
Cayley graphs, Quasi-isometry, Amenable, Trees, paths...

Overview

2 failed conjectures

1. **Burnside's Problem:** Does every finitely generated infinite group contain $\mathbb{Z}$ as a subgroup?
2. **The Von Neumann conjecture:** Does every nonamenable finitely generated group contain the free group $\mathbb{F}_2$?

2 failed conjectures

1. **Burnside's Problem:** Does every finitely generated infinite group contain $\mathbb{Z}$ as a subgroup?
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Figure: Alfred Tarski. Source: Wikipedia

Translation-like actions, a geometric messias

Definition (Translation-like action)

An action by Γ on X is translation-like if it is **free**, and satisfies for all g,

$$(\cdot g) \approx \mathrm{Id}_X \,.$$

Or equivalently, $\{d(x \cdot g, x) | x \in X\}$ is bounded for all g.

Translation-like actions, a geometric messias

Theorem (Geometric Burnside Problem)

A finitely generated group G is infinite if and only if it admits a ***translation-like action by*** $\mathbb{Z}$.

Theorem (Geometric Von Neumann Theorem)

A finitely generated group G is nonamenable if and only if it admits a ***translation-like action by*** *the free group* $\mathbb{F}_2$.

Transitive translation-like actions

Theorem (Burnside Problem transitive version)

*G is infinite **and has at most 2 ends** if and only if it admits a **transitive** translation-like action by $\mathbb{Z}$.*

Theorem (Geometric Von Neumann Theorem)

*G is nonamenable if and only if it admits a **transitive** translation-like action by the free group $\mathbb{F}_2$.*

Transitive translation-like actions

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Theorem (Geometric Von Neumann Theorem)

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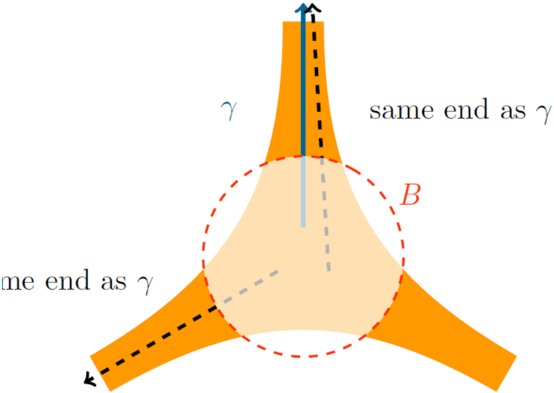
Definition (Ends)

”Ends are infinite components you can separate by removing a finite part.”

Drawing ends



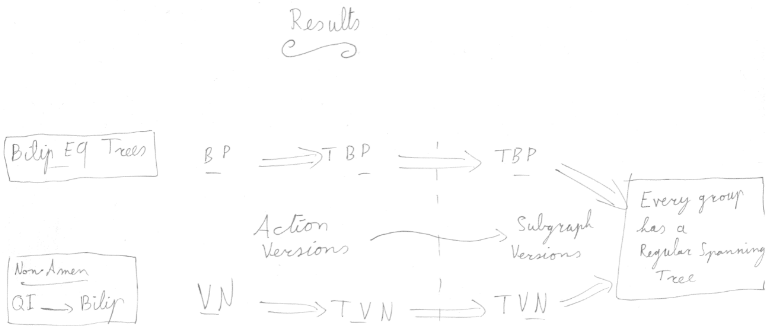


Figure: The results of the day and their relations. (T)BP: (transitive) geometric Burnside problem. VN: geometric Von Neumann theorem.

Stop talking about actions, Start talking about subgraphs

Theorem

- *Let $H \curvearrowright G$ be a translation-like action.*
- *Take* $\mathsf{Cay}(H, U)$
- *then there exists a Cayley graph* $\mathsf{Cay}(G, W)$ *of* G

Stop talking about actions, Start talking about subgraphs

Theorem

- *Let $H \curvearrowright G$ be a translation-like action.*
- *Take* $\mathrm{Cay}(H, U)$
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  ... containing a spanning subgraph Φ whose connected components are isomorphic to $\mathrm{Cay}(H, U)$.

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- *Let* $H \curvearrowright G$ *be a translation-like action.*
- *Take* $\mathrm{Cay}(H, U)$
- *then there exists a Cayley graph* $\mathrm{Cay}(G, W)$ *of* G
  ... containing a spanning subgraph Φ *whose connected components are isomorphic to* $\mathrm{Cay}(H, U)$.
- *In particular, if the action is transitive, then*

$$\Phi \cong \mathrm{Cay}(H, U).$$

Stop talking about actions, Start talking about subgraphs

$$\Phi = (G, \{\{g, g \cdot u\} | g \in G, u \in U\}).$$

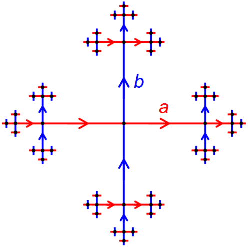

Figure: The Cayley graph of $\mathbb{F}_2$.

Stop talking about actions, Start talking about subgraphs

- Translation-like action $\Rightarrow$ Subgraph

Stop talking about actions, Start talking about subgraphs

- Translation-like action $\Rightarrow$ Subgraph
- Bilipschitz equivalence $\Rightarrow$ Subgraph

Burnside in graph theoretic setting

- G has a transitive translation-like action by $\mathbb{Z}$.

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  $\Rightarrow$ we have a spanning subgraph $\mathbb{Z} \subset \mathsf{Cay}(G, S)$.

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Burnside in graph theoretic setting

- G has a transitive translation-like action by $\mathbb{Z}$.
  $\Rightarrow$ we have a spanning subgraph $\mathbb{Z} \subset \mathrm{Cay}(G, S)$.
  $\Rightarrow$ Hamiltonian path.

Theorem (Hamiltonian paths in Cayley graphs)

Let G be a finitely generated group with at most two ends, then there exists a Cayley graph $\mathrm{Cay}(G, S)$ of G allowing a bi-infinite Hamiltonian path.

Von Neumann in graph theoretic setting

Theorem (Transitive geometric Von Neumann theorem)

Let G be a finitely generated group and $k \in \mathbb{Z}_{\geq 3}$, then the following statements are equivalent.

1. *G is non-amenable.*
2. *There exists a k-regular spanning tree $\Phi \subset \mathrm{Cay}(G, W)$.*

$$\Phi \cong T_k$$

.

Bilipschitz equivalence of trees

Theorem (Bilipschitz equivalence of non-amenable trees)

Let T, T' be two trees with uniformly bounded valencies[1]. Let all valencies be equal or larger than 3, then $T \sim_{Bilip} T'$.

Main theorem of the day

We have the following

- If G has at most two ends, G admits a 2-regular spanning tree.
- If G in nonamenable, G admits a k-regular spanning tree for all $k \geq 3$.

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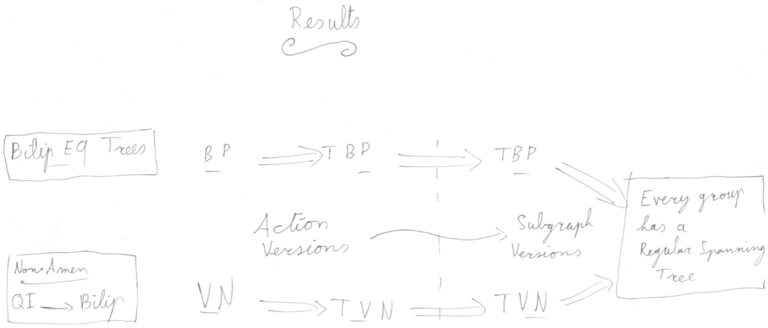


Figure: The results of the day and their relations.
(T)BP: (transitive) geometric Burnside problem.
VN: geometric Von Neumann theorem.

Examples of the theorem

$$\begin{tikzcd}
\cdots & ba^{-2} \arrow[l, dotted] & ba^{-1} \arrow[l, "a"'] & b \arrow[l, "a"'] & ba \arrow[l, "a"'] & ba^{2} \arrow[l, "a"'] \arrow[r, dotted] & \cdots \\
\cdots & a^{-2} \arrow[l, dotted] \arrow[u, "b"] \arrow[r, "a"] & a^{-1} \arrow[u, "b"] \arrow[r, "a"] & 1 \arrow[u, "b"] \arrow[r, "a"] & a \arrow[u, "b"] \arrow[r, "a"] & a^{2} \arrow[u, "b"] \arrow[r, dotted] & \cdots
\end{tikzcd}$$

Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

Examples of the theorem

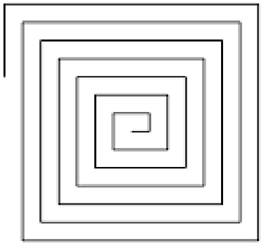


Figure: $\mathbb{Z}^2$ Hamiltonian path

Open questions

- Does this hold for every Cayley graph?
  - If G amenable, OPEN
  - If G non-amenable, NO

counterexample

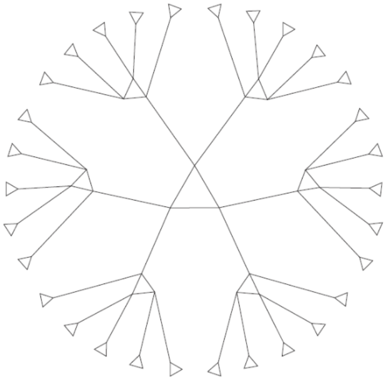


Figure: Representation of $G = \mathbb{Z} * \mathbb{Z}_3$.

Overview

Theorem (Bilipschitz equivalence of non-amenable trees)

Let T, T' be two trees with uniformly bounded valencies[2]*. Let all valencies be equal or larger than 3, then $T \sim_{Bilip} T'$.*

Basic concepts in trees

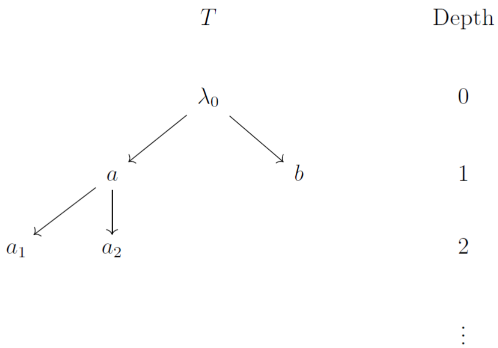
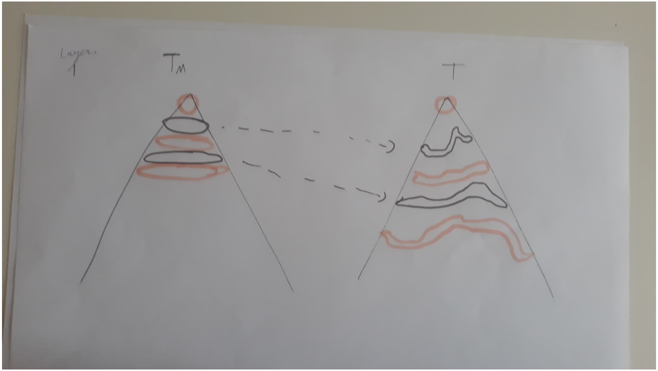


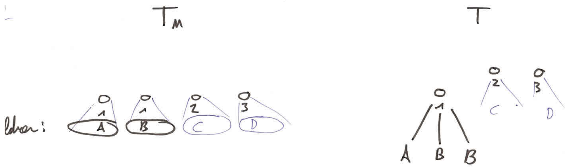
Figure: The "family" tree

Here comes the drawing of layers

- It suffices to show that $T \sim_{\text{Bilip}} T_M$ for all $M > D$
- Nonamenable Quasi-isometric spaces are bilipschitz equivalent.



How to construct a quasi-isometry



Overview

Theorem (Transitive geometric Von Neumann theorem)

Let G be a finitely generated group and $k \in \mathbb{Z}_{\geq 3}$, then the following statements are equivalent.

1. *G is non-amenable.*
2. $\exists \operatorname{Cay}(G, W)$ *with a spanning subgraph* $\Phi \subset \operatorname{Cay}(G, W)$ *so*
$$\Phi \cong \bigsqcup T_k.$$
3. $\exists \operatorname{Cay}(G, W)$ *with a k-regular spanning tree* $\Phi' \subset \operatorname{Cay}(G, W)$
$$\Phi' \cong T_k.$$

Lemma

Let G be a finitely generated group and $k \in \mathbb{Z}_{\geq 3}$, if we have that:

- *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a spanning subgraph $\Phi \subset \mathrm{Cay}(G, W)$ such that*

$$\Phi \cong \bigsqcup T_k.$$

Then it holds that:

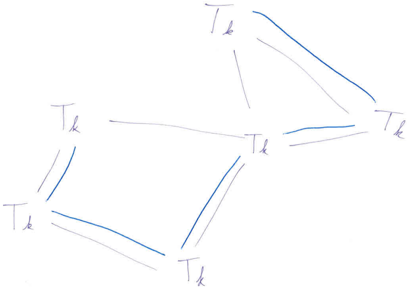
- *There exists a Cayley graph $\mathrm{Cay}(G, W)$ of G such that there exists a k-regular spanning tree $\Phi' \subset \mathrm{Cay}(G, W)$.*

$$\Phi' \cong T_k.$$

Begin proof

Start with the spanning subgraph $\Phi \cong \bigsqcup T_k$.

Drawing of Φ and Φ'



Construction of Φ'

- Add edges to make Φ a spanning tree.
- Use the bilipschitz equivalence of trees.

Some more questions?

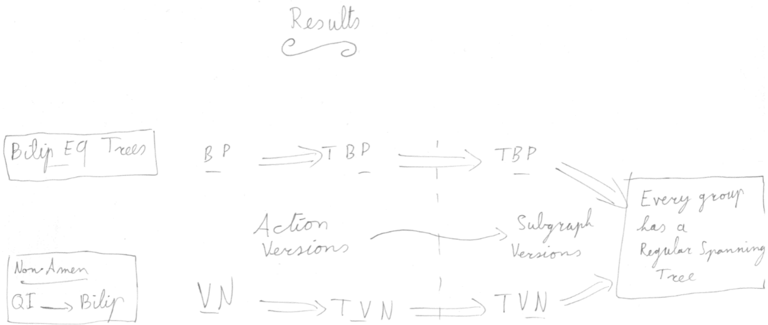


Figure: The results of the day and their relations. (T)BP: (transitive) geometric Burnside problem. VN: geometric Von Neumann theorem.

Proof of the transitive Burnside problem

The transitive Burnside problem

Theorem (The transitive Geometric Burnside problem)

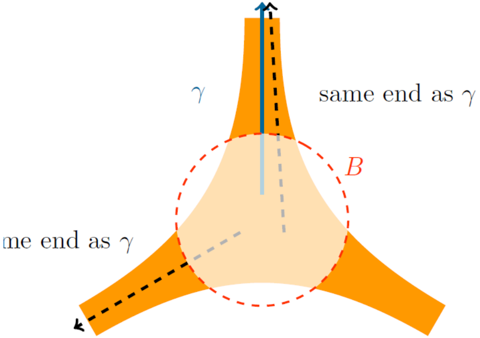
Let G have at most two ends, then there exists a Cayley graph $\mathsf{Cay}(G, S)$ of G allowing a bi-infinite Hamiltonian path.

The transitive Burnside problem

Theorem (The transitive Geometric Burnside problem)

Let Γ be a connected infinite graph with $\deg(v) \leq D$, then Γ is bilipschitz equivalent to a graph Γ' admitting a bi-infinite Hamiltonian path P' if and only if Γ has at most two ends.

If G has more ends



If G has at most 2 ends

Outline

- Construct P hitting
  - At least 1 time at every vertex
  - At most 2 times every edge.

Outline

- Construct P hitting
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    $\Rightarrow$ At most D times at every vertex

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- Construct P hitting
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  First 1 end and then 2 ends.

Outline

- Construct P hitting
    - At least 1 time at every vertex
    - At most 2 times every edge.
      $\Rightarrow$ At most D times at every vertex

  First 1 end and then 2 ends.
- Construct a Hamiltonian path P'. By deleting the duplicated vertices in P.

Existence of Eulerian paths

Theorem (Existence of Eulerian paths)

A countably infinite connected multigraph[3] Γ *admits an Eulerian path if and only if the following conditions are satisfied:*

I. *The degree of every vertex is* ***even****.* (or infinite)

II. Γ *has at most 2 ends.*

III. *Removing a finite subgraph* $\Phi \subset \Gamma$ *with* ***even*** *valencies, we have that* $\Gamma - E(\Phi)$ *has* ***one infinite connected component****.*

Construct P

- Let Γ have one end.
- Double all edges.

Existence of Eulerian paths

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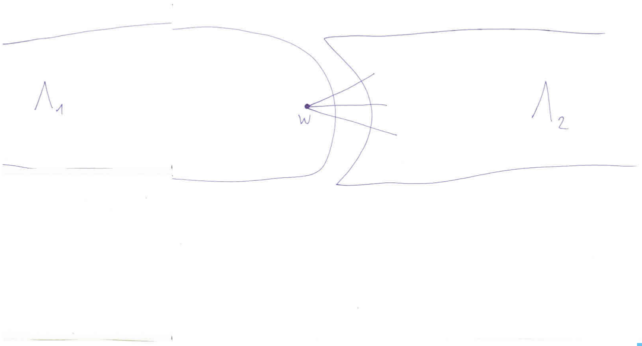
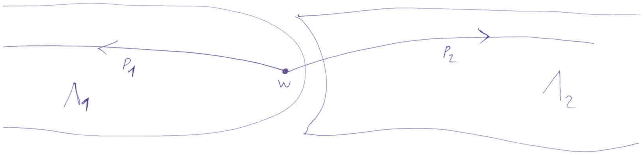
Construct P

- Let Γ have two ends.

Figure 1





p_1
w
Λ_1
p_2
Λ_2

Double all edges not in P_1 and P_2. Then every vertex has even degree.

Existence of Eulerian paths

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A countably infinite connected multigraph[3] Γ *admits an Eulerian path if and only if the following conditions are satisfied:*

I. *The degree of every vertex is **even**. (or infinite)*

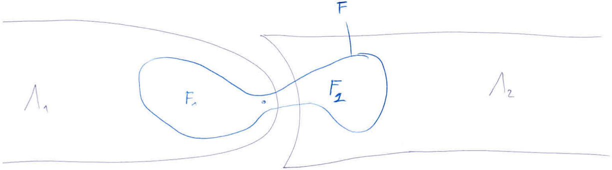
II. Γ *has at most 2 ends.*

III. *Removing a finite subgraph* $\Phi \subset \Gamma$ *with **even** valencies, we have that* $\Gamma - E(\Phi)$ *has **one infinite connected component**.*

Satisfying the third point: Figure 4

$$\sum \deg v = 2 \cdot \text{ number of edges}$$

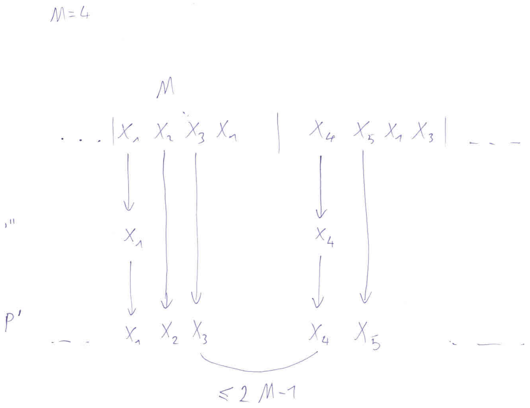
Hence w has even degree.
w is the only edge in Λ_i to have odd degree.



We have found P

We have found a path P

Construction of P'



Why can we do this and why do we need this

- Hall
- This way two consecutive elements $y - n, y_{n+1}$ are at most $2M - 1$ edges away along P.

Hall's selection theorem

Intuitively: "If you can insert every finite part. You can insert everything."

Construct Γ'

We then define the $2M - 1$-bilipschitz equivalent graph Γ' as

$$(V(\Gamma), \{\{v, v'\} | d_\Gamma(v, v') \leq 2M - 1\}).$$