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On Translation-like Actions, Amenability and Homology on Discrete Spaces

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Topics of the day

- We study finitely generated infinite groups

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  by representing them as graphs (=**Cayley graphs**)

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- Main topic: Regular spanning trees in Cayley graphs
- Extra topic: The Uniformly Finite Homology

Regular spanning trees

Theorem: Regular trees of Cayley graphs

Let G be a finitely generated infinite group, then there exists a Cayley graph $\mathsf{Cay}(G, S)$ of G allowing a regular spanning tree.

Sources

[7] Clara Löh. *Geometric group theory*. Springer, 2017.

[8] Brandon Seward. Burnside's problem, spanning trees and tilings. *Geometry & Topology*, 18(1):179–210, 2014.

[9] Kevin Whyte. Amenability, bilipschitz equivalence, and the von neumann conjecture. *arXiv preprint math/9704202*, 2008.

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Overview

Cayley graphs Examples

- the n-dimensional grids $\mathrm{Cay}(\mathbb{Z}^n, \{e_1, ..., e_n\})$

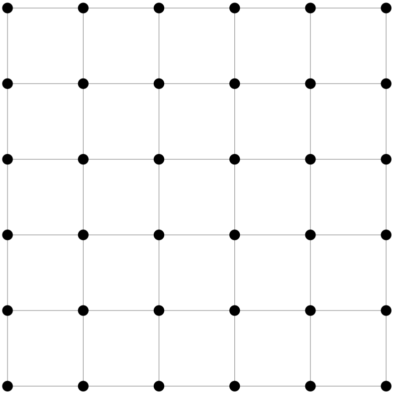

Figure: A picture of (a part of) $\mathrm{Cay}(\mathbb{Z}^2, \{(1,0),(0,1)\})$.

Cayley graph of $\mathbb{F}_2$

- $\mathrm{Cay}(\mathbb{F}_2, \{a, b\})$

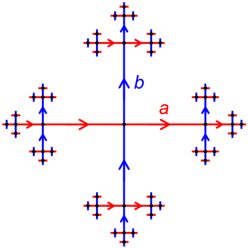

Figure: The Cayley graph of $\mathbb{F}_2$.

- Note that $\mathbb{F}_2$ is a 4-regular spanning tree
- The **degree** or **valency** of a vertex is the number of adjacent edges.

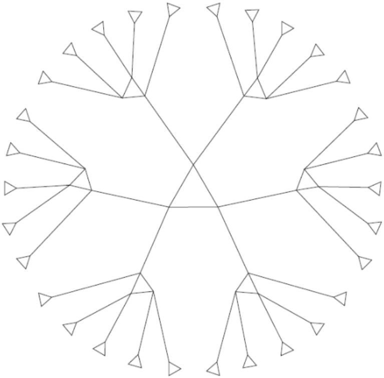

Figure: Representation of $G = \mathbb{Z} * \mathbb{Z}_3$.

Cayley graphs $D_\infty = \langle a, b | b^2, abab \rangle$

$$\begin{tikzcd}
\dots \arrow[r, dashed, leftarrow] & ba^{-2} & ba^{-1} \arrow[l, "a"'] & b \arrow[l, "a"'] & ba \arrow[l, "a"'] & ba^2 \arrow[l, "a"'] \arrow[r, dashed] & \dots \\
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\end{tikzcd}$$

Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

$$\dots \dashleftarrow bab = a^{-1} \xrightarrow{b} ab \xrightarrow{ab} 1 \xrightarrow{b} b \xrightarrow{ab} a \dashrightarrow \dots$$

Figure: The graph $\mathrm{Cay}(D_\infty, \{ab, b\})$.

A connected graph is a metric space

$$
\begin{tikzcd}
\dots & ba^{-2} \arrow[l, dotted] & ba^{-1} \arrow[l, "a"'] & b \arrow[l, "a"'] & ba \arrow[l, "a"'] & ba^2 \arrow[l, "a"'] \arrow[r, dotted] & \dots \\
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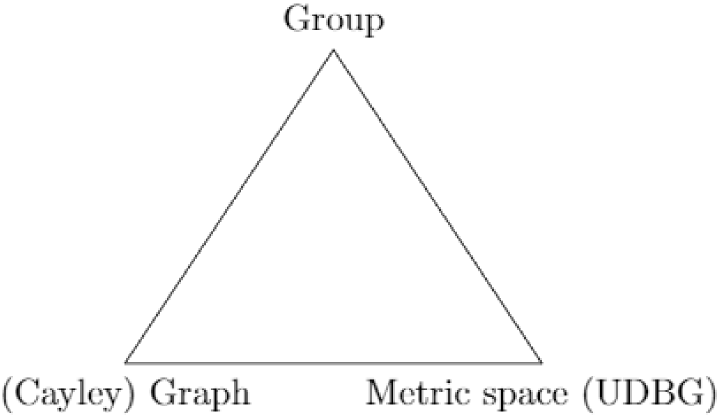
\end{tikzcd}
$$

Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

Holy Trinity (Drie-eenheid)



Definition amenable

Definition (Amenable)

A discrete metric space X is amenable if and only if it contains a Følner sequence. A Følner sequence consists of finite sets F_n (growing balls) such that for any $R > 0$:

$$\lim \frac{|\partial_R F_n|}{|F_n|} = 0.$$

(Volume $>>$ boundary)

Amenability

- Examples are $\mathbb{Z}^n$ and $\mathbb{R}^n$
- Volume ball of size $R \approx R^n$.
- Surface/Boundary ball of size $R \approx R^{n-1}$.

Amenability

- Examples are $\mathbb{Z}^n$ and $\mathbb{R}^n$
- Volume ball of size $R \approx R^n$.
- Surface/Boundary ball of size $R \approx R^{n-1}$.
- Why is Usain Bolt quicker than Blake.

Nonamenable group $\mathbb{F}_2$

$\mathbb{F}_2$ is nonamenable as for every finite set F

$$|\partial F| \geq |F|.$$

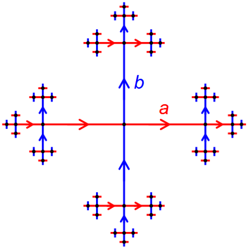


Figure: The Cayley graph of $\mathbb{F}_2$.

Definition Regular spanning trees

Definition (Regular trees)

Let $k \in \mathbb{N}$, we define the k-regular tree T_k as the tree where each vertex has valency k.

Definition (Spanning subgraph)

Let Γ be a graph. A spanning subgraph $\Phi \subset \Gamma$ satisfies

$$V(\Gamma) = V(\Phi).$$

Overview

2 failed conjectures

1. **Burnside's Problem:** Does every finitely generated infinite group contain $\mathbb{Z}$ as a subgroup?
2. **The Von Neumann conjecture:** Does every nonamenable finitely generated group contain the free group $\mathbb{F}_2$?

2 failed conjectures

1. **Burnside's Problem:** Does every finitely generated infinite group contain $\mathbb{Z}$ as a subgroup?
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Figure: Alfred Tarski. Source: Wikipedia

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Translation-like actions, a geometric messias

Forget the **group structure** and look at its **metric**!

Definition (Translation-like action)

An action by Γ on X is translation-like if it is **free**, and satisfies for all g,

$$(g\cdot) \approx \mathrm{Id}_X \,.$$

Or equivalently, $\{d(g \cdot x, x) | x \in X\}$ is bounded for all g.

Translation-like actions, a geometric messias

Theorem (Geometric Burnside Problem)

A finitely generated group G is infinite if and only if it admits a ***translation-like action by*** $\mathbb{Z}$.

Theorem (Geometric Von Neumann Theorem)

A finitely generated group G is nonamenable if and only if it admits a ***translation-like action by*** *the free group* $\mathbb{F}_2$.

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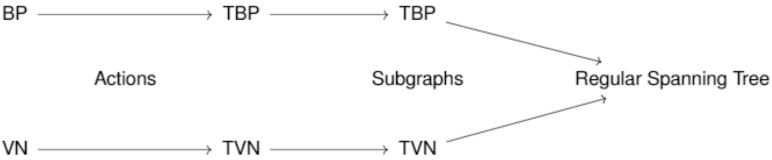

Figure: The results of the day and their relations.
(T)BP: (transitive) geometric Burnside problem.
VN: geometric Von Neumann theorem.

Transitive translation-like actions

Theorem (Burnside Problem transitive version)

*G is infinite **and has at most 2 ends** if and only if it admits a **transitive** translation-like action by $\mathbb{Z}$.*

Theorem (Geometric Von Neumann Theorem)

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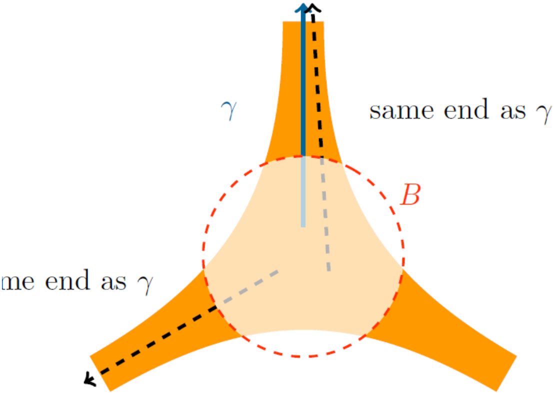
Definition (Ends)

"Ends are infinite components you can separate by removing a finite part."

Drawing ends



Actions $\to$ *Subgraphs*

Let us say $(1\cdot)$ moves every element to the right.

$$
\begin{tikzcd}
\dots & ba^{-2} \arrow[l, dashed] & ba^{-1} \arrow[l, "a"'] & b \arrow[l, "a"'] & ba \arrow[l, "a"'] & ba^2 \arrow[l, "a"'] \arrow[r, dashed] & \dots \\
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Figure: The graph $\mathrm{Cay}(D_\infty, \{a, b\})$.

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$$

Figure: Action on $\mathbb{Z} \curvearrowright D_\infty$.

Construction of the Subgraph

For an action $H \curvearrowright \mathrm{Cay}(G, S)$ and $\mathrm{Cay}(H, U)$,
we hence construct a graph

$$\Phi = (G, \{\{g, u \cdot g\} | g \in G, u \in U\}).$$

Burnside using Subgraphs

- G has a transitive translation-like action by $\mathbb{Z}$.

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Burnside using Subgraphs

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  $\Rightarrow$ we have a spanning subgraph $\mathbb{Z} \subset \mathsf{Cay}(G, S)$.

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  $\Rightarrow$ 2-regular spanning tree

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Burnside using Subgraphs

- *G* has a transitive translation-like action by $\mathbb{Z}$.
  $\Rightarrow$ we have a spanning subgraph $\mathbb{Z} \subset \mathrm{Cay}(G, S)$.
  $\Rightarrow$ 2-regular spanning tree

Theorem (2-regular spanning trees $\mathbb{Z}$ in Cayley graphs)

*Let G be a finitely generated group with at most two ends, then there exists a Cayley graph $\mathrm{Cay}(G, S)$ of G allowing a 2-**regular spanning tree**.*

Von Neumann using Subgraphs

Theorem (Transitive geometric Von Neumann theorem)

Let G be a finitely generated group and $k \in \mathbb{Z}_{\geq 3}$, then the following statements are equivalent.

1. *G is non-amenable.*
2. *There exists a 4-regular spanning tree $\Phi \subset \mathsf{Cay}(G, W)$.*

$$\Phi \cong \mathbb{F}_2.$$

(We can replace $\mathbb{F}_2$ by any regular tree T_k)

Main theorem of the day

We have the following

- If G has at most two ends, G admits a 2-regular spanning tree.

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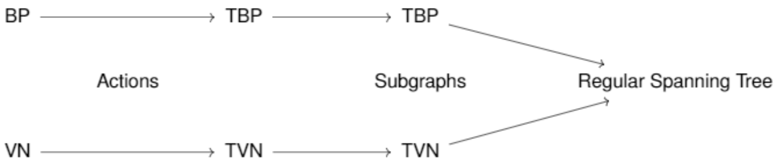

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Examples of the theorem

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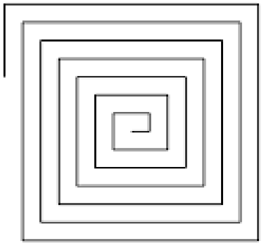


Figure: $\mathbb{Z}^2$ Hamiltonian path

Overview

The Normal 0-homology H_0

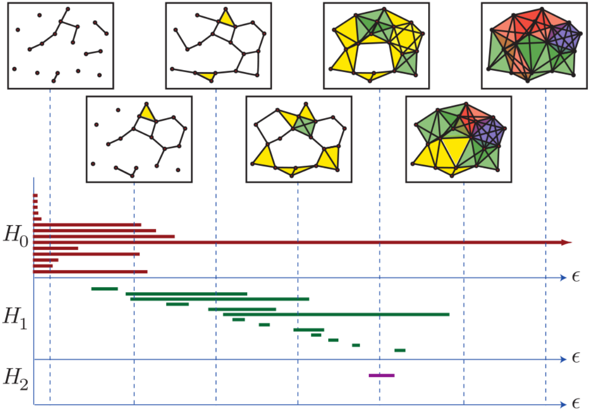
Let (X, E) be a graph, define

- 0-chains C_0 as sums of vertices $\mathbb{Z}[X]$.
- 1-chains C_1 as sums of directed edges $\mathbb{Z}[E]$.
- The differential $\partial([x, y]) = [y] - [x]$. We call $\operatorname{Im} \partial = B_0$ the 0-boundaries
- The 0-homology

  $$H_0 = \frac{C_0}{B_0},$$

  So we have the equality of classes $[x] = [y]$.
- The homology allows us to move points around.

Barcodes



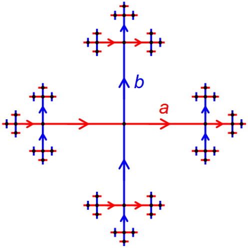


Figure: The Cayley graph of F_2.[**?**]

- $3[e] - 5[aba] + 4[a^{-1}b^{-1}a] = 2[e]$

Normal homology $H_\bullet(X)$

Theorem

Let X be a group, then the homology $H_\bullet(X)$ as $\mathbb{Z}$-module is given by

$$H_0 \cong \mathbb{Z};$$
$$H_k = 0 \quad \forall k > 0.$$

The uniformly finite H_0^{uf}

- In the uniformly finite homology we allow infinite sums of both vertices and edges.
- However, every element

$$c = \sum_{x \in X} c_x [x] \in H_0^{uf}(X)$$

must satisfy,

$$\|c\|_\infty = \sup |c_x| < \infty.$$

Theorem

Let $S \subset X$, then we define the cycle (or class)

$$[S] := \sum_{s \in S} [s].$$

The set $\{[S] | S \subset X\}$ is a generating set of $C_0^{uf}(X)$ (and of $H_0^{uf}(X)$).

The uniformly finite H_0^{uf}

Take for instance $X = \langle t \rangle$ to be the infinite cyclic group then $[e]$ is a boundary. Take an infinite path going to e

$$b = \sum_{t^n n \in \mathbb{N}} [t^{n+1}, t^n].$$

Then $[e] = \partial b$.

$$e \longrightarrow t \longrightarrow t^2 \longrightarrow t^3 \longrightarrow t^4 \dashrightarrow \ldots$$

Figure: The 1-chain $-b$, taking e along an infinite path.

Overview

When is $[X] = 0$?

Theorem

X is nonamenable if and only if $H_0^{uf}(X) = 0$.

Let X be amenable

Lemma

Let X be amenable then $H_0^{uf}(X) \neq 0$.

Proof.

- We will prove $[X] \neq 0$
- Let there be a $b \in C_1(X)$ such that $\partial b + [X] = 0$.
- Følner sequence $\lim_{n\to\infty} \frac{|\partial_r F_n|}{|F_n|} = 0$.
- Then the average number of paths going to a point is $|F_n|/|\partial_r F_n| \to \infty$. Contradiction!

$\square$

F_2 has zero uniformly finite homology

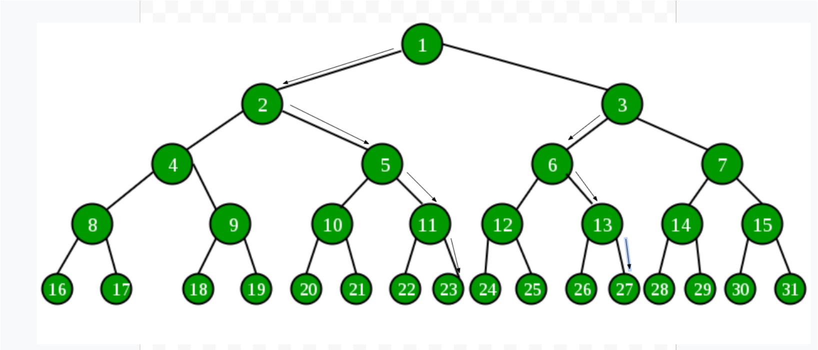


Figure: binary tree paths

Questions?