1 Modelling Still Air Rotation

1.1 A classical Hess-Smith Method

A panel code was developed with the aim to model the effects of streamline curvature on the airfoil pressure distribution exactly. The panel code base closely follows existing literature to establish the straight incomming flow model, and in particular the approach described in chapter 4 of ref.[1], and was extended by the author to model streamline curvature. A classical Hess-Smith approach was followed such that the airfoil was panelled with:

1. Zero order source line segments over each panel
2. Zero order vortex line segments over each panel
3. All vortices have the same strenght

Using the linearity of the differential operator, a linear system of equations was established to force the solution to satisfy two conditions:

1. Impermeability: flow may not cross the airfoil surface, which translates in a still atmosphere. This consists in imposing that the normal compoment of the speed vector in the surface reference frame must be null at the panel control points
2. Kutta Condition: an approximate condition was considered to impose flow tangency at the trailing edge, which consisted in imposing that the tangential velocity of the upper and lower panel control points just next to the leading edge is the same[1]

A numerical implementation which is believed to be reasonably accurate was found, with the following features:

- Control points were located in the middle of each panel
- The self-induction of panels was controlled to ensure we always solve the outer flow problem, by shifting the control points outwards of $1e-8$

The details behind the procedure are fairly interesting and partially described in the code, but their description does not fit the timeframe available for this report. In any case the value of the code is essentially pedagogical, as it is a fairly inneficient approach whose main focus was on highlighting all the steps in constructing the linear system and postprocessing matrices.

[1] Ref. [1], suggests that this approach is acceptable as long as the trailing edge wedge angle is not too small, and that the last panel sizes are the same on each side. This condition is certainly not the best one, but it is easy to implement, and the objective of this code was to study the efffect of a slightly curved path, not trailing edge flow.

1.1.1 The Impermeability condition

The impermeability condition on the airfoil surface is expressed as a zero normal velocity at the panel control points. The speed at the control point of panel k is written as the sum of three contributions:

$$\vec{U^k} = \vec{U^k}_{movement} + \vec{U^k}_{sources} + \vec{U^k}_{vortices} \tag{1}$$

The normal component is easily obtained through the dot product of the speed vector with the unit vector normal to the panel $\vec{e^k_\perp}$:

$$\vec{U^k} \cdot \vec{e^k_\perp} = 0 = \left(\vec{U^k}_{movement} + \vec{U^k}_{sources} + \vec{U^k}_{vortices}\right) \cdot \vec{e^k_\perp} \tag{2}$$

whereby we can write the impermeability condition as:

$$-\left(\vec{U^k}_{sources} + \vec{U^k}_{vortices}\right) \cdot \vec{e^k_\perp} = \vec{U^k}_{movement} \cdot \vec{e^k_\perp} \tag{3}$$

Considering that the speed contributions from the singularities can be expressed as a linear combination of the source and vortice strenghts, we can rewrite the right hand side as a matrix multiplication to obtain a standard linear system form:

$$\begin{bmatrix} u^{1,1}_{q\perp} & \dots & u^{1,k}_{q\perp} & \dots & u^{1,N}_{q\perp} & u^{1}_{\Gamma\perp} \\ \vdots & \ddots & & & \vdots & \vdots \\ u^{k,1}_{q\perp} & \dots & u^{k,k}_{q\perp} & \dots & u^{k,N}_{q\perp} & u^{k}_{\Gamma\perp} \\ \vdots & & & \ddots & \vdots & \vdots \\ u^{N,1}_{q\perp} & \dots & u^{N,k}_{q\perp} & \dots & u^{N,N}_{q\perp} & u^{N}_{\Gamma\perp} \end{bmatrix} \begin{bmatrix} q^1 \\ \vdots \\ q^k \\ \vdots \\ q^N \end{bmatrix} = \begin{bmatrix} \vec{U^1}_{movement} \cdot \vec{e^1_\perp} \\ \vdots \\ \vec{U^k}_{movement} \cdot \vec{e^k_\perp} \\ \vdots \\ \vec{U^N}_{movement} \cdot \vec{e^1_\perp} \end{bmatrix} \tag{4}$$

1.1.2 The Kutta condition

The speed at the two panels next to the trailing edge can be written as:

$$\vec{U}^{TEup} \cdot \vec{e}^{TEup}_{\parallel} = \left(\vec{U}^{TEup}_{movement} + \vec{U}^{TEup}_{sources} + \vec{U}^{TEup}_{vortices}\right) \cdot \vec{e}^{TEup}_{\parallel} \tag{5}$$

$$\vec{U}^{TEdown} \cdot \vec{e}^{TEdown}_{\parallel} = \left(\vec{U}^{TEdown}_{movement} + \vec{U}^{TEdown}_{sources} + \vec{U}^{TEdown}_{vortices}\right) \cdot \vec{e}^{TEdown}_{\parallel} \tag{6}$$

Whereby the approximate Kutta condition, same tangential speed component on each side of the trailing edge:

$$\vec{U}^{TEup} \cdot \vec{e}^{TEup}_{\parallel} = \vec{U}^{TEdown} \cdot \vec{e}^{TEdown}_{\parallel} \tag{7}$$

which is easily rewritten as:

$$\begin{aligned} \left(\vec{U}^{TEup}_{sources} + \vec{U}^{TEup}_{vortices}\right) \cdot \vec{e}^{TEup}_{\parallel} - \left(\vec{U}^{TEdown}_{sources} + \vec{U}^{TEdown}_{vortices}\right) \cdot \vec{e}^{TEdown}_{\parallel} &= \dots \\ \dots = \dots \vec{U}^{TEup}_{movement} \cdot \vec{e}^{TEup}_{\parallel} - \vec{U}^{TEdown}_{movement} \cdot \vec{e}^{TEdown}_{\parallel} & \end{aligned} \tag{8}$$

or as a matricial multiplication:

$$\begin{bmatrix} \left(u_{q\perp}^{TEup,1} - u_{q\perp}^{TEdown,1}\right) & \dots & \left(u_{q\perp}^{TEup,k} - u_{q\perp}^{TEdown,k}\right) & \dots & \left(u_{q\perp}^{TEup,N} - u_{q\perp}^{TEdown,N}\right) \end{bmatrix} \cdot \dots$$

$$\dots \cdot \begin{bmatrix} q^1 \\ \vdots \\ q^k \\ \vdots \\ q^N \\ \Gamma \end{bmatrix} = \begin{bmatrix} \vec{U}_{movement}^{TEup} \cdot \vec{e}_{\parallel}^{TEup} - \vec{U}_{movement}^{TEdown} \cdot \vec{e}_{\parallel}^{TEdown} \end{bmatrix} \tag{9}$$

1.1.3 Setting up a linear system

Applying simultaneously the impermeability condition and the kutta condition, we get a closed linear system of $N+1$ equations:

$$\begin{bmatrix} u_{q\perp}^{1,1} & \dots & u_{q\perp}^{1,k} & \dots & u_{q\perp}^{1,N} & u_{\Gamma\perp}^{1} \\ \vdots & \ddots & & & \vdots & \vdots \\ u_{q\perp}^{k,1} & \dots & u_{q\perp}^{k,k} & \dots & u_{q\perp}^{k,N} & u_{\Gamma\perp}^{k} \\ \vdots & & & \ddots & \vdots & \vdots \\ u_{q\perp}^{N,1} & \dots & u_{q\perp}^{N,k} & \dots & u_{q\perp}^{N,N} & u_{\Gamma\perp}^{N} \\ \dots & \dots & \dots & Kutta & \dots & \dots \end{bmatrix} \begin{bmatrix} q^1 \\ \vdots \\ q^k \\ \vdots \\ q^N \\ \Gamma \end{bmatrix} = \begin{bmatrix} \vec{U^1}_{movement} \cdot \vec{e}_{\perp}^{1} \\ \vdots \\ \vec{U^k}_{movement} \cdot \vec{e}_{\perp}^{k} \\ \vdots \\ \vec{U^N}_{movement} \cdot \vec{e}_{\perp}^{N} \\ \vec{U}_{movement}^{TEup} \cdot \vec{e}_{\parallel}^{TEup} - \vec{U}_{movement}^{TEdown} \cdot \vec{e}_{\parallel}^{TEdown} \end{bmatrix} \tag{10}$$

The problem is solved by finding the intensity of each source and the strength of the circulation through Gaussian elimination. The influence matrix depends solely on the geometry, and is generated on the code initialization[2].

1.2 Boundary Conditions

The problem of flow in straight or steady curved path is essentially the same from a physical point of view, and the only differences arise in the boundary conditions, specially the impermeability condition.

1.2.1 Flow in straight path

In a straight movement, corresponding to a uniform free stream, the movement vector is constant in space, and depends only on angle of attack:

$$\vec{U^k}_{movement} = -\cos\alpha \vec{e_x} - \sin\alpha \vec{e_y} \tag{11}$$

[2] which corresponds to the instanciation of an *inviscid_panel_case* object

1.2.2 Flow in curved path

On the other hand, on a curved path, each panel is forced with a different free stream component, due to the effect of rotation.

We start by recalling our frame, in which the unit chord airfoil leading edge coincides with the origin $(0,0)$ and the trailing edge lies on the x axis $(1,0)$. We can now define the center of rotation as a function of the chord over radius ratio[3] , and choose the airfoil attachment point to coincide with the quarter chord point, $x_{attachment} = x_{center} = 1/4$:

$$\vec{p}_{center} = \left(x_{center} \; , \; \frac{1}{c/R} \right) = \left(\frac{1}{4} \; , \; -\frac{1}{c/R} \right) \tag{12}$$

In order to define the perceived speed at an arbitrary point $\vec{p} = (x, y)$, we write the vector between this point and the center of rotation $\vec{r}$:

$$\vec{r} = \vec{p} - \vec{p}_{center} = \left(x - \frac{1}{4} \; , \; y + \frac{1}{c/R} \right) \tag{13}$$

from which the radius of rotation r is easily obtained as:

$$r = \sqrt{\left(x - \frac{1}{4} \right)^2 + \left(y + \frac{1}{c/R} \right)^2} \tag{14}$$

and the radial unit vector comes naturally as:

$$\vec{e}_r = \frac{\left(x - \frac{1}{4} \; , \; y + \frac{1}{c/R} \right)}{\sqrt{\left(x - \frac{1}{4} \right)^2 + \left(y + \frac{1}{c/R} \right)^2}} \tag{15}$$

given all movement takes place in the azimuthal direction, we use the radial unit vector to derive the azimuthal unit vector $\vec{e}_\theta$ from the condition that these two vectors must be normal to each other and have unit norm:

$$\vec{e}_\theta \cdot \vec{e}_r = 0 \qquad \Rightarrow \qquad \vec{e}_\theta = \frac{\left(-y - \frac{1}{c/R} \; , \; x - \frac{1}{4} \right)}{\sqrt{\left(x - \frac{1}{4} \right)^2 + \left(y + \frac{1}{c/R} \right)^2}} \tag{16}$$

We are now able to write the perceived speed at any point recalling that in solid rotation the speed magnitude is proportional to the distance to the center of rotation:

$$\vec{U}_{mov} = r\vec{e}_\theta = \left(-y - \frac{1}{c/R} \; , \; x - \frac{1}{4} \right) \tag{17}$$

which is written at each panel $\vec{U^k}_{mov}$ as a function of its control point coordinates $\left(x_{cp}^k, y_{cp}^k \right)$:

$$\vec{U^k}_{mov} = \left(-y_{cp}^k - \frac{1}{c/R} \; , \; x_{cp}^k - \frac{1}{4} \right) \tag{18}$$

and used as boundary condition.

[3] Or alternatively the solidity σ

References

[1] W.H. Mason , Applied Aerodynamics Course Notes, Virginia Tech, 1997