Modelling airfoil flaps with an orthogonal curvilinear coordinate transformation

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Abstract

We follow Goldstein (1938) in our exploration of Lamé's (1859) formidable work on curvilinear coordinate systems to define an conformal (angle-preserving) airfoil transformation parametrized by an arbitrary chordline deformation interpolant. There is nothing fundamentally new in this work, but the author hopes this piece will serve as advocacy material for the formidable potential of orthogonal curvilinear coordinate tranformations.

We propose an hybrid analytical-numerical implementation which works as a prototyping code to support many of our short term activities: continuous flap modelling for VI-code (e.g. R/Xfoil) analysis and airfoil optimization, generation of equivalent shapes to account for inhomogeneous inflow cases in standard panel codes (e.g. curved inflow in a VAWT case), thin mesh generation and deformation for hybrid Lagrangian-Eulerian approaches (e.g. PhyFLOW) or dynamic differential operator modification in ALE FV codes applied to fluid-structure interaction (in the longer term!).

Contents

1 Flap rotation

1.1 Hinge line deformation

1.1.1 Solid Body Rotation

Consider a flap rotated as a rigid body by an angle θ around its hinge. The deformed hinge line will be given as a piecewise linear function inflected at the hinge position x_h:

$$f_{(x)} = \begin{cases} c & , \quad 0 < x < x_h \\ c + m\left(x - x_{hinge}\right) & , \quad x_h \leq x \end{cases}$$

where m is the slope of the deformed hinge line, directly related to the flap angle θ:

$$m = \tan\theta$$

and the first derivative of the hinge line will written:

$$\frac{df}{dx} = \begin{cases} 0 & , \quad x < x_{hinge} \\ m & , \quad x \geq x_{hinge} \end{cases}$$

clearly showing that the "pure" solid body rotation introduces a first derivative discontinuity, and hence a “corner” in the hinge line.

1.1.2 Smoothing Solid Body Rotation

To eliminate 1^{st} derivative discontinuities we will start by developping a softened , C^2 continuous form of the hingeline deformation function f whose softness will depend on a knee parameterδ:

$$f_{(x)} = \begin{cases} c & , \quad x < x_h - \delta \\ p_{(x)} & , \quad x_h - \delta \leq x < x_h + \delta \\ c + m\left(x - x_h\right) & , \quad x_h + \delta \leq x \end{cases}$$

and $p_{(x)}$ will be chosen to be a smooth function such that f preserves C^2 continuity. For this purpose, we set that $p_{(x)}$ will abide to the following boundary conditions:

x	p	$\frac{dp}{dx}$	$\frac{d^2p}{dx^2}$
$x_h - \delta$	c	0	0
$x_h + \delta$	$c + m\delta$	m	0

Table 1: g-eta-x function boundary conditions

1.1.3 Looking for a C2-smooth knee function

The p function can also be written in terms of a C^2 smooth function g, a scaling δm, and a coordinate tranformation $\eta_{(x)}$

$$p_{(x)} = 2\delta m g_{\left(\eta_{(x)}\right)} + c$$

with the coordinate transformation $\eta_{(x)}$ defined as:

$$\eta_{(x)} = \frac{x - (x_h - \delta)}{(x_h + \delta) - (x_h - \delta)} = \frac{x - x_h}{2\delta} + \frac{1}{2}$$

in which case the boundary conditions can be restated as:

x	η	g	$\frac{\partial g}{\partial \eta}$	$\frac{\partial^2 g}{\partial \eta^2}$
$x_h - \delta$	0	0	0	0
$x_h + \delta$	1	$\frac{1}{2}$	1	0

Table 2: g-eta-x function boundary conditions

Considering these boundary conditions, we decide to look for a suitable expression for g in the form a polynomial[1]. The 6 boundary conditions produce 6 equations binding the g function, and so, assuming the conditions are independent, there will be one and only on 6th order, 5th degree, polynomial satisfying all six boundary conditions.

To look for that polynomial, we will first write it as a dot product between its six coefficients and the canonical polynomial base elements:

$$g_{(\eta)} = \sum_{i=0}^{5} a_i \eta^i = \begin{bmatrix} 1 & \eta & \eta^2 & \eta^3 & \eta^4 & \eta^5 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

We can now write the 1^{st} and 2^{nd} derivatives of g in a similar form:

$$\frac{\partial g}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\sum_{i=0}^{5} a_i \eta^i\right) = \sum_{i=0}^{5} a_i \frac{\partial}{\partial \eta}\left(\eta^i\right) = \begin{bmatrix} 0 & 1 & 2\eta & 3\eta^2 & 4\eta^3 & 5\eta^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

[1]Other forms could also be chosen: trigonometric function series, exponentials, and many others. But my intuition tells me polynomials are usually a good starting point for this sort of problems!

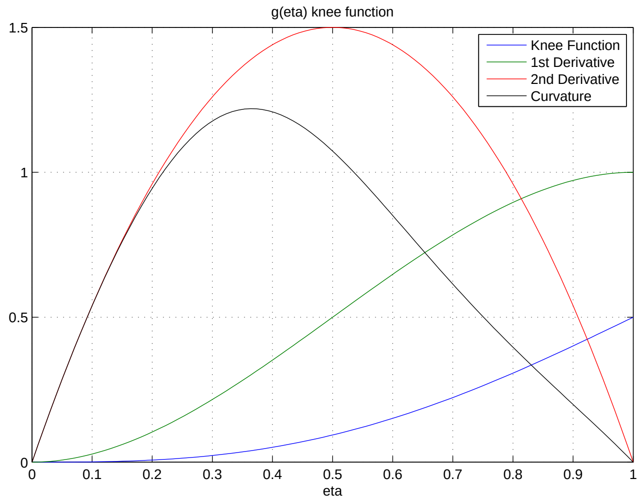


Figure 1: The g-eta knee function

$$\frac{\partial^2 g_{(\eta)}}{\partial \eta^2} = \sum_{i=0}^{5} a_i \frac{\partial^2}{\partial \eta^2}\left(\eta^i\right) = \begin{bmatrix} 0 & 0 & 2 & 6\eta & 12\eta^2 & 20\eta^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

providing us for a straightforward way to write a linear system :

$$\begin{bmatrix} \left[g_{(\eta=0)}\right] \\ \left[\frac{\partial g}{\partial \eta}_{(\eta=0)}\right] \\ \left[\frac{\partial^2 g}{\partial \eta^2}_{(\eta=0)}\right] \\ \left[g_{(\eta=1)}\right] \\ \left[\frac{\partial g}{\partial \eta}_{(\eta=1)}\right] \\ \left[\frac{\partial^2 g}{\partial \eta^2}_{(\eta=1)}\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$$

and express it in matrix form in terms of the coefficients:

$$[M]\{A\} = \{B\} \qquad \Leftrightarrow \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}$$

Fortunately, the matrix has rank 6, and this means that:

1. The boundary conditions are indeed independent from each other
2. The matrix is invertible and the linear system has a unique solution

The solution is obtained by inverting the matrix numerically[2], and has a very clean form:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1/2 \\ 0 \end{bmatrix}$$

We can write the g function and its derivatives explicitly as:

$$g_{(\eta)} = \eta^3 - \frac{\eta^4}{2} \qquad , \qquad \frac{\partial g}{\partial \eta} = 3\eta^2 - 2\eta^3 \qquad , \qquad \frac{\partial^2 g}{\partial \eta^2} = 6\eta - 6\eta^2$$

So that the curvature of the $g - \eta$'s graphline is written as:

$$\kappa_g = \frac{\frac{\partial^2 g}{\partial \eta^2}}{\left(1 + \left(\frac{\partial g}{\partial \eta}\right)^2\right)^{\frac{3}{2}}} = \frac{6\eta - 6\eta^2}{\left((3\eta^2 - 2\eta^3)^2 + 1\right)^{\frac{3}{2}}}$$

for which there exists an analytical primitive in η:

$$\int \kappa_g d\eta = \frac{\left(3\eta^2 - 2\eta^3\right)\sqrt{\left(3\eta^2 - 2\eta^3\right)^2 + 1}}{4\eta^6 - 12\eta^5 + 9\eta^4 + 1}$$

which will facilitate the construction of an orthogonal curvilinear coordinate transformation.

1.2 Scaling the smooth knee function

We recall the definition of the η coordinate transformation to write its jacobian

$$\frac{\partial \eta}{\partial x} = \frac{1}{2\delta}$$

so that we can state the dimensional p function and its 1^{st} derivative in terms of $g's$ derivatives:

$$p_{(x)} = 2\delta m g_{\left(\eta_{(x)}\right)} + c$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \eta}\frac{\partial \eta}{\partial x} = 2\delta m \frac{\partial g}{\partial \eta}\frac{1}{2\delta} = m\frac{\partial g}{\partial \eta} \qquad , \qquad m, \delta \perp x, \eta$$

[2]Manual gaussian elimination would also work on this system!

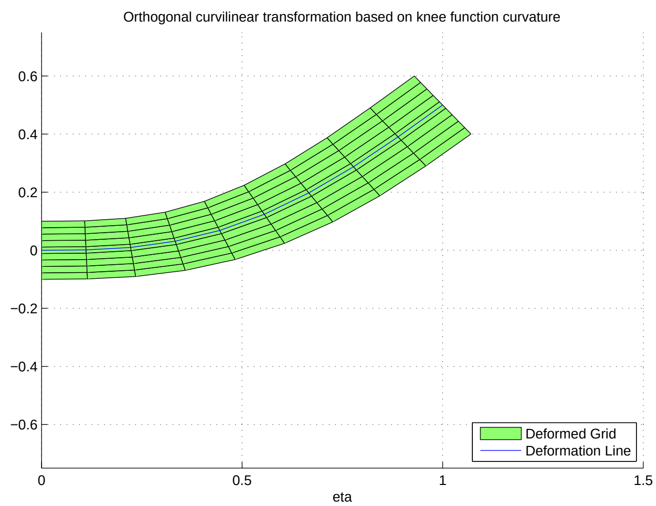


Figure 2: An orthogonal curvilienear transformation defined from the g-eta knee function curvature primitive in eta *(not renormalized)*

and for the 2^{nd} derivative we get:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left(m\frac{\partial g}{\partial \eta}\right) = m\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial \eta}\right) = m\frac{\partial^2 g}{\partial \eta^2}\frac{\partial \eta}{\partial x}$$

$$= \frac{m}{2\delta}\frac{\partial^2 g}{\partial \eta^2} = \frac{m}{2\delta}\left(6\eta - 6\eta^2\right)$$

So the curvature of the $p-x$ graphline will be given as:

$$\kappa^p_{(x)} = \frac{\frac{\partial^2 p}{\partial x^2}}{\left(1+\left(\frac{\partial p}{\partial x}\right)^2\right)^{\frac{3}{2}}} = \frac{\frac{m}{2\delta}\frac{\partial^2 g}{\partial \eta^2}}{\left(1+\left(m\frac{\partial g}{\partial \eta}\right)^2\right)^{\frac{3}{2}}} = \frac{1}{2\delta}\frac{m\frac{\partial^2 g}{\partial \eta^2}}{\left(1+\left(m\frac{\partial g}{\partial \eta}\right)^2\right)^{\frac{3}{2}}}$$

$$= \frac{1}{2\delta}\frac{m\left(6\eta - 6\eta^2\right)}{\left(m^2\left(3\eta^2 - 2\eta^3\right)^2 + 1\right)^{\frac{3}{2}}}$$

Before we take the x primitive of κ_p we observe that:

$$d\eta = d\left(\frac{x - x_1}{2\delta}\right) = \frac{dx}{2\delta} \quad \Leftrightarrow \quad dx = 2\delta d\eta$$

So that we can write:

$$K^p_{(x)} = \int \kappa^p dx = \int \kappa^p 2\delta d\eta = \int \frac{1}{2\delta} \frac{m \frac{\partial^2 g}{\partial \eta^2}}{\left(1 + \left(m \frac{\partial g}{\partial \eta}\right)^2\right)^{\frac{3}{2}}} 2\delta d\eta = \int \frac{m \frac{\partial^2 g}{\partial \eta^2}}{\left(1 + \left(m \frac{\partial g}{\partial \eta}\right)^2\right)^{\frac{3}{2}}} d\eta$$

and find an analytical primitive[3]:

$$K^p_{(x)} = \int \kappa^p dx = \frac{m\left(3\eta^2 - 2\eta^3\right)\sqrt{m^2\left(3\eta^2 - 2\eta^3\right)^2 + 1}}{4m^2\eta^6 - 12m^2\eta^5 + 9m^2\eta^4 + 1} + C_{kp} \qquad , \quad \eta = \eta_{(x)}$$

1.3 The smoothed hinge line deformation function

Following our definition for the smoothed hinge line deformation function $f \in C^2$ we can write:

$$f_{(x)} = \begin{cases} c & , \quad x < x_h - \delta \\ 2\delta m g_{\left(\eta_{(x)}\right)} + c & , \quad x_h - \delta \leq x < x_h + \delta \\ c + m\left(x - x_{hinge}\right) & , \quad x_h + \delta \leq x \end{cases}$$

together with its first derivative:

$$\frac{df}{dx} = \begin{cases} 0 & , \quad x < x_h - \delta \\ m \left.\frac{\partial g}{\partial \eta}\right|_{\eta_{(x)}} & , \quad x_h - \delta \leq x < x_h + \delta \\ m & , \quad x_h + \delta \leq x \end{cases}$$

and its second derivative:

$$\frac{d^2 f}{dx} = \begin{cases} 0 & , \quad x < x_h - \delta \\ \frac{m}{2\delta} \left.\frac{\partial^2 g}{\partial \eta^2}\right|_{\eta_{(x)}} & , \quad x_h - \delta \leq x < x_h + \delta \\ 0 & , \quad x_h + \delta \leq x \end{cases}$$

which implies that the curvate of the $f - x$ line will be null everywhere except in the $[x_h - \delta\,,\, x_h + \delta]$ interval, so that we can write:

$$\kappa^f_{(x)} = \begin{cases} 0 & , \quad x < x_h - \delta \\ \kappa^p_{(x)} & , \quad x_h - \delta \leq x < x_h + \delta \\ 0 & , \quad x_h + \delta \leq x \end{cases}$$

which therefore admits a straightforward analytical primitive:

$$K^f_{(x)} = \begin{cases} 0 & , \quad x < x_h - \delta \\ K^p_{(x)} & , \quad x_h - \delta \leq x < x_h + \delta \\ K^p_{(x_h + \delta)} & , \quad x_h + \delta \leq x \end{cases}$$

[3] With the symbolic toolbox!

where the constant C_{kp} of the $K^p_{(x)}$ function will be chosen such that:

$$K^p_{(x_h-\delta)} = 0$$

whereby we can write

$$\eta_{(x_h-\delta)} = 0 \quad \Rightarrow \quad K^p_{(x_h-\delta)} = 0 + C_{kp} \quad \Rightarrow \quad C_{kp} = 0$$

so K^p is explicitly written as:

$$K^p_{(x)} = \frac{m\left(3\eta^2 - 2\eta^3\right)\sqrt{m^2\left(3\eta^2 - 2\eta^3\right)^2 + 1}}{4m^2\eta^6 - 12m^2\eta^5 + 9m^2\eta^4 + 1} \qquad , \quad \eta = \eta_{(x)}$$

2 The Orthogonal Curvilinear Transformation

We follow Goldstein (1938) in our exploration of Lamé's (1859) formidable work on curvilinear coordinate systems, to define a transformation:

$$T : (x, y) \to (s, n)$$

from it's Jacobian:

$$ds = h_1 dx \qquad , \qquad dn = h_2 dy$$
$$\begin{bmatrix} ds \\ dn \end{bmatrix} = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

with the h functions given in terms of the reference line curvature:

$$h_1 = h_{1(x)} = 1 + \kappa y \qquad\qquad h_2 = 1$$

We will now try to find an explicit expression for the transformation T by integrating the above definition in terms of differential elements. This is easier for the second coordinate so that is where we will start to illustrate the procedure:

$$n = \int dn = \int h_2 dy \qquad \Rightarrow n = y + C_{yn}$$

where $C_{yn}$is a constant such that:

$$(n = 0 \Leftrightarrow y = 0) \quad \Rightarrow \quad C_{yn} = 0 \quad \Rightarrow \quad n = y$$

And we are now ready to attack the first coordinate:

$$ds = h_1 dx = (1 + \kappa y)\, dx \quad \Leftrightarrow \quad \int ds = \int (1 + \kappa y)\, dx$$

$$\Leftrightarrow \quad s = x + y \int \kappa dx \qquad , \quad y \perp x$$

Defining

$$K_{(x)} = \int \kappa dx$$

we can write:

$$s_{(x,y)} = x + yK_{(x)}$$

So the T transformation can be written as:

$$\begin{bmatrix} s \\ n \end{bmatrix} \equiv T_{(x,y)} = \begin{bmatrix} x + yK_{(x)} \\ y \end{bmatrix}$$

2.1 For the particular case of a flapped airfoil

In this case we can use the expression for $K^f_{(x)}$ of the deformed hingeline:

$$K^f_{(x)} = \begin{cases} 0 & , \quad x < x_h - \delta \\ K^p_{(x)} & , \quad x_h - \delta \leq x < x_h + \delta \\ K^p_{(x_h+\delta)} & , \quad x_h + \delta \leq x \end{cases}$$

where

$$K^p_{(x)} = \frac{m\left(3\eta^2 - 2\eta^3\right)\sqrt{m^2\left(3\eta^2 - 2\eta^3\right)^2 + 1}}{4m^2\eta^6 - 12m^2\eta^5 + 9m^2\eta^4 + 1} \qquad , \quad \eta = \eta_{(x)}$$

$$\eta_{(x)} = \frac{x - x_h}{2\delta} + \frac{1}{2} \qquad , \qquad m = \tan\theta \qquad , \qquad \delta = \text{chosen!}$$

Which provides an effective way to deform airfoil shapes and meshes to account for airfoil flapping, as shown on figure 3, generated with our first code implementation, which is fully vectorized, and therefore fast for a Matlab code.

2.2 Towards a general approach

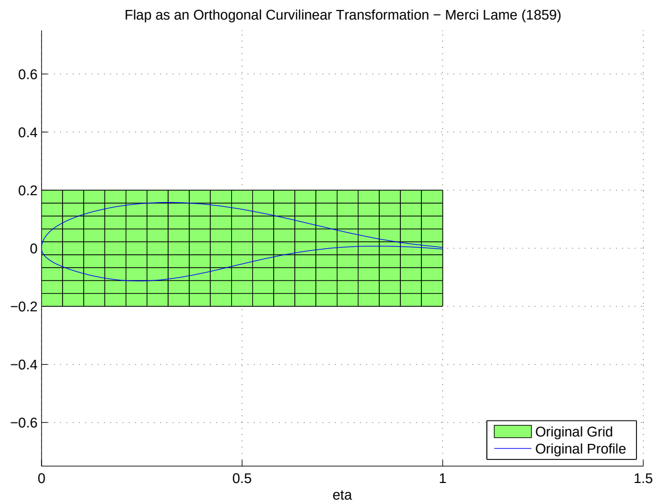
In general, the curvature of a graph line for any function f is written as:

$$\kappa_g = \frac{\frac{\partial^2 f}{\partial x^2}}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)^{\frac{3}{2}}}$$

and this expression has a general primitive in x given as[4]:

$$K = \int \kappa dx = \int \frac{\frac{\partial^2 f}{\partial x^2}}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)^{\frac{3}{2}}} dx = \frac{\frac{\partial f}{\partial x}}{\left(1 + \left(\frac{\partial f}{\partial x}\right)^2\right)^{\frac{1}{2}}} + C$$

[4] Computed with Wolfram Alpha, even though it would not be impossible to see it by hand!



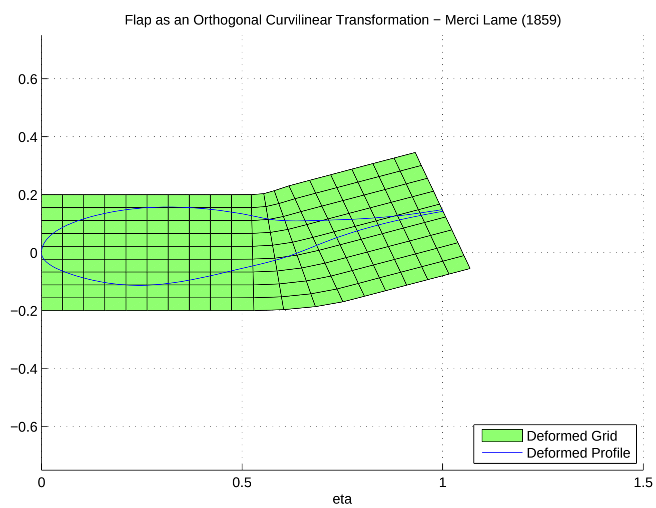


Figure 3: Original and Deformed airfoils on Mesh (not renormalized, purely afine transformation)

And this opens many possibilities, namely to calculate transformations along arbitrarly defined C^2 continuous lines for which we have the first and second derivatives[5].

In fact, arbitrary deformations can be defined based on a set of points describing an arbitraty.

3 Renormalizing the Hinge Line

The hinge line depicted in figure 2 does not have the same lenght as the original hingeline. However, on a real flapping airfoil the hinge line retains its lenght and the chord may change slightly [6].

It is therefore desirable to enhance our transformation by renormalizing the hinge line. Our last code implementation incorporates an approximate renormalization of the hinge line, and the details are described in the code comments.

The current implementation is very simple, but has some limitations from an analytical standpoint:

- The purely affine character is lost, but the renormalized transformation remains quasi afine
- The renormalized transformation introduces a single 2nd derivative discontinuitie at the hinge point.

As such, that the current implementation is only accurate for moderate flap angles:

1. Errors are negligible at 10degrees
2. Noticeable but small at 20degrees
3. Relevant at 45 degrees

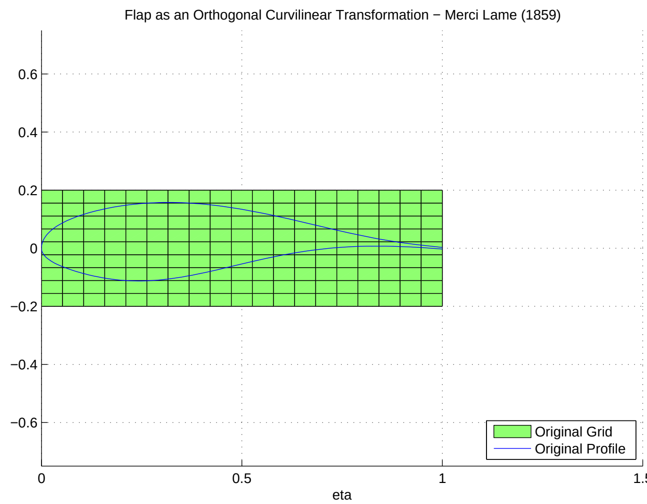
A generalized, fully C^2 version valid for large flapping angles may could be developped later. We have ideas on how to do that while implementing an adaptative smoothing parameter to deal with very thick airfoils, but that did not seem necessary at this stage.

Figures 4 and 5 show the the renormalized transformation. The circle of figure 5 highlights that the Trailing Edge position moves in the same way as it would with a real flap, because the lenght of the total chordline is preserved.

The purely afine transformation did not preserve this important feature, nor the smooth transformations that are currently in use!

[5] In fact, the 1st derivative will be sufficient whenever the constant can be calculated otherwise, for example by matching two points of the T transformation.

[6] if we define it as the distance from the trailing edge to the leading edge, which becomes debatable!



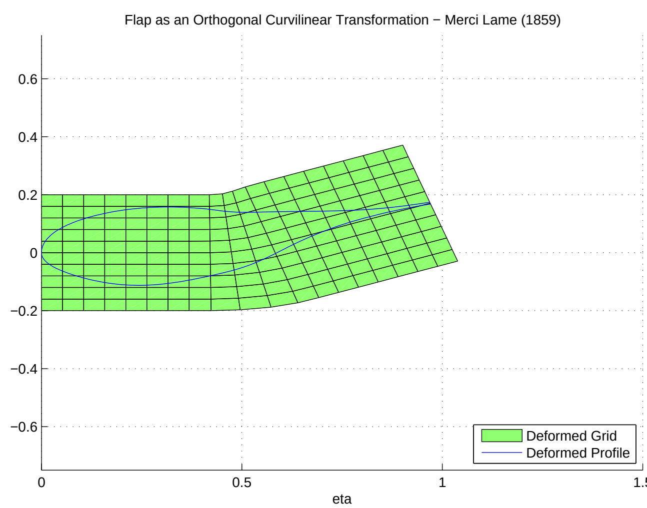


Figure 4: Original and Deformed airfoils on Mesh (not renormalized, purely afine transformation)

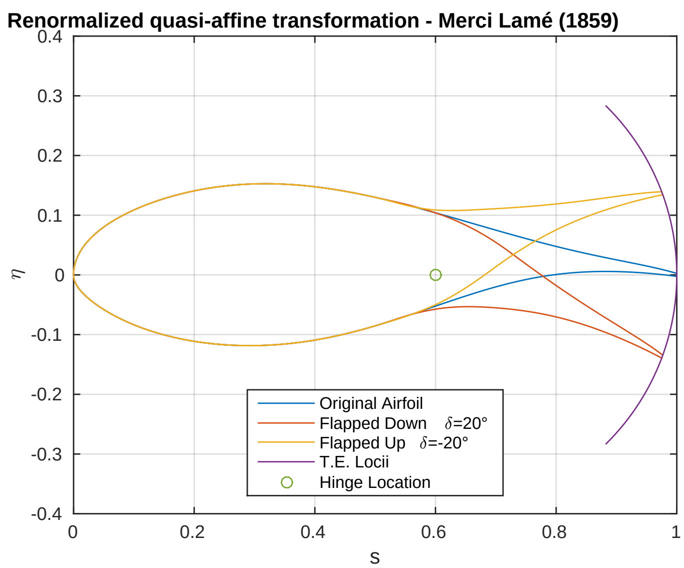


Figure 5: Renormalized Quasi Affine Flapping Airfoil Transformation, a practical example