1 Quantification of code accuracy

The accuracy of numerical predictions is quantified with a Least Squares norm regularized for the extent of data coverage. Here is an example for lift polars, where C_l^{exp} denotes the measured lift coefficient and C_l^{sim} denotes the simulated value:

$$\epsilon_i^{C_l} = (\alpha_{max} - \alpha_{min})^{-1} \left(\int_{\alpha_{min}}^{\alpha_{max}} \left(C_l^{sim} - C_l^{exp} \right)^2 d\alpha \right)^{\frac{1}{2}}$$

The same approach is adopted for drag and moment coefficient polars, when available. A similar norm is adopted for boundary layer runs, but the integration proceeds along the streamwise coordinate until the end of the dataset at $x = L$:

$$\epsilon_i^{\theta} = \frac{1}{L} \left(\int_0^L \left(\theta_{(x)}^{sim} - \theta_{(x)}^{exp} \right)^2 dx \right)^{\frac{1}{2}}$$

Error measures for each quantity are then defined

2 Parametrization of the Skin Friction Relation

The skin friction closure relation is a function of the form:

$$C_f = f_{(H,Re_\theta)}^{C_f^0}$$

This function has three key features:

1. It tends to infinity as H approaches one:

$$\lim_{H \to 1} C_f = +\infty$$

2. It is positive for share factors below separation $H < H_{sep}$ and negative afterwards:

$$\exists H_{sep} = f_{(Re_\theta)} \ : \ \begin{cases} C_f > 0 & H < H_{sep} \\ C_f < 0 & H > H_{sep} \end{cases}$$

3. And it tends to a nearly constant value in deep separation:

$$\exists C_f^{dsep} = f_{(Re_\theta)} \ : \ \lim_{H \to \infty} C_{f(H,Re_\theta)} = C_f^{dsep}$$

Values of the shape factor at which separation occurs (H_{sep}) and the skin friction in deeply separated C_f^{dsep} flows are not precisely known. A good parametrization must let them be varied by the minimization algorithm, and that is indirectly achieved by the following expression:

$$C_{f(H,Re_\theta,A_i)}^{mod} = S_{(H,Re_\theta,A_i)}^{dim} \left(f_{(H,Re_\theta)}^{C_f^0} + \delta^{C_f} \right) - \delta^{C_f} \qquad , \quad \delta^{C_f} \in \mathbb{R}$$

Where δ^{C_f} is a constant property of the parametrization that enables variations in the shape factor at which the onset of separation occurs, whereas the general shape of the closure relation is taylored by the $S^{dim}_{()}$. The dimensional shape function maps the region for which the closure relation is being taylored into a unit square, over which shape of the curve is taylored with a classic Bernstein polynomial approach.

$$S^{dim}_{(H,Re_\theta,A_i)} = B^{dxx}_{\left(\eta^H_{(H)},\eta^{RT}_{(Re_\theta)},A_i\right)}$$

$$with \quad \eta^H = \begin{cases} 0 & H < u_{ub} \\ \frac{H-H_{lb}}{H_{ub}-H_{lb}} & otherwise \\ 1 & H > u_{ub} \end{cases} \qquad and \quad \eta^{Re_\theta} = \begin{cases} 0 & Re < u_{ub} \\ \ln\left(\frac{Re}{Re_{min}}\right) / \ln\left(\frac{Re_{max}}{Re_{min}}\right) & otherwise \\ 1 & H > u_{ub} \end{cases}$$

3 Bernstein polynomials

In the future, we will use Bernstein polynomials of arbitrary order. However, for this early phase of the work we will restrict our study to Bernstein polynomials of order 6 and degree 5. The basis of Bernstein polynomials of order 6 and degree 5 is written as:

$$\begin{aligned} B^{d5r0}_{(x)} &= x^5 \\ B^{d5r1}_{(x)} &= 5x^4\left(1-x\right) \\ B^{d5r2}_{(x)} &= 10x^3\left(1-x\right)^2 \\ B^{d5r3}_{(x)} &= 10x^2\left(1-x\right)^3 \\ B^{d5r4}_{(x)} &= 5x\left(1-x\right)^4 \\ B^{d5r5}_{(x)} &= \left(1-x\right)^5 \end{aligned} \qquad \begin{aligned} \tfrac{\partial}{\partial x}\left(B^{d5r0}\right) &= 5x^4 \\ \tfrac{\partial}{\partial x}\left(B^{d5r1}\right) &= 5\left(4x^3\left(1-x\right)-x^4\right) \\ \tfrac{\partial}{\partial x}\left(B^{d5r2}\right) &= 10\left(3x^2\left(1-x\right)^2-2x^3\left(1-x\right)\right) \\ \tfrac{\partial}{\partial x}\left(B^{d5r3}\right) &= 10\left(2x\left(1-x\right)^3-3x^2\left(1-x\right)^2\right) \\ \tfrac{\partial}{\partial x}\left(B^{d5r4}\right) &= 5\left(\left(1-x\right)^4-4x\left(1-x\right)^3\right) \\ \tfrac{\partial}{\partial x}\left(B^{d5r5}\right) &= -5\left(1-x\right)^4 \end{aligned}$$

Let us now define the 6 bernstein polynomial (shape) coefficients:

$$A_1, A_2, A_3, A_4, A_5, A_6$$

Used to construct arbitrary polynomials by linear combination of the Bernstein basis:

$$B^{d5}_{(x,A_i)} = A_1 B^{d5r0}_{(x)} + A_2 B^{d5r1}_{(x)} + A_3 B^{d5r2}_{(x)} + A_4 B^{d5r3}_{(x)} + A_5 B^{d5r4}_{(x)} + A_6 B^{d5r5}_{(x)}$$

The derivative of B^{d5} to the x-coordinate is written as:

$$\frac{\partial}{\partial x}\left(B^{d5}_{(x,A_i)}\right) = A_1\frac{\partial}{\partial x}\left(B^{d5r0}_{(x)}\right) + A_2\frac{\partial}{\partial x}\left(B^{d5r1}_{(x)}\right) + A_3\frac{\partial}{\partial x}\left(B^{d5r2}_{(x)}\right) + A_4\frac{\partial}{\partial x}\left(B^{d5r3}_{(x)}\right) + A_5\frac{\partial}{\partial x}\left(B^{d5r4}_{(x)}\right) + A_6\frac{\partial}{\partial x}\left(\right.$$

4 Shape Function

For now, we will only apply changes as a function of the shape factor H. Let us then write:

$$S^{dim}_{(H,Re_\theta,A_i)} = B^{d5}_{\left(\eta^H_{(H)},A_i\right)} \qquad with \qquad with \quad \eta^H = \begin{cases} 0 & H < u_{ub} \\ \frac{H-H_{lb}}{H_{ub}-H_{lb}} & otherwise \\ 1 & H > u_{ub} \end{cases}$$

So that we can write:

$$\frac{\partial S}{\partial H} = \frac{\partial}{\partial H}\left(B^{d5}_{\left(\eta^H_{(H)},A_i\right)}\right) = \frac{\partial}{\partial \eta}\left(B^{d5}_{\left(\eta^H_{(H)},A_i\right)}\right)\frac{\partial \eta}{\partial H} \qquad with \qquad x = \eta^H$$

So:

$$\frac{\partial}{\partial \eta}\left(B^{d5}_{\left(\eta^H_{(H)},A_i\right)}\right) = \frac{\partial}{\partial x}\left(B^{d5}_{(x,A_i)}\right)$$

And:

$$\frac{\partial \eta}{\partial H} = \begin{cases} \frac{\partial}{\partial H}(0) & H < u_{ub} \\ \frac{\partial}{\partial H}\left(\frac{H-H_{lb}}{H_{ub}-H_{lb}}\right) & otherwise \\ \frac{\partial}{\partial H}(1) & H > u_{ub} \end{cases} = \begin{cases} 0 & H < u_{ub} \\ \frac{1}{H_{ub}-H_{lb}} & otherwise \\ 0 & H > u_{ub} \end{cases}$$

Whereby:

$$\frac{\partial S}{\partial H} = \frac{\partial}{\partial x}\left(B^{d5}_{(x,A_i)}\right)\frac{\partial \eta}{\partial H}$$

5 Cf Parametrization Summary

The bernstein polynomials are given as:

$$B^{d5}_{(x,A_i)} = A_1 B^{d5r0}_{(x)} + A_2 B^{d5r1}_{(x)} + A_3 B^{d5r2}_{(x)} + A_4 B^{d5r3}_{(x)} + A_5 B^{d5r4}_{(x)} + A_6 B^{d5r5}_{(x)}$$

Their derivatives come as:

$$\frac{\partial}{\partial x}\left(B^{d5}_{(x,A_i)}\right) = A_1\frac{\partial}{\partial x}\left(B^{d5r0}_{(x)}\right) + A_2\frac{\partial}{\partial x}\left(B^{d5r1}_{(x)}\right) + A_3\frac{\partial}{\partial x}\left(B^{d5r2}_{(x)}\right) + A_4\frac{\partial}{\partial x}\left(B^{d5r3}_{(x)}\right) + A_5\frac{\partial}{\partial x}\left(B^{d5r4}_{(x)}\right) + A_6\frac{\partial}{\partial x}\Big($$

The map from x to H is given as:

$$x = \begin{cases} 0 & H < u_{ub} \\ \frac{H-H_{lb}}{H_{ub}-H_{lb}} & otherwise \\ 1 & H > u_{ub} \end{cases}$$

The jacobian of the transformation is given as:

$$\frac{\partial x}{\partial H} = \begin{cases} 0 & H < u_{ub} \\ \frac{1}{H_{ub}-H_{lb}} & otherwise \\ 0 & H > u_{ub} \end{cases}$$

The dimensional shape function is given as:

$$S^{d5}_{(H,A_i)} = B^{d5}_{\left(\eta^{H}_{(H)},A_i\right)}$$

And it derivative is given as:

$$\frac{\partial S^{d5}}{\partial H} = \frac{\partial}{\partial x}\left(B^{d5}_{(x,A_i)}\right)\frac{\partial x}{\partial H}$$

The new skin friction correlation is given as:

$$C^{mod}_{f(H,Re_\theta,A_i)} = S^{d5}_{(H,A_i)}\left(f^{C^0_f}_{(H,Re_\theta)} + \delta^{C_f}\right) - \delta^{C_f} \qquad , \quad \delta^{C_f} \in \mathbb{R}$$

Its derivative to H is given as:

$$\frac{\partial}{\partial H}\left(C^{mod}_{f(H,Re_\theta,A_i)}\right) = \frac{\partial S^{d5}}{\partial H}\left(f^{C^0_f}_{(H,Re_\theta)} + \delta^{C_f}\right) + S^{d5}_{(H,A_i)}\frac{\partial}{\partial H}\left(f^{C^0_f}_{(H,Re_\theta)}\right)$$

Derivatives to other variables are simpler. Here, the example for Re_θ:

$$\frac{\partial}{\partial Re_\theta}\left(C^{mod}_{f(H,Re_\theta,A_i)}\right) = S^{d5}_{(H,A_i)}\frac{\partial}{\partial Re_\theta}\left(f^{C^0_f}_{(H,Re_\theta)}\right)$$

Done!

5.1 For paper

The skin friction correlation is parametrized by combining the original correlation $\left(C^0_f = f^{C^0_f}_{(H,Re_\theta)}\right)$ with a shape function $S^{d5}_{(H,A_i)}$ whose shape depends on a collection of coefficients A_i determined by the optimization algorithm:

$$C^{mod}_{f(H,Re_\theta,A_i)} = S^{d5}_{(H,A_i)}\left(f^{C^0_f}_{(H,Re_\theta)} + \delta^{C_f}\right) - \delta^{C_f} \qquad , \quad \delta^{C_f} \in \mathbb{R}$$

The shape function consists of a linear combination of 5th degree Bernstein polynomials and $\left(\delta^{C_f},\, H_{lb},\, H_{ub}\right)$ are constants that affect the parametrization scope:

$$B^{d5}_{(x,A_i)} = \sum_{i=0}^{i=5} A_{i+1} B^{5i}_{(x)} \qquad with \qquad \left\{ \begin{array}{l} B^{Mi}_{(x)} = \left(\begin{array}{c} M \\ i \end{array}\right) x^i\,(1-x)^{M-i} \\ x = \frac{H-H_{lb}}{H_{ub}-H_{lb}} \end{array} \right.$$

6 Hstar Parametrization

The key feature here is to preserve the limit value for a collapsing boundary layer:

$$\lim_{H\to 1} H^* = 2$$

This is easily by setting the lower limit of the H intervention region at $H_{min} = 1$ (which we would do anyway), fixing the first Bernstein coefficient of to $A_1^H = 1$ (it is simply not provided as a free parameter to the optimizer) and making a classic CST approach (no offset needed here). So we write:

$$H^{*mod}_{(H,Re_\theta,A_i)} = S^{d5}_{(H,A_i)} \left(f^{H^{*0}}_{(H,Re_\theta)} \right)$$

Whereby the derivative to H comes from the chain rule:

$$\frac{\partial}{\partial H} \left(H^{*mod}_{(H,Re_\theta,A_i)} \right) = \frac{\partial}{\partial H} \left(S^{d5}_{(H,A_i)} \left(f^{H^{*0}}_{(H,Re_\theta)} \right) \right)$$

$$= \frac{\partial}{\partial H} \left(S^{d5}_{(H,A_i)} \right) \left(f^{H^{*0}}_{(H,Re_\theta)} \right) + S^{d5}_{(H,A_i)} \frac{\partial}{\partial H} \left(f^{H^{*0}}_{(H,Re_\theta)} \right)$$

And the other derivatives come more simply, here the example for Re_θ:

$$\frac{\partial}{\partial Re_\theta} \left(H^{*mod}_{(H,Re_\theta,A_i)} \right) = \frac{\partial}{\partial Re_\theta} \left(S^{d5}_{(H,A_i)} \left(f^{H^{*0}}_{(H,Re_\theta)} \right) \right) = S^{d5}_{(H,A_i)} \frac{\partial}{\partial Re_\theta} \left(f^{H^{*0}}_{(H,Re_\theta)} \right)$$

6.1 For paper