Modelling the effect of DBD plasma actuators on Boundary Layer Development

Gael de Oliveira, Ricardo Pereira

February 15, 2017

"Il semble que le travail des ingénieurs, des dessinateurs, des calculateurs du bureau d'études ne soit ainsi en apparence, que de polir et d'effacer, d'alléger ce raccord, d'équilibrer cette aile, jusqu'à ce qu'il n'y ait plus une aile accrochée à un fuselage, mais une forme parfaitement épanouie, enfin dégagée de sa gangue, une sorte d'ensemble spontané, mystérieusement lié, et de la meme qualité que celle du poéme."

Antoine de Saint Exupéry

Terres des Hommes

Contents

Part I

The Boundary Layer Equations with Force Terms

In part I, we will deduce the Von Karmann Integral Boundary layer equations with an additional force term. The path from the Navier Stokes to the Integral equations follows classical literature, and we sticked quite closely to the deduction path highlighted in the MIT lecture notes of reference [1] (classical boundary layer approach).

We added an extra force term (original work) and struggled with some algebraic steps in a more explicit way, but in terms of rationale this part (I) is essentially straightforward.

Chapter 1

Boundary Layer Equations

1.1 The Navier Stokes Equations

The Navier Stokes equations for bidimensional flow are written explicitly as:

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + \frac{1}{\rho}F_x$$

$$\frac{\partial V}{\partial T} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) + \frac{1}{\rho}F_y$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

Where $\boldsymbol{F} = (F_x, F_y)$ is a volume force acting on the flow, expressed in $\left[N/m^2\right]$ per span meter for 2D flows.

1.2 The Boundary Layer PDEs

1.2.1 Scaling the Variables

To obtain the Boundary Layer equations we introduce the following scales for adimensionalization:

$$\begin{cases} x = \frac{X}{L} & y = \frac{Y}{\delta} \\ u = \frac{U}{U_e} & v = \frac{V}{U_e}\frac{L}{\delta} \\ p = \frac{P}{\rho U_e^2} & t = \frac{U_e}{L}T \end{cases}$$

Where L is the tangential length scale (usually the chord) and δ is the normal lenght scale, corresponding to the boundary layer thickness. The above relations can be inverted:

$$\begin{cases} X = xL & Y = y\delta \\ U = uU_e & V = v\frac{U_e\delta}{L} \\ P = \rho U_e^2 p & T = t\frac{L}{U_e} \end{cases}$$

So that they be fed back into the Navier Stokes equations to convert them to adimensional form.

1.2.2 The first momentum equation

The first momentum equation can be rewritten in terms of adimensional coordinatesb by feeding in the scaled variables:

$$\frac{\partial\left(uU_e\right)}{\partial\left(t\frac{L}{U_e}\right)}+\left(uU_e\right)\frac{\partial\left(uU_e\right)}{\partial\left(xL\right)}+\left(v\frac{U_e\delta}{L}\right)\frac{\partial\left(uU_e\right)}{\partial\left(y\delta\right)}=$$

$$=-\frac{1}{\rho}\frac{\partial\left(\rho U_e^2 p\right)}{\partial\left(xL\right)}+\nu\left(\frac{\partial^2\left(uU_e\right)}{\partial\left(xL\right)^2}+\frac{\partial^2\left(uU_e\right)}{\partial\left(y\delta\right)^2}\right)+\frac{1}{\rho}F_x$$

Recalling that differentiation is a linear operation and assuming[1] $U_e \perp (x,y)$, we can write:

$$\frac{U_e^2}{L}\frac{\partial u}{\partial t}+\left(\frac{U_e^2}{L}\right)u\frac{\partial u}{\partial x}+\left(\frac{U_e^2}{L}\right)v\frac{\partial u}{\partial y}=-\frac{U_e^2}{L}\frac{\partial p}{\partial x}+\nu\left(\frac{U_e}{L^2}\frac{\partial^2 u}{\partial x^2}+\frac{U_e}{\delta^2}\frac{\partial^2 u}{\partial y^2}\right)+\frac{1}{\rho}F_x$$

Multiplying both sides by $\frac{L}{U_e^2}$ we get:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\nu}{U_e L}\left(\frac{\partial^2 u}{\partial x^2}+\frac{L^2}{\delta^2}\frac{\partial^2 u}{\partial y^2}\right)+\frac{L}{\rho U_e^2}F_x$$

Where we identify the Reynolds Number:

$$Re=\frac{U_e L}{\nu}\qquad\Rightarrow\qquad\frac{1}{Re}=\frac{\nu}{U_e L}$$

So that the 1^{st} momentum equation reaches its usual adimensional form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{L^2}{\delta^2}\frac{\partial^2 u}{\partial y^2}\right)+\frac{L}{\rho U_e^2}F_x$$

1.2.3 The 2nd momentum equation

The second momentum equation is adimensionalized with the same procedure:

$$\frac{\partial\left(v\frac{U_e\delta}{L}\right)}{\partial\left(t\frac{L}{U_e}\right)}+\left(uU_e\right)\frac{\partial\left(v\frac{U_e\delta}{L}\right)}{\partial\left(xL\right)}+\left(v\frac{U_e\delta}{L}\right)\frac{\partial\left(v\frac{U_e\delta}{L}\right)}{\partial\left(y\delta\right)}=$$

$$=-\frac{1}{\rho}\frac{\partial\left(\rho U_e^2 p\right)}{\partial\left(y\delta\right)}+\nu\left(\frac{\partial^2\left(v\frac{U_e\delta}{L}\right)}{\partial\left(xL\right)^2}+\frac{\partial^2\left(v\frac{U_e\delta}{L}\right)}{\partial\left(y\delta\right)^2}\right)+\frac{1}{\rho}F_y$$

[1] This step is only exact for no external pressure gradient, but typical pressure gradients are sufficiently small for this assumption to be reasonable (except in the case of strong interactions like shocks!)

Exploiting the linear nature of differentiation again, we get:

$$\left(\frac{U_e^2\delta}{L^2}\right)\frac{\partial v}{\partial t}+\left(\frac{U_e^2\delta}{L^2}\right)u\frac{\partial v}{\partial x}+\left(\frac{U_e^2\delta}{L^2}\right)v\frac{\partial v}{\partial y}=$$

$$=-\frac{U_e^2}{\delta}\frac{\partial p}{\partial y}+\nu\left(\frac{U_e\delta}{L}\frac{1}{L^2}\frac{\partial^2 v}{\partial x^2}+\frac{U_e\delta}{L}\frac{1}{\delta^2}\frac{\partial^2 v}{\partial y^2}\right)+\frac{1}{\rho}F_y$$

Multiplying both sides by $\frac{L^2}{U_e^2\delta}$ and reworking we get:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{L^2}{\delta^2}\frac{\partial p}{\partial y}+\frac{\nu}{U_eL}\left(\frac{\partial^2 v}{\partial x^2}+\frac{L^2}{\delta^2}\frac{\partial^2 v}{\partial y^2}\right)+\frac{L^2}{U_e^2\delta}\frac{1}{\rho}F_y$$

Where we identify the Reynolds number again:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\left(\frac{L}{\delta}\right)^2\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\left(\frac{L}{\delta}\right)^2\frac{\partial^2 v}{\partial y^2}\right)+\frac{L^2}{U_e^2\delta}\frac{1}{\rho}F_y$$

1.2.4 Boundary Layer Approximations

All manipulations conducted up to this point were exact, and did not introduce any loss of generality into the Navier Stokes equations. In this section, we will introduce results from order of magnitude analysis, so that we can approximate the Navier Stokes system of PDEs with the so-called boundary layer PDEs:

$$\frac{\nu}{U_eL}\left(\frac{L}{\delta}\right)^2=O\left(1\right)\qquad\Rightarrow\qquad O\left(\delta\right)=O\left(\frac{L}{\sqrt{Re}}\right)$$

Substituting the approximate relation for $\delta\simeq\frac{L}{\sqrt{Re}}$ into the first momentum equation we get a new exact scalling [2]:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{L}{\rho U_e^2}F_x$$

For the second momentum equation we obtain[3]:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\left(\frac{L}{\frac{L}{\sqrt{Re}}}\right)^2\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\left(\frac{L}{\frac{L}{\sqrt{Re}}}\right)^2\frac{\partial^2 v}{\partial y^2}\right)+\frac{L^2}{\rho U_e^2}\frac{1}{\left(\frac{L}{\sqrt{Re}}\right)}F_y$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-Re\frac{\partial p}{\partial y}+\frac{1}{Re}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{L\sqrt{Re}}{\rho U_e^2}F_y$$

We will now attempt to take the limit of the equations as $Re\to\infty$, which induces loss of exactness. For this purpose we group terms in Re while taking

[2] The relation $\delta\simeq\frac{L}{\sqrt{Re}}$ is an approximation, but the momentum equation remains exact, it is just scaled differently than in the previous steps!

[3] Again, this is just a change of scaling, and not an approximation strictu-sensu!

the care to keep our force terms, as their magnitude remains unknown at this stage[4]:

$$Re\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=Re\left(-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{L}{\rho U_e^2}F_x\right)+\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=Re\left(-\frac{\partial p}{\partial y}+\frac{1}{\sqrt{Re}}\frac{L}{\rho U_e^2}F_y\right)+\frac{1}{Re}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$$

Takin the limit as $Re\to\infty$ we get the Adimensional Boundary Layer PDEs:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{L}{\rho U_e^2}F_x$$

$$-\frac{\partial p}{\partial y}+\frac{1}{\sqrt{Re}}\frac{L}{\rho U_e^2}F_y=0$$

Which can be rewritten into dimensional, non scaled coordinates, provided that we exploit the relation $\delta=\frac{L}{\sqrt{Re}}$ to redefine our scallings as:

$$\begin{cases} x=\frac{X}{L} & y=\frac{Y}{L}\sqrt{Re}\\ u=\frac{U}{U_e} & v=\frac{V}{U_e}\sqrt{Re}\\ p=\frac{P}{\rho U_e^2} & t=\frac{U_e}{L}T \end{cases} \tag{1.1}$$

Feeding back into the Adimensional Boundary Layer PDEs, we recover the Dimensional Boundary Layer PDEs[5]:

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=-\frac{1}{\rho}\frac{\partial P}{\partial X}+\nu\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{1}{\rho}F_x$$

$$-\frac{\partial P}{\partial Y}+F_y=0$$

Where mass continuity is kept (for now!):

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$$

[4]The force terms will be considered relevant, even if they are not in the right scaling yet.

[5]Intermediate steps in original deduction, to be found in the plasma_rationale18.pdf file, since Gael's external Hdd died and the lyx/latex files were lost!

Chapter 2

Integral Boundary Layer Equations

As[1] we have reached the Boundary Layer PDEs we are ready to move towards Von Karman's approximation with the integral boundary layer ODEs.

2.1 Defining integral variables

We define the dimensional displacement thickness δ_1

$$\delta_1 = \int_0^\infty \left(1 - \frac{U}{U_e}\right) dY$$

the dimensional momentum thickness δ_2:

$$\delta_2 = \int_0^\infty \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dY$$

and the dimensional energy thickness[2] δ_3:

$$\delta_3 = \int_0^\infty \frac{U}{U_e}\left(1 - \frac{U^2}{U_e^2}\right) dY$$

Similarly, we define the skin friction as:

$$\tau_w = \mu \left(\frac{\partial U}{\partial Y}\right)_{y=0}$$

[1]Note: in this document I will not reproduce the part on the matching of the outer flow here, but it is a good idea to insert this in appendix!

[2]Notice that, for $U \leq U_e$ we have $\delta_2 < \delta_1$ but $\delta_2 < \delta_3$ (counter-intuitive).

And the rate of energy dissipation in the boundary layer due to the action of viscosity D as[3]:

$$D = \int_0^\infty \mu \left(\frac{\partial U}{\partial Y} \right)^2 dY$$

2.2 The integral momentum equation

2.2.1 Preparing for the integration

We will now take the 1^{st}momentum equation of the inner flow matched with the outer flow, written in dimensional coordinates[4]:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = U_e \frac{\partial U_e}{\partial X} + \nu \frac{\partial^2 U}{\partial Y^2} + \frac{1}{\rho} F_x$$

To rework this expression, we take mass continuity of the inner flow written in dimensional coordinates:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

And multiply it by $(U_e - U)$ to obtain:

$$(U_e - U) \frac{\partial U}{\partial X} + (U_e - U) \frac{\partial V}{\partial Y} = 0$$

Adding this form of the continuity equation to the 1^{st} momentum equation and reordering, we get:

$$-\nu \frac{\partial^2 U}{\partial Y^2} = U_e \frac{\partial U_e}{\partial X} - U \frac{\partial U}{\partial X} + (U_e - U) \frac{\partial U}{\partial X} - V \frac{\partial U}{\partial Y} + (U_e - U) \frac{\partial V}{\partial Y} + \frac{1}{\rho} F_x \quad (2.1)$$

2.2.2 Reworking first derivatives in Y into integrands

We will now rework the terms with Y derivatives from equation 2.1, to condense them into a simple integrand:

$$-V \frac{\partial U}{\partial Y} + (U_e - U) \frac{\partial V}{\partial Y} = U_e \frac{\partial V}{\partial Y} - V \frac{\partial U}{\partial Y} - U \frac{\partial V}{\partial Y} = U_e \frac{\partial V}{\partial Y} - \left(V \frac{\partial U}{\partial Y} + U \frac{\partial V}{\partial Y} \right)$$

considering that our assumptions/definitions state that $\frac{\partial U_e}{\partial Y} = 0$ we can add a term multiplied by it when that helps us:

$$= U_e \frac{\partial V}{\partial Y} + V \underbrace{\frac{\partial U_e}{\partial Y}}_{=0} - \left(V \frac{\partial U}{\partial Y} + U \frac{\partial V}{\partial Y} \right) = \left(U_e \frac{\partial V}{\partial Y} + V \frac{\partial U_e}{\partial Y} \right) - \left(V \frac{\partial U}{\partial Y} + U \frac{\partial V}{\partial Y} \right)$$

[3]The integral is finite because $\frac{\partial U}{\partial Y} \to 0$ as we move away from the wall $Y \to \infty$

[4]Section on matching will need to be added at the end of the boundary layer PDE deduction. Work from plasma_rationale18.pdf globally valid but needs some cleaning!

where we notice that the terms we isolated in parentheses correspond to derivatives of multiplications:

$$= \frac{\partial}{\partial Y}\left(U_e V\right) - \frac{\partial}{\partial Y}\left(UV\right) = \frac{\partial}{\partial Y}\left(U_e V - UV\right)$$

2.2.3 Reworking first derivatives in X into integrands

The terms with the first X derivatives from equation 2.1 can be reworked to the ressemble the integrands of the thicknesses:

$$U_e \frac{\partial U_e}{\partial X} - U\frac{\partial U}{\partial X} + \left(U_e - U\right)\frac{\partial U}{\partial X} = U_e \frac{\partial U_e}{\partial X} - 2U\frac{\partial U}{\partial X} + U_e\frac{\partial U}{\partial X}$$

Noticing that $2U\frac{\partial U}{\partial X} = \frac{\partial}{\partial X}\left(U^2\right)$, we are able to write:

$$= U_e \frac{\partial U_e}{\partial X} - \frac{\partial}{\partial X}\left(U^2\right) + U_e\frac{\partial U}{\partial X}$$

adding and subtracting $U\frac{\partial U_e}{\partial X}$ to create integration groups:

$$= U_e \frac{\partial U_e}{\partial X} - \frac{\partial}{\partial X}\left(U^2\right) + \left(U_e\frac{\partial U}{\partial X} + U\frac{\partial U_e}{\partial X}\right) - U\frac{\partial U_e}{\partial X}$$

$$= U_e \frac{\partial U_e}{\partial X} - \frac{\partial}{\partial X}\left(U^2\right) + \frac{\partial}{\partial X}\left(U_e U\right) - U\frac{\partial U_e}{\partial X}$$

$$= \left(U_e - U\right)\frac{\partial U_e}{\partial X} + \frac{\partial}{\partial X}\left(U_e U - U^2\right)$$

2.2.4 Integrating into the momentum equation

We can finally rewrite equation 2.1, mass and momentum, in an integrable form:

$$-\nu\frac{\partial^2 U}{\partial Y^2} = \left(U_e - U\right)\frac{\partial U_e}{\partial X} + \frac{\partial}{\partial X}\left(U_e U - U^2\right) + \frac{\partial}{\partial Y}\left(U_e V - UV\right) + \frac{1}{\rho}F_x$$

and proceed to the integration along the Y coordinate:

$$\int_0^\infty \left(-\nu\frac{\partial^2 U}{\partial Y^2}\right) dY =$$

$$\int_0^\infty \left(U_e - U\right)\frac{\partial U_e}{\partial X} dY + \int_0^\infty \frac{\partial}{\partial X}\left(U_e U - U^2\right) dY$$

$$+ \int_0^\infty \frac{\partial}{\partial Y}\left(U_e V - UV\right) dY + \int_0^\infty \left(\frac{1}{\rho}F_x\right) dY$$

To rework the first integral of the right hand side, we notice that $U_e \perp Y \Rightarrow \frac{\partial U_e}{\partial X}\perp Y$ and put the displacement thickness in evidence:

$$\int_0^\infty \left(U_e - U\right)\frac{\partial U_e}{\partial X} dY = \frac{\partial U_e}{\partial X} U_e \int_0^\infty \left(1 - \frac{U}{U_e}\right) dY = \frac{\partial U_e}{\partial X} U_e \delta_1$$

For the second integral we recall that $X \perp Y$ and assume that the integrals are smooth and defined[5] to identify the momentum thickness δ_2:

$$\int_0^\infty \frac{\partial}{\partial X}\left(U_e U - U^2\right) dY = \frac{\partial}{\partial X}\left(U_e^2 \int_0^\infty \left(\frac{U}{U_e}\left(1 - \frac{U}{U_e}\right)\right) dY\right)$$
$$= \frac{\partial}{\partial X}\left(U_e^2 \delta_2\right)$$

For the third integral we will exploit the no slip condition[6] on the surface $U|_{Y=0} = 0$ and the fact that $(U - U_e) \to 0$ as we move away from the body surface:

$$\int_0^\infty \frac{\partial}{\partial Y}\left(U_e V - UV\right) dY = \left[U_e V - UV\right]_{Y=0}^{Y\to\infty} = \left[U_e V - UV\right]_{Y=0}^{Y\to\infty} = -\left.U_e V\right|_{Y=0}$$

Which covers situations in which there is suction or transpiration on the body surface, as long as no-slip remains[7].

Finally, to handle the left hand side integral we assume $\nu \perp Y$ and recall that, by definition, shear stresses vanish as we move out of the boundary layer:

$$\int_0^\infty \left(-\nu \frac{\partial^2 U}{\partial Y^2}\right) dY = \left[-\nu \frac{\partial U}{\partial Y}\right]_{Y=0}^{Y\to\infty} = -\nu \left.\frac{\partial U}{\partial Y}\right|_{Y\to\infty} + \nu \left.\frac{\partial U}{\partial Y}\right|_{Y=0} = \nu \left.\frac{\partial U}{\partial Y}\right|_{Y=0}$$
$$= \frac{1}{\rho}\left(\mu \left.\frac{\partial U}{\partial Y}\right|_{Y=0}\right) = \frac{\tau_w}{\rho}$$

Replacing the integrals into the momentum (and mass) equation, we get:

$$\frac{\tau_w}{\rho} = \frac{\partial U_e}{\partial X} U_e \delta_1 + \frac{\partial}{\partial X}\left(U_e^2 \delta_2\right) - \left.U_e V\right|_{Y=0} + \int_0^\infty \left(\frac{1}{\rho} F_x\right) dY \qquad (2.2)$$

Which is the *Integral Momentum Boundary Layer Equation*[8], also known as the first *Von Karman Integral Equation.* We have added the transpiration and plasma terms, but will leave the reworking of the plasma term for a latter stage to avoid early loss of generality.

---

[5]Which is a fair assumption in the absence of shocks, forbiden anyway in this formulation! Needed to be able to swap the integral and derivative orders!

[6]Would it make sense to consider that the plasma actuator approximates a situation in which the no slip condition breaks? probably not but nice to think about it!

[7]A slip condition could be inserted easily at this stage leading to a $(UV)|_{Y=0}$ term. This term would only appear when suction/transpiration is not null, and would be much much smaller than the $U_e V|_{Y=0}$ term in any case.

[8]Even though it is a composition of mass and momentum, just like the so-called energy equations (shape factor or not!)

2.3 The integral energy equation

We depart again from continuity and X momentum of the inner flow written in dimensional coordinates:

$$-\nu \frac{\partial^2 U}{\partial Y^2} = U_e \frac{\partial U_e}{\partial X} - U \frac{\partial U}{\partial X} - V \frac{\partial U}{\partial Y} + \frac{1}{\rho} F_x$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

But this time, we multiply the momentum equation by $2U$ and the continuity equation by $\left(U_e^2 - U^2\right)$:

$$-2\nu U \frac{\partial^2 U}{\partial Y^2} = 2UU_e \frac{\partial U_e}{\partial X} - 2U^2 \frac{\partial U}{\partial X} - 2UV \frac{\partial U}{\partial Y} + \frac{2U}{\rho} F_x$$

$$\left(U_e^2 - U^2\right) \frac{\partial U}{\partial X} + \left(U_e^2 - U^2\right) \frac{\partial V}{\partial Y} = 0$$

Adding the two equations, we get:

$$-2\nu U \frac{\partial^2 U}{\partial Y^2} = 2UU_e \frac{\partial U_e}{\partial X} - 2U^2 \frac{\partial U}{\partial X} + \left(U_e^2 - U^2\right) \frac{\partial U}{\partial X} - 2UV \frac{\partial U}{\partial Y} + \left(U_e^2 - U^2\right) \frac{\partial V}{\partial Y} + \frac{2U}{\rho} F_x \tag{2.3}$$

2.3.1 Reworking first derivatives in X into integrands

We start by reordering the terms X derivatives and exploit the distributive property while inverting the chain rule on the differentiation of powers:

$$2UU_e \frac{\partial U_e}{\partial X} - 2U^2 \frac{\partial U}{\partial X} + \left(U_e^2 - U^2\right) \frac{\partial U}{\partial X} = U \left(2U_e \frac{\partial U_e}{\partial X}\right) + U_e^2 \frac{\partial U}{\partial X} - 3U^2 \frac{\partial U}{\partial X}$$

$$= \left(U \frac{\partial}{\partial X}\left(U_e^2\right) + U_e^2 \frac{\partial U}{\partial X}\right) - \frac{\partial}{\partial X}\left(U^3\right) = \frac{\partial}{\partial X}\left(U_e^2 U\right) - \frac{\partial}{\partial X}\left(U^3\right)$$

$$= \frac{\partial}{\partial X}\left(U_e^2 U - U^3\right) = \frac{\partial}{\partial X}\left(U_e^3 \underbrace{\left(\frac{U}{U_e}\left(1 - \frac{U^2}{U_e^2}\right)\right)}_{\delta_3 \text{integrand}}\right)$$

2.3.2 Reworking first derivatives in Y into integrands

To find thickness integrands from the Y derivative terms we rely on the fact that $U_e^2 \perp Y$:

$$-2UV \frac{\partial U}{\partial Y} + \left(U_e^2 - U^2\right) \frac{\partial V}{\partial Y} = U_e^2 \frac{\partial V}{\partial Y} - \left(U^2 \frac{\partial V}{\partial Y} + V\left(2U \frac{\partial U}{\partial Y}\right)\right)$$

$$= U_e^2 \frac{\partial V}{\partial Y} - \left(U^2 \frac{\partial V}{\partial Y} + V \frac{\partial U^2}{\partial Y} \right) = \frac{\partial}{\partial Y} \left(V U_e^2 \right) - \frac{\partial}{\partial Y} \left(V U^2 \right)$$

$$= U_e^2 \frac{\partial}{\partial Y} \left(V \left(1 - \frac{U^2}{U_e^2} \right) \right)$$

2.3.3 Reworking the left hand side

To rework the left hand side, we rely upon a differentiation identity based on the chain rule and follow an approach similar to that of integration by parts[9]:

$$\frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right) = \frac{\partial U}{\partial Y} \frac{\partial U}{\partial Y} + U \frac{\partial^2 U}{\partial Y^2} \quad \Leftrightarrow \quad U \frac{\partial^2 U}{\partial Y^2} = - \left(\frac{\partial U}{\partial Y} \right)^2 + \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right)$$

feeding this into the left hand side of equation 2.3 we get:

$$-2\nu U \frac{\partial^2 U}{\partial Y^2} = 2\nu \left(\frac{\partial U}{\partial Y} \right)^2 - 2\nu \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right)$$

2.3.4 Integrating into the energy equation

Feeding the reworked terms into our energy equation we get:

$$2\nu \left(\frac{\partial U}{\partial Y} \right)^2 - 2\nu \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right) =$$

$$= \frac{\partial}{\partial X} \left(U_e^3 \left(\frac{U}{U_e} \left(1 - \frac{U^2}{U_e^2} \right) \right) \right) + U_e^2 \frac{\partial}{\partial Y} \left(V \left(1 - \frac{U^2}{U_e^2} \right) \right) + \frac{2U}{\rho} F_x$$

Integrating over Y from the wall to infinity yields:

$$\int_0^\infty 2\nu \left(\frac{\partial U}{\partial Y} \right)^2 dY - 2\nu \int_0^\infty \frac{\partial}{\partial Y} \left(U \frac{\partial U}{\partial Y} \right) dY =$$

$$= \int_0^\infty \frac{\partial}{\partial X} \left(U_e^3 \left(\frac{U}{U_e} \left(1 - \frac{U^2}{U_e^2} \right) \right) \right) dY \quad +$$

$$+ U_e^2 \int_0^\infty \frac{\partial}{\partial Y} \left(V \left(1 - \frac{U^2}{U_e^2} \right) \right) dY + \int_0^\infty \left(\frac{2U}{\rho} F_x \right) dY$$

Reworking the first LHS term, we identify the dissipation rate D:

$$\int_0^\infty 2\nu \left(\frac{\partial U}{\partial Y} \right)^2 dY = \frac{2}{\rho} \int_0^\infty \mu \left(\frac{\partial U}{\partial Y} \right)^2 dY = \frac{2D}{\rho}$$

[9] It took some observation of our reference's development to see this! In fact we are "just" preparing an integration by parts... Von Karman was a smart guy!

We show that the second LHS term vanishes as a consequence of no-slip $U|_{Y=0} = 0$ and the absence of shear outside the boundary layer $\lim\limits_{Y\to\infty} \frac{\partial U}{\partial Y} = 0$:

$$2\nu \int_0^\infty \frac{\partial}{\partial Y}\left(U\frac{\partial U}{\partial Y}\right) dY = 2\nu \left[U\frac{\partial U}{\partial Y}\right]_{Y=0}^{Y\to\infty} = 2\nu\,(0-0) = 0$$

Proceeding to the left hand side's first term, we recall that $X \perp Y$ and put the energy thickness in evidence in the first term:

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial X}\left(U_e^3\left(\frac{U}{U_e}\left(1-\frac{U^2}{U_e^2}\right)\right)\right) dY &= \frac{\partial}{\partial X}\left(U_e^3\int_0^\infty \left(\frac{U}{U_e}\left(1-\frac{U^2}{U_e^2}\right)\right) dY\right)\\ &= \frac{\partial}{\partial X}\left(U_e^3\delta_3\right)\end{aligned}$$

To look at the second term of the right hand side:

$$U_e^2\int_0^\infty \frac{\partial}{\partial Y}\left(V\left(1-\frac{U^2}{U_e^2}\right)\right) dY = U_e^2\left[V\left(1-\frac{U^2}{U_e^2}\right)\right]_{Y=0}^{Y\to\infty}$$

we rely on the boundary conditions:

$$\lim_{Y\to\infty}\frac{U^2}{U_e^2} = 1 \qquad , \qquad U|_{Y=0} = 0$$

to conclude that:

$$U_e^2\left[V\left(1-\frac{U^2}{U_e^2}\right)\right]_{Y=0}^{Y\to\infty} = -\left.U_e^2 V\right|_{Y=0}$$

So that we are finally able to write the Integral Boundary Layer Energy Equation[10]:

$$\frac{2D}{\rho} = \frac{\partial}{\partial X}\left(U_e^3\delta_3\right) - \left.U_e^2 V\right|_{Y=0} + \int_0^\infty \left(\frac{2U}{\rho}F_x\right) dY \qquad (2.4)$$

Where we have kept the force term unprocessed again to retain generality.

<<WARNING: minus on suction term just corrected, must be rivered down through the document>>

[10] Which was derived by combining momentum and mass with different coefficients than for the momentum equation (also from momentum and mass!)

Chapter 3

Adimensional Integral Boundary Layer Equations

3.1 Adimensional Groups

We define the skin friction coefficient:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2}$$

And the dissipatoin coefficient:

$$C_D = \frac{D}{\rho U_e^3}$$

Together with the shape factors[1]:

$$H_{12} = \frac{\delta_1}{\delta_2} \qquad , \qquad H_{32} = \frac{\delta_3}{\delta_2}$$

We will further define the adimensionalized edge velocity:

$$u_e = \frac{U_e}{U_\infty}$$

And an adimensional suction velocity[2]:

$$v_0 = \left.\frac{V}{U_\infty}\right|_{Y=0}$$

Finally, we will also use the $x = \frac{X}{L}$ scaling of the boundary layer PDEs deduction, and define the adimensional thicknesses:

$$\delta^* = \frac{\delta_1}{L} \qquad , \qquad \theta = \frac{\delta_2}{L} \qquad , \qquad \theta^* = \frac{\delta_3}{L}$$

where L is the longitudinal lenght scale (usually the chord for airfoils).

[1] Notice that both $H_{12} > 1$ and $H_{32} > 1$, as we have $\delta_2 < \delta_1$ but $\delta_2 < \delta_3$ (counter-intuitive).

[2] With a different scaling than the v's of the boundary layer deduction

3.2 Adimensional Momentum Equation

We will now adimensionalize the Von Karmann Integral Momentum Equation, 2.2 using the groups we just defined. To start, divide both sides by U_e^2

$$\frac{\tau_w}{\rho} = \frac{\partial U_e}{\partial X} U_e \delta_1 + \frac{\partial}{\partial X}\left(U_e^2 \delta_2\right) - U_e V|_{Y=0} + \int_0^{\infty} \left(\frac{1}{\rho} F_x\right) dY$$

$$\frac{\tau_w}{\rho U_e^2} = \frac{\delta_1}{U_e}\frac{\partial U_e}{\partial X} + \frac{1}{U_e^2}\frac{\partial}{\partial X}\left(U_e^2 \delta_2\right) - \frac{1}{U_e^2} U_e V|_{Y=0} + \frac{1}{U_e^2}\int_0^{\infty} \left(\frac{1}{\rho} F_x\right) dY$$

$$\frac{\tau_w}{\rho U_e^2} = \frac{\delta_1}{U_e}\frac{\partial U_e}{\partial X} + \frac{1}{U_e^2}\frac{\partial}{\partial X}\left(U_e^2 \delta_2\right) - \left.\frac{V}{U_e}\right|_{Y=0} + \int_0^{\infty} \left(\frac{1}{\rho U_e^2} F_x\right) dY$$

Reworking term by term, we start by identifying the skin friction term:

$$\frac{\tau_w}{\rho U_e^2} = \frac{1}{2}\frac{\tau_w}{\frac{1}{2}\rho U_e^2} = \frac{C_f}{2}$$

The second term of the right side will combine with the first term:

$$\frac{1}{U_e^2}\frac{\partial}{\partial X}\left(U_e^2 \delta_2\right) = \frac{1}{U_e^2}\left(U_e^2 \frac{\partial \delta_2}{\partial X} + 2\delta_2 U_e \frac{\partial U_e}{\partial X}\right) = \frac{\partial \delta_2}{\partial X} + 2\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X}$$

Inserting:

$$\frac{\delta_1}{U_e}\frac{\partial U_e}{\partial X} + 2\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + \frac{\partial \delta_2}{\partial X} = \left(\delta_1 + 2\delta_2\right)\frac{1}{U_e}\frac{\partial U_e}{\partial X} + \frac{\partial \delta_2}{\partial X}$$

$$= \left(\frac{\delta_1}{\delta_2} + 2\frac{\delta_2}{\delta_2}\right)\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + \frac{\partial \delta_2}{\partial X} = \left(H_{12} + 2\right)\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + \frac{\partial \delta_2}{\partial X}$$

The suction term will be reworked by keeping in mind that in general $U_e \neq U_\infty$:

$$\left.\frac{V}{U_e}\right|_{Y=0} = \frac{U_\infty}{U_\infty}\left.\frac{V}{U_e}\right|_{Y=0} = \frac{\left.\frac{V}{U_\infty}\right|_{Y=0}}{\frac{U_e}{U_\infty}} = \frac{v_o}{u_e}$$

So that we can now rewrite the momentum equation as:

$$\frac{C_f}{2} = \left(H_{12} + 2\right)\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + \frac{\partial \delta_2}{\partial X} - \frac{v_o}{u_e} + \int_0^{\infty}\left(\frac{1}{\rho U_e^2} F_x\right) dY \tag{3.1}$$

Using the $x = \frac{X}{L}$ scaling of the boundary layer PDEs deduction, and considering that $U_\infty \perp U_e$, we find it useful to write:

$$\frac{C_f}{2} = \left(H_{12} + 2\right)\frac{\delta_2}{U_e}\frac{U_\infty}{U_\infty}\frac{\partial U_e}{\partial\left(xL\right)} + \frac{\partial \delta_2}{\partial\left(xL\right)} - \frac{v_o}{u_e} + \int_0^{\infty}\left(\frac{1}{\rho U_e^2} F_x\right) dY$$

Reworking, we get:

$$\frac{C_f}{2} = (H_{12}+2)\,\frac{\frac{\delta_2}{L}}{\frac{U_e}{U_\infty}}\,\frac{\partial\left(\frac{U_e}{U_\infty}\right)}{\partial\,(x)} + \frac{\partial\left(\frac{\delta_2}{L}\right)}{\partial\,(x)} - \frac{v_o}{u_e} + \int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right) dY$$

And substituting with the adimensional groups for the edge speed and the thicknesses, we obtain:

$$\frac{C_f}{2} = (H_{12}+2)\,\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} + \frac{\partial \theta}{\partial x} - \frac{v_o}{u_e} + \int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY$$

Which is the form of the equation that is solved and discretized in R/Xfoil and in our Matlab Solver.

3.3 Adimensional Energy Equation

We will now follow a similar procedure for the integral energy equation, 2.4, and start by dividing both sides by U_e^3 and recalling that $U_e^3 \perp Y$:

$$2\underbrace{\frac{D}{\rho U_e^3}}_{C_D} = \frac{1}{U_e^3}\frac{\partial}{\partial X}\left(U_e^3\delta_3\right) - \underbrace{\frac{V}{U_e}\bigg|_{Y=0}}_{\frac{v_o}{u_e}} + \frac{1}{U_e^3}\int_0^\infty \left(\frac{2U}{\rho}F_x\right) dY$$

The left hand side term is easily related with the dissipation coefficient and the suction term has the same form as in the momentum equation.

$$2C_D = \frac{1}{U_e^3}\frac{\partial}{\partial X}\left(U_e^3\delta_3\right) - \frac{v_o}{u_e} + \frac{1}{U_e^3}\int_0^\infty \left(\frac{2U}{\rho}F_x\right) dY \tag{3.2}$$

However, first right hand side term is harder to rework if we want derivatives of only H_{32} and U_e :

$$\begin{aligned}\frac{1}{U_e^3}\frac{\partial}{\partial X}\left(U_e^3\delta_3\right) &= \frac{1}{U_e^3}\left(U_e^3\frac{\partial \delta_3}{\partial X} + \delta_3\frac{\partial}{\partial X}\left(U_e^3\right)\right)\\ &= \frac{1}{U_e^3}\left(U_e^3\frac{\partial \delta_3}{\partial X} + 3U_e^2\delta_3\frac{\partial U_e}{\partial X}\right) = \frac{\partial \delta_3}{\partial X} + 3\frac{\delta_3}{U_e}\frac{\partial U_e}{\partial X}\end{aligned} \tag{3.3}$$

Before we proceed, we find it useful to study the relation between $\frac{\partial \delta_3}{\partial X}$ and $\frac{\partial H_{32}}{\partial X}$:

$$\frac{\partial H_{32}}{\partial X} = \frac{\partial}{\partial X}\left(\frac{\delta_3}{\delta_2}\right) = \delta_3\frac{\partial}{\partial X}\left(\frac{1}{\delta_2}\right) + \frac{1}{\delta_2}\frac{\partial \delta_3}{\partial X} = -\frac{1}{\delta_2}\frac{\delta_3}{\delta_2}\frac{\partial \delta_2}{\partial X} + \frac{1}{\delta_2}\frac{\partial \delta_3}{\partial X}$$

$$\Leftrightarrow \qquad \frac{\partial \delta_3}{\partial X} = \delta_2\frac{\partial H_{32}}{\partial X} + H_{32}\frac{\partial \delta_2}{\partial X} \tag{3.4}$$

So we just managed to express our $\frac{\partial \delta_3}{\partial X}$ term as the sum of a $\frac{\partial H_{32}}{\partial X}$ term and a $\frac{\partial \delta_2}{\partial X}$. To reach our goal of having only $\frac{\partial H_{32}}{\partial X}$ and $\frac{\partial U_e}{\partial X}$ terms, we make a little trick and resort to the integral momentum equation to relate $\frac{\partial \delta_2}{\partial X}$ with $\frac{\partial U_e}{\partial X}$ with an algebraic relation, as in expression 3.1:

$$\frac{C_f}{2} = (H_{12}+2)\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + \frac{\partial \delta_2}{\partial X} - \frac{v_o}{u_e} + \int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right)dY$$

$$\frac{\partial \delta_2}{\partial X} = \frac{C_f}{2} - (H_{12}+2)\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + \frac{v_o}{u_e} - \int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right)dY$$

Inserting this into expression 3.4, we get:

$$\frac{\partial \delta_3}{\partial X} = \delta_2\frac{\partial H_{32}}{\partial X} + H_{32}\frac{C_f}{2} - (H_{12}+2)H_{32}\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + H_{32}\frac{v_o}{u_e} - H_{32}\int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right)dY$$

Inserting this results into expression 3.3, we write:

$$\frac{1}{U_e^3}\frac{\partial}{\partial X}\left(U_e^3\delta_3\right) = \frac{\partial \delta_3}{\partial X} + 3\frac{\delta_3}{U_e}\frac{\partial U_e}{\partial X}$$

$$= \delta_2\frac{\partial H_{32}}{\partial X} + H_{32}\frac{C_f}{2} - (H_{12}+2)H_{32}\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + H_{32}\frac{v_o}{u_e} - H_{32}\int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right)dY + 3\frac{\delta_3}{U_e}\frac{\partial U_e}{\partial X}$$

$$= \delta_2\frac{\partial H_{32}}{\partial X} + H_{32}\frac{C_f}{2} + (1-H_{12})H_{32}\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} + H_{32}\frac{v_o}{u_e} - H_{32}\int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right)dY$$

We are now able to rewrite the energy equation, 3.2, in terms of $\frac{\partial U_e}{\partial X}$ and $\frac{\partial H_{32}}{\partial X}$ derivatives:

$$2C_D = \delta_2\frac{\partial H_{32}}{\partial X} + H_{32}\frac{C_f}{2} + (1-H_{12})H_{32}\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} \quad + \quad ...$$

$$+H_{32}\frac{v_o}{u_e} - H_{32}\int_0^\infty \left(\frac{1}{\rho U_e^2}F_x\right)dY - \frac{v_o}{u_e} + \frac{1}{U_e^3}\int_0^\infty \left(\frac{2U}{\rho}F_x\right)dY$$

Grouping terms, we get:

$$2C_D - H_{32}\frac{C_f}{2} = \delta_2\frac{\partial H_{32}}{\partial X} + (1-H_{12})H_{32}\frac{\delta_2}{U_e}\frac{\partial U_e}{\partial X} \quad + \quad ...$$

$$+(H_{32}-1)\frac{v_o}{u_e} + \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right)dY - H_{32}\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right)dY$$

We are finally able to complete the adimensionalization, using the $X = xL$ scaling and the adimensionalized edge velocity $u_e = \frac{U_e}{U_\infty}$:

$$2C_D - H_{32}\frac{C_f}{2} = \left(\frac{\delta_2}{L}\right)\frac{\partial H_{32}}{\partial x} + (1-H_{12})H_{32}\frac{\left(\frac{\delta_2}{L}\right)}{\left(\frac{U_e}{U_\infty}\right)}\frac{\partial\left(\frac{U_e}{U_\infty}\right)}{\partial x} \quad + \quad ...$$

$$+(H_{32}-1)\frac{v_o}{u_e} + \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right)dY - H_{32}\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right)dY$$

to identify the adimensional thicknesses we defined in the previous section:

$$2C_D - H_{32}\frac{C_f}{2} = \theta\frac{\partial H_{32}}{\partial x} + (1 - H_{12})\, H_{32}\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} \quad + \quad ...$$

$$+ (H_{32} - 1)\frac{v_o}{u_e} + \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right) dY - H_{32}\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY$$

Which is the final equation, and can be reordered in a way amenable to integration:

$$\frac{\partial H_{32}}{\partial x} = \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} +$$

$$+ (H_{12} - 1)\frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x} - \frac{1}{\theta}(H_{32} - 1)\frac{v_o}{u_e} +$$

$$+\frac{1}{\theta}\left(H_{32}\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY - \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right) dY\right)$$

At this stage, we are nearly ready to rework the equations into an integrable form, but need to treat the plasma terms first.

Part II

Plasma Specific Force Term Treatment

Chapter 4

Plasma Force Coefficients

4.1 Force Momentum Coefficient

The force field term appears in the Von Karman integral momentum equations:

$$\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY$$

This term is adimensional and can be rewritten in terms of the adimensional groups we have been using, and keeping in mind that in general $U_e \neq U_\infty$:

$$\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY = \int_0^\infty \left(\frac{U_\infty^2}{U_\infty^2}\frac{F_x}{\rho U_e^2}\right) dY = \int_0^\infty \left(\frac{U_\infty^2}{U_e^2}\frac{F_x}{\rho U_\infty^2}\right) dY$$

$$= \int_0^\infty \left(\frac{1}{\left(\frac{U_e^2}{U_\infty^2}\right)}\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}\frac{F_x}{\rho U_\infty^2}\right) dY = \int_0^\infty \left(\frac{1}{2u_e^2}\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\right) dY$$

$$= \frac{1}{2u_e^2}\int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\right) dY$$

We are now ready to define the force momentum coefficient C_{FM}:

$$C_{FM} = \int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\right) dY \qquad \Rightarrow \qquad \int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY = \frac{C_{FM}}{2u_e^2}$$

verify that it is really adimensional:

$$\left[\int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\right) dY\right] = [m]\frac{\left[kg \cdot m \cdot s^{-2}/\left(m^2 m\right)\right]}{\left[kg \cdot /\left(m^2 m\right)\right]\left[m^2 \cdot s^{-2}\right]} = [m]\frac{\left[m \cdot s^{-2}\right]}{\left[m^2 \cdot s^{-2}\right]} = \frac{\left[m^2 \cdot s^{-2}\right]}{\left[m^2 \cdot s^{-2}\right]} = adim.$$

and rewrite the adimensional integral momentum equation, without loss of generality:

$$\frac{C_f}{2} = (H_{12}+2)\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} + \frac{\partial \theta}{\partial x} - \frac{v_o}{u_e} + \frac{C_{FM}}{2u_e^2} \tag{4.1}$$

4.2 Force Energy Coefficient

As a consequence of the insertion of the Integral Momentum Equation in the Integral Energy Equation, the force terms appear twice in the Energy Shape Factor equation:

$$H_{32}\int_0^\infty \left(\frac{F_x}{\rho U_e^2}\right) dY - \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right) dY$$

It is straightforward to identify the force term from the Integral Momentum Equation and rewrite the above statement:

$$H_{32}\frac{C_{FM}}{2u_e^2} - \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right) dY$$

The second term can also be rewritten in terms of adimensional groups, using $u = \frac{U}{U_\infty}$ as in the boundary layer PDEs deduction and recalling the $u_e \perp Y$ to get the edge velocity out of the integration:

$$\int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right) dY = \int_0^\infty \left(\frac{U_\infty^2}{U_\infty^2}\frac{2F_x}{\rho U_e^2}\frac{U_\infty}{U_\infty}\frac{U}{U_e}\right) dY$$

$$= \int_0^\infty \left(\frac{1}{\left(\frac{U_e^2}{U_\infty^2}\right)}\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\frac{\left(\frac{U}{U_\infty}\right)}{\left(\frac{U_e}{U_\infty}\right)}\right) dY = \int_0^\infty \left(\frac{1}{u_e^2}\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\frac{u}{u_e}\right) dY$$

$$= \frac{1}{u_e^3}\int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}u\right) dY$$

So that we can now define the force energy coefficient C_{FE}:

$$C_{FE} = \int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}u\right) dY \qquad \Rightarrow \qquad \int_0^\infty \left(\frac{2F_x}{\rho U_e^2}\frac{U}{U_e}\right) dY = \frac{C_{FE}}{u_e^3}$$

And verify that it is adimensional:

$$\left[\int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}u\right) dY\right] = [m]\frac{\left[kg\cdot m\cdot s^{-2}/\left(m^2 m\right)\right]}{\left[kg/\left(m^2 m\right)\right]\left[m^2\cdot s^{-2}\right]} = [m]\frac{\left[m\cdot s^{-2}\right]}{\left[m^2\cdot s^{-2}\right]} = \frac{\left[m^2\cdot s^{-2}\right]}{\left[m^2\cdot s^{-2}\right]} = adim.$$

So that the force terms of the energy shape factor equation become:

$$H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}$$

And we can finally rewrite the energy shape factor equation in terms of these groups, without loss of generality:

$$\frac{\partial H_{32}}{\partial x} = \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} + \underbrace{(H_{12}-1)\frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x}}_{Pressure\,Gradient} - \frac{1}{\theta}\underbrace{(H_{32}+1)\frac{v_o}{u_e}}_{Suction} + \frac{1}{\theta}\underbrace{\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right)}_{Plasma} \tag{4.2}$$

Chapter 5

The Plasma Force Field

5.1 Typical Descriptors

The force field exerted on the flow by a single DBD plasma actuator can be described in terms of fource variables:

1. The total force exerted by the actuator per unit span, F_T^p in $[N/m]$
2. The thickness of the force field, T^p in $[m]$
3. The lenght of the force field, L^p in $[m]$
4. The starting location of the force field, X_0^p in m

And a weighting function describing the field's shape, which is a property of the plasma actuator and remains fairly independent from the flow field. Pereira, Marius and Ragni showed that the plasma field has nearly eliptical equiforce lines.

5.1.1 Adimensional Force Field Lenght

The lenght of the force field can be rewritten in adimensional form with the longitudinal direction length scale, L:

$$l^p = \frac{L^p}{L} \qquad , \quad [adim.]$$

Similarly, the starting stance can be adimensionalized with the longitudinal lenght scale:

$$x_0^p = \frac{X_0^p}{L} \qquad , \quad [adim.]$$

5.1.2 Adimensional Force Field Thickness

The ratio between the thickness of the force field and the boundary layer thickness describes which parts of the boundary layer are affected by the plasma. It is therefore expected to have a key role in the energy balance of the boundary layer, and therefore drive changes in θ^* directly while driving the δ^*changes indirectly. On the other hand, momentum changes are expected to be driven essentially by total plasma force. The thickness scaling is therefore critical to the success of the modelling approach:

$$\bar{t}^p = \frac{T^p}{\delta} = \frac{t^p}{\bar{\delta}} \qquad , \quad [adim.]$$

$$with \qquad t^p = \frac{T^p}{L} \qquad and \qquad \bar{\delta} = \frac{\delta}{L}$$

Where t^p is adimensional but scaled in the longitudinal coordinate, as an intermediate step towards the right, normal scalling $\bar{t}^p$. An alternative scaling for the height take the form a height Reynolds number:

$$Re_{t^p} = \frac{U_e t^p}{\nu} \qquad , \quad [adim.]$$

However, the arguments for the Re_{t^p} approach are fairly weak[1]! The boundary layer velocity profile is stretched accross δ and as such the scaling $\bar{t}^p \simeq \frac{t}{\delta}$ gives the best indication of which part of the boundary layer are spanned by the plasma force field.

The shape of the boundary layer is defined by Re_θ and H, but not its exact height, which comes as function of these through Head's H_1 shape parameter, which can be used to recover the scaling variable:

$$H_1 = \frac{\bar{\delta} - \delta^*}{\theta} \qquad , \qquad \bar{\delta} = \frac{\delta}{L}$$

Reworking:

$$\begin{aligned} H_1 = \frac{\bar{\delta} - \delta^*}{\theta} &= \frac{\bar{\delta}}{\theta} - \frac{\delta^*}{\theta} = \frac{\bar{\delta}}{\theta} - H_{12} \\ \Leftrightarrow \qquad & \bar{\delta} = \theta \left(H_1 + H_{12} \right) \\ \Leftrightarrow \qquad & \delta = \delta_2 \left(H_1 + H_{12} \right) \end{aligned}$$

we obtain an expression with which we can rewrite the adimensionalization of the plasma field thickness as:

$$\bar{t}^p = \frac{T^p}{\delta_2 \left(H_1 + H_{12} \right)} = \frac{t^p}{\theta \left(H_1 + H_{12} \right)}$$

This is the key parameter that defines the relative size of the plasma thickness to the size of the boundary layer, as will be discussed in the next section.

And therefore reach get the change to rewrite the Force Momentum and Energy Coefficients in a more compact form:

[1] The only arguments in favor of Re is that it looks more like Re_θ and it is always cool to make a new Reynolds something!

5.2 Force Field Topology

The total force exerted by the actuator on the flow is the integral of the force field over the field action area:

$$F_T^p = \int_{X_0^P}^{X_0^P + L^p} \int_0^{T^p} (F_x)\, dY\, dX$$

Before we proceed, we will define the average dimensional density of the plasma force field, ϕ^p, such that:

$$\phi_x^p = \frac{\int_{X_0^P}^{X_0^P + L^p} \int_0^{T^p} (F_x)\, dY\, dX}{\int_{X_0^P}^{X_0^P + L^p} \int_0^{T^p} dY\, dX}$$

$$\Leftrightarrow \qquad \phi_x^p = \frac{F_T^p}{L^p T^p}$$

Where it is wise to stress that:

$$(\phi_x^p, F_T^p, L^p, T^p) \quad \perp \quad (X, Y)$$

before pointing that the force field density F_x and the average field densityϕ_x^p naturally have the same units:

$$[\phi_x^p] = [F_x] = N/\left(m^2 m\right)$$

5.2.1 Weighting Functions

We will now attempt to express the force field density in terms of weighting functions, following the work of [Marius, Pereira, Ragni], who stated that the field isolines could be approximated with semi-ellipses. For this purpose, we will define two weighting functions. One for the longitudinal coordinate:

$$w^x_{(x,x_0,a)} = \begin{cases} \frac{\pi}{2} sin\left(\frac{\pi(x-x_0)}{a}\right) & , \quad \left(\frac{x-x_0}{a}\right) \in [0,1] \\ 0 & , \quad otherwise \end{cases}$$

and another for the normal coordinate[2]

$$w^y_{(y,b)} = \begin{cases} \frac{\pi}{2} sin\left(\pi\left(\frac{y}{2b} + \frac{1}{2}\right)\right) & , \quad \frac{y}{b} \in [0,1] \\ 0 & , \quad otherwise \end{cases} \tag{5.1}$$

These functions were chosen to satisfy the identities:

$$w^x_{(x,x_0,a)} = w^x_{\left(\frac{x}{c},\frac{x_0}{c},\frac{a}{c}\right)} \qquad , \quad \forall\, y, a, c \in \mathbb{R}$$

[2]Check in WolframAlpha: int(pi/(2*b) * sin(pi * (y/(2*b) +1/2)), dy, 0, b)

$$w^y_{(y,b)} = f\left(\frac{y}{b}\right) \qquad \Rightarrow \qquad w^y_{(y,b)} = w^y_{\left(\frac{y}{C},\frac{b}{C}\right)} \qquad , \quad \forall\, y,b,c \in \mathbb{R} \tag{5.2}$$

such that they would have unit lenght integrals over a broad class of paths:

$$\int_{X_0^p}^{X_0^p+L^p} w^x_{\left(X,X_0^p,L^p\right)} dX = L^p \qquad , \quad \forall\, L^p\,,\, X_0^p\, \in \mathbb{R}$$

$$\int_0^{T_p} w^y_{(Y,T_p)} dY = T^p \qquad , \quad \forall\, T^p\, \in \mathbb{R} \tag{5.3}$$

in any reference frame[3]:

$$\int_{x_0^p}^{x_0^p+l^p} w^x_{\left(x,x_0^p,l^p\right)} dx = l^p \qquad , \quad \forall\, l^p\,,\, x_0^p\, \in \mathbb{R}$$

$$\int_0^{\bar{t}^p} w^y_{(y,\bar{t}^p)} dy = \bar{t}^p \qquad , \quad \forall\, \bar{t}^p \in [0,1] \subset \mathbb{R}$$

where we recall that y was scaled as $y = \frac{Y}{\delta}$, using the normal lenght scale. The average field density and the weighting functions can be combined to write:

$$\int_{X_0^P}^{X_0^P+L^p} \int_0^{T^p} \phi_x^p w^y_{(Y,T_p)} w^x_{\left(X,X_0^p,a\right)} dY\, dX$$

which we can rework into the total actuator force by exploiting the distributive property and independence of variables:

$$\phi_x^p \int_{X_0^P}^{X_0^P+L^p} w^x_{\left(X,X_0^p,a\right)} \left(\int_0^{T^p} w^y_{(Y,T_p)} dY \right) dX =$$

$$= \phi_x^p \underbrace{\left(\int_0^{T^p} w^y_{(Y,T_p)} dY \right)}_{T^p} \underbrace{\left(\int_{X_0^P}^{X_0^P+L^p} w^x_{\left(X,X_0^p,a\right)}\, dX \right)}_{L^p} = \phi_x^p T^p L^p = F_T^p$$

thereby showing that it is consistent to represent the field density as:

$$F_x = \phi_x^p w^y_{(Y,T_p)} w^x_{\left(X,X_0^p,a\right)} \tag{5.4}$$

[3]This last sentence is pretty much a pleonasm (or a Lapalissade;). Still, it is an opportunity to stress that the property holds on any reference frame, but obviously not accross frame transformations, as the Jacobian would then introduce a compensation that shifts the value of the function. It would not be the same expression, but, anyway, confusion on this point could arise easily (I talk for myself!).

5.3 Revisiting the Force Coefficients

5.3.1 Force Momentum Coefficient

The force momentum coefficient can be rewritten in terms of the plasma characteristics by inserting expression 5.4 in its definition:

$$C_{FM} = \int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2} \right) dY = \int_0^\infty \left(\frac{\phi_x^p w^y_{(Y,T_p)} w^x_{\left(X,X_0^p,L^p\right)}}{\frac{1}{2}\rho U_\infty^2} \right) dY$$

Recalling that $\frac{\phi_x^p}{\frac{1}{2}\rho U_\infty^2} \perp Y$ and $w^x_{\left(X,X_0^p,a\right)}$, we can take these terms out of the integral, to write

$$= w^x_{\left(X,X_0^p,L^p\right)} \left(\frac{\phi_x^p}{\frac{1}{2}\rho U_\infty^2} \right) \int_0^\infty \left(w^y_{(Y,T_p)} \right) dY$$

Considering that $w^y_{(Y,T_p)} = 0$ for $Y > T^p$, and the integral properties of the weighting function, we have:

$$\int_0^\infty \left(w^y_{(Y,T_p)} \right) dY = \int_0^{T_p} \left(w^y_{(Y,T_p)} \right) dY = T^p$$

Whereby we rewrite the force momentum coefficient as:

$$C_{FM(X)} = w^x_{\left(X,X_0^p,L^p\right)} \left(\frac{\phi_x^p}{\frac{1}{2}\rho U_\infty^2} \right) T^p = w^x_{\left(X,X_0^p,L^p\right)} \left(\frac{\phi_x^p T^p}{\frac{1}{2}\rho U_\infty^2} \right)$$

Considering that:

$$w^x_{\left(X,X_0^p,L^p\right)} = w^x_{\left(x,x_0^p,l^p\right)}$$

and definning the average force momentum coefficient as:

$$C_{\phi_x^p} = \frac{\phi_x^p T^p}{\frac{1}{2}\rho U_\infty^2}$$

we are able to rewrite the Force Momentum Coefficient in terms of adimensional groups:

$$C_{FM(x)} = w^x_{\left(x,x_0^p,l^p\right)} C_{\phi_x^p}$$

Finally, we point that the Average Force Momentum Coefficient is independent of space and can therefore be understood as a property of the plasma actuator, unlike the Force Momentum Coefficient!

5.3.2 Force Energy Coefficient

The force energy coefficient C_{FE} depends on the combined effect of the plasma field and the boundary layer speed profile at the point of actuation. As such, it is harder to rework than the Force Momentum Coefficient, but we will start in the same way, by feeding expression 5.4 into its definition:

$$C_{FE} = \int_0^\infty \left(\frac{\phi_x^p w^y_{(Y,T_p)} w^x_{\left(X,X_0^p,a\right)}}{\frac{1}{2}\rho U_\infty^2} u \right) dY = w^x_{\left(X,X_0^p,a\right)} \frac{\phi_x^p}{\frac{1}{2}\rho U_\infty^2} \int_0^\infty \left(w^y_{(Y,T_p)} u \right) dY$$

$$= w^x_{\left(X,X_0^p,a\right)} \frac{1}{T^p} \frac{\phi_x^p T^p}{\frac{1}{2}\rho U_\infty^2} \int_0^\infty \left(w^y_{(Y,T_p)} u \right) dY = w^x_{\left(X,X_0^p,a\right)} C_{\phi_x^p} \int_0^\infty \left(\frac{w^y_{(Y,T_p)} u}{T^p} \right) dY$$

Where we used $C_{\phi_x^p} = \frac{\phi_x^p T^p}{\frac{1}{2}\rho U_\infty^2}$ and $w^x_{\left(X,X_0^p,a\right)} \perp Y$ to rework the integral. We will now look in greater detail at the integral term, in an attemp to identify the right approach to model it with a new closure relation. In line with classical boundary layer theory, it would be natural to scale the normal coordinate with the boundary layer thickness. However, as we will attempt to use the Swafford profiles, which are scaled with the momentum thickness, we will define a new scalling consistent with this profile family, $\eta = \frac{Y}{\delta_2}$

We proceed by writting the differential of the $\eta = \frac{Y}{\delta_2}$ scalling:

$$\eta = \frac{Y}{\delta_2} = \frac{\left(\frac{Y}{L}\right)}{\left(\frac{\delta_2}{L}\right)} = \frac{Y}{\theta L} \qquad \Leftrightarrow \qquad Y = \theta L \eta$$

$$\Rightarrow \qquad dY = d\left(\theta L \eta\right) = \left(\theta L\right) d\eta + \left(\eta L\right) d\theta$$

We are interested in the particular case of an integration along a path that follows the Y coordinate, all other things being equal. Because L is constant and $\theta \perp Y$, we can therefore consider $d\theta = 0$, and hence $dY = \left(\theta L\right) d\eta$ so that we can rewrite the integral of the C_{FE} coefficient as:

$$\int_0^\infty \left(\frac{w^y_{(Y,T_p)} u}{T^p} \right) dY = \int_0^\infty \left(\frac{w^y_{(Y,T_p)} u}{T^p} \right) \left(\theta L\right) d\eta = \int_0^\infty \left(\frac{w^y_{(Y,T_p)} u}{\frac{\left(\frac{T^p}{L}\right)}{\theta}} \right) d\eta \quad (5.5)$$

We will now define the Momentum Scaled Plasma Thickness:

$$t_\theta^p = \frac{T^p}{\delta_2} = \frac{\left(\frac{T^p}{L}\right)}{\theta} = \frac{t^p}{\theta}$$

Finally, we will exploit the weighting function identity highlighted in expression 5.2, to write:

$$w^y_{(Y,T_p)} = w^y_{\left(\frac{Y}{\delta_2}, \frac{T_p}{\delta_2}\right)} = w^y_{\left(\eta, t_\theta^p\right)}$$

and the fact that:

$$w^y_{\left(\eta, t_\theta^p\right)} = 0 \quad , \quad \forall \, \frac{\eta}{t_\theta^p} > 1$$

$$\Rightarrow \qquad \frac{w^y_{\left(\eta,t^p_\theta\right)}u}{t^p_\theta} = 0 \quad , \quad \forall\, \eta > t^p_\theta$$

We can rewrite expression 5.5 as a definite integral written entirely in terms of adimensional groups:

$$\int_0^\infty \left(\frac{w^y_{(Y,T_p)}u}{\frac{\left(\frac{T^p}{L}\right)}{\theta}} \right) d\eta = \int_0^{t^p_\theta} \left(\frac{w^y_{\left(\eta,t^p_\theta\right)}u}{t^p_\theta} \right) d\eta$$

This integral drives the effect of the plasma on the Energy of the boundary layer, and hence on the shape factors (H and H^*). It does not depend explicitly[4] on x, therefore making it a good candidate to look for a closure relation. We will call it the Energy Interaction Coefficient C_{EI} and use it to rewrite the Force Energy Coefficient in a convenient form:

$$C_{FE} = w^x_{\left(X,X^p_0,a\right)} C_{\phi^p_x} \int_0^\infty \left(\frac{w^y_{(Y,T_p)}u}{T^p} \right) dY = w^x_{\left(X,X^p_0,a\right)} C_{\phi^p_x} C_{EI}$$

$$with \qquad C_{EI} = \int_0^{t^p_\theta} \left(\frac{w^y_{\left(\eta,t^p_\theta\right)}u_{(\eta)}}{t^p_\theta} \right) d\eta \tag{5.6}$$

A similar process conducted with the $y = \frac{Y}{\delta}$ scaling would lead to a similar form:

$$C_{EI} = \int_0^{\bar{t}^p} \left(w^y_{(y,\bar{t}^p)} \frac{u_{(y)}}{\bar{t}^p} \right) dy = \int_0^{t^p_\theta} \left(\frac{w^y_{\left(\eta,t^p_\theta\right)}u_{(\eta)}}{t^p_\theta} \right) d\eta$$

It is clear that theδ scaling preserves geometric similarity, but the θ scaling is harder to interpret, and there almost seems to be an inconsistency between the two scalings. Even so, both scalings reach as high as each other in the boundary layer and, most importantly, preserve the information contained in the input vector. In fact:

- Considering that the velocity profile is a function of (H, Re_θ)
- Recalling the existence of a closure in (H, Re_θ) for Head's shape factor

We notice the implication:

$$\begin{cases} H_1 = f\left(H, Re_\theta\right) \\ \delta = \delta_2\left(H_1 + H\right) \end{cases} \qquad \Rightarrow \qquad \exists \left(g : \mathbb{R}^2 \to \mathbb{R}\right) \,:\, \frac{\delta}{\delta_2} = g\left(H, Re_\theta\right)$$

and observe that the existence of g such that $\bar{t}^p = \frac{t^p_\theta}{g(H,Re_\theta)}$ implies that:

$$\exists \,\left(G : \mathbb{R}^3 \to \mathbb{R}^3\right) \;:\; (H, Re_\theta, \bar{t}^p) = G\left(H, Re_\theta, t^p_\theta\right)$$

[4]But it depends implicitly on x. As we will show later, it depends on H, Re_θ and $\bar{t}^p$, where H and Re_θ depend on x.

Which means that it is equivalent to define Energy Interaction Coefficient closure as either :

$$C_{EI} = f\left(H, Re_\theta, t_\theta^p\right) \qquad or \qquad C_{EI} = \bar{f}\left(H, Re_\theta, \bar{t}^p\right) \tag{5.7}$$

because the two closures can be written as compositions of closures of the remaining variables:

$$f = \bar{f} \circ G$$

As such, despite the loss of geometric similarity, the $\eta = \frac{Y}{\delta_2}$ scaling does preserve information and hence forms a valid basis to write a closure relation. From a practical standpoint, the fact that the momentum thickness admits a very robust definition outweights the advantages derived from the instinctive nature of the geometric thickness as a scalling variable.

Chapter 6

The Energy Interaction Coefficient

6.1 Closure Strategy

In the last section, we showed that the Force Energy Coefficient C_{FE} could be written in terms of an Energy Interaction Coefficient C_{EI} . The Energy Interaction Coefficient is the convolution integral of the plasma force field normal weighting function with the boundary layer velocity profile, and we showed that it is a function of three variables:

$$(H, Re_\theta, t_\theta^p)$$

The integrand was dominated by the velocity profile of the boundary layer, which can either be estimated from:

- Experimental Data, such as Pereira's PIV results
- A quasi analytical profile family, like the Swafford profiles of reference [2]

For both sources, it is impractical, if not impossible[1], to obtain an exact analytical expression for the Energy Interaction Coefficient Integral. As such we will evaluate the integral over a wide range of conditions and use this dataset to design a compact fit and create a closure relation in the first form of expression 5.7:

$$C_{EI} = f\,(H, Re_\theta, t_\theta^p)$$

We will start by using the Swafford profile as a closure tool and we will later compare our results against the experimental profiles obtained from Pereira's PIV experiments.

[1]Exactness would not be extremely relevant in this context anyway, given that the Swafford profiles, like all Turbulent Boundary Layer profiles are approximations themselves.

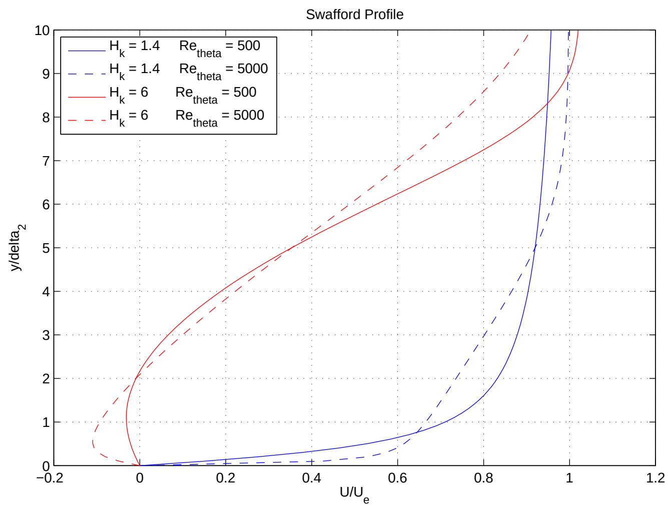


Figure 6.1: Swafford Profile for various shape factors, H, and momentum thickness Reynolds, Re_θ

6.2 The Swafford Profile

Swafford built upont Whitfield's work to propose a family of Analytical Approximated Turbulent Boundary Layer profiles that transition smoothly from attached to separated flow. These profiles are intimately connected with the closure relations used in R/Xfoil and therefore provides a consistent baseline to generate the Energy Interaction Coefficient dataset. The evaluation procedure for Swaffords prodiles was copied from reference into table 6.2 to ensure notation consistency.

6.2.1 Numerical Integration

The Swafford Profile, as any analitical turbulent boundary layer profile, is hard to integrate numerically. The strong gradients of the near wall region and the presence of multiple scales present a challenge that seems hard to overcome for all variable step integration methods, even of higher order (such as the Gauss Kronberg scheme). This is specially true for low shape factors (≈ 1.4) where the sublayer region is thinnest, and variable step methods exhibited unpredictable, inconsistent behaviour.

As such, after many experiments, it was found that, despite its simplicity, the most reliable integration method was a naive mid-point Riemann sum integration scheme (Kahan summation could bring additional improvements).

The integration methods were assessed while checking the consistency of

	Computational Step	Comment
1	H and Re_θ provided as Inputs	$u_e^+ = \sqrt{\frac{2}{\lvert c_f \rvert}}$
2	$c_f = \frac{0.3e^{-1.33H}}{(\log_{10}(Re_\theta))^{1.74+0.31H}} + \frac{1.1}{10^4}\left(\tanh\left(4 - \frac{H}{0.875}\right) - 1\right)$	
3	$s = \frac{c_f}{\lvert c_f \rvert}$	*Fudge at zero for stability*
4	$\frac{U}{U_e}_{(2)} = \frac{1}{1.95}\left(\operatorname{atanh}\left(\frac{8.5-H}{7.5}\right) - 0.364\right)$	$\frac{U}{U_e} : \left(\eta = \frac{Y}{\delta_2} = 2\right)$
5	$\frac{U}{U_e}_{(5)} = 0.155 + 0.795\operatorname{sech}\left(0.51\left(H - 1.95\right)\right)$	$\frac{U}{U_e} : \left(\frac{Y}{\delta_2} = 5\right)$
6	$g_{(2)} = \frac{\frac{U}{U_e}_{(2)} - \frac{s}{0.09u_e^+}\operatorname{atan}\left(0.18\frac{Re_\theta}{u_e^+}\right)}{1 - \frac{s\pi}{0.18u_e^+}}$	$g = g_{\left(\eta = \frac{Y}{\delta_2}\right)}$
7	$g_{(5)} = \frac{\frac{U}{U_e}_{(5)} - \frac{s}{0.09u_e^+}\operatorname{atan}\left(0.45\frac{Re_\theta}{u_e^+}\right)}{1 - \frac{s\pi}{0.18u_e^+}}$	
8	$b = \frac{\ln\left(\frac{\operatorname{atanh}\left(g_{(2)}^2\right)}{\operatorname{atanh}\left(g_{(5)}^2\right)}\right)}{\ln\left(\frac{2}{5}\right)}$	
9	$a = \frac{\operatorname{atanh}\left(g_{(2)}^2\right)}{2^b}$	
10	$u^+ = \frac{s}{0.09}\operatorname{atan}\left(0.09y^+\right) + \left(u_e^+ - \frac{s\pi}{0.18u_e^+}\right)\sqrt{\tanh\left(a\eta^b\right)}$	

Figure 6.2: Swafford Profile Generation, transcribed from ref. [2]

the generated profile's shape factor and momentum thickness against the inputs. Errors between 1% and 10% were observed on all methods, but the naive scheme performed consistently better whenever integrations were conducted on 10 million or more points for constant spacing and a million or more points for cosine spacing.

It is not clear wether the remaining errors come from:

- Inconsistencies in the profile definition
- The profile generator implementation
- The numerical integration schemes

The authors checked the profile generator implementation several times, and therefore tend to believe that the remaining errors in the shape factor and momentum thickness estimation stem from numerical integration artifacts.

It is reasonable to expect that some numerical cancelation may occur while summing the contributions of the many integration subintervals. Grouped sum methods, such as Kahan or Knuth summation (eventually with varying bit depth[2]), should be tried to confirm or infirm this intuition if more complex closure relations are to be developped in the future.

Even so, the integral of the Energy Interaction Coefficient C_{EI} seems far easier than the thicknesses integrals: a sensitivity study to determine how many

[2]Avaialable in Matlab through the Xsum package from MatlabCentral.

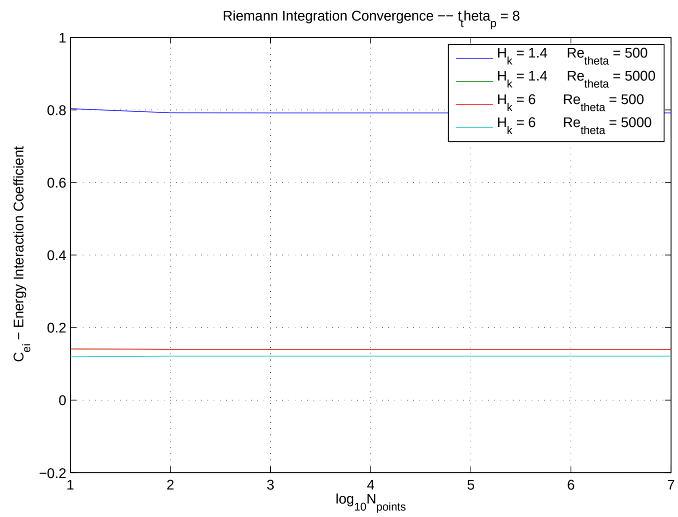


Figure 6.3: Convergence of Energy Interaction Coefficient with Riemann Scheme

integration points were needed was conducted accross a variety of $(H, Re_\theta, t_\theta^p)$ combinations, and showed that the naive midpoint Riemann integration along 1000 equally space points was a proper choice to evaluate C_{EI}.

6.2.2 Low Shape Factor Indefinition

The Swafford profile's greatest strength is its ability to represent both attached and separated flow reasonably, providing a smooth, plausible transtion between the two regimes. On the other hand, its behaviour for very small shape factors is less robust. Indeed, it is well defined for all Re_θ values down to shape factors slightly below 1.4, but ceases to exist for shape factors below $1.3 - 1.4$ whenever Re_θ tends towards small values[3].

Fortunately, lower shape factor values are rarelly observed in practical flows. In fact, unless suction or other boundary layer energisation techniques are applied, smaller shape factors could only arise in well developed accelerating flows, that is, under a favorable pressure gradient. Even in the favorable pressure gradient before the suction peak of airfoils, these shape factors seem quite unusual, implying that the Swafford's profile indefinition in this region should not be a major cause of concern.

Still, when definning the closure relation, it is important to ensure that it can be evaluated far enough to provide the right asymptotic behaviour at low shape factors. Indeed, iterative solvers such as R/Xfoil often request the evaluation of the closure relations for very small shape factors as they hop around unphysical

[3]Imaginary components appear in the calculation of the coefficients.

cases throughout the convergence process. This is a well known issue in the development of closure relations, and the necessary robustness can often be achieved through a mix of extrapolation for moderate shape factors, say in the $H \in [1.2, 1.4]$ interval, and fudging for lower values.

6.2.3 Unit Shape Factor Limit

For the particular case of the Energy Interaction Coefficient, C_{EI}, it is possible to determine the value of the integral for $H = 1$ analically. To do so, we start by arguing that a boundary layer with a shape factor of H concentrates all the shear on a singular vorticity sheet at the body surface, thereby falling back to a slip boundary condition on the surface (mathematically speaking). The velocity profile is straight in this case, and we write:

$$\begin{cases} \frac{U}{U_e} = 1 \; : \; 0^+ < Y \\ \lim_{Y \to 0^-} \frac{U}{U_e} = 0 \end{cases} \qquad , \quad H = 1$$

$$\Rightarrow \qquad u_{(\eta)} = 1 \qquad , \quad (H = 1 \, \wedge \, \eta > 0)$$

We feed this definition of the profile into the Energy Interaction Coefficient definition, from expression 5.6:

$$C_{EI} = \int_0^{t_\theta^p} \left(\frac{w^y_{(\eta, t_\theta^p)} u_{(\eta)}}{t_\theta^p} \right) d\eta = \frac{1}{t_\theta^p} \int_{0^+}^{t_\theta^p} \left(w^y_{(\eta, t_\theta^p)} \right) d\eta \qquad , \quad H = 1$$

Recalling that the integral identities of the weitghting functions, as in expression 5.3, are valid over any reference frame:

$$\int_0^{t_\theta^p} w^y_{(\eta, t_\theta^p)} d\eta = t_\theta^p \qquad , \quad \forall \, t_\theta^p \in \mathbb{R}$$

We write:

$$C_{EI} = \frac{t_\theta^p}{t_\theta^p} = 1 \qquad , \quad \begin{cases} H = 1 \\ \forall \, Re_\theta \in \mathbb{R} \\ \forall \, t_\theta^p \in \mathbb{R} \end{cases}$$

which will drive the behaviour of our closure for low shape factors and also form a plausible upper bound for all C_{EI} values.

6.3 Closure Dataset

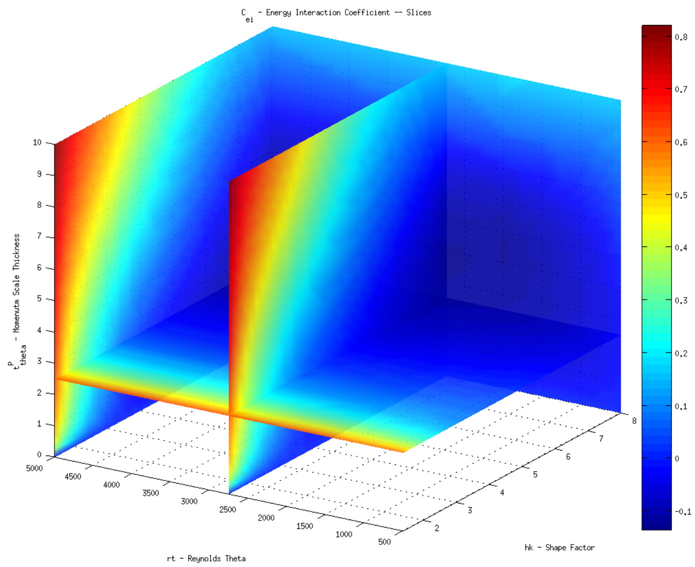
The closure dataset was generated with 100 points in every direction over the three variables, with the following ranges:

- Shape Factors H from 1.4 to 8 in 40(100) equally spaced steps

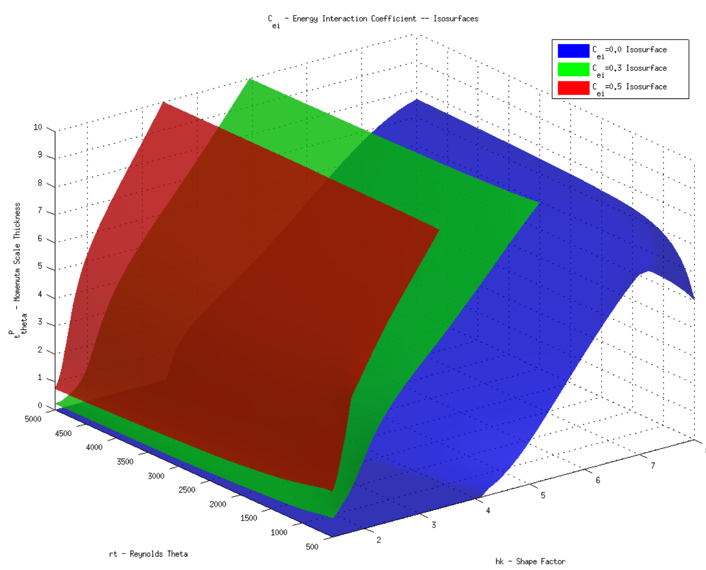
- Momentum Thickness Reynolds Re_θ from 500 to 5000 in 40(100) equally spaced steps
- Momentum Scaled Plasma Thickness t^p_θ from 0 to 10 in 40(100) equally spaced steps

6.4 Closure Relation

<<< This is the part in which we write a very nice expression to fit the closure dataset and name it the Pereira-Oliveira Closure Relation>>>

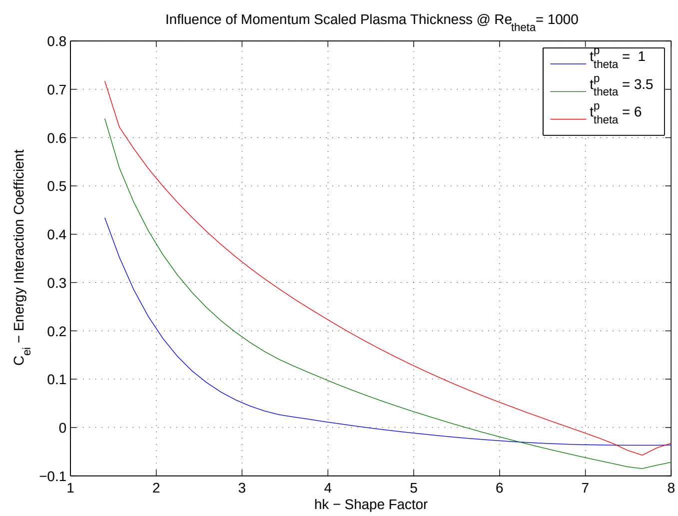


(a) Slices

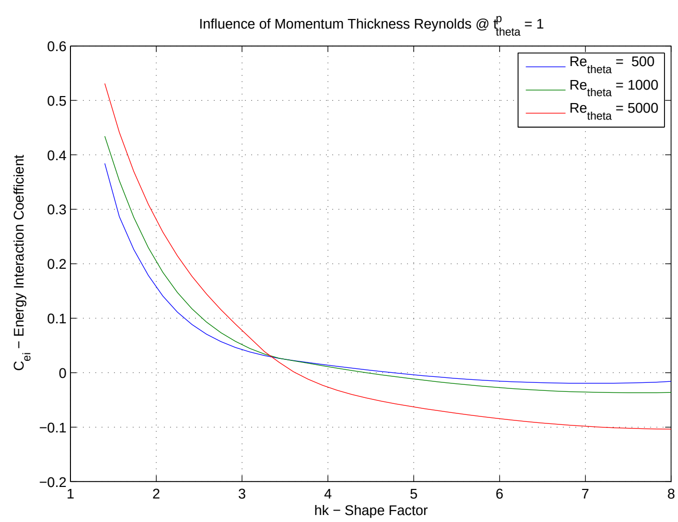


(b) Isosurfaces

Figure 6.4: C_{EI} closure dataset visualizations



(a) Influence of Plasma Force Field Thickness



(b) Influence of Momentum Thickness Reynolds

Figure 6.5: C_{EI} closure dataset cuts

Part III

Integrating the Boundary Layer Equations

Chapter 7

The integral boundary layer ODEs

We depart by reordering and adding the pressure gradient compressibility terms to the Integral Boundary Layer equations for consistency with R/Xfoil's formulation[1]. The momentum equation from expression 4.1 turns into:

$$\frac{\partial\theta}{\partial x} = \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right)\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} - \frac{C_{FM}}{2u_e^2}$$

The Integral Energy Shape Factor Equation in adimensional terms, as written in expression 4.2:

$$\frac{\partial H_{32}}{\partial x} = \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}}H_{12} - 1\right)\frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right)\frac{v_o}{u_e} + \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right)$$

These two equations are complemented by the shear lag equation, described in reference []:

$$\frac{dC_\tau}{d\xi} = \frac{C_\tau K_C}{\delta}\left(C_{\tau_{EQ}}^{1/2} - C_\tau^{1/2}\right)$$

and used to model the effect of turbulence history on the dissipation coefficient. In its present form, our system has the set $\{\theta, H^*, C_\tau\}$as unknowns of our system, while $\frac{dU_e}{d\xi}$ and M_e are prescribed inputs[2].

7.1 Closure and RHS Evaluation

Closure relations are needed to complete the definition of the equations and gain the ability to compute the right hand sides of the boundary layer equations as

[1] The suction and plasma terms would also have compressibility components, but they were ignored for now

[2] If it weren't we would need four equations.

they were written in the last section. The closure relations are just algebraic relations between the intermediate variables and the unknowns, but their format still has a major impact on the formulation of the numerical problem. We will use a similar set of closure relations as R/Xfoil, of which we will provide an overview in this section, with an emphasis on their dependencies.

7.1.1 Shape Factor Closures

The shape factor closure relations:

Variable	Rfoil	Definition	Dependencies
Kinematic Shape Parameter	HK	H_k	$f\left(H, M\right)$
Energy Shape Parameter	HST	$H^* = \frac{\theta^*}{\theta}$	$f(H_k, Re_\theta, M)$
Density Shape Parameter	HCT	$H^{**} = \frac{\delta^{**}}{\theta}$	$f\left(H_k, M\right)$
Head's Shape Parameter	HH1CAL	$H_1 = \frac{\delta - \delta^*}{\theta}$	$f\left(H_k\right)$

7.1.2 Slip and Friction Closures

The slip and friction closure relations:

Variable	Rfoil	Definition	Dependencies
Skin Friction Coefficient	CFT	C_f	$f\left(H_k, Re_\theta, M\right)$
Slip Speed	USG	U_S	$f\left(H, H_k, H^*\right)$

7.1.3 Composite Closures

And the composite closure relations:

Variable	Rfoil	Definition	Dependencies
Equilibrium Shear Stress Coefficient	CTAUZERO	C_τ^{EQ}	$f\left(H, H_k, H^*, U_S\right)$
Dissipation Term	DIT	$\frac{2C_d}{H^*}$	$f\left(H^*, U_s, C_f, C_\tau\right)$

7.1.4 Expected Inputs

The closure relations we just highlighted are sufficient to evaluate the RHS term from three variables provided as inputs:

$$\{\theta, H, C_\tau\} \qquad or \qquad \{\theta, \delta^*, C_\tau\}$$

But not from the three ODE unknowns $\{\theta, H^*, C_\tau\}$ unless we invert two closure relations:

$$H^* = f\left(H_k, Re_\theta, M\right) \qquad and \qquad H_k = f\left(H\right)$$

to recover H from H^*. The $H_k = f\left(H\right)$ is easy to invert, but the $H^* = f\left(H_k, Re_\theta, M\right)$ relation is not monotonous, and therefore only admits local pseudo-inverses (numerical or analytical). This approach is cumbersome and following it would lead to a cumbersome, innefective code that does not lend itselft well to modification. We will therefore choose another approach to circumvent the RHS evaluation problem, by transforming the ODE system to use more convenient unknowns.

7.2 ODE integration

Ordinary differential equation system can be integrated numerically in both implicit and explicit forms. As will becomes clear in section Our system of equations

We present the two approaches and discuss their advantages and weaknesses for the solution of the integral boundary layer ODE system.

7.2.1 Implicit Solvers

Implicit solvers solve problems of the type:

$$0 = f\left(\boldsymbol{y}, \frac{\partial \boldsymbol{y}}{\partial \xi}, \xi\right) \qquad , \quad \begin{cases} \boldsymbol{y} & \in \mathbb{R}^n \\ \xi & \in \mathbb{R} \\ f & : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \end{cases}$$

with initial conditions provided as input:

$$\begin{cases} \xi = \xi_0 \\ \boldsymbol{y}_0 = \boldsymbol{y}|_{\xi=\xi_0} \\ \left.\frac{\partial \boldsymbol{y}}{\partial \xi}\right|_{\xi=\xi_0} : \quad f\left(\boldsymbol{y}_0, \left.\frac{\partial \boldsymbol{y}}{\partial \xi}\right|_{\xi=\xi_0}, \xi_0\right) = 0 \end{cases}$$

where the initial condition for the derivatives must be consistent and is usually be determined numerically.

7.2.2 Explicit Solvers

Explicit solvers rely on the cannonical form:

$$\frac{\partial \boldsymbol{y}}{\partial \xi} = f\left(\boldsymbol{y}, \xi\right) \qquad , \quad \begin{cases} \boldsymbol{y} & \in \mathbb{R}^n \\ \xi & \in \mathbb{R} \\ f & : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \end{cases}$$

with initial conditions provided as input:

$$\begin{cases} \xi = \xi_0 \\ \boldsymbol{y}_0 = \boldsymbol{y}|_{\xi=\xi_0} \end{cases}$$

to solve the problem and output a sequence of $(\xi, \boldsymbol{y})$ pairs until a certain integration limit ξ_M.

7.2.3 Solution Strategy

Most well known ODE integration algorithms were designed with explicit solvers in mind, and implicit solvers are less common and mature[3]. We will therefore focus our efforts on the use of explicit solvers and re-formulate our problem such that the boundary layer equations can be rewritten in this format:

$$\begin{bmatrix} \frac{\partial y_1}{\partial \xi} \\ \frac{\partial y_2}{\partial \xi} \\ \frac{\partial y_3}{\partial \xi} \end{bmatrix} = \begin{bmatrix} f_1\left(\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}, \xi\right) \\ f_2\left(\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}, \xi\right) \\ f_3\left(\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}, \xi\right) \end{bmatrix}$$

[3]There is an implicit solver in matlab (ode15i) but is far less specialized than the family of explicit solvers (eg. ode45, ode23, ode15)

Chapter 8

Formulating the Direct Problem

$$\frac{\partial \theta}{\partial x} = \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right) \frac{\theta}{u_e} \frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} - \frac{C_{FM}}{2u_e^2}$$

The Integral Energy Shape Factor Equation in adimensional terms, as written in expression 4.2:

$$\frac{\partial H_{32}}{\partial x} = \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}} H_{12} - 1\right) \frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right)\frac{v_o}{u_e} + \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right)$$

The integral boundary layer equations can be rewritten as a matricial system of equations as:

$$\underbrace{\begin{bmatrix} \frac{\partial \theta}{\partial x} \\ \frac{\partial H^*}{\partial x} \\ \frac{\partial C_\tau}{\partial x} \end{bmatrix}}_{\frac{\partial \boldsymbol{y}}{\partial \xi}} = \underbrace{\begin{bmatrix} \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right) \frac{\theta}{u_e} \frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} - \frac{C_{FM}}{2u_e^2} \\ \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}} H_{12} - 1\right) \frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right)\frac{v_o}{u_e} + \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right) \\ \frac{C_\tau}{\delta} K_C \left(C_{\tau_{EQ}}^{1/2} - C_\tau^{1/2}\right) \end{bmatrix}}_{f(\boldsymbol{y}, x)}$$

From now on, will call the $\frac{\partial \boldsymbol{y}}{\partial \xi}$ term the LHS and the $f\left(\boldsymbol{y}, \xi\right)$ term the RHS.

8.1 Transforming the ODE system

We apply a transformation on the ODE system, so that we can work with three convenient unknowns[1]:

$$\{\theta, H, C_\tau\}$$

[1]Drela works with $\{\theta, \delta^*, C_\tau\}$ but we choose to work with $\{\theta, H, C_\tau\}$ as it seems more stable numerically and less error prone for our formulation.

To do so we depart from the closure relation for the energy shape factor[2]:

$$H^* = f\left(H_k, Re_\theta, M\right)$$

$$dH^* = \frac{\partial H^*}{\partial H_k} dH_k + \frac{\partial H^*}{\partial Re_\theta} dRe_\theta + \frac{\partial H^*}{\partial M} dM$$

The definition of the momentum thickness Reynolds[3]:

$$Re_\theta = \frac{U_e \theta}{\nu_e} \qquad \Rightarrow dRe_\theta = \left(\frac{U_e}{\nu_e}\right) d\theta + \theta d\left(\frac{U_e}{\nu_e}\right)$$

And the kinematic shape factor closure relation:

$$H_k = f\left(H, M\right) \qquad \Rightarrow \qquad dH_k = \frac{\partial H_k}{\partial H} dH + \frac{\partial H_k}{\partial M} dM$$

Inserting these two relations progressively into the differential of the H^*closure relation, we get:

$$dH^* = \frac{\partial H^*}{\partial H_k}\left(\frac{\partial H_k}{\partial H} dH + \frac{\partial H_k}{\partial M} dM\right) + \frac{\partial H^*}{\partial Re_\theta}\left(\left(\frac{U_e}{\nu_e}\right) d\theta + \theta d\left(\frac{U_e}{\nu_e}\right)\right) + \frac{\partial H^*}{\partial M} dM$$

Reordering:

$$dH^* = \frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H} dH + \left(\frac{U_e}{\nu_e}\right)\frac{\partial H^*}{\partial Re_\theta} d\theta + \theta \frac{\partial H^*}{\partial Re_\theta} d\left(\frac{U_e}{\nu_e}\right) + \left(\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial M} + \frac{\partial H^*}{\partial M}\right) dM$$

Switching sides we can define the differential operator for dH:

$$\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H} dH = dH^* - \left(\frac{U_e}{\nu_e}\right)\frac{\partial H^*}{\partial Re_\theta} d\theta - \theta \frac{\partial H^*}{\partial Re_\theta} d\left(\frac{U_e}{\nu_e}\right) - \left(\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial M} + \frac{\partial H^*}{\partial M}\right) dM$$

Dividing both sides by $\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}$ we get[4]:

$$dH = \frac{1}{\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}} dH^* + \frac{-\left(\frac{U_e}{\nu_e}\right)\frac{\partial H^*}{\partial Re_\theta}}{\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}} d\theta + \frac{-\theta\frac{\partial H^*}{\partial Re_\theta}}{\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}} d\left(\frac{U_e}{\nu_e}\right) + \frac{-\left(\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial M} + \frac{\partial H^*}{\partial M}\right)}{\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}} dM$$

Defining some auxilliary variables:

$$\lambda^{H^*} = 1 \qquad , \qquad \lambda^\theta = -\left(\frac{U_e}{\nu_e}\right)\frac{\partial H^*}{\partial Re_\theta}$$

[2]This step and its consequences need to be modified when new dependencies are introduced on H^*.

[3]We will consider that ν_e is constant (ok for incompressible!)

[4]This expression becomes numerically instable near separation! It is not yet clear wether it is analytically singular or not. If the singularity is only numerical, it may eventually be circumvented by fudging $\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}$ to $100eps$ as it tends to zero!

$$\lambda^{(U_e/\nu_e)} = -\theta\frac{\partial H^*}{\partial Re_\theta} \qquad , \qquad \lambda^M = -\left(\frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial M} + \frac{\partial H^*}{\partial M}\right)$$

$$\gamma = \frac{\partial H^*}{\partial H_k}\frac{\partial H_k}{\partial H}$$

We can rewrite the shape factor's differential operator in a compact form with some chances of being numerically stable:

$$dH = \frac{1}{\gamma}\left(\lambda^{H^*}dH^* + \lambda^\theta d\theta + \lambda^{(U_e/\nu_e)}d\left(\frac{U_e}{\nu_e}\right) + \lambda^M dM\right)$$

Dividing both sides by $d\xi$ we get the transformation we were looking for:

$$\frac{dH}{d\xi} = \frac{1}{\gamma}\left(\lambda^{H^*}\frac{dH^*}{d\xi} + \lambda^\theta\frac{d\theta}{d\xi} + \lambda^{(U_e/\nu_e)}\frac{d\left(\frac{U_e}{\nu_e}\right)}{d\xi} + \lambda^M\frac{dM}{d\xi}\right)$$

Which will not be stable near separation, given that $\frac{\partial H^*}{\partial H_k} \to 0$ around $H_k = 4$, thereby meaning that the displacement thickness gets a tendency to blow up, the so-called "Goldstein Singularity"[5].

Using the transformation above, we can write the so-called direct system. To integrate around separation, we would need to define another transformation, to obtain the so-called inverse system, for which δ^* or H are prescribed and U_e is set as an unknown!

8.2 Rewritting the ODE in a convenient matrix format

We will now rewrite the ODE of the beginning in a convenient matrix format[6]:

$$\begin{bmatrix} \frac{\partial\theta}{\partial\xi} \\ \frac{\partial H^*}{\partial\xi} \\ \frac{\partial C_\tau}{\partial\xi} \\ \frac{d\left(\frac{U_e}{\nu_e}\right)}{d\xi} \\ \frac{dM}{d\xi} \end{bmatrix} = \quad \cdot \quad \ldots$$

$$\cdot \begin{bmatrix} \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right)\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} - \frac{C_{FM}}{2u_e^2} \\ \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}}H_{12} - 1\right)\frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right)\frac{v_o}{u_e} + \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right) \\ \frac{C_\tau}{\delta}K_C\left(C_{\tau_{EQ}}^{1/2} - C_\tau^{1/2}\right) \\ \frac{d\left(\frac{U_e}{\nu_e}\right)}{d\xi} \\ \frac{dM}{d\xi} \end{bmatrix}$$

[5] Verify that this is really the Goldstein singularity (there are two of them!)

[6] Only the first three equations are independent!

Which lends itselft well to rewritte the transformation of the shape factor differential as a dot product:

$$\frac{dH}{d\xi} = \frac{1}{\gamma}\left[\begin{array}{ccccc} \lambda^{\theta} & \lambda^{H^*} & 0 & \lambda^{(U_e/\nu_e)} & \lambda^{M} \end{array}\right]\left[\begin{array}{c} \frac{d\theta}{d\xi} \\ \frac{dH^*}{d\xi} \\ \frac{\partial C_\tau}{\partial\xi} \\ \frac{d\left(\frac{U_e}{\nu_e}\right)}{d\xi} \\ \frac{dM}{d\xi} \end{array}\right]$$

So that we can define a transformation matrix for the ODE system:

$$\left[\begin{array}{c} \frac{\partial\theta}{\partial\xi} \\ \frac{dH}{d\xi} \\ \frac{\partial C_\tau}{\partial\xi} \end{array}\right] = \frac{1}{\gamma}\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \lambda^{\theta} & \lambda^{H^*} & 0 & \lambda^{(U_e/\nu_e)} & \lambda^{M} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]\left[\begin{array}{c} \frac{d\theta}{d\xi} \\ \frac{dH^*}{d\xi} \\ \frac{\partial C_\tau}{\partial\xi} \\ \frac{d\left(\frac{U_e}{\nu_e}\right)}{d\xi} \\ \frac{dM}{d\xi} \end{array}\right]$$

To yield:

$$\left[\begin{array}{c} \frac{\partial\theta}{\partial x} \\ \frac{dH}{dx} \\ \frac{\partial C_\tau}{\partial x} \end{array}\right] = \frac{1}{\gamma}\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \lambda^{\theta} & \lambda^{H^*} & 0 & \lambda^{(U_e/\nu_e)} & \lambda^{M} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad \cdot \qquad \ldots$$
$$\cdot \left[\begin{array}{c} \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right)\frac{\theta}{u_e}\frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} - \frac{C_{FM}}{2u_e^2} \\ \frac{2C_D}{\theta} - \frac{H_{32}}{\theta}\frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}}H_{12} - 1\right)\frac{H_{32}}{u_e}\frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right)\frac{v_o}{u_e} + \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right) \\ \frac{C_\tau}{\delta}K_C\left(C_{\tau_{EQ}}^{1/2} - C_\tau^{1/2}\right) \\ \frac{d\left(\frac{U_e}{\nu_e}\right)}{dx} \\ \frac{dM}{dx} \end{array}\right] \tag{8.1}$$

which is a convenient form of the direct problem, compatible with all modern explicit solver implementations.

8.3 Plasma Specific Numerical Issues

In the previous section, we formulated the direct system of integral boundary layer equations, expression 8.1, for the purpose of numerical integration of attached boundarty layers. Even though the plasma terms do not change the structure and nature of the integral boundary layer ODEs, they introduce some additional challenges.

In particular, the lenght of the plasma force field is usually a few orders of magnitude smaller than the longitudinal lenght scales of the boundary layer, leading to a situation in which classical variable step ODE solvers[7] may fail

[7]Such as Matlab's ode45 or ode23 solvers.

to capture the plasma region completely, leading to inconsistent, unpredictable results. There are three ways to tackle this challenge:

1. Managing classical ODE solvers, such as Matlab's ode45 solver, with event functions to highlight the plasma actuation regions

2. Resorting to stiff ODE solvers, such as Matlab's ode15s solver

3. Force classical ODE solvers, such as Matlab's ode45 solver, to run with small, fixed predetermined steps to enfore a sufficient resolution of the plasma actuation area

The two last strategies were briefly attempted, providing sufficient insight to decide that the model validation would be conducted with a fixed step method. Indeed, despite their computational inneficiency, when used with small step sizes, fixed step methods provide the most robust numerical solution procedure, ensuring numerical accuracy consistency accross a variety of cases.

Chapter 9

The Lumped Plasma Actuator Approximation

9.1 Quantifying Boundary Layer Perturbations

High resolution fixed step methods, such as those discussed in section 8.3, are adequate for reasearch and development purposes, but their inneficiency becomes problematic in design and production contexts[1].

In this section, we will attempt to overcome this challenge by providing designers with a set of compact expressions to assess the effect of a DBD plasma actuator on a boundary layer.

We will develop our insight into the the effect of the plasma actuator on the boundary layer by comparing changes in boundary layer variables accross the actuation region:

$$\varepsilon^{P} = \Delta^{p\,on} - \Delta^{p\,off}$$

Where we follow expression 8.1 to define $\Delta^{p\,on}_{(\ldots)}$ as the evolution of the boundary layer with a force field

$$\Delta^{p\,on} = \begin{bmatrix} \Delta^{p\,on}_{\theta} \\ \Delta^{p\,on}_{H} \end{bmatrix} = \int_{x_0^p}^{x_0^p + l^p} \left[\begin{array}{c} \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right) \frac{\theta}{u_e} \frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} - \frac{C_{FM}}{2u_e^2} \\ \frac{2C_D}{\theta} - \frac{H_{32}}{\theta} \frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}} H_{12} - 1\right) \frac{H_{32}}{u_e} \frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right) \frac{v_o}{u_e} + \frac{1}{\theta}\left(H_{32} \frac{C_{FM}}{2u_e^2} - \frac{C_F}{u_e^3}\right. \end{array} \right.$$

and $\Delta^{p\,off}_{(\ldots)}$ as the evolution without the force field:

$$\Delta^{p\,off} = \begin{bmatrix} \Delta^{p\,off}_{\theta} \\ \Delta^{p\,off}_{H} \end{bmatrix} = \int_{x_0^p}^{x_0^p + l^p} \begin{bmatrix} \frac{C_f}{2} - \left(H_{12} + 2 - M_e^2\right) \frac{\theta}{u_e} \frac{\partial u_e}{\partial x} + \frac{v_o}{u_e} \\ \frac{2C_D}{\theta} - \frac{H_{32}}{\theta} \frac{C_f}{2} + \left(\frac{2H^{**}}{H_{32}} H_{12} - 1\right) \frac{H_{32}}{u_e} \frac{\partial u_e}{\partial x} - \frac{1}{\theta}\left(H_{32} - 1\right) \frac{v_o}{u_e} \end{bmatrix} dx$$

[1]Such as for the implementation of plasmas in Rfoil

So that we can use the distributive property of the integral to write:

$$\varepsilon^P = \begin{bmatrix} \varepsilon^p_\theta \\ \varepsilon^p_H \end{bmatrix} = \Delta^{p\,on} - \Delta^{p\,off} = \int_{x_0^p}^{x_0^p+l^p} \begin{bmatrix} -\frac{C_{FM}}{2u_e^2} \\ \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right) \end{bmatrix} dx \quad (9.1)$$

9.1.1 Dealing with closure relation perturbations

Expression 9.1 ignores perturbations in closure relations, which can be specially relevant for the skin friction. More sophisticated expressions may be derived following a similar approach once closure relation perturbations will be modeled.

9.2 Lumped Plasma Approximation

Building upon the observation that the plasma force field length is usually much smaller than the boundary layer's longitudinal lenght scale, we will explore the case in which the lenght of the force field tends towards zero:

$$\lim_{l^p\to 0^+} \varepsilon^P = \lim_{l^p\to 0^+} \int_{x_0^p}^{x_0^p+l^p} \begin{bmatrix} -\frac{C_{FM}}{2u_e^2} \\ \frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right) \end{bmatrix} dx$$

We will call this particular the Lumped Plasma Approximation. Using this approximation, the above expression can be reworked by inserting the Force Momentum Coefficient definition in the first term:

$$\lim_{l^p\to 0^+} \varepsilon^P_\theta = \lim_{l^p\to 0^+} \int_{x_0^p}^{x_0^p+l^p} \left(-\frac{C_{FM}}{2u_e^2}\right) dx = -\frac{1}{2u_e^2} \lim_{l^p\to 0^+} \int_{x_0^p}^{x_0^p+l^p} \left(C_{\phi^p_x} w^x_{(x,x_0^p,l^p)}\right) dx \quad (9.2)$$

We notice the limit indetermination, and put in evidence that:

$$\lim_{l^p\to 0^+} C_{\phi^p_x} = \infty \qquad , \qquad \lim_{l^p\to 0^+} \left(\int_{x_0^p}^{x_0^p+l^p} w^x_{(x,x_0^p,l^p)} dx\right) = 0$$

Before we recall that, by definition:

$$C_{\phi^p_x} = \frac{\phi^p_x T^p}{\frac{1}{2}\rho U_\infty^2} \qquad , \quad with \quad \phi^p_x = \frac{F^p_T}{L^p T^p}$$

$$\Rightarrow \qquad C_{\phi^p_x} = \frac{T^p}{\frac{1}{2}\rho U_\infty^2}\frac{F^p_T}{L^p T^p} = \frac{F^p_T}{\frac{1}{2}\rho U_\infty^2 L}\frac{1}{l^p}$$

and infer[2] that:

$$\lim_{l^p\to 0^+} \int_{x_0^p}^{x_0^p+l^p} \frac{w^x_{(x,x_0^p,l^p)}}{l^p} dx = 1$$

[2]We will not proove this fact, but it is quite easy to accept!

so that we can rewrite expression 9.2 as:

$$-\frac{1}{2u_e^2}\lim_{l^p\to 0}\int_{x_0^p}^{x_0^p+l^p}\left(\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}\frac{1}{l^p}w^x_{\left(x,x_0^p,l^p\right)}\right)dx = -\frac{1}{2u_e^2}\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}\underbrace{\lim_{l^p\to 0}\int_{x_0^p}^{x_0^p+l^p}\left(\frac{w^x_{\left(x,x_0^p,l^p\right)}}{l^p}\right)dx}_{=1}$$

$$\Rightarrow \qquad \varepsilon_\theta^p = \int_{x_0^p}^{x_0^p+l^p}\left(-\frac{C_{FM}}{2u_e^2}\right)dx = -\frac{1}{2u_e^2}\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L} \qquad , \quad l^p \to 0^+$$

Proceeding in a similar fashion for the Force Energy Coefficient C_{FE}, we obtain:

$$\lim_{l^p\to 0^+}\int_{x_0^p}^{x_0^p+l^p}\left(\frac{C_{FE}}{u_e^3}\right)dx = \lim_{l^p\to 0^+}\int_{x_0^p}^{x_0^p+l^p}\left(\frac{w^x_{\left(x,x_0^p,l^p\right)}C_{\phi_x^p}C_{EI}}{u_e^3}\right)dx$$

$$= \frac{C_{EI}}{u_e^3}\lim_{l^p\to 0^+}\int_{x_0^p}^{x_0^p+l^p}\left(w^x_{\left(x,x_0^p,l^p\right)}C_{\phi_x^p}\right)dx = \frac{C_{EI}}{u_e^3}\lim_{l^p\to 0^+}\int_{x_0^p}^{x_0^p+l^p}\left(\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}\frac{1}{l^p}w^x_{\left(x,x_0^p,l^p\right)}\right)dx$$

$$= \frac{C_{EI}}{u_e^3}\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}\underbrace{\lim_{l^p\to 0^+}\int_{x_0^p}^{x_0^p+l^p}\left(\frac{w^x_{\left(x,x_0^p,l^p\right)}}{l^p}\right)dx}_{=1} = \frac{C_{EI}}{u_e^3}\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}$$

To reach:

$$\varepsilon_H^p = \int_{x_0^p}^{x_0^p+l^p}\frac{1}{\theta}\left(H_{32}\frac{C_{FM}}{2u_e^2} - \frac{C_{FE}}{u_e^3}\right)dx = \frac{1}{\theta}\frac{H_{32}}{2u_e^2}\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L} - \frac{1}{\theta}\frac{C_{EI}}{u_e^3}\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L} \qquad , \quad l^p \to 0^+$$

9.2.1 Compact Perturbation Model

It is practical to define an additional adimensional group, the Total Plasma Force Coefficient, C_{FT}:

$$C_{FT} = \frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}$$

So that the plasma perturbation expressions derived in this section can be written in a compact form based upon adimensional groups:

$$\begin{cases} \varepsilon_\theta^p = -\frac{C_{FT}}{2u_e^2} \\ \varepsilon_H^p = \frac{1}{\theta}\left(\frac{H_{32}}{2u_e^2}C_{FT} - \frac{C_{EI}}{u_e^3}C_{FT}\right) \end{cases} \qquad , \quad l^p \to 0^+$$

And expressed in terms of closure relations:

$$\begin{cases} \varepsilon_\theta^p = -\frac{C_{FT}}{2u_e^2} \\ \varepsilon_H^p = \frac{C_{FT}}{\theta}\left(\frac{H^*_{\left(H_k,Re_\theta,M\right)}}{2u_e^2} - \frac{C_{EI\left(H,Re_\theta,t_\theta^p\right)}}{u_e^3}\right) \end{cases} \qquad , \quad l^p \to 0^+ \tag{9.3}$$

Expression 9.3 can develop designer insight by highlighting that, whenever the lumped plasma approximation is valid:

- Momentum thickness perturbations are independent from the force field thickness and boundary layer state and depend only the total force coefficient, C_{FT}
- Shape factor (and displacement thickness) perturbations depend on the state of the boundary layer just before it is perturbed by the plasma actuator

These considerations can provide a good basis to and tune the Energy Interaction Coefficient (C_{EI}) closure relation against experimental results.

9.2.2 Tuning the Energy Interaction Closure

Using The lumped plasma approximation can function a tool to tune the Energy Interaction Coefficient closure relation, $C_{EI\left(H,Re_\theta,t^p_\theta\right)}$, from experimental results.

For this purpose, it is convenient to rewrite expression 9.3 in terms of thickness changes:

$$\varepsilon^p_H = \frac{1}{\theta}\left(\underbrace{\frac{C_{FT}}{2u_e^2}}_{-\varepsilon^p_\theta} H^*_{(H_k,Re_\theta,M)} - \underbrace{\frac{C_{FT}}{2u_e^2}}_{-\varepsilon^p_\theta} \frac{C_{EI\left(H,Re_\theta,t^p_\theta\right)}}{u_e}\right) = \frac{\varepsilon^p_\theta}{\theta}\frac{C_{EI\left(H,Re_\theta,t^p_\theta\right)}}{u_e} - \frac{\varepsilon^p_\theta}{\theta} H^*_{(H_k,Re_\theta,M)}$$

$$\Leftrightarrow \qquad C_{EI\left(H,Re_\theta,t^p_\theta\right)} = u_e\left(\theta\frac{\varepsilon^p_H}{\varepsilon^p_\theta} + H^*_{(H_k,Re_\theta,M)}\right)$$

Part IV

Towards a Skin Friction Correction

In part IV we will share our reflections on a path to derive a closure relation correction for the skin friction. We will first attempt to devise a simple correction in the spirit of Merchant's work on suction. For this purpose, we will consider a plasma flow between moving flat plates, solve it analyticall and obtain a skin friction correction. However, the results obtained with this first approach do not match experiment very well, and it is very questionable from an analytical standpoint.

As such, we move on to start a larger analytical effort, and consider the free-shear laminar flow over a flat plate with a continuous, infinite plasma actuator. We derive a similarity solution for a particular class of these flows and solve it, creating our own "mini-"Blasius/Falkner-Skan flow case and solution (very "mini"!).

Chapter 10

Plasma Driven Flow between Flat Plates

We will follow an approach similar to the one of Merchant when he derived a skin friction correction for the effect of distributed suction. Despite its simplicity and questionable assumptions, this approach lead to excellent results for the suction case. Furthermore, it lends itselft well to account for the effect of the Plasma force field, which is a more difficult case.

Our approach follows four key steps:

1. Consider a fully developped, force field driven flow between two infinite flat plates whose spacing is equal to the force field thickness, T_p, in two limit cases:
    (a) When the bottom flat plate (on which the plasma actuator sits) is stopped, and the upper flat plate can move freely (so that it has zero skin friction)
    (b) When the two flat plates are stopped, but the problem remains asymetric because only bottom flat plate holds a plasma actuator
2. Solve the above problem analitically, using a body force defined through the DBD plasma force field normal weighting function, w^Y
3. Rework the flow field description into a simple expression for the skin friction of a fully developped, plasma driven flow
4. Correct the unperturbed boundary layer skin friction by adding the skin friction from the plasma driven flow, as if the two flows did not interact with each other

This approach has obvious limitations, but it still provides a relevant first step towards the modelling of skin friction perturbations caused by DBD plasma force fields.

Throughout the derivation, it will become clear that this approach can probably be extended to a full class of equilibrium flows by building upon Blasius's semi-infinite flat plate solution strategy. Such a strategy would definitely be more elegant and improve greatly on the theoretical solidity of the model.

The two limit cases serve as error bounds and significant physical phenomena would also be ignored by a Blasius like approach:

- Non-equilibrium conditions are likely to introduce even greater errors than those introduced by the flat plate approach
- Effective viscosity perturbations due to turbulence and plasma induced thermodynamic effects would also constrain the validity of a Blasius like approach

As such, the authors decided to leave more sophisticated approaches for later work, in which they hope to propose a general formulation for the inclusion of perturbation effects in boundary layer velocity profiles.

10.1 Problem Statement

Let us consider the fully developped flow between two infinite flat plates whose spacing is equal to the plasma force field thickness. We start by restating the Navier Stokes equations while restricting ourselves to steady flow and a single force field component:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + \frac{1}{\rho}F_x$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

And infer that the flow admits two key geometric scales:

$$O\left[X\right] = L \qquad , \qquad O\left[Y\right] = T^p$$

$$with \quad T^p \ll L$$

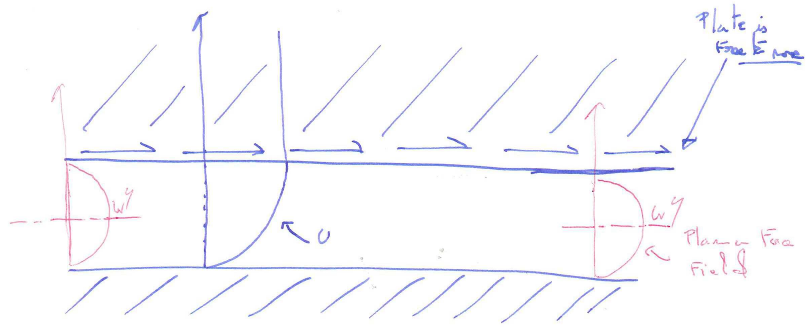
And define an unknown scale for the flow speed in the longitudinal direction:

$$O\left[U\right] = \bar{U}$$

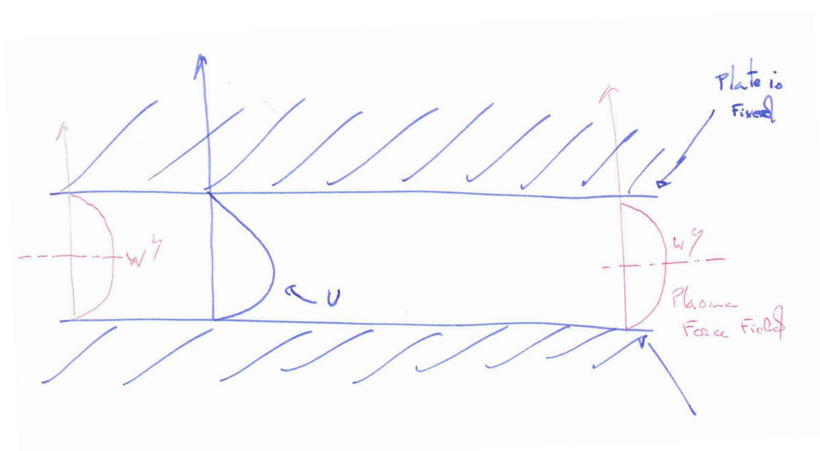
So that we can pursue an order of magnitude analysis on the continuity equation:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \qquad \Rightarrow \qquad O\left[\frac{\partial U}{\partial X}\right] = O\left[\frac{\partial V}{\partial Y}\right]$$

$$\Leftrightarrow \quad \frac{O\left[U\right]}{O\left[X\right]} = \frac{O\left[V\right]}{O\left[Y\right]} \quad \Leftrightarrow \quad \frac{\bar{U}}{L} = \frac{O\left[V\right]}{T^p} \Leftrightarrow \quad \frac{T^p}{L}\bar{U} = O\left[V\right]$$



(a) Free Top Plate Limit



(b) Fixed Plates Limit

Figure 10.1: Plasma driven flow between infinite flat plates

With this result, we define a set of scaled variables for our problem[1]:

$$\xi = \frac{X}{L} \quad \Rightarrow \quad X = L\xi \quad , \quad \eta = \frac{Y}{T^P} \quad \Rightarrow \quad Y = \eta T^P$$

$$\tilde{U} = \frac{U}{\bar{U}} \quad \Rightarrow \quad U = \tilde{U}\bar{U} \quad , \quad \tilde{V} = \frac{V}{O\left[V\right]} = \frac{V}{\frac{T^p}{L}\bar{U}} \quad \Rightarrow \quad V = \frac{T^p}{L}\bar{U}\tilde{V}$$

So that the continuity equation can be rewritten as:

$$\frac{\bar{U}}{L}\left(\frac{\partial\tilde{U}}{\partial\xi} + \frac{\partial\tilde{V}}{\partial\eta}\right) = 0$$

Our ultimate goal[2], is to identify the leading terms of the 1^{st} momentum equation to obtain an acurate, approximate description of the flow that admits an analytical solution. However, before we attack this equation, we need to reach a scale for the pressure.

10.1.1 Scaling the Second Momentum Equation

The second momentum equation can be used to obtain a scale for the pressure, by feeding in the scales we already defined, and pursuing an order of magnitude on the scaled equation:

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right)$$

$$\left(\tilde{U}\bar{U}\right)\frac{\partial\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(L\xi\right)} + \left(\frac{T^p}{L}\bar{U}\tilde{V}\right)\frac{\partial\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(\eta T^P\right)} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(\frac{\partial^2\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(L\xi\right)^2} + \frac{\partial^2\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(\eta T^P\right)^2}\right)$$

Reworking the convective terms, we obtain:

$$\left(\tilde{U}\bar{U}\right)\frac{\partial\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(L\xi\right)} = \left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{U}\frac{\partial\tilde{V}}{\partial\xi}$$

$$\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)\frac{\partial\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(\eta T^P\right)} = \left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{V}\frac{\partial\left(\tilde{V}\right)}{\partial\left(\eta\right)}$$

Proceeding to the viscous terms, we get:

$$\frac{\partial^2\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(L\xi\right)^2} = \left(\frac{T^p}{L^3}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\xi^2}$$

[1] The game here consists in dividing each dimensional variable by its order of magnitude, so that all scaled variables end up with unit order of magnitude!

[2] C'est l'objectif spécifique, pas le global, en fait! Anw bé yan!

$$\frac{\partial^2\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)}{\partial\left(\eta T^P\right)^2}=\left(\frac{1}{T^PL}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\eta^2}$$

So that the second momentum equation can be rewritten without loss of exactitude as:

$$\left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{U}\frac{\partial\tilde{V}}{\partial\xi}+\left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{V}\frac{\partial\left(\tilde{V}\right)}{\partial\left(\eta\right)}=-\frac{1}{\rho}\frac{\partial P}{\partial Y}+\nu\left(\left(\frac{T^p}{L^3}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\xi^2}+\left(\frac{1}{T^PL}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\eta^2}\right) \tag{10.1}$$

Exploiting the fact that all scaled variables have unit order of magnitude to reach to estimate the magnitude of the convective terms:

$$O\left[\left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{U}\frac{\partial\tilde{V}}{\partial\xi}+\left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{V}\frac{\partial\left(\tilde{V}\right)}{\partial\left(\eta\right)}\right]=O\left[\left(\frac{T^p}{L^2}\bar{U}^2\right)\right]$$

We will now recall a general property of the order of magnitude operation:

$$O\left[a+b\right]\approx O\left[a\right]+O\left[b\right]\approx\max\left(O\left[a\right],O\left[b\right]\right)\qquad\forall\,a,b\in\mathbb{R}$$

To assess the order of magnitude of viscous terms:

$$O\left[\left(\frac{T^p}{L^3}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\xi^2}+\left(\frac{1}{T^PL}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\eta^2}\right]=\max\left(O\left[\left(\frac{T^p}{L^3}\bar{U}\right)\right],O\left[\frac{1}{T^PL}\bar{U}\right]\right)$$

Recalling that the lenght of the flat plate is much larger than the thickness of the force field:

$$T^P\ll L\qquad\Rightarrow\qquad\left[\left(\frac{T^p}{L^3}\bar{U}\right)\right]\ll O\left[\frac{1}{T^PL}\bar{U}\right]$$

Whereby we get:

$$O\left[\left(\frac{T^p}{L^3}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\xi^2}+\left(\frac{1}{T^PL}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\eta^2}\right]=O\left[\frac{1}{T^PL}\bar{U}\right]$$

So that when we apply the order of magnitude operation on expression 10.1, we get:

$$O\left[\frac{1}{\rho}\frac{\partial P}{\partial Y}\right]=O\left[\left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{U}\frac{\partial\tilde{V}}{\partial\xi}+\left(\frac{T^p}{L^2}\bar{U}^2\right)\tilde{V}\frac{\partial\left(\tilde{V}\right)}{\partial\left(\eta\right)}\right]+O\left[\nu\left(\left(\frac{T^p}{L^3}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\xi^2}+\left(\frac{1}{T^PL}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\eta^2}\right)\right]$$

Leading to:

$$\frac{1}{O\left[\rho T^P\right]}O\left[P\right]=O\left[\left(\frac{T^p}{L^2}\bar{U}^2\right)\right]+O\left[\frac{\nu}{T^PL}\bar{U}\right]$$

Whereby:

$$O\left[P\right] = O\left[\rho\frac{T_p^2}{L^2}\bar{U}^2\right] + O\left[\frac{\mu}{L}\bar{U}\right] = \frac{1}{L}\left(O\left[\rho\frac{T_p^2}{L^2}\bar{U}^2\right] + O\left[\mu\bar{U}\right]\right)$$

So that we write:

$$\bar{P} = O\left[P\right] = \frac{1}{L}\max\left(\rho\frac{T_p^2}{L}\bar{U}^2\ ,\ \mu\bar{U}\right)$$

And define the scaled pressure as:

$$\tilde{P} = \frac{P}{\bar{P}} \quad \Rightarrow \quad P = \bar{P}\tilde{P} \qquad ,\ such\,that \quad O\left[\tilde{P}\right] = 1$$

It is not necessary to resolve the maximum for now, as the ultimate outcome will be same in either case because the order of magnitude of the pressure always decreases when L tends towards infinity << i still don't know what happens (in general) if we also take the limit $\mu \to 0$, but I guess we can live with that for now! we are in the boundary layer, il y a meme pas de probleme!>> :

$$\lim_{L\to\infty} \bar{P} = 0 \tag{10.2}$$

The importance of this property will become clear in the next sections as we will explore the case in which the lenght of the flat becomes very large. For this purpose, it is handy to rework the second momentum equation, expression 10.1, as to put the division by the lenght scale in evidence:

$$-\frac{\bar{P}}{\rho T^P}\frac{\partial\tilde{P}}{\partial\eta} = \frac{1}{L}\left(\left(\frac{T^p}{L}\bar{U}^2\right)\tilde{U}\frac{\partial\tilde{V}}{\partial\xi} + \left(\frac{T^p}{L}\bar{U}^2\right)\tilde{V}\frac{\partial\left(\tilde{V}\right)}{\partial\left(\eta\right)} - \nu\left(\left(\frac{T^p}{L^2}\bar{U}\right)\frac{\partial^2\tilde{V}}{\partial\xi^2} + \left(\frac{\bar{U}}{T^P}\right)\frac{\partial^2\tilde{V}}{\partial\eta^2}\right)\right) \tag{10.3}$$

10.1.2 Scaling the First Momentum Equation

Feeding these results into the first momentum equation, while restricting ourselves to steady flow $\left(\frac{\partial U}{\partial T} = 0\right)$:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + \frac{1}{\rho}F_x$$

$$\left(\tilde{U}\bar{U}\right)\frac{\partial\left(\tilde{U}\bar{U}\right)}{\partial\left(L\xi\right)} + \left(\frac{T^p}{L}\bar{U}\tilde{V}\right)\frac{\partial\left(\tilde{U}\bar{U}\right)}{\partial\left(\eta T^P\right)} = -\frac{1}{\rho}\frac{\partial\left(\bar{P}\tilde{P}\right)}{\partial\left(L\xi\right)} + \nu\left(\frac{\partial^2\left(\tilde{U}\bar{U}\right)}{\partial\left(L\xi\right)^2} + \frac{\partial^2\left(\tilde{U}\bar{U}\right)}{\partial\left(\eta T^P\right)^2}\right) + \frac{1}{\rho}F_x$$

Reworking the convective terms, we obtain:

$$\left(\tilde{U}\bar{U}\right)\frac{\partial\left(\tilde{U}\bar{U}\right)}{\partial\left(L\xi\right)} = \frac{\bar{U}^2}{L}\tilde{U}\frac{\partial\tilde{U}}{\partial\xi}$$

$$\left(\frac{T^p}{L}\bar{U}\tilde{V}\right)\frac{\partial\left(\tilde{U}\bar{U}\right)}{\partial\left(\eta T^P\right)}=\frac{\bar{U}^2}{L}\tilde{V}\frac{\partial\tilde{U}}{\partial\eta}$$

And proceed to the viscous terms:

$$\frac{\partial^2\left(\tilde{U}\bar{U}\right)}{\partial\left(L\xi\right)^2}=\frac{\bar{U}}{L^2}\frac{\partial^2\tilde{U}}{\partial\xi^2}$$

$$\frac{\partial^2\left(\tilde{U}\bar{U}\right)}{\partial\left(\eta T^P\right)^2}=\frac{\bar{U}}{T_p^2}\frac{\partial^2\tilde{U}}{\partial\eta^2}$$

Before wrapping up with the pressure gradient term:

$$-\frac{1}{\rho}\frac{\partial\left(\bar{P}\tilde{P}\right)}{\partial\left(L\xi\right)}=-\frac{1}{\rho}\frac{\bar{P}}{L}\frac{\partial\tilde{P}}{\partial\xi}$$

So that the first momentum equation can now be rewritten as:

$$\frac{\bar{U}^2}{L}\tilde{U}\frac{\partial\tilde{U}}{\partial\xi}+\frac{\bar{U}^2}{L}\tilde{V}\frac{\partial\tilde{U}}{\partial\eta}=-\frac{1}{\rho}\frac{\bar{P}}{L}\frac{\partial\tilde{P}}{\partial\xi}+\nu\left(\frac{\bar{U}}{L^2}\frac{\partial^2\tilde{U}}{\partial\xi^2}+\frac{\bar{U}}{T_p^2}\frac{\partial^2\tilde{U}}{\partial\eta^2}\right)+\frac{1}{\rho}F_x$$

Reordering to grouping the terms that are divided by L, we get our last exact version of the x momentum equation:

$$\frac{1}{L}\left(\bar{U}^2\tilde{U}\frac{\partial\tilde{U}}{\partial\xi}+\bar{U}^2\tilde{V}\frac{\partial\tilde{U}}{\partial\eta}+\frac{\bar{P}}{\rho}\frac{\partial\tilde{P}}{\partial\xi}-\nu\frac{\bar{U}}{L}\frac{\partial^2\tilde{U}}{\partial\xi^2}\right)=\nu\frac{\bar{U}}{T_p^2}\frac{\partial^2\tilde{U}}{\partial\eta^2}+\frac{1}{\rho}F_x \qquad (10.4)$$

10.2 Asymptotic Solution for Fully Developped Flow

10.2.1 The fully developped flow limit

We will now take the limit of our system of equations when the length of the flat plate tends towards infinity. Starting with the limit of the continuity equation, we observe that it becomes singular, and our system therefore looses an equation:

$$\lim_{L\to\infty}\left\{\frac{\bar{U}}{L}\left(\frac{\partial\tilde{U}}{\partial\xi}+\frac{\partial\tilde{V}}{\partial\eta}\right)=0\right\}\qquad\Leftrightarrow\qquad 0=0$$

Proceeding with the first momentum equation, expression 10.4, we obtain the key equation of our problem:

$$\lim_{L\to\infty}\left\{\frac{1}{L}\left(\bar{U}^2\tilde{U}\frac{\partial\tilde{U}}{\partial\xi}+\bar{U}^2\tilde{V}\frac{\partial\tilde{U}}{\partial\eta}+\frac{\bar{P}}{\rho}\frac{\partial\tilde{P}}{\partial\xi}-\nu\frac{\bar{U}}{L}\frac{\partial^2\tilde{U}}{\partial\xi^2}\right)=\nu\frac{\bar{U}}{T_p^2}\frac{\partial^2\tilde{U}}{\partial\eta^2}+\frac{1}{\rho}F_x\right\}$$

$$\Leftrightarrow \quad 0 = \nu \frac{\bar{U}}{T_p^2} \frac{\partial^2 \tilde{U}}{\partial \eta^2} + \frac{1}{\rho} F_x$$

$$\Leftrightarrow \quad 0 = \nu \frac{\partial^2 U}{\partial Y^2} + \frac{1}{\rho} F_x \tag{10.5}$$

Before we take the limit of the second momentum equation, expression 10.3, we will recall expression 10.2, stating that the limit of the pressure scalling vanishes as the flat plate lenght tends to infinity:

$$\lim_{L \to \infty} \bar{P} = 0$$

So that we can write that the 2^{nd} momentum equation is also singular to first order:

$$\lim_{L \to \infty} \left\{ -\frac{\bar{P}}{\rho T^P} \frac{\partial \tilde{P}}{\partial \eta} = \qquad \dots \right.$$

$$\left. = \frac{1}{L} \left(\left(\frac{T^p}{L} \bar{U}^2 \right) \left(\tilde{U} \frac{\partial \tilde{V}}{\partial \xi} + \tilde{V} \frac{\partial \left(\tilde{V} \right)}{\partial \left(\eta \right)} \right) - \nu \left(\left(\frac{T^p}{L^2} \bar{U} \right) \frac{\partial^2 \tilde{V}}{\partial \xi^2} + \left(\frac{\bar{U}}{T^P} \right) \frac{\partial^2 \tilde{V}}{\partial \eta^2} \right) \right) \right\}$$

$$\Leftrightarrow \quad 0 = 0$$

At this stage, it is worth noting that it would probably be possible to build a more sophisticated model by following an Asymptotic Matched Expansions approach of higher order, in which systems of equations would be obtained for the L^0, L^{-1} and L^{-2} orders, or a more interesting scaling parameter could be chosen. The system we obtained corresponds to the leading term of the asymptotic expansion, and it is not matched, because we expand in a single parameter, but it seems enough for a first model.

10.2.2 Integrating the momentum equation

Our system of equations collapsed into a single equation, expression 10.5, which can be reworked to put the shear stress in evidence:

$$\frac{\partial}{\partial Y} \left(\mu \frac{\partial U}{\partial Y} \right) + F_x = 0 \qquad , \qquad \frac{\partial \tau}{\partial Y} + F_x = 0$$

Integrating once, we obtain:

$$\int \frac{\partial}{\partial Y} \left(\mu \frac{\partial U}{\partial Y} \right) dY + \int F_x dY = 0$$

$$\Leftrightarrow \quad \mu \frac{\partial U}{\partial Y} + \int F_x dY = C^\tau \tag{10.6}$$

Where $C^{\tau}\perp Y$ is an integration constant, and the force field is reworked with expression 5.4, while recalling that $w^{x}_{\left(X,X_0^p,a\right)}\perp Y$ we write[3]:

$$\int F_x dY = \int \left(\phi^p_x w^y_{(Y,T_p)} w^x_{\left(X,X_0^p,a\right)}\right) dY = \phi^p_x w^x_{\left(X,X_0^p,L_p\right)} \int \left(w^y_{(Y,T_p)}\right) dY$$

10.2.2.1 Weighting Function Primitives

The plasma force normal weighting function, expression 5.1, can be rewritten as:

$$w^y_{(y,b)} = \begin{cases} \frac{\pi}{2}\cos\left(\frac{\pi y}{2b}\right) & , \quad \frac{y}{b} \in [0,1] \\ 0 & , \quad otherwise \end{cases} \tag{10.7}$$

so that it is integrated analytically:

$$\omega^y_{(y,b)} = \int w^y_{(y,b)} dY = \begin{cases} b\sin\left(\frac{\pi y}{2b}\right) & , \quad \frac{y}{b} \in [0,1] \\ b & , \quad otherwise \end{cases} \tag{10.8}$$

and care was taken to choose the integration constants so that the primitive would be continuous. Proceeding in a similar way for the second primitive, we write:

$$\Omega^y_{(y,b)} = \int \omega^y_{(y,b)} dY = \begin{cases} -\frac{2b^2}{\pi}\cos\left(\frac{\pi y}{2b}\right) & , \quad \frac{y}{b} \in [0,1] \\ 0 & , \quad otherwise \end{cases} \tag{10.9}$$

10.2.2.2 Shear Stress Field

The shear stress field can now be rewrittenby feeding the weighting function primitive, expression 10.8, into the shear stress equation of expression 10.6:

$$\tau + \phi^p_x w^x_{\left(X,X_0^p,a\right)} \omega^y_{(Y,T_p)} = C^{\tau}$$

Which also provides and explicit expression for the shear rate field:

$$\begin{aligned} \frac{\partial U}{\partial Y} &= \frac{C^{\tau}}{\mu} - \frac{\phi^p_x w^x_{\left(X,X_0^p,L_p\right)}}{\mu} \omega^y_{(Y,T_p)} \\ &= \frac{C^{\tau}}{\mu} - \frac{\phi^p_x w^x_{\left(X,X_0^p,L_p\right)}}{\mu} \begin{cases} T_p \sin\left(\frac{\pi Y}{2T_p}\right) & , \quad \frac{Y}{T_p} \in [0,1] \\ T_p & , \quad otherwise \end{cases} \end{aligned} \tag{10.10}$$

Observing this expression helps us notice that the first primitive of the weighting function is zero at the wall $\omega^y_{(Y=0,L_p)} = 0$, whereby the shear field expression reduces to a convenient form at the wall:

$$\tau\big|_{Y=0} + \phi^p_x w^x_{\left(X,X_0^p,L_p\right)} \underbrace{\omega^y_{(Y,T_p)}\Big|_{Y=0}}_{=0} = C^{\tau}$$

[3] The fully developped flow assumption supposes that$w^X \perp X$, but we argue that we are close to this situation if $T^p \ll L$.

$$\Rightarrow \qquad \tau_w^p = C^\tau \tag{10.11}$$

Where we defined the skin friction of the plasma driven flow as $\tau_w^p = \tau|_{Y=0}$.

10.2.2.3 Velocity Field

The velocity field is obtained explictly by integrating expression 10.10, leading to:

$$\int \frac{\partial U}{\partial Y} dY = \frac{C^\tau}{\mu} \int dY - \frac{\phi_x^p w^x_{(X,X_0^p,L_p)}}{\mu} \int \omega^y_{(Y,L_p)} dy$$

Which is reworked with the second primitive of the weighting function, expression 10.9, leading to:

$$\begin{aligned} U &= C^U + \frac{C^\tau}{\mu} Y - \frac{\phi_x^p w^x_{(X,X_0^p,L_p)}}{\mu} \Omega^y_{(Y,T_p)} \\ &= C^U + \frac{C^\tau}{\mu} Y - \frac{\phi_x^p w^x_{(X,X_0^p,L_p)}}{\mu} \begin{cases} -\frac{2T_p^2}{\pi} \cos\left(\frac{\pi Y}{2T_p}\right) & , \quad \frac{Y}{T_p} \in [0,1] \\ 0 & , \quad otherwise \end{cases} \end{aligned} \tag{10.12}$$

Which is a general solution of the problem we formulated in section 8.3, and can be matched to any set of two independent boundary conditions through the two integration constants C^τ and C^U.

10.2.3 Free Plate Limit Case

In the free upper flat plate limit case, the boundary conditions are written as:

$$U|_{Y=0} = 0 \qquad , \qquad \tau|_{Y=T_p} = 0$$

Where the first boundary condition embodies no slip at the bottom flat plate and the second boundary condition means that the top flat plate is free to move without opposing any resistance to the flow. To understand the meaning of this condition, it is wise to observe expression 10.10 to notice that:

$$\lim_{Y\to\infty} \tau = \tau|_{Y=T_p}$$

10.2.3.1 No Slip on Bottom Plate

The first condition translates directly into a value for the C^U integration constant, by writting the velocity field of expression 10.12 at the wall:

$$0 = U|_{Y=0} = C^U + \frac{C^\tau}{\mu}(0) + \frac{\phi_x^p w^x_{(X,X_0^p,L_p)}}{\mu} \frac{2T_p^2}{\pi} \underbrace{\cos\left(\frac{\pi(0)}{2T_p}\right)}_{=1}$$

$$\Leftrightarrow \quad 0 = C^U + \frac{\phi_x^p w^x_{\left(X,X_0^p,L_p\right)}}{\mu} \frac{2}{\pi} T_p^2$$

$$\Leftrightarrow \quad C^U = -\frac{\phi_x^p w^x_{\left(X,X_0^p,L_p\right)}}{\mu} \frac{2}{\pi} T_p^2 \tag{10.13}$$

10.2.3.2 No Stress on Top Plate

The C^τ integration constant is determined directly the second boundary condition by evaluating the shear stress field, expression 10.10, at the upper plate:

$$\underbrace{\mu \frac{\partial U}{\partial Y}\bigg|_{Y=T_p}}_{=\tau|_{Y=T_p}} = C^\tau - \phi_x^p w^x_{\left(X,X_0^p,L_p\right)} \underbrace{\omega^y_{(Y,T_p)}\bigg|_{Y=T_p}}_{=T_p}$$

$$\Leftrightarrow \quad C^\tau = \phi_x^p w^x_{\left(X,X_0^p,L_p\right)} T_p \tag{10.14}$$

10.2.4 Fixed Plate Limit Case

In the fixed top flat plate limit case, the boundary conditions are written as:

$$U|_{Y=0} = 0 \qquad , \qquad U|_{Y=T_p} = 0$$

Meaning that no slip is applied on both sides, whereby the first boundary condition and the C^U integration constant remains unchanged[4]:

$$C^U = -\frac{\phi_x^p w^x_{\left(X,X_0^p,L_p\right)}}{\mu} \frac{2}{\pi} T_p^2 \tag{10.15}$$

10.2.4.1 No Slip on Top Plate

The second boundary condition is now imposed by evaluating the velovity field, expression 10.12, at the top plate height:

$$\underbrace{U|_{Y=T_p}}_{=0} = C^U + \frac{C^\tau}{\mu}(T_p) - \frac{\phi_x^p w^x_{\left(X,X_0^p,L_p\right)}}{\mu} \underbrace{\Omega^y_{(Y,T_p)}\bigg|_{Y=T_p}}_{=0}$$

$$\Leftrightarrow \quad 0 = C^U + \frac{C^\tau}{\mu}(T_p)$$

$$\Leftrightarrow \quad C^\tau = -\frac{\mu}{T_p} C^U$$

[4]For both cases, the boundary conditions are not only indepent but also orthoghonal/uncoupled in their relation to the integration constants!

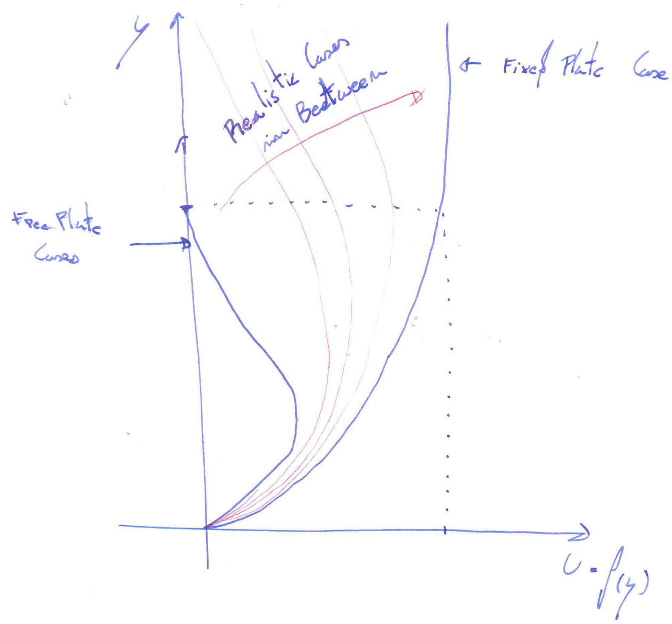


Figure 10.2: Free and Fixed Plate Limit Cases

Inserting the expression for C^U, we get a new expression for the C^τ integration constant:

$$C^\tau = \frac{2}{\pi}\left(\phi_x^p w^x_{(X,X_0^p,L_p)} T_p\right) \tag{10.16}$$

To interpret this statement, we recall from expression 10.11, that the C^τ integration constant is equal to the skin friction on the lower plate τ_w^p. The skin friction expression has a similar form for the fixed plate case, expression 10.16, and the free plate case, expression 10.14. In fact the expressions only differ by a factor $^2/_\pi$, meaning that the skin friction is 30% smaller $(1 - {}^2/_\pi)$ than for the fixed case.

10.3 A simple Skin Friction Correction

10.3.1 Postulates

We postulate that unbounded plasma driven flows along (single) flat plates must have a flow field that lies somewhere between the Free plate and the Fixed plate limit case flows. Figure 10.2 illustrates this statement.

If this holds, then we can also defend that the skin friction of a fully developed unbounded plasma driven flow in equilibirum is likely to be bounded by the skin frictions of these limit cases:

$$\frac{2}{\pi}\left(\phi_x^p w^x_{\left(X,X_0^p,L_p\right)} T_p\right) < \tau_w^p < \phi_x^p w^x_{\left(X,X_0^p,L_p\right)} T_p$$

that is, between the values of expressions 10.16 and 10.14.

Furthermore, we argue that the skin friction only reaches its maximum value when the flow is fully developped. As such, we expect that the skin friction of a developping plasma driven flow will always be smaller than the skin frictin of the corresponding (fully) developped flow.

Given that the plasma driven flows observed on DBD plasma actuators usually have very moderate scale ratios :

$$\frac{T_p}{L_p} \approx \frac{2.5}{17.5} \approx 0.14$$

it is expectable that these flows remain quite far from equilbrium, and therefore introduce smaller skin friction perturbations than those predicted from fully developped flow models. Considering these arguments, we propose to establish a skin friction perturbation correction based upon the skin friction of the fixed plate flow case.

The skin friction of a boundary layer perturbed by a plasma actuator, τ_w, can now be approximated by summing the skin friction of the unperturbed boundary layer, τ_w^0 to the skin friction of the plasma driven flow between fixed plates,τ_w^p:

$$\underbrace{\tau_w^T}_{total} = \underbrace{\tau_w^0}_{unperturbed} + \underbrace{\tau_w^p}_{fixed} \tag{10.17}$$

10.3.2 Coefficient Form

We will now rework expression 10.17 to obtain a compact skin friction coefficient correction formula. Inserting expression 10.16, we get:

$$\tau_w^T = \tau_w^0 + \frac{2}{\pi}\left(\phi_x^p w^x_{\left(X,X_0^p,L_p\right)} T_p\right)$$

Divididing both sides to identify the skin friction coefficient adimensional groups:

$$\underbrace{\frac{\tau_w^T}{\frac{1}{2}\rho U_e^2}}_{C_f^T} = \underbrace{\frac{\tau_w^0}{\frac{1}{2}\rho U_e^2}}_{C_f^0} + \underbrace{\frac{2}{\pi}\frac{\phi_x^p w^x_{\left(X,X_0^p,L_p\right)} T_p}{\frac{1}{2}\rho U_e^2}}_{C_f^p}$$

Reworking the plasma term to express it in terms of our usual adimensional groups, we write:

$$C_f^p = \frac{2}{\pi}\frac{\phi_x^p w^x_{(X,X_0^p,L_p)} T_p}{\frac{1}{2}\rho U_e^2} = \frac{2}{\pi}\frac{1}{\underbrace{\left(\frac{U_e^2}{U_\infty^2}\right)}_{u_e^2}}\underbrace{\frac{\overbrace{\phi_x^p T_p}^{C_{\phi_x^p}}}{\frac{1}{2}\rho U_\infty^2} w^x_{(x,x_0^p,l_p)}}_{C_{FM(x)}}$$

$$\Leftrightarrow \quad C_f^p = \frac{2}{\pi}\frac{C_{FM}}{u_e^2}$$

So that the the total skin friction in a boundary layer can now be written as:

$$\underbrace{C_f^T}_{total} = \underbrace{C_f^0}_{unperturbed} + \frac{2}{\pi}\frac{C_{FM}}{u_e^2} \tag{10.18}$$

Which is the expression implemented in our model.

Chapter 11

Similarity Solution of Laminar, Free Shear Plasma Flow

In this chapter, we review the work of Blasius and Falkner-Skan in order to inspire ourselves and identify a class of similarity solutions for the laminar flow on flat plate with a plasma field. This is still work under progress.

11.1 Towards the Similarity Equations

11.1.1 Similarity Postulate

We postulate that there exist boundary layer flows such that:

$$U = U_e f_{(\eta)}$$

Where we have defined δ as the boundary layer thickness:

$$\eta = \frac{Y}{\delta} \qquad , \qquad \delta = b_{(x)} \perp Y$$

Implying that:

$$\frac{\partial \eta}{\partial Y} = \frac{1}{\delta} \qquad \frac{\partial \eta}{\partial X} = -\frac{Y}{\delta^2}\frac{\partial \delta}{\partial X}$$

And assuming η is sufficiently smooth, the inverse theorem to leads to:

$$\frac{\partial Y}{\partial \eta} = \left[\frac{\partial \eta}{\partial Y}\right]^{-1} = \delta$$

Whereby:

$$\frac{\partial}{\partial X}\left(f_{(\eta)}\right) = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial X} \qquad , \qquad \frac{\partial}{\partial Y}\left(f_{(\eta)}\right) = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial Y}$$

So that we can write the x-derivative:

$$\frac{\partial U}{\partial X} = \frac{\partial}{\partial X}\left(U_e f_{(\eta)}\right) = f_{(\eta)}\frac{\partial}{\partial X}\left(U_e\right) + U_e\frac{\partial}{\partial X}\left(f_{(\eta)}\right) = f_{(\eta)}\frac{\partial U_e}{\partial X} + U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial X}$$

$$= f_{(\eta)}\frac{\partial U_e}{\partial X} - U_e\frac{\partial f}{\partial \eta}\left(\frac{Y}{\delta^2}\frac{\partial \delta}{\partial X}\right) = f_{(\eta)}\frac{\partial U_e}{\partial X} - \eta\frac{\partial f}{\partial \eta}\frac{U_e}{\delta}\frac{\partial \delta}{\partial X}$$

And also the y-derivative, by exploiting $U_e \perp Y$:

$$\frac{\partial U}{\partial Y} = \frac{\partial}{\partial Y}\left(U_e f_{(\eta)}\right) = U_e\frac{\partial}{\partial Y}\left(f_{(\eta)}\right) = U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial Y} = \frac{U_e}{\delta}\frac{\partial f}{\partial \eta}$$

11.1.2 Continuity

Recalling continuity:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

We can feed in the expression for $\frac{\partial U}{\partial X}$ to get a relation for $\frac{\partial V}{\partial Y}$:

$$f_{(\eta)}\frac{\partial U_e}{\partial X} + U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial X} + \frac{\partial V}{\partial Y} = 0$$

$$\frac{\partial V}{\partial Y} = -\left(f_{(\eta)}\frac{\partial U_e}{\partial X} + U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial X}\right)$$

Recalling that:

$$\frac{\partial V}{\partial Y} = \frac{\partial V}{\partial \eta}\frac{\partial \eta}{\partial Y}$$

We further rework continuity into:

$$\frac{\partial V}{\partial \eta}\frac{\partial \eta}{\partial Y} = -\left(f_{(\eta)}\frac{\partial U_e}{\partial X} + U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial X}\right)$$

And use the inverse theorem to write:

$$\frac{\partial V}{\partial \eta} = -\frac{\partial Y}{\partial \eta}\left(f_{(\eta)}\frac{\partial U_e}{\partial X} + U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial X}\right)$$

At which stage it is useful to feed in the expressions we obtained for η:

$$\frac{\partial V}{\partial \eta} = -\delta\left(f_{(\eta)}\frac{\partial U_e}{\partial X} + U_e\frac{\partial f}{\partial \eta}\left(-\frac{Y}{\delta^2}\frac{\partial \delta}{\partial X}\right)\right)$$

$$\frac{\partial V}{\partial \eta} = -\delta\left(f_{(\eta)}\frac{\partial U_e}{\partial X} - \frac{U_e Y}{\delta^2}\frac{\partial f}{\partial \eta}\left(\frac{\partial \delta}{\partial X}\right)\right)$$

$$\frac{\partial V}{\partial \eta} = U_e\frac{Y}{\delta}\frac{\partial f}{\partial \eta}\frac{\partial \delta}{\partial X} - \delta f_{(\eta)}\frac{\partial U_e}{\partial X}$$

$$\frac{\partial V}{\partial \eta} = U_e \frac{\partial \delta}{\partial X} \eta \frac{\partial f}{\partial \eta} - \delta \frac{\partial U_e}{\partial X} f_{(\eta)}$$

Integrating in η:

$$\int_0^\eta \frac{\partial V}{\partial \eta} d\eta = \int_0^\eta \left(U_e \frac{\partial \delta}{\partial X} \eta \frac{\partial f}{\partial \eta} - \delta \frac{\partial U_e}{\partial X} f_{(\eta)} \right) d\eta$$

We now notice that, as long as V satisfies minimum smoothness requirements:

$$V_{(\eta)} = V_0 + \int_0^\eta \frac{\partial V}{\partial \eta} d\eta \qquad \Rightarrow \qquad \Leftrightarrow \quad V_{(\eta)} - V_0 = \int_0^\eta \frac{\partial V}{\partial \eta} d\eta$$

So that we get an expression for V that must be matched to the wall transpiration boundary condition:

$$V - V_0 = \int_0^\eta \left(U_e \frac{\partial \delta}{\partial X} \eta \frac{\partial f}{\partial \eta} - \delta \frac{\partial U_e}{\partial X} f_{(\eta)} \right) d\eta$$

And if we assume that δ and U_e are taken as independent variables[1]:

$$V = U_e \frac{\partial \delta}{\partial X} \int_0^\eta \left(\eta \frac{\partial f}{\partial \eta} \right) d\eta - \delta \frac{\partial U_e}{\partial X} \int_0^\eta \left(f_{(\eta)} \right) d\eta + C_\eta \qquad , \quad C_\eta : V_{(\eta=0)} = V_0$$

$$C_\eta : V_{(\eta=0)} = V_0 \quad \Rightarrow \quad C_\eta = V_0$$

<< get rid of C_eta for clarity, as we always have $C_\eta = V_0$, so use V_0 (suction/transpiration speed) all along! >>

It makes sense to rework the integral with the chain rule:

$$\frac{\partial}{\partial \eta}\left(\eta f_{(\eta)}\right) = \underbrace{\frac{\partial \eta}{\partial \eta}}_{=1} f_{(\eta)} + \eta \frac{\partial f}{\partial \eta} = f_{(\eta)} + \eta \frac{\partial f}{\partial \eta}$$

$$\Leftrightarrow \quad \eta \frac{\partial f}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\eta f_{(\eta)}\right) - f_{(\eta)}$$

$$\Leftrightarrow \quad \int \left(\eta \frac{\partial f}{\partial \eta} \right) d\eta = \int \left(\frac{\partial}{\partial \eta}\left(\eta f_{(\eta)}\right) \right) d\eta - \int f_{(\eta)} d\eta$$

$$\Leftrightarrow \quad \int \left(\eta \frac{\partial f}{\partial \eta} \right) d\eta = \eta f_{(\eta)} - \int \left(f_{(\eta)} \right) d\eta$$

Finally, defining $g_{(\eta)}$ as the integral of $f_{(\eta)}$ over the $[0, \eta]$ interval, we get:

$$g_{(\eta)} = \int_0^\eta \left(f_{(\eta)} \right) d\eta \qquad , \qquad f_{(\eta)} = \frac{\partial g}{\partial \eta}$$

[1] Meaning that δ and U_e define η but do not depend on it!

Notice that, provided minimal smoothness requirements are met by f, the definition of g implies that:

$$g_{(0)} = \int_0^0 \left(f_{(\eta)}\right) d\eta = 0 \tag{11.1}$$

So that V can be be written as:

$$V = U_e \frac{\partial \delta}{\partial X} \left(\eta f_{(\eta)} - \int \left(f_{(\eta)}\right) d\eta \right) - \delta \frac{\partial U_e}{\partial X} \int_0^\eta \left(f_{(\eta)}\right) d\eta + C_\eta$$

$$V = U_e \frac{\partial \delta}{\partial X} \left(\eta \frac{\partial g}{\partial \eta} - g_{(\eta)} \right) - \delta \frac{\partial U_e}{\partial X} g_{(\eta)} + C_\eta$$

$$V = U_e \frac{\partial \delta}{\partial X} \eta \frac{\partial g}{\partial \eta} - \left(\delta \frac{\partial U_e}{\partial X} + U_e \frac{\partial \delta}{\partial X} \right) g_{(\eta)} + C_\eta$$

Noticing that:

$$\frac{\partial}{\partial X} \left(\delta \frac{\partial U_e}{\partial X} + U_e \frac{\partial \delta}{\partial X} \right) = \frac{\partial}{\partial X} \left(\delta U_e \right)$$

We finally get:

$$V = U_e \frac{\partial \delta}{\partial X} \eta \frac{\partial g}{\partial \eta} - \frac{\partial}{\partial X} \left(\delta U_e \right) g_{(\eta)} + C_\eta$$

A similar result could have been obtained using the stream function, which seems the most favored approach in literature. That approach retains sufficient generality because the stream function exists as a unique function[2] as long the flow field is solenoidal, i.e. divergence-free or when continuity holds in incompressible flow.

Even so, despite the additional algebraic burden that it implies, we chose to use an explicit formulation because:

- We had doubts as to wether there were hidden assumptions in the recovery of the stream-function when integrating the tangential speed
- Emphasis was placed on this question, as we wanted our deduction to remain independent from any particular assumptions on the boundary layer growth pattern $\delta_{(x)}$, $\frac{\partial \delta}{\partial X}$ and external flow speed distribution $\frac{\partial U_e}{\partial X}$.
- A thorough review of Schwartz's Theorem on mixed derivatives would probably provide a sound theoretical foundation to answer these questions in a more elegant way

Still, it should be noted that, as soon as we have written V from U through continuity, we have introduced an isogeometric constraint imposing the flow field's solenoidal nature.

This formulation should therefore be completely equivalent to a stream-function based one, as that is exactly what the stream function does to collapse the 2D flow description into a single variable!

[2] Ahah! All functions are unique! but do they have equal rights?

11.1.3 Momentum

Let us now observe the first momentum equation of the Boundary Layer PDEs:

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=U_e\frac{\partial U_e}{\partial X}+\nu\frac{\partial^2 U}{\partial Y^2}+\frac{1}{\rho}F_x$$

Before we rework it, let us redefine U in terms of g:

$$U=U_e f_{(\eta)}=U_e\frac{\partial g}{\partial \eta}$$

And do the same for its first derivatives:

$$\frac{\partial U}{\partial X}=f_{(\eta)}\frac{\partial U_e}{\partial X}-\eta\frac{\partial f}{\partial \eta}\frac{U_e}{\delta}\frac{\partial \delta}{\partial X}=\frac{\partial g}{\partial \eta}\frac{\partial U_e}{\partial X}-\eta\frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}\frac{\partial \delta}{\partial X}$$

$$\frac{\partial U}{\partial Y}=U_e\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial Y}=\frac{U_e}{\delta}\frac{\partial f}{\partial \eta}=\frac{U_e}{\delta}\frac{\partial^2 g}{\partial \eta^2}$$

Before we attack the second y derivative:

$$\frac{\partial^2 U}{\partial Y^2}=\frac{\partial}{\partial Y}\left(\frac{U_e}{\delta}\frac{\partial^2 g}{\partial \eta^2}\right)=\frac{U_e}{\delta}\frac{\partial}{\partial Y}\left(\frac{\partial^2 g}{\partial \eta^2}\right)+\frac{\partial^2 g}{\partial \eta^2}\frac{\partial}{\partial Y}\left(\frac{U_e}{\delta}\right)$$

Recalling that $U_e, \delta \perp y$, we get:

$$\frac{\partial^2 U}{\partial Y^2}=\frac{U_e}{\delta}\frac{\partial}{\partial Y}\left(\frac{\partial^2 g}{\partial \eta^2}\right)=\frac{U_e}{\delta}\frac{\partial}{\partial \eta}\left(\frac{\partial^2 g}{\partial \eta^2}\right)\frac{\partial \eta}{\partial Y}$$

$$=\frac{U_e}{\delta}\frac{\partial^3 g}{\partial \eta^3}\frac{\partial \eta}{\partial Y}=\frac{U_e}{\delta^2}\frac{\partial^3 g}{\partial \eta^3}$$

So that we can now rework the momentum equation term by term! We start with the first convective term:

$$U\frac{\partial U}{\partial X}=\left(U_e\frac{\partial g}{\partial \eta}\right)\left(\frac{\partial g}{\partial \eta}\frac{\partial U_e}{\partial X}-\eta\frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}\frac{\partial \delta}{\partial X}\right)=U_e\left(\frac{\partial g}{\partial \eta}\right)^2\frac{\partial U_e}{\partial X}-\eta\frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2}\frac{U_e^2}{\delta}\frac{\partial \delta}{\partial X}$$

And proceed to the second convective term:

$$V\frac{\partial U}{\partial Y}=\left(U_e\frac{\partial \delta}{\partial X}\eta\frac{\partial g}{\partial \eta}-\frac{\partial}{\partial X}\left(\delta U_e\right)g_{(\eta)}+C_\eta\right)\left(\frac{U_e}{\delta}\frac{\partial^2 g}{\partial \eta^2}\right)=$$

$$=\eta\frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2}\frac{U_e^2}{\delta}\frac{\partial \delta}{\partial X}-g_{(\eta)}\frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}\frac{\partial}{\partial X}\left(\delta U_e\right)+\frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}C_\eta$$

So that the two convective terms turn into:

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=$$

$$= U_e \left(\frac{\partial g}{\partial \eta}\right)^2 \frac{\partial U_e}{\partial X} \underbrace{-\eta \frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2}\frac{U_e^2}{\delta}\frac{\partial \delta}{\partial X} + \eta \frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \eta^2}\frac{U_e^2}{\delta}\frac{\partial \delta}{\partial X}}_{=0} - g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}\frac{\partial}{\partial X}(\delta U_e) + \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta} C_\eta$$

$$= \left(\frac{\partial g}{\partial \eta}\right)^2 U_e \frac{\partial U_e}{\partial X} - g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}\frac{\partial}{\partial X}(\delta U_e) + \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta} C_\eta$$

So we can finally write the complete momentum equation:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = U_e \frac{\partial U_e}{\partial X} + \nu \frac{\partial^2 U}{\partial Y^2} + \frac{1}{\rho} F_x$$

$$\Leftrightarrow \qquad \left(\frac{\partial g}{\partial \eta}\right)^2 U_e \frac{\partial U_e}{\partial X} - g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta}\frac{\partial}{\partial X}(\delta U_e) + \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta} C_\eta =$$

$$= U_e \frac{\partial U_e}{\partial X} + \nu \frac{U_e}{\delta^2}\frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x$$

11.1.4 Particular Cases

11.1.4.1 Zero Pressure Gradient Case

For the particular case of zero pressure gradient, we can write:

$$\frac{\partial U_e}{\partial X} = 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial X}(\delta U_e) = U_e \frac{\partial \delta}{\partial X}$$

So that the momentum equation collapses to:

$$\Leftrightarrow \qquad -g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2}\frac{U_e^2}{\delta}\frac{\partial \delta}{\partial X} + \frac{\partial^2 g}{\partial \eta^2}\frac{U_e}{\delta} C_\eta =$$

$$= \nu \frac{U_e}{\delta^2}\frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x$$

11.1.4.2 No Gradient, No Transpiration Case

We will now restrain ourselves to the case in which there is no suction or transpiration at the wall, to impose that:

$$C_\eta = V_0 = 0$$

So that our equation collapses into:

$$\Leftrightarrow \qquad -g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2}\frac{U_e^2}{\delta}\frac{\partial \delta}{\partial X} = \nu \frac{U_e}{\delta^2}\frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x \tag{11.2}$$

11.2 Estimating Boundary Layer Growth

11.2.1 Order of Magnitude Analysis

We start by restating the Navier Stokes equations while restricting ourselves to steady flow and a single force field component:

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=U_e\frac{\partial U_e}{\partial X}+\nu\frac{\partial^2 U}{\partial Y^2}+\frac{1}{\rho}F_x$$
$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0$$

And infer that the flow admits two key geometric scales:

$$O\left[X\right]=L \qquad , \qquad O\left[Y\right]=\delta$$

And define an unknown scale for the flow speed in the longitudinal direction:

$$O\left[U\right]=\bar{U}$$

11.2.1.1 Continuity Equation

We will now pursue an order of magnitude analysis on the continuity equation to obtain an estimate of the order of magnitude of the vertical speed component:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \qquad \Rightarrow \qquad O\left[\frac{\partial U}{\partial X}\right]=O\left[\frac{\partial V}{\partial Y}\right]$$
$$\Leftrightarrow \quad \frac{O\left[U\right]}{O\left[X\right]}=\frac{O\left[V\right]}{O\left[Y\right]} \quad \Leftrightarrow \quad \frac{\bar{U}}{L}=\frac{O\left[V\right]}{\delta} \Leftrightarrow \quad \frac{\delta}{L}\bar{U}=O\left[V\right]$$

11.2.1.2 Momentum Equation

The order of magnitude operation can now be applied to the x momentum equation:

$$O\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=O\left[U_e\frac{\partial U_e}{\partial X}\right]+O\left[\nu\frac{\partial^2 U}{\partial Y^2}\right]+O\left[\frac{1}{\rho}F_x\right]$$

We start by reworking the convective terms:

$$O\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=O\left[\bar{U}\frac{\bar{U}}{L}+\bar{U}\frac{\bar{U}}{L}\right]=\frac{\bar{U}^2}{L}$$

<< this part may or may not be kept>>
It is worth noting that the classical deductions consider

$$O\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=\frac{\bar{U}^2}{L}$$

But we would be lead to think this should be:

$$O\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=2\frac{\bar{U}^2}{L}$$

Does this mean that to get:

$$O\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=\frac{U_e^2}{L}$$

we should make a (fundamented) guess in the style:

$$\bar{U}=U_e\sqrt{2}$$

on grounds that U varies between 0 and U_e, and $\bar{U}$ should be a representative value of that range? maybe a game with Bernoulli would be welcome? $\frac{1}{2}\rho U^2$ may give some cool terms (but conservation of energy does not hold in the boundary layer, and hence classical bernoulli also does not, but the pressure equation with the vorticity terms does)!

<< end of temporary part>>

The pressure gradient cannot be estimated from the order of magnitude analysis as it an independent, prescribed variable with a wide dynamic range. So we only take U_e out of the order of magnitude operator:

$$O\left[U_e\frac{\partial U_e}{\partial X}\right]=U_eO\left[\frac{\partial U_e}{\partial X}\right]$$

The viscous term can be estimated quite simply:

$$O\left[\nu\frac{\partial^2 U}{\partial Y^2}\right]=\nu\frac{O\left[U\right]}{\left(O\left[Y\right]\right)^2}=\nu\frac{\bar{U}}{\delta^2}$$

To rework the plasma term, we will need a representative value of the plasma force density in the boundary layer. We will assume that the plasma is all contained in the boundary layer, and take the average force value, which we already defined at a previous stage:

$$O\left[F_x\right]=\frac{\int_{X_0^P}^{X_0^P+L^p}\int_0^{T^p}\left(F_x\right)dY\,dX}{\int_{X_0^P}^{X_0^P+L^p}\int_0^{T^p}dY\,dX}=\phi_x^p=\frac{F_T^p}{L^pT^p}$$

So that we can write:

$$O\left[\frac{1}{\rho}F_x\right]=\frac{\phi_x^p}{\rho}$$

<< change this to average force in boundary layer:

$$O\left[F_x\right]=\frac{\int_0^\delta\left(F_x\right)dY}{\int_0^\delta dY}=\frac{\int_0^\delta F_xdY}{\delta}$$

We have been working with:

$$F_x = \phi_x^p w^y_{(Y,T_p)} w^x_{\left(X,X_0^p,a\right)}$$

But will define a new parameter γ_x^p for the force density in the longitudinal direction, and dependent in X but not Y:

$$\gamma_x^p = \phi_x^p w^x_{\left(X,X_0^p,a\right)} \perp Y$$

So that we can introduce the notation:

$$F_x = \gamma_x^p w^y_{(Y,T_p)} \tag{11.3}$$

And hence

$$\int_0^\delta F_x dY = \int_0^\delta \gamma_x^p w^y_{(Y,T_p)} dY = \gamma_x^p \int_0^\delta w^y_{(Y,T_p)} dY$$

Assuming that the plasma is contained entirely in the boundary layer:

$$T_p < \delta$$

We can write:

$$\int_0^\delta w^y_{(Y,T_p)} dY = \int_0^{T_p} w^y_{(Y,T_p)} dY = T_p \qquad , \quad T_p < \delta$$

$$\Rightarrow \qquad \int_0^\delta F_x dY = \gamma_x^p T_p$$

So that we get:

$$O\left[F_x\right] = \frac{\int_0^\delta (F_x)\, dY}{\int_0^\delta dY} = \frac{\int_0^\delta F_x dY}{\delta} = \frac{\gamma_x^p T_p}{\delta} \quad \perp \quad Y$$

And recall the scaling:

$$\bar{t}_p = \frac{T_p}{\delta}$$

So that we write:

$$O\left[F_x\right] = \gamma_x^p \bar{t}_p \quad \perp \quad Y$$

And

$$O\left[\frac{1}{\rho} F_x\right] = \frac{\gamma_x^p \bar{t}_p}{\rho} \quad \perp \quad Y$$

But dependent on X, both through δ and γ_x^p.

>>

Whereby the complete order of magnitude version of the momentum equation becomes:

$$\frac{\bar{U}^2}{L} = U_e O\left[\frac{\partial U_e}{\partial X}\right] + \nu \frac{\bar{U}}{\delta^2} + \frac{\gamma_x^p T_p}{\delta \rho} \tag{11.4}$$

11.2.2 Particular Cases

11.2.2.1 Flat Plate with No Plasma

In the Flat plate case, there is no pressure gradient:

$$\frac{\partial U_e}{\partial X} = 0$$

Further discarding the growth plasma term, the order of magnitude version of the momentum equation, expression 11.4, collapses into:

$$\frac{\bar{U}^2}{L} = \nu \frac{\bar{U}}{\delta^2}$$

From which the classical boundary layer thickness expression can be recovered:

$$\begin{aligned} \frac{1}{v}\frac{\bar{U}}{L} = \frac{1}{\delta^2} \qquad &\Leftrightarrow \qquad v\frac{L}{\bar{U}} = \delta^2 \\ \Leftrightarrow \qquad \left(\frac{\delta}{L}\right)^2 &= \frac{v}{\bar{U}L} = \frac{1}{\frac{UL}{\nu}} = \frac{1}{Re_L} \\ \Leftrightarrow \qquad \frac{\delta}{L} = \frac{1}{\sqrt{Re_L}} \qquad &\delta = \frac{L}{\sqrt{Re_L}} \end{aligned}$$

And turn into the expression used for the Blasius expression if we take the X coordinate as the longitudinal lenght scale:

$$\delta = \frac{X}{\sqrt{Re_X}} \tag{11.5}$$

And further write the derivative of the boundary layer thickness scale to the X direction:

$$\begin{aligned} \frac{\partial \delta}{\partial X} &= \frac{\partial}{\partial X}\left(\frac{X}{\sqrt{\frac{U}{\nu}}\sqrt{X}}\right) = \sqrt{\frac{\nu}{U}}\frac{\partial}{\partial X}\left(\sqrt{X}\right) = \sqrt{\frac{\nu}{U}}\frac{1}{2}\frac{1}{\sqrt{X}} \\ &= \sqrt{\frac{\nu}{UX}}\frac{1}{2} = \frac{1}{2}\frac{1}{\sqrt{\frac{UX}{\nu}}} = \frac{1}{2\sqrt{Re_X}} \end{aligned}$$

Finally, it is also useful to relate Re_δ to Re_X:

$$\begin{aligned} Re_\delta &= \frac{U_e\delta}{\nu} = \frac{U_e}{\nu}\frac{X}{\sqrt{Re_X}} = \frac{U_e}{\nu}\frac{X}{\sqrt{\frac{U_eX}{\nu}}} = \frac{U_e}{\nu}\frac{X}{\sqrt{X}}\frac{\sqrt{\nu}}{\sqrt{U_e}} \\ &= \frac{U_e}{\sqrt{U_e}}\frac{X}{\sqrt{X}}\frac{\sqrt{\nu}}{\nu} = \sqrt{U_e}\sqrt{X}\frac{1}{\sqrt{\nu}} = \sqrt{\frac{U_eX}{\nu}} = \sqrt{Re_X} \end{aligned}$$

Meaning that for the very particular case of a flat plate, with no pressure gradient, no suction and no plasma, we have:

$$Re_\delta = \sqrt{Re_X} \tag{11.6}$$

11.2.2.2 Flat Plate with Plasma

We depart again from expression 11.4 and drop the pressure gradient but keep the plasma term:

$$\frac{\bar{U}^2}{L} = \nu \frac{\bar{U}}{\delta^2} + \frac{\gamma_x^p \bar{t}_p}{\rho}$$

So that we put the boundary layer thickness in evidence:

$$\nu \frac{\bar{U}}{\delta^2} = \frac{\bar{U}^2}{L} - \frac{\gamma_x^p \bar{t}_p}{\rho}$$

$$\frac{1}{\delta^2} = \frac{1}{\bar{U}\nu}\left(\frac{\bar{U}^2}{L} - \frac{\gamma_x^p \bar{t}_p}{\rho}\right)$$

$$\delta^2 = \frac{1}{\frac{1}{\bar{U}\nu}\left(\frac{\bar{U}^2}{L} - \frac{\gamma_x^p \bar{t}_p}{\rho}\right)}$$

$$\delta^2 = \frac{\bar{U}\nu}{\left(\frac{\bar{U}^2}{L} - \frac{\phi_x^p}{\rho}\right)} = \frac{\bar{U}\nu}{\bar{U}^2\left(\frac{1}{L} - \bar{t}_p \frac{\gamma_x^p}{\rho \bar{U}^2}\right)}$$

$$\delta^2 = \frac{\nu}{\bar{U}} \frac{1}{\left(\frac{1}{L} - \bar{t}_p \frac{\gamma_x^p}{\rho \bar{U}^2}\right)}$$

Whereby we can write:

$$\delta = \sqrt{\frac{\nu}{\bar{U}} \frac{1}{\left(\frac{1}{L} - \bar{t}_p \frac{\gamma_x^p}{\rho \bar{U}^2}\right)}}$$

Choosing the X coodinate as the lenght scale again, we get:

$$\delta_{(X)} = \sqrt{\frac{\nu}{\bar{U}} \frac{1}{\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)}} = \left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)^{-\frac{1}{2}} \tag{11.7}$$

Please note that even though $\bar{t}_p$ depends on δ:

$$\bar{t}_p = \frac{T_p}{\delta}$$

We are considering that the input of interest is $\bar{t}_p$ and not T_p, and further constrain ourselves to situations in which:

$$T_p : \qquad \frac{\partial \bar{t}_p}{\partial \delta} = 0 \Rightarrow \frac{\partial \bar{t}_p}{\partial x} = 0$$

This assumption constrains us to a particular class of equilibrium flows, and there may exist some equilibrium flows that we are ruling out with this constraint. Still, it reduces complexity in a significant way, without loss of exactness but with some loss of generality.

Considering that we constrained ourselves to $\frac{\partial \bar{t}_p}{\partial X} = 0$, we compute the spatial rate of growth as [3]:

$$\frac{\partial \delta}{\partial X} = \frac{\partial}{\partial X}\left(\left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}}\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)^{-\frac{1}{2}}\right) = \left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}} \frac{\partial}{\partial X}\left(\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)^{-\frac{1}{2}}\right)$$

$$= -\frac{1}{2}\left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}}\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)^{-\frac{3}{2}} \frac{\partial}{\partial X}\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)$$

$$= -\frac{1}{2}\left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}}\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)^{-\frac{3}{2}} \frac{\partial}{\partial X}\left(\frac{1}{X}\right)$$

$$= \frac{1}{2}\left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}}\left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho \bar{U}^2}\right)^{-\frac{3}{2}}\left(\frac{1}{X^2}\right)$$

We don't rework further, as this form seems promising for simplifications and plasma longitudinal force distribution criteria for similarity!

11.3 Profile Solutions

11.3.1 Preliminaries

Before we embark into the exploration of different simlar flow solutions, we review some linkages between $f_{(\eta)} = g_{(\eta)}$ and key boundary layer properties:

- The boundary layer thicknesses δ_1, δ_2 and δ_3
- The shape factors H_{12} and H_{32}
- The skin friction

The two next sections would fit better in other places, from a narrative point of view, but this would lead to difficulties for the author to identify which parts of the postprocessing are general which parts are specific to the particular flow cases (Blasius, Plasma on Flat Plate, etc).

11.3.1.1 Skin Friction

By definition, the skin friction is written as:

$$\tau_w = \mu\left(\frac{\partial U}{\partial Y}\right)_{y=0} \qquad , \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2}$$

[3]Mbalax rocks! Exploring the luso-senegalese community: philip monteiro! and other gelongal pearls Abdel kader feat doug e tee !

Recalling that:

$$\frac{\partial U}{\partial Y} = \frac{U_e}{\delta}\frac{\partial f}{\partial \eta} \qquad \Rightarrow \qquad \tau_w = \mu \frac{U_e}{\delta}\left.\frac{\partial f}{\partial \eta}\right|_{\eta=0} = \mu \frac{U_e}{\delta} g^{,,}_{(\eta=0)}$$

Provides and expression to write the skin friction coefficient from the similarity profile:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_e^2} = \frac{\mu \frac{U_e}{\delta}}{\frac{1}{2}\rho U_e^2} g^{,,}_{(\eta=0)} = \frac{2}{U_e \delta}\frac{\mu}{\rho} g^{,,}_{(\eta=0)} = \frac{2\nu}{U_e \delta} g^{,,}_{(\eta=0)} \tag{11.8}$$

11.3.1.2 Boundary Layer Thicknesses

Before we present numerical results, it is useful to review how the boundary layer thicknesses can be related with the numerical results. Recalling the similarity assumption and normal coordinate scalling:

$$U = U_e f_{(\eta)} \qquad , \qquad \frac{\partial Y}{\partial \eta} = \left[\frac{\partial \eta}{\partial Y}\right]^{-1} = \delta$$

Implying that:

$$\frac{U}{U_e} = f_{(\eta)} \qquad , \qquad dY = \frac{\partial Y}{\partial \eta} d\eta$$

Given that $\delta \perp Y$, the displacement thickness is reworked into:

$$\delta_1 = \int_0^\infty \left(1 - \frac{U}{U_e}\right) dY$$

$$\delta_1 = \int_0^\infty \left(1 - f_{(\eta)}\right) \frac{\partial Y}{\partial \eta} d\eta$$

$$\frac{\delta_1}{\frac{\partial Y}{\partial \eta}} = \int_0^\infty \left(1 - f_{(\eta)}\right) d\eta$$

$$\bar{\delta}_1 \equiv \frac{\delta_1}{\delta} = \int_0^\infty \left(1 - f_{(\eta)}\right) d\eta$$

Similarty the momentum thickness is rewritten into:

$$\bar{\delta}_2 \equiv \frac{\delta_2}{\delta} = \int_0^\infty f_{(\eta)} \left(1 - f_{(\eta)}\right) d\eta$$

And the energy thickness:

$$\bar{\delta}_3 \equiv \frac{\delta_3}{\delta} = \int_0^\infty f_{(\eta)} \left(1 - f_{(\eta)}^2\right) d\eta$$

Whereby the shape factors turn into:

$$H = H_{12} = \frac{\delta_1}{\delta_2} = \frac{\frac{\delta_1}{\delta}}{\frac{\delta_2}{\delta}} \qquad , \qquad H_{32} = \frac{\delta_3}{\delta_2} = \frac{\frac{\delta_3}{\delta}}{\frac{\delta_2}{\delta}}$$

11.3.1.3 Relating Different Reynolds

As we will obtaine numerical results for $\bar{\delta}_2$:

$$\bar{\delta}_2 = \frac{\delta_2}{\delta} \qquad \Leftrightarrow \qquad \delta_2 = \bar{\delta}_2 \delta$$

It is nice to feed this into the definition of Re_θ:

$$Re_\theta = \frac{U_e \delta_2}{\nu} = \frac{U_e \delta}{\nu} \bar{\delta}_2 = \bar{\delta}_2 Re_\delta \tag{11.9}$$

To obtain a handy expression for reworking skin friction correlations.

11.3.2 Blasius: Free Shear Similar Flow on Flat Plate

11.3.2.1 Problem Formulation

The classic Blasius solution uses dimensional analysis results with the x coordinate as the lenght scale to state:

$$\delta = \frac{X}{\sqrt{Re_X}}$$

For which we also computed the X:

$$\frac{\partial \delta}{\partial X} = \frac{1}{2\sqrt{Re_X}}$$

Before we feed this into equation 11.2, we notice that this expression is only valid in the case of no plasma force, and therefore drop the plasma term to move towards the classical Blasius solution:

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \frac{U_e^2}{\delta} \frac{\partial \delta}{\partial X} = \nu \frac{U_e}{\delta^2} \frac{\partial^3 g}{\partial \eta^3}$$
$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} U_e^2 \frac{\partial \delta}{\partial X} = \nu \frac{U_e}{\delta} \frac{\partial^3 g}{\partial \eta^3}$$

Sp that we can now feed the boundary layer progression rules:

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} U_e^2 \left(\frac{1}{2\sqrt{Re_X}} \right) = \nu \frac{U_e}{\left(\frac{X}{\sqrt{Re_X}} \right)} \frac{\partial^3 g}{\partial \eta^3}$$

Which is reworked, through an error prone process[4], into:

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} = \frac{\nu}{U_e} \frac{2\sqrt{Re_X}}{\left(\frac{X}{\sqrt{Re_X}} \right)} \frac{\partial^3 g}{\partial \eta^3}$$

[4]This looks simple, but I had to do some 3 or 4 times, strangely, the first and last both had the right result even though I did not recall what the result should look like! Thank you self-cancelling errors, you are my best friends!

$$2\frac{\partial^3 g}{\partial \eta^3} + g_{(\eta)}\frac{\partial^2 g}{\partial \eta^2} = 0$$

Which is also rewritten in the classic notation of Blasius solutions as they are presented in books[5]:

$$g''' + \frac{1}{2}gg'' = 0$$

11.3.2.2 Numerical Method

We will now inspire ourselves from ref[6] to recall that any N-order ODE can be rewritten into a system of first order ODEs. Let us set:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} g \\ g' \\ g'' \end{bmatrix} \qquad \textit{and} \qquad \boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} g' \\ g'' \\ g''' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_3' \end{bmatrix}$$

Our system would then be restated as:

$$g''' = -\frac{1}{2}gg'' = -\frac{1}{2}y_1 y_3$$

So that we would write:

$$\boldsymbol{y}' = \begin{bmatrix} y_2 \\ y_3 \\ -\frac{1}{2}y_1 y_3 \end{bmatrix}$$

And attempt to solve the problem under the following boundary conditions:

- As shown in expression 11.1, the definition of g implies that unless f is extremely non-smooth:
$$g_{(\eta=0)} = 0$$
- No Slip translates in a constraint for the first derivative at the wall:
$$U|_{\eta=0} = 0 \qquad \Rightarrow \qquad f_{(\eta=0)} = g'_{(\eta=0)} = 0$$
- The fact that perturbations must vanish as we move away from the wall implies that :
$$\lim_{\eta\to\infty} f_{(\eta)} = \lim_{\eta\to\infty} g'_{(\eta)} = 1$$

[5]Several books use f to represent what we call g, and g for what we call f and h for the second derivatives!

[6]Analytical Approximations to the Solution of the Blasius Equation By J. Y. Parlange, R. D. Braddock, and G. Sander, Brisbane, Australia

It is now clear that our boundary conditions are not defined at a single point, $\eta = 0$, but at two different "points", $\eta = 0$ and $\eta \to \infty$.

We do not have a boundary condition for the second derivative at the wall: $g''_{(\eta=0)}$ is directly proportional to the skin friction, which is one of the most important unknowns we are solving for.

The ODE problem must therefore satify boundary conditions defined at two different points, meaning it is a Two Point Boundary Value Problem, as opposed to the more common Initial Value problems such as the marching of the Von Karmann Intergral Equations.

We will recast our problem into a canonical form so that we can solve it with mainstram BVP algorithms.

If our problem has N first order equations [7] the the Boundary conditions are defined at two points η_A and η_B, then we can formulate a residual function R^{BC}:

$$R^{BC} : \mathbb{R}^{NxN} \to \mathbb{R}$$

Such that:

$$R^{BC}_{(\boldsymbol{y}^A,\boldsymbol{y}^B)} \equiv 0 \qquad iff \quad BC\ are\ met$$

It is further desirable that $R^{BC} \in C^1$. So, that for this problem it becomes meaningful to set:

$$\eta_A = 0 \qquad y_1^A = g_{(\eta=0)} = 0 \qquad y_2^A = g'_{(\eta=0)} = 0$$

$$\eta_B \to \infty \qquad y_2^B = g'_{(\eta_B)} = 1$$

Which corresponds to a system of equations:

$$\begin{bmatrix} y_1^A \\ y_2^A \\ y_2^B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{bmatrix} y_1^A \\ y_2^A \\ y_2^B \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

So that is seems meaningful to write our residual function [8]:

$$R^{BC}_{(\boldsymbol{y}^A,\boldsymbol{y}^B)} = \begin{bmatrix} y_1^A \\ y_2^A \\ y_2^B \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[7] Equivalent to state that our one equation non-linear problem is N^{th} order.

[8] Notice that other residual function choises would be possible, as for example:

$$R^{BC}_{(\boldsymbol{y}^A,\boldsymbol{y}^B)} = \begin{bmatrix} (y_1^A)^2 \\ (y_2^A)^2 \\ (y_2^B - 1)^2 \end{bmatrix}$$

Which would also be a legitimate choice, and could eventually provide better convergence!

Variable	Numerical	Literature
$2g''_{(0)}$	0.66412	
$\int_0^\infty \left(1 - f_{(\eta)}\right) d\eta$	1.7208	
$\int_0^\infty f_{(\eta)} \left(1 - f_{(\eta)}\right) d\eta$	0.6641	
$\int_0^\infty f_{(\eta)} \left(1 - f^2_{(\eta)}\right) d\eta$	1.0443	
H	2.5911	
H^*	1.5726	

Table 11.1: Blasius Equation Solutions: Key Variables

11.3.2.3 Numerical Results

Numerical results were obtained from with the BVP4C Two-Point Boundary Value Problem solver, for an integration range $\eta = [0, 10]$ with 10000 points. A simple, informal, convergence assesment showed that results are well converged (changes from 5 to 10, but not from 10 to 20).

Early solutions relied on series expansions in terms of $\alpha = g''_{(0)}$ and well known numerical approaches include those of Topfer, Howarth. The series expansion approach could be a very nice approach to obtain quasi-analytical closed form solutions for different levels of plasma forcing!

11.3.2.4 Skin Friction

For the particular case of the Blasius boundary layer growth case, we can combine expression 11.8 with our boundary layer growth law, expression 11.5:

$$C_f = \frac{2\nu}{U_e \delta} g''_{(\eta=0)} \qquad , \qquad \delta = \frac{X}{\sqrt{Re_X}}$$

To yield:

$$C_f = \frac{2\nu}{U_e \delta} g''_{(\eta=0)} = \frac{2\nu}{U_e \left(\frac{X}{\sqrt{Re_X}}\right)} g''_{(\eta=0)} = 2\frac{\nu}{U_e X}\sqrt{Re_X}\, g''_{(\eta=0)}$$

$$= \frac{2}{\frac{U_e X}{\nu}} \sqrt{Re_X}\, g''_{(\eta=0)} = 2\frac{\sqrt{Re_X}}{Re_X} g''_{(\eta=0)} = \frac{2}{\sqrt{Re_X}} g''_{(\eta=0)}$$

Considering that our numerical results provide $g''_{(0)} = 0.3321$, we therefore recover the well known Blasius result with our own numerical results:

$$C_f = \frac{0.66412}{\sqrt{Re_X}}$$

It is also possible to rework this relation in order to transform the Re_X dependency into an Re_θ. Recalling expression 11.6 and feeding it into expression 11.9:

$$Re_\delta = \sqrt{Re_X}$$

$$Re_\theta = \bar{\delta}_2 \sqrt{Re_X}$$

we get:

$$\Rightarrow \qquad \sqrt{Re_X} = \frac{Re_\theta}{\bar{\delta}_2}$$

So that the skin friction expression, can be rewritten as:

$$C_f = \frac{2}{\sqrt{Re_X}} g''_{(\eta=0)} = \frac{2}{\frac{Re_\theta}{\bar{\delta}_2}} g''_{(\eta=0)} = \frac{2\bar{\delta}_2 g''_{(\eta=0)}}{Re_\theta}$$

Replacing with the numerical values, we get:

$$C_f == \frac{2\,(0.6641)\,(0.3321)}{Re_\theta} = \frac{0.4410}{Re_\theta}$$

I have seen this result before (Brederode?) but can't find the reference back to cross check coefficient validity!

11.3.3 Plasma Actuated Free Shear Similar Flow

11.3.3.1 Problem Formulation

We depart again from expression 11.2:

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \frac{U_e^2}{\delta} \frac{\partial \delta}{\partial X} = \nu \frac{U_e}{\delta^2} \frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x$$

But before we insert the flat plate boundary layer growth expressions for the plasma case:

$$\delta_{(X)} = \left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{1}{2}}$$

$$\frac{\partial \delta}{\partial X} = \frac{1}{2} \left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{3}{2}} \left(\frac{1}{X^2}\right)$$

Let us first rework some intermediary terms that will arise[9]:

$$\frac{U_e^2}{\delta} \frac{\partial \delta}{\partial X} = \frac{U_e^2}{\left(\left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{1}{2}}\right)} \left(\frac{1}{2} \left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{3}{2}} \left(\frac{1}{X^2}\right)\right)$$

$$= U_e^2 \left(\frac{\nu}{U_e}\right)^{-\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{\frac{1}{2}} \left(\frac{1}{2} \left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{3}{2}} \left(\frac{1}{X^2}\right)\right)$$

[9] Groupe Beugue Baye

Exploiting the cancelations, we get:

$$= \frac{U_e^2}{2} \underbrace{\left(\frac{\nu}{U_e}\right)^{-\frac{1}{2}} \left(\frac{\nu}{U_e}\right)^{\frac{1}{2}}}_{=1} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{3}{2}} \left(\frac{1}{X^2}\right)$$

$$= \frac{U_e^2}{2} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{3}{2}+\frac{1}{2}} \left(\frac{1}{X^2}\right)$$

$$= \frac{U_e^2}{2} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-1} \left(\frac{1}{X^2}\right)$$

Let us now rework an intermediary term from the right hand side:

$$\frac{1}{\delta^2} = \frac{1}{\left(\left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-\frac{1}{2}}\right)^2} = \frac{1}{\left(\frac{\nu}{U_e}\right)^{1} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-1}} = \left(\frac{U_e}{\nu}\right) \left(\frac{1}{X} - \frac{T_p}{\delta} \frac{\gamma_x^p}{\rho U_e^2}\right)$$

We are now ready to rework expression 11.4:

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \frac{U_e^2}{\delta} \frac{\partial \delta}{\partial X} = \nu \frac{U_e}{\delta^2} \frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \frac{U_e^2}{\delta} \frac{\partial \delta}{\partial X} = \nu U_e \frac{1}{\delta^2} \frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \left(\frac{U_e^2}{2} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^{-1} \left(\frac{1}{X^2}\right)\right) = \nu U_e \left(\left(\frac{U_e}{\nu}\right) \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)\right) \frac{\partial^3 g}{\partial \eta^3} + \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \frac{U_e^2}{2} \left(\frac{1}{X^2}\right) = U_e^2 \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \frac{\partial^3 g}{\partial \eta^3} + \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} \frac{U_e^2}{2} \left(\frac{1}{X^2}\right) = U_e^2 \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^2 \frac{\partial^3 g}{\partial \eta^3} + \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} = 2X^2 \frac{U_e^2}{U_e^2} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^2 \frac{\partial^3 g}{\partial \eta^3} + 2\frac{X^2}{U_e^2} \left(\frac{1}{X} - \frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} = 2 \left(\frac{X}{X} - X\frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^2 \frac{\partial^3 g}{\partial \eta^3} + 2\frac{X}{U_e^2} \left(\frac{X}{X} - X\frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \frac{1}{\rho} F_x$$

$$-g_{(\eta)} \frac{\partial^2 g}{\partial \eta^2} = 2 \left(1 - X\frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right)^2 \frac{\partial^3 g}{\partial \eta^3} + 2\frac{X}{U_e^2} \left(1 - X\frac{\bar{t}_p \gamma_x^p}{\rho U_e^2}\right) \frac{1}{\rho} F_x$$

We will now rewrite F_x in terms of the notation introduced in expression 11.3 and use a scaling for the plasma height:

$$F_x = \gamma_x^p w^y_{(Y,T_p)} \qquad , \qquad \bar{t}_p = \frac{T_p}{\delta}$$

So that we rewrite our equation as:

$$-g_{(\eta)}\frac{\partial^2 g}{\partial\eta^2}=2\left(1-X\frac{\bar{t}_p\gamma_x^p}{\rho U_e^2}\right)^2\frac{\partial^3 g}{\partial\eta^3}+2\frac{X}{U_e^2}\left(1-X\frac{\bar{t}_p\gamma_x^p}{\rho U_e^2}\right)\frac{1}{\rho}\gamma_x^p w^y_{(Y,T_p)}$$

Which can be reworded to put the $X\gamma_x^p$ terms in evidence:

$$-g_{(\eta)}\frac{\partial^2 g}{\partial\eta^2}=2\left(1-X\bar{t}_p\frac{\gamma_x^p}{\rho U_e^2}\right)^2\frac{\partial^3 g}{\partial\eta^3}+2X\frac{\gamma_x^p w^y_{(Y,T_p)}}{\rho U_e^2}\left(1-X\bar{t}_p\frac{\gamma_x^p}{\rho U_e^2}\right)$$

$$-g_{(\eta)}\frac{\partial^2 g}{\partial\eta^2}=2\left(1-\bar{t}_p\frac{X\gamma_x^p}{\rho U_e^2}\right)^2\frac{\partial^3 g}{\partial\eta^3}+2\frac{X\gamma_x^p}{\rho U_e^2}w^y_{(Y,T_p)}\left(1-\bar{t}_p\frac{X\gamma_x^p}{\rho U_e^2}\right)$$

So that this equation becomes independent of X, if we set:

$$F_X \quad : \quad X\gamma_x^p\perp X$$

And also obey to the constraint/rule we set earlier:

$$T_p \quad : \quad \bar{t}_p\perp X$$

We can therefore define an adimensional force equilibrium parameter [10]:

$$\lambda^{eq}_{FX}=\bar{t}_p\frac{X\gamma_x^p}{\rho U_e^2} \tag{11.10}$$

And recall an interesting property of the weighting function:

$$w^y_{(Y,T_p)}=w^y_{\left(\frac{Y}{\delta},\frac{T_p}{\delta}\right)}=w^y_{(\eta,\bar{t}_p)}$$

To inspire ourselves and define a scaled weighting function:

$$\omega_{(\eta)}=\frac{w^y_{(\eta,\bar{t}_p)}}{\bar{t}_p} \tag{11.11}$$

Notice that other, more practical equilibrium flows could be defined if only we were able to find groups such that:

$$\frac{U_e}{\nu}\frac{\partial\delta}{\partial X}\delta \quad \perp \quad U_e, X$$

$$\frac{1}{U_e^2}\frac{\delta}{\frac{\partial\delta}{\partial X}}\frac{F_x}{\rho} \quad \perp \quad U_e, X$$

Useful or not, we have identified a class of plasma driven free shear equilibrium flows, for which our profile equation is written as[11]:

$$-g_{(\eta)}\frac{\partial^2 g}{\partial\eta^2}=2\left(1-\lambda^{eq}_{FX}\right)^2\frac{\partial^3 g}{\partial\eta^3}+2\omega_{(\eta)}\lambda^{eq}_{FX}\left(1-\lambda^{eq}_{FX}\right) \tag{11.12}$$

And subject to the usual boundary conditions:

[10] Which is purely theoretical, because we cannot really make these force distributions in real life!

[11] Duggy Tee "Fulani"

- Consistency:

$$g_{(\eta=0)} = 0$$

- No-slip:

$$\left.\frac{\partial g}{\partial \eta}\right|_{\eta=0} = 0$$

- Vanishing Perturbation as $\eta \to \infty$ [12]:

$$\lim_{\eta\to\infty} \frac{\partial g}{\partial \eta} = 1$$

Again, useful or not, this document is about mental masturbation, which is a form a utility that is not meant to be captured by the welfarist school, because everyone knows neoclassical economists are sad people[13].

11.3.3.2 Numerical Solution

Following the same path as for the numerical solution of the Blasius equation, we will now recast our sweet equation, expression 11.12, into the book notation:

$$-gg'' = 2\left(1-\lambda\right)^2 g''' + 2\omega\lambda\left(1-\lambda\right)$$

And put the third derivative of g in evidence:

$$-gg'' - 2\omega\lambda\left(1-\lambda\right) = 2\left(1-\lambda\right)^2 g'''$$
$$2\left(1-\lambda\right)^2 g''' = -\left(gg'' + 2\omega\lambda\left(1-\lambda\right)\right)$$
$$g''' = -\frac{gg'' + 2\omega\lambda\left(1-\lambda\right)}{2\left(1-\lambda\right)^2}$$

To notice that this expression falls back to Blasius when $\lambda = 0$. Further defining the vectors:

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} g \\ g' \\ g'' \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} g' \\ g'' \\ g''' \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \\ y_3' \end{bmatrix}$$

The third derivative can be rewritten as:

$$y_3' = -\frac{y_1 y_3 + 2\omega\lambda\left(1-\lambda\right)}{2\left(1-\lambda\right)^2}$$

---

[12] In an matched expansions approach (méthode des expansions complémentaires) the matching condition would probably provide a slightly different boundary condition, something like $\lim_{\eta\to\infty} \frac{\partial g}{\partial \eta} = 1$, so that would be a very interesting case to explore to derive corrections, but we will leave that for a later stage!

[13] Vieux Birahim Sall - Ho Maniko

So that our sweet equation can be expressed as a set of three first order nonlinear ODEs:

$$\boldsymbol{y}' = \begin{bmatrix} y_2 \\ y_3 \\ -\frac{y_1 y_3 + 2\omega\lambda(1-\lambda)}{2(1-\lambda)^2} \end{bmatrix}$$

Finally, let us not forget that:

$$\lambda = \lambda_{FX}^{eq} \perp \eta \qquad , \qquad \omega = \omega_{(\eta)}$$

The solution seems stable and well converged, using the BVP4C solver with 1000 points over integration ranges $\eta \in [0, 10]$. Some refinement[14] occurs using $\eta \in [0, 20]$ with 2000 points, specially when computing separated profiles, for example with $\lambda = -0.6$ and $\bar{t}_p = 0.5$.

11.3.3.3 Boundary Layer Growth

For the particular case of a boundary on a flat plate with plasma but no pressure gradient, we get, we have found that boundary layer growth was represented through expression 11.7:

$$\delta_{(X)} = \left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}} \left(\frac{1}{X} - \bar{t}_p \frac{\gamma_x^p}{\rho U_e^2}\right)^{-\frac{1}{2}}$$

We recall the definition of the plasma equilibrium term of expression 11.10:

$$\lambda_{FX}^{eq} = \bar{t}_p \frac{X\gamma_x^p}{\rho U_e^2}$$

To put it in evidence in the boundary layer growth expression as:

$$\delta_{(X)} = \left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}} \left(\frac{1}{X}\left(1 - \bar{t}_p \frac{X\gamma_x^p}{\rho U_e^2}\right)\right)^{-\frac{1}{2}}$$

And inject it to obtain:

$$\delta_{(X)} == \left(\frac{\nu}{\bar{U}}\right)^{\frac{1}{2}} \left(\frac{1}{X}\left(1 - \lambda_{FX}^{eq}\right)\right)^{-\frac{1}{2}}$$

So that we can write:

$$\delta_{(X)} = \left(\frac{\nu}{U_e}\right)^{\frac{1}{2}} X^{\frac{1}{2}} \left(1 - \lambda_{FX}^{eq}\right)^{-\frac{1}{2}} \tag{11.13}$$

$$\delta_{(X)} = \frac{X}{\sqrt{Re_X}} \frac{1}{\sqrt{1 - \lambda_{FX}^{eq}}}$$

[14]Idrissa Diop - Nobel, very simplistic, but interesting in terms of graphical setup, Gelongal are definitely "comercial oriented" but display impressive production means and manage all western graphical codes! Le Sénégal c'est comme chez vous, Farima a raison!

Which is a far more elegant expression that the one we departed from! It is now also meaningful to explore the relation between Re_δ and Re_x:

$$Re_\delta = \frac{U_e \delta}{\nu} = \frac{U_e}{\nu}\frac{X}{\sqrt{Re_X}}\frac{1}{\sqrt{1-\lambda_{FX}^{eq}}}$$

After some, cumbersome rewritting in comments, :

$$Re_\delta = \frac{\sqrt{Re_X}}{\sqrt{1-\lambda_{FX}^{eq}}} \tag{11.14}$$

Which collapses to expression 4 for the Blasius case($\lambda_{FX}^{eq} = 0$), prompting good hopes that no errors were made in this path (inchallah!).

11.3.3.4 Skin Friction

Recalling expression 11.8, which was deduced to process the Blasius results and was the last one before generality was lost, the skin friction comes as:

$$C_f = \frac{2\nu}{U_e \delta} g''_{(\eta=0)}$$

Which is reworked into:

$$C_f = \frac{2}{\frac{U_e \delta}{\nu}} g''_{(\eta=0)} = \frac{2}{Re_\delta} g''_{(\eta=0)}$$

So that we can insert expression 11.14 into this definition to obtain a compact expression for the skin friction:

$$C_f = \frac{2}{\left(\frac{\sqrt{Re_X}}{\sqrt{1-\lambda_{FX}^{eq}}}\right)} g''_{(\eta=0)} = 2\frac{\sqrt{1-\lambda_{FX}^{eq}}}{\sqrt{Re_X}} g''_{(\eta=0)}$$

Which also falls back to the Blasius case expression when there is no plasma force, and exhibits a nice form! Finally, because $\bar{\delta}_2$ is also a numerical result, we recall expressions 11.9 and 11.14:

$$Re_\theta = \bar{\delta}_2 Re_\delta \qquad , \qquad Re_\delta = \frac{\sqrt{Re_X}}{\sqrt{1-\lambda_{FX}^{eq}}}$$

And combine them to obtain:

$$Re_\theta = \bar{\delta}_2 \frac{\sqrt{Re_X}}{\sqrt{1-\lambda_{FX}^{eq}}} \tag{11.15}$$

$$\frac{\bar{\delta}_2}{Re_\theta} = \frac{\sqrt{1-\lambda_{FX}^{eq}}}{\sqrt{Re_X}}$$

Which is inserted into the skin friction expression to obtain:

$$C_f = \frac{2\bar{\delta}_2 g''_{(\eta=0)}}{Re_\theta}$$

Which is identical as the expression obtained for the Blasius case, thanks to cancelations! (does this cancellation always occur?), but now $g''_{(\eta=0)}$ is not a constant anymore, but a function of $\bar{t}_p$ and λ^{eq}_{FX}.

11.3.3.5 Recasting the inputs

We have written the terms describing the plasma under the assumption that we would be able to make any plasma strenght and thickness distribution that would be needed to obtain an equilibrium boundary layer (for laminar flow!), with constant $\bar{t}_p$ and λ^{eq}_{FX}.

However, now we would like to compute these parameters for any plasma actuator on any flat plate boundary layer, and without caring about the location on the plate. That, is we want to be able to calculate $\bar{t}_p$ and λ^{eq}_{FX} without using the X variable. Ideally, we would have the following inputs:

$$\left\{\frac{U_e}{\nu}\ ,\ \delta_2\ ,\ T_p\ ,\ \gamma^p_x\right\}$$

To obtain:

$$\{\bar{t}_p\ ,\ \lambda^{eq}_{FX}\}$$

We will attempt to inspire ourselves on this path by observing the expressions for these parameters:

$$\frac{\lambda^{eq}_{FX}}{\bar{t}_p} = \frac{X\gamma^p_x}{\rho U_e^2} \qquad , \qquad \bar{t}_p = \frac{T_p}{\delta}$$

maybe it is nicer to rework the plasma equilibrium parameter λ^{eq}_{FX} into this:

$$\lambda^{eq}_{FX} = \frac{T_p}{\delta}\frac{X\gamma^p_x}{\rho U_e^2} = \left(\frac{X}{\delta}\right)\frac{T_p\gamma^p_x}{\rho U_e^2} \tag{11.16}$$

and attempt to find an expression for $\frac{X}{\delta}$ and re-express it in term of Re_θ. We recall expressions 11.5:

$$\delta = \frac{X}{\sqrt{Re_X}}\frac{1}{\sqrt{1-\lambda^{eq}_{FX}}}$$

Rearrange it into:

$$\frac{\delta}{X} = \frac{1}{\sqrt{Re_X}}\frac{1}{\sqrt{1-\lambda^{eq}_{FX}}} \tag{11.17}$$

And use expression 11.15:

$$Re_\theta = \bar{\delta}_2 \frac{\sqrt{Re_X}}{\sqrt{1-\lambda_{FX}^{eq}}}$$

To rewrite $\sqrt{Re_x}$ as:

$$\sqrt{Re_X} = \frac{Re_\theta}{\bar{\delta}_2}\sqrt{1-\lambda_{FX}^{eq}}$$

So that we combine it with expression11.17 to yield:

$$\frac{\delta}{X} = \frac{1}{\sqrt{Re_X}}\frac{1}{\sqrt{1-\lambda_{FX}^{eq}}} = \frac{1}{\left(\frac{Re_\theta}{\bar{\delta}_2}\sqrt{1-\lambda_{FX}^{eq}}\right)}\frac{1}{\sqrt{1-\lambda_{FX}^{eq}}} = \frac{\bar{\delta}_2}{(Re_\theta)}\frac{1}{\sqrt{1-\lambda_{FX}^{eq}}\sqrt{1-\lambda_{FX}^{eq}}}$$

$$\frac{\delta}{X} = \frac{\bar{\delta}_2}{Re_\theta}\frac{1}{1-\lambda_{FX}^{eq}}$$

$$\frac{X}{\delta} = \frac{Re_\theta}{\bar{\delta}_2}\left(1-\lambda_{FX}^{eq}\right)$$

Which can be inserted into expression 11.17, to yield an equation that looses all its dependence in X:

$$\lambda_{FX}^{eq} = \left(\frac{X}{\delta}\right)\frac{T_p\gamma_x^p}{\rho U_e^2} \qquad \Leftrightarrow \lambda_{FX}^{eq} = \left(\frac{Re_\theta}{\bar{\delta}_2}\left(1-\lambda_{FX}^{eq}\right)\right)\frac{T_p\gamma_x^p}{\rho U_e^2}$$

Attempting to solve, we pursue our manipulation with a shorthand$\lambda = \lambda_{FX}^{eq}$:

$$\lambda_{FX}^{eq} = \left(1-\lambda_{FX}^{eq}\right)\frac{Re_\theta}{\bar{\delta}_2}\frac{T_p\gamma_x^p}{\rho U_e^2}$$

$$\lambda_{FX}^{eq} = \frac{\frac{T_p\gamma_x^p}{\rho U_e^2}}{\frac{\bar{\delta}_2}{Re_\theta}+\frac{T_p\gamma_x^p}{\rho U_e^2}} \tag{11.18}$$

It is important to recall that $\bar{\delta}_2$ is a function of λ and t_p – implying that expression 11.18 is not explicit, and that λ will have to be determined iteratively, because we do not have an analytical expression for $\bar{\delta}_2(\lambda, t_p)$. Furthermore, even when we fit the data with an analytical expression, it is unlikely that the explicit solution of equation 11.18 will be tractable.

To rework the scaled plasma thickness we write:

$$\bar{\delta}_2 = \frac{\delta_2}{\delta} \qquad \Leftrightarrow \qquad \delta = \frac{\delta_2}{\bar{\delta}_2}$$

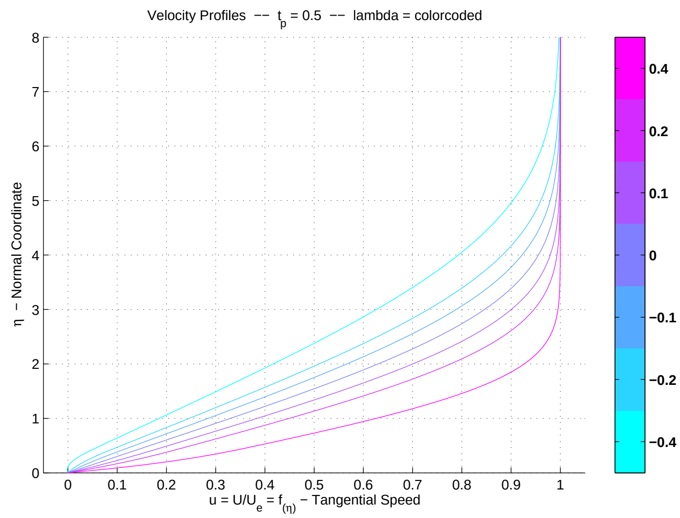
So that we can write:

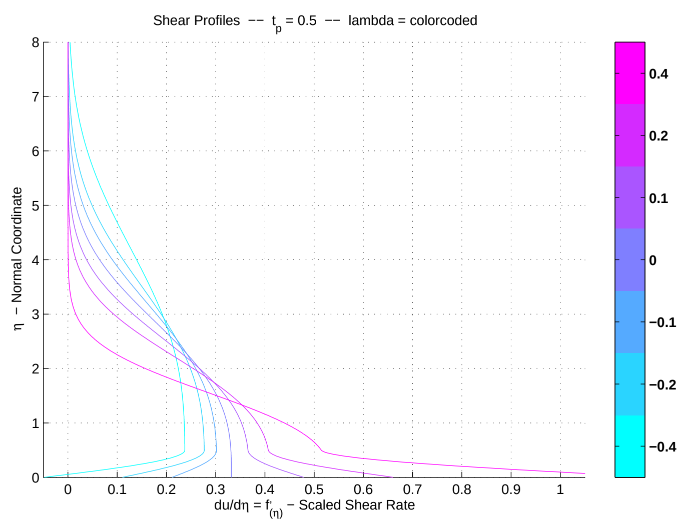
$$\bar{t}_p = \frac{T_p}{\delta} = \frac{T_p}{\left(\frac{\delta_2}{\bar{\delta}_2}\right)} = \bar{\delta}_2\frac{T_p}{\delta_2} = \bar{\delta}_2\frac{Re_{T_p}}{Re_\theta}$$

The solution procedure will be highlighted soon, a simple iterative matching loop with a neat residual formulation should suffice. Other options would be to insert the parameter in the BVP problem loop (i think there is an option doing something similar to that at the BL level...)

11.3.3.6 Numerical Results

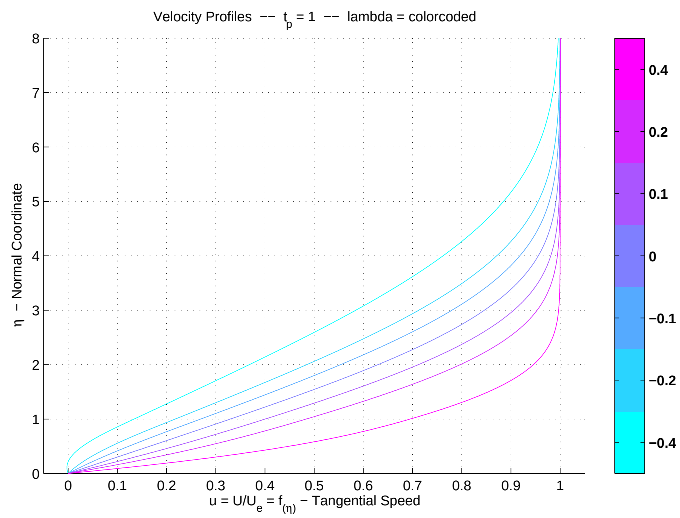
In this section, we present some numerical results for various combinations of λ and $\bar{t}_p$:

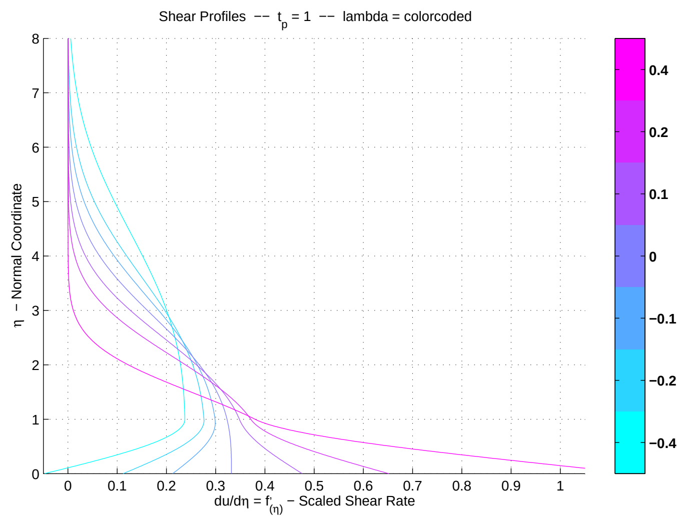




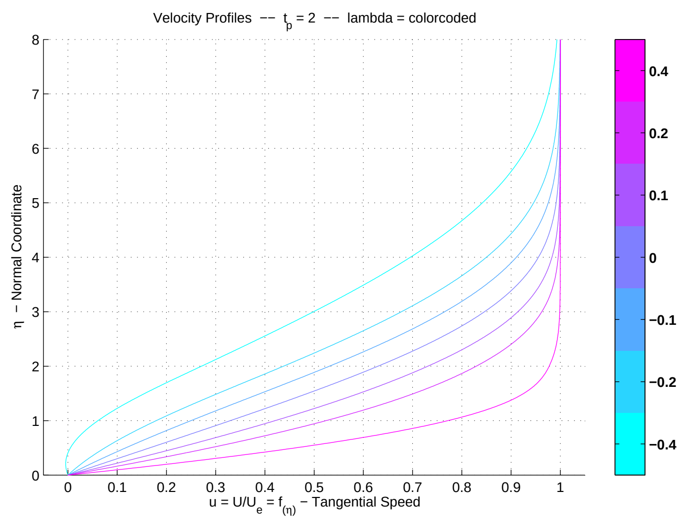
lambda	t_p	2*f_prime	dstr	theta	h12	h_str
-0.4000	+0.5000	-0.0967	+2.6256	+0.9305	+2.8216	+1.5720
-0.2000	+0.5000	+0.2212	+2.1695	+0.7985	+2.7169	+1.5712
-0.1000	+0.5000	+0.4229	+1.9440	+0.7317	+2.6570	+1.5715
+0.0000	+0.5000	+0.6641	+1.7208	+0.6641	+2.5911	+1.5726
+0.1000	+0.5000	+0.9576	+1.5003	+0.5957	+2.5186	+1.5748
+0.2000	+0.5000	+1.3221	+1.2832	+0.5261	+2.4388	+1.5786

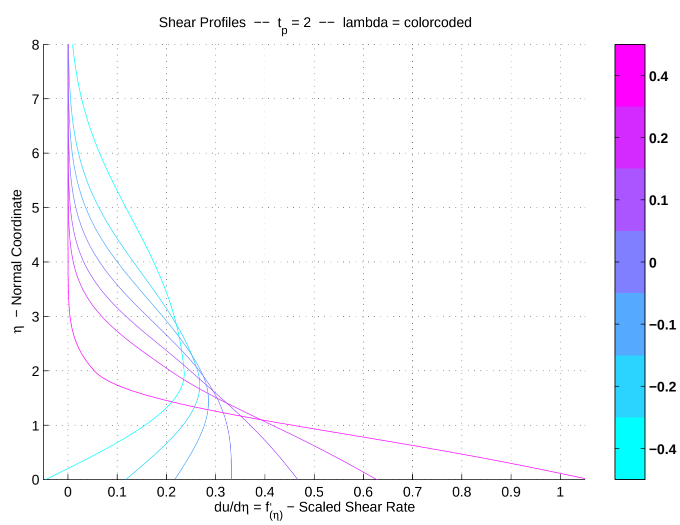
+0.4000	+0.5000	+2.3959	+0.8633	+0.3825	+2.2568	+1.5946





lambda	t_p	2*f_prime	dstr	theta	h12	h_str
-0.4000	+1.0000	-0.0955	+2.8364	+0.9327	+3.0410	+1.5704
-0.2000	+1.0000	+0.2242	+2.2635	+0.8029	+2.8192	+1.5677
-0.1000	+1.0000	+0.4254	+1.9880	+0.7348	+2.7055	+1.5690
+0.0000	+1.0000	+0.6641	+1.7208	+0.6641	+2.5911	+1.5726
+0.1000	+1.0000	+0.9514	+1.4632	+0.5906	+2.4773	+1.5791
+0.2000	+1.0000	+1.3032	+1.2165	+0.5140	+2.3666	+1.5891
+0.4000	+1.0000	+2.3064	+0.7632	+0.3503	+2.1788	+1.6203





lambda	t_p	2*f_prime	dstr	theta	h12	h_str
-0.4000	+2.0000	-0.0911	+3.2366	+0.9414	+3.4381	+1.5648
-0.2000	+2.0000	+0.2350	+2.4183	+0.8184	+2.9550	+1.5578
-0.1000	+2.0000	+0.4339	+2.0552	+0.7450	+2.7587	+1.5628
+0.0000	+2.0000	+0.6641	+1.7208	+0.6641	+2.5911	+1.5726
+0.1000	+2.0000	+0.9338	+1.4145	+0.5764	+2.4541	+1.5862
+0.2000	+2.0000	+1.2541	+1.1360	+0.4826	+2.3538	+1.6013
+0.4000	+2.0000	+2.1199	+0.6630	+0.2832	+2.3415	+1.6091

Appendix A

Notation

A.1 Scales and Reference Frames

Throughout the text, we have defined many reference frames and scales:

Variable	Definition	Description	Units
L	$-$	Longitudinal Lenght Scale	m
U_e	$-$	Edge Velocity	m/s
X	$-$	Dimensional Longitudinal Coordinate	m
x	$\frac{X}{L}$	Adimensional Longitudinal Coordinate	$-$
Y	$-$	Dimensional Normal Coordinate	m
y	$\frac{Y}{\delta} \simeq \frac{Y}{L}\sqrt{Re}$	Adimensional Normal Coordinate	$-$
η	$\frac{Y}{\delta_2} = \frac{\left(\frac{Y}{L}\right)}{\left(\frac{\delta_2}{L}\right)} = \frac{Y}{\theta L}$	Momentum Scaled Normal Coordinate	$-$
U	$-$	Dimensional Longitudinal Speed	m/s
u	$\frac{U}{U_e}$	Adimensional Longitudinal Speed	$-$
V	$-$	Dimensional Normal Speed	m/s
v	$\frac{V}{U_e}\frac{L}{\delta} \simeq \frac{V}{U_e}\sqrt{Re}$	Adimensional Normal Speed	$-$
P	$-$	Dimensional Pressure	N/m^2
p	$\frac{P}{\rho U_e^2}$	Adimensional Pressure	$-$
T	$-$	Time Coordinate	s
t	$\frac{U_e}{L}T$	Adimensional Time Coordinate	$-$

A.2 Boundary layer groups

The scales were used to define both dimensional and adimensional groups to describe the boundary layer. These are pretty usual, but it is useful to recall definitions and state our conventions:

Variable	Definition	Description	Units
δ	$Y : U = 0.995U_e$	Dimensional Boundary Layer Thickness	$[m]$
$\bar{\delta}$	$\frac{\delta}{L}$	Adimensional Boundary Layer Thickness	–
δ_1	$\int_0^\infty \left(1 - \frac{U}{U_e}\right) dY$	Dimensional Displacement Thickness	$[m]$
δ^*	$\frac{\delta_1}{L}$	Adimensional Displacement Thickness	–
δ_2	$\int_0^\infty \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dY$	Dimensional Momentum Thickness	$[m]$
θ	$\frac{\delta_2}{L}$	Adimensional Momentum Thickness	–
Re_θ	$\frac{U_e\delta_2}{\nu_e} = \frac{U_e L}{\nu_e}\theta$	Momentum Thickness Reynolds	–
δ_3	$\int_0^\infty \frac{U}{U_e}\left(1 - \frac{U^2}{U_e^2}\right) dY$	Dimensional Energy Thickness	$[m]$
θ^*	$\frac{\delta_3}{L}$	Adimensional Energy Thickness	–
τ_w	$\mu\left(\frac{\partial U}{\partial Y}\right)_{y=0}$	Skin Friction	
C_f	$\frac{\tau_w}{\frac{1}{2}\rho U_e^2}$	Skin Friction Coefficient	–
D	$\int_0^\infty \mu\left(\frac{\partial U}{\partial Y}\right)^2 dY$	Dissipation Rate	
C_D	$\frac{D}{\rho U_e^3}$	Dissipation Rate Coefficient	–
$H = H_{12}$	$\frac{\delta_1}{\delta_2} = \frac{\delta^*}{\theta}$	Shape Factor	–
$H^* = H_{32}$	$\frac{\delta_3}{\delta_2} = \frac{\theta^*}{\theta}$	Energy Shape Factor	–
H_1	$\frac{\delta - \delta^*}{\theta}$	Head's Shape Factor	–
U_∞	–	Unperturbed Free Stream Velocity	$[m/s]$
U_e	–	Edge Velocity	$[m/s]$
u_e	$\frac{U_e}{U_\infty}$	Adimensionalized Edge Velocity	–
V_0	$V\|_{Y=0}$	Suction Velocity	$[m/s]$
v_0	$\left.\frac{V}{U_\infty}\right\|_{Y=0}$	Adimensional suction velocity	–

A.3 Plasma Specific Variables

We complemented classical work with plasma and force term specific variables:

Variable	Definition	Description	Units
F_T^p	–	Actuator Force per unit span	N/m
L^p	–	Dimensional Force Field Lenght	m
l^p	$\frac{L^p}{L}$	Adimensinal Force Field Lenght	–
X_0^p	–	Dimensional Force Field Start	m
x_0^p	$\frac{X_0^p}{L}$	Adimensional Force Field Start	m
T^p	–	Dimensional Force Field Thickness	m
t^p	$\frac{T^p}{L}$	Adimensional Force Field Thickness	–
$\bar{t}^p$	$\frac{t^p}{\delta} = \frac{T^p}{\delta}$	Scaled Force Field Thickness	–
t_θ^p	$\frac{T^p}{\delta_2} = \frac{t^p}{\theta}$	Momentum-Scaled Force Field Thickness	–
F_x	–	Dimensional Force Field Density	$N/\left(m^2 m\right)$
ϕ_x^p	$\frac{F_T^p}{L^p T^p}$	Average dimensional force field density	$N/\left(m^2 m\right)$
C_{FM}	$\int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2}\right) dY$	Force Momentum Coefficient	–
C_{FE}	$\int_0^\infty \left(\frac{F_x}{\frac{1}{2}\rho U_\infty^2} u\right) dY$	Force Energy Coefficient	–
$C_{\phi_x^p}$	$\frac{\phi_x^p T^p}{\frac{1}{2}\rho U_\infty^2}$	Average Force Momentum Coefficient	–
C_{EI}	$\int_0^{\bar{t}^p} \left(\frac{w^y_{(y,\bar{t}^p)} u_{(y)}}{\bar{t}^p}\right) dy$	Energy Interaction Coefficient	–
C_{FT}	$\frac{F_T^p}{\frac{1}{2}\rho U_\infty^2 L}$	Total Plasma Force Coefficient	–

Finally, for the :

Bibliography

[1] C. C. Mei, Notes on 1.63 Advanced Environmental Fluid Mechanics, MIT, 2001

[2] Swafford, T.W., Analytical Approximation of Two-Dimensional Sperated Turbulent Boundary-Layer Velocity Profiles, AIAA Journal, Technical Notes, Vol.21, No.6, June 1983

[3] Vasco de Brederode, Fundamentos de Aerodinamica Incompressivel, Edicao do Autor,