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Theory and Methodology

# Transformation of multidepot multisalesmen problem to the standard travelling salesman problem

Yang GuoXing

*Department of Information Engineering, Beijing Metallurgical Management Cadres Institute, Beijing 100024, China*

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**Abstract**

One type of multidepot multisalesmen problem can be transformed into a standard travelling salesman problem by copying depots and introducing imaginary depots.

*Keywords:* Travelling salesman; Multidepot multisalesmen problem; Imaginary depot

## 1. Introduction

A truck company has three depots in the northeast of China. There are $m1$, $m2$, and $m3$ trucks in depot 1, 2, and 3 respectively. They are often sent from the three depots to accomplish a special task, and then return to the three depots in a limited time. Every depot retains the original number of trucks.

The task contains $n$ subtasks. The subtask $i$ ($i = 1, 2, \dots, n$) is to take freight from $A_i$ to $B_i$. The distance from $A_i$ to $B_i$ is $L_i$. The distance $c(i, j)$ from task $i$ to task $j$ is the distance from $B_i$ to $A_j$, and the distance $c(j, i)$ from task $j$ to task $i$ is the distance from $B_j$ to $A_i$ as shown in Fig. 1.

We define $L_i$ and $L_j$ as loaded distances, $c(i, j)$ and $c(j, i)$ as unloaded distances. Because the loaded distances cannot be optimized, the problem is to find a set of routes of minimum total unloaded distance for the trucks to finish the task and which do not violate the above limitations. As the time limitation is too tight, all the trucks should be used.

For the sake of convenience, the subtask $i$ can be regarded as one point $i$ while the distance $L_i$ from $A_i$ to $B_i$ as the weight of point $i$. This is shown in Fig. 2. If the time constraint were ignored, the problem would be a multidepot multisalesmen problem.

There are a lot of effective algorithms for the standard TSP. For the single depot multisalesmen problem, Laport and Nobert [5] present a cutting planes method. Ali and Kennington [2] proved a duality based branch-and-bound algorithm. Gavish and Srikanth [3] provide a branch-and-bound algorithm which uses a different relax-

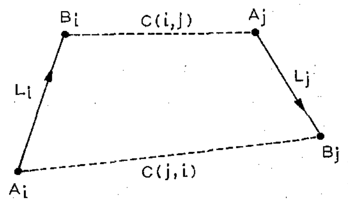


Fig. 1. The subtasks.

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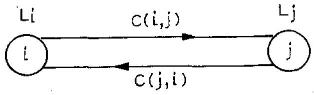


Fig. 2. The equivalent problem of Fig. 1.

ation. Bellmore and Hong [2] present a transformation to the standard travelling salesman problem.

There are two variants of the multidepot multisalesmen problem. One is that every salesman must return to its original depot. Laporte and Nobert [6] proved an algorithm for this problem. They transformed this problem into an assignment problem and then solved the transformed model. But the transformation is not complete because the transformed model contains nonassignment constraints. Another type is that the salesmen do not have to return to its original depot. Yang [7] presents a branch-and-bound algorithm based on a multiple $m$-tree and no transformations to a standard travelling salesman problem are known. This paper will transform this problem into the standard travelling salesman problem, so that any algorithm for the standard TSP can be used to solve the transformed problem.

## 2. Transformation

For the sake of convenience, the problem can be described as follows: Given $k$ depot cities (node $1, 2, \ldots, k$), $n-k$ customer cities (node $k+1, k+2, \ldots, n$), and $m(i)$ salesmen in depot $i$, find for the salesmen a set of routes of minimum total distance, so that each customer city is visited exactly once, while all salesmen start from and end at the depots, and the number of salesmen in any depots do not change after the travel. The distance from $i$ to $j$ is $c(i, j)$ $(i, j = 1, 2, \ldots, n)$.

In a graph concept, the multidepot multisalesmen problem can be stated as follows: A graph $G(N, A)$ consists of a set $N$ of nodes indexed by $1, 2, \ldots, n$ together with a set $A$ of arcs $(i, j)$ for $i, j \in N$, $i \neq j$. An arc $(i, j)$ is said to be directed from node $i$ to node $j$. Associated with each arc $(i, j)$ is a real number $c(i, j)$ representing a distance measure of the arc. There are $m(i)$ insert arcs and $m(i)$ outward arcs incident with node $i$ $(i = 1, 2, \ldots, k)$, one insert arc and one outward arc incident with node $k+1, k+2, \ldots, n$, and every route starts from and ends at nodes $1, 2, \ldots, k$. Minimize the total distance of the routes.

We transform this problem to the standard travelling salesman problem in an expanded graph $G'(N', A')$. The set $N'$ of nodes in $G'(N', A')$ is obtained by adding $k$ imaginary nodes, which are labeled by $1', 2', \ldots, k'$, and $m(i)-1$ copies of node $i$, labeled by $i_2, i_3, \ldots, i_{m(i)}$ for $i = 1, 2, \ldots, k$. The set $A'$ of arcs in $G'(N', A')$ contains (1) every arc in $A$, except the arcs between nodes $1, 2, \ldots, k$; (2) the arcs between copy nodes and customer nodes; (3) the arcs from customer nodes to imaginary nodes; and (4) the arcs from an imaginary node to the next imaginary node and next depot node. The measured distance $d(i, j)$ of each arc in $A'$ is given by:

(1) $d(i, j) = c(i, j)$,
for $i, j = k+1, \ldots, n$.

(2) $d(i, j) = c(i, j)$ and $d(j, i) = c(j, i)$,
for $i = 1, 2, \ldots, k$; $j = k+1, \ldots, n$.
$d(i_r, j) = c(i, j)$ and $d(j, i_r) = c(j, i)$,
for $i = 1, 2, \ldots, k$; $j = k+1, \ldots, n$;
$r = 2, 3, \ldots, m(i)$.

(3) $d(j, i') = c(j, i)$,
for $i = 1, 2, \ldots, k$; $j = k+1, \ldots, n$.

(4) $d(i', (i+1)') = 0$,
for $i = 1, 2, \ldots, k-1$.
$d(i', i+1) = 0$,
for $i = 1, 2, \ldots, k-1$.
$d(k', 1) = 0$.

The arcs from imaginary nodes with zero distance are called imaginary arcs. The imaginary arcs of a three depot problem are illustrated in Fig. 3.

**Theorem 1.** *For every solution of the multidepot multisalesmen problem on $G(N, A)$, a corresponding tour on $G'(N', A')$ with the same total distance can be constructed.*

**Proof.** We prove the theorem with mathematics induction.

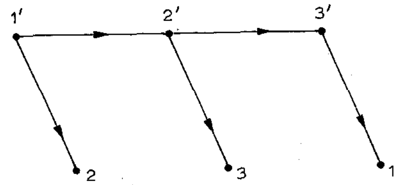


Fig. 3. The arcs from the imaginary nodes.

(1) If $k = 2$, the conclusion is proved as follows: If no salesman starts from depot 1 and returns to depot 2, there must be $m(1)$ cycles joined with depot 1 and $m(2)$ cycles joined with depot 2. Regard depot 1 as $m(1)+1$ nodes numbered 1, $1_2, 1_3, \ldots, 1_{m(1)}$, and 1′, then $m(1)$ piece of the route be cut by $1_2, 1_3, \ldots, 1_{m(1)}$ between 1 and 1′ corresponding to the $m(1)$ cycles. Regard depot 2 as $m(2)+1$ nodes numbered 2, $2_2, 2_3, \ldots, 2_{m(2)}$, and 2′ then $m(2)$ piece of the route be cut by $2_2, 2_3, \ldots, 2_{m(2)}$ between 2 and 2′ corresponding to the $m(2)$ cycles. Link 1′ to 2 and 2′ to 1. The tour on $G'(N', A')$ is constructed. As $c(j, 1') = c(j, 1)$, $c(j, 2') = c(j, 2)$, for $j = k+1, \ldots, n$, and $c(1', 2) = c(2', 1) = 0$, the total distance of the tour on $G'(N', A')$ is equivalent to that of its corresponding $\Sigma m(i)$ tours on $G(N, A)$.

If some salesmen start from depot 1 and return to depot 2, the Euler circle exists. As there are $m(1)$ insert arcs and $m(1)$ outward arcs incident to depot 1, depot 1 must appear $m(1)$ times in the Euler circle named 1, $1_2, 1_3, \ldots,$ and $1_{m(1)}$. Depot 2 must appear $m(2)$ times named $2, 2_2, 2_3, \ldots,$ and $2_{m(2)}$. Link 1′ to 2′ and add them before 1. The tour on $G'(N', A')$ is constructed. As $c(j, 1') = c(j, 1)$, for $j = k+1, \ldots, n$, and $c(1', 2') = c(2', 1) = 0$, the total distance of the tour on $G'(N', A')$ is equivalent to that of its corresponding $\Sigma m(i)$ tours on $G(N, A)$.

(2) If $k = n - 1$, the conclusion is right. Then the correctness for $k = n$ can be proved as follows:

If $k = n - 1$, the conclusion is right, the tour on $G'(N', A')$ with $n - 1$ depots has been found and the node before node 1 must be $(n-1)'$ (see Fig. 3).

If no salesman starts from depot $n$ and returns to the others, there must be $m(n)$ circles joined with depot $n$. Regard depot $n$ as $m(n)+1$ nodes numbered $n, n_2, n_3, \ldots, n_{m(n)}$ and $n'$. Then $m(n)$ piece of route is cut by $n_2, n_3, \ldots, n_{m(n)}$ between $n$ and $n'$ corresponding to the $m(n)$ circles. Break the link of $(n-1)'$ to 1 and join $(n-1)'$ to $n$, $n'$ to 1. The tour on $G'(N', A')$ is constructed. As $c((n-1)', 1) = c((n-1)', n) = c(n', 1) = 0$, the total distance of the tour on $G'(A', N')$ is equivalent to that of its corresponding $\Sigma m(i)$ tours on $G(N, A)$.

If some salesmen start from depot $n$ and return to the others, the Euler circle exists. Depot $n$ must appear $m(n)$ times named $n, n_2, n_3, \ldots, m_{m(n)}$. Add $n'$ between $(n-1)'$ and 1. The tour on $G'(N', A')$ is constructed. As $c((n-1)', n') = c(n', 1) = 0$, the total distance of the tour on $G'(A', N')$ is equivalent to that of its corresponding $\Sigma m(i)$ tours on $G(N, A)$.

**Theorem 2.** *For every tour on $G'(N', A')$, there is a set of corresponding tours on $G(N, A)$ with the same total distance.*

**Proof.** Remove the imaginary arcs with 0 distance and collect the nodes $i, i_2, \ldots, i_{m(i)}$ and $i'$ to one node $i$, for $i = 1, 2, \ldots, k$. The corresponding set of tours is constructed.

**Theorem 3.** *If a tour on $G'(N', A')$ is a minimum cost tour, then the corresponding tours on $G(N, A)$ also form a minimum cost set of tours.*

This theorem is obvious from Theorem 1 and Theorem 2.

As an example, $n = 8$, $k = 2$, $m(1) = 2$, $m(2) = 2$ is illustrated in Fig. 4. We will show that a single tour on $G'(N', A')$ is equivalent to $\Sigma m(i)$ routes in $G(N, A)$. Fig. 4a shows that every salesman returns to its original depot. It presents that one salesman from depot 1 visits city 3 and returns to depot 1, another salesman from depot 1 visits city 7, city 6 and returns to depot 1. One salesman from depot 2 visits city 4, city 5 and returns to depot 2. Another salesman from depot 2 visits city 8 and returns to depot 2. As 1′ and 2′ are imaginary depots and $d(1', 2) = d(2', 1) = 0$, they do not influence the total distance, but they link depot 1 and depot 2 to a single tour, so the

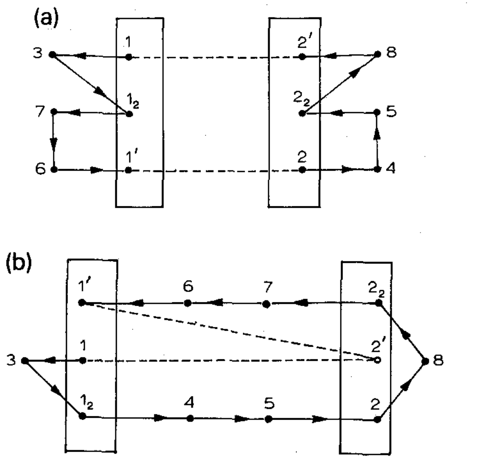


Fig. 4. The tours on the transformed graph.

imaginary depots must be included in this situation.

Another situation is shown in Fig. 4b. In this figure, one salesman starts from depot 1 visits city 3 and returns to depot 1. Another salesman from depot 1 visits city 4, city 5 and returns to depot 2. One salesman starts from depot 2 visits city 8 and returns to depot 2. Another salesman from depot 2 visits city 7, city 6 and returns to depot 1.

## 3. Improving the transformation

For the transformed TSP, the arcs that are incident to the copies of one depot have the same cost. In order to suppress the degeneracy of the TSP, some arcs can be removed from $G'(N', A')$ (Jonker and Volgenant [4]). For depot $i$ ($i = 1, 2, \dots, k$), we select $m(i) - 1$ arbitrary nondepot nodes named $s_2, s_3, \dots, s_{m(i)}$. We can prove that the arcs $(s_j, i_l)$ can be removed for $j = 2, 3, \dots, m(i)$ and $l = 2, 3, \dots, m(i)$, $l \neq j$.

**Theorem 4.** *For the transformed TSP, there is an optimal solution not containing the arcs* $(s_j, i_l)$, *for* $i = 1, 2, \dots, k$; $j = 2, 3, \dots, m(i)$, *and* $l = 2, 3, \dots, m(i)$, $l \neq j$.

**Proof.** Suppose there is an arc $(s_j, i_l)$, for $l \neq j$, in the optimal solution. As $i_l$ and $i_j$ are copied from the same depot $i$, we can exchange copy depots $i_l$ and $i_j$, not influencing the distance. Then the new arc is $(s_j, i_j)$.

Any algorithm of the TSP can be used to solve the transformed TSP. In order to solve the constrained problem, give a special mark to the depots, copy depots and imaginary depots. In the solving process, guarantee that the total weight of the nodes between any two marked nodes does not exceed the salesman's capacity. The distance between any two special marked nodes does not violate the distance limitation of the salesman.

## 4. Conclusion

By copying depots and introducing imaginary depots, one type of the multidepot multisalesmen problem can be transformed into the standard asymmetric travelling salesman problem with $\Sigma m(i)$ more cities. Any algorithm of the TSP can be used to solve the transformed TSP.

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