Fuzzy Shortest-Path Network Problems With Uncertain Edge Weights

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This paper presents two new types of fuzzy shortest-path network problems. We consider the edge weight of the network as uncertain, which means that it is either imprecise or unknown. Thus, the first type of fuzzy shortest-path problem uses triangular fuzzy numbers for the imprecise problem. The second type uses level $(1-\beta, 1-\alpha)$ interval-valued fuzzy numbers, which are based on past statistical data corresponding to the confidence intervals of the edge weights for the unknown problem. The main results obtained from this study are: (1) using triangular fuzzy numbers and a signed distance ranking method to obtain Theorem 1, and (2) using level $(1-\beta, 1-\alpha)$ interval-valued fuzzy numbers, combining statistics with fuzzy sets and a signed distance ranking method to obtain Theorem 2. We conclude that the shortest paths in the fuzzy sense obtained from Theorems 1 and 2 correspond to the actual paths in the network, and the fuzzy shortest-path problem is an extension of the crisp case.

Keywords: triangular fuzzy number, level $(1-\beta, 1-\alpha)$ interval-valued fuzzy number, shortest-path network problem, fuzzy shortest-path network problem, signed-distance ranking

1. INTRODUCTION

One of the fundamental problems in network theory is finding shortest paths in a network. Over the past several years this problem has often been posed as a subset of other optimization problems [6]. Consider the following scenario. A motorist wishes to find the shortest possible route from Chicago to Boston. Given a road map of the United States on which the distance between each pair of successive intersections is marked, how can we determine the shortest route [3]? Obviously, the shortest-path problem requires determining the shortest route between a source and a destination in a transportation network. In some applications, the numbers associated with the edges of networks may represent characteristics other than lengths, and we may want the optimum paths, where optimum can be defined by different criteria. Nevertheless, the shortest-path problem is the most common problem in the whole class of optimum path problems.

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Shortest-path algorithms can usually be modified slightly to find other optimum paths, for example, in computer networks.

The analysis of fuzzy counterparts of the shortest-path problem appears to have become a popular task in recent years [8]. The main advantage, compared to the nonfuzzy problem formulation, is that the decision-maker (DM) is not forced into a precise formulation. Indeed, there is much vaguely formulated information or imprecisely quantified physical data in real world descriptions. Therefore, fuzziness can be introduced into a network in a variety of ways, e.g., through edge capacities, edge weights, vertex restrictions, or arc lengths [1, 2, 10-12]. Dubois and Prade [4] first introduced the fuzzy shortest-path problem in 1980. The major drawback of this fuzzy shortest-path problem is the lack of interpretation. That is, a fuzzy shortest path is found but it may not correspond to an actual path in the network. This problem is circumvented with new models based on fuzzy shortest paths and multiple objectives presented by Klein [8]. He developed a hybrid multi-criteria algorithm based on fuzzy dynamic programming (DP), i.e., $f(N) = (1, 1, 1, \ldots, 1)$, $f(i) = \underset{(i,j)\in E}{dom}(e_{ij} \tilde{+} f(j))$, to solve these models. Klein analyzed the fuzzy shortest-path algorithms in terms of general fuzzy mathematical programming. Nevertheless, the proposed approach did not seem to be an extension of the crisp counterpart. Mares and Horak [11] proposed that the uncertainty connected with the input data of a network can be described and investigated by means of fuzzy sets and fuzzy quantities theory. They showed that after summing fuzzy quantities, it is possible to derive the uncertainty connected with more complex paths and reserves. Chanas and Kolodziejczyk [2] analyzed the maximum flow in a network in which an excess over the beforehand fixed quota of arc capacity is admissible. This problem is represented as a partially fuzzy linear programming task. They then presented an algorithm for searching maximum flow assuming integer values of flows on network arcs. Okada and Soper [12] introduced an order relation between fuzzy number based on "fuzzy min" concept and that a nondominated path or Pareto Optimal path from the specified node to every other node is defined. They then proposed an algorithm for solving fuzzy shortest path problems based on the multiple-labeling method for a multicriteria shortest path problem.

In this study, we investigated two new types of fuzzy shortest-path problems, which are different from previous methods mentioned in the literature [1, 2, 4, 8, 10-12]. In our work, we considered each edge weight of the network as uncertain, which means that it is either imprecise or unknown. Consequently, we first use triangular fuzzy numbers for the imprecise problem, and then use a signed distance ranking method to defuzzify the fuzzy numbers to obtain the first type of shortest-path problem in the fuzzy sense. Second, we use level $(1 - \beta, 1 - \alpha)$ interval-valued fuzzy numbers for the unknown problem. Each level $(1 - \beta, 1 - \alpha)$ interval-valued fuzzy number is based on past statistical data, which corresponds to the confidence intervals of the edge weights. After applying the signed distance ranking method to defuzzify the level $(1 - \beta, 1 - \alpha)$ interval-valued fuzzy numbers, we obtain the second type of shortest-path problem in the fuzzy sense. A dynamic programming recursion and a tabular calculation method were developed in this study to solve these problems. The main results from our study are that the fuzzy shortest paths obtained from our models correspond to actual paths in the network, and that the fuzzy shortest-path problems are an extension of the crisp problem. In addition, we propose an approach to combine statistics with fuzzy sets for obtaining a fuzzy shortest-path model. This approach will be very useful for solving practical problems.

The paper is organized as follows. Section 2 outlines the definitions of a signed distance ranking method for fuzzy numbers and for level (λ, α) interval-valued fuzzy numbers. In section 3 we formulate the shortest-path problem and discuss its fuzzy counterpart. Section 4 involves the first type of fuzzy shortest-path problem in which Theorem 1 is presented and an illustrative example is given. In section 5, we examine the second type of fuzzy shortest-path problem based on past statistical data. We also present Theorem 2 and give an illustrative example. Section 6 discusses the main results of this work. Finally, we state our conclusion in section 7.

2. PREREQUISITES

While considering the fuzzy shortest-path problem, some prerequisites are needed to deal with level (λ, ρ) interval-valued fuzzy numbers, which are given below.

Definition 1: A fuzzy set $\tilde{b}_\lambda$ defined on $R = (-\infty, \infty)$, which has the following membership function called a *level* λ *fuzzy point*, $0 < \lambda \leq 1$.

$$\mu_{\tilde{b}_\lambda}(x) = \begin{cases} \lambda, & x = b \\ 0, & otherwise \end{cases} \tag{1}$$

Definition 2: A fuzzy set $[a_\alpha, b_\alpha]$, where $0 \leq \alpha \leq 1$ and defined on R, which has the following membership function called a *level* α *fuzzy interval*:

$$\mu_{[a_\alpha, b_\alpha]}(x) = \begin{cases} \alpha, a \leq x \leq b \\ 0, otherwise \end{cases} \tag{2}$$

Definition 3: The level λ triangular fuzzy number $\tilde{A}$, $0 < \lambda \leq 1$, denoted by $\tilde{A} = (a, b, c, \lambda)$, is a fuzzy set defined on R with the membership function defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{\lambda(x-a)}{b-a}, a \leq x \leq b \\ \dfrac{\lambda(c-x)}{c-b}, b \leq x \leq c \\ \\ 0, otherwise \end{cases} \tag{3}$$

Let the family of all level λ fuzzy numbers be denoted by $F_N(\lambda) = \{(a, b, c; \lambda) \mid \forall a < b < c, a, b, c \in R\}$, $0 < \lambda \leq 1$. In addition, let $(a, b, c; 1)$ be the triangular fuzzy number and denoted by (a, b, c).

Definition 4: A fuzzy set $\tilde{A}$ defined on $R = (-\infty, \infty)$, which has the following membership function is called an *interval-valued fuzzy set*,

$$\tilde{A} = \{(x, [\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)])\}, x \in R, \quad 0 \leq \mu_{\tilde{A}^L}(x) \leq \mu_{\tilde{A}^U}(x) \leq 1 .$$

Symbolically $\tilde{A}$ is denoted by $[\tilde{A}^L, \tilde{A}^U]$, $\forall x \in R$. The membership grade of x in $\tilde{A}$ lies in the interval $[\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)]$. Obviously, the largest grade is $\mu_{\tilde{A}^U}(x)$ and the smallest is $\mu_{\tilde{A}^L}(x)$. In particular, if $\tilde{A}^L = (a, b, c; \lambda)$, $\tilde{A}^U = (p, b, q; \rho)$, $0 < \lambda \le \rho \le 1$, and $p < a < b < c < q$, we obtain $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] = [(a, b, c; \lambda), (p, b, q; \rho)]$, which is called the level (λ, ρ) interval-valued fuzzy number (see Fig. 1). Then the membership function of $\tilde{A} = [\tilde{A}^L, \tilde{A}^U]$ can be defined as

$$\mu_{\tilde{A}^L}(x) = \begin{cases} \dfrac{\lambda(x-a)}{b-a}, a \le x \le b \\ \dfrac{\lambda(c-x)}{c-b}, b \le x \le c \\ 0, otherwise \end{cases} \tag{4}$$

$$\mu_{\tilde{A}^U}(x) = \begin{cases} \dfrac{\rho(x-b)}{b-p}, p \le x \le b \\ \dfrac{\rho(q-x)}{q-b}, b \le x \le q \\ 0, otherwise \end{cases} \tag{5}$$

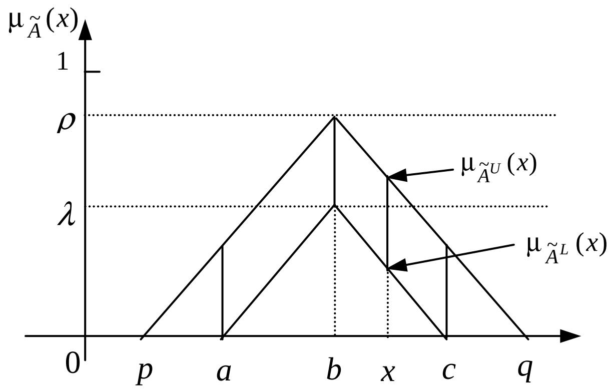

Fig. 1. An interval-valued fuzzy number $\tilde{A}$.

Let the family of all level (λ, ρ) interval-valued fuzzy numbers be denoted by $F_{IN}(\lambda, \rho) = \{[(a, b, c; \lambda), (p, b, q; \rho)] \mid \forall p < a < b < c < q, a, b, c, p, q \in R\}$, $0 < \lambda < \rho \le 1$. We obtain the following property of binary operation from [7, 15].

Property 1. Let $\tilde{A} = (a, b, c; \lambda)$ and $\tilde{B} = (p, q, r; \lambda) \in F_N(\lambda)$. We have $\tilde{A} \oplus \tilde{B} = (a + p, b + q, c + r, \lambda) \in F_N(\lambda)$.

Definition 5: Let $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] = [(a_1, b_1, c_1; \lambda), (p_1, b_1, q_1; \rho)]$ and $\tilde{B} = [\tilde{B}^L, \tilde{B}^U] =$

$[(a_2, b_2, c_2; \lambda), (p_2, b_2, q_2; \rho)] \in F_{IN}(\lambda, \rho)$ be interval-valued fuzzy numbers. The binary operation $\oplus$ is defined by $\tilde{A} \oplus \tilde{B} = [\tilde{A}^L \oplus \tilde{B}^L, \tilde{A}^U \oplus \tilde{B}^U]$.

Property 2. Let $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] = [(a_1, b_1, c_1; \lambda), (p_1, b_1, q_1; \rho)]$ and $\tilde{B} = [\tilde{B}^L, \tilde{B}^U] = [(a_2, b_2, c_2; \lambda), (p_2, b_2, q_2; \rho)] \in F_{IN}(\lambda, \rho)$ be interval-valued fuzzy numbers. Then we have $\tilde{A} \oplus \tilde{B} = [(a_1 + a_2, b_1 + b_2, c_1 + c_2; \lambda), (p_1 + p_2, b_1 + b_2, r_1 + r_2; \rho)]$.

Before defining the ranking of level (λ, ρ) fuzzy numbers on $F_{IN}(\lambda, \rho)$, we first consider the definition of the signed distance on R [14].

Definition 6: Let $d^*(d, 0) = b$, $b, 0 \in$ R.

Geometrically, $0 < b$ means that b lies to the right of the origin 0, and the distance between b and 0 is denoted by $b = d^*(b, 0)$. Similarly, $b < 0$ means that b lies to the left of 0 and the distance between b and 0 is denoted by $-b = -d^*(b, 0)$. Therefore $d^*(b, 0)$ denotes the signed distance of b, which is measured from 0.

Let $\tilde{A} = [\tilde{A}^L, \tilde{A}^U] = [(a, b, c; \lambda), (p, b, q; \rho)] \in F_{IN}(\lambda, \rho)$. Now we consider the ordering of level (λ, ρ) interval-valued fuzzy numbers defined on $F_{IN}(\lambda, \rho)$.

From Fig. 2 we can see that an α-cut of level (λ, ρ) interval-valued fuzzy number $\tilde{A}$ is determined as follows:

For $0 \le \alpha \le \lambda$, the α-cut is $\{[A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)]\}$; otherwise, for $\lambda \le \alpha \le \rho$, the α-cut is $[A_l^U(\alpha)_\alpha, A_r^U(\alpha)]$. (6)

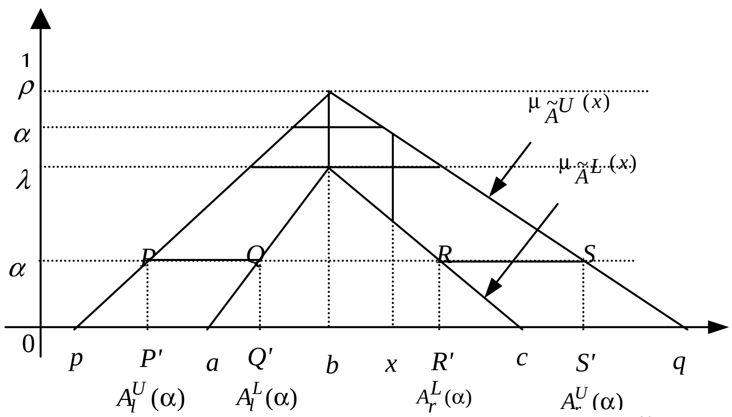

Fig. 2. The α-cut of level (λ, ρ) interval-valued fuzzy number $\tilde{A}$.

Then we derive the following equations from (4) and (5).

For $0 \le \alpha \le \lambda$,

$$A_l^L(\alpha) = a + (b-a)\frac{\alpha}{\lambda}, \quad A_r^L(\alpha) = c - (c-b)\frac{\alpha}{\lambda},$$

$$A_l^U(\alpha) = p + (b-p)\frac{\alpha}{\rho}, \text{ and } A_r^U(\alpha) = q - (q-b)\frac{\alpha}{\rho}. \tag{7}$$

For $\lambda \le \alpha \le \rho$,

$$A_l^U(\alpha) = p + (b-p)\frac{\alpha}{\rho} \text{ and } A_r^U(\alpha) = q - (q-b)\frac{\alpha}{\rho}. \tag{8}$$

Next consider the signed distance defined on $F_{IN}(\lambda, \rho)$. For any closed interval $[a, b]$, the signed distance of $[a, b]$ measured from 0 is defined by $d^*([a, b], 0) = \frac{1}{2}(a+b)$. For any two disjoint closed intervals $[a, b]$ and $[c, d]$, the signed distance of $[a, b] \cup [c, d]$ measured from 0 is defined by $d^*([a, b] \cup [c, d], 0) = \frac{1}{2}(d^*([a, b], 0) + d^*([c, d], 0))$. We can see that for each $\alpha \in [0, \lambda]$, $[A_l^L(\alpha), A_r^L(\alpha)]$ and $[A_l^L(\alpha)_\alpha, A_r^L(\alpha)_\alpha]$ are a one-to-one mapping. Similarly, for each $\alpha \in [0, \rho]$, $[A_l^U(\alpha), A_r^U(\alpha)]$ and $[A_l^U(\alpha)_\alpha, A_r^U(\alpha)_\alpha]$ are also a one-to-one mapping. Therefore, from Fig. 2, we can define the signed distance of $\tilde{\tilde{A}}$ from $\tilde{0}_1$ (y-axis) by $d^\circ(\tilde{\tilde{A}}, \tilde{0}_1) = \frac{1}{\lambda}\int_0^\lambda d^*([A_l^U(\alpha), A_l^L(\alpha)] \cup [A_r^L(\alpha), A_r^U(\alpha)], 0)\, da + \frac{1}{\rho+\lambda}\int_\lambda^\rho d^*([A_l^U(\alpha), A_r^U(\alpha)], 0) d\alpha.$

Property 3. Let $\tilde{\tilde{A}} = [(a, b, c; \lambda), (p, b, q; \rho)] \in F_{IN}(\lambda, \rho)$, $0 < \lambda < \rho \le 1$. Then the signed distance from $\tilde{\tilde{A}}$ measured from 0 is defined by $d^\circ(\tilde{\tilde{A}}, \tilde{0}_1) = \frac{1}{8}(6b + a + c + 4p + 4q + \frac{3\lambda}{\rho}(2b - p - q))$.

Property 4. Let $\tilde{\tilde{A}} = [(a_1, b_1, c_1; \lambda), (p_1, b_1, q_1; \rho)]$ and $\tilde{\tilde{B}} = [(a_2, b_2, c_2; \lambda), (p_2, b_2, q_2; \rho)] \in F_{IN}(\lambda, \rho)$. Then we have the binary operation $d^\circ(\tilde{\tilde{A}} \oplus \tilde{\tilde{B}}, \tilde{0}_1) = d^\circ(\tilde{\tilde{A}}, \tilde{0}_1) + d^\circ(\tilde{\tilde{B}}, \tilde{0}_1)$.

Definition 7: Let $\tilde{\tilde{A}} = [(a_1, b_1, c_1; \lambda), (p_1, b_1, r_1; \rho)]$ and $\tilde{\tilde{B}} = [(a_2, b_2, c_2; \lambda), (p_2, b_2, r_2; \rho)] \in F_{IN}(\lambda, \rho)$, for $0 < \lambda < \rho \le 1$. The rankings of interval-valued fuzzy numbers on $F_{IN}(\lambda, \rho)$ are defined by

$$\tilde{\tilde{A}} \prec \tilde{\tilde{B}} \text{ iff } d^\circ(\tilde{\tilde{A}}, \tilde{0}_1) < d^\circ(\tilde{\tilde{B}}, \tilde{0}_1)$$
$$\tilde{\tilde{A}} \approx \tilde{\tilde{B}} \text{ iff } d^\circ(\tilde{\tilde{A}}, \tilde{0}_1) = d^\circ(\tilde{\tilde{B}}, \tilde{0}_1).$$

For the family of all level λ fuzzy numbers $F_N(\lambda)$, we have the following similar results.

Definition 8: For each $\lambda \in (0, 1]$ and $\tilde{\tilde{C}} = (a, b, c; \lambda) \in F_N(\lambda)$, the signed distance of $\tilde{A}$ measured from $\tilde{0}_1$ is defined by $d(\tilde{\tilde{C}}, \tilde{0}_1) = \frac{1}{4}(2b + a + c)$.

Property 5. Let $\tilde{A} = (a, b, c; \lambda)$ and $\tilde{B} = (p, q, r; \lambda) \in F_N(\lambda)$. Then we obtain the binary operation $d(\tilde{A} \oplus \tilde{B}, \tilde{0}_1) = d(\tilde{A}, \tilde{0}_1) + d(\tilde{B}, \tilde{0}_1)$.

Definition 9: Let $\tilde{A} = (a, b, c; \lambda)$ and $\tilde{B} = (p, q, r; \lambda) \in F_N(\lambda)$. For $\lambda \in (0, 1)$, the rankings of level λ fuzzy numbers on $F_N(\lambda)$ are defined by

$$\tilde{A} \prec \tilde{B} \text{ iff } d(\tilde{A}, \tilde{0}_1) < d(\tilde{B}, \tilde{0}_1)$$
$$\tilde{A} \approx \tilde{B} \text{ iff } d(\tilde{A}, \tilde{0}_1) = d(\tilde{B}, \tilde{0}_1).$$

3. FORMULATION OF SHORTEST-PATH NETWORK PROBLEMS

3.1 Crisp Shortest-Path Network Problem

In the shortest-path problem, each edge in the network has a number, which is called the edge weight. The length of a path is the sum of all of the edge weights in the path. There are usually many paths between a pair of nodes, for instance nodes *s* and *t*, but only a path with the minimum length is the shortest path from *s* to *t*. In some applications, edge weights can be interpreted as measurements other than distances. For example, edge weights are often used to represent time, cost, penalties, or any other quantity that accumulates linearly along a path and that one wishes to minimize. As we stated in section 1, the shortest-path problem is the most common problem in network flow problems. The shortest-path algorithm can usually be modified to find other optimum paths. Accordingly, we shall focus on the traditional shortest-path problem in networks. Generally, there are three kinds of shortest-path problems in a network.

(1) The shortest path from one source node to one destination node,
(2) The shortest paths from one source node to all other nodes,
(3) The shortest paths between all pairs of nodes.

Dijkstra's algorithm [5] was designed to solve Problem (2). Problem (3) can be solved by running Dijkstra's algorithm once from each vertex or by using Floyd's algorithm [9] only. Since all algorithms for solving Problem (1) and Problem (2) are essentially the same, we will discuss only the problem of finding the shortest path from one source node to one destination. This problem can be stated as follows. A network is an acyclic directed graph $G = (V, E)$ with a weight function $w: E \to R$ mapping edges to real-valued numbers. The set V refers to the vertices of the graph and the set E refers to the edges of the graph. The length of a path from vertex i to vertex j, $p = < i, i_1, \ldots, j>$, is the sum of the weights of its constituent edges. The problem is to find a shortest path from a given source vertex $s \in V$ to a given destination vertex $n \in V$ in a network. However, the problem of finding shortest paths is defined only if the network does not contain a negative cycle [3, 6]. Note that a network can have some directed edges with negative weights and yet does not contain a negative cycle.

According to Bellman's equation [9], a DP formulation for the shortest-path problem can be given as follows. Given a network with an acyclic directed graph $G = (V, E)$ with n vertices numbered from 1 to n such that 1 is the source and n is the destination. Then, we have

$$f(n) = 0$$
$$f(i) = \min_{i<j}\{ c_{ij} + f(j) \mid < i, j > \in E\} . \tag{9}$$

Here c_{ij} is the weight of the directed edge <i, j>, and $f(i)$ is the length of the shortest path from vertex i to vertex n (see Fig. 3). Note that it is relatively easy to solve the shortest-path problem while all of the edge weights are nonnegative, and the $O(n^2)$ greedy algorithm for this case is Dijkstra's [5].

Example 1. Consider the network shown in Fig. 3 where the numbers are edge weights.

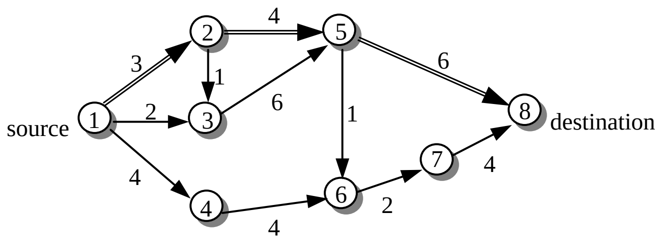

Fig. 3. The network and its shortest path for Example 1.

Fig. 3 shows that $V = \{j \mid j = 1, 2, \ldots, 8\}$, $E = \{$ <1, 2>, <1, 3>, <1, 4>, … , <7, 8>$\}$, and $c_{12} = 3$, $c_{13} = 2$, $c_{14} = 4$, …, $c_{78} = 4$. The shortest path from vertex 1 to vertex 8 is $1 \to 2 \to 5 \to 8$ which has a total length $f(1) = 13$ (hours).

3. 2 Fuzzy Shortest-Path Problem

The fuzzy shortest-path problem was first introduced by Dubois and Prade [4]. Generally, fuzziness can be introduced into the network in a variety of ways, e.g., through edge capacities, edge weights, or vertex restrictions. In real life situations, some unexpected events may occur so that the edge weight in the network may change slightly. Hence, in most situations, the weight can only be estimated within a certain interval. Consequently, it is very difficult for a DM to give a single precise number to represent each edge weight in the network. Because of this interval estimation feature, the representation of each edge weight can be more realistically and naturally achieved through the use of a fuzzy number. As a result, the fuzzy approach seems much more natural to us for solving the shortest-path problem in practice.

In [8], two specific categories for fuzzy shortest-path problems are discussed. The first involves each edge weight depicted as a fuzzy number, and the second depicts the length of a path as a fuzzy number with each edge in the network having a membership value. In [4], Dubois and Prade discussed the solution to the first problem using an extended sum with Floyd's and Ford's algorithms used to solve the problem. The major drawback in this solution is that a fuzzy shortest path can be found, but a shortest path with that length may not exist. However, this problem is circumvented with the new models based on fuzzy shortest paths and multiple objectives presented by Klein [8]. He presented a hybrid multi-criteria DP recursion, $f(N) = (1, 1, 1, \ldots, 1)$ and $f(i) = \underset{(i,j)\in E}{dom}(e_{ij} \tilde{+} f(j))$, to solve the models. In the next two sections, we will present two new types of fuzzy shortest-path problems in networks.

4. FUZZY SHORTEST-PATH PROBLEM BASED ON FUZZY NUMBERS

In this section, the essential problem we consider is that the edge weight in the network, c_{ij}, is imprecise. The edge weight should be expressed using fuzzy linguistics, such as "around four miles", for practical situations. Therefore, the fuzzy linguistics used for imprecise edge weight lead to the use of triangular fuzzy number

$$\tilde{c}_{ij} = (c_{ij} - \Delta_{ij1}, c_{ij}, c_{ij} + \Delta_{ij2}), \tag{10}$$

where $0 < \Delta_{ij1} < c_{ij1}$, $0 < \Delta_{ij2}$ (see Fig. 4), for a suitable representation. Note that Δ_{ij1} and Δ_{ij2} should be determined by the DM.

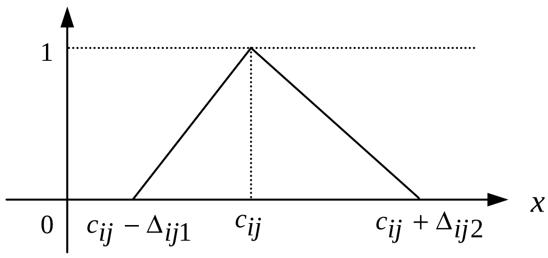

Fig. 4. The fuzzy number $\tilde{c}_{ij}$.

For $\lambda = 1$, from Definition 8, we obtain

$$d(\tilde{c}_{ij}, \tilde{0}_1) = c_{ij} + \tfrac{1}{4}\Delta_{ij} \; (= c_{ij}^*), \tag{11}$$

where $\Delta_{ij} = \Delta_{ij2} - \Delta_{ij1}$. This is the signed distance of $\tilde{c}_{ij}$ measured from $\tilde{0}_1$. Since $c_{ij}^* = d(\tilde{c}_{ij}, \tilde{0}_1) = \frac{1}{4}(3c_{ij} + \Delta_{ij2}) + \frac{1}{4}(c_{ij} - \Delta_{ij1}) > 0$, we conclude that $d(\tilde{c}_{ij}, \tilde{0}_1)$ is a positive distance measured from $\tilde{0}_1$ to $\tilde{c}_{ij}$, and c_{ij}^* is also a positive number measured from 0. In (11), when $\Delta_{ij1} = \Delta_{ij2}$, we obtain $c_{ij}^* = c_{ij}$. Thus the fuzzy problem becomes crisp. We call $c_{ij}^* = c_{ij} + \frac{1}{4}\Delta_{ij}$ an estimate of the edge weight $<i, j>$ in the fuzzy sense. From Fig. 3 and (9), the solution of DP can be derived as follows:

$$f(8) = 0,\; f(7) = \min_{7<j}\{c_{7j} + f(j) \,|< 7, j >\in E\} = c_{78},$$

$$f(6) = \min_{6<j}\{c_{6j} + f(j) \,|< 6, j >\in E\} = c_{67} + f(7) = c_{67} + c_{78},$$

$$f(5) = \min_{5<j}\{c_{5j} + f(j) \,|< 5, j >\in E\} = \min\{c_{56} + f(6), c_{58} + f(8)\} = c_{58}.$$

Similarly, we obtain

$f(4) = c_{46} + c_{67} + c_{78}$, $f(3) = c_{35} + c_{58}$, $f(2) = c_{25} + c_{58}$, and
$f(1) = \min\{c_{12} + f(2), c_{13} + f(3), c_{14} + f(4)\} = c_{12} + c_{25} + c_{58}$.

The mean for each $f(i)$, $i = 1, \ldots, 7$, is a sum of c_{ij}, $<i, j> \in E$. Because there are finite paths from node 1 to node n in a network, we conclude that there are also finite paths from node i to node n in the network. Thus there must exist a path $p = <i, i_1, i_2, \ldots, i_{m(i)}, n>$ (i.e. $<i, i_1>, <i_1, i_2>, \ldots, <i_{m(i)}, n> \in E$) for $f(i) = c_{ii_1} + c_{i_1i_2} + \ldots + c_{i_{m(i)}n}$. Note that $f(i)$ is the length of the shortest path from vertex i to vertex n. We then derive inequalities from $f(i)$ as

$$c_{ii_1} + c_{i_1i_2} + \ldots + c_{i_{m(i)}n} \leq c_{ik_1} + c_{k_1k_2} + \ldots + c_{k_{p(k)}n}, \tag{12}$$

where at least one equal sign holds for all possible paths, $p = <i, k_1, k_2, \ldots, k_{p(k)}, n>$, from vertex i to vertex n. In summary, this is

$$f(i) = \min\{c_{ik_1} + c_{ik_2} + \ldots + c_{k_{p(k)}n} \mid \text{ for all paths } p = <i, k_1, k_2, \ldots, k_{p(k)}, n>\}.$$

The DM should choose appropriate values for parameters to satisfy

$$\Delta_{ii_1} + \Delta_{i_1i_2} + \ldots + \Delta_{i_{m(i)}n} \leq \Delta_{ik_1} + \Delta_{k_1k_2} + \ldots + \Delta_{k_{p(k)}n}, \tag{13}$$

where at least one equal sign holds. Adding one quarter of (13) to (12), according to (11), Definition 8, and Property 5, we obtain

$$d(\tilde{c}_{ii_1} \oplus \tilde{c}_{i_1i_2} \oplus \ldots \oplus \tilde{c}_{i_{m(i)}n}) \leq d(\tilde{c}_{ik_1} \oplus \tilde{c}_{k_1k_2} \oplus \ldots \oplus \tilde{c}_{k_{p(k)}n}), \tag{14}$$

where at least one equal sign holds. From Definition 9, we can see that (14) is equivalent to

$$\tilde{c}_{ii_1} \oplus \tilde{c}_{i_1i_2} \oplus \ldots \oplus \tilde{c}_{i_{m(i)}n} \precsim \tilde{c}_{ik_1} \oplus \tilde{c}_{k_1k_2} \oplus \ldots \oplus \tilde{c}_{k_{p(k)}n}, \tag{15}$$

where at least one $\approx$ holds for all possible paths from vertex i to vertex n. Obviously, (15) is obtained from fuzzifying (12) and taking (13) as a fuzzified condition. Note that the relations $\prec$ and $\approx$ in (15) are the rankings defined in $F_N(1)$ (see Definition 9). From Definition 8 and Property 5, we obtain $d(\tilde{c}_{ii_1} \oplus \tilde{c}_{i_1i_2} \oplus \ldots \oplus \tilde{c}_{i_{m(i)}n}, \tilde{0}_1) = d(\tilde{c}_{ii_1}, \tilde{0}_1) + d(\tilde{c}_{i_1i_2}, \tilde{0}_1) + \ldots + d(\tilde{c}_{i_{m(i)}n}, \tilde{0}_1) = c^*_{ii_1} + c^*_{i_1i_2} + \ldots + c^*_{k_{p(k)}n}$. Similarly, we obtain $d(\tilde{c}_{ik_1} \oplus \tilde{c}_{k_1k_2} \oplus \ldots \oplus \tilde{c}_{k_{p(k)}n}, \tilde{0}_1) = c^*_{ik_1} + c^*_{k_1k_2} + \ldots + c^*_{k_{p(k)}n}$. (16)

Then, from (11), (15), (16), and Definition 9, we derive the following inequalities:

$$c^*_{ii_1} + c^*_{i_1i_2} + \ldots + c^*_{i_{m(i)}n} \leq c^*_{ik_1} + c^*_{k_1k_2} + \ldots + c^*_{k_{p(k)}n}, \tag{17}$$

where at least one equal sign holds for all possible paths from vertex i to vertex n. Let $f^*(i)$ be the length of the shortest path from vertex i to vertex n in network $G = (V, E)$ with $\{c^*_{ij} \mid <i, j> \in E\}$. From (17), we obtain $f^*(i) = c^*_{ii_1} + c^*_{i_1i_2} + \ldots + c^*_{i_{m(i)}n}$. Similarly, we can obtain $f^*(j) = c^*_{jj_1} + c^*_{j_1j_2} + \ldots + c^*_{j_{m(j)}n}$. (18)

We rewrite (9) as follows: For any fixed i, $f(i) \leq c_{ij} + f(j), \forall i < j$, $<i, j> \in E$, where at least one equal sign holds. Then,

$$c_{ii_1} + c_{i_1 i_2} + \ldots + c_{i_{m(i)} n} \leq c_{ij} + c_{jj_1} + c_{j_1 j_2} + \ldots + c_{j_{m(j)} n}, \forall i < j, <i, j> \in E, \tag{19}$$

where at least one equal sign holds. The DM should choose appropriate values for parameters to satisfy

$$\Delta_{ii_1} + \Delta_{i_1 i_2} + \ldots + \Delta_{i_{m(i)} n} \leq \Delta_{ij} + \Delta_{jj_1} + \Delta_{j_1 j_2} + \ldots + \Delta_{j_{m(j)} n}, \forall i < j, <i, j> \in E, \tag{20}$$

where at least one equal sign holds. From (19) and (20), we obtain

$$\tilde{c}_{ii_1} \oplus \tilde{c}_{i_1 i_2} \oplus \ldots \oplus \tilde{c}_{i_{m(i)} n} \precsim \tilde{c}_{ij} \oplus \tilde{c}_{jj_1} \oplus \tilde{c}_{j_1 j_2} \oplus \ldots \oplus \tilde{c}_{j_{m(j)} n}, \forall i < j, <i, j> \in E, \tag{21}$$

where at least one $\approx$ holds. From Definitions 8 and 9, Property 5, and (11), we obtain

$$c^*_{ii_1} + c^*_{i_1 i_2} + \ldots + c^*_{i_{m(i)} n} \leq c^*_{ij} + c^*_{jj_1} + c^*_{j_1 j_2} + \ldots + c^*_{j_{m(j)} n}, \forall i < j, <i, j> \in E, \tag{22}$$

where at least one equal sign holds.

Remark 1. Note that (19) and (20) have the same relation property.

From (18) and (22), the DP recursion of the first type of shortest-path problem in the fuzzy sense can be given by $f^*(i) = \min_{i<j}\{c^*_{ij} + f^*(j) \mid <i, j> \in E\}$ and $f^*(n) = 0$. Finally, we summarize the above description in the following Theorem.

Theorem 1. Consider a network $G = (V, E)$ with n vertices numbered from 1 to n with edge weights $\{c_{ij} \mid <i, j> \in E\}$. An estimate of the edge weight, c^*_{ij}, is based on a triangular fuzzy number from (10), and is defined by $c^*_{ij} = c_{ij} + \frac{1}{4}(\Delta_{ij2} - \Delta_{ij1}) = c_{ij} + \frac{1}{4}\Delta_{ij}$, where Δ_{ij1} and Δ_{ij2} are parameters whose values are determined by the DM so as to satisfy (20), thus creating a set of edge weights in the fuzzy sense, $\{c^*_{ij} \mid <i, j> \in E\}$. The DP recursion of the first type of shortest-path problem in the fuzzy sense is given by

$$f^*(i) = \min_{i<j}\{c^*_{ij} + f^*(j) \mid <i, j> \in E\}, \text{ and} \tag{23}$$

$$f^*(n) = 0,$$

where $f^*(i)$ is the length of the shortest path in the fuzzy sense from vertex i to vertex n.

Notice that when Δ_{ij2} and Δ_{ij1} for each edge $<i, j> \in E$ in Theorem 1, we obtain $c^*_{ij} = c_{ij}$, and as a result, the fuzzy shortest-path problem becomes a crisp problem.

Example 2. This example is a continuation of Example 1. In this paper, we introduce a tabular method for evaluating the dynamic programming recursion of Example 1. The tabular method is stated as follows. First, we put the edge weights from Fig. 3 into the correct entries in Table 1, i.e., $c_{12} = 3$, $c_{13} = 2$, $c_{14} = 4$, …, $c_{67} = 2$, and $c_{78} = 4$. Next, we calculate $f(j)$ in reverse order of $j = 7, 6, \ldots, 2, 1$ (see Table 1).

Initially, $f(8) = 0$. We start with $j = 7$; see the $i = 7$ row. Since $c_{78} = 4$, check the $j = 8$ column, obtaining $f(8) = 0$. Hence, we obtain $f(7) = c_{78} + f(8) = 4 + 0 = 4$. Next, $j = 6$, see the $i = 6$ row. Since $c_{67} = 2$, check the $j = 7$ column, obtaining $f(7) = 4$. We obtain $f(6) = c_{67} + f(7) = 2 + 4 = 6$. Then, $j = 5$; see the $i = 5$ row. Since there are two entries, c_{56} and c_{58}, in that row, we obtain $f(5) = \min\{c_{56} + f(6), c_{58} + f(8)\} = \min\{1 + 6, 6 + 0\} = 6$. Using the same process we have $f(4) = 10$, $f(3) = 12$, and $f(2) = 10$. Finally, $j = 1$, see row $i = 1$. Since there are three entries, c_{12}, c_{13}, and c_{14} in that row, we obtain $f(1) = \min\{c_{12} + f(2), c_{13} + f(3), c_{14} + f(4)\} = \min\{3 + 10, 2 + 12, 4 + 10\} = 13$. In summary, the shortest path obtained from $f(1) = c_{12} + f(2) = c_{12} + c_{25} + f(5) = c_{12} + c_{25} + c_{58} + f(8) = c_{12} + c_{25} + c_{58}$ is 1, 2, 5, 8 with length 13.

Table 1. Tabular method for finding the shortest path in the crisp case.

$f(j)$	13	10	12	10	6	6	4	0	i
j	1	2	3	4	5	6	7	8	
		3	2	4					1
c_{12} c_{13}			1		4				2
					6				3
c_{23}						4			4
						1		6	5
							2		6
c_{67}								4	7
c_{78}									8

Now consider the fuzzy case. We look for inequalities that satisfy (19). Clearly, any row with two or more entries in Table 1 entails an inequality. Hence, we get rows $i = 1$, 2, and 5 and obtain the following inequalities:

when $i = 1$, $c_{12} + f(2) < c_{13} + f(3)$, i.e., $c_{12} + c_{25} + c_{58} < c_{13} + c_{35} + c_{58}$,
or $c_{12} + f(2) < c_{14} + f(4)$, i.e., $c_{12} + c_{25} + c_{58} < c_{14} + c_{46} + c_{67} + c_{78}$.
when $i = 2$, $c_{25} + f(5) < c_{23} + f(3)$, i.e., $c_{25} + c_{58} < c_{23} + c_{35} + c_{58}$.
when $i = 5$, $c_{58} + f(8) < c_{56} + f(6)$, i.e., $c_{58} < c_{56} + c_{67} + c_{78}$.

Then, according to Remark 1, the parameters of (20) based on the above inequalities are derived as

$$\Delta_{12} + \Delta_{25} + \Delta_{58} < \Delta_{13} + \Delta_{35} + \Delta_{58},$$
$$\Delta_{12} + \Delta_{25} + \Delta_{58} < \Delta_{14} + \Delta_{46} + \Delta_{67} + \Delta_{78},$$
$$\Delta_{25} + \Delta_{58} < \Delta_{23} + \Delta_{35} + \Delta_{58}, \text{ and}$$
$$\Delta_{58} < \Delta_{56} + \Delta_{67} + \Delta_{78}. \qquad (*)$$

If the DM chooses the values of parameters: $\Delta_{12} = 0.5$, $\Delta_{13} = 1$, $\Delta_{14} = 0.8$, $\Delta_{23} = 0.9$, $\Delta_{25} = 1$, $\Delta_{35} = 1.2$, $\Delta_{46} = 1.3$, $\Delta_{56} = 1.5$, $\Delta_{58} = 0.8$, $\Delta_{67} = 1$, and $\Delta_{78} = 0.9$, to satisfy the conditions in (*), then the fuzzy numbers in (10) are determined as

$\tilde{c}_{12} = (3-0.2, 3, 3+0.7)$, $\tilde{c}_{13} = (2-0.5, 2, 2+1.5)$, $\tilde{c}_{14} = (4-0.2, 4, 4+1)$,
$\tilde{c}_{23} = (1-0.3, 1, 1+1.2)$, $\tilde{c}_{25} = (4-1, 4, 4+2)$, $\tilde{c}_{35} = (6-0.3, 6, 6+1.5)$,
$\tilde{c}_{46} = (4-0.2, 4, 4+1.5)$, $\tilde{c}_{56} = (1-0.5, 1, 1+2)$, $\tilde{c}_{58} = (6-0.3, 6, 6+1.1)$,
$\tilde{c}_{67} = (2-1, 2, 2+2)$, and $\tilde{c}_{78} = (4-0.2, 4, 4+1.1)$.

From Theorem 1 we obtain the following estimate of the edge weights in the fuzzy sense: $c^*_{12} = 3.125$, $c^*_{13} = 2.25$, $c^*_{14} = 4.2$, $c^*_{23} = 1.225$, $c^*_{25} = 4.25$, $c^*_{35} = 6.3$, $c^*_{46} = 4.325$, $c^*_{56} = 1.375$, $c^*_{58} = 6.2$, $c^*_{67} = 2.25$, and $c^*_{78} = 4.225$. Fig. 5 shows the fuzzy network $G = (V, E)$ with $\{c^*_{ij} \mid <i, j> \in E\}$.

The tabular method for finding the shortest path in a fuzzy network is given in Table 2. Using the same approach as in Table 1, the shortest path in the fuzzy sense obtained from $f^*(1) = c_{12} + f(2) = c^*_{12} + c^*_{25} + f^*(5) = c^*_{12} + c^*_{25} + c^*_{58} + f^*(8) = c^*_{12} + c^*_{25} + c^*_{58}$, is 1, 2, 5, 8 with length 13.575. The shortest path in the fuzzy sense is longer than the crisp shortest path by $\frac{f^*(1) - f(1)}{f(1)} \times 100 = 4.42\%$.

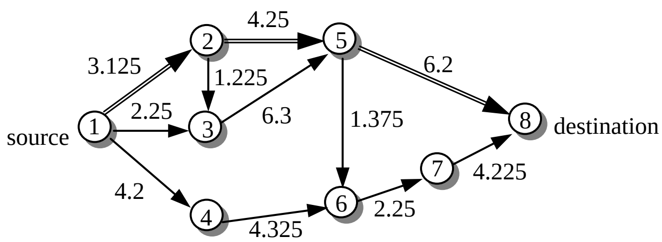

Fig. 5. The fuzzy network $G = (V, E)$ with $\{c^*_{ij} \mid <i, j> \in E\}$.

Table 2. Finding the shortest path in a fuzzy case.

$f^*(j)$	13.575	10.45	12.5	10.8	6.2	6.475	4.225	0	i
j	1	2	3	4	5	6	7	8	
		3.125	2.25	4.2					1
			1.225		4.25				2
					6.3				3
						4.325			4
						1.375		6.2	5
							2.25		6
								4.225	7
									8

Notice that the length of the shortest path in the fuzzy sense obtained from Theorem 1 is related to the tolerable range of each edge weight $[c_{ij} - \Delta_{ij1}, c_{ij} + \Delta_{ij2}]$ as well as the parameter values that are determined by the DM.

5. FUZZY SHORTEST-PATH PROBLEM BASED ON STATISTICAL DATA AND LEVEL (1−β, 1−α) INTERVAL-VALUED FUZZY NUMBERS

The fuzzy shortest-path problem also occurs when the edge weights of the network, c_{ij}, are unknown. In this section, we propose an approach in which the DM can use past statistical data to estimate the value of c_{ij} and solve the fuzzy shortest-path problem. The problem can be stated as follows. Let c_{ijq}, q = 1, 2, … , n, be n samples of c_{ij}. We obtain the mean $\bar{c}_{ij} = \frac{1}{n}\sum_{q=1}^{n} c_{ijq}$ and variance $s_{jk}^2 = \frac{1}{n-1}\sum_{q=1}^{n}(c_{ijq} - \bar{c}_{ij})^2$ for a network $G = (V, E)$ with $\{\bar{c}_{ij} \mid <i, j> \in E\}$. Then the problem can be formulated as the following dynamic programming recursion:

$$\bar{f}(n) = 0$$
$$\bar{f}(i) = \min_{i<j}\{\bar{c}_{ij} + \bar{f}(j) \mid <i, j> \in E\}, \tag{24}$$

where $\bar{f}(i)$ is the length of the shortest path from vertex i to vertex n in the network G = (V, E) with $\{\bar{c}_{ij} \mid <i, j> \in E\}$.

Substituting c_{ij} for $\bar{c}_{ij}$ in (12) leads to $\bar{f}(i) = \min\{\bar{c}_{ik_1} + \bar{c}_{k_1k_2} + \ldots + \bar{c}_{k_{p(k)}n} \mid$ for all paths p = < i, k_1, k_2, …, $k_{p(i)}$, n>, from vertex i to vertex n}. Let $\bar{f}(i) = \bar{c}_{ii'_1} + \bar{c}_{i'_1i'_2} + \ldots + \bar{c}_{i'_{m(I')}n}$ and $\bar{f}(j) = \bar{c}_{jj'_1} + \bar{c}_{j'_1j'_2} + \ldots + \bar{c}_{j'_{m(j')}n}$. Rewriting (24) as $\bar{f}(i) \le \bar{c}_{ij} + \bar{f}(j)$, $\forall i < j$, $<i, j> \in E$, where at least one equal sign holds. This yields

$$\bar{c}_{ii'_1} + \bar{c}_{i'_1i'_2} + \ldots + \bar{c}_{i'_{m(I')}n} \le \bar{c}_{ij} + \bar{c}_{jj'_1} + \bar{c}_{j'_1j'_2} + \ldots + \bar{c}_{j'_{m(j')}n}, \tag{25}$$

where at least one equal sign holds. If we let the unknown c_{ij} be a population of the edge weights, $\bar{c}_{ij}$ will be a point estimate of c_{ij}. However, a point estimate has an accuracy problem so we use a confidence-interval estimate instead. The formula for the (1 − α) × 100% confidence interval of c_{ij} from [13] is given by $[\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})s_{ij}^*,\ \bar{c}_{ij} + t_{n-1}(\alpha_{ij2})s_{ij}^*]$, where $s_{ij}^2 = \frac{1}{n-1}\sum_{q=1}^{n}(c_{ijq} - \bar{c}_{ij})^2$, $s_{ij}^* = \frac{s_{ij}}{\sqrt{n}}$, $0 < \alpha_{ijk} < 1$, k = 1, 2, $\alpha_{ij1} + \alpha_{ij2} = \alpha$, and $0 < \alpha < 1$. Let T be the t-distribution with n – 1 degrees of freedom, and let $t_{n-1}(\alpha_{ijk})$, k = 1, 2, be the constant that satisfies

$$P(T > t_{n-1}(\alpha_{ijk})) = \alpha_{ijk}. \tag{26}$$

Since the t-distribution with n – 1 degrees of freedom is symmetrical on the y-axis, we obtain

$$P(\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})\frac{s_{ij}}{\sqrt{n}} \le c_{ij} \le \bar{c}_{ij} + t_{n-1}(\alpha_{ij2})\frac{s_{ij}}{\sqrt{n}})$$
$$= P(-t_{n-1}(\alpha_{ij1}) \le \frac{\sqrt{n}(c_{ij} - \bar{c}_{ij})}{s_{ij}} \le 0) + P(0 \le \frac{\sqrt{n}(c_{ij} - \bar{c}_{ij})}{s_{ij}} \le t_{n-1}(\alpha_{ij2}))$$
$$= \frac{1}{2} - \alpha_{ij1} + \frac{1}{2} - \alpha_{ij2} = 1 - \alpha.$$

Then the $(1-\alpha)\times 100\%$ confidence interval of c_{ij} is given by

$$[\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})s^*_{ij},\ \bar{c}_{ij} + t_{n-1}(\alpha_{ij2})s^*_{ij}].$$

Since the interval does not represent a single value, we use the following level $(1-\alpha)$ fuzzy number that corresponds to that interval,

$$\tilde{\tilde{c}}^U_{ij} = (\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})s^*_{ij}, \bar{c}_{ij}, \bar{c}_{ij} + t_{n-1}(\alpha_{ij2})s^*_{ij}; 1-\alpha). \tag{27}$$

Note that the membership grade of $\bar{c}_{ij}$ is not always at $(1-\alpha)$ during any given time period due to statistical variations. We assume that the membership grade of $\bar{c}_{ij}$ lies in an interval $[1-\beta, 1-\alpha]$, $0 < 1-\beta < 1-\alpha < 1$. Then we should consider level $(1-\beta, 1-\alpha)$ interval-valued fuzzy numbers. Let $0<\alpha<\beta<1$, $0<\alpha_{ijk}<\beta_{ijk}<1$, $k = 1, 2$, and $\beta_{ij1} + \beta_{ij2} = \beta$. The $(1-\beta)\times 100\%$ confidence interval of c_{ij} is $[\bar{c}_{ij} - t_{n-1}(\beta_{ij1})s^*_{ij}, \bar{c}_{ij} + t_{n-1}(\beta_{ij2})s^*_{ij}]$. Similarly, we derive the level $(1-\beta)$ fuzzy number as

$$\tilde{\tilde{c}}^L_{ij} = (\bar{c}_{ij} - t_{n-1}(\beta_{ij1})s^*_{ij}, \bar{c}_{ij}, \bar{c}_{ij} + t_{n-1}(\beta_{ij2})s^*_{ij}; 1-\beta). \tag{28}$$

Accordingly, the level $(1-\beta, 1-\alpha)$ interval-valued fuzzy number can be derived from (27) and (28),

$$\tilde{\tilde{c}}_{ij} = [\tilde{\tilde{c}}^L_{ij}, \tilde{\tilde{c}}^U_{ij}]. \tag{29}$$

Fig. 6 shows the level $(1-\beta, 1-\alpha)$ interval-valued fuzzy number $\tilde{\tilde{c}}_{ij}$. Since $0 < \alpha_{ijk} < \beta_{ijk} < 1$, we have $t_{n-1}(\beta_{ijk}) < t_{n-1}(\alpha_{ijk})$, $k = 1, 2$. Then we obtain $0 < \bar{c}_{ij} - t_{n-1}(\alpha_{ij1})s^*_{ij} < \bar{c}_{ij} - t_{n-1}(\beta_{ij1})s^*_{ij} < \bar{c}_{ij} < \bar{c}_{ij} + t_{n-1}(\beta_{ij2})s^*_{ij} < \bar{c}_{jk} + t_{n-1}(\alpha_{ij2})s^*_{ij}$. (30)

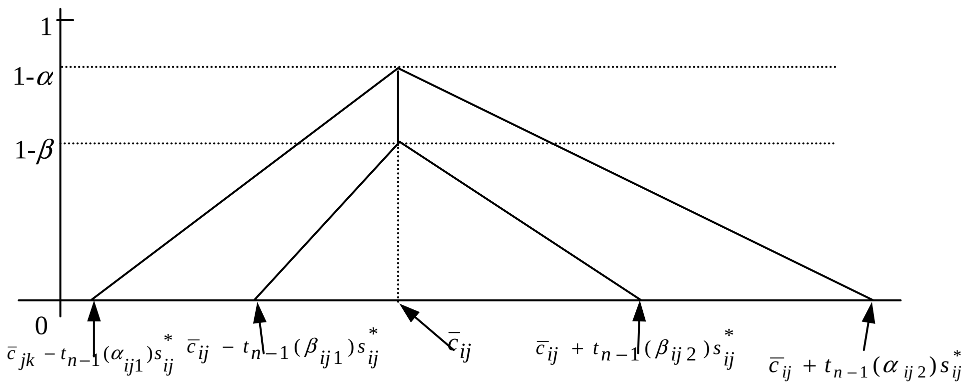

Fig. 6. The level $(1-\beta, 1-\alpha)$ interval-valued fuzzy number $\tilde{\tilde{c}}_{ij}$.

From Property 3, we obtain $d^\circ(\tilde{\tilde{c}}_{ij}, \tilde{0}_1) = 2\bar{c}_{jk} + \frac{1}{8}\{(t_{n-1}(\beta_{ij2}) - t_{n-1}(\beta_{ij1}))\, s^*_{ij} + (4-3\lambda)(t_{n-1}(\alpha_{ij2}) - t_{n-1}(\alpha_{ij1}))\, s^*_{ij}\}$, where $\lambda = \frac{1-\beta}{1-\alpha}$.

Let $c^{\circ}_{ij} = \frac{1}{2} d^{\circ}(\tilde{\tilde{c}}_{ij}, \tilde{0}_1) = \bar{c}_{ij} + \frac{1}{16}\mu_{ij}$, (31)

where $\mu_{ij} = (t_{n-1}(\beta_{ij2}) - t_{n-1}(\beta_{ij1})\, s^*_{ij} + (4 - 3\lambda)(t_{n-1}(\alpha_{ij2}) - t_{n-1}(\alpha_{ij1})\, s^*_{ij}$. From (30), $0 < i < 1$, and $0 < 4 - 3\lambda$, we obtain $c^{\circ}_{ij} = \frac{1}{16}(11\bar{c}_{ij} + t_{n-1}(\beta_{ij2})\ s^*_{ij} + (4 - 3\lambda)t_{n-1}(\alpha_{ij2})\, s^*_{ij}) + \frac{1}{16}(\bar{c}_{ij} - t_{n-1}(\beta_{ij1})\, s^*_{ij}) + \frac{1}{16}(4 - 3\lambda)(\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})\, s^*_{ij}) + \frac{3\lambda}{16}\bar{c}_{ij} > 0$. Obviously, c°_{ij} is a positive number measured from 0, and $c^{\circ}_{ij} \in [\bar{c}_{ij} - t_{n-1}(\alpha_{ij_1})s^*_{ij},\ \bar{c}_{ij} + t_{n-1}(\alpha_{ij_2})s^*_{ij}]$. Hence, c°_{ij} is an estimate of the edge weight c_{ij} in the fuzzy sense. In particular, if $\alpha_{ij1} = \alpha_{ij2}$ and $\beta_{ij1} = \beta_{ij2}$, then we have $t_{n-1}(\alpha_{ij1}) = t_{n-1}(\alpha_{ij2})$ and $t_{n-1}(\beta_{ij1}) = t_{n-1}(\beta_{ij2})$, which leads to $c^{\circ}_{ij} = \bar{c}_{ij}$. The DM should choose appropriate values for parameters $\alpha_{ii'_1k}, \alpha_{i'_1i'_2k}, \ldots, \alpha_{i'_{m(i')}nk}, \alpha_{ijk}, \alpha_{jj'_1k}, \ldots, \alpha_{j'_{m(j')}nk}, \beta_{ii'_1k}, \ldots, \beta_{i'_{m(i)}nk}, \beta_{ijk}, \beta_{jj'_1k}, \ldots, \beta_{j'_{m(j')}nk}$, $k = 1, 2$, to satisfy the following equalities

$$\mu_{ii'_1} + \mu_{i'_1i'_2} + \ldots + \mu_{i'_{m(i')}n} \leq \mu_{ij} + \mu_{jj'_1} + \mu_{j'_1j'_2} + \ldots + \mu_{j'_{m(j')}n},\ \forall i < j, <i, j> \in E \tag{32}$$

Similar to (21) of section 4, after adding one sixteenth of (32) to (25), we derive

$$d^{\circ}(\tilde{\tilde{c}}_{ii'_1} \oplus \tilde{\tilde{c}}_{i'_1i'_2} \oplus \ldots \oplus \tilde{\tilde{c}}_{i'_{m(i')}n}) \ \leq\ d^{\circ}(\tilde{\tilde{c}}_{ij} \oplus \tilde{\tilde{c}}_{jj'_1} \oplus \tilde{\tilde{c}}_{j'_1j'_2} \oplus \ldots \oplus \tilde{\tilde{c}}_{j'_{m(j')}n}), \tag{33}$$

where at least one equal sign holds. From Properties 3 and 4, and Definition 7, (33) is equivalent to

$$\tilde{\tilde{c}}_{ii'_1} \oplus \tilde{\tilde{c}}_{i'_1i'_2} \oplus \ldots \oplus \tilde{\tilde{c}}_{i'_{m(i')}n} \ \precsim\ \tilde{\tilde{c}}_{ij} \oplus \tilde{\tilde{c}}_{jj'_1} \oplus \tilde{\tilde{c}}_{j'_1j'_2} \oplus \ldots \oplus \tilde{\tilde{c}}_{j'_{m(j')}n}, \tag{34}$$

where at least one $\approx$ holds. Note that the relations $\prec$ and $\approx$ are the rankings defined on $F_{IN}(1 - \beta, 1 - \alpha)$ (see Definition 7). From Properties 3 and 4, Definition 7, (33), and (34), we obtain

$$c^{\circ}_{ii'_1} + c^{\circ}_{i'_1i'_2} + \ \ldots + c^{\circ}_{i'_{m(i')}n} \ \leq\ c^{\circ}_{ij} + c^{\circ}_{jj'_1} + c^{\circ}_{j'_1j'_2} + \ldots + c^{\circ}_{j'_{m(j')}n},\ \forall i < j, <i, j> \in E, \tag{35}$$

where at least one equal sign holds.

Remark 2. Note that (25) and (32) have the same relation property.

Let $f^{\circ}(i)$ be the length of the shortest path from vertex i to vertex n in a network $G = (V, E)$ with $\{c^{\circ}_{ij} \mid <i, j> \in E\}$. Similar to (18), we obtain

$$f^{\circ}(i) = c^{\circ}_{ii'_1} + c^{\circ}_{i'_1i'_2} + \ \ldots + c^{\circ}_{i'_{m(i')}n} \quad \text{and} \quad f^{\circ}(j) = c^{\circ}_{jj'_1} + c^{\circ}_{j'_1j'_2} + \ \ldots + c^{\circ}_{j'_{m(j')}n}. \tag{36}$$

From (35) and (36), the DP recursion of the second type of shortest-path problem in the fuzzy sense is given by $f^{\circ}(i) = \min_{i<j}\{c^{\circ}_{ij} + f^{\circ}(j) \mid <i, j> \in E\}$ and $f^{\circ}(n) = 0$. We now summarize the above description in Theorem 2.

Theorem 2. Consider a network $G = (V, E)$ with n vertices numbered from 1 to n with edge weights $\{c_{ij} \mid <i, j> \in E\}$. An estimate of the edge weight, c°_{ij}, which is based on the level $(1-\beta, 1-\alpha)$ interval-valued fuzzy number given by (29), is defined by $c_{ij}^{\circ} = \bar{c}_{ij} + \frac{1}{16}\{(t_{n\text{-}1}(\beta_{ij2}) - t_{n\text{-}1}(\beta_{ij1}))\, s_{ij}^{*} + (4-3\lambda)(t_{n\text{-}1}(\alpha_{ij2}) - t_{n\text{-}1}(\alpha_{ij1}))\, s_{ij}^{*}\} = \bar{c}_{ij} + \frac{1}{16}\mu_{ij}$, $\lambda = \frac{1-\beta}{1-\alpha}$. A set of edge weights in the fuzzy sense is given by $\{c_{ij}^{\circ} \mid < i, j > \in E\}$. The DP recursion of the second type of shortest-path problem in the fuzzy sense is then given by

$$f^{\circ}(i) = \min_{i<j}\{c_{ij}^{\circ} + f^{\circ}(j) \mid < i, j > \in E\}, \text{ and}$$
$$f^{\circ}(n) = 0,$$

where $f^{\circ}(i)$ is the length of the shortest path in the fuzzy sense from vertex i to vertex n.

Example 3. The problem is the same as Example 1. Suppose we have the following point estimates from past statistical data: $\bar{c}_{12} = 3.12$, $\bar{c}_{13} = 2.05$, $\bar{c}_{14} = 4.5$, $\bar{c}_{23} = 0.95$, $\bar{c}_{25} = 3.9$, $\bar{c}_{35} = 6.1$, $\bar{c}_{46} = 3.95$, $\bar{c}_{56} = 1.1$, $\bar{c}_{58} = 6.25$, $\bar{c}_{67} = 2.4$, $\bar{c}_{78} = 4.3$, $s_{12}^{*} = 1.5$, $s_{13}^{*} = 0.8$, $s_{14}^{*} = 1.6$, $s_{23}^{*} = 0.5$, $s_{25}^{*} = 2$, $s_{35}^{*} = 2.5$, $s_{46}^{*} = 0.9$, $s_{56}^{*} = 0.7$, $s_{58}^{*} = 1.8$, $s_{67}^{*} = 0.8$, and $s_{78}^{*} = 1.9$. In Table 3, we put the edge weights, $\bar{c}_{ij}$, into the correct entries of the Table. Next, we calculate *f*(*j*) using the same tabular method as in Example 2.

The tabular method for finding the shortest path in the fuzzy sense is shown in Table 3. Using the same approach as in Table 1, the shortest path based on statistical data obtained from $\bar{f}(1) = \bar{c}_{12} + \bar{f}(2) = \bar{c}_{12} + \bar{c}_{25} + \bar{f}(5) = \bar{c}_{12} + \bar{c}_{25} + \bar{c}_{58} + \bar{f}(8) = \bar{c}_{12} + \bar{c}_{25} + \bar{c}_{58}$, is 1, 2, 5, 8 with length 13.27.

Table 3. Finding the shortest path in the fuzzy sense.

f	13.27	10.15	12.35	10.65	6.25	6.70	4.30	0	*i*
j	1	2	3	4	5	6	7	8	
		3.12	2.05	4.5					1
			0.95		3.9				2
					6.1				3
						3.95			4
						1.1		6.25	5
							2.4		6
								4.3	7
									8

Now consider the fuzzy case. Similar to Example 2, a row with two or more entries implies an inequality. After searching Table 3 for inequalities, we obtain rows i = 1, 2, and 5, and hence, the following inequalities which satisfy (25):

When $i = 1$, $\bar{c}_{12} + \bar{f}(2) < \bar{c}_{13} + \bar{f}(3)$, i.e., $\bar{c}_{12} + \bar{c}_{25} + \bar{c}_{58} < \bar{c}_{13} + \bar{c}_{35} + \bar{c}_{58}$,
or $\bar{c}_{12} + \bar{f}(2) < \bar{c}_{14} + \bar{f}(4)$, i.e., $\bar{c}_{12} + \bar{c}_{25} + \bar{c}_{58} < \bar{c}_{14} + \bar{c}_{46} + \bar{c}_{67} + \bar{c}_{78}$.
When $i = 2$, $\bar{c}_{25} + \bar{f}(5) < \bar{c}_{23} + \bar{f}(3)$, i.e., $\bar{c}_{25} + \bar{c}_{58} < \bar{c}_{23} + \bar{c}_{35} + \bar{c}_{58}$.
When $i = 5$, $\bar{c}_{58} + \bar{f}(8) < \bar{c}_{56} + \bar{f}(6)$, i.e., $\bar{c}_{58} < \bar{c}_{56} + \bar{c}_{67} + \bar{c}_{78}$.

According to Remark 2, the parameters of (32) based on the above inequalities are derived as

$\mu_{12} + \mu_{25} + \mu_{58} < \mu_{13} + \mu_{35} + \mu_{58}$, (37)
$\mu_{12} + \mu_{25} + \mu_{58} < \mu_{14} + \mu_{46} + \mu_{67} + \mu_{78}$, (38)
$\mu_{25} + \mu_{58} < \mu_{23} + \mu_{35} + \mu_{58}$, and (39)
$\mu_{58} < \mu_{56} + \mu_{67} + \mu_{78}$. (40)

Let $\alpha = 0.05$, $\alpha_{121} = \alpha_{251} = \alpha_{581} = 0.03$, $\alpha_{122} = \alpha_{252} = \alpha_{582} = 0.02$, $\alpha_{131} = \alpha_{141} = \alpha_{231} = \alpha_{351} = \alpha_{461} = \alpha_{561} = \alpha_{671} = \alpha_{781} = 0.035$, and $\alpha_{132} = \alpha_{142} = \alpha_{232} = \alpha_{352} = \alpha_{462} = \alpha_{562} = \alpha_{672} = \alpha_{782} = 0.015$. We can see that $\alpha_{ij1} + \alpha_{ij2} = 0.05 = \alpha$. From the Student's t-distribution table with 29 degrees of freedom [13], we have $t_{29}(0.03) = 1.9758$, $t_{29}(0.02) = 2.184$, $t_{29}(0.035) = 1.9066$, and $t_{29}(0.015) = 2.323$.

Let $\beta = 0.1$. Since $0 < \alpha_{ijk} < \beta_{ijk} < 1$, $k = 1, 2$, we let $\beta_{121} = \beta_{251} = \beta_{581} = 0.055$, $\beta_{122} = \beta_{252} = \beta_{582} = 0.045$, $\beta_{131} = \beta_{141} = \beta_{231} = \beta_{351} = \beta_{461} = \beta_{561} = \beta_{671} = \beta_{781} = 0.07$, and $\beta_{132} = \beta_{142} = \beta_{232} = \beta_{352} = \beta_{462} = \beta_{562} = \beta_{672} = \beta_{782} = 0.03$. Then we have $\beta_{ij1} + \beta_{ij2} = 0.1 = \beta$. According to the Student' t-distribution with 29 degrees of freedom, we have $t_{29}(0.055) = 1.6602$, $t_{29}(0.045) = 1.7682$, $t_{29}(0.07) = 1.5438$, and $t_{29}(0.03) = 1.9758$.

Since $\alpha = 0.05$ and $\beta = 0.1$, we obtain $\lambda = \frac{1-\beta}{1-\alpha} = 0.9474$. Note that $\mu_{ij} = (t_{n-1}(\beta_{ij2}) - t_{n-1}(\beta_{ij1})\, s_{ij}^* + (4 - 3\lambda)(t_{n-1}(\alpha_{ij2}) - t_{n-1}(\alpha_{ij1}))\, s_{ij}^*$. Then we obtain $\mu_{12} = 0.5236$, $\mu_{13} = 0.7313$, $\mu_{14} = 1.4626$, $\mu_{23} = 0.4570$, $\mu_{25} = 0.6981$, $\mu_{35} = 2.2850$, $\mu_{46} = 0.8226$, $\mu_{56} = 0.6398$, $\mu_{58} = 0.6284$, $\mu_{67} = 0.7312$, and $\mu_{78} = 1.7366$. Because the above parameters satisfy (35)-(38), we derive the level (0.9, 0.95) interval-valued fuzzy number $\tilde{\tilde{c}}_{ij} = [\tilde{\tilde{c}}_{ij}^L, \tilde{\tilde{c}}_{ij}^U]$, where $\tilde{\tilde{c}}_{ij}^L = (\bar{c}_{ij} - t_{n-1}(\beta_{ij1})s_{ij}^*, \bar{c}_{ij}, \bar{c}_{ij} + t_{n-1}(\beta_{ij2})s_{ij}^*; 0.9)$, and $\tilde{\tilde{c}}_{ij}^U = (\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})s_{ij}^*, \bar{c}_{ij}, \bar{c}_{ij} + t_{n-1}(\alpha_{ij2})s_{ij}^*; 0.95)$. From Theorem 2, we obtain $c_{12}^\circ = 3.1527$, $c_{13}^\circ = 2.0957$, $c_{14}^\circ = 4.5914$, $c_{23}^\circ = 0.9786$, $c_{25}^\circ = 3.9436$, $c_{35}^\circ = 6.2428$, $c_{46}^\circ = 4.0014$, $c_{56}^\circ = 1.1399$, $c_{58}^\circ = 6.2892$, $c_{67}^\circ = 2.4457$, and $c_{78}^\circ = 4.4085$. Thus the fuzzy network G = (V, E) with $\{c_{ij}^\circ \mid <i, j> \in E\}$ is constructed, as shown in Fig. 7. The tabular method for finding the shortest path is shown in Table 4. Using the same approach as in Table 1, the shortest path in the fuzzy sense obtained from $f^\circ(1) = c_{12}^\circ + f^\circ(2) = c_{12}^\circ + c_{25}^\circ + f^\circ(5) = c_{12}^\circ + c_{25}^\circ + c_{58}^\circ + f^\circ(8) = c_{12}^\circ + c_{25}^\circ + c_{58}^\circ$, is 1, 2, 5, 8 with length 13.3855. The shortest path in the fuzzy sense is longer than the shortest path based on statistical data by $\frac{f^\circ(1) - \bar{f}(1)}{\bar{f}(1)} \times 100\% = 0.87\%$.

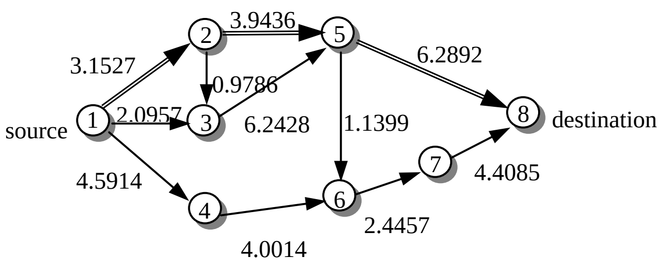

Fig. 7. The fuzzy network $G = (V, E)$ with $\{c_{ij}^\circ \mid <i, j> \in E\}$.

Notice that the length of the shortest path in the fuzzy sense obtained from Theorem 2 is related to the values of parameters μ_{ij} in (32), which are determined by the DM.

6. DISCUSSION

This study has produced the following significant results for the fuzzy shortest-path problem in network.

Table 4. Finding the shortest path in the fuzzy sense.

$f^{\circ}(j)$	13.3855	10.2328	12.532	10.8556	6.2892	6.8542	4.4085	0	i
j	1	2	3	4	5	6	7	8	
		3.1527	2.0957	4.5914					1
			0.9786		3.9436				2
					6.2428				3
						4.0014			4
						1.1399		6.2892	5
							2.4457		6
								4.4085	7
									8

(1) In Theorem 1, if $\Delta_{ij1} = \Delta_{ij2}$ for each $<i, j> \in E$, then $c_{ij}^{*} = c_{ij}$. This means that the set of fuzzy edge weights, $\{c_{ij}^{*} \mid < i, j > \in E\}$, becomes the set of crisp edge weights, $\{c_{ij} \mid < i, j > \in E\}$. We conclude, therefore, that the crisp shortest-path problem is a special case of the fuzzy shortest-path problem.

(2) Also in Theorem 1, if $\Delta_{ij1} < \Delta_{ij2}$ for each $<i, j> \in E$, then $c_{ij}^{*} > c_{ij}$. This means that the fuzzy shortest path is longer than the crisp shortest path. Conversely, if $\Delta_{ij1} > \Delta_{ij2}$ for each $<i, j> \in E$, then $c_{ij}^{*} < c_{ij}$, and so the fuzzy shortest path is shorter than the crisp path.

(3) In Theorem 2, if $\beta_{ij1} = \beta_{ij2}$ for each $<i, j> \in E$ and $\beta = 1$, we obtain $c_{ij}^{\circ} = \bar{c}_{ij} + \frac{1}{4}(t_{n-1}(\alpha_{ij2}) - t_{n-1}(\alpha_{ij1}))\, s_{ij}^{*}$. Here, c_{ij}° is the defuzzified value of level $(1 - \alpha)$ fuzzy number $\tilde{\tilde{c}}_{ij} = (\bar{c}_{ij} - t_{n-1}(\alpha_{ij1})\, s_{ij}^{*},\ \bar{c}_{ij},\ \bar{c}_{ij} + t_{n-1}(\alpha_{ij2})\, s_{ij}^{*}; (1-\alpha)$. Let $c_{ij}^{**} = \bar{c}_{ij} + \frac{1}{4}(t_{n-1}(\alpha_{ij2}) - t_{n-1}(\alpha_{ij1}))\, s_{ij}^{*}$. Thus, Theorem 3, a special case of Theorem 2, can be derived as follows.

Theorem 3. In Theorem 2, when $\beta_{ij1} = \beta_{ij2}$ for each $<i, j> \in E$ and $\beta = 1$, we have the following fuzzy shortest-path problem, which is derived from level $(1 - \alpha)$ fuzzy numbers,

$$f^{**}(i) = \min_{i<j}\{c_{ij}^{**} + f^{**}(j) \mid < i, j > \in E\}, \text{ and}$$

$$f^{**}(n) = 0,$$

where $f^{**}(i)$ is the length of the shortest path in the fuzzy sense from vertices i to n in the network $G = (V, E)$ with $\{c_{ij}^{**} \mid <i, j> \in E\}$.

(4) Again, in Theorem 2, if $\alpha_{ij1} = \alpha_{ij2}$ and $\beta_{ij1} = \beta_{ij2}$ for each $<i, j> \in E$, then $c_{ij}^{\circ} = \bar{c}_{ij}$. Thus Theorem 2 becomes the shortest-path network problem $G = (V, E)$ with the edge weights $\{\bar{c}_{ij} \mid <i, j> \in E\}$, where $\bar{c}_{ij}$ is a point estimate.

(5) Consider the choice for the values of parameters α_{ijk} and β_{ijk}, $k = 1, 2$, in Theorem 2. Let c_{ijq}, $q = 1, \ldots, n$, be n samples of c_{ij} and let $\bar{c}_{ij} = \frac{1}{n}\sum_{q=1}^{n} c_{ijq}$ be the mean. Let $A = \{q \mid c_{ijq} < \bar{c}_{ij}, q = 1, \ldots, n\}$ and $B = \{q \mid c_{ijq} \geq \bar{c}_{ij}, q = 1, \ldots, n\}$. Let the number of elements in A and B be denoted by $n(A)$ and $n(B)$, respectively. Then the variances are $s_{jkA}^2 = \frac{1}{n(A)}\sum_{q \in A}(x_{ijq} - \bar{x}_{ij})^2$ and $s_{jkB}^2 = \frac{1}{n(B)}\sum_{q \in B}(x_{ijq} - \bar{x}_{ij})^2$. If $s_{jkB}^2 < s_{jkA}^2$, consequently, the DM should choose $\alpha_{ij1} < \alpha_{ij2}$ and $\beta_{ij1} < \beta_{ij2}$ satisfying $0 < \alpha_{ijk} < \beta_{ijk} < 1$, $k = 1, 2$. This implies $t_{n-1}(\alpha_{ij2}) < t_{n-1}(\alpha_{ij1})$ and $t_{n-1}(\beta_{ij2}) < t_{n-1}(\beta_{ij1})$. This result is interpreted in Fig. 6 as follows. When $t_{n-1}(\alpha_{ij2}) < t_{n-1}(\alpha_{ij1})$ and $t_{n-1}(\beta_{ij2}) < t_{n-1}(\beta_{ij1})$, two triangles go to the left side. On the contrary, when $t_{n-1}(\alpha_{ij2}) > t_{n-1}(\alpha_{ij1})$ and $t_{n-1}(\beta_{ij2}) > t_{n-1}(\beta_{ij1})$, i.e., $s_{jkB}^2 > s_{jkA}^2$, two triangles go to the right side.

7. CONCLUSIONS

In this paper, two types of fuzzy shortest-path problems in networks were presented. The first problem, which uses triangular fuzzy numbers to represent imprecise edge weights, was discussed in section 4, while section 5 discussed the second problem. Each level $(1 - \beta, 1 - \alpha)$ interval-valued fuzzy number represents the confidence interval for the unknown edge weight, which is obtained based on past statistical data. A DP recursion formulation and a tabular calculation method were developed to solve these problems. Examples were given to illustrate their solutions. The proposed approach combining statistics with fuzzy sets for obtaining the fuzzy shortest-path in a network is very useful for solving practical problems.

APPENDIX

Comparing Klein's Method With Ours

The hybrid multi-criteria DP recursion proposed by Klein [8] is

$$f(N) = (1, 1, 1, \ldots, 1),$$

$$f(i) = \underset{(i,j) \in E}{dom}(e_{ij} \tilde{+} f(j)), \tag{41}$$

where $e_{ij} = (\mu_1(i, j), \mu_2(i, j), \ldots, \mu_R(i, j))$, is the path from i to j in the network. The operator $\tilde{+}$ represents the combinatorial sum, and *dom* is the domination operator based on "more is better". The combinatorial sum of two-tuples is defined as follows. Let Z = min{$\mu_x(j, k)$, $\mu_y(k, q)$}. Then = $e_{j,k} \tilde{+} e_{k,q}$ where the i-th element of the R-tuple $e_{j,q}$ is given by

$$(e_{j,q})^i = \max_{x+y=i}(Z) = \mu_i(j, q).$$

The recursion in (41) yields the set of nondominated paths from source 1 to destination N. The fuzzy shortest path length is then defined by

$$\tilde{P}_{1,N} = \{1/(\max_i(\mu_1^i(1, N)), \ldots, K/(\max_i(\mu_k^i(1, N)), \ldots, R/(\max_i(\mu_R^i(1, N))\} \quad (42)$$

where $\mu_k^i(s,t)$ represents the membership grade in fuzzy set K of the path from vertex s to vertex t given by the ith nondominated R-tuple.

The major differences between [8] and this paper are as follows:

(1) The edge weights and recurrence formula used by Klein are different from those used in this paper. In Klein's method, there is a path associated with each fuzzy set and its membership function for edge weights. For example, assume that each edge can take a length of 1, 2, and 3. The three-tuple associated with each edge gives the membership value of the edge in each of the fuzzy sets 1, 2, and 3 (see example in [8]). With the defined relationships between paths and their respective edges, a variety of objectives for the model can be considered. These models are then solved using a hybrid multi-criteria DP in (41). In this study, however, we presented two types of fuzzy shortest-path problems. In section 4, we used the triangular fuzzy number, $\tilde{c}_{ij} = (c_{ij} - \Delta_{ij1}, c_{ij}, c_{ij} + \Delta_{ij2})$, to represent the imprecise edge weight c_{ij} as the first type of fuzzy shortest-path problem. In section 5, the level $(1 - \beta, 1 - \alpha)$ interval-valued fuzzy number based on past statistical data, $\tilde{\tilde{c}}_{ij} = [\tilde{\tilde{c}}_{ij}^L, \tilde{\tilde{c}}_{ij}^U]$, is used to represent the unknown edge weight c_{ij} as the second type of fuzzy shortest-path problem. The recurrence formulas used in this paper are

$$\tilde{f}(n) = \tilde{0}_1, \quad \tilde{f}(i) = \min_{i<j}\{\tilde{c}_{ij} + \tilde{f}(j) \mid <i, j> \in E\} \text{ and}$$

$$\tilde{f}(n) = \tilde{0}_1, \quad \tilde{f}(i) = \min_{i<j}\{\tilde{\tilde{c}}_{ij} + \tilde{f}(j) \mid <i, j> \in E\},$$

where $\tilde{f}(i)$ is the fuzzy shortest path distance from i to N.

(2) The viewpoint of fuzzy shortest-path in [8] differs from that in this paper. In Klein's work, a DM could determine the length of the fuzzy shortest path using a certain membership grade in (42) (see example in [8]). The path obtained will correspond to a path in the original network and is formed by the general backtracking techniques of DP in (41). In this paper, the DM cannot determine the length of the fuzzy shortest path using membership grade. Two theorems were proposed to find the shortest path in the fuzzy sense. The fuzzy shortest paths obtained from our theorems correspond to actual paths in the network, and the fuzzy shortest-path problems are an extension of the original problem. The recurrence formula for obtaining fuzzy shortest path in Theorem 1 is $f^*(i) = \min_{i<j}\{c_{ij}^* + f^*(j) \mid <i, j> \in E\}$ and $f^*(n) = 0$, where $c_{ij}^* = c_{ij} +$

$\frac{1}{4}(\Delta_{ij2} - \Delta_{ij1}) = c_{ij} + \frac{1}{4}\Delta_{ij}$. Meanwhile, the recurrence formula for obtaining fuzzy shortest path in Theorem 2 is $f^\circ(i) = \min_{i<j}\{c^\circ_{ij} + f^\circ(j) \mid <i, j> \in E\}$, and $f^\circ(n) = 0$, where $c^\circ_{ij} = \bar{c}_{ij} + \frac{1}{16}\{(t_{n-1}(\beta_{ij2}) - t_{n-1}(\beta_{ij1}))\, s^*_{ij} + (4 - 3\lambda)(t_{n-1}(\alpha_{ij2}) - t_{n-1}(\alpha_{ij1}))\, s^*_{ij}\}$.

(3) In [8], the primary result is to develop algorithms for fuzzy shortest-path models based on DP procedures and multi-criteria optimization. In this paper, an approach that combines fuzzy sets with statistics to solve the fuzzy shortest-path problem with unknown edge weights is presented. The proposed approach is very useful for solving practical problems. The primary result obtained from this study is a theorem through which the shortest-path in the fuzzy sense is obtained based on statistical confidence-interval estimates for the uncertain edge weights problem.

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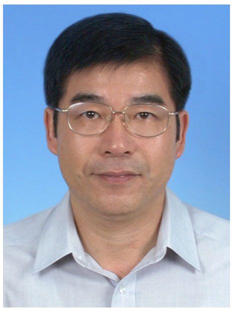

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