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Theory and Methodology

A stochastic and dynamic model for the single-vehicle pick-up and delivery problem

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Abstract

In this paper a stochastic and dynamic model for the Pick-up and Delivery Problem is developed and analyzed. Demands for service arrive according to a Poisson process in time. The pick-up locations of the demands are independent and uniformly distributed over a service region. A single vehicle must transport the demands from the pick-up to the delivery location. Once a demand has been picked up it can only be dropped off at its desired delivery location. The delivery locations are independent and uniformly distributed over the region, and they are independent of the pick-up locations. The objective is to minimize the expected time in the system for the demands. Unit-capacity vehicle and multiple-capacity vehicle variations are considered. For each variation, bounds on the performance of the routing policies are derived for light and heavy traffic. The policies are analyzed using both analytical methods and simulation.

Keywords: Stochastic, dynamic vehicle routing; Queueing

1. Introduction

In this paper the single-vehicle *Dynamic Pick-up and Delivery Problem* (DPDP) is developed and analyzed. Requests for service arrive in time according to a Poisson process in some given service region. The pick-up locations of the customers are independent and uniformly distributed in the region. Each customer requires to be transported to a delivery location. The delivery locations are also independent and uniformly distributed within the region. The pick-up locations of the customers are independent of their delivery locations.

A single service vehicle must transport the customers from their pick-up locations to their delivery locations. The vehicle travels according to the Euclidean metric, and its velocity is constant. Two variations of the single-vehicle DPDP are considered: the *unit*-capacity vehicle and the *multiple*-capacity vehicle.

The objective of the problem is to minimize the expected time the demands spend in the system, and *not* to minimize the expected distance that the vehicle travels. There are often costs associated

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with the time a demand spends in system, which are not directly addressed by focusing on minimizing the expected travel distance. Note that these costs may not be quantifiable, and that to some extent minimizing the time in system will tend to keep the travel distance from being high. The customers can only be dropped off at their desired delivery locations in the DPDP.

A similar vehicle routing problem that was analyzed in a dynamic and stochastic framework is the *Dynamic Traveling Repairman Problem* (DTRP) introduced by Bertsimas and van Ryzin [1]. Other results for the DTRP have been developed in [2,5]. The primary difference between the DTRP and the DPDP is that in the DTRP the vehicle spends time at the location of each demand to service it, and does not transport demands. Thus in the DTRP, the vehicle does *not* change location while servicing a demand, but the vehicle *does* change location in the DPDP during service. The relationship between the DTRP and the DPDP is similar to the relationship between the TSP and the *Stacker Crane Problem* [4].

The dynamic formulation of vehicle routing problems was discussed by Psaraftis in [6], where the *Dynamic Traveling Salesman Problem* was defined. In the Dynamic Traveling Salesman Problem, demands for service arrive to each node in a graph according to a Poisson process. The demands must be serviced on-site. The objective of the problem is to maximize the expected number of demands serviced within some period of time or to minimize the expected time in system for the demands.

There is a large body for knowledge on vehicle routing and queueing theory, but the area where these two fields interface is not as well developed. The DPDP attempts to investigate systems that are subject to the effects of queueing phenomena and vehicle routing simultaneously [8]. The problem is applicable to taxicab companies and limousine services. These services require the transportation of demands from pick-up locations to delivery locations. The Dynamic Pick-up and Delivery Problem can be used for the analysis of such urban transportation systems.

In Section 2, several preliminary results are briefly discussed. In Section 3, the DPDP is defined and the notation is introduced. In Section 4, the optimal routing policy in light traffic is derived for the unit-capacity vehicle DPDP, and several routing policies are analyzed in heavy traffic. In Section 5, routing policies for the multiple-capacity vehicle DPDP are considered. The performance of these policies is compared in Section 6, and some concluding remarks are presented in Section 7.

2. Probabilistic and queueing background

Some well-known results employed in the analysis are briefly discussed.

1. Given a sequence of independent identically distributed random variables $Z_i (i = 1, 2, \dots, M)$ and a random positive integer M, independent of all of the Z_i's, let

$$U = \sum_{i=1}^{M} Z_i. \tag{1}$$

It is known [7] that

$$V[U] = V[M]E^2[Z] + E[M]V[Z], \tag{2}$$

where $E[M]$ and $E[Z]$ denote the expected values, and $V[M]$ and $V[Z]$ denote the variances of M and Z, respectively.

2. For an M/G/1 queue the expected waiting time in queue [3] is

$$W = \frac{\lambda E[S^2]}{2(1-\rho)} = \frac{\lambda(V[S] + E^2[S])}{2(1-\lambda E[S])}, \tag{3}$$

where λ is the arrival rate, $E[S]$ is the mean, $E[S^2]$ is the second moment, and $V[S]$ is the variance of the service time distribution. The traffic intensity is $\rho = \lambda E[S]$.

3. For a GI/G/1 queue, an upper bound for the expected waiting time in queue is

$$W \leqslant \frac{\lambda(V[A] + V[S])}{2(1-\rho)}, \tag{4}$$

where $V[A]$ is the variance of the interarrival time, $V[S]$ is the variance of the service time, λ is the arrival rate, and ρ is the traffic intensity [3]. The bound becomes asymptotically exact as $\rho \to 1$.

4. If L_n is the length of the optimal Traveling Salesman Problem (TSP) tour through n independent, uniformly distributed points in a convex region of area A, then

$$\lim_{n\to\infty} E\left[\frac{L_n}{\sqrt{n}}\right] = \beta\sqrt{A}, \tag{5}$$

and

$$\lim_{n\to\infty} \frac{V[L_n]}{n} = 0, \tag{6}$$

where β is the TSP constant ($\beta \approx 0.712$) [4].

5. Let Y_1 be the distance between two independent uniformly distributed points in the unit square. Let Y_2 be the distance between one uniformly distributed point in the unit square and the center of the square. From geometrical probability [1,3]: $c_1 = E[Y_1] \simeq 0.52, c_2 = E[Y_1^2] \simeq 0.33, c_3 = E[Y_2] \simeq 0.383, c_4 = E[Y_2^2] \simeq 0.167, \sigma_1^2 = V[Y_1] \simeq 0.06$.

3. Problem definition and notation

A service vehicle is located in a square region $\mathscr{A}$, of area A. (The results can be readily extended for any convex bounded region $\mathscr{A}$.) Demands for service arrive according to a Poisson process in time with arrival rate λ. The demand pick-up locations are independent and uniformly distributed in the service region. Each demand must be transported from its pick-up location to its delivery location. The delivery locations are also independent and uniformly distributed throughout the service region, and are independent of the pick-up locations. The service vehicle travels at a constant velocity v and is either of unit-capacity or of multiple-capacity. The traffic intensity is defined as $\rho = \lambda\bar{s}$, where $\bar{s}$ is the expected time for the vehicle to travel from the pick-up location of a demand directly to its delivery location. The distance between points is determined according to a Euclidean metric.

The system is analyzed under steady state conditions. The objective is to minimize T, the expected time in the system for the demands in steady state. The expected time demands spend waiting in queue in steady state is W, and the expected number of demands waiting in queue in steady state is N. The value of T resulting from an optimal policy, when one exists, is T^*.

4. The single-vehicle unit-capacity DPDP

In the first version of the problem considered, a single vehicle transports one demand at a time. Two operating conditions will be examined: light traffic ($\lambda \to 0$) and heavy traffic $\left(\lambda \to v/c_1\sqrt{A}\right)$.

4.1. Light traffic conditions

4.1.1. A lower bound on performance

Let X_0^* be the location of the *stochastic median*, which is defined as

$$X_0^* = \arg\ \min\{E[||X - X_0||]\ \forall X_0 \in \mathscr{A}\}, \tag{7}$$

where X is a uniformly distributed point in $\mathscr{A}$.

Proposition 1. *A lower bound on the expected time demands spend in the system is*

$$T^* \geqslant \frac{1}{v}E[||X - X_0^*||] + \bar{s}. \tag{8}$$

This is derived using a method similar to that for the DTRP in [1]. Intuitively, Eq. (8) can be obtained by noting that the time a demand spends in the system includes the time for the vehicle to travel to the pick-up location of that demand (that is the time that the vehicle travels empty plus the time the vehicle travels while carrying other demands) and the time to service that demand.

Corollary 1. *If $\mathscr{A}$ is a square of area A, then $E[||X - X_0^*||]$ is the expected distance between one uniformly distributed point in the square and the center of the square. $\bar{s}$ is the distance between two independent uniformly distributed points in the square. Thus,*

$$T^* \geqslant c_3\frac{\sqrt{A}}{v} + c_1\frac{\sqrt{A}}{v} \simeq 0.90\frac{\sqrt{A}}{v}. \tag{9}$$

4.1.2. The first-come–first-served policy

In the *First-Come–First-Served* (FCFS) Policy the vehicle services demands in the order of arrival. If the vehicle drops off a demand and there are no waiting demands, the vehicle waits at the last demand's delivery location. Note that in this system successive service times are not independent, and so any use of M/G/1 results is an approximation.

The expected time in the system for a demand is

$$T_{\mathrm{FCFS}} \simeq \frac{\lambda(2c_2A/v^2 + 2E[Y_1Y_3])}{2\left(1 - 2\lambda c_1 \frac{\sqrt{A}}{v}\right)} + 2c_1\frac{\sqrt{A}}{v}. \tag{10}$$

This is derived by modeling the system as an M/G/1 queue using Eq. (3). The service time for this model includes two components: Y_1, the time for the vehicle to travel empty from the previous delivery location to the current pick-up location, and Y_3, the time for the vehicle to transport the demand to its delivery location.

The first moment for this service time is the expected time to travel from a pick-up location to a delivery location plus the expected time to travel from a delivery location to the next pick-up location. It follows that

$$E[Y_1 + Y_3] = c_1\frac{\sqrt{A}}{v} + c_1\frac{\sqrt{A}}{v} = 2c_1\frac{\sqrt{A}}{v}, \tag{11}$$

and the second moment is

$$\begin{aligned} E\left[(Y_1 + Y_3)^2\right] &= E[Y_1^2] + 2E[Y_1Y_3] + E[Y_3^2] \\ &= 2\frac{c_2A}{v} + 2E[Y_1Y_3]. \end{aligned} \tag{12}$$

Consecutive service times are not independent. This can be seen for the case where the distance between the pick-up and delivery points for a demand is close to $\sqrt{2A}$, which implies that the next pick-up to delivery distance will tend to be long. The results for the M/G/1 queue are used as an approximation. Traffic intensity is very light, so the queueing effects will be small. It follows that the absolute difference between the approximate expected time in system and the actual expected time in system will be quite small.

Taking the ratio of the expected time of (10) to the bound of (9) results in

$$\lim_{\lambda\to 0}\frac{T_{\mathrm{FCFS}}}{T^*} \leqslant \frac{2c_1\sqrt{A}/v}{c_3\sqrt{A}/v + c_1\sqrt{A}/v} \simeq 1.15. \tag{13}$$

4.1.3. The stochastic queue median policy

The *Stochastic Queue Median* (SQM) Policy is similar to the FCFS Policy with one difference: after *each* demand is delivered the vehicle returns to a depot located at the stochastic median of the region. If there are no waiting demands, the vehicle waits at the depot.

The expected time in the system for a demand is

$$T_{\mathrm{SQM}} = \frac{\lambda E[(Y_2 + Y_4 + Y_5)^2]}{2(1 - \lambda E[Y_2 + Y_4 + Y_5])} + c_3\frac{\sqrt{A}}{v} + c_1\frac{\sqrt{A}}{v}, \tag{14}$$

where Y_2 is the time to travel from the stochastic median of the region to a uniformly distributed point in the region, Y_4 is the time to travel between two independent and uniformly distributed points in the region, and Y_5 is the time to travel from a uniformly distributed point to the stochastic median of the region. Eq. (14) is derived by modeling the system as an M/G/1 queue. For each demand, the vehicle must travel from the depot to the demand's pick-up location, then to demand's delivery location, and finally return to the depot.

Taking the ratio of the expected time of Eq. (14) to the bound of (8) results in

$$\lim_{\lambda\to 0}\frac{T_{\mathrm{SQM}}}{T^*} \leqslant \frac{c_3\sqrt{A}/v + c_1\sqrt{A}/v}{c_3\sqrt{A}/v + c_1\sqrt{A}/v} = 1. \tag{15}$$

Therefore, this policy becomes asymptotically optimal policy in light traffic. This policy quickly becomes unstable as λ increases. No policy can affect the expected pick-up to delivery $(P \to D)$ distance, so good policies must strive to decrease the expected delivery to pick-up $(D \to P)$ distance as the number of demands in the system increase. The SQM Policy does not take advantage of this, so it quickly becomes unstable as λ increases.

4.2. Heavy traffic conditions

In the unit-capacity vehicle formulation, the time devoted to service a particular demand is the time it takes the vehicle to travel empty to the demand's pick-up location plus the time it takes the vehicle to transport the demand to its delivery location. The time the vehicle travels empty is affected by the policy used. The time it takes the vehicle to deliver a demand is the time it takes the vehicle to travel between two independent uniformly distributed points. Thus the mean time to service a demand is bounded from below by $c_1\sqrt{A}/v$. Therefore, a necessary condition for stability is

$$\lambda c_1 \frac{\sqrt{A}}{v} < 1. \tag{16}$$

The sufficiency of (16) will now be addressed.

4.2.1. The existence of a stable policy

First, a useful lemma is established. Let N_1 and N_2 be independent binomial random variables with parameters n and p.

Lemma 1. *$E[|N_1 - N_2|]$ is* $\mathrm{O}(\sqrt{n})$.

Proof. It is known that $E[N_1] = E[N_2] = np$, and $V[N_1] = V[N_2] = np(1-p)$. Thus, $E[N_1 - N_2] = 0$, and $V[N_1 - N_2] = 2np(1-p)$. Note that if r is any positive real number, $P(-r < N_1 - N_2 < r) = P(|N_1 - N_2| < r)$. Then from Chebyshev's inequality [9], $P(-k\sqrt{V[N_1 - N_2]} < N_1 - N_2 < k\sqrt{V[N_1 - N_2]}) \geqslant 1 - 1/k^2$, and the following can be obtained:

$$\begin{aligned} E[|N_1 - N_2|] &\leqslant \sum_{k=2}^{\infty}\left[\left(1-\frac{1}{k^2}\right) - \left(1-\frac{1}{(k-1)^2}\right)\right]k\sqrt{V[N_1 - N_2]} \\ &= \left[1 + \sum_{k=1}^{\infty}\frac{1}{k^2}\right]\sqrt{V[N_1 - N_2]}. \end{aligned} \tag{17}$$

Thus,

$$E[|N_1 - N_2|] \leqslant \left[\left(1 + \frac{1}{6}\pi^2\right)\sqrt{2p(1-p)}\right]\sqrt{n}. \quad \square \tag{18}$$

The policy that is used to establish the sufficiency of the stability condition ($\lambda c_1\sqrt{A}/v < 1$) is the *Sector Touring* Policy which is defined as follows: The service region is divided into m^2 subregions, referred to as *sectors*. Each sector is a square of area A/m^2. Demands are grouped into sets composed of n customers that arrive consecutively. For each set of n demands, a tour is constructed through those demands. For each set, there must be $n-1$ connections made from delivery locations to pick-up locations within a service tour. The service tours are constructed to maximize the numbers delivery to pick-up connections where both the delivery point and the pick-up point are located in the same sector. For this policy, an optimal service tour is a tour where the number of within-sector delivery to pick-up connections is equal to the maximum possible over all of the tours. If there are multiple optimal tours, a tour is randomly selected from the optimal tours. The vehicle services the sets in FCFS order. After servicing a set, the vehicle returns to the center of the region. If there are not any complete sets in the system, the vehicle waits at the center of the region until there is a complete set.

Lemma 2. *If $\lambda c_1\sqrt{A}/v < 1$ then there exist values for n and m for the Sector Touring Policy such that the system is stable.*

Proof. The system can be viewed as a GI/G/1 queue. A complete set of n customers constitutes an arrival to the system. Thus, the interarrival distribution is nth-order Erlang. The service distribution is the time it takes the server to travel to the first item in the tour for a set, plus the time to service the n demands in a set, plus the time to return to the center of the region. Note that the service times are independent because the server returns to the center of the region after each set is serviced.

In order for the system to be stable, the expected interarrival time for sets must be longer than the expected service time for sets. Since n customer arrivals must occur for a set to arrive, and n customer services occur each time a set is serviced, it follows that for the system to be stable the expected interarrival time for customers has to be greater than the expected time to service a set divided by n.

Immediately after the arrival of the final demand in a randomly selected set, randomly select a sector. Call the set that just arrived s_0 and the selected sector m_0. Only points in s_0 will be considered for the remainder of the proof. Let N_{in} be the total number of customers with delivery locations in m_0. Let N_{out} be the total number of customers with pick-up locations in m_0. It follows that N_{in} and N_{out} are independent random variables with binomial distributions with parameters n and p, where n is the number of items is a set and $p = 1/m^2$. From Lemma 1, $E[|N_{\text{in}} - N_{\text{out}}|]$ is $O(\sqrt{n})$. Let

$$[N_{\text{in}} - N_{\text{out}}]^+ = \begin{cases} N_{\text{in}} - N_{\text{out}}, & N_{\text{in}} - N_{\text{out}} > 0, \\ 0, & N_{\text{in}} - N_{\text{out}} \leqslant 0. \end{cases}$$

Thus $E[[N_{\text{in}} - N_{\text{out}}]^+] < E[|N_{\text{in}} - N_{\text{out}}|]$, and so $E[[N_{\text{in}} - N_{\text{out}}]^+]$ is $O(\sqrt{n})$. This means that the expected number of delivery locations in m_0 that can be matched with a pick-up location in m_0 is $O(n)$ but not $o(n)$, while the expected number that cannot be matched is $O(\sqrt{n})$.

Let $T_{D(m_0)\to P}$ be the total time traveling from the delivery points in m_0 to pick-up points. Also let N_{m_0} be the number of $D \to P$ connections within m_0 and $\tilde{N}_{m_0}$ be the number of connections where the delivery point is in m_0 but the pick-up point is not in m_0. Then the total number of delivery points in m_0 is $N_{m_0} + \tilde{N}_{m_0}$. The time to travel from a delivery location in m_0 to a pick-up location in m_0 is bounded from above by $\sqrt{2A}/mv$. The time to travel from a pick-up location in m_0 to a delivery location outside of m_0 is bounded from above by $\sqrt{2A}/v$. Thus,

$$T_{D(m_0)\to P} \leqslant N_{m_0} \frac{\sqrt{2A}}{mv} + \tilde{N}_{m_0} \frac{\sqrt{2A}}{v}. \tag{19}$$

Note that $(1/n)E[N_{m_0} + \tilde{N}_{m_0}] = 1/m^2$, and thus $(1/n)E[N_{m_0}] < 1/m^2$. Since $E[N_{m_0}]$ is $O(n)$ but not $o(n)$, and $E[\tilde{N}_{m_0}]$ is $O(\sqrt{n})$, it follows that

$$\lim_{n\to\infty} E\left[\tfrac{T_{D(m_0)\to P}}{n}\right] = \lim_{n\to\infty} \frac{1}{n} E\left[T_{D(m_0)\to P}\right] \leqslant \frac{1}{m^2} \frac{\sqrt{2A}}{mv}. \tag{20}$$

Let $T_{D\to P}$ be the total time traveling from the delivery points in all sectors to pick-up points. Further, let $T_{P\to D}$ be the total time traveling from the pick-up points in all sectors to delivery points. Note that

$$E\left[\frac{T_{D\to P}}{n}\right] = \sum_{m_0\in\mathscr{S}} E\left[\frac{T_{D(m_0)\to P}}{n}\right] = m^2 E\left[\frac{T_{D(m_0)\to P}}{n}\right], \tag{21}$$

where $\mathscr{S}$ is the set of sectors. Using (20) then gives

$$\lim_{n\to\infty} E\left[\frac{T_{D\to P}}{n}\right] \leqslant \frac{\sqrt{2A}}{mv}. \tag{22}$$

Thus, for sufficiently large values for m and n, $E\left[T_{D\to P/n}\right]$ can be made arbitrarily close to zero.

Further, let T_c be the time to travel from the center to the first pick-up point in s_0 plus the time to travel from the last delivery point to the center. Then $E[T_c/n]$ is bounded from above by $\sqrt{2A}/vn$, and $E[T_{P\to D}/n]$ is always $c_1\sqrt{A}/v$. Therefore, there exist values for n and m such that the expected service time for a demand in s_0, and thus for a randomly selected set, can be made arbitrarily close to $c_1\sqrt{A}/v$. It follows that when $\lambda < v/c_1\sqrt{A}$ there exist values for n and m such that the expected service time is less than the expected inter-arrival time, and the system is stable. □

Proposition 2. *A necessary and sufficient condition for the single-vehicle unit-capacity system to be stable is* $\lambda c_1\sqrt{A}/v < 1$.

Proof. When $\lambda c_1\sqrt{A}/v < 1$, by Lemma 2 there exist values for n and m for the Sector Touring Policy such that it is stable. Therefore, whenever $\lambda c_1\sqrt{A}/v < 1$, there exists a stable policy. Thus the condition $\lambda c_1\sqrt{A}/v < 1$ is sufficient for stability. In the discussion for (16), it was argued that the condition is necessary. □

4.2.2. A heavy traffic lower bound

A lower bound that is useful in heavy traffic ($\lambda \to v/c_1\sqrt{A}$) is provided in the following proposition.

Proposition 3. *A lower bound for the expected time in the system is*

$$T^* \geqslant \frac{\gamma^2\lambda A}{v^2(1-\rho)^2} - \frac{1-2\rho}{2\lambda}, \tag{23}$$

where γ *is the constant* $\sqrt{2}/3\pi$.

Proof. This proof parallels the proof in [1]. Let $\mathscr{E}$ be any set of J points in $\mathscr{A}$, where J is a discrete random variable with a known expected value. Let X be a uniformly distributed point in $\mathscr{A}$ independent of $\mathscr{E}$. Let $Z^* \equiv \min_{x\in\mathscr{E}} \|X - x\|$. Then it can be established that

$$E[Z^*] \geqslant \frac{2\sqrt{A}}{3\sqrt{\pi E[J]}}. \tag{24}$$

This provides a lower bound on the expected distance to travel to a demand. Consider an arbitrarily tagged demand. There are three possible sources of locations from which the vehicle could travel to pick-up the demand. These include the delivery locations of demands in the system when the tagged demand arrives, the vehicle's location when the tagged demand arrives, and the delivery locations of demands that arrive after the tagged demand arrives. This gives N locations because of PASTA [10], plus 1 for the vehicle's location, plus N more due to Little's Law. Then, $E[J] = 2N + 1$. Recall that $\bar{s}$ is the expected time to transport a demand from its pick-up location to its delivery location, and let $\overline{t_{\mathrm{p}}}$ be the expected time to travel from the delivery location of a demand to the pick-up location of the next demand that will be serviced. Eq. (24) can be used to bound $\overline{t_{\mathrm{p}}}$. Using $\bar{s} + \overline{t_{\mathrm{p}}} \leqslant 1/\lambda$, and Little's Law, the proposition can be established. □

4.2.3. The sectoring policy

The *Sectoring* Policy is defined as follows: The service region is divided into m^2 subregions, referred to as *sectors*. Each sector is a square of area A/m^2. There are several selection rules employed to determine which demand to pick-up. After a vehicle delivers a demand, the next demand that is picked-up is the oldest demand whose pick-up location is in the sector where the vehicle just delivered the demand. If the vehicle delivers a demand in an empty sector, a random demand is selected from the system for the next pick-up. If there are no demands in the system, the server will pick-up the first demand that arrives. After any demand is picked-up, it is immediately delivered.

This policy was analyzed because analogous policies performed well for the DTRP [1]. The Sectoring Policy is also similar to the Sector Touring Policy, which was used in proving the sufficiency of the stability conditions. It will be shown that the Sectoring Policy performs poorly for the DPDP.

In the analysis of this policy, the following definitions are used.

Definition 1. A demand is said to be *in a sector* when the pick-up location for that demand is in that sector.

Definition 2. The vehicle is said to be *interested in a demand* when the vehicle selects that demand to be the next demand to be serviced.

Definition 3. The *primary sector* is a sector that is randomly selected for the purpose of analyzing the policy.

Definition 4. A *sector interarrival time* is the time between consecutive arrivals of demands to the primary sector.

Definition 5. A *sector service time* begins when the vehicle is first interested in a demand in the primary sector, and ends when the next demand is dropped off in the primary sector. Note that many individual demands can be serviced during one *sector service time*, so that a sector service time contains many individual service times for individual demands. A sector service time is not composed solely of individual service times, but can include additional components when sectors are empty.

Proposition 4. *For the Sectoring Policy, a heavy traffic lower bound on the expected time in the system is*

$$\lim_{\rho\to 1} T(1-\rho)^3 \geqslant \frac{9c_1 v}{4\sqrt{A}}, \tag{25}$$

where $\rho = \lambda c_1 \sqrt{A}/v$.

Proof. Let P_s be the steady state probability that the primary sector is empty when the vehicle enters that sector. Every time the vehicle drops a demand off in an empty sector, the expected service time for the next demand is larger than the expected service time for a random service. Therefore, using $P_s = 0$ in the analysis provides a lower bound for the expected waiting time, and that is sufficient to show that the policy has poor performance.

Let W_{sec} be the expected waiting time in queue of a demand in a particular sector. Let t_{p} be the time for the vehicle to go from one demand's delivery location to the next demand's pick-up location, and let $\overline{t_{\text{p}}}$ be its expected value. Let t_{d} be the time for the vehicle to go from the demand's pick-up location to its delivery location, and let $\overline{t_{\text{d}}}$ be its expected value. Then

$$T = W_{\text{sec}} + \overline{t_{\text{p}}} + \overline{t_{\text{d}}}. \tag{26}$$

The arrival rate in the primary sector, λ_{sec}, is

$$\lambda_{\text{sec}} = \frac{\lambda}{m^2}. \tag{27}$$

Let K be the number of individual demands that are serviced during a sector service time of the primary sector. By the definitions, only *one* demand *inside* the primary sector is serviced during a sector service time. Also by the definitions, any number of demands can be serviced during a sector service time that are located *outside* of the primary sector. Let s_{sec} be the sector service time. Let $\overline{K} = E[K]$, and it follows that

$$\bar{s}_{\text{sec}} = \overline{K}(\overline{t_{\text{p}}} + \overline{t_{\text{d}}}), \tag{28}$$

where $\bar{s}_{\text{sec}}$ is the expected sector service time for a sector. Note that this is not true for the condition where a specific sector is given to be the primary sector.

Applying Eq. (2) yields

$$V[s_{\text{sec}}] = V[K](\overline{t_{\text{d}}} + \overline{t_{\text{p}}})^2 + E[K]V[t_{\text{d}} + t_{\text{p}}]. \tag{29}$$

Note that the probability that the delivery location of a demand is in a particular sector is $1/m^2$, and that each demand has a probability of $1/m^2$ of concluding a sector service time. Therefore, K has a geometric distribution with $p = 1/m^2$. Thus

$$\overline{K} = \frac{1}{p} = m^2, \tag{30}$$

and

$$V[K] = \frac{1-p}{p^2} = \frac{1 - 1/m^2}{(1/m^2)^2} = (m^4 - m^2). \tag{31}$$

The distance between a demand's pick-up location and delivery location is the distance between two i.i.d. points in a square of area A. Therefore,

$$\overline{t_{\text{d}}} = c_1 \frac{\sqrt{A}}{v}. \tag{32}$$

The distance between a demand's delivery location and the next demand's pick-up location is the distance between two i.i.d. points in a square of area A/m^2. Therefore,

$$\overline{t_{\text{p}}} = c_1 \frac{\sqrt{A}}{mv}. \tag{33}$$

It follows that

$$V[t_{\text{d}} + t_{\text{p}}] \simeq \frac{A}{v^2}\sigma_1^2 + \frac{A}{v^2 m^2}\sigma_1^2. \tag{34}$$

This is only approximate since t_{d} and t_{p} are not strictly independent. The error is equal to the covariance between t_{d} and t_{p}. (It is expected that this covariance term will decrease as the number of sectors becomes large.)

It follows that

$$\bar{s}_{\text{sec}} = m^2 \left(c_1 \frac{\sqrt{A}}{v} + c_1 \frac{\sqrt{A}}{mv} \right), \tag{35}$$

and Eq. (29) becomes

$$V[s_{\text{sec}}] = (m^4 - m^2)\left[c_1 \frac{\sqrt{A}}{v}\left(1 + \frac{1}{m}\right)\right]^2 + m^2 \frac{A\sigma_1^2}{v^2}\left(1 + \frac{1}{m^2}\right). \tag{36}$$

Eqs. (3), (35) and (36) are then combined, and calculus is used to find the optimal value of m:

$$m^* \simeq \frac{3}{2(1-\rho)}, \tag{37}$$

where $\rho = c_1(\sqrt{A}/v)\lambda$. As $\rho \to 1$, then from (37) $m^* \to \infty$. Using m^* and considering only the dominant term yields (25). $\square$

From (37) the optimal number of sectors, $(m^*)^2$, is of the order $(1-\rho)^{-2}$. Applying Little's Law to (25) shows that the number of demands in queue is at least of order $(1-\rho)^{-3}$, justifying the assumption that $P_s = 0$ in heavy traffic. Note that the lower bound of (23) is of the order $(1-\rho)^{-2}$ while the lower bound on the Sectoring Policy is of the order $(1-\rho)^{-3}$. Therefore, this policy is not within a constant factor of the lower bound.

4.2.4. The stacker crane policy

In this policy, demands are grouped into contiguous sets as they arrive. Each set contains n demands. The vehicle services the sets in first-come–first-served order. The vehicle follows an optimal tour of the demands in each set that minimizes the time to service the set. Since the vehicle has constant velocity, this translates into minimizing the total travel distance. The optimal value of n will then be determined.

The problem of constructing tours is known in the literature as the *Stacker Crane Problem* [4] and is a variation of the *Traveling Salesman Problem* (TSP). The crucial difference is that not every point in this tour can be reached from every other point; when the vehicle goes to a pick-up location, it must immediately go to the delivery location for that demand. The service of n demands results in a tour of $2n$ points that must be visited (one pick-up and one delivery point for every demand), but there are restrictions on the order in which these points can be visited.

Alternatively, the Stacker Crane Problem can be viewed as an asymmetric TSP by considering each pick-up and delivery pair as a single point. Unfortunately, no probabilistic results exist for the expected distance or for the variance of the distance for the Stacker Crane Problem. The individual $P \rightarrow D$ arcs will not change in length regardless of the order in which demands are serviced. The expected distances for the $D \rightarrow P$ arcs are dependent on the order that the demands are serviced. Therefore, the results for the probabilistic TSP, presented in Eqs. (5) and (6), were used as an approximation for the distance of the $D \rightarrow P$ sections of the tour.

If the TSP results of Eqs. (5) and (6) are used for the portions of the Stacker Crane Problem that consist of the delivery to pick-up segments, then in heavy traffic the expected time in system can be shown to be a function of the order $(1-\rho)^{-2}$. This suggests that this policy is of the order $(1-\rho)^{-2}$. Since Eqs. (5) and (6) have not been proven for the Stacker Crane Problem variation, numerical methods were used to derive the performance of the policy.

This policy was analyzed using simulation for several values of λ. Multiple values of n, the size of the set, were used for each value of λ in order to determine an optimal value for n as a function of λ. The value of n that resulted in the minimum mean time in the system was then used in the analysis (Fig. 1). The mean time in system is relatively insensitive to the exact value of n selected close to the minimum.

Once the optimal values of n were determined, two basic models were evaluated: one based on $\lambda(1-\rho)^{-2}$ and one based on $(1-\rho)^{-2}$. Regressions were performed for both, which produced almost identical results. The use of $\lambda(1-\rho)^{-2}$ is closer in form to the bound of (23), and it was thus preferred. The coefficients for a quadratic regression were calculated (Table 1). For simplicity, in deriving this relationship it was assumed that the service area is the unit square $(A = 1)$ and the velocity of the vehicle is $v = 1$. The value of R^2 was 0.9995. It follows that the time in the system is correlated with $\lambda(1-\rho)^{-2}$ in the tested range (Fig. 2).

Because the Stacker Crane Problem is NP-hard, the simulation uses a heuristic to determine the optimal tour through the points of a set, the *Large Arcs Algorithm* [4]. This algorithm is only guaranteed to produce tours within 3 times the length of the optimal tour. It is an indication of the robustness of this policy that the simulation results are within a constant factor of the bound of (23) even though suboptimal tours are used.

4.2.5. The nearest neighbor policy

In the *Nearest Neighbor* Policy, after a demand has been delivered, the vehicle proceeds to the nearest pick-up location. Because the policy makes all decisions based on the pick-up locations, the selection process causes the distribution of the remaining pick-up locations to be non-uniform.

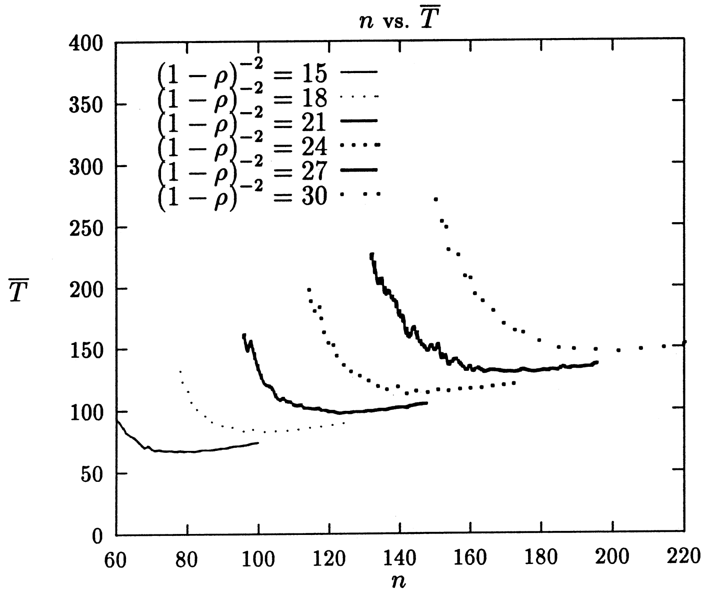

Fig. 1. Mean time in system versus set size for the Stacker Crane Policy for different values of $(1-\rho)^{-2}$.

Since the delivery locations are independent of the pick-up locations, their distribution is unaffected by this policy. The delivery locations remain independent and uniformly distributed over the service region.

Thus, the location of the vehicle when selecting the next demand to be serviced is uniformly distributed and independent of the pick-up locations. This selection process favors pick-up locations that are not close to the boundaries of the service region. This is due to the fact that a uniformly distributed point has a higher probability of being within some radial distance r from a point that is in the center of the service region than being within r of a point that is located close to the boundary of the region. Therefore, the distribution of the remaining pick-up locations will not be uniform. In other words, while the location of an arrival is uniformly distributed, the location of an arbitrary customer in the queue is not uniformly distributed. There are no known methods to analytically determine and express the actual distribution of these locations. This was the motivating factor in using simulation to analyze this policy.

A simulation model was developed for the Nearest Neighbor Policy. A sample of the results is presented in Fig. 3, where the pick-up locations of the demands in the system at the end of a trial are shown. As was typical for the trials, there are regions of very high density and considerable regions that are devoid of demands. As expected, the high density regions have a tendency to be toward the corners and the edges of the region.

The results from a number of trials were combined and tested for uniformity. A chi-square statistic was calculated for goodness-of-fit to a uniform distribution over the entire region, with a resulting p value that was less than 10^{-9}.

Table 1
Quadratic regression for $\lambda(1-\rho)^{-2}$ and T

Parameter	Coefficient estimate	Std. error	t Statistic
$\lambda(1-\rho)^{-2}$	3.7	0.683	5.388
$\sqrt{\lambda}(1-\rho)^{-1}$	−7.4	7.89	−0.944
Constant	22.4		

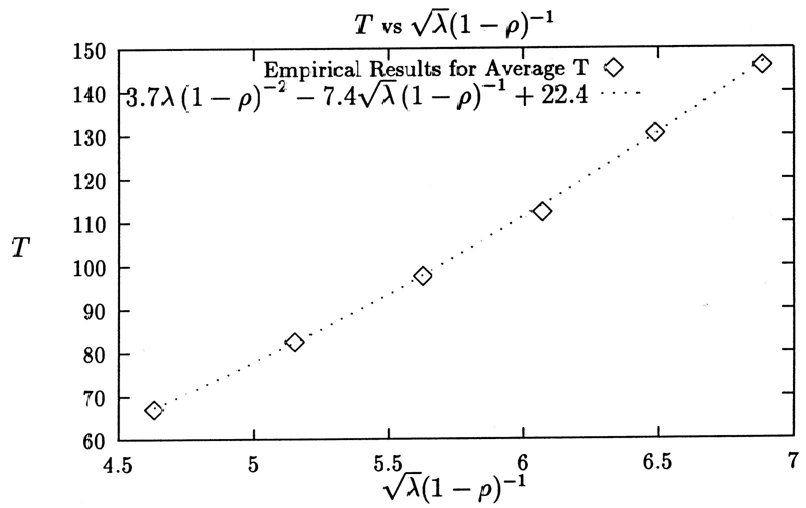

Fig. 2. Regression for the results of the Stacker Crane Policy.

Empirical trials were performed to obtain the relationship between the expected number of demands in the system and the arrival rate, and hence the relationship between the expected time in the system and the arrival rate (Fig. 4). The coefficients of a quadratic regression of N against $(1-\rho)^{-2}$ were calculated (Table 2). The value for R^2 was 0.9997. In determining this relationship, both v and A had values of unity. From Little's Law, it was thus established that T also increases at the rate of $(1-\rho)^{-2}$, which is the same as the order of the lower bound. Therefore, the Nearest Neighbor Policy appears to perform within a constant factor guarantee of the optimal.

5. The single-vehicle multiple-capacity DPDP

The case of the multiple-capacity vehicle is now considered. Due to the multiple-capacity, the vehicle is no longer required to go directly from the pick-up location of a demand to the delivery location of that demand. It can inter-mix pick-ups and deliveries according to any pattern, the only constraint being the vehicle capacity. The restriction that demands can only be dropped off at their desired delivery locations is maintained.

Under light traffic conditions (i.e., $\lambda \to 0$), the additional capacity will seldom be needed because the system will only rarely contain more than one demand. Thus the results are the same as for the unit-capacity formulation. Therefore, the light traffic condition was not analyzed further. The single-vehicle multiple-capacity system was analyzed in heavy traffic. For this formulation of the problem, heavy traffic implies that

$$\lambda \to \frac{\kappa v}{c_1\sqrt{A}}, \tag{38}$$

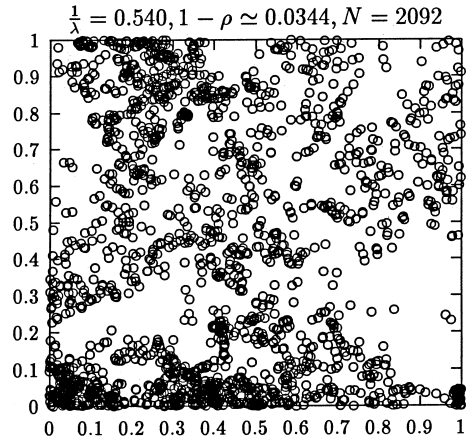

Fig. 3. Distribution of pick-up locations of demands in system (trial 1).

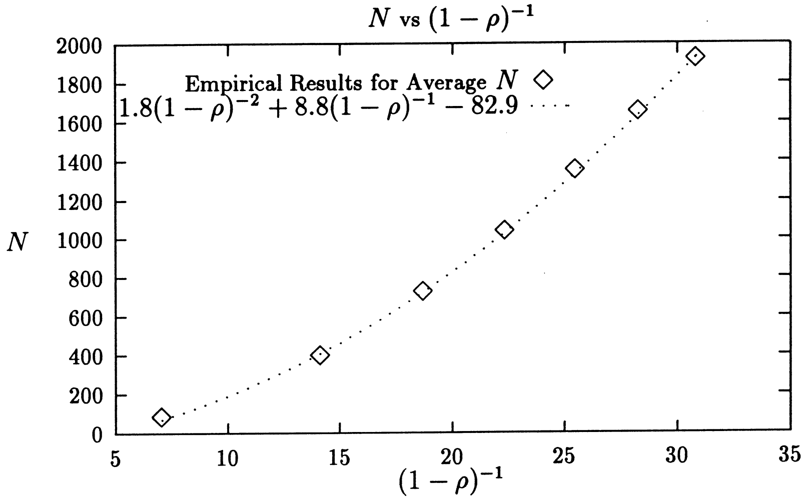

Fig. 4. Regression for the results of the nearest neighbor policy.

where k is the capacity of the vehicle. The maximum rate at which the vehicle could be servicing demands is obtained if the following conditions are met at all times:

1. The vehicle is carrying κ demands (i.e., operating at full capacity).
2. The delivery locations of all κ demands are collinear along the current direction of travel. In other words, all the demands are being transported directly towards their delivery locations.

If these conditions are met, κ demands would be serviced simultaneously by the vehicle, at velocity v, for an expected distance of $c_1\sqrt{A}$. Therefore, the maximum rate of service is $v\kappa/c_1\sqrt{A}$ and the condition $\lambda < \kappa v/c_1\sqrt{A}$ is necessary for stability.

This condition can be seen to be sufficient by using a modification of the Sector Touring Policy. Assume the sectors are numbered from 1 to m^2. Label each customer according to the customer's pick-up and delivery sectors, i.e., a customer type (i,j) has a pick-up location in sector i and a delivery location in sector j $(i,j = 1,\ldots,m^2)$. Create groups for each of the m^4 customer types with κ customers per group.

Consider each group as a single "super demand" and apply the Sector Touring Policy with the following differences. First, the interarrival times for groups are κth order Erlang instead of being exponential. Second, there is additional time spent picking up demands in a sector and delivering demands in a sector. However, the time to do both will still go to zero as m goes to infinity. Third, individual customers will have to wait while their sets form; the expected time for this $(1/\lambda)(\kappa(n-1)/2)m^4$. The expected time in system for a customer will end up being longer due to this effect, but it will still be finite, and thus the system is stable. These differences have no effect on the proof of Proposition 2, and thus it is straightfor-

Table 2
Quadratic regression for $(1-\rho)^{-2}$ and N

Parameter	Coefficient estimate	Std. error	t Statistic
$(1-\rho)^{-2}$	1.8	0.098	18.7
$(1-\rho)^{-1}$	8.8	3.79	2.32
Constant	−82.9		

ward to modify Proposition 2 to apply to the multiple-capacity case. The remainder of the paper will concentrate on a vehicle where the capacity is not limited. This is often the case with mail and small parcel delivery services, where there is enough room in the vehicle to accommodate a very large number of requests at the same time.

5.1. A heavy traffic lower bound

A lower bound that is useful in heavy traffic for the multiple-capacity case is provided by the following proposition.

Proposition 5. *A lower bound for the system waiting time of demands is*

$$T^* > \frac{4A}{9\pi v^2}\lambda - \frac{1}{4\lambda}. \tag{39}$$

Proof. Let W_α be the expected time for the vehicle to travel directly to the pick-up location of a demand from its previous location (either at a delivery or at a pick-up point), and W_β be the expected time for the vehicle to travel directly to the delivery location of a demand from its previous location. For stability

$$W_\alpha + W_\beta < \frac{1}{\lambda}. \tag{40}$$

Let N_q be the expected number of customers waiting in queue to be picked up. Recall that N is the expected number of customers in the system, and $N = \lambda T$. Randomly tag a customer. The vehicle can travel to the pick-up location of the tagged customer from any one of the following points:

- the pick-up location of any customer waiting in queue to be picked up when the tagged customer arrives;
- the delivery location of any customer in the system when the tagged customer arrives;
- the location of the server when the tagged customer arrives;
- the pick-up location of any customer that arrives while the tagged customer is waiting in queue to be picked up;
- the delivery location of any customer that arrives while the tagged customer is waiting in queue to be picked up.

The expected number of pick-up locations of customers waiting in queue to be picked up when the tagged customer arrives is N_q because of PASTA [10]. Also by PASTA, the expected number of delivery locations of customers in the system when the tagged customer arrives is N. Naturally, there is only one location for the server. Finally, from Little's Law, both the expected number of pick-up locations and the expected number of delivery locations of customers that arrive while the tagged customer is waiting in queue to be picked up are N_q. Then, for Eq. (24),

$$E(J) = N_q + N + 1 + N_q + N_q = 3N_q + N + 1.$$

This gives

$$W_\alpha \geqslant \frac{2\sqrt{A}}{3v\sqrt{\pi(3N_q + N + 1)}} \geqslant \frac{2\sqrt{A}}{3v\sqrt{\pi(4N + 1)}}. \tag{41}$$

Similarly, the vehicle can travel to the delivery location of a tagged customer from any one of the following points:

- the pick-up location of any customer waiting in queue to be picked up when the tagged customer arrives;
- the delivery location of any customer in the system when the tagged customer arrives;
- the pick-up location of the tagged customer;
- the pick-up location of any customer that arrives while the tagged customer is in the system;
- the delivery location of any customer that arrives while the tagged customer is in the system.

Thus $E[J] = N_q + 3N + 1$ by similar reasoning as above. This gives

$$W_\beta \geqslant \frac{2\sqrt{A}}{3v\sqrt{\pi(N_q + 3N + 1)}} \geqslant \frac{2\sqrt{A}}{3v\sqrt{\pi(4N + 1)}}. \tag{42}$$

Using (40) with the bounds for W_α and W_β yields the proposition. □

5.2. The DPDP with unlimited capacity

5.2.1. The dual traveling salesman problem (TSP) policy

The following definitions are used.

Definition 6. A *set* is a collection of n demands, where n is a given integer. The properties that the demands in a set possess depend on the routing policy that is being employed.

Definition 7. During a *pick-up tour* the vehicle picks up n demands consecutively and does not travel to any delivery locations.

Definition 8. During a *delivery tour* the vehicle delivers n demands consecutively and does not travel to any pick-up locations.

In the *Dual TSP* Policy, as demands arrive they are grouped into sets. The sets are serviced in first-come–first-served order. For each set, the vehicle first determines an optimal tour through the pick-up locations of the demands in that set and, after picking up all n demands in the tour, the vehicle determines an optimal tour through the delivery locations. The vehicle services the demands in the order determined by these tours. The starting points of the pick-up tour and of the delivery tour are selected randomly. For tractability reasons, the tours are treated as complete cycles, beginning and ending at the same point. Since it is expected that the number of points in a tour has to become large to maintain stability as the arrival rate increases, the additional distance that the vehicle would travel is small.

Proposition 6. *For the expected time in the system for the Dual TSP Policy in heavy traffic,*

$$\lim_{\lambda\to\infty} \frac{T}{\lambda} \leqslant \left(\frac{K_1^2}{2} + \frac{3K_1}{2}\right)\frac{\beta^2 A}{v^2}, \tag{43}$$

where $K_1 = (5+\sqrt{57})/4 \simeq 3.137$.

Proof. The expected total time that a demand spends in the system can be divided into six components: W_{f}, the expected waiting time for the set to finish forming, W_{s}, the expected waiting time for the set in queue, W_1, the expected time for the vehicle to travel to the first point in the pick-up tour, W_{p}, the expected time for the pick-up tour to be performed, W_2, the expected time for the vehicle to travel to the first point in the delivery tour, and W_{d}, the expected time for the vehicle to deliver demands. Then, the expected time in the system can be expressed as

$$T = W_{\mathrm{f}} + W_{\mathrm{s}} + W_1 + W_{\mathrm{p}} + W_2 + W_{\mathrm{d}}. \tag{44}$$

The expected time a demand waits while its set finishes forming is simply the expected number of additional demands that must arrive times the expected interarrival time:

$$W_{\mathrm{f}} = \frac{n-1}{2\lambda}. \tag{45}$$

The interarrival time of sets has an nth order Erland distribution with parameter λ. Using (4), the expected time a set waits in the queue of sets is bound by

$$W_{\mathrm{s}} \leqslant \frac{(\lambda/n)(n/\lambda^2 + (1/v^2)V[L_n] + (1/v^2)V[L_n] + 2\sigma_1^2 A/v^2)}{2[1-(\lambda/n)((1/v)E[L_n] + (1/v)E[L_n] + 2c_1\sqrt{A}/v)]}. \tag{46}$$

This bound is asymptotically exact for high traffic intensity.

Because of the random procedure in selecting the initial points of the tours,

$$W_1 = c_1\frac{\sqrt{A}}{v}. \tag{47}$$

Because the vehicle determines the order of visitation for the pick-up tour by forming the TSP tour, the expected distance that the vehicle must travel in the pick-up tour is $E[L_n]$. A demand must wait for the entire tour. This waiting has two components: waiting time spent at the pick-up location before the vehicle picks up the demand and waiting time spent in the vehicle while the vehicle picks up the remaining demands in the tour. For this analysis, it is not important to distinguish between these two components. The expected time the demand waits during the pick-up tour is

$$W_{\rm p} = \frac{1}{v}E[L_n]. \tag{48}$$

The expected time for the vehicle to travel from the last point in the pick-up tour to the first point in the delivery tour is the expected time to travel between two independent and uniformly distributed points. It follows that

$$W_2 = c_1\frac{\sqrt{A}}{v}. \tag{49}$$

A demand has to wait only for the fraction of the delivery tour that the demand is in the service vehicle. Since a demand is equally likely to be serviced in any position in the tour, it follows that the demand is expected to wait for one-half of the tour. Therefore,

$$W_{\rm d} = \frac{2}{2v}E[L_n]. \tag{50}$$

Eqs. (45)–(50) are then combined. As λ becomes large, the number of demands in a set, n, becomes large in order to reduce the expected distance traveled between demands so that the system stays stable. It follows that Eqs. (5) and (6) can then be used. Finally, calculus is used to minimize with respect to n, and the optimal value of n is obtained for large λ

$$\lim_{\lambda\to\infty}\frac{n^*}{\lambda^2} \simeq \frac{K_1^2\beta^2 A}{v^2}, \tag{51}$$

where $K_1 \simeq 3.137$ and $\beta \simeq 0.72$. This confirms our assumption that, as λ becomes large, n also becomes large. Using this result to obtain an approximation for n, with the appropriate simplifications, yields (43). □

5.2.2. Variations of the dual TSP policy

Two variations of the Dual TSP Policy were analyzed. These were the *Partially Sectored Dual TSP* Policy, and the *Sectored Dual TSP* Policy. These policies divide $\mathscr{A}$ into m^2 square sectors of area A/m^2 each. For the Partially Sectored Dual TSP Policy, a set consists of n demands that have their pick-up locations in the same sector; the delivery locations are not considered in forming sets. For the Sectored Dual TSP Policy, a set consists of n demands that have their pick-up locations in the same sector *and* their delivery locations in the same sector. These can be two different sectors.

For both policies, sets are served in first-come-first-served order. To service a set, the vehicle first performs a TSP tour of the pick-up locations, and then the vehicle performs the delivery tour. Both the starting point for the pick-up tour and the starting point for the delivery tour are randomly selected.

For the Partially Sectored Dual TSP Policy, the length of the pick-up tour is reduced compared with the *Dual TSP* Policy by a factor of m. For the Sectored Dual TSP Policy the length of the pick-up tour and the length of the delivery tour are both reduced compared with the Dual TSP Policy by a factor of m.

With some simplifying and letting m and n be large, an approximate bound for the Partially Sectored Dual TSP Policy is

$$\begin{aligned} T \leqslant{} & \frac{m^2(n-1)}{2\lambda} \\ & + \frac{n}{2\lambda[1-\lambda(\beta\sqrt{A}/mv\sqrt{n}+\beta\sqrt{A}/v\sqrt{n}+2c_1\sqrt{A}/v)]} \\ & + \left(\frac{1}{2}+\frac{1}{m}\right)\frac{\sqrt{A}}{v}\beta\sqrt{n}+2c_1\frac{\sqrt{A}}{v} \end{aligned} \tag{52}$$

and for the Sectored Dual TSP Policy is

$$\begin{aligned} T \leqslant{} & \frac{m^4(n-1)}{2\lambda} \\ & + \frac{n}{2\lambda(1-2\lambda\beta\sqrt{A}/mv\sqrt{n}-2\lambda c_1\sqrt{A}/nv)}+\frac{\beta\sqrt{An}}{mv} \\ & + \frac{\beta\sqrt{An}}{2mv}+2c_1\frac{\sqrt{A}}{v}. \end{aligned} \tag{53}$$

Both m and n are constrained to be positive integers and to take values that maintain stability. The optimal values of n and m were empirically estimated for both policies. The result for both was that the optimal value of m was 2. Therefore, m cannot be treated as large, and the approximations used were not appropriate.

More realistic estimates were also obtained. Simulations were performed to obtain empirically based estimates. Using these empirically based models, the optimal values of m and n were obtained. The result for both policies was that the

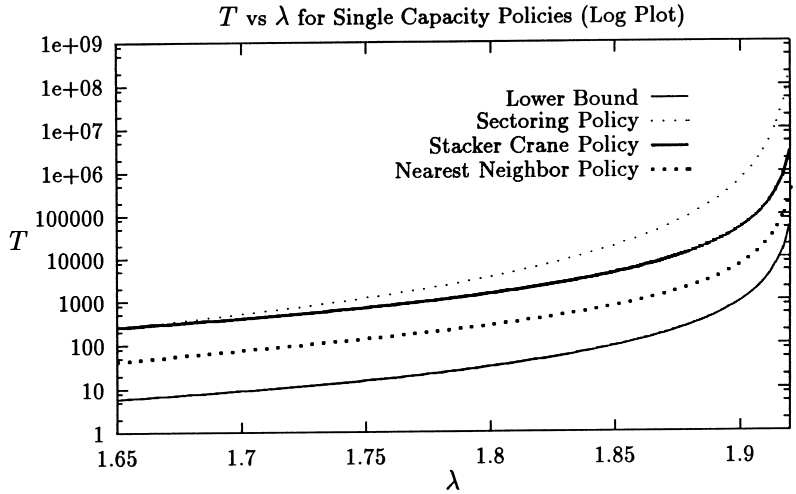

Fig. 5. Performance of routing policies for the unit-capacity vehicle in heavy traffic (log scale).

optimal value of m was 1. This causes both policies to be the same as the Dual TSP Policy. No further work was done as the sectoring used in this policy does not seem to lead to an improvement over the non-sectored Dual TSP Policy as long as n could be increased.

The reason for the lack of improvement of the Partially Sectored Dual TSP Policy can be seen by noting that the expected number of demands *that have not yet been assigned to sets* is $\frac{1}{2}(n-1)m^2$. This portion of demands in the queue is based solely on the parameters n and m and is independent of λ. Noting that the expected time in the system increases as the expected number of demands in queue increases, it follows that the impact of increasing unassigned demands in queue is too large to be offset by the reduction in the time of the pick-up tour. A similar effect occurs in the Sectored Dual TSP Policy, except that the expected number of demands that have not yet been assigned to sets is $\frac{1}{2}(n-1)m^4$.

5.3. The multiple-capacity nearest neighbor policy

For the *Multiple-Capacity Nearest Neighbor* Policy, *valid* delivery locations are the delivery locations for demands that have been picked-up but not delivered, i.e., demands in the vehicle. In this policy, after a demand has been delivered, the vehicle proceeds to the nearest pick-up or valid delivery location. Because the policy makes all decisions based on the pick-up and delivery locations, the selection process causes the distribution of the remaining pick-up locations and delivery locations to be non-uniform.

There are no known methods to analytically determine and express the actual distribution of these locations. Thus, a simulation model was developed. Empirical trials were performed to obtain the relationship between the expected time in the system and the arrival rate. The coefficients of a linear regression of T against λ were calculated (Table 3). The value for R^2 was 0.9969. In deter-

Table 3
Quadratic regression for λ and T

Parameter	Coefficient estimate	Std. error	t Statistic
λ	2.3	0.0097	240
Constant	−0.18		

mining this relationship, it was assumed that both $v = 1$ and $A = 1$. The linear regression is of the same order as the lower bound in (39). Therefore, the Multiple-Capacity Nearest Neighbor Policy appears to perform within a constant factor guarantee of the optimal.

6. Policy comparison

Graphs were obtained to compare the performance of the various routing policies. These are based on the various bounds and/or approximations that were developed for the policies. Fig. 5 presents the unit-capacity policies that were analyzed in heavy traffic as $\lambda \to v/c_1\sqrt{A}$. The mean time in the system is in logarithmic scale. The graph illustrates that for large values of λ, the Nearest Neighbor Policy appears to be superior based on the empirically derived equation for the performance of this system. Moreover, the Stacker Crane Policy is better than the Sectoring Policy. The lower bound of (23) is also shown in the graph.

Fig. 6 presents the performance of policies for the multiple-capacity vehicle. Because the sectored policies have no benefit over the Dual TSP Policy, only the lower bound, the Multiple-Capacity Nearest Neighbor Policy, and the Dual TSP Policy are shown.

7. Summary and conclusions

In this paper the Dynamic Pick-up and Delivery Problem (DPDP) was introduced and analyzed. The DPDP problem was defined for a unit-capacity service vehicle and a multiple-capacity service vehicle. A lower bound was obtained for the performance of on-line policies for both case in light traffic conditions, and the behavior of two policies was examined.

A bound of the order $(1-\rho)^{-2}$ was obtained for the performance of the unit-capacity system in heavy traffic conditions. Three routing policies were developed, the Sectoring Policy, the Stacker Crane Policy, and the Nearest Neighbor Policy. The Sectoring Policy is tractable, but its performance is bad being at best proportional to $(1-\rho)^{-3}$. The Stacker Crane Policy was analyzed using simulation, and its performance is of the order $(1-\rho)^{-2}$. The Nearest Neighbor Policy is not tractable, so simulation was employed. It also was of the order $(1-\rho)^{-2}$. The Nearest Neighbor Policy performs better than the Stacker Crane Policy for higher values for λ.

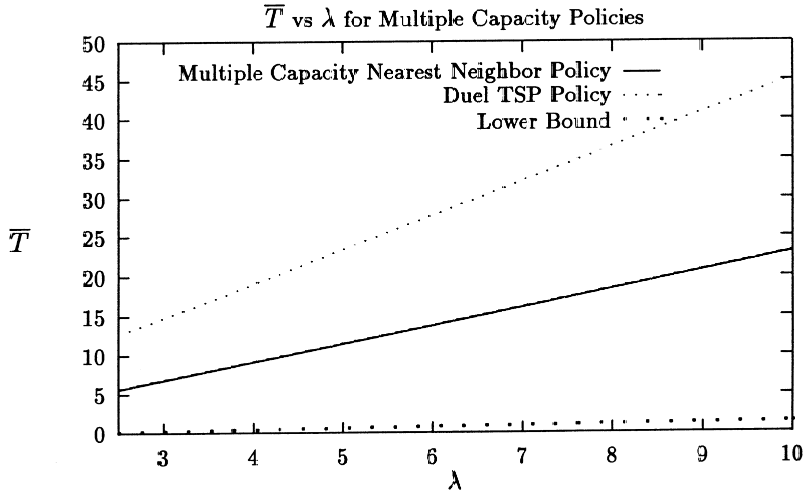

Fig. 6. Performance of routing policies for the multiple-capacity vehicle in heavy traffic.

For the multiple-capacity vehicle in heavy traffic, a lower bound proportional to λ was derived. Note that the bound was derived independent of the capacity of the vehicle. This indicates that as the arrival rate increases, the effect on the expected time in system cannot always be offset by an increase in capacity. This is different from common non-spatial queues, such as the $M/M/k$. The *Dual Traveling Salesman* Policy was developed for this case. The policy has performance proportional to λ and proportional to the lower bound. The Multiple-Capacity Nearest Neighbor Policy also has performance proportional to λ, and performed better than the Dual Traveling Salesman Policy.

Two variations of the Dual Traveling Salesman Policy were analyzed that utilized sectoring. It was determined that these policies perform best when the whole service region is a single sector, in which case the policies reduce to the Dual Traveling Salesman Policy. Therefore, sectoring of this type is not beneficial.

There are several areas that are open for future research. Tighter bounds could be derived for the heavy traffic case, and optimal policies could be developed. Also, bounds and policies could be developed for vehicles with limited capacities. Finally, multiple vehicle formulations should be pursued, without the limitation that customers can only be dropped off at their desired destinations. This would permit various forms of relaying customers by multiple servers.

Acknowledgements

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