Assignment5

Question 1: VAR example

A. Please look at the log. B. The VAR model that I want to estimate is:

$$\begin{bmatrix} 1 & 0 & 0 \\ \beta_{21} & 1 & 0 \\ \beta_{31} & \beta_{32} & 1 \end{bmatrix} \begin{bmatrix} P_{1t} \\ Y_{2t} \\ FFR_{3t} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{23} & \gamma_{33} \end{bmatrix} \begin{bmatrix} P_{1,t-1} \\ Y_{2,t-1} \\ FFR_{3,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{1t} \end{bmatrix} \tag{1}$$

As we can see, I put the change in the Federal Funds Rate (FFR) as the thrid equation as we saw in class. The reason is that by putting the FFR last we can say that the change in inflation growth rate and output affect monetary policy because of policy rules are act the same way as Taylor-type rules However, there is a lot of ambiguity as to the order of the change in inflation and the change in output. In this case, I decided to say that the CPI affects output. My reasoning is that when there are low periods of inflation there is a tendency for output growth to be lower and vice versa. Therefore, output growth is affected contemporaneously by the CPI. However, the reverse is true as well, but as we see later on the ordering of these two variables do not really matter.

C.What I assume about the contemporaneous effects of the variables is that the change in the CPI is not affected contemporaneously by changes in the output nor by changes in hte FFR. Similarly, the change in output is affected by changes in the inflation but not in changes of the FFR. My short run restrictions are represented by the outer left matrix that is lower triangular.

D. Please look at the Appendix.

E. THe part of the resuslts that are due to the assumptions that I have made are graphs d, g and h (going left to right i.e a,b,c next line d,e,f etc. . .) that the contemporaneous effect is zero.

F. A shock of the inflation growth rate has a relatively significant effect on the infaltion growth rate but seems to die out after 5 steps. There is a negative significant effect of the shock on GDP output that is persistent, but there is no significant affect on the FFR. A shock to the change in output leads to an increase in infaltion that is persistent, whereas the effect on the change of output dies out. The shock also sees an increase in the FFR but it also dies out. Interestingly enough, a shock in the FFR is insignificant on output growth but there does seem to be a small effect on the inflation growth that is persistent. The effect of a shock in the FFR on the FFR dies out.

G. Look at the appendix. The short run restrictions that we are imposing is that the change in output is not contemporaneously affected by changes in the CPI nor by changes in the FFR, whereas now the CPI is contemporaneously affected by changes in the output growth and not by the changes in the FFR. The graphs between this ordering and the ordering above are nearly identical. The only changes that we see are in graphs b and d, where the "intercept" at step 0 changes its position, which is relfecting the new short run restrictions that we have made.

H. The results are essentially identical meaning that our main conclusions remain.

Question 2: General VAR

A. We often use VARs in macro time series estimation instead of IVs and single-equation estimations methods because of endogeniety issues. The VARs is a good a way to work with these endogeniety issues. To illustrate this, let's look at the following example. Let:

$$y_{1t} = \frac{M}{P} = \text{real money balance} \tag{2}$$
$$y_{2t} = \text{Real GNP} \tag{3}$$

The money demand function is:

$$y_{1t} = \gamma_{10} + \beta_{12} y_{2t} + \gamma_{11} y_{1,t-1} + \gamma_{12} y_{2,t-1} + \epsilon_{1t} \tag{4}$$

In this function, ϵ_{1t} encompossases all other factors, β_{12} is the elasticity of real money balances with respect to real income and the lagged terms allow for a different long-run elasticity. As we can see in the function, money demand,y_{1t}, is a function of the money supply, y_{2t}. Similarly, the function of money supply is dependent on the money demand:

$$y_{2t} = \gamma_{20} + \beta_{21} y_{1t} + \gamma_{21} y_{1,t-1} + \gamma_{22} y_{2,t-1} + \epsilon_{2t} \tag{5}$$

What we are interested in these equations are the β_{12} and β_{21} coefficients but the problem is there is an inverse realtionship between the two equations. y_{2t} is affected by y_{1t} and vice versa. This leads us to have inconsistent β coefficients if we were to estimate them via OLS as they are endegenous. Since there is a correlation between income and money, as evident from the equations, we are unable to infer any causality. This is where the VAR comes in.

If we estimate both equations together, ϵ_{1t} and ϵ_{2t} may include omitted variables. In a fully specified strucutral model ϵ_{1t} and ϵ_{2t} represnt exogenous shocks, in this case money demand shock and productivity shock respectively. What this means is that if ϵ_{1t} is indeed an exogenous shock, a change in it will affect y_{1t} and also y_{2t} via β_{21}. This means that we can theoreticaly identify β_{21} but there is the identification problem due to exogeneity. To run them together, we simply the VAR.

B. Using the example above, the structural VAR is:

$$\begin{bmatrix} 1 & -\beta_{12} \\ -\beta_{22} & 1 \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \gamma_{10} \\ \gamma_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \tag{6}$$

In matrix notation:

$$B_{2x2} y_{t_{2x1}} = \Gamma_{0_{2x1}} + \Gamma_{1_{2x2}} y_{t-1_{2x1}} + \epsilon_{t_{2x1}} \tag{7}$$

C. The reduced form VAR is written as:

$$y_t = B^{-1}\Gamma_0 + B^{-1}\Gamma_1 y_{t-1} + B^{-1}\epsilon_t \tag{8}$$
$$= C + \Phi y_{t-1} + e_t \tag{9}$$

The reason as to why we need to define and estimate the reduced form version is so that we can get the structural parameters, the parameters that we want. In the structural VAR model, we are unable to run any of the equations because we do not know the B matrix as it is a matrix of the coefficients that we want since we do not know what they are!

D. What we want to identify is the B^{-1} by looking at the error term and the structural errors. We know that there is linear relationship between the forecast error e_t and ϵ_t, so if we were to solve the identification problem then we can estimate our coefficients. However, if we have no additional restrictions we end up with the following.

$$\begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} = \frac{1}{1 - \beta_{12}\beta_{21}} \begin{bmatrix} \epsilon_{1t} + \beta_{12}\epsilon_{2t} \\ \beta_{21}\epsilon_{1t} + \epsilon_{2t} \end{bmatrix} \tag{10}$$

Now we need to define variance-covariance of the forecast errors:

$$E[e_t] = 0, \text{as the structural errors mean is 0} \tag{11}$$
$$E[e_t e_t'] = E[B^{-1}\epsilon_t\epsilon_t'(B^{-1})'] \tag{12}$$
$$= B^{-1}E[\epsilon_t\epsilon_t'](B^{-1})' \tag{13}$$
$$= B^{-1}D(B^{-1})' = \Omega\Omega \qquad = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \tag{14}$$

Note that $\omega_{12} = \omega_{21}$ as Ω is a non-diagonal variance-Covariance matrix as the forecast errors are affected by shocks. The D matrix is the variance-covariance matrix with positive elements and is a diagonal, as we assume that the shocks are independent of each other. However, if we look a the structural VAR, equation(1), there are 10 parameters, compared to the reduced form VAR(1), equation (2), there are only 9 parameters. From that, we are unable to identify β_{12},$\beta_2 1$ from ω_{12} as we only have the correlation of income with the the forecast and money with the forecast. Consequently, we need additional restrictions to identify the model, i.e short-run, long-run or sign restrictions.

E. The way that I would the short-run restrictions is by setting one of the structural shocks to zero. By doing so the number of parameters from the structural VAR and the reduced form VAR will be the same, allowing us to solve the identification problem. The Cholesky states that for any positive definite symmetric matrix, there exists a uniquely decomposed triangular factorization

$$\Omega = T\Delta T' \tag{15}$$

where Δ is a diagonal positive elements and T is a lower triangular matrix with 1s on the diagonal. Remember that we made these assumptions:

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \tag{16}$$
$$D = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \tag{17}$$
$$\Omega = B^{-1}D(B^{-1})' \tag{18}$$

So if we were to say that B^{-1} is lower tringular then we could say that:

$$T = B^{-1} \tag{19}$$
$$\Delta = D \tag{20}$$
$$\Omega = B^{-1}D(B^{-1})' \tag{21}$$

This would imply that the reduced form VAR and Cholesky gives us the structural VAR if B^{-1} is lower triangular. But what does it mean for B^{-1} to be lower triangular? Well it means that β_{12} must be equal to 0, so as to have the variables on the right side of the diagonal of the matrix equal to 0.

F. Now remember:

$$\begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} = \frac{1}{1-\beta_{12}\beta_{21}} \begin{bmatrix} \epsilon_{1t} + \beta_{12}\epsilon_{2t} \\ \beta_{21}\epsilon_{1t} + \epsilon_{2t} \end{bmatrix} \tag{22}$$

What happens if $\beta_{12} = 0$?

$$\begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} = \frac{1}{1-\beta_{12}\beta_{21}} \begin{bmatrix} \epsilon_{1t} + \beta_{12}\epsilon_{2t} \\ \beta_{21}\epsilon_{1t} + \epsilon_{2t} \end{bmatrix} \qquad = \begin{bmatrix} \epsilon_{1t} + \beta_{12}\epsilon_{2t} \\ \beta_{21}\epsilon_{1t} + \epsilon_{2t} \end{bmatrix} = \begin{bmatrix} \epsilon_{1t} \\ \beta_{21}\epsilon_{1t} + \epsilon_{2t} \end{bmatrix} \tag{23}$$

We can now identify ϵs from the forecast errors recursively!

$$e_{1t} = \epsilon_{1t} \tag{24}$$
$$e_{2t} = \epsilon_{2t} + \beta_{21}\epsilon_{1t} \tag{25}$$

G. The IRF returns the dynamic response of a shock on all variables in the VAR system. Recall that our reduced form VAR is given by:

$$y_t = C + \Phi y_{t-1} + e_t \tag{26}$$

In addition, the effect of e_t on y_{t+s} we call ψ_s the effect $B^{-1}\epsilon_t$ on $y_t + s$. SO the effect of ϵ_t on y_{t+s} is $\theta_s = \psi_s B^{-1}$.

So the vector MA in terms of structural shock can be written as:

$$\begin{aligned} y_t &= \mu + e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + ..., \text{the reduced form} && (27)\\ &= \mu + B^{-1}\epsilon_t + \Psi_1 B^{-1}\epsilon_{t-1} + \Psi_2 B^{-1}\epsilon_{t-2} + ... && (28)\\ &= \mu + \Theta_0 epsilon_t + \Theta_1 epsilon_{t-1} + \Theta_2 epsilon_{t-2} + ... && (29)\end{aligned}$$

So, if we know Ψ and the matrix B we can determine Θ. So that gives us the structural IRF. WE know that

$$\Theta_s = \Psi_s B^{-1} \tag{30}$$

$$\begin{bmatrix} \Theta_{11,s} & \Theta_{12,s} \\ \Theta_{21,s} & \Theta_{22,s} \end{bmatrix} = \begin{bmatrix} \psi_{11,s} & \psi 12, s \\ \psi 21, s & \psi 22, s \end{bmatrix} \frac{1}{1-\beta_{12}\beta_{21}} \begin{bmatrix} 1 & \beta_{12} \\ \beta_{21} & 1 \end{bmatrix} \tag{31}$$

Therefore,

$$\theta_{21,s} = \frac{1}{1-\beta_{12}\beta_{21}} (\psi 21, s + \beta_{21}\psi 22, s) \tag{32}$$

H.To calculate the effect of the structural shock in equation 1 on the left hand side variable of equation 1, 3 periods ahead we do the following.

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \Theta_{11,0} & \Theta_{12,0} \\ \Theta_{21,0} & \Theta_{22,0} \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} + \begin{bmatrix} \Theta_{11,1} & \Theta_{12,1} \\ \Theta_{21,1} & \Theta_{22,1} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Theta_{11,2} & \Theta_{12,2} \\ \Theta_{21,2} & \Theta_{22,2} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-2} \\ \epsilon_{2,t-2} \end{bmatrix} + \begin{bmatrix} \Theta_{11,3} & \Theta_{12,3} \\ \Theta_{21,3} & \Theta_{22,3} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-3} \\ \epsilon_{2,t-3} \end{bmatrix} \tag{33}$$

We want to calculate $\frac{\delta y_{1,t+3}}{\delta \epsilon_{1,t}}$. We use the stationarity of the time series to say $\frac{\delta y_{1,t}}{\delta \epsilon_{1,t-3}}$ respectively. So, we want to find $\theta_{11,3}$. WE know that Θ_3 is equal to:

$$\begin{aligned} \begin{bmatrix} \Theta_{11,3} & \Theta_{12,3} \\ \Theta_{21,3} & \Theta_{22,3} \end{bmatrix} &= \Psi_3 B^{-1} && (34)\\ &= \begin{bmatrix} \psi_{11,3} & \psi 12, 3 \\ \psi 21, 3 & \psi 22, 3 \end{bmatrix} \frac{1}{1-\beta_{12}\beta_{21}} \begin{bmatrix} 1 & \beta_{12} \\ \beta_{21} & 1 \end{bmatrix} && (35)\end{aligned}$$

So,

$$\theta_{11,3} = \frac{1}{1-\beta_{12}\beta_{21}} (\psi_{11,3} + \beta_{12}\psi_{12,3}) \tag{36}$$

Similarly, the effect of the structural shock in equation 2 on the left hand side variable of equation 1 3 periods ahead is $theta_{12,3}$. From above,

$$\theta_{12,3} = \frac{1}{1-\beta_{12}\beta_{21}} (\psi_{12,3} + \beta_{21}\psi_{11,3}) \tag{37}$$

Since we assume that B^{-1} is lower triangular, meaning that $\beta_{12} = 0$, we have that $\Theta_{11,3} = \psi_{11,3}$ and $\theta_{12,3} = \psi_{12,3} + \beta_{21}\psi_{11,3}$