Ex 1)

$$\mathcal{L}(X \mid Y=0) = \mathcal{U}([0, \theta]), \quad \theta \in (0,1)$$

$$\mathcal{L}(X \mid Y=1) = \mathcal{U}([0, 1])$$

$$p = \mathbb{P}(Y=1)$$

$$\eta(x) = \mathbb{P}(Y=1 \mid X=x).$$

D'après le cours, on a

$$\eta(x) = \frac{p f_+(x)}{p f_+(x) + (1-p) f_-(x)}$$ avec f_+ & f_- densités cond. aux classes.

$$= \frac{p \, \mathbb{1}_{[0,1]}(x)}{p \, \mathbb{1}_{[0,1]}(x) + (1-p) \frac{\mathbb{1}_{[0,\theta]}(x)}{\theta}}$$

- Si $\theta < x < 1$, $\eta(x) = p/p = 1$
- Si $1 < x \vee x < 0$ (indéfini)
- Si $0 \leqslant x \leqslant \theta$, $\eta(x) = \frac{p}{p + \frac{(1-p)}{\theta}} = \frac{\theta p}{\theta p + 1 - p}$

Donc, $\eta(x) = \begin{cases} 1 & \text{si } \theta < x \\ \dfrac{\theta p}{\theta p + 1 - p} & \text{si } 0 \leq x \leq \theta \end{cases}$

Si $\theta = 1/2$, $\eta(x) = \begin{cases} 1 & \frac{1}{2} < x \\ \dfrac{p/2}{p/2 + 1 - p} & 0 \leq x \leq \frac{1}{2} \end{cases}$

⚠ Dans le cours, $y \in \{-1, 1\}$!

Ex 2

$\mathcal{L}_X = P_X$, $\mathcal{L}_{Y|X=x} = \mathcal{B}er(\eta(x))$, $\eta(x) = \dfrac{x}{x+\theta}$, $\forall \theta > 0$.

Par définition, $g^*(x) = \mathbb{1}_{\{\eta(x) > 1/2\}}$

Dans notre cas, $\eta(x) > \frac{1}{2} \iff \dfrac{x}{x+\theta} > \frac{1}{2}$

$\iff \dfrac{2x}{x+\theta} > 1 \iff x + \theta < 2x$

$\iff \theta < x$

Donc, $g^*(x) = \mathbb{1}_{\{x > \theta\}}$.

Par définition, $L(g^*) = L^* = \mathbb{E}\left[\min\{\eta(x) ; 1 - \eta(x)\}\right]$

$$= \mathbb{E}\left[\mathbb{1}_{\{x \leq \theta\}} \eta(x) + \mathbb{1}_{\{x > \theta\}} (1 - \eta(x))\right]$$

Si $x \leq \theta$, alors $\eta(x) \leq \frac{1}{2}$
donc $\min\{\eta(x) ; 1 - \eta(x)\} = \eta(x)$.

On a la densité de la distribution marginale P_X :

$$P_X : \mathbb{R} \to \mathbb{R}^+$$
$$x \mapsto \frac{\mathbb{1}_{[0, \alpha\theta]}}{\alpha\theta}, \quad \forall \alpha > 1.$$

On peut donc réécrire l'espérance :

$$L^* = \int_{\mathbb{R}} \mathbb{1}_{\{x \leq \theta\}} \eta(x) + \mathbb{1}_{\{x > \theta\}} (1 - \eta(x)) \, dP_X(x)$$

$$= \int_{\mathbb{R}} \left[\mathbb{1}_{\{x \leq \theta\}} \eta(x) + \mathbb{1}_{\{x < \theta\}} (1 - \eta(x))\right] P(x) \, dx$$

$$= \frac{1}{\alpha\theta} \int_0^{\alpha\theta} \mathbb{1}_{\{x \leq \theta\}} \eta(x) + \mathbb{1}_{\{x < \theta\}} (1 - \eta(x)) \, dx$$

$$L^* = \frac{1}{\alpha\theta}\left(\int_0^{\theta} \eta(x)\,dx + \int_{\theta}^{\alpha\theta} 1-\eta(x)\,dx\right)$$

$$= \frac{1}{\alpha\theta}\left(\int_0^{\theta} 1-\left(1-\eta(x)\right)dx + \int_{\theta}^{\alpha\theta} 1-\eta(x)\,dx\right)$$

$$= \frac{1}{\alpha\theta}\left(\int_0^{\theta} 1-\frac{\theta}{x+\theta}\,dx + \int_{\theta}^{\alpha\theta} \frac{\theta}{x+\theta}\,dx\right)$$

$$= \frac{1}{\alpha\theta}\left(\left[x-\theta\ln(x+\theta)\right]_0^{\theta} + \left[\theta\ln(x+\theta)\right]_{\theta}^{\alpha\theta}\right)$$

$$= \frac{1}{\alpha\theta}\left(\theta-\theta\ln(2\theta)+\theta\ln(\theta)+\theta\ln(\alpha\theta+\theta)-\theta\ln(2\theta)\right)$$

$$= \frac{1}{\alpha\theta}\left(\theta-2\theta\ln(2\theta)+\theta\ln\theta+\theta\ln\left(\theta(\alpha+1)\right)\right)$$

$$= \frac{1}{\alpha\cancel{\theta}}\left(\theta\left(1-2\ln(2\theta)+2\ln\theta+\ln(\alpha+1)\right)\right)$$

$$= \frac{1}{\alpha}\left(1+2\ln\left(\frac{\theta}{2\theta}\right)+\ln(\alpha+1)\right)$$

$$= \frac{1}{\alpha}\left(1-2\ln 2+\ln(\alpha+1)\right)$$

On peut calculer la dérivée :

$$\frac{\partial L^*}{\partial \alpha} = \frac{1}{\alpha} \frac{1}{\alpha+1} - \left(1 - 2\ln 2 + \ln \alpha + 1\right) \frac{1}{\alpha^2}$$

$$= \frac{1}{\alpha^2} \left(2 \ln 2 - \ln \alpha + 1 - 1\right) + \frac{1}{\alpha} \frac{1}{\alpha+1}$$

Si on développe, on voit que

$$\frac{\partial L^*}{\partial \alpha} > 0 \iff \frac{2^{2\alpha+2}}{e} > (\alpha+1)^{\alpha+1}.$$

Par croissance comparée, on sait que, à partir d'un certain α^*, $(\alpha^*+1)^{\alpha^*+1}$ domine $\frac{2^{2(\alpha^*+1)}}{e}$. Si $\alpha = 1$, $\frac{\partial L^*}{\partial \alpha} > 0$ et si $\alpha = 2$, $\frac{\partial L^*}{e} < 0$. Donc $\alpha^* \in]1, 2[$.

On peut calculer $\alpha^* \approx 1{,}80$.

On peut même se rapporter à $h = (a+1) e^{1/a+1}$ (aucune solution analytique).

$\boxed{Ex\ 3}$

$X = (T, U, V)$ où T, U, V iid et $T, U, V \sim \mathcal{E}(1)$.

$Y = \mathbb{1}_{\{T+U+V<\theta\}}$ $\forall \theta > 0$.

$\mathcal{L}_X = \mathcal{E}(1)^{\otimes}$

1) V n'est pas observé

$$\eta(x) = \mathbb{P}(Y=1 \mid X=x) = \mathbb{P}(T+U+V<\theta \mid T, U)$$

$$= \mathbb{P}(V<\theta-T-U \mid T, U)$$

On fixe $T=t$ et $U=u$. On a

$$\eta(x) = \mathbb{E}_V\left[\mathbb{1}_{\{V<\theta-t-u\}}\right]$$

$$= \int_0^{\theta-t-u} \underbrace{\mathbb{1}_{\{t+u<\theta\}}}_{\text{Si } t+u>\theta,\ \forall v \in \mathbb{R}^+,\ t+u+v>\theta} e^{-v}\, dv$$

$$= \mathbb{1}_{\{t+u<\theta\}} \left[-e^{-v}\right]_0^{\theta-t-u} = \mathbb{1}_{\{t+u<\theta\}} \left(1-e^{-(\theta-t-u)}\right)$$

On calcule le classifieur de Bayes :

$$g^*(x) = \mathbb{1}_{\{\eta(x) > 1/2\}}$$

$$\eta(x) > \frac{1}{2} \iff \begin{cases} t+u \leq \theta \\ e^{-(\theta - t - u)} < \frac{1}{2} \end{cases} \iff \begin{cases} t+u \leq \theta \\ \theta - t - u > \ln 2 \end{cases}$$

$$\iff \theta > t+u+\ln 2$$

$$g^*(x) = \mathbb{1}\{t+u < \theta - \ln 2\}$$

On peut calculer l'erreur de Bayes :

$$L^* = \mathbb{E}_{t,u}\left[\min\{\eta(x) ; 1 - \eta(x)\}\right]$$

$$= \mathbb{E}_{t,u}\left[\mathbb{1}_{\{t+u \geq \theta - \ln 2\}}\eta(x) + \mathbb{1}_{\{t+u < \theta - \ln 2\}}(1 - \eta(x))\right]$$

Calculons la densité de $Z = T + U$:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_T(u)\, f_U(z-u)\, du$$

$$f_Z(z) = \int_{-\infty}^{+\infty} \mathbb{1}_{\mathbb{R}_+}(k)\, \mathbb{1}_{\mathbb{R}_+}(z-k)\, e^{-k} e^{-(z-k)}\, dk$$

$$= \int_0^z e^{-k} e^{-(z-k)}\, dk$$

$$= \int_0^z e^{-z}\, dk = e^{-z} \lambda([0,z]) = z e^{-z}.$$

On peut donc ré-écrire l'espérance :

$$L^* = \mathbb{E}_Z\left[\mathbb{1}_{\{Z \geq \theta - \ln 2\}}\, \eta(Z) + \mathbb{1}_{\{Z < \theta - \ln 2\}}\, (1 - \eta(Z))\right]$$

avec $\eta(z) = \left(1 - e^{-\theta + z}\right) \mathbb{1}_{\{z < \theta\}}$.

$$L^* = \int_0^{\theta - \ln 2} (1 - \eta(z))\, f_Z(z)\, dz + \int_{\theta - \ln 2}^{\theta} \eta(z)\, f_Z(z)\, dz$$

$$= \int_0^{\theta - \ln 2} e^{-\theta + z}\, z e^{-z}\, dz + \int_{\theta - \ln 2}^{\theta} \left(1 - e^{-\theta + z}\right) z e^{-z}\, dz$$

$$= \int_0^{\theta - \ln 2} e^{-\theta} z\, dz + \int_{\theta - \ln 2}^{\theta} z e^{-z} - z e^{-\theta}\, dz$$

$$= e^{-\theta}\left[\frac{1}{2} z^2\right]_0^{\theta - \ln 2} + \boxed{\left[-z e^{-z}\right]_{\theta - \ln 2}^{\theta} + \left[-e^{-z}\right]_{\theta - \ln 2}^{\theta}} - e^{-\theta}\left[\frac{1}{2} z^2\right]_{\theta - \ln 2}^{\theta}$$

$$\int z e^{-z} = \int_a^b z \left(-e^{-z}\right)' dz$$

$$= \left[-z e^{-z}\right]_a^b - \int_a^b -e^{-z} \, dz$$

$$= \left[-z e^{-z}\right]_a^b + \left[-e^{-z}\right]_a^b$$

$$L^* = e^{-\theta} \frac{(\theta - \ln 2)^2}{2} - \theta e^{-\theta} + (\theta - \ln 2) e^{-(\theta - \ln 2)} - e^{-\theta} + e^{-(\theta - \ln 2)}$$

$$- \frac{\theta^2}{2} e^{-\theta} + \frac{(\theta - \ln 2)^2}{2} e^{-\theta}$$

$$= -e^{-\theta}\left(1 + \theta + \frac{\theta^2}{2} - (\theta - \ln 2)^2\right) + e^{-(\theta - \ln 2)}\left(1 + (\theta - \ln 2)\right)$$

$$= -e^{-\theta}\left(1 + \theta + \frac{\theta^2}{2} - (\theta - \ln 2)^2\right) + 2e^{-\theta}\left(1 + \theta - \ln 2\right)$$

$$= e^{-\theta}\left(-1 - \theta - \frac{\theta^2}{2} + (\theta - \ln 2)^2 + 2 + 2\theta - 2\ln 2\right)$$

$$= e^{-\theta}\left(-1 - \theta - \frac{\theta^2}{2} + \theta^2 - 2\theta \ln 2 + \ln^2 2 + 2 + 2\theta - 2\ln 2\right)$$

$$= e^{-\theta}\left(1 + \theta + \frac{\theta^2}{2} - 2\ln 2 (1 + \theta) + \ln^2 2\right)$$

Si $\theta = 9$, $L^* \approx 0,0046$.

2) Seulement T est observé.

$$\eta(x) = \mathbb{P}(T + U + V < \theta \mid T)$$

$$= \mathbb{P}(U + V < \theta - T)$$

On pose $z = U + V$.

$$\eta(x) = \int_0^{\theta - t} z e^{-z} \, dz = \left[-z e^{-z}\right]_0^{\theta - t} - \left[e^{-z}\right]_0^{\theta - t}$$

$$= -(\theta - t) e^{-(\theta - t)} - e^{-(\theta - t)} + 1$$

$$= 1 + (-(\theta - t) - 1) e^{-(\theta - t)}$$

$$= 1 - (\theta - t + 1) e^{-(\theta - t)}$$

$$\eta(x) > \frac{1}{2} \iff \begin{cases} \theta \leq t \\ 1 - (\theta - t + 1) e^{-(\theta - t)} > 1/2 \end{cases}$$

$$\iff \begin{cases} \theta \leq t \\ (\theta - t + 1) e^{-(\theta - t)} < 1/2 \end{cases}$$

$$\Leftrightarrow e^{-(\theta-t)} < \frac{1}{2(\theta-t+1)}$$

$$\Leftrightarrow t-\theta < -\log\left(2(\theta-t+1)\right)$$

$$\Leftrightarrow t-\theta < -\log 2 - \log(\theta-t+1)$$

$$\Leftrightarrow \theta-t-\log(\theta-t+1) > \log 2$$

$$\eta(t) > \tfrac{1}{2} \Leftrightarrow \theta-t-\log(\theta-t+1) > \log 2$$

$$g^*(t) = \mathbb{1}_{\{\theta-t-\log(\theta-t+1) > \log 2\}}$$

$$L^* = \mathbb{E}\left[\min\{\eta(t) \,;\, 1-\eta(t)\}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{\theta-t-\log(\theta-t+1) < \log 2\}}\eta(t) + \mathbb{1}_{\{\theta-t-\log(\theta-t+1) > \log 2\}}(1-\eta(t))\right]$$

$$= \int_0^\theta \left[\mathbb{1}_{\{\dots\}}\, 1-(\theta-t+1)e^{-(\theta-t)} + \mathbb{1}_{\{\dots\}}(\theta-t+1)e^{-(\theta-t)}\right] e^{-t}\,dt$$

Si $\theta = 9$

$L^* \approx 0{,}0059$ Plus grand que lorsque T et U sont observés.

3) Aucun n'est observé.

On pose $Z = T + U$ et $S = V + Z$. On calcule la densité de S,
$$f_S(s) = \int_0^{+\infty} z e^{-z} e^{-(s-z)} \mathbb{1}_{\mathbb{R}_+}(s-z)\, dz$$
$$= \int_0^s z e^{-s}\, dz = e^{-s}\left[\frac{z^2}{2}\right]_0^s = \frac{s^2 e^{-s}}{2}$$

On calcule $\eta(x) = \mathbb{P}(Y=1 \mid X=x) = \mathbb{P}(S < \theta)$
$$= \int_0^\theta \frac{s^2 e^{-s}}{2}\, ds$$
$$= \left[\frac{-s^2+2s+2}{2} e^{-s}\right]_0^\theta = 1 - \frac{e^{-\theta}}{2}(\theta^2+2\theta+2)$$

$$\eta(x) \geq \frac{1}{2} \iff e^{-\theta}(\theta^2+2\theta+2) < 1$$

$$g^* = \mathbb{1}\{e^{-\theta}(\theta^2+2\theta+2) < 1\}$$

$$L^* = \mathbb{E}\left[\min\{\eta(x) ; 1-\eta(x)\}\right]$$
$$= \mathbb{E}\left[\mathbb{1}\{e^{-\theta}(\theta^2+2\theta+2) \geq 1\}\, \eta(x) + \mathbb{1}_{\{\dots\}}(1-\eta(x))\right]$$

Si $\theta = 9$,

$L^* \approx 0{,}0062$ Encore plus grand que q° 2 !

Ex 4

1) $$f^{+}(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right)$$

$$f^{-}(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1} (x-\mu_-)\right)$$

$$\eta = \frac{p f_+}{p f_+ + (1-p) f_-}$$

Donc, $$\eta(x) = \frac{\frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right)}{\frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right) + \frac{1-p}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1} (x-\mu_-)\right)}$$

On pose $\pi_{\{+,-\}} = (2\pi)^{d/2} |\Sigma_{\{+,-\}}|^{1/2}$ et

$$e_{\{+,-\}} = \exp\left(-\frac{1}{2}\left((x-\mu_{\{+,-\}})^T \Sigma_{\{+,-\}}^{-1} (x-\mu_{\{+,-\}})\right)\right)$$

Étudions $\frac{p e_+}{\pi_+} + \frac{(1-p) e_-}{\pi_-}$:

$$\frac{\pi_- p\, e_+}{\pi_+ \pi_-} + \frac{(1-p)\pi_+ e_-}{\pi_+ \pi_-} = \frac{\pi_- p\, e_+ + (1-p)\pi_+ e_-}{\pi_+ \pi_-}$$

$$= \frac{p\pi_- e_+ + \pi_+ e_- - p\pi_+ e_-}{\pi_+ \pi_-}$$

$$= \frac{p\pi_- e_+ + (1-p)\pi_+ e_-}{\pi_+ \pi_-}$$

$$\eta(x) = \frac{p e_+}{\cancel{\pi_+}} \frac{\cancel{\pi_+} \pi_-}{p\pi_- e_+ + (1-p)\pi_+ e_-} = \frac{\cancel{p} e_+ \pi_-}{\cancel{p}\left(\pi_- e_+ + \left(\frac{1-p}{p}\right)\pi_+ e_-\right)}$$

$$= \frac{e_+ \pi_-}{\pi_- e_+ + \left(\frac{1-p}{p}\right)\pi_+ e_-}$$

$$= \frac{e_+ \pi_-}{e_+ \pi_- \left(1 + \frac{1-p}{p} \frac{\pi_+ e_-}{\pi_- e_+}\right)}$$

$$= \frac{1}{1 + \frac{1-p}{p} \frac{(2\pi)^{d/2} |\Sigma_+|^{1/2}}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \frac{\exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1} (x-\mu_-)\right)}{\exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right)}}$$

$$= \frac{1}{1 + \frac{1-p}{p} \left|\frac{\Sigma_+}{\Sigma_-}\right|^{1/2} \exp\left(-\frac{1}{2}\left((x-\mu_-)^T \Sigma_-^{-1} (x-\mu_-) - (x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right)\right)}$$

2) $\log\left(\frac{\eta_\theta(x)}{1-\eta_\theta(x)}\right) = \theta^T x$

$$\Rightarrow \frac{\eta_\theta(x)}{1-\eta_\theta(x)} = e^{\theta^T x}$$

On peut donc écrire :

$$\eta_\theta(x)\left(1+e^{\theta^T x}\right) = \eta_\theta(x) + \frac{\eta_\theta^2(x)}{1-\eta_\theta(x)}$$

$$= \frac{\eta_\theta(x) - \eta_\theta^2(x) + \eta_\theta^2(x)}{1-\eta_\theta(x)} = \frac{\eta_\theta(x)}{1-\eta_\theta(x)} = e^{\theta^T x}$$

On obtient :

$$\eta_\theta(x) = \frac{e^{\theta^T x}}{1+e^{\theta^T x}} = \frac{1}{1+e^{-\theta^T x}}.$$

Par définition, on sait que $\eta_\theta(x) = \frac{p f_+}{f_x}$, avec $f_x = p f_+ + (1-p) f_-$.

On a donc $\begin{cases} f_+ = 1/p \\ f_- = \frac{f_x - p f_+}{1-p} = \frac{1+e^{-\theta^T x} - 1}{1-p} = \frac{e^{-\theta^T x}}{1-p} \end{cases}$

Ex 5)

1) $w \in]0,1[$.

$$L_w(g) = 2\,\mathbb{E}_{X,Y}\Big[(1-w)\,\mathbb{1}_{\{Y=1\}}\,\mathbb{1}_{\{g(X)=-1\}} + w\,\mathbb{1}_{\{Y=-1\}}\,\mathbb{1}_{\{g(X)=1\}}\Big]$$

$$= \mathbb{E}_{X,Y}\Big[(1-w)\,\underbrace{\mathbb{E}_Y\big[\mathbb{1}_{\{Y=1\}}\,|\,X\big]}_{\text{Law of total expectation}}\,\mathbb{1}_{\{g(X)=-1\}} + w\,\mathbb{E}_Y\big[\mathbb{1}_{\{Y=-1\}}\,|\,X\big]\,\mathbb{1}_{\{g(X)=1\}}\Big]$$

$$= \mathbb{E}\Big[(1-w)\,\eta(X)\,\mathbb{1}_{\{g(X)=-1\}} + w\,(1-\eta(X))\,\mathbb{1}_{\{g(X)=1\}}\Big]$$

Donc, si $g^*(X) = 1 \iff w(1-\eta(X)) \leq (1-w)\,\eta(X)$

(g^* minimise L. Donc, si $g^*(X)=1$, $w(1-\eta(X))$ est plus petit que $(1-w)\,\eta(X)$)

$$\iff w - w\,\eta(X) \leq \eta(X) - w\,\eta(X)$$

$$\iff w \leq \eta(X)$$

Donc, $g^*(X) = 2\,\mathbb{1}_{\{w \leq \eta(X)\}} - 1$.

Calculons L^* :

$$L_w(g^*) = 2\mathbb{E}\left[\mathbb{1}_{\{\eta(X) < w\}}(1-w)\eta(X) + \mathbb{1}_{\{w \leq \eta(X)\}} w(1-\eta(X))\right]$$

$$= 2\mathbb{E}\left[\left(\mathbb{1}_{\{\eta(X) < w\}} + \mathbb{1}_{\{w \leq \eta(X)\}}\right)\min\left\{(1-w)\eta(X) ; w(1-\eta(X))\right\}\right]$$

$$= 2\mathbb{E}\left[\min\left\{(1-w)\eta(X) ; w(1-\eta(X))\right\}\right]$$

2) $u \in \,]0,1[$

On cherche g tq. $g \in \underset{g}{\operatorname{argmin}}\, \mathbb{P}(Y \neq g(X))$

s.t. $\mathbb{P}(g(X) = 1) = u$.

On peut trouver un q tq $\mathbb{P}(\eta(X) > q) = u$.

Posons $q = q_{1-u}(\eta(X))$ $\left(\mathbb{P}(\eta(X) \leq q) = 1-u \iff \underbrace{1 - \mathbb{P}(\eta(X) \leq q)}_{= u} \iff \mathbb{P}(\eta(X) > q) = u \right)$

On a $L_u(g) = \mathbb{E}\left[\mathbb{1}_{\{g(X) = -1\}}\eta(X) + \mathbb{1}_{\{g(X) = 1\}}(1-\eta(X))\right]$

$$= \mathbb{E}\Big[(\eta(X) - q)\, \mathbb{1}_{\{g(X)=-1\}} + q\, \mathbb{1}_{\{g(X)=-1\}} + (q - \eta(X))\, \mathbb{1}_{\{g(X)=1\}} + (1-q)\, \mathbb{1}_{\{g(X)=1\}} \Big]$$

Or,

- $\mathbb{E}\left[q\, \mathbb{1}_{\{g(X)=-1\}} \right] = q\, \mathbb{P}(g(X) \neq 1) = q(1-u)$

- $\mathbb{E}\left[(1-q)\, \mathbb{1}_{\{g(X)=1\}} \right] = (1-q)\, \mathbb{P}(g(X)=1) = (1-q)u$

On peut donc écrire

$$L_u(g) = \mathbb{E}\Big[(\eta(X) - q)\, \mathbb{1}_{\{g(X)=-1\}} + (q - \eta(X))\, \mathbb{1}_{\{g(X)=1\}} \Big] + q(1-u) + (1-q)u.$$

Par définition, $g^*(X) = 1 \iff q - \eta(X) < \eta(X) - q$

$$\iff -2\eta(X) < -2q$$

$$\iff q < \eta(X)$$

Donc, $g^*(X) = 2\, \mathbb{1}_{\{q < \eta(X)\}} - 1$ et, $\mathbb{P}(g^*(X) = 1) = \mathbb{P}(\eta(X) > q) = u$.

$$L_u^* = \mathbb{E}\Big[\min\{ \eta(X) - q \,;\, q - \eta(X) \} \Big] + q(1-u) + (1-q)u.$$

3) Ⓡ est une valeur spécifique de rejet.

$\gamma \in]0, \frac{1}{2}[$.

$$L_\gamma^R(g) = \mathbb{P}(Y \neq g(X), g(X) \neq R) + \gamma \mathbb{P}(g(X) = R)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{g(X)=-1\}} \eta(X) + \mathbb{1}_{\{g(X)=1\}}(1-\eta(X))\right] + \gamma \mathbb{P}(g(X) = R)$$

$$= \mathbb{E}\left[\mathbb{1}_{\{g(X)=-1\}} \eta(X) + \mathbb{1}_{\{g(X)=1\}}(1-\eta(X)) + \gamma \mathbb{1}_{\{g(X)=R\}}\right]$$

Si $g^*(X) = 1 \iff 1-\eta(X) < \eta(X) \wedge 1-\eta(X) < \gamma$

$g^*(X) = -1 \iff \eta(X) < 1-\eta(X) \wedge \eta(X) < \gamma$

$g^*(X) = R \iff \gamma < \eta(X) \wedge \gamma < 1-\eta(X)$

Donc

$$g^*(X) = \begin{cases} 1 & \text{si } 1-\eta(X) < \eta(X) \wedge 1-\eta(X) < \gamma \\ -1 & \text{si } \eta(X) < 1-\eta(X) \wedge \eta(X) < \gamma \\ R & \text{si } \gamma < \eta(X) \wedge \gamma < 1-\eta(X) \end{cases}$$

$\forall \gamma \in]0, \frac{1}{2}[$

$\gamma \leq 1-\gamma$.

Donc $\eta(X) < \gamma$

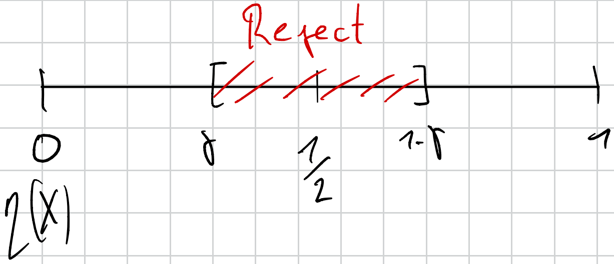
$\Rightarrow 1-\eta(X) > 1-\gamma > \gamma > \eta(X)$

et $1-\eta(X) < \gamma$

$\iff -\eta(X) < \gamma - 1$

$\Rightarrow \eta(X) > 1-\gamma > \gamma > 1-\eta(X)$

$$= \begin{cases} 1 & \text{si } 1-\eta(X) < \gamma \\ -1 & \text{si } \eta(X) < \gamma \\ R & \text{si } \gamma < \eta(X) < 1-\gamma \end{cases}$$


Reject
0
γ
1/2
1-γ
1
Z(X)

Ex 8

5) $A(f) = \mathbb{E}\left[\max\{0, 1 - Y f(X)\}\right]$

$$= \mathbb{E}\left[\max\{0, 1 - f(X)\}\mathbb{1}_{\{Y=1\}} + \max\{0, 1 + f(X)\}\mathbb{1}_{\{Y=-1\}}\right]$$

$$= \mathbb{E}\left[\max\{0, 1 - f(X)\}\,\mathbb{E}[\mathbb{1}_{\{Y=1\}}|X] + \max\{0, 1+f(X)\}\,\mathbb{E}[\mathbb{1}_{\{Y=-1\}}|X]\right]$$

$$= \mathbb{E}\left[\max\{0, 1-f(X)\}\eta(X) + \max\{0, 1+f(X)\}(1-\eta(X))\right]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{f(X)\leq 1\}}(1-f(X))\,\eta(X)$$

$$+ \mathbb{1}_{\{f(X)\geq -1\}}(1+f(X))(1-\eta(X))\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{f(X)<-1\}}(1-f(X))\,\eta(X) \quad > 2\eta(X) \quad \text{si: } f(X) < -1$$

$2\min\{\eta(X), 1-\eta(X)\} \leq$ $+ \mathbb{1}_{\{-1\leq f(X)\leq 1\}}(1-f(X))\,\eta(X) + (1+f(X))(1-\eta(X)) \leq 2\max\{\eta(X), 1-\eta(X)\}$

si: $f(X) \in [-1, 1]$.

$$+ \underbrace{\mathbb{1}_{\{f(X)>1\}}(1+f(X))(1-\eta(X))}_{> 2(1-\eta(X)) \quad \text{si: } f(X) > 1}\Big]$$

$$f(X) = 2\,\mathbb{1}_{\{1-\eta(X) < \eta(X)\}} - 1$$

$$= 2\,\mathbb{1}_{\{\frac{1}{2} < \eta(X)\}}$$

Disjonction de cas :

Fixons X.

Si $\eta(X) < 1 - \eta(X)$:

si $f(X) < -1$:

$$(1 - f(X))\eta(X) > 2\eta(X)$$

si $f(X) > 1$:

$$(1 + f(x))(1 - \eta(X)) > 2(1 - \eta(X)) > 2\eta(X)$$

Si $f(X) \in [-1, 1]$:

$$2\eta(X) \leq (1 - f(X))\,\eta(X) + (1 + f(X))(1 - \eta(X)) \leq 2(1 - \eta(X))$$

$2\eta(X)$ minore donc ces 3 termes. $f(X) = -1$ minimise donc l'intégrande pour un X fixé si $\eta(X) < 1 - \eta(X)$.

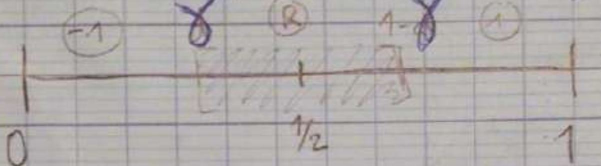
De la même manière, si $1 - \eta(X) < \eta(X)$, $2(1 - \eta(X))$ minore les 3 termes. $f(X) = -1$ minimise donc l'intégrande pour un X fixé si $1 - \eta(X) \leq \eta(X)$.

Donc, $f(X) = 2\,\mathbb{1}_{\{\eta(X) > \frac{1}{2}\}} - 1$.

$$\cancel{g^*(X) = \begin{cases} -1 & \text{if } 1-\eta(X) > \eta(X) \ \& \ \ldots \\ 1 & \text{if } \eta(X) > 1-\eta(X) \ \& \ \ldots \\ (R) & \end{cases}}$$

$$g^*(X) = \begin{cases} -1 & \text{si } \eta(X) < 1-\eta(X) \ \& \ \eta(X) < d \\ 1 & \text{si } 1-\eta(X) < \eta(X) \ \& \ 1-\eta(X) < d \\ (R) & \text{si } d < \eta(X) \ \& \ d < 1-\eta(X) \end{cases}$$

$$= \begin{cases} -1 & \text{if } \eta(X) < d \\ 1 & \text{if } \eta(X) > 1-d \\ (R) & \text{if } \gamma < \eta(X) < 1-\gamma \end{cases}$$

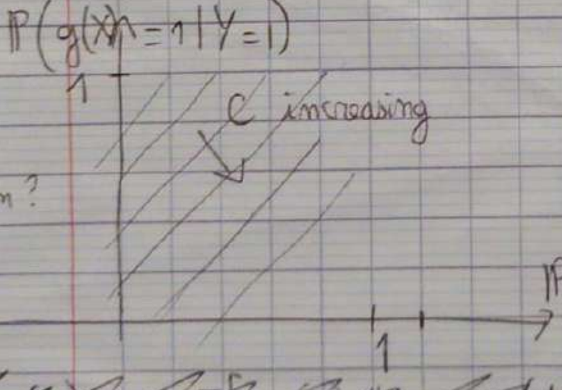


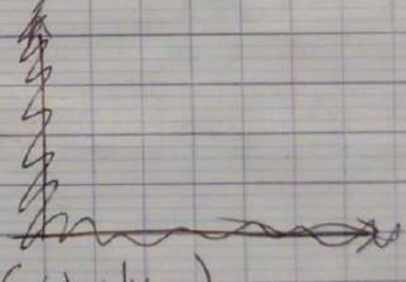
$$L(g^*) = \mathbb{E}\left[\min\{\eta(X); 1-\eta(X); \gamma\}\right]$$

(6) 1) $L_\omega(g) = \mathbb{E}\left[2\omega(Y) \cdot \mathbb{1}_{\{Y \neq g(X)\}}\right]$

$: g \mapsto \mathbb{P}(\{g(X)=1 \mid Y=0\})$ FP

$: g \mapsto \mathbb{P}(\{g(X)=1 \mid Y=1\})$ TP





(*) interpretation ?

$$\cancel{L_\omega(g) = 2\,\mathbb{E}\left[\omega(0)\mathbb{1}_{\{g(X)=1\}}(1-\eta(X)) + \omega(1)\mathbb{1}_{\{g(X)=0\}}\eta(X)\right]}$$

$K_n(\ldots$

$\Rightarrow Y\varepsilon_i \sim Ber(1/2)$

$$L_\omega(g) = C = 2\,\mathbb{E}\big[\mathbb{E}[\omega(Y)\mathbb{1}_{\{Y\neq g(X)\}} \mid Y]\big]$$

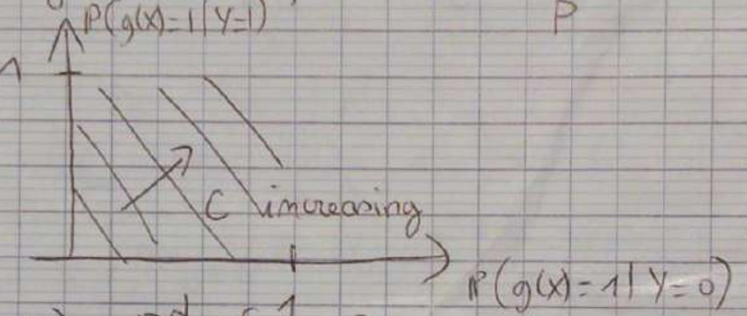
$$= 2\,\mathbb{E}\big[\mathbb{1}_{\{Y=0\}}\,\omega(0)\,\mathbb{P}(g(X)=1 \mid Y=0) + \mathbb{1}_{\{Y=1\}}\,\omega(1)\,\mathbb{P}(g(X)=0 \mid Y=1)\big]$$

$$= 2\omega(0)\mathbb{P}(Y=0)\mathbb{P}(g(X)=1 \mid Y=0) + 2\omega(1)\mathbb{P}(Y=1)\big(1-\mathbb{P}(g(X)=1 \mid Y=1)\big)$$

$$\rightarrow \mathbb{P}(g(X)=1 \mid Y=1) = 1 - \frac{C - 2\omega(0)(1-p)\mathbb{P}(g(X)=1 \mid Y=0)}{2\,\omega(1)\,p}$$

2) $\mathbb{P}(g(X)=1) = C = \mathbb{P}(g(X)=1 \mid Y=0)\,\mathbb{P}(Y=0) + \mathbb{P}(g(X)=1 \mid Y=1)\,\mathbb{P}(Y=1)$

$$\mathbb{P}(g(X)=1 \mid Y=1) = \frac{C - (1-p)\mathbb{P}(g(X)=1 \mid Y=0)}{p}$$



7) $(X,Y) \in \mathbb{R}^d \times \{-1,1\}$

$\mathbb{E}[\mathbb{1}\{-Yf(X)>0\}]$

Avant : $L(g) = \mathbb{E}[\mathbb{1}_{\{-Yg(X)>0\}}]$, soft classifier : $f : \mathcal{X} \to \mathbb{R}$

↳ associated classifier $g = \operatorname{sgn}(f)$

Mtn : $A(f) = \mathbb{E}[\varphi(-Yf(X))]$, φ cost function

↳ convex optim

1) $R(g) = \mathbb{E}\left[\left(Y - g(X) \right)^2 \right]$

$= \mathbb{E}\left[4 \left(\mathbb{1}_{\{Y=-1\}} \eta(X) + \mathbb{1}_{\{Y=-1\}} (1-\eta(X)) \right) \right]$

$g^*(x) = 2 \mathbb{1}_{\{\eta(x) > \frac{1}{2}\}} - 1$

2) $R(f) = \mathbb{E}\left[\left(Y - f(X) \right)^2 \right]$

$\cancel{= \mathbb{E}\left[\mathbb{1}_{\{Y=-1\}} \eta(X) (1+f(X))^2 + \mathbb{1}_{\{Y=1\}} (1-\eta(X)) (1-f(X))^2 \right]}$

$\cancel{g^*(x) = 1 \iff (1-\eta(x))(1-f(x))^2 < \eta(x)(1+f(x))^2}$

$= \mathbb{E}\left[\mathbb{E}\left[Y^2 - 2Y f(X) + f(X)^2 \mid X \right] \right]$

$= \mathbb{E}\left[1 - 2\mathbb{E}[Y \mid X] f(X) + f(X)^2 \right]$

$= \mathbb{E}\left[1 - 2(2\eta(X) - 1) f(X) + f(X)^2 \right]$

$h(u) = 1 - 2(2\eta - 1)u + u^2$

$h'(u) = -2(2\eta - 1) + 2u$

$\Rightarrow u^* = 2\eta - 1$

Donc $f^*(X) = 2\eta(X) - 1$

3) $A(f) = \mathbb{E}\left[e^{-Yf(X)} \right]$

$= \mathbb{E}\left[e^{-f(X)} \eta(X) + e^{f(X)} (1-\eta(X)) \right]$

$h(u) = e^{-u}\eta + e^{u}(1-\eta)$

$h'(u) = -e^{-u}\eta + e^{u}(1-\eta)$

$h'(u) = 0 \iff \eta = e^{2u}(1-\eta)$

$\iff u = \frac{1}{2} \ln\left(\frac{1-\eta}{\eta} \right)$

$f^*(X) = \frac{1}{2} \ln\left(\frac{1-\eta(X)}{\eta(X)} \right)$

(1) Si $Y \sim Ber(p)$, ...

4) $A(f) = \mathbb{E}\left[\log\left(1 + e^{-Yf(X)}\right)\right]$

$$= \mathbb{E}\left[\log\left(1+e^{-f(X)}\right)\eta(X) + \log\left(1+e^{f(X)}\right)(1-\eta(X))\right]$$

$$h(u) = \log(1+e^{-u})\,\eta + \log(1+e^{u})(1-\eta)$$

$$h'(u) = \frac{-e^{-u}}{1+e^{-u}}\,\eta + \frac{e^{u}}{1+e^{u}}(1-\eta)$$

$$h'(u) = 0 \iff \frac{e^{u}}{1+e^{u}}(1-\eta) = \frac{1}{e^{u}+1}\,\eta$$

$$\iff u = \log\left(\frac{\eta}{1-\eta}\right)$$

$$f^*(X) = \log\left(\frac{\eta(X)}{1-\eta(X)}\right)$$

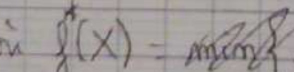
SVM 5) $A(f) = \mathbb{E}\left[\max\{0, 1 - Yf(X)\}\right]$

$$= \mathbb{E}\left\{\max\{0, 1-f(X)\}\,\eta(X) + \max\{0, 1+f(X)\}(1-\eta(X))\right\}$$

$$= \mathbb{E}\left[\mathbb{1}_{\{f(X)\leq 1\}}(1-f(X))\,\eta(X) + \mathbb{1}_{\{f(X)\geq -1\}}(1+f(X))(1-\eta(X))\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{f(X)\leq -1\}}(1-f(X))\,\eta(X) + \mathbb{1}_{\{-1\leq f(X)\leq 1\}}(1+f(X) - 2\eta(X)f(X)) + \mathbb{1}_{\{f(X)\geq 1\}}(1-\eta(X) + (1-\eta(X))f(X))\right]$$

Fonction affine par morceaux, continue



d'où $f^*(X) = \cancel{\min\{2(1-\eta(X)), 2\eta(X)\}}$

$$= \mathbb{1}_{\{\eta(X) > \frac{1}{2}\}} - \mathbb{1}_{\{\eta(X) < \frac{1}{2}\}}$$

ranking

(8) $(X,Y), (X',Y') \in \mathbb{R}^d \times \mathcal{Y}$

1) $L(\pi) = \mathbb{P}((Y-Y')\pi(X,X') < 0)$

$= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{(Y-Y')\pi(X,X')<0\}} \mid X,X']]$

$= \mathbb{E}[\mathbb{E}[\mathbb{1}_{\{Y-Y'<0\}} \mid X,X'] \mathbb{1}(\pi(X,X') = 1)$
$+ \mathbb{E}[\mathbb{1}_{\{Y-Y'>0\}} \mid X,X'] \mathbb{1}(\pi(X,X') = -1)]$

$= \mathbb{E}[e_-(X,X') \mathbb{1}(\pi(X,X') = 1) + e_+(X,X') \mathbb{1}(\pi(X,X') = -1)]$

$\pi^*(X,X') = 2 \, \mathbb{1}_{\{e_-(X,X') < e_+(X,X')\}} - 1$

$L(\pi^*) = \mathbb{P}((Y-Y')\pi^*(X,X') < 0)$
$= \mathbb{E}[\min\{e_-(X,X'), e_+(X,X')\}]$

$L(\pi) - L^* = \mathbb{E}[e_+ \mathbb{1}_{\{\pi=-1\}} + e_- \mathbb{1}_{\{\pi=1\}} - e_+ \mathbb{1}_{\{\pi^*=-1\}} - e_- \mathbb{1}_{\{\pi^*=1\}}]$

$= \mathbb{E}[e_+ (\mathbb{1}_{\{\pi=-1\}} - \mathbb{1}_{\{\pi^*=-1\}}) + e_-(\mathbb{1}_{\{\pi=1\}} - \mathbb{1}_{\{\pi^*=1\}})]$

$= \mathbb{E}[(e_+ - e_-)(\mathbb{1}_{\{\pi=-1\}} - \mathbb{1}_{\{\pi^*=-1\}})]$

Or $\mathbb{1}_{\{\pi=-1\}} - \mathbb{1}_{\{\pi^*=-1\}} = \mathbb{1}_{\{\pi \neq \pi^*\}} \operatorname{sgn}(e_+ - e_-)$

Donc $L(\pi) - L^* = \mathbb{E}[|e_+ - e_-| \mathbb{1}_{\{\pi \neq \pi^*\}}]$

$Y \sim Ber(0)$, $\varepsilon_i \sim Ber(1/2) \Rightarrow Y\varepsilon_i \sim Ber(1/2)$

2) $\rho_+(X, X') = \mathbb{P}(Y - Y' > 0 \mid X, X')$

$= \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{Y=1} \mathbb{1}_{Y'=-1} \mid X, X']\right]$

$= \eta(X)(1 - \eta(X'))$

$\rho_-(X, X') = (1 - \eta(X))\eta(X')$

$\eta = \frac{1}{2}, 0, 1$

3) $\mathcal{Y} = \mathbb{R}$, $Y = m(X) + \sigma(X)N$

$\rho_+(X, X') = \mathbb{P}(\underbrace{m(X) + \sigma(X)N - m(X') - \sigma(X')N'}_{Z} > 0 \mid X, X')$

$Z \sim \mathcal{N}(m(X) - m(X'), \sigma^2(X) + \sigma^2(X'))$

$\rho_+(X, X') = 1 - F_Z(0)$

$\rho_- = F_Z(0)$

pb si $m(X) \approx m(X')$