

Ex 1)

$$\mathcal{L}(X|Y=0) = \mathcal{U}([0, \theta]), \quad \theta \in (0,1)$$

$$\mathcal{L}(X|Y=1) = \mathcal{U}([0, 1])$$

$$p = P(Y=1)$$

$$\varphi(x) = P(Y=1 | X=x).$$

D'après le cours, on a

$$\varphi(x) = \frac{p f_+(x)}{p f_+(x) + (1-p) f_-(x)} \quad \text{avec } f_+ \text{ & } f_- \text{ densités cond. aux classes.}$$

$$= \frac{p \mathbf{1}_{[0,1]}(x)}{p \mathbf{1}_{[0,1]}(x) + (1-p) \frac{\mathbf{1}_{[0,\theta]}(x)}{\theta}}.$$

Si $\theta < x < 1$, $\varphi(x) = p/p = 1$

Si $x < 0$ ou $x > 1$, $\varphi(x) = 0$ (indéfini)

Si $0 \leq x \leq \theta$, $\varphi(x) = \frac{p}{p + \frac{(1-p)}{\theta}} = \frac{\theta p}{\theta p + 1 - p}$

$$\text{Donc, } \eta(x) = \begin{cases} 1 & \text{si } \theta < x \\ \frac{\theta p}{\theta p + 1-p} & \text{si } 0 < x \leq \theta \end{cases}$$

$$\text{Si } \theta = 1/2, \quad \eta(x) = \begin{cases} 1 & \frac{1}{2} < x \\ \frac{p_1}{p_1 + 1-p} & 0 < x \leq \frac{1}{2} \end{cases}$$

Ex 2)

⚠ Dans le cours, $y \in \{-1, 1\}\!$

$$L_x = P_x, \quad L_{y_1|x=x} = \text{Ber}(\eta(x)), \quad \eta(x) = \frac{x}{x+\theta},$$

$$\forall \theta > 0.$$

$$\text{Par définition, } g^*(x) = \mathbb{1}_{\{\eta(x) > y_1\}}$$

$$\text{Dans notre cas, } \eta(x) > \frac{1}{2} (\Rightarrow \frac{x}{x+\theta} > \frac{1}{2})$$

$$(\Rightarrow \frac{2x}{x+\theta} > 1 \Rightarrow x + \theta < 2x \\ \Rightarrow \theta < x)$$

$$\text{Donc, } g^*(x) = \mathbb{1}_{\{x \geq \theta\}}.$$

$$\text{Par définition, } L(g^*) = L^* = \mathbb{E} \left[\min \{ g(x), 1 - g(x) \} \right]$$

$$= \mathbb{E} \left[\mathbb{1}_{\{x \leq \theta\}} g(x) + \mathbb{1}_{\{x > \theta\}} (1 - g(x)) \right]$$

Si $x \leq \theta$, alors $g(x) \leq \frac{1}{2}$
donc $\min \{ g(x), 1 - g(x) \} = g(x)$.

On a la densité de la distribution marginale P_x :

$$P_x: \mathbb{R} \rightarrow \mathbb{R}^+$$

$$x \mapsto \frac{\mathbb{1}_{[0, x\theta]}}{\alpha \theta}, \quad \forall x > 1.$$

On peut donc écrire l'espérance:

$$L^* = \int_{\mathbb{R}} \mathbb{1}_{\{x \leq \theta\}} g(x) + \mathbb{1}_{\{x > \theta\}} (1 - g(x)) dP_x(x)$$

$$= \int_{\mathbb{R}} \left[\mathbb{1}_{\{x \leq \theta\}} g(x) + \mathbb{1}_{\{x > \theta\}} (1 - g(x)) \right] P(x) dx$$

$$= \frac{1}{\alpha \theta} \int_0^{x\theta} \mathbb{1}_{\{x \leq \theta\}} g(x) + \mathbb{1}_{\{x > \theta\}} (1 - g(x)) dx$$

$$L^* = \frac{1}{\alpha\theta} \left(\int_0^\alpha 2(x) dx + \int_\theta^{\alpha\theta} 1 - 2(x) dx \right)$$

$$= \frac{1}{\alpha\theta} \left(\int_0^\theta 1 - (1 - 2(x)) dx + \int_\theta^{\alpha\theta} 2(x) dx \right)$$

$$= \frac{1}{\alpha\theta} \left(\int_0^\theta 2 - \frac{\theta}{x+\theta} dx + \int_\theta^{\alpha\theta} \frac{\theta}{x+\theta} dx \right)$$

$$= \frac{1}{\alpha\theta} \left(\left[x - \theta \ln(x+\theta) \right]_0^\theta + \left[\theta \ln(x+\theta) \right]_\theta^{\alpha\theta} \right)$$

$$= \frac{1}{\alpha\theta} \left(\theta - \theta \ln(2\theta) + \theta \ln(\theta) + \theta \ln(\alpha\theta + \theta) - \theta \ln(2\theta) \right)$$

$$= \frac{1}{\alpha\theta} \left(\theta - 2\theta \ln(2\theta) + \theta \ln \theta + \theta \ln(\theta(\alpha+1)) \right)$$

$$= \frac{1}{\alpha\theta} \left(\theta \left(1 - 2 \ln(2\theta) + 2 \ln \theta + \ln(\alpha+1) \right) \right)$$

$$= \frac{1}{\alpha} \left(1 + 2 \ln \left(\frac{\theta}{2\theta} \right) + \ln(\alpha+1) \right)$$

$$= \frac{1}{\alpha} \left(1 - 2 \ln 2 + \ln(\alpha+1) \right)$$

On peut calculer la dérivée :

$$\begin{aligned}\frac{\partial L^*}{\partial \alpha} &= \frac{1}{\alpha} \frac{1}{\alpha+1} - \left(1 - 2\ln 2 + \ln \alpha + 1\right) \frac{1}{\alpha^2} \\ &= \frac{1}{\alpha^2} \left(2\ln 2 - \ln \alpha + 1\right) + \frac{1}{\alpha} \frac{1}{\alpha+1}\end{aligned}$$

Si on développe, on voit que

$$\frac{\partial L^*}{\partial \alpha} > 0 \iff \frac{2^{2\alpha+2}}{e} > (\alpha+1)^{\alpha+1}.$$

Par croissance comparée, on sait que, à partir d'un certain α^* , $(\alpha^*+1)^{\alpha^*+1}$ domine $\frac{2^{2(\alpha^*+1)}}{e}$. Si $\alpha = 1$, $\frac{\partial L^*}{\partial \alpha} > 0$ et si $\alpha = 2$, $\frac{\partial L^*}{\partial \alpha} < 0$. Donc $\alpha^* \in]1, 2[$.

On peut calculer $\alpha^* \approx 1,80$.

On peut même se rappeler si $h = (\alpha+1) e^{1/\alpha+1}$
(aucune solution analytique).

E_x 3]

$X = (T, U, V)$ ou T, U, V iid et $T, U, V \sim E(1)$.

$$Y = \mathbb{1}_{\{T+U+V < \theta\}} \quad \forall \theta > 0.$$

$$\mathcal{L}_x = E(1)^{\otimes 3}$$

1) V n'est pas observé

$$\begin{aligned} g(x) &= P(Y=1 \mid X=x) = P(T+U+V < \theta \mid T, U) \\ &= P(V < \theta - T - U \mid T, U) \end{aligned}$$

On fixe $T=t$ et $U=u$. On a

$$\begin{aligned} g(x) &= \mathbb{E}\left[\mathbb{1}_{\{V < \theta - t - u\}}\right] \\ &= \int_0^{\theta-t-u} \underbrace{\mathbb{1}_{\{t+u < \theta\}}}_{\text{S: } t+u > \theta, \forall v \in \mathbb{R}^+} e^{-v} dv \end{aligned}$$

$$\begin{aligned} &= \mathbb{1}_{\{t+u < \theta\}} \left[-e^{-v} \right]_0^{\theta-t-u} = \mathbb{1}_{\{t+u < \theta\}} \\ &\quad (1 - e^{-(\theta-t-u)}) \end{aligned}$$

On calcule le classifieur de Bayes :

$$g^*(x) = \mathbb{1}_{\{\gamma(x) > \gamma_1\}}$$

$$\gamma(x) > \frac{1}{2} \quad (\Rightarrow) \quad \begin{cases} t+n \leq \theta \\ e^{-(\theta-t-n)} < \frac{1}{2} \end{cases} \quad (\Rightarrow) \quad \begin{cases} t+n \leq \theta \\ \theta - t - n > \ln 2 \end{cases}$$

$$(\Rightarrow) \quad \theta > t + n + \ln 2$$

$$g^*(x) = \mathbb{1}_{\{t+n < \theta - \ln 2\}}$$

On peut calculer l'erreur de Bayes :

$$L^* = \mathbb{E}_{t,n} \left[\min \{ \gamma(x) ; 1 - \gamma(x) \} \right]$$

$$= \mathbb{E}_{t,n} \left[\mathbb{1}_{\{t+n \geq \theta - \ln 2\}} \gamma(x) + \mathbb{1}_{\{t+n < \theta - \ln 2\}} (1 - \gamma(x)) \right]$$

Calculons la densité de $Z = T + U$:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_T(u) f_U(z-u) du$$

$$\begin{aligned}
 f_t(z) &= \int_{-\infty}^{+\infty} \mathbb{1}_{\mathbb{R}_+}(k) \prod_{k>z} (z-k) e^{-k} e^{-(z-k)} dk \\
 &= \int_0^z e^{-k} e^{-(z-k)} dk \\
 &= \int_0^z e^{-2} dk = e^{-z} \lambda([0,z]) = z e^{-z} -
 \end{aligned}$$

On peut donc ré-écrire l'espérance :

$$L^* = \mathbb{E}\left[\frac{1}{z} \left(\mathbb{1}_{\{z \geq \theta - \ln 2\}} \varrho(z) + \mathbb{1}_{\{z < \theta - \ln 2\}} (1 - \varrho(z)) \right) \right]$$

$$\text{avec } \varrho(z) = (1 - e^{-\theta+z}) \mathbb{1}_{\{z < \theta\}}.$$

$$\begin{aligned}
 L^* &= \int_0^{\theta - \ln 2} (1 - \varrho(z)) f_t(z) dz + \int_{\theta - \ln 2}^\theta \varrho(z) f_t(z) dz \\
 &= \int_0^{\theta - \ln 2} e^{-\theta+z} z e^{-z} dz + \int_{\theta - \ln 2}^\theta (1 - e^{-\theta+z}) z e^{-z} dz \\
 &= \int_0^{\theta - \ln 2} e^{-\theta} z dz + \int_{\theta - \ln 2}^\theta z e^{-z} - z e^{-\theta} dz
 \end{aligned}$$

$$= e^{-\theta} \left[\frac{1}{2} z^2 \right]_0^{\theta - \ln 2} + \underbrace{[-z e^{-z}]_{\theta - \ln 2}^\theta}_{\theta - \ln 2} + \underbrace{[-e^{-z}]_{\theta - \ln 2}^\theta}_{\theta - \ln 2} - e^{-\theta} \left[\frac{1}{2} z^2 \right]_{\theta - \ln 2}^\theta$$

$$\begin{aligned}
 \int z e^{-z} dz &= \int_a^b z (e^{-z})' dz \\
 &= [-ze^{-z}]_a^b - \int_a^b -e^{-z} dz \\
 &= [-ze^{-z}]_a^b + [-e^{-z}]_a^b
 \end{aligned}$$

$$\begin{aligned}
 U^* &= e^{-\theta} \frac{(\theta - \ln 2)^2}{2} - \theta e^{-\theta} + (\theta - \ln 2) e^{-(\theta - \ln 2)} - e^{-\theta} + e^{-(\theta - \ln 2)} \\
 &\quad - \frac{\theta^2}{2} e^{-\theta} + \frac{(\theta - \ln 2)^2}{2} e^{-\theta} \\
 &= -e^{-\theta} \left(1 + \theta + \frac{\theta^2}{2} - (\theta - \ln 2)^2 \right) + e^{-(\theta - \ln 2)} \left(1 + (\theta - \ln 2) \right) \\
 &= -e^{-\theta} \left(1 + \theta + \frac{\theta^2}{2} - (\theta - \ln 2)^2 \right) + 2e^{-\theta} (\theta - \ln 2) \\
 &= e^{-\theta} \left(-1 - \theta - \frac{\theta^2}{2} + (\theta - \ln 2)^2 + 2 + 2\theta - 2\ln 2 \right) \\
 &= e^{-\theta} \left(-1 - \theta - \frac{\theta^2}{2} + \theta^2 - 2\theta\ln 2 + \ln^2 2 + 2 + 2\theta - 2\ln 2 \right) \\
 &= e^{-\theta} \left(1 + \theta + \frac{\theta^2}{2} - 2\ln 2 (1 + \theta) + \ln^2 2 \right)
 \end{aligned}$$

$$S: \Theta = 9, L^* \approx 0,006.$$

2) Seulement T est observé.

$$\varrho(x) = P(T+U+V < \theta | T)$$

$$= P(U+V < \theta - t)$$

On pose Z = U+V.

$$\begin{aligned} \varrho(z) &= \int_0^{\theta-t} z e^{-t} dt = [z e^{-z}]_0^{\theta-t} - [e^{-z}]_0^{\theta-t} \\ &= -(\theta-t) e^{-(\theta-t)} - e^{-(\theta-t)} + 1 \\ &= 1 + ((\theta-t)-1) e^{-(\theta-t)} \\ &= 1 - (\theta-t+1) e^{-(\theta-t)} \end{aligned}$$

$$\varrho(x) > \frac{1}{2} \Leftrightarrow \begin{cases} \theta < t \\ 1 - (\theta-t+1) e^{-(\theta-t)} > \frac{1}{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} \theta < t \\ (\theta-t+1) e^{-(\theta-t)} < \frac{1}{2} \end{cases}$$

$$\Leftrightarrow e^{-(\theta-t)} < \frac{1}{2(\theta-t+1)}$$

$$\Leftrightarrow t - \theta < -\log(2(\theta-t+1))$$

$$\Leftrightarrow t - \theta < -\log 2 - \log(\theta-t+1)$$

$$\Leftrightarrow \theta - t - \log(\theta-t+1) > \log 2$$

$$g(t) > \frac{1}{2} \Leftrightarrow \theta - t - \log(\theta-t+1) > \log 2$$

$$g^*(t) = \mathbb{1}_{\{\theta - t - \log(\theta-t+1) > \log 2\}}$$

$$L^* = \mathbb{E} \left[\min \{ g(t) ; 1 - g(t) \} \right]$$

$$= \mathbb{E} \left[\mathbb{1}_{\{\theta - t - \log(\theta-t+1) \leq \log 2\}} + \mathbb{1}_{\{\theta - t - \log(\theta-t+1) > \log 2\}} \right]$$

$$= \int_0^\theta \left[\mathbb{1}_{\{.. \}} 1 - (\theta-t+1)e^{-(\theta-t)} + \mathbb{1}_{\{..\}} (\theta-t+1)e^{-(\theta-t)} \right] e^{-t} dt$$

Si $\theta = 9$

$$L^* \approx 0,0059$$

Plus grand que lorsqu'en T et U sont observés.

3) Aucun n'est observé.

On pose $Z = T + U$ et $S = V + Z$. On calcule la densité de S , $f_S(s) = \int_0^{+\infty} ze^{-z} e^{-(s-z)} \mathbb{1}_{\mathbb{R}_+}(s-z) dz$

$$= \int_0^s ze^{-s} dz = e^{-s} \left[\frac{z^2}{2} \right]_0^s = \frac{s^2 e^{-s}}{2}$$

On calcule $\gamma(x) = P(Y=1 | X=x) = P(S < \theta)$

$$= \int_0^\theta \frac{s^2 e^{-s}}{2} ds$$
$$= \left[-\frac{s^2 + 2s + 2}{2} e^{-s} \right]_0^\theta = 1 - \frac{e^{-\theta}}{2} (\theta^2 + 2\theta + 2)$$

$$\gamma(x) \rightarrow \frac{1}{2} \quad (\Rightarrow e^{-\theta} (\theta^2 + 2\theta + 2) \leq 1)$$

$$g^* = \mathbb{1}\{e^{-\theta} (\theta^2 + 2\theta + 2) \leq 1\}$$

$$L^* = \mathbb{E}[\min \{ \gamma(x) ; 1 - \gamma(x) \}]$$

$$= \mathbb{E}\left[\mathbb{1}\{e^{-\theta} (\theta^2 + 2\theta + 2) \geq 1\} \gamma(x) + \mathbb{1}\{\dots\} (1 - \gamma(x))\right]$$

Si $\theta = 3$,

$$L^* \approx 0,0062 \quad \text{Encore plus grand que } q^* L !$$

Ex 1

$$1) f^+(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_+)^T \Sigma_+^{-1} (x - \mu_+)\right)$$

$$f^-(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_-)^T \Sigma_-^{-1} (x - \mu_-)\right)$$

$$2) \frac{p f_+}{p f_+ + (1-p) f_-} = \frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu_+)^T \Sigma_+^{-1} (x - \mu_+)\right)$$

$$\text{Donc, } 2(x) = \frac{(2\pi)^{d/2} |\Sigma_+|^{1/2}}{(2\pi)^{d/2} |\Sigma_+|^{1/2} e^{-\frac{1}{2}(x - \mu_+)^T \Sigma_+^{-1} (x - \mu_+)} + (2\pi)^{d/2} |\Sigma_-|^{1/2} e^{-\frac{1}{2}(x - \mu_-)^T \Sigma_-^{-1} (x - \mu_-)}}$$

$$\text{On pose } \pi_{\{+, -\}} = (2\pi)^{d/2} |\Sigma_{\{+, -\}}|^{1/2} \text{ et}$$

$$e_{\{+, -\}} = \exp\left(-\frac{1}{2} ((x - \mu_{\{+, -\}})^T \Sigma_{\{+, -\}}^{-1} (x - \mu_{\{+, -\}}))\right)$$

Etudions $\frac{p e_+}{\pi_+} + \frac{(1-p) e_-}{\pi_-}$:

$$\begin{aligned}
 \frac{\pi_+ - p}{\pi_+ + \pi_-} e_+ + \frac{(1-p)\pi_+ + c_-}{\pi_+ + \pi_-} &= \frac{\pi_+ - p e_+ + (1-p)\pi_+ e_-}{\pi_+ + \pi_-} \\
 &= \underbrace{p \pi_- e_+}_{\pi_+ + \pi_-} + \underbrace{\pi_+ e_- - p \pi_+ e_-}_{\pi_+ + \pi_-} \\
 &= \frac{p \pi_- e_+ + (1-p)\pi_+ e_-}{\pi_+ + \pi_-}
 \end{aligned}$$

$$\begin{aligned}
 \eta(x) &= \frac{p e_+}{\pi_+} \frac{\pi_+ \pi_-}{p \pi_- e_+ + (1-p)\pi_+ e_-} = \frac{p e_+ \pi_-}{p(\pi_- e_+ + (1-p)\pi_+ e_-)} \\
 &\quad \underbrace{e_+ \pi_-}_{e_+ \pi_- \left(1 + \frac{1-p}{p} \frac{\pi_+ e_-}{\pi_- e_+}\right)} \\
 &= \frac{1}{1 + \frac{1-p}{p} \frac{(2\pi)^{d/2} |\sum_+|^{1/2} \exp(-\frac{1}{2} (x - \mu_+)^T \sum_+^{-1} (x - \mu_+))}{(2\pi)^{d/2} |\sum_-|^{1/2} \exp(-\frac{1}{2} (x - \mu_-)^T \sum_-^{-1} (x - \mu_-))}} \\
 &= \frac{1}{1 + \frac{1-p}{p} \left| \frac{\sum_+}{\sum_-} \right|^{1/2} \exp\left(-\frac{1}{2} ((x - \mu_+)^T \sum_+^{-1} (x - \mu_+) - (x - \mu_-)^T \sum_-^{-1} (x - \mu_-))\right)}
 \end{aligned}$$

$$2) \log \left(\frac{\varrho(x)}{1-\varrho(x)} \right) = \theta^T x$$

$$\Rightarrow \frac{\varrho(x)}{1-\varrho(x)} = e^{\theta^T x}$$

On peut donc écrire :

$$\varrho(x) \left(1 + e^{\theta^T x} \right) = \varrho(x) + \frac{\varrho^2(x)}{1-\varrho(x)}$$

$$= \frac{\varrho(x) - \varrho^2(x) + \varrho^2(x)}{1-\varrho(x)} = \frac{\varrho(x)}{1-\varrho(x)} = e^{\theta^T x}$$

On obtient :

$$\varrho(x) = \frac{e^{\theta^T x}}{1+e^{\theta^T x}} = \frac{1}{1+e^{-\theta^T x}}.$$

Par définition, on sait que $\varrho(x) = \frac{p f_+}{f_x}$,
avec $f_x = p f_+ + (1-p) f_-$.

$$\text{On a donc } \begin{cases} f_+ = \frac{1}{p} \\ f_- = \frac{f_x - p f_+}{1-p} = \frac{1+e^{-\theta^T x}-1}{1-p} = \frac{e^{-\theta^T x}}{1-p} \end{cases}$$

Ex 5)

1) $w \in [0, 1]$.

$$L_w(q_x) = 2 \mathbb{E}_{x,y} \left[(1-w) \mathbb{1}_{\{y=1\}} \mathbb{1}_{\{g(x)=-1\}} + w \mathbb{1}_{\{y=-1\}} \mathbb{1}_{\{g(x)=1\}} \right]$$

$$= \mathbb{E}_{x,y} \left[(1-w) \underbrace{\mathbb{E}_y \left[\mathbb{1}_{\{y=1\}} | x \right] \mathbb{1}_{\{g(x)=-1\}}} + w \mathbb{E}_y \left[\mathbb{1}_{\{y=-1\}} | x \right] \mathbb{1}_{\{g(x)=1\}} \right]$$

Law of total expectation

$$= \mathbb{E} \left[(1-w) \eta(x) \mathbb{1}_{\{g(x)=-1\}} + w (1-\eta(x)) \mathbb{1}_{\{g(x)=1\}} \right]$$

Donc, si $g^*(x) = 1 \Leftrightarrow w(1-\eta(x)) \leq (1-w)\eta(x)$

(g^* minimise L . Donc, si $g^*(x)=1$, $w(1-\eta(x))$ est plus petit que $w\eta(x)$)

$$\Leftrightarrow w - w\eta(x) \leq \eta(x) - w\eta(x)$$

$$\Leftrightarrow w \leq \eta(x)$$

Donc, $g^*(x) = 2 \mathbb{1}_{\{w < \eta(x)\}} - 1$.

Calculons $L_w(g^*)$

$$\begin{aligned} L_w(g^*) &= 2 \mathbb{E} \left[\mathbb{1}_{\{\eta(x) < w\}} (1-w) \eta(x) + \mathbb{1}_{\{w \leq \eta(x)\}} w (1-\eta(x)) \right] \\ &= 2 \mathbb{E} \left[\left(\mathbb{1}_{\{\eta(x) < w\}} + \mathbb{1}_{\{w \leq \eta(x)\}} \right) \min \left\{ (1-w) \eta(x), w (1-\eta(x)) \right\} \right] \\ &= 2 \mathbb{E} \left[\min \left\{ (1-w) \eta(x), w (1-\eta(x)) \right\} \right] \end{aligned}$$

$$2) \quad \mu \in [0, 1]$$

On cherche g tq. $g \in \underset{g}{\operatorname{argmin}} \mathbb{P}(Y \neq g(X))$

$$\text{s.t. } \mathbb{P}(g(X) = 1) = \mu.$$

On peut trouver un q tq $\mathbb{P}(\eta(X) > q) = \mu$.

$$\begin{aligned} \text{Posons } q &= q_{1-\mu}(\eta(X)) \quad \left(\mathbb{P}(\eta(X) \leq q) = 1-\mu \Leftrightarrow q = \mathbb{P}(\eta(X) \leq q) \right. \\ &\quad \left. = \mu \right) \\ &\quad \Leftrightarrow \mathbb{P}(\eta(X) > q) = \mu \end{aligned}$$

$$\text{On a } L_\mu(g) = \mathbb{E} \left[\mathbb{1}_{\{g(X) = -1\}} \eta(X) + \mathbb{1}_{\{g(X) = 1\}} \eta(X) \right]$$

$$= \mathbb{E} \left[(\eta(X) - q) \mathbb{1}_{\{g(X)=1\}} + q \mathbb{1}_{\{g(X)=1\}} + (q - \eta(X)) \mathbb{1}_{\{g(X)=1\}} (1-q) \mathbb{1}_{\{g(X)=1\}} \right]$$

Or,

$$- \mathbb{E} \left[q \mathbb{1}_{\{g(X)=1\}} \right] = q \mathbb{P}(g(X) \neq 1) = q(1-n)$$

$$- \mathbb{E} \left[(1-q) \mathbb{1}_{\{g(X)=1\}} \right] = (1-q) \mathbb{P}(g(X)=1) = (1-q)n$$

On peut donc écrire

$$\text{L}_n(q) = \mathbb{E} \left[(\eta(X) - q) \mathbb{1}_{\{g(X)=1\}} + (q - \eta(X)) \mathbb{1}_{\{g(X)=1\}} \right] + q(1-n) + (1-q)n.$$

Par définition, $g^*(X) = 1 \Leftrightarrow q < \eta(X) \leq q$

$$\Leftrightarrow -2\eta(X) < -2q$$

$$\Leftrightarrow q < \eta(X)$$

Donc, $g^*(X) = 2 \mathbb{1}_{\{q < \eta(X)\}} - 1$ et, $\mathbb{P}(g^*(X)=1) = \mathbb{P}(\eta(X) > q) = n$.

$$\text{L}_n^* = \mathbb{E} \left[\min \left\{ \eta(X) - q ; q - \eta(X) \right\} \right] + q(1-n) + (1-q)n.$$

3) ① est une valeur spécifique de rejet.
 $\gamma \in]0, \frac{1}{2}[$.

$$\begin{aligned}
 L_{\gamma}^q(g) &= P(Y \neq g(X), g(X) \neq \textcircled{1}) + \gamma P(g(X) = \textcircled{1}) \\
 &= E[\mathbb{1}_{\{g(X)=-1\}} \gamma^2(X) + \mathbb{1}_{\{g(X)=1\}} (1-\gamma^2(X))] + \gamma P(g(X) = \textcircled{1}) \\
 &= E[\mathbb{1}_{\{g(X)=-1\}} \gamma^2(X) + \mathbb{1}_{\{g(X)=1\}} (1-\gamma^2(X)) + \gamma \mathbb{1}_{\{g(X) = \textcircled{1}\}}]
 \end{aligned}$$

$\Leftrightarrow g^+(X) = 1 \Leftrightarrow 1 - \gamma^2(X) < \gamma(X) \wedge 1 - \gamma^2(X) < \gamma$
 $\Leftrightarrow g^+(X) = -1 \Leftrightarrow \gamma^2(X) < 1 - \gamma^2(X) \wedge \gamma(X) < \gamma$
 $\Leftrightarrow g^+(X) = \textcircled{1} \Leftrightarrow \gamma < \gamma^2(X) \wedge \gamma < 1 - \gamma^2(X)$

Donc

$$g^+(X) = \begin{cases} 1 & \text{si } 1 - \gamma^2(X) < \gamma^2(X) \wedge 1 - \gamma^2(X) < \gamma \\ -1 & \text{si } \gamma^2(X) < 1 - \gamma^2(X) \wedge \gamma(X) < \gamma \\ \textcircled{1} & \text{si } \gamma < \gamma^2(X) \wedge \gamma < 1 - \gamma^2(X) \end{cases}$$

$\forall t \in]0, \frac{1}{2}[$

$$\gamma \leq 1 - \gamma$$

$$\text{Donc } g^+(X) < \gamma$$

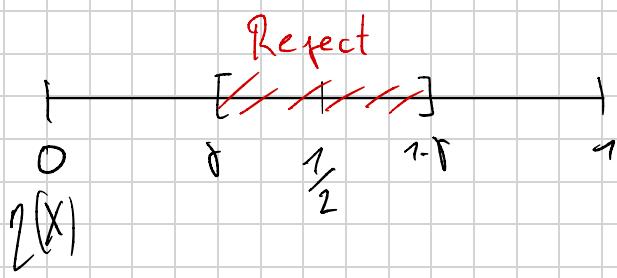
$$\Rightarrow \gamma^2(X) > \gamma > \gamma^2(X)$$

$$\text{et } 1 - \gamma^2(X) < \gamma$$

$$\Leftrightarrow -\gamma^2(X) < \gamma - 1$$

$$\Rightarrow \gamma^2(X) > 1 - \gamma > \gamma > \gamma^2(X)$$

$$= \begin{cases} 1 & \text{si } 1 - \gamma^2(X) < \gamma \\ -1 & \text{si } \gamma^2(X) < \gamma \\ \textcircled{1} & \text{si } \gamma < \gamma^2(X) < 1 - \gamma^2(X) \end{cases}$$



$$L(g) = \mathbb{E} \left[\gamma(x) \mathbb{1}_{\{g(x) = -1\}} + (1 - \gamma(x)) \mathbb{1}_{\{g(x) = 1\}} \right]$$

$$L^* = \mathbb{E} \left[\min \{ \gamma(x); 1 - \gamma(x) \} \right]$$

$$\begin{aligned} L(g) - L^* &= \mathbb{E} \left[\gamma(x) \mathbb{1}_{\{g(x) = -1\}} + (1 - \gamma(x)) \mathbb{1}_{\{g(x) = 1\}} \right. \\ &\quad \left. - \min \{ \gamma(x); 1 - \gamma(x) \} \right] \\ &= \mathbb{E} \left[\gamma(x) \cdot (1 - \gamma(x)) \mathbb{1}_{\{g(x) = -1 \wedge g^*(x) = 1\}} \right. \\ &\quad \left. + (1 - \gamma(x)) \cdot \gamma(x) \mathbb{1}_{\{g(x) = 1 \wedge g^*(x) = -1\}} \right] \\ &= \mathbb{E} \left[\mathbb{1}_{\{g^*(x) \neq g(x)\}} \left((2\gamma(x) - 1) \mathbb{1}_{\{g(x) = -1\}} \right. \right. \\ &\quad \left. \left. + (1 - 2\gamma(x)) \mathbb{1}_{\{g(x) = 1\}} \right) \right] \end{aligned}$$

$$= \mathbb{E} \left[\mathbb{1}_{\{\dots\}} \left(|2\gamma(x) - 1| \right) \right]$$

$$g^* \text{ est le classifier} = 2 \mathbb{E} \left[\mathbb{1}_{\{g(x) \neq g^*(x)\}} |2(x) - \frac{1}{2}| \right]$$

de Bayes. Donc, si:

$g^*(x) \neq g(x)$ alors

$$(2\gamma(x) - 1) \mathbb{1}_{\{g(x) = -1\}} > 0$$

$$\text{et } (1 - 2\gamma(x)) \mathbb{1}_{\{g(x) = 1\}} > 0$$

Master 2

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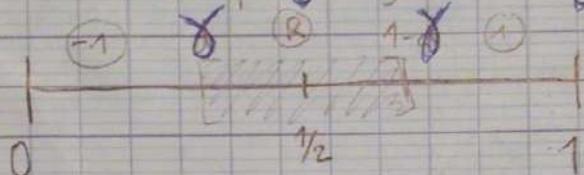
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$$g^*(x) = \begin{cases} -1 & \text{if } 1-\eta(x) \geq \eta(x) \in [1-\eta(x), \eta(x)] \\ 1 & \text{if } \eta(x) \geq 1-\eta(x) \in [\eta(x), 1-\eta(x)] \end{cases}$$

$$\begin{aligned} g^*(x) &= \begin{cases} -1 & \text{if } \eta(x) \leq 1-\eta(x) \text{ & } \eta(x) < d \\ 1 & \text{if } 1-\eta(x) \leq \eta(x) \leq 1-\eta(x) \leq d \\ \textcircled{R} & \text{if } d \leq \eta(x) \text{ & } d \leq 1-\eta(x) \end{cases} \\ &= \begin{cases} -1 & \text{if } \eta(x) < d \\ 1 & \text{if } \eta(x) \geq 1-d \\ \textcircled{R} & \text{if } \textcircled{X} < \eta(x) < 1-\textcircled{X} \end{cases} \end{aligned}$$



$$L(g^*) = \mathbb{E}[\min\{\eta(x); 1-\eta(x)\}]$$

$$\textcircled{6} \quad i) L_\omega(g) = \mathbb{E}[2\omega(y) \cdot \mathbf{1}_{\{Y \neq g(x)\}}]$$

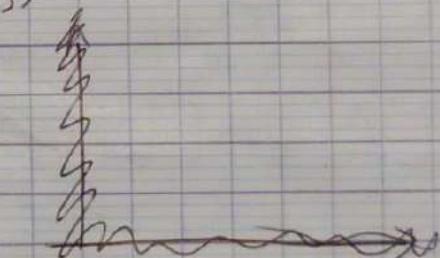
$$\textcircled{*}: g \mapsto P(\{g(x)=1 | Y=0\}) \quad \text{FP}$$

$$\textcircled{*}: g \mapsto P(\{g(x)=1 | Y=1\}) \quad \text{TP}$$

$$P(g(x)=1 | Y=1)$$

1
C increasing

$\textcircled{*}$
interpretation?



$$P(g(x)=1 | Y=0)$$

$$P(g(x)=1 | Y=1)$$

$$(L_\omega(g) - 2\mathbb{E}\omega(0)) \mathbf{1}_{\{g(x)=0\}} + (1-\mathbb{E}\omega(1)) \mathbf{1}_{\{g(x)=1\}}$$

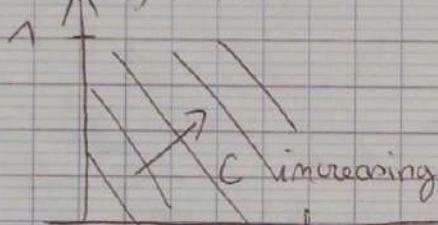
$$K_n(\mathcal{D}) \rightarrow 0$$

$$\begin{aligned}
 L_\omega(g) &= C = 2 \mathbb{E}[\mathbb{E}[\omega(Y) \mathbf{1}_{\{Y+g(X)\}} | Y]] \\
 &= 2 \mathbb{E}\left[\mathbf{1}_{\{Y=0\}} \omega(0) P(g(X)=1 | Y=0) + \mathbf{1}_{\{Y=1\}} \omega(1) P(g(X)=0 | Y=1)\right] \\
 &= 2\omega(0)P(Y=0)P(g(X)=1 | Y=0) + 2\omega(1)P(Y=1)(1-P(g(X)=1 | Y=1))
 \end{aligned}$$

$$\rightarrow P(g(X)=1 | Y=1) = 1 - \frac{C - 2\omega(0)(1-p)P(g(X)=1 | Y=0)}{2\omega(1)p}$$

$$2) P(g(X)=1) = C = P(g(X)=1 | Y=0) P(Y=0)$$

$$P(g(X)=1 | Y=1) = \frac{C - (1-p)P(g(X)=1 | Y=0)}{p}$$



$$\textcircled{7} \quad (X, Y) \in \mathbb{R}^d \times \{-1, 1\}$$

$$\mathbb{E}[\mathbf{1}_{\{-Yf(x)>0\}}]$$

Front: $L(g) = \mathbb{E}[\mathbf{1}_{\{-Yg(x)>0\}}]$, soft classifier: $f: X \rightarrow \mathbb{R}$

Mtn: $A(f) = \mathbb{E}[\Psi(-Yf(x))]$, Ψ cost function
associated classifier $g = \text{sgn}(f)$
↳ convex optim

$$1) R(g) = E[(y - g(x))^2]$$

$$= E[4(1_{\{Y=-1\}}\gamma(x) + 1_{\{Y=1\}}(1-\gamma(x)))]$$

$$g^*(x) = 2 \cdot 1_{\{\gamma(x) > \frac{1}{2}\}} - 1$$

$$2) R(f) = E[(y - f(x))^2]$$
~~$$= E[1_{\{Y=-1\}}\gamma(x)(1+f(x))^2 + 1_{\{Y=1\}}(1-\gamma(x))(1-f(x))^2]$$~~
~~$$g^*(x) = 1 \Leftrightarrow (1-\gamma(x))(1-f(x)) < \gamma(x)(1+f(x))^2$$~~

$$= E[E[Y^2 - 2Yf(x) + f(x)^2 | X]]$$

$$= E[1 - 2E[Y|X]f(x) + f(x)^2]$$

$$= E[1 - 2(2\gamma(x)-1)f(x) + f(x)^2]$$

$$\begin{aligned} h(u) &= 1 - 2(2\gamma - 1)u + u^2 \\ h'(u) &= -2(2\gamma - 1) + 2u \\ \Rightarrow u^* &= 2\gamma - 1 \end{aligned}$$

Donc $f^*(x) = 2\gamma(x) - 1$

$$3) A(f) = E[e^{-Yf(x)}]$$

$$= E[e^{-f(x)\gamma(x)} + e^{f(x)(1-\gamma(x))}]$$

$$h(u) = e^{-u}\gamma + e^u(1-\gamma)$$

$$h'(u) = -e^{-u}\gamma + e^u(1-\gamma)$$

$$h'(u) = 0 \Leftrightarrow \gamma - e^{2u}(1-\gamma)$$

$$\Leftrightarrow u = \frac{1}{2} \ln\left(\frac{1-\gamma}{\gamma}\right)$$

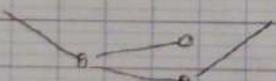
$$f^*(x) = \frac{1}{2} \ln\left(\frac{1-\gamma(x)}{\gamma(x)}\right)$$

$$\begin{aligned}
 4) A(f) &= E \left[\log \left(1 + e^{-Y f(x)} \right) \right] \\
 &= E \left[\log \left(1 + e^{-f(x)} \right) \eta(x) + \log \left(1 + e^{f(x)} \right) \right. \\
 &\quad \left. (1 - \eta(x)) \right] \\
 h(u) &= \log \left(1 + e^{-u} \right) \eta + \log \left(1 + e^u \right) (1 - \eta) \\
 h'(u) &= \frac{-e^{-u}}{1 + e^{-u}} \eta + \frac{e^u}{1 + e^u} (1 - \eta) \\
 h'(u) = 0 &\Rightarrow \frac{e^u}{1 + e^u} (1 - \eta) = \frac{1}{e^u + 1} \eta \\
 &\Rightarrow u = \log \left(\frac{\eta}{1 - \eta} \right) \\
 f^*(x) &= \log \left(\frac{\eta(x)}{1 - \eta(x)} \right)
 \end{aligned}$$

SVM 5) $A(f) = E \left[\max \{0, 1 - Y f(x)\} \right]$

$$\begin{aligned}
 &= E \left[\max \{0, 1 - f(x)\} \eta(x) + \max \{0, 1 + f(x)\} \right. \\
 &\quad \left. (1 - \eta(x)) \right] \\
 &= E \left[1_{\{f(x) \leq 1\}} \eta(x) + 1_{\{f(x) > 1\}} (1 - \eta(x)) \right] \\
 f^*(x) &= E \left[1_{\{f(x) \leq 1\}} (1 - f(x)) \eta(x) + 1_{\{f(x) > 1\}} (1 + f(x)) \right. \\
 &\quad \left. - 2 \eta(x) f(x) + 1_{\{f(x) \geq 1\}} (1 - \eta(x) + (1 - \eta(x)) f(x)) \right]
 \end{aligned}$$

Fonction affine par morceaux, continue



$$\begin{aligned}
 \text{donc } f^*(x) &= \min \left\{ 0, \max \{0, 1 - \eta(x)\}, 2\eta(x) \right\} \\
 &= 1_{\{\eta(x) > \frac{1}{2}\}} - 1_{\{\eta(x) < \frac{1}{2}\}}
 \end{aligned}$$

prob) Let T, T_1, T_2 be classes of

$$Q_n(\text{conv}(T)) = R_n(T)$$

(7)

kernel classes

$$x \in \mathcal{B}_n(C) \Leftrightarrow x^T x \geq \frac{1}{n} \forall i \in S$$

kernel function with feature
 $\phi(x), \phi(x')$ where $x \mapsto X_1, \dots, X_n$
 define the Union
 of functions $H = \{f \mapsto \int f d\mu\}$

works) Consider no i.i.d.
 R is k-Lipschitz and define,

$$\|f\| \leq \int |f| d\mu \leq \int g d\mu$$

then $\int f d\mu = \int g d\mu$
 for average

show that on $\mathcal{X} = \{x\}$

ranking

$$(8) \quad (x, y), (x', y') \in \mathbb{R}^d \times \mathcal{Y}$$

$$1) \quad L(\pi) = P((y - y') \pi(x, x') < 0)$$

$$= E[E[1_{\{(y-y')\pi(x,x') < 0\}} | x]]$$

$$= E[E[1_{\{(y-y')\pi(x,x') < 0\}} | x] \mathbb{P}(\pi(x, x') = 1)] + E[1_{\{(y-y') > 0\}} | x] \mathbb{P}(\pi(x, x') = -1)]$$

$$= E[e_-(x, x') \mathbb{P}(\pi(x, x') = 1) + e_+(x, x') \mathbb{P}(\pi(x, x') = -1)]$$

~~$e_-(x, x') = e^{-\pi(x, x')}$~~

$$\pi^*(x, x') = 2 \cdot 1_{\{e_-(x, x') < e_+(x, x')\}} - 1$$

$$L(\pi^*) = P((y - y') \pi^*(x, x') < 0)$$

$$= E[\min\{e_-(x, x'), e_+(x, x')\}]$$

$$L(\pi) - L^* = E[e_+ 1_{\{\pi=-1\}} + e_- 1_{\{\pi=1\}} - e_+ 1_{\{\pi^*=-1\}} - e_- 1_{\{\pi^*=1\}}]$$

$$= E[e_+ (1_{\{\pi=-1\}} - 1_{\{\pi^*=-1\}}) + e_- (1_{\{\pi=1\}} - 1_{\{\pi^*=1\}})]$$

$$= E[(e_+ - e_-)(1_{\{\pi=-1\}} - 1_{\{\pi^*=-1\}})]$$

$$\text{On } 1_{\{\pi=-1\}} - 1_{\{\pi^*=-1\}} = 1_{\{\pi \neq \pi^*\}} \text{ sgn}(e_+ - e_-)$$

$$\text{Donc } L(\pi) - L^* = E[(e_+ - e_-) 1_{\{\pi \neq \pi^*\}}]$$

$$\text{Ber}(0), \varepsilon_i \sim \text{Ber}(y_2) \Rightarrow y\varepsilon_i \sim \text{Ber}(y_2)$$

$$2) \rho_+(x, x') = P(y - y' > 0 | x, x')$$

$$= E[\mathbb{E}[\mathbb{1}_{y=1} \mathbb{1}_{y=-1} | x, x']]$$

$$= \eta(x)(1 - \eta(x'))$$

$$\rho_-(x, x') = (1 - \eta(x))\eta(x')$$

$$\eta = \frac{1}{2}, 0, 1$$

$$3) Y = \mathbb{R}, Y = m(x) + \sigma(x)N$$

$$\rho_+(x, x') = P(\underbrace{m(x) + \sigma(x)N - m(x') - \sigma(x')N'}_{Z} > 0 | x, x')$$

$$= E[\mathbb{E}[\mathbb{1}_{Z>0} | x, x']]$$

$$Z \sim N(m(x) - m(x'), \sigma^2(x) + \sigma^2(x'))$$

$$\rho_+(x, x') = 1 - F_Z(0)$$

$$\rho_- = F_Z(0)$$

pb si $m(x) \approx m(x')$