Exercise 03:
Leaky Integrate & Fire Neuron

Theoretical Neuroscience I

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1 Simulation of a leaky integrate-and-fire neuron

Question:
Plot the input current and the membrane voltage for a leaky integrate-and-fire neuron for a constant input current, an input current of low frequency and a linearly increasing input current. Use the following parameter values:

$$r_m = 1.5\mathrm{M}\Omega\,\mathrm{mm}^2 \qquad c_m = 20\mathrm{nF}\,\mathrm{mm}^{-2} \qquad E_L = V_{\mathrm{reset}} = V_0 = -65\mathrm{mV} \qquad V_{\mathrm{thresh}} = -50\mathrm{mV}$$

Also compute the inter-spike-intervals.

Answer:

- Constant input current (Figure 1): The membrane voltage rises exponentially until it reaches the threshold value of -50 mV. The neuron fires and the membrane voltage gets reset to V_{reset}. As the input current does not change this is repeated over and over. The inter-spike-intervals are all equal, i.e. the spike rate is constant.
- Low frequency input current (Figure 2): With the added frequency the input current does not stay large enough for long enough for the membrane voltage to cross the threshold value to fire. As a result there are no spikes, and the membrane voltage behaves the same as for the single compartment model up to an offset of E_L.
- Ramping input current (Figure 3): The input current slowly increases over time. The membrane current first rises slowly, until it reaches V_{thresh} for the first time. The neuron fires and the membrane voltage is reset, then rises again, as the input current is still high enough to elicit another spike and continues to rise. After every spike the gradient of the membrane voltage is higher than after the last, because the input current is higher. Consequently the inter-spike-intervals also get shorter over time. This can also be seen in Figure 4, where the inter-spike-intervals are plotted against the spike number. Apparently the inter-spike-interval length decreases exponentially.

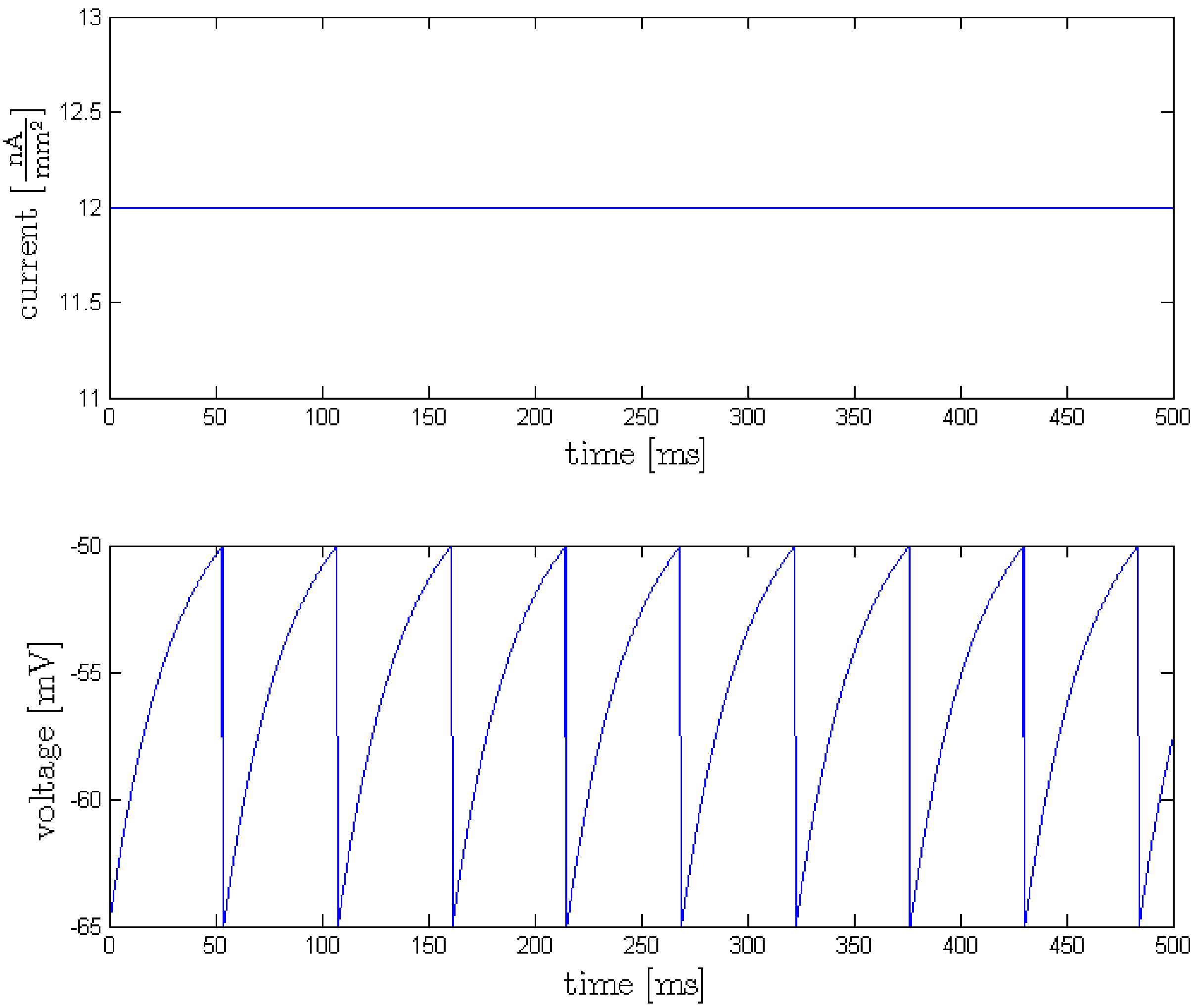

Figure 1: Top: The constant input current of 12 $\mathrm{nA\,mm^{-2}}$ plotted against time. Bottom: The resulting membrane voltage over time.

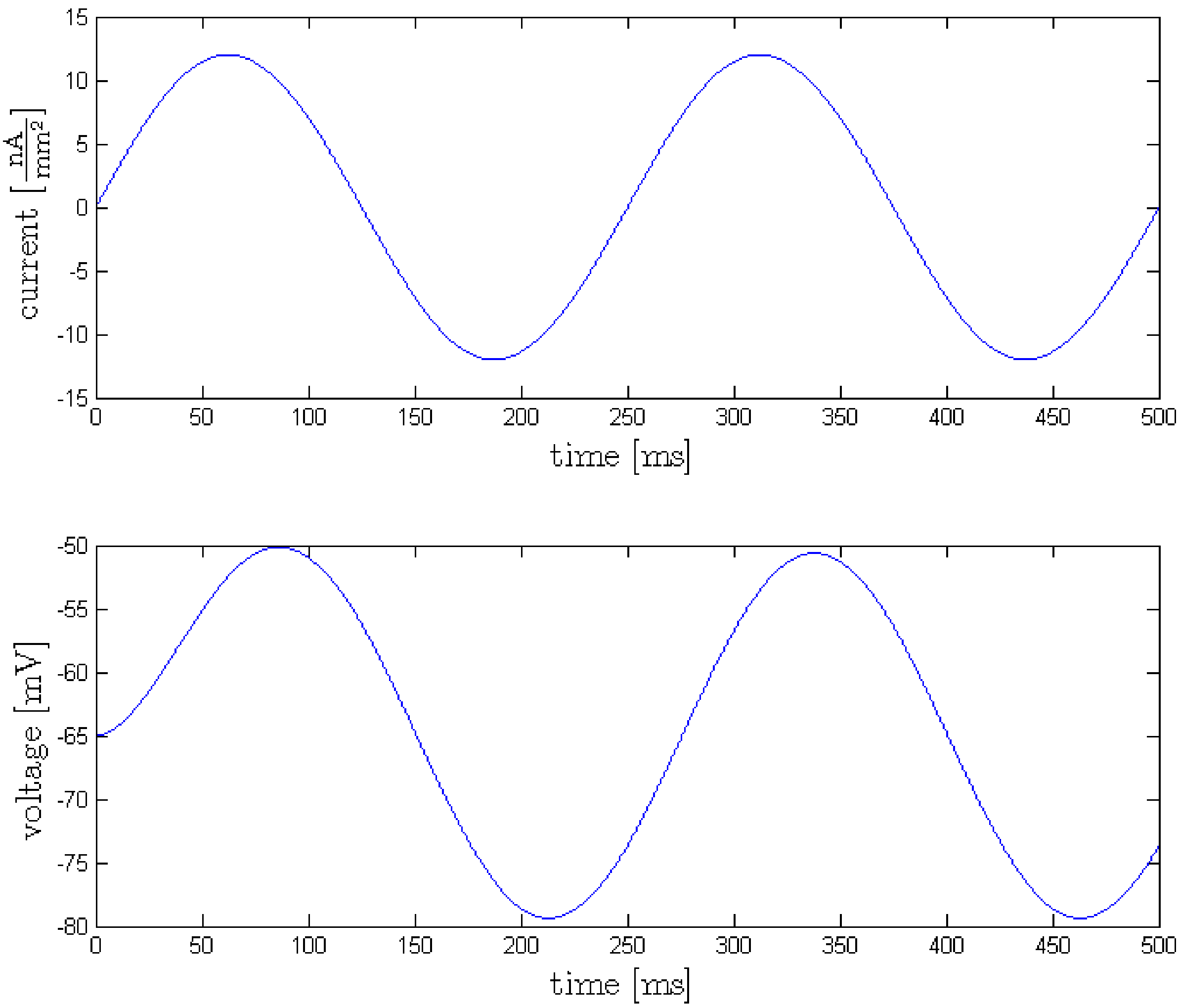

Figure 2: Top: The low frequency (4 Hz) input current $i_e(t) = 12\text{nA}\,\text{mm}^{-2} \cdot \sin(2\pi 4t)$ plotted against time. Bottom: The resulting membrane voltage over time.

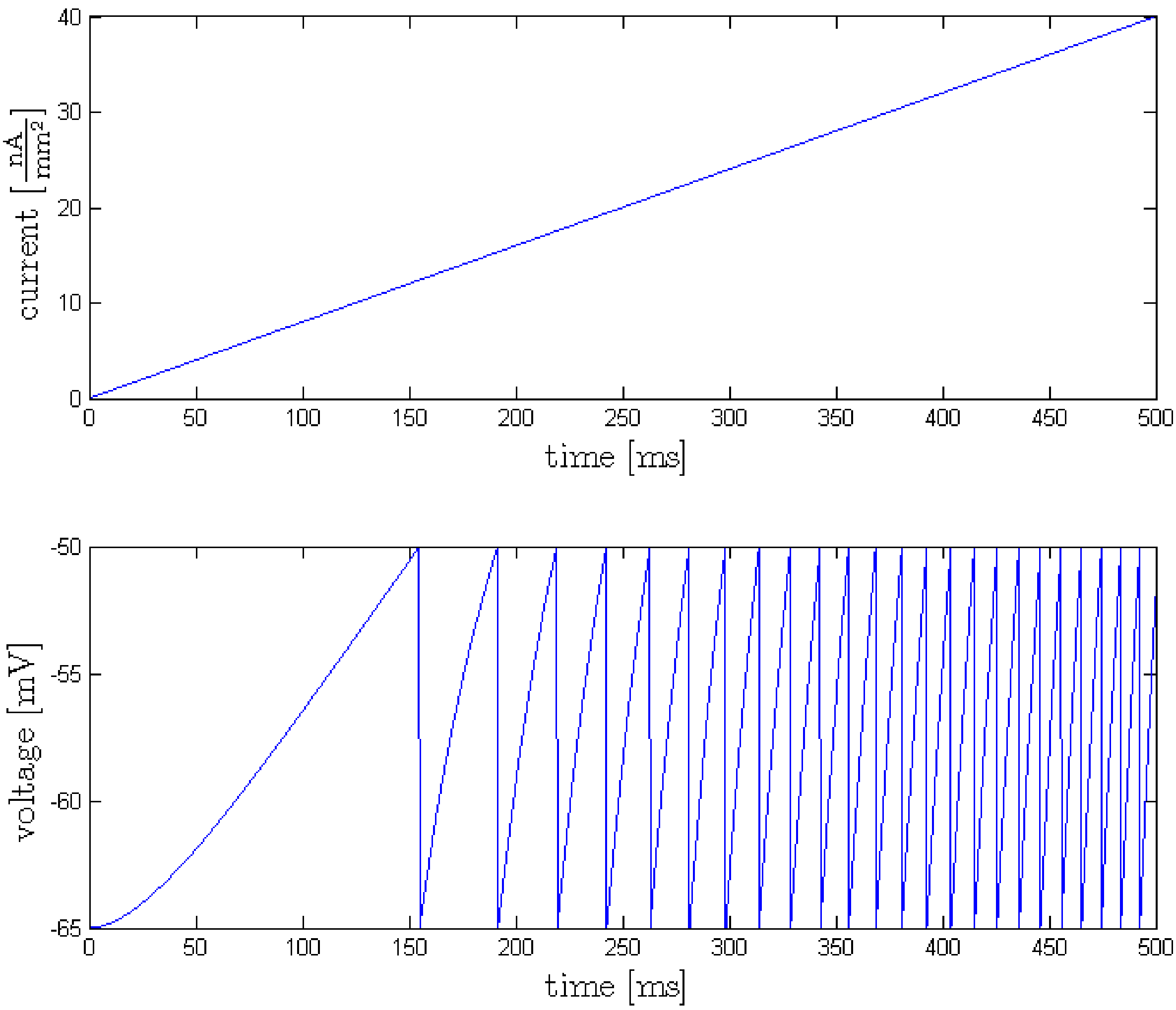

Figure 3: Top: The ramping input current $i_e(t) = 12\text{nA}\,\text{mm}^{-2} \cdot t/150\text{ms}$ plotted against time. Bottom: The resulting membrane voltage over time.

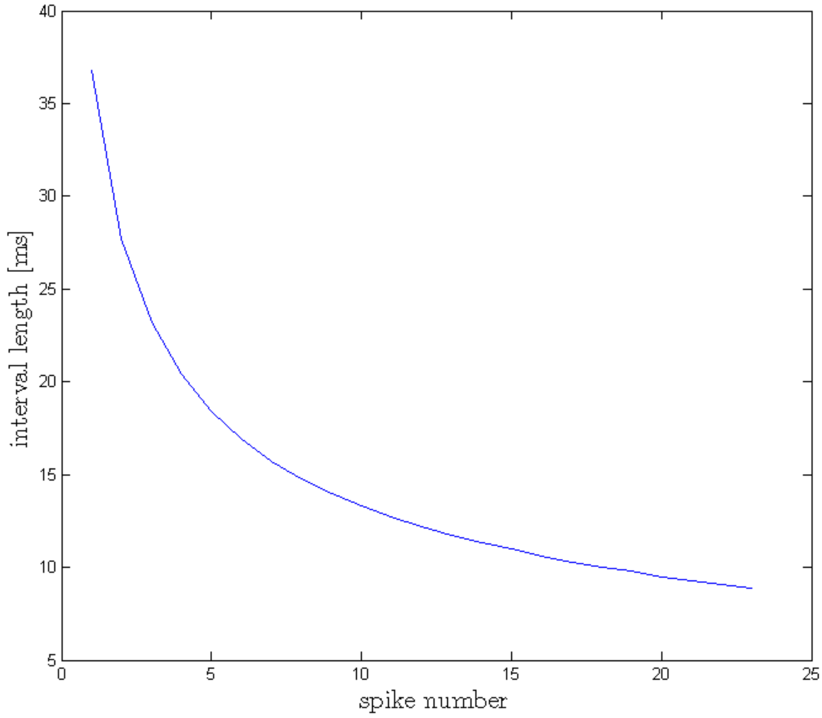

Figure 4: The inter-spike-interval length as a function of the number of spikes generated so far for the ramping input current.

2 Optional assignment

Question:
Analytically solve the dynamical equation

$$\frac{\mathrm{d}V_m(t)}{\mathrm{d}t} = \frac{1}{\tau_m}\left(r_m \cdot i_e(t) + E_L - V_m(t)\right)$$

over a time interval Δt, to derive the numerical iteration rule

$$V_m(t+\Delta t) = V_m(t)\exp\left(-\frac{\Delta t}{\tau_m}\right) + \left(i_e(t)\cdot r_m + E_L\right)\left(1-\exp\left(-\frac{\Delta t}{\tau_m}\right)\right).$$

Answer:
Like in the single compartment model, the membrane potential relaxes towards an equilibrium potential as long as the membrane potential is below the threshold potential. Since we assume $i_e(t)$ to be constant over small intervals, we can say that $V_\infty = r_m \cdot i_e(t) + E_L$ and replace it in our dynamical equation and the numerical iteration rule for easier readability.

$$\frac{\mathrm{d}V_m(t)}{\mathrm{d}t} = \frac{1}{\tau_m}\left(V_\infty - V_m(t)\right)$$

$$V_m(t+\Delta t) = V_m(t)\exp\left(-\frac{\Delta t}{\tau_m}\right) + V_\infty\left(1-\exp\left(-\frac{\Delta t}{\tau_m}\right)\right)$$

With some rearrangement we obtain:

$$V_m(t+\Delta t) = V_\infty + e^{-\frac{\Delta t}{\tau_m}}\left(V_m(t) - V_\infty\right)$$

We rearrange the terms in the dynamical equation to separate the voltage and time differentials:

$$\frac{1}{V_\infty - V_m(t)}\,\mathrm{d}V_m(t) = \frac{1}{\tau_m}\,\mathrm{d}t$$

Now we can integrate over an interval $[t', t' + \Delta t]$.

$$\begin{aligned}
\Leftrightarrow \qquad && \int_{t'}^{t'+\Delta t} \frac{1}{V_\infty - V_m(t)}\,\mathrm{d}V_m(t) &= \int_{t'}^{t'+\Delta t} \frac{1}{\tau_m}\,\mathrm{d}t \\
\Leftrightarrow \qquad && \left[-\ln\left(V_\infty - V_m(t)\right)\right]_{t'}^{t'+\Delta t} &= \left[\frac{t}{\tau_m}\right]_{t'}^{t'+\Delta t} \\
\Leftrightarrow \qquad && -\ln\big(V_\infty - V_m(t'+\Delta t)\big) + \ln\big(V_\infty - V_m(t')\big) &= \frac{t'+\Delta t}{\tau_m} - \frac{t'}{\tau_m}
\end{aligned}$$

Exponentiate e with the equation…

$$\Leftrightarrow \qquad \frac{V_\infty - V_m(t')}{V_\infty - V_m(t'+\Delta t)} = e^{\frac{\Delta t}{\tau_m}}$$

…and do some simple rearrangement to solve for $V_m(t' + \Delta t)$:

$$\Leftrightarrow \qquad V_m(t'+\Delta t) = V_\infty + e^{-\frac{\Delta t}{\tau_m}}\big(V_m(t') - V_\infty\big)$$

$\square$