Discrete Applied Mathematics 1 (1979) 89–103

THE MIXED POSTMAN PROBLEM

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The mixed postman problem is formulated as an integer linear program. A one to one correspondence is established between the extreme points of the linear programming polyhedron and prime assigned Euler networks. A prime assigned Euler network gives rise to a set of length equivalent prime postman tours. Some prime postman tour is optimal. Thus, it is sufficient to search the extreme points of the polyhedron to find an optimal postman tour.

1. Introduction

The postman problem is the problem of finding a shortest length tour in a connected network where each edge and arc is required to be traversed at least once. This would be the problem faced by a postman delivering mail from a truck along street segments, or the problem faced by the delivery of other municipal services such as street sweeping or snow removal.

The underlying network is either totally undirected, totally directed, or mixed. When the network is totally undirected, the resulting problem is the well-known Chinese postman problem. It has recently been examined in detail by Edmonds and Johnson [5] and others [11, 12], and solution methods are given. The case for which the network is totally directed, which has been referred to as the directed Chinese postman problem, has also been examined [2, 5, 12] and is easily solved.

However, when the network is a mixed network no exact solution procedure, in general, solves the postman problem (except integer linear programming techniques). In this paper, we address this most general case of the postman problem, namely, the mixed postman problem (note that the undirected and directed cases are subsumed by the mixed case).

In Section 2 we define what we call an assigned Euler network. Such a network yields a postman tour for the original network. In Section 3 we define a prime assigned Euler network. Some prime assigned Euler network is optimal and so it is enough to find a shortest prime assigned Euler network. It is shown in Section 4 that there is a one-to-one correspondence between prime assigned Euler networks and the extreme points of a certain convex polyhedron. Therefore, in searching for an optimal postman tour it is sufficient to search the extreme points of the specified convex polyhedron. In Section 5 we relate our results to several special cases of the postman problem. Although this paper does not investigate algorithms based upon the results of Section 4, we do suggest possible strategies in Section 6.

2. Problem description and formulation

The postman problem is the problem of finding the shortest length tour in a connected network where each edge and arc is traversed at least once. Let $G=(N, E, A)$ be a strongly connected network[1] consisting of a set of nodes N, a set of undirected edges E, and a set of directed arcs A. It is assumed that G contains no loops but may have multiple edges and arcs connecting any pair of nodes.[2] Each edge and arc, i.e. each link, has a non-negative length. A traversable path from node n_i to n_k is a sequence of links which allows traversal from n_i to n_k. A tour in G is a traversable path that begins and ends at the same node.

A postman tour T in G is a tour in G which contains each link at least once. The postman problem is solved by finding a minimum length postman tour in G. An Euler tour is a postman tour which contains each link exactly once. Of course, if G contains an Euler tour, then this tour is an optimal postman tour.

The existence of an Euler tour on a network G was characterized by Ford and Fulkerson [6].[3] The degree of a node is the number of links incident to it. An arc which is outgoing from a node is positively incident to that node, and an arc which is incoming to a node is said to be negatively incident to that node. A node is called pseudo-symmetric if the number of positively incident arcs equals the number of negatively incident arcs. A network is said to be symmetric if each node is pseudo-symmetric.

Theorem 2.1 (Ford and Fulkerson [6]). *An Euler tour exists on a connected network G if and only if*

(i) *every node $n \in N$ has even degree, and*

(ii) *for every $X \subseteq N$, the difference between the number of arcs from X to $\bar{X}$ and the number of arcs from $\bar{X}$ to X is less than or equal to the number of edges joining X and $\bar{X}$, where $\bar{X}$ is the compliment of X.*

Note that if G is symmetric then it contains an Euler tour.

If G is not Euler, then a postman tour T on G will require that some of the links be traversed more than once. Corresponding to such a T, we can construct a

[1]. A network G is said to be strongly connected if for every $n_i, n_k \in N$ $(n_i \neq n_k)$, there exists a traversable sequence from n_i to n_k. Edmonds and Johnson [5] show that a postman tour exists on G if and only if G is strongly connected.

[2]. A loop is an edge or an arc that connects a node to itself. For the purposes of finding an optimal postman tour it can be ignored.

[3]. Their result generalized earlier results for directed graphs ($E=\emptyset$) and undirected graphs ($A=\emptyset$).

unique directed network $G(T) \equiv (N, A(T))$ from G and T as follows. For each time that a link in G is traversed, place in $A(T)$ a parallel arc, where the direction assigned to the arc corresponds to the direction of traversal on the link. By construction, $G(T)$ contains an Euler tour which corresponds to the postman tour T on G. We will refer to $G(T)$ as an assigned Euler network of G. An optimal postman tour T yields a minimal length assigned Euler network, where the length of the network is defined to be the sum of the lengths of the links of the network.

In general, an assigned Euler network does not have to be constructed from a postman tour on G. To consider this, suppose $A(G)$ is the arc set obtained from the union of the arcs A of G and the set of arcs formed by all possible directed copies of edges in E. A network $G' = (N, A')$ is said to be an assigned Euler network of G if

(i) G' is Euler,

(ii) A' is composed of arcs from the set $A(G)$, and

(iii) each link of G has at least one paralled arc copy in A'.

For an assigned Euler network G' of G we can find an Euler tour on G' and use this to form in the obvious way a postman tour T on G. Then, $G' = G(T)$ for some postman tour T. Note that for a postman tour T on G, the network $G(T)$ is unique, but that there may exist several postman tours corresponding to a given Euler network G'.[4]

Since a postman tour T on G can be constructed from an assigned Euler network of G, it follows that the solution procedure for the postman problem can be divided into two parts [5]:

(1) Determine (i) the number of copies of each link of G and (ii) determine for the edges the appropriate assigned direction for the copies necessary to form an assigned Euler network of minimal length.

(2) Find an Euler tour on the assigned Euler network. From this construct a postman tour on G.

Finding an Euler tour on an Euler network is a straightforward process and there are algorithms for this [5, 7, 10]. Thus, the postman problem is solved when an optimal G' is found.

The problem of finding a minimal length assigned Euler network of G can be formulated as an integer linear programming problem (ILP). For the purposes of this formulation, we introduce the following notation. Let $\varepsilon = |E|$ and $\alpha = |A|$, where $|\cdot|$ denotes cardinality. Arbitrarily label the edges of G from 1 to ε and the arcs of G from $\varepsilon + 1$ to $\varepsilon + \alpha$. Let $e^i(l^i)$ refer to the ith edge (link) and $a^{\varepsilon+i}(l^{\varepsilon+i})$ refer to the ith arc (labelled as link $\varepsilon + i$).

To construct the network $G' = (N, A')$, we introduce the following non-negative variables to count how many times each edge and arc appears in A'. Let e^i connect nodes n_j and n_k, $j < k$. Let the corresponding directed arcs in $A(G)$ be denoted as a^{2i-1} and $\bar{a}^{2i}$, with assigned direction from n_j to n_k and from n_k to n_j,

4. The number of Euler tours on a directed Euler graph is computable by a formula given in [7].

respectively. Then, let the edge variables x_{2i-1} and x_{2i} count the respective number of parallel copies of $\bar{a}^{2i-1}$ and $\bar{a}^{2i}$ that will be in A'. For arc $a^{\varepsilon+i} \in A$, let its image in $A(G)$ be denoted as $\bar{a}^{2\varepsilon+i}$. Then, let the arc variable $x_{2\varepsilon+i}$ count the number of parallel copies of $\bar{a}^{2\varepsilon+i}$ that will be in A'.

To consider the length of the network G', let the length of each edge $e^i \in E$ be given by d_{2i}, where $d_{2i} \geqslant 0$, and define $d_{2i-1} = d_{2i}$, and let the length of each arc $a^{\varepsilon+i}$ be given by $d_{2\varepsilon+i}$, where $d_{2\varepsilon+i} \geqslant 0$.

To formulate the mixed postman problem as in ILP, the arc variables are required to be at least one, and the sum of the two opposite edge variables are required to be at least one. The requirement that G' be an Euler network is expressed by equating the number of arcs and directed edges leaving each node to the number entering. An arbitrary one of these flow equations is redundant and so the equation corresponding to the last labelled node is omitted. Define the numbers l, m, and n to be $l = \varepsilon + \alpha$, $m = |N| - 1$, and $n = 2\varepsilon + \alpha$.

The ILP formulation of the mixed postman problem (MPP) is then as follows

$$\text{minimize } \sum_{j=1}^{n} d_j x_j, \tag{1}$$

subject to

$$\sum_{j \in \Gamma(n_i)} x_j - \sum_{j \in \Gamma^{-1}(n_i)} x_j = 0, \quad i = 1, \ldots, m, \tag{2}$$

$$x_{2i-1} + x_{2i} \geqslant 1, \quad i = 1, \ldots, \varepsilon, \tag{3}$$

$$x_{2\varepsilon+i} \geqslant 1, \quad i = 1, \ldots, \alpha, \tag{4}$$

$$x_j \geqslant 0, \quad j = 1, \ldots, n, \tag{5}$$

$$x_j \text{ integer}, \quad j = 1, \ldots, n \tag{6}$$

where $\Gamma(n_i)$ is the set of all indices of arcs in $A(G)$ that are positively incident to node n_i, and $\Gamma^{-1}(n_i)$ is the set of all indices of arcs in $A(G)$ that are negatively incident to node n_i. Later we will work implicitly with the surplus variables of Eq. (3) and (4). These will be denoted as s_i, for $i = 1, \ldots, l$.

Once feasible x_j, $j = 1, \ldots, n$, are found by solving (2)–(6), a unique assigned Euler network can be constructed and a postman tour on G from the assigned Euler network can be determined.

The contribution of this paper is the establishment of a one-to-one correspondence between the extreme points of the polyhedron given by Eq (2)–(5), and a set of assigned Euler networks which satisfy certain necessary conditions for optimality. An optimal Euler network is contained in this set. It is also shown that the extreme points of the polyhedron given by (2)–(5) possess a near integer property that easily yields the x_j values of a corresponding postman tour.

3. Prime assigned Euler networks

In the previous section we have shown that the postman problem on G is equivalent to the problem of finding a minimum length assigned Euler network G'. If we are given an assigned Euler network G' and G' contains no shorter length network which is itself an assigned Euler network for G, then G' is in some sense of minimum length, although not necessarily optimal. This motivates consideration of a set of assigned Euler networks which we call prime assigned Euler networks.

Let G' be an assigned Euler network and T some postman tour on G such that $G(T) = G'$. Let S be a traversable path in T from node n_j to node n_k such that the image of S in G' contains at least one set of arcs which form a directed path from n_j to n_k. The largest number of such arc-disjoint paths in G' is called the number of assigned copies of S in G'.

Definition 3.1. An assigned Euler network G' for a network G is said to be prime when it satisfies the following conditions:

(i) No proper partial network of G' is an assigned Euler network for G.

(ii) If G' contains several copies of an edge, then either all are assigned the same direction, or G' contains exactly two copies of an edge and both copies are oppositely assigned.

(iii) Let S^1 and S^2 be two different sets of edges and arcs in G, each forming a traversable path between the same pair of nodes (either in the same or opposite direction). If G' has only one assigned copy of each of S^1 or S^2, then some edge or arc in S^1 or S^2 is copied in G' only once. If G' has two or more copies of S^1, then some edge or arc of S^2 is copied in G' only once.

We now show that some prime assigned Euler network is optimal, and hence it is sufficient to find a shortest prime assigned Euler network.

Theorem 3.1. *Some prime assigned Euler network is optimal.*

Proof. Suppose G^* is a minimum length assigned Euler network for G. Let $d(\cdot)$ denote the length function.

Condition (ii). For any edge e in G whose length is positive, G^* satisfies condition (ii), because if G^* does not, then oppositely assigned copies of e in G^* may be removed in pairs from G^* while still retaining the degree and pseudosymmetry of the adjacent nodes. The resulting network G^{2*} has $d(G^{2*}) < d(G^*)$, which is a contradiction. Suppose G contains one or more edges of zero length and suppose G^* does not satisfy condition (ii). Then, the above described removal procedure performed on edges of zero length results in a network G^{2*} which satisfies condition (ii) and for which $d(G^{2*}) = d(G^*)$.

For the case above where G^* does satisfy condition (ii), define $G^{2*} = G^*$. Then, in either case we have an assigned Euler network G^{2*} which satisfies condition (ii) for all edges and for which $d(G^{2*}) = d(G^*)$.

For the proof of condition (iii), it is important to note that for a traversable path S in G an assigned Euler network G' can contain exactly one assigned copy of S and yet contain two copies of each edge and arc in S. For example, suppose S contains one edge and one arc and G' contains two assigned copies of the edge, one in each direction, and two copies of the arc. Note that G' contains exactly one assigned copy of S.

Condition (iii). Let S^1 and S^2 be two traversable paths in G between the same pair of nodes.

Case 1. S^1 and S^2 are oriented in the same direction. (a) If $d(S^1) < d(S^2)$ (or vice versa), then G^{2*} satisfies condition (iii), because if it does not, then an assigned copy of S^2 can be removed from G^{2*} and replaced with an assigned copy of S^1. This procedure maintains the degree and symmetry of all nodes. The resulting network G^{3*} has $d(G^{3*}) < d(G^*)$, which is a contradiction. (b) If $d(S^1) = d(S^2)$, then the above described removal procedure may be performed until no more assigned copies of S^2 can be removed from G^{2*} without violating the requirement that each link of G has at least one parallel arc copy in G^{2*}. Performing this procedure on all such pairs of paths results in an assigned Euler network G^{3*} which satisfies condition (ii), condition (iii), Case 1 and for which $d(G^{3*}) = d(G^*)$.

For subcase (a) above if we define $G^{3*} = G^{2*}$, then the previous comment about G^{3*} applies here also.

Case 2. S^1 and S^2 are oriented in opposite directions. (a) If $d(S^1) + d(S^2) > 0$, then G^3 satisfies condition (iii) because if it does not, then a directed tour of links formed from an assigned copy of S^1 and an assigned copy of S^2 can be removed from G^{3*} while still maintaining the degree and pseudosymmetry of all nodes. The resulting network G^{4*} has $d(G^{4*}) < d(G^*)$, which is a contradiction. (b) If $d(S^1) = 0 = d(S^2)$, the above removal procedure may be performed with the same restrictions as that for Case 1 above. Performing this procedure on all such pairs of paths results in an assigned Euler network G^{4*} which satisfies both conditions (ii) and (iii) and for which $d(G^{4*}) = d(G^*)$. The satisfaction of conditions (ii) and (iii) for all edges and pairs of traversable paths in G implies that condition (i) is satisfied. Hence G^{4*} is prime. Since $d(G^{4*}) = d(G^*)$, it follows that G^{4*} is optimal, and hence the proof that some prime assigned Euler network is optimal.

From Theorem 3.1 it is sufficient to consider only prime assigned Euler networks in searching for a minimum length assigned Euler network. The importance of this comes from the result, to be given in the next section, that there is a one-to-one correspondence between prime assigned Euler networks and the extreme points of the polyhedron given by Eqs. (2)–(5).

4. An extreme point property

Consider the ILP given in Eqs. (1)–(6). If we convert the inequalities to equations, then in matrix notation the problem can be written

$$\text{minimize } d'x, \tag{7}$$

subject to

$$P\binom{x}{s}=\begin{bmatrix} C & O \\ U & -I \end{bmatrix}\binom{x}{s}=\binom{0}{1}=b, \tag{8}$$

$$x, s \geqslant 0, \tag{9}$$

$$x, s \text{ integer} \tag{10}$$

where C is a matrix of dimension $m \times n$ corresponding to Eq. (2), s is a vector of surplus variables having one variable per link, U is a matrix of dimension $l \times n$ and I is the identity matrix of dimension l, where U and $-I$ correspond to Eqs. (3) and (4), and 1 is a vector of ones having dimension l.

The jth column of P, C, and U is denoted, respectively, as P^j, C^j, and U^j. Let u_k be a unit vector having a "+1" in position k. Then, note that for $j \leqslant 2\varepsilon$ and $k=\lceil \frac{1}{2}j \rceil$ where $\lceil \cdot \rceil$ means the smallest integer greater than or equal to its argument that

$$P^j=\begin{bmatrix} C^j \\ u_k \end{bmatrix}$$

and

$$P^{2k-1}=\begin{bmatrix} C^{2k-1} \\ u_k \end{bmatrix}, \quad \text{and} \quad P^{2k}=\begin{bmatrix} -C^{2k-1} \\ u_k \end{bmatrix}.$$

The columns P^{2k-1} and P^{2k} for $k \leqslant \varepsilon$ will be referred to as the columns of an edge pair (specifically, for edge e^k).

We will refer to a *chain* as being a traversable path from node n_i to n_j having no repeated link. It is well known [6] that K is a chain from node n_i to n_j if and only if

$$\sum_{l^h \in K} C^{k_h} = C^t \tag{11}$$

where k_h is the approximate superscript for link l^h having the desired assignment of direction, and t corresponds to a link from node n_i to n_j.

We now wish to investigate the properties of extreme points of the polyhedron given by Eqs. (2)–(5), or in matrix notation, by Eqs. (8) and (9). This can clearly be done by considering basic feasible solutions to Eqs. (8) and (9). Let B be an $(m+l)\times(m+l)$ submatrix of P. Necessary conditions for B to be a basic matrix are given by the following lemmas.

Lemma 4.1. *Suppose the matrix B contains an edge column of P. Then, B is a basis matrix of* (8) *only if the other column of the edge pair and corresponding surplus column are not both in B.*

Proof. Suppose that B is a basis containing columns P^{2k-1}, P^{2k}, and P^{n+k}, where $k \leqslant \varepsilon$. clearly,

$$P^{n+k} = -\tfrac{1}{2}(P^{2k-1} + P^{2k})$$

which contradicts the linear independence of these columns.

Lemma 4.2. *Let K^1 and K^2 be two chains connecting two nodes such that K^1 and K^2 contain some different links. Let B contain the appropriate columns of P associated with the links of K^1 and K^2. Then, B is a basis matrix only if there is at least one link of $K^1 \cup K^2$ such that either*

(i) *the link is an edge of G and B contains neither the other column of the edge pair nor the corresponding surplus column, or*

(ii) *the link is an arc of G and B does not contain the corresponding surplus column.*

Proof. Assume that the conditions of Lemma 4.1 are satisfied (or else B is not a basis).

Case (a) K^1 and K^2 have the same orientation.

Suppose that no edge or arc column corresponding to K^1 or K^2 satisfies the conditions of the theorem. For a given edge or arc in K^i, let P_i^f be its appropriate column in B, and P_i^s its corresponding surplus column, or the other column of the edge pair, as given in B. If $P_i^{f_l}$ and $P_i^{s_l}$ are the columns of an edge pair, then let

$$\lambda_i^{f_l} = \tfrac{1}{2} \quad \text{and} \quad \lambda_i^{s_l} = -\tfrac{1}{2}.$$

If $P_i^{f_l}$ is an edge or arc column and $P_i^{s_l}$ is the corresponding surplus column, then let

$$\lambda_i^{f_l} = 1 \quad \text{and} \quad \lambda_i^{s_l} = 1.$$

(Notice by Lemma 4.1 that no edge has both the other column of the edge pair and the corresponding surplus column in B.) By Eq. (11) and the structure of each column of P, we readily have the

$$\sum_{l' \in K^1} (\lambda_1^{f_{l'}} \cdot P_1^{f_{l'}} + \lambda_1^{s_{l'}} \cdot P_1^{s_{l'}}) = \sum_{l' \in K^2} (\lambda_2^{f_{l'}} \cdot P_2^{f_{l'}} + \lambda_2^{f_{l'}} P_2^{s_{l'}}),$$

which contradicts the linear independence of the columns of B.

Case (b) K^1 and K^2 have opposite orientation.

The proof follows a similar procedure to that of Case (a).

Lemma 4.3. *Matrix B is a feasible basis matrix only if B contains an edge column and arc column for each edge and arc of the network G.*

Proof. Trivial.

We can now give a correspondence between prime assigned Euler networks and matrices which satisfy the conditions of Lemmas 4.1–4.3. We will first show that any feasible basis matrix gives rise to a prime assigned Euler network. For this, we need the following theorem.

Theorem 4.1. *Each extreme point of the polyhedron given by* (8)–(9) *has component values of* 0, $\frac{1}{2}$, *or some positive integer. If an edge pair of variables is basic, then these variables can take on only the values* 0, $\frac{1}{2}$, *or* 1, *and furthermore, if one edge variable has the value* $\frac{1}{2}$ *then its edge pair also has the value* $\frac{1}{2}$. *All other edge variables, arc variables and surplus variables are integral.*

Proof. Let B be a feasible basis matrix for (8)–(9). Permute and partition B to look like

$$B=\left[\begin{array}{cccc:c} C_1 & C_3 & -C_1 & C_2 & 0 \\ I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ \hdashline 0 & I & 0 & 0 & -I \end{array}\right] \equiv \begin{bmatrix} B_1 & 0 \\ B_2 & -I \end{bmatrix} \tag{12}$$

where

(i) the submatrix $[C_1, I, 0, 0]$ contains edge columns that have their corresponding edge pair columns in B, and these columns are contained in the submatrix $[-C_1, I, 0, 0]$,

(ii) the submatrix $[C_3, 0, 0, I]$ contains edge and arc columns which have their associated surplus columns in B, and these columns are contained in the submatrix $[0, 0, 0, -I]$, and

(iii) the submatrix $[C_2, 0, I, 0]$ consists of all the remaining edge and arc columns. Clearly, some of these columns and rows may be vacuous. For example, if B contains no edge column whose edge pair column is also in B, then the submatrices in (i) above are vacuous.

Let x_1, x_4, x_2, x_3, and s be the vectors corresponding to the column groupings in B; namely, x_1 and x_2 are the vectors corresponding to the edge pairs in B, x_4 is the vector of edge and arc variables having their associated surplus columns in B, and x_3 is the vector of remaining edge and arc variables of B. We must solve

$$B\cdot\begin{pmatrix} x_1 \\ x_4 \\ x_2 \\ x_3 \\ s \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{13}$$

$$x_1, x_2, x_3, x_4, s \geqslant 0.$$

We have from (12) and (13)

$$
\begin{aligned}
C_1(x_1-x_2)+C_3x_4+C_2x_3&=0,\\
x_1+x_2\qquad\qquad\qquad&=1,\\
x_3&=1,\\
x_4-s\quad&=1,\\
x_1,x_2,x_3,x_4,s\geq 0.&
\end{aligned}
\tag{14}
$$

Define $y=x_1-x_2$. Then, the first three equations of (14) become

$$
\begin{aligned}
[C_1,C_3]\binom{y}{x_4}&=-C_2\cdot 1,\\
x_1+x_2&=1.
\end{aligned}
\tag{15}
$$

Let $\det(B)$ be the determinant of B. From the partitioning of B in (12) it is clear that the $\det(B)=\pm\det(B_1)=\pm\det([2C_1,C_3])\neq 0$ since B is a basis matrix (note that $[C_1,C_3]$ is a square matrix). The $\det([2C_1,C_3])\neq 0$ implies that the $\det([C_1,C_3])\neq 0$, and hence it follows from the total unimodularity of the matrix C that the $\det([C_1,C_3])=\pm 1$.

Since $[C_1,C_3]$ is unimodular, it follows that $-[C_1,C_3]^{-1}\cdot C_2\cdot 1$ is integral. Let $-[C_1,C_3]^{-1}\cdot C_2\cdot 1=(z_1,z_2)^t$. Then, from (15)

$$
\begin{aligned}
y=x_1-x_2&=z_1\\
x_1+x_2&=1,\\
x_4&=z_2,\\
x_1,x_2,x_4&\geq 0
\end{aligned}
\tag{16}
$$

where z_1 and z_2 are integral. From (16) we have only three possibilities for each edge pair in B:

$$x_{1,i}=0,\qquad x_{2,i}=1,\qquad z_i=-1$$

or

$$x_{1,i}=1,\qquad x_{2,i}=0,\qquad z_i=1$$

or

$$x_{1,i}=\tfrac{1}{2},\qquad x_{2,i}=\tfrac{1}{2},\qquad z_i=0$$

where it is clear from the notation that $x_{1,i}$ and $x_{2,i}$ are an edge pair of variables. We already have that $x_3=1$. Finally, since z_2 is integral, it immediately follows that s is integral. This completes the proof of the theorem.

Let $\bar{x}_j$ be the components of a basic feasible solution where all values of $\frac{1}{2}$ are rounded up to 1. Notice that these x_j values satisfy Eq. (2)–(6), and consequently are the traversal numbers for a postman tour, or equivalently are the number of assigned copies of edges and arcs for a unique assigned Euler network $G(T)$ for G. Construct the network $G(T)$ in the normal manner from the x_j values. Then, $G(T)$ satisfies the following theorem.

Theorem 4.2. *An assigned Euler network formed by rounding up to* 1 *the* $\frac{1}{2}$ *values of an extreme point to Eqs.* (2)–(5) *is a prime assigned Euler network.*

Proof. Since the positive $\bar{x}_j$ are associated with basic columns of P, using Lemmas 4.1–4.3 it is straightforward to show that Definition 3.1 is satisfied.

It should seem reasonable that we can also go the other direction; that is. that a prime assigned Euler network will give rise to a basic feasible solution to Eqs. (8)–(9). We now show this.

Let $G(T)$ be a prime assigned Euler network for G, and recall that $G(T)$ satisfies the conditions of Definition 3.1. Construct the matrix Q from the columns of the matrix P as follows.

(1) For each edge of $G(T)$ place in Q the edge column from P which corresponds to the assignment of the edge.

(2) For each arc in $G(T)$ place in Q the arc column from P.

(3) If more than one copy of any assigned edge or arc is in $G(T)$, then place the corresponding surplus column in Q.

(4) If $G(T)$ contains two copies of an edge, where exactly one of them is assigned one direction and one the other direction, then place in Q both columns of the edge pair.

We note that Q may not be square. The matrix Q is referred to as the associated matrix of $G(T)$. The correspondence between prime assigned Euler networks and matrices which satisfy the conditions of Lemmas 4.1 and 4.3 is given by the following result.

Lemma 4.4. *Let Q be the associated matrix of $G(T)$. Then, Q satisfies the conditions of Lemmas* 4.1 *to* 4.3.

Proof. By the construction of Q and Definition 3.1 it is readily apparent that Lemma 4.1 and Lemma 4.2 are satisfied. Finally, since $G(T)$ is an assigned Euler network for G, Lemma 4.3 is satisfied.

Since the matrix Q satisfies the conditions of Lemmas 4.1–4.3, the result which one might reasonably conjecture is given by the following theorems.

Theorem 4.3. *Let $G(T)$ be a prime assigned Euler network. Then, the columns of Q are linearly independent.*

Proof. Suppose that $G(T)$ is a prime assigned Euler network. Let Q be the associated matrix of $G(T)$, where the columns of Q are $P^{h_1}, \ldots, P^{h_f}$.

Suppose that the columns of Q are linearly dependent. Then, there exist scalar λ_i not all zero such that

$$\lambda_1 P^{h_1} + \cdots + \lambda_f P^{h_f} = 0. \tag{17}$$

Since the surplus columns of P are linearly independent, there is at least one $\lambda_i \neq 0$ corresponding to an edge or arc column. Renumber the index set in (17) so that

(i) this first $\lambda_i \neq 0$ becomes λ_1,

(ii) $\lambda_2, \ldots, \lambda_s$ are the remaining non-zero scalars, and

(iii) the edge and arc columns for $\lambda_1, \ldots, \lambda_s$ are first r columns (note that $r \geqslant 2$), and consequently, assuming $s - r > 0$, the remaining columns for $\lambda_{r+1}, \ldots, \lambda_s$ are surplus columns. Define $s - r \equiv t$, and let the surplus columns $P^{h_{r+i}}$, $i = 1, \ldots, t$, be given by

$$P^{h_{r+i}} = \begin{bmatrix} 0 \\ \cdots \\ -u_{k_i} \end{bmatrix}, \qquad i \in I_t.$$

Then, from (17), it follows that

$$\lambda_1 C^{h_1} = -(\lambda_2 C^{h_2} + \cdots + \lambda_r C^{h_r}), \tag{18}$$

$$\lambda_1 U^{h_1} = -(\lambda_2 U^{h_2} + \cdots + \lambda_r U^{h_r}) + \lambda_{r+1} u_{k_1} + \cdots + \lambda_{r+t} u_{k_t} \tag{19}$$

for $\lambda_i \neq 0$, $i = 1, \ldots, r+t$, and where U^i is the ith column of the submatrix U of the matrix P (recall Eq. (8)).

Note that for $p \neq q$, U^p and U^q are linearly dependent if and only if P^p and P^q are the columns of an edge pair. Thus, for (19) to be satisfied with $\lambda_i \neq 0$, $i = 1, \ldots, r+t$, it follows that

(i) for each edge column P^{h_i}, $i \in I_r$, either

(a) $U^{h_i} = U^{h_k}$ for some $k \in I_r$, in which case P^{h_i} and P^{h_k} are the columns of an edge pair, or

(b) for some $j \in I_t$, $U^{h_i} = u_{k_j}$, in which case $P^{h_{r+j}}$ is the surplus column for P^{h_i}, and

(ii) for each arc column P^{h_i}, for some $j \in I_t$, $U^{h_i} = u_{k_j}$, and so $P^{h_{r+j}}$ is the surplus column of P^{h_i}.

Column P^{h_1} corresponds to an edge, or arc, connecting a pair of nodes. Suppose this pair of nodes is n_w and n_z. Let us denote this single edge, or single arc, path as K^1. It is straightforward to show that (18) is satisfied if and only if columns $P^{h_2}, \ldots, P^{h_r}$ correspond to the edges and arcs of a traversable path K^2 connecting nodes n_w and n_z. Since $G(T)$ is a prime assigned Euler network, K^1 and K^2 must satisfy the conditions of Definition 3.1. Thus, by the construction of Q, the conditions (i) and (ii) of the previous paragraph cannot hold for each edge and arc column in Q. This is a contradiction, and hence it follows that the columns of Q are linearly independent.

Theorem 4.4. *There exists a unique extreme point solution x to* (2)–(5) *corresponding to every prime assigned Euler network of G.*

Proof. Let $G(T)$ be a prime assigned Euler network for G. Let Q be the matrix associated with $G(T)$ and let x_j be the values associated with $G(T)$; namely, x_j are the number of assigned copies of edges and arcs of G in $G(T)$. If there are any $x_{2i-1}=x_{2i}=1$ for $i\leqslant\varepsilon$, then set these values to $\frac{1}{2}$. Let x_k, for $k=n+1,\ldots,n+l$ be such as to satisfy (3) and (4). Then $x=(x_j)$ is a solution to (2)–(5), and the columns of Q correspond to the positive x_j. By Theorem 4.3, the columns of Q are linearly independent, and therefore x is an extreme point solution to (2)–(5).

Note that if Q is a square matrix that x is a unique non-degenerate basic feasible solution. If Q is not square, then there are columns of P which together with the columns of Q form a feasible basis matrix for x. In this case, a number of different degenerate basic feasible solutions correspond to the extreme point x.

We now give the main result of this paper. Note that the following theorem is an immediate consequence of Theorems 4.2 and 4.4.

Theorem 4.5. *The network $G(T)$ is a prime assigned Euler network for G if and only if the x_j values which give the number of assigned copies of edges and arcs of $G(T)$ are equal to the values given by some extreme point x of Eqs.* (2)–(5) *where all the $\frac{1}{2}$ values of the extreme point are rounded up to* 1.

Note that Theorem 4.5 gives a one-to-one correspondence between the extreme points of (2)–(5) and prime assigned Euler networks for G. Theorem 4.5 together with the fact that a prime assigned Euler network for G yields a postman tour for G implies that it is sufficient to search the extreme points of the polyhedron given by Eqs. (2)–(5) to determine an optimal postman tour on G.

For the purposes of the discussion above, it was assumed that $G=(N,E,A)$ was a mixed network. Clearly, the results apply for $G=(N,E)$ an undirected network, and for $G=(N,A)$ a directed network.

5. Some implications

In this section we discuss some of the implications of the results of the last few sections to special cases of the postman problem.

By Theorem 4.5, a feasible solution to Eqs. (2)–(5) guarantees an extreme point solution which in turn leads to a postman tour. Conversely, a postman tour can be used to obtain a feasible solution to (2)–(5). Thus (see footnote 1), a network G is strongly connected if and only if (2)–(5) has a solution.

Suppose for the network $G=(N,E,A)$ that $E=\emptyset$. By Theorem 4.1, the resulting directed postman problem has only integral solutions. In fact, the resulting problem is just a minimum cost flow problem. When $A=\emptyset$, the resulting problem is the Chinese postman problem. Theorem 4.1 implies that Eq. (6) can be replaced by x_i 0 or 1. The linear programming relaxation of this problem would have Eq (3) replaced by $x_{2i-1}+x_{2i}=1$ for $i=1,\ldots,\varepsilon$.

Let G be any connected network having an Euler tour. Let T be such a tour on G. Then $G(T)$ contains the same number of links as G and each associated edge and arc variable will be 0 or 1. By Theorem 4.5 we have that (1)–(5) has an optimal solution consisting of only 0–1 components. Conversely, if (1)–(5) has an optimal solution consisting of only 0–1 components, no link of G is copied more than once to give the associated assigned Euler network G'. With G' we obtain a postman tour T on G where $G(T)=G'$. The tour T must be an Euler tour since it is a postman tour with no link traversed more than once.

6. Possible solution strategies

It is appropriate to note here that Papadimitriou [13] has proven that the mixed postman problem is NP-complete. Thus, in the sense of Edmonds, the problem is a hard one to solve computationally. Certainly, searching the extreme points of a convex polyhedron may be inefficient at best, and computationally intractable at worst. However, we feel that the extreme point result is theoretically interesting. Although we do not investigate algorithms that could be obtained from the results, we do suggest possible solution strategies that perhaps could be the basis of good heuristics.

As stated earlier, to solve the postman problem it is sufficient to search the extreme points of the polyhedron given by Eqs. (2)–(5). There are methods for searching extreme points that appear to work well even in settings similar to the postman problem. For example, Andrew, Hoffman, and Krabeck [1] have given results similar in spirit to Theorem 4.5 for the set covering problem. Lemke, Salkin, and Spielberg [9] have proposed an implicit enumeration algorithm for searching the extreme points and have reported moderate success for problems having up to 200 rows and 500 variables.

Along these lines it should be noted the Eqs (2)–(5) have a generalized upper bounding structure [4] which allows for easier basis inversion. On the other hand, the postman problem is quite degenerate and usually many bases correspond to a given extreme point. Thus moving to new extreme points may be difficult.

Along different lines, the results of Theorem 4.1 can be used to convert problem (1)–(6) to a mixed integer program where one edge variable of each edge pair is required to be integral. A slight modification can be made to this formulation by introducing one binary variable for each edge with all other variables allowed to be continuous. In either case Bender's [3] partitioning method may prove useful.

Acknowledgements

The authors would like to recognize Thomas L. Morin for first suggesting the postman problem as a topic for the first author's dissertion [8] and for many valuable comments.

References

[1] G. Andrew, T. Hoffman, and C. Krabeck, On the generalized set covering problem, Control Data Corporation, 1968.
[2] E.J. Beltrami and L.D. Bodin, Networks and vehicle routing for municipal waste collection, Networks 4 (1974) 65–94.
[3] J.F. Benders, Partitioning procedures for solving mixed-variables programming problems, Numerische Mathematick 4 (1962) 238–252.
[4] G.B. Dantzig and R.M. Van Slyke, Generalized upper bounding techniques, J. Computer Sys. Sci 1 (1967), 213–226.
[5] Edmonds and E.L. Johnson, Matching, Euler Tours and the Chinese Postman, Math. Programming 5 (1973) 88–124.
[6] L.R. Ford and D.R. Fulkerson, Flows in Networks (Princeton University Press, Princeton, NJ, 1962).
[7] F. Harary, Graph Theory, (Addison-Wesley, Reading, MA, 1969).
[8] C.H. Kappauf, The postman problem, Ph. D. Dissertation, Northwestern University, Evanston, ILL, 1975.
[9] C.E. Lemke, H.M. Salkin, and K. Spielberg, Set covering by single branch enumeration with linear programming sub-problems, Operations Res. 19 (1971) 988–1022.
[10] C.W. Marshall, Applied Graph Theory (Wiley Interscience, New York, 1971).
[11] K.G. Murty, The symmetric assignment problem, ORC 67-12 Operations Research Center, University of California, Berkeley, 1967.
[12] C.S. Orloff, A fundamental problem in vehicle routing, Networks 4 (1974), 35–64.
[13] C.H. Papadimitriou, On the complexity of edge-traversing, J.C.M. 23 (3) (1976) 544–554.