3

Tours and Matchings

3.1 Eulerian graphs

The first proper problem in graph theory was the Königsberg bridge problem. In general, this problem concerns of travelling in a graph such that one tries to avoid using any edge twice. In practice these eulerian problems occur, for instance, in optimizing distribution networks – such as delivering mail, where in order to save time each street should be travelled only once. The same problem occurs in mechanical graph plotting, where one avoids lifting the pen off the paper while drawing the lines.

Euler tours

DEFINITION. A walk $W = e_1e_2 \dots e_n$ is a **trail**, if $e_i \neq e_j$ for all $i \neq j$. An **Euler trail** of a graph G is a trail that visits every edge once. A connected graph G is **eulerian**, if it has a closed trail containing every edge of G. Such a trail is called an **Euler tour**.

Notice that if $W = e_1e_2 \dots e_n$ is an Euler tour (and so $E_G = \{e_1, e_2, \dots, e_n\}$), also $e_ie_{i+1} \dots e_ne_1 \dots e_{i-1}$ is an Euler tour for all $i \in [1, n]$. A complete proof of the following **Euler's Theorem** was first given by HIERHOLZER in 1873.

Theorem 3.1 (EULER **(1736),** HIERHOLZER **(1873)).** *A connected graph G is eulerian if and only if every vertex has an even degree.*

Proof. ($\Rightarrow$) Suppose $W\colon u \xrightarrow{\star} u$ is an Euler tour. Let v ($\neq u$) be a vertex that occurs k times in W. Every time an edge arrives at v, another edge departs from v, and therefore $d_G(v) = 2k$. Also, $d_G(u)$ is even, since W starts and ends at u.

($\Leftarrow$) Assume G is a nontrivial connected graph such that $d_G(v)$ is even for all $v \in G$. Let

$$W = e_1e_2 \dots e_n\colon v_0 \xrightarrow{\star} v_n \quad \text{with} \quad e_i = v_{i-1}v_i$$

be a longest trail in G. It follows that all $e = v_nw \in G$ are among the edges of W, for, otherwise, W could be prolonged to We. In particular, $v_0 = v_n$, that is, W is a closed trail. (Indeed, if it were $v_n \neq v_0$ and v_n occurs k times in W, then $d_G(v_n) = 2(k-1)+1$ and that would be odd.)

If W is not an Euler tour, then, since G is connected, there exists an edge $f = v_iu \in G$ for some i, which is not in W. However, now

$$e_{i+1} \dots e_ne_1 \dots e_if$$

is a trail in G, and it is longer than W. This contradiction to the choice of W proves the claim. □

Example 3.1. The k-cube Q_k is eulerian for even integers k, because Q_k is k-regular.

Theorem 3.2. *A connected graph has an Euler trail if and only if it has at most two vertices of odd degree.*

Proof. If G has an Euler trail $u \xrightarrow{\star} v$, then, as in the proof of Theorem 3.1, each vertex $w \notin \{u, v\}$ has an even degree.

Assume then that G is connected and has at most two vertices of odd degree. If G has no vertices of odd degree then, by Theorem 3.1, G has an Euler trail. Otherwise, by the handshaking lemma, every graph has an even number of vertices with odd degree, and therefore G has exactly two such vertices, say u and v. Let H be a graph obtained from G by adding a vertex w, and the edges uw and vw. In H every vertex has an even degree, and hence it has an Euler tour, say $u \xrightarrow{\star} v \to w \to u$. Here the beginning part $u \xrightarrow{\star} v$ is an Euler trail of G. □

The Chinese postman

The following problem is due to GUAN MEIGU (1962). Consider a village, where a postman wishes to plan his route to save the legs, but still every street has to be walked through. This problem is akin to Euler's problem and to the shortest path problem.

Let G be a graph with a weight function $\alpha\colon E_G \to \mathbb{R}^+$. The **Chinese postman problem** is to find a minimum weighted tour in G (starting from a given vertex, the post office).

If G is *eulerian*, then any Euler tour will do as a solution, because such a tour traverses each edge exactly once and this is the best one can do. In this case the weight of the optimal tour is the total weight of the graph G, and there is a good algorithm for finding such a tour:

Fleury's algorithm:

- Let $v_0 \in G$ be a chosen vertex, and let W_0 be the trivial path on v_0.
- Repeat the following procedure for $i = 1, 2, \ldots$ as long as possible: suppose a trail $W_i = e_1 e_2 \ldots e_i$ has been constructed, where $e_j = v_{j-1} v_j$.
  Choose an edge e_{i+1} ($\neq e_j$ for $j \in [1, i]$) so that

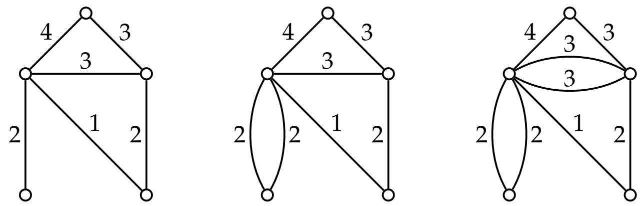
(i) e_{i+1} has an end v_i, and
(ii) e_{i+1} is not a bridge of $G_i = G - \{e_1, \ldots, e_i\}$, unless there is no alternative.

Notice that, as is natural, the weights $\alpha(e)$ play no role in the eulerian case.

Theorem 3.3. *If G is eulerian, then any trail of G constructed by Fleury's algorithm is an Euler tour of G.*

Proof. Exercise. □

If G is *not eulerian*, the poor postman has to walk at least one street twice. This happens, *e.g.*, if one of the streets is a dead end, and in general if there is a street corner of an odd number of streets. We can attack this case by reducing it to the eulerian case as follows. An edge $e = uv$ will be **duplicated**, if it is added to G parallel to an existing edge $e' = uv$ with the same weight, $\alpha(e') = \alpha(e)$.



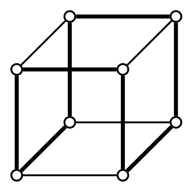
Above we have duplicated two edges. The rightmost multigraph is eulerian.

There is a good algorithm by EDMONDS AND JOHNSON (1973) for the construction of an optimal eulerian supergraph by duplications. Unfortunately, this algorithm is somewhat complicated, and we shall skip it.

3.2 Hamiltonian graphs

In the connector problem we reduced the cost of a spanning graph to its minimum. There are different problems, where the cost is measured by an active user of the graph. For instance, in the **travelling salesman problem** a person is supposed to visit each town in his district, and this he should do in such a way that saves time and money. Obviously, he should plan the travel so as to visit each town once, and so that the overall flight time is as short as possible. In terms of graphs, he is looking for a minimum weighted Hamilton cycle of a graph, the vertices of which are the towns and the weights on the edges are the flight times. Unlike for the shortest path and the connector problems no efficient reliable algorithm is known for the travelling salesman problem. Indeed, it is widely believed that no practical algorithm exists for this problem.

Hamilton cycles



DEFINITION. A path P of a graph G is a **Hamilton path**, if P visits every vertex of G once. Similarly, a cycle C is a **Hamilton cycle**, if it visits each vertex once. A graph is **hamiltonian**, if it has a Hamilton cycle.

Note that if $C : u_1 \to u_2 \to \cdots \to u_n$ is a Hamilton cycle, then so is $u_i \to \dots u_n \to u_1 \to \dots u_{i-1}$ for each $i \in [1, n]$, and thus we can choose where to start the cycle.

Example 3.2. It is obvious that each K_n is hamiltonian whenever $n \geq 3$. Also, as is easily seen, $K_{n,m}$ is hamiltonian if and only if $n = m \geq 2$. Indeed, let $K_{n,m}$ have a