Prob-Stat(MA20205)/CT

Fill in the blanks (Numerical)

Date of Exam : 27th Oct, 2024

Time : SLOT A

Duration : 50min

No of questions:

Type: Random-sequential (navigation allowed)

Each question carries 01 marks

October 23, 2024

1A. Let X_1 and X_2 be independent random variables, where X_i follows an exponential distribution with mean $2i$ for $i = 1, 2$. Find $P(X_1 < X_2)$.

(answer should be correct up to three decimal places)

Answer: 0.667

Range: 0.005

Solution:

$$P(X_1 < X_2) = \int_0^{\infty} \int_0^{x_2} \frac{1}{2} e^{-\frac{x_1}{2}} \frac{1}{4} e^{-\frac{x_2}{4}} dx_1 dx_2 = \frac{2}{3}.$$

2A. The lifetimes (in years) X and Y of two components of an electronic device have the joint pdf:

$$\begin{aligned} f(x,y) &= \frac{2}{\sqrt{2\pi}} exp\Big\{-\frac{x^2}{2} - 2y\Big\}, \qquad -\infty < x < \infty, y \geq 0 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Find $P(X < 0, Y > \frac{1}{2},)$

(answer should be correct up to three decimal places)

Answer: 0.1839

Rage: 0.005

Solution: Clearly X and Y are independent. And $Y \sim$ exponential with mean $\frac{1}{2}$, and $X \sim N(0,1)$. Hence $P(Y > \frac{1}{2}, X < 0) = P(Y > \frac{1}{2})P(X < 0) = \frac{1}{2e} = \frac{1}{e} \times \Phi(0) = \frac{1}{2e}$

3A. Let (X, Y) have the joint density function given by

$$\begin{aligned} f(x,y) &= 1, \quad 0 < |y| < x < 1 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Find $P(0.2 < X < 0.4)$.

(answer should be correct up to two decimal places)

ANS 0.12.

Range: 0.01

Solution: $f(X) = 2x, \quad 0 < x < 1$. So $P(0.2 < X < 0.4) = \int_{0.2}^{0.4} \ 2x \ dx = 0.12$.

4A. Let (X, Y) have bivariate normal distribution with density function

$$f(x,y) = \frac{3}{4\pi\sqrt{2}}\ exp\left[-\frac{9}{16}\left((x-1)^2 - \frac{2}{3}(x-1)(y-1) + (y-1)^2\right)\right].$$

Find the coefficient of correlation between X and Y.

(answer should be correct up to three decimal places)

ANS 0.333

Range: 0.005

Solution: Comparing with the pdf of a bivariate normal distribution, we get $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\rho = \frac{1}{3}$.

5A. Let (X, Y) have bivariate normal distribution with density function

$$f(x,y) = \frac{3}{4\pi\sqrt{2}}\ exp\left[-\frac{9}{16}\left((x-1)^2 - \frac{2}{3}(x-1)(y-1) + (y-1)^2\right)\right].$$

Find $P\left(\frac{2}{3} < X < 2 \;\middle|\; Y = 2\right)$.

(answer should be correct up to three decimal places)

ANS 0.5222

Range: 0.005

Solution: Comparing with the pdf of a bivariate normal distribution, we get $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\rho = \frac{1}{3}$.

The conditional distribution od $X|Y = 2$ is $N\left(\frac{4}{3},\ \frac{8}{9}\right)$.

$P\left(\frac{2}{3} < X < 2 \;\middle|\; Y = 2\right) = \Phi(-\frac{1}{\sqrt{2}}) - \Phi(-\frac{1}{\sqrt{2}}) = 2\Phi(0.71) - 1 = 2 \times 0.7611 - 1 = 0.5222$

6A. Let (X, Y) have bivariate normal distribution with density function

$$f(x, y) = \frac{3}{4\pi\sqrt{2}}\ exp\left[-\frac{9}{16}\left((x-1)^2 - \frac{2}{3}(x-1)(y-1) + (y-1)^2\right)\right].$$

Find $P(2 < 2X + 3Y < 8)$., when $\Phi(0.25) = 0.5987$. (answer should be correct up to three decimal places)

ANS 0.6826

Range: 0.005

Solution: Comparing with the pdf of a bivariate normal distribution, we get $\mu_1 = \mu_2 = 1$, $\sigma_1 = \sigma_2 = 1$, $\rho = \frac{1}{3}$.

$2X + 3Y \sim N(5, 9)$.

$P(2 < 2X + 3Y < 8) = \Phi(1) - \Phi(-1) = 2 \times 0.8413 - 1 = 0.6826$.

7A. Let (X, Y) be a continuous bivariate random variable with the joint pdf

$$\begin{aligned} f(x,y) &= \frac{1}{2}(x+y)\ exp\Big\{-(x+y)\Big\}\ , \quad x>0,\ y>0 \\ &= 0\ , \quad \text{otherwise.} \end{aligned}$$

Find the value of the conditional density of $X \Big| Y = 1$ at $x = 1$.
(answer should be correct up to three decimal places)

ANS 0.3678.

Range: 0.005

Solution: The marginal pdf of Y is $f_Y(y) = \frac{1}{2}(y+1)\ e^{-y},\ y > 0$.

The conditional pdf of $X|Y = y$ is $f_{X|Y=y}(x) = \frac{x+y}{1+y}\ e^{-x},\ x > 0$.

So $f_{X|y=1}(1) = e^{-1} = 0.3678$

8A. Let Y given $T = t$ follow a Poisson distribution with mean t and let T follow an Exponential distribution with mean 0.5. Find $Var(Y)$.

(answer should be correct up to two decimal places)

ANS: 0.75

Range: 0.01

Solution: $Var(Y) = Var\Big(E(Y|T)\Big) + E\Big(Var(Y|T)\Big) = Var(T) + E(T) = 0.25 + 0.5 = 0.75.$

9A. Let (X, Y) be a continuous bivariate random variable with the joint pdf

$$\begin{aligned} f(x,y) &= y\ e^{-y(x+1)}\ , \quad x>0, y>0 \\ &= 0\ , \quad \text{otherwise.} \end{aligned}$$

Find $E(Y|X=3)$.

ANS: 0.5

Range: 0.00

Solution: The marginal pdf of X is $f_X(x) = (x+1)^{-2}, \quad x > 0$.

The conditional pdf of $Y|X=x$ is

$f_{Y|X=x}(y|x) = y(x+1)^2\ e^{-y(x+1)}, \quad y > 0$ for $x > 0$.

$f_{Y|X=3}(y) = 16ye^{-4y}, y > 0$.

$E(Y|X=3) = \frac{2}{4} = 0.5$.

10A. Let X and Y be continuous random variables with the joint pdf

$$\begin{aligned} f(x,y) &= cx^2y, \quad -y < x < 1, 0 < y < 1, \\ &= 0\ , \qquad \text{otherwise.} \end{aligned}$$

where c is a constant. Find the value of c.

(answer should be correct up to three decimal places)

ANSWER : 30/7= 4.2857

ERROR RANGE: 0.005

Solution: $1 = \int_0^1 \int_{-y}^1 cx^2y dxdy = 7c/30$. `Hence` $c = \frac{30}{7}$.

11A. Let X and Y be discrete random variables with the the joint probability mass function

$$\begin{aligned} p(x,y) &= \frac{(x^2+y^2)}{25}, \quad x=1,2;\ y=0,1,2 \\ &= 0\ , \quad \text{elsewhere.} \end{aligned}$$

Find $P(|X-Y|=1)$.

(answer should be correct up to two decimal places)

ANS: 0.44

Range: 0.01

Solution: $P(|X-Y|=1) = P((X,Y)=(1,0)) + P((X,Y)=(2,1)) + P((X,Y)=(1,2))$ $= \frac{11}{25} = 0.44$.

12A. Let X and Y be discrete random variables with the the joint probability mass function

$$\begin{aligned} p(x,y) &= \frac{(x+y)}{15}, \quad x=0,1,2;\ y=1,2 \\ &= 0\ , \quad \text{elsewhere.} \end{aligned}$$

Find $P(X \leq 1|Y=1)$. (answer should be correct up to one decimal place)

ANS: 0.5

Range: 0.00

Solution: $P((X,Y)=(0,1)) = \frac{1}{15}, P((X,Y)=(1,1)) = \frac{2}{15}, P((X,Y)=(2,1)) = \frac{3}{15}$.
$P(Y=1) = \frac{6}{15}$.
So $P(X \leq 1|Y=1) = 0.5$.