How do we get those "magic" equations?

Victor Gafiatulin

Take a look at the matrices generated by applying functions min and max to natural numbers in plane XY (X goes left to right, Y — up to bottom, both X and Y start from 1):

min:

$$\begin{matrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & n-2 & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & n-1 & n-1 \\ 1 & 2 & 3 & \cdots & n-2 & n-1 & n \end{matrix}$$

max:

$$\begin{matrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 2 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 3 & 3 & 3 & \cdots & n-2 & n-1 & n \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ n-2 & n-2 & n-2 & \cdots & n-2 & n-1 & n \\ n-1 & n-1 & n-1 & \cdots & n-1 & n-1 & n \\ n & n & n & \cdots & n & n & n \end{matrix}$$

For sumin observe diagonals starting with the main diagonal and going down into the lower triangle:

Main diagonal:

$$[1, 2, ...\ n]$$

Each next is one element shorter:

$$[1, 2, ...\ n-1],\ [1, 2, ...\ n-2],\ ...,\ [2],\ [1]$$

Let's start summation over lower triangle diagonal-wise starting with the main diagonal:

$$\sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i + ... + \sum_{i=1}^{2} i + \sum_{i=1}^{1} i =$$

$$\sum_{i=1}^{n} \frac{i(i+1)}{2} =$$

$$\frac{1}{2}(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i) = \frac{1}{2}(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}) =$$

$$\frac{n(n+1)(n+2)}{6}$$

To get sum over the whole matrix we need to multiply the result by 2 (to account for both upper and lower triangles) and subtract $\frac{n(n+1)}{2}$, since we counted the main diagonal while summing over both upper and lower triangles:

$$2 \cdot \frac{n(n+1(n+2))}{6} - \frac{n(n+1)}{2} =$$

$$\frac{n(n+1)(2n+1)}{6}$$

Similarly for sumax:

Main diagonal:

$$[1, 2, ...\ n]$$

Each next is again one element shorter:

$$[2, 3, ...\ n],\ [3, 4...\ n],\ ...,\ [n-1, n],\ [n]$$

Moving to summation we can present them as following:

$$\text{sum } [1, 2, \ldots\ n] + (\text{sum } [1, 2, \ldots\ n] \text{ - sum } [1])$$
$$+ (\text{sum } [1, 2, \ldots\ n] \text{ - sum } [1, 2]\ ) + \ldots + (\text{sum } [1, 2, \ldots\ n] \text{ - sum } [1, 2, \ldots\ n-2])$$
$$+ (\text{sum } [1, 2, \ldots\ n] \text{ - sum } [1, 2, \ldots\ n-1])$$

Consequently:

$$\sum_{i=0}^{n-1}(\frac{n(n+1)}{2} - \sum_{j=1}^{i} j) =$$

$$\frac{n^2(n+1)}{2} - \sum_{i=0}^{n-1}\sum_{j=1}^{i} j =$$

$$\frac{n^2(n+1)}{2} - \sum_{i=0}^{n-1}\frac{i(i+1)}{2} =$$

$$\frac{n^2(n+1)}{2} - \frac{1}{2}(\frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2}) =$$

$$\frac{n(n+1)(2n+1)}{6}$$

Multiplying by 2 and subtracting $\frac{n(n+1)}{2}$ we get

$$\frac{n(n+1)(4n-1)}{6}$$