System of Linear Equations

$\{ Ax = B \}$

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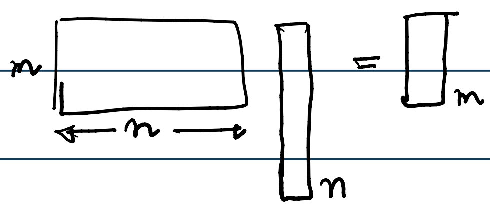
$Ax = b$ $\qquad A_{m\times n}$, rank $= r$

4 Possibilities

① $r = m = n$ {Square, Invertible} $\qquad A_1 x = b$ has 1 solution for every B

② $r = m < n$ $\qquad$ $C(A)$ spans $\mathbb{R}^m \Rightarrow$ sol^n for every b

More unknowns



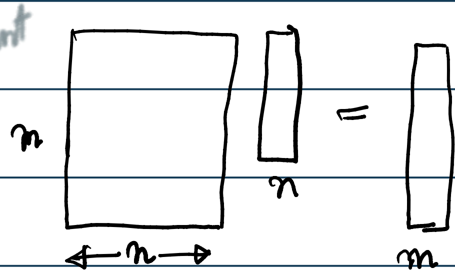
$A_2 x = b$ $\qquad$ {∞ sol^{n}s}

$\dim(N(A_2)) = n - r > 0$

Hence any sol^n can be extended by adding more from null space.

③ $r = n < m$

Inconsistent Equations $\Downarrow$ 0 sol^n

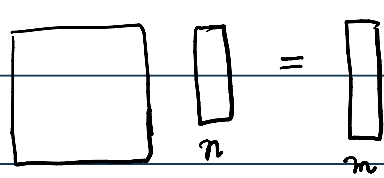


$A_3 x = b$ has either 1 or 0 solutions.

when b is in the column space, there is 1 solution

when $b \notin C(A)$, no solution.

④ $r < n$, $r < m$



$A_4 x = b$ has either 0 or ∞ solutions.

when $b \notin C(A)$, no sol^n.

Because $r < m$, it doesn't span $\mathbb{R}^m$,

$\dim(N(A_4)) = n - r > 0$, Thus any sol^n can be extended to get ∞ sol^n.

Difficulties with $Ax=b$

ⓐ Ordinary elimination might not work.
There may be too many equations & no solⁿ.

ⓑ A square matrix might be singular.

ⓒ A may be extremely ill-conditioned or too large.

SUGGESTIONS

0. Every matrix $A = U\Sigma V^T$ has a pseudoinverse
$$A^+ = V\Sigma^+ U^T$$

1. If A is square & invertible, with reasonable size, & its' condition number $\frac{\sigma_1}{\sigma_n}$ is not large. Elimination will succeed (possibly with row exchanges). We have
$$PA = LU \quad \text{or} \quad A = LU$$

2. Suppose $m > n = r$: There are too many equations $Ax=b$ to expect a solution. If the columns of A are independent, and not too ill-conditioned, then we solve the normal equations $A^TA\hat{x} = A^Tb$

we solve the normal equations $A^T A \hat{x} = A^T b$ to find the least squares solution $\hat{x}$

Vector b is probably not in the column space of A, & $Ax = b$ is probably impossible. $A\hat{x}$ is the projection of b onto that column space.

3. Suppose $m < n$. Now equation $Ax = b$ has many solutions, if it has one. A has nonzero nullspace. The solution 'x' is underdetermined. We want to choose the best 'x' for our purpose. Two possible choices of x^+ and x_1:

$x = x^+ = A^+ b$. The pseudo inverse A^+ gives the minimum ℓ^2 norm solution with nullspace component $= 0$

$x = x_1 =$ minimum ℓ^1 norm solution. This sol^n is often sparse (many zero components). {Related to 'basis pursuit'}

4. Columns of A are badly conditioned. Ratio $\frac{\sigma_1}{\sigma_r}$ is too large.

Usual sol^n → Orthogonalize Columns.

5. A may be nearly singular. In this case, A^TA will have a very large inverse. We then solve.

$\det(A)=0$

$$\text{Minimize} \quad \|Ax-b\|^2 + \delta^2\|x\|^2$$

$$\text{Solve} \quad (A^TA + \delta^2 I)\, x_\delta = A^Tb$$

6. If A is too big, we have to resort to randomized methods.