TEXT BOOKS:
i) Higher Engineering Mathematics by B V Ramana ., Tata McGraw Hill.
ii) Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers.
iii) Advanced Engineering Mathematics by Kreyszig, John Wiley & Sons.

REFERENCE BOOKS:
i)Advanced Engineering Mathematics by R.K Jain & S R K Iyenger, Narosa Publishers.
ii)Advanced Engineering Mathematics by Michael Green Berg, Pearson Publishers .
iii)Engineering Mathematics by N.P Bali and Manish Goyal.

Course Outcomes: After learning the concepts of this paper the student will be able to
1.Analyze the solution of the system of linear equations and to find the Eigen values and Eigen vectors of a matrix.
2.Find the extreme values of functions of two variables with / without constraints.
3.Solve first and higher order differential equations.
4.Solve first order linear and non-linear partial differential equations.
5.Solve differential equations with initial conditions using Laplac

UNIT-I

MATRICES

Matrix : A system of mn numbers real (or) complex arranged in the form of an ordered set of 'm' rows, each row consisting of an ordered set of 'n' numbers between [ ] (or) ( ) (or) || || is called a matrix of order m xn.

Eg: $\begin{bmatrix} a_{11} & a_{12}.........a_{1n} \\ a_{21} & a_{12}.........a_{2n} \\ .... & \\ .... & \\ a_{m1} & a_{m2}.......a_{mn} \end{bmatrix}_{m \times n}$ $= [a_{ij}]_{mxn}$ where $1 \le i \le m$, $1 \le j \le n$.

Order of the Matrix: The number of rows and columns represents the order of the matrix. It is denoted by mxn, where m is number of rows and n is number of columns.

Types of Matrices:

Row Matrix: A Matrix having only one row is called a "Row Matrix".

Eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

Column Matrix: A Matrix having only one column is called a "Column Matrix" .

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

Null Matrix: A= $\left[a_{ij}\right]_{mxn}$ such that $a_{ij}=0$ $\forall$ i and j. Then A is called a "Zero Matrix". It is denoted by Omxn.

Eg: $O_{2x3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rectangular Matrix: If A=$\left[a_{ij}\right]_{mxn}$, and $m \neq n$ then the matrix A is called a "Rectangular Matrix"

Eg : $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

Square Matrix: If A=$\left[a_{ij}\right]_{mxn}$ and m = n then A is called a "Square Matrix".

Eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is a 2x2 matrix

Lower Triangular Matrix: A square Matrix $A_{nXn}=\left[a_{ij}\right]_{nxn}$ is said to be lower Triangular of $a_{ij}=0$ if i <j i.e. if all the elements above the principle diagonal are zeros.

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is a Lower triangular matrix

Upper Triangular Matrix: A square Matrix $A=\left[a_{ij}\right]_{nxn}$ is said to be upper triangular of $a_{ij}=0$ if i.>j. i.e. all the elements below the principle diagonal are zeros.

Eg: $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an Upper triangular matrix

Triangle Matrix: A square matrix which is either lower triangular or upper triangular is called a triangle matrix.

Principal Diagonal of a Matrix: In a square matrix, the set of all a_{ij}, for which i = j are called principal diagonal elements. The line joining the principal diagonal elements is called principal diagonal.

Note: Principal diagonal exists only in a square matrix.

Diagonal elements in a matrix: A= $[a_{ij}]_{nxn}$, the elements a_{ij} of A for which i = j.

i.e. a_{11}, a_{22}….a_{nn} are called the diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

Diagonal Matrix: A Square Matrix is said to be diagonal matrix, if a_{ij} = 0 for i$\neq$j i.e. all the elements except the principal diagonal elements are zeros.

Note: 1. Diagonal matrix is both lower and upper triangular.

2. If d_1, d_2…….d_n are the diagonal elements in a diagonal matrix it can be represented as **diag** $[d_1, d_2,……., d_n]$

Eg : A = diag (3,1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Scalar Matrix: A diagonal matrix whose leading diagonal elements are equal is called a "Scalar Matrix". Eg : A= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit/Identity Matrix: If A = $[a_{ij}]_{nxn}$ such that a_{ij}=1 for i = j, and a_{ij}=0 for $i \neq j$ then A is called a "Identity Matrix" or Unit matrix. It is denoted by I_n

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Trace of Matrix: The sum of all the diagonal elements of a square matrix A is called Trace of a matrix A, and is denoted by Trace A or tr A.

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then tr A = a+b+c

Singular & Non Singular Matrices: A square matrix A is said to be "Singular" if the determinant of $|A| = 0$, Otherwise A is said to be "Non-singular".

Note: 1. Only non-singular matrices possess inverse.

2. The product of non-singular matrices is also non-singular.

Inverse of a Matrix: Let A be a non-singular matrix of order n if there exist a matrix B such that AB=BA=I then B is called the inverse of A and is denoted by A^{-1}.

If inverse of a matrix exist, it is said to be invertible.

Note: 1. The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$.

2 .Every Invertible matrix has unique inverse**.**

3. If A, B are two invertible square matrices then AB is also invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

4. $A^{-1} = \frac{AdjA}{\det A}$ $\quad$ *where* $detA \neq 0$,

Theorem: The inverse of a Matrix if exists is Unique.

Note: 1. $(A^{-1})^{-1} = A$ 2. $I^{-1} = I$

Theorem: If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^{1})^{-1} = (A^{-1})^{1}$

Sub Matrix: - A matrix obtained by deleting some of the rows or columns or both from the given matrix is called a sub matrix of the given matrix.

Eg: Let A = $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2x3}$ is a sub matrix of A obtained by deleting first row and 4^{th} column of A.

Minor of a Matrix: Let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Eg: A = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ be a 4x3 matrix

Here B = $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2

And C = $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'

det C = 2(7-12)-1(21-10)+(18-5) = -9

Properties of trace of a matrix: Let A and B be two square matrices and λ be any scalar

1) tr (λ A) = λ (tr A) ; 2) tr(A+B) = trA + trB ; 3) tr (AB) = tr(BA)

Idempotent Matrix: A square matrix A Such that $A^2=A$ then A is called "Idempotent Matrix".

Eg: A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Involutary Matrix: A square matrix A such that $A^2 = I$ then A is called an Involutary Matrix.

Eg: A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Nilpotent Matrix: A square matrix A is said to be Nilpotent if there exists a + ve integer n such that $A^n = 0$ here the least n is called the Index of the Nilpotent Matrix.

Eg: A = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose of a Matrix: The matrix obtained by interchanging rows and columns of the given matrix A is called as transpose of the given matrix A. It is denoted by A^T or A^1

Eg: A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties of transpose of a matrix: If A and B are two matrices and A^T , B^T are their transposes then

1) $\left(A^T\right)^T = A$; 2) $\left(A+B\right)^T = A^T + B^T$; 3) $\left(KA\right)^T = KA^T$; 4) $\left(AB\right)^T = B^T A^T$

Symmetric Matrix: A square matrix A is said to be symmetric if $A^T = A$

If $A = \left[a_{ij}\right]_{nxn}$ $then\ A^T = \left[a_{ji}\right]nxn$ where $a_{ij} = a_{ji}$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

Skew-Symmetric Matrix: A square matrix A is said to be Skew symmetric If $A^T = -A$.

If $A = \left[a_{ij}\right]_{nxn}$ *then* $A^T = \left[a_{ji}\right]_{nxn}$ *where* $a_{ij} = -a_{ji}$.

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Note: All the principle diagonal elements of a skew symmetric matrix are always zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

Theorem: Every square matrix can be expressed uniquely as the sum of symmetric and skew symmetric matrices.

Proof: Let A be a square matrix, A = $\frac{1}{2}(A+A) = \frac{1}{2}\left(A + A^T + A - A^T\right) =$

$\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$ = P + Q, where P = $\frac{1}{2}\left(A+A^T\right)$; Q = $\frac{1}{2}\left(A-A^T\right)$

Thus every square matrix can be expressed as a sum of two matrices.

Consider $P^T = \left[\frac{1}{2}\left(A+A^T\right)\right]^T = \frac{1}{2}\left(A+A^T\right)^T = \frac{1}{2}\left(A^T+\left(A^T\right)^T\right) = \frac{1}{2}\left(A+A^T\right)$ =P, since $P^T = P$,

P is symmetric

Consider $Q^T = \left[\frac{1}{2}\left(A-A^T\right)\right]^T = \frac{1}{2}\left(A-A^T\right)^T = \frac{1}{2}\left(A^T-\left(A^T\right)^T\right) = -\frac{1}{2}\left(A-A^T\right) = -Q$

Since $Q^T = -Q$, Q is Skew-symmetric.

To prove the representation is unique: Let A= R+S $\rightarrow (1)$ be the representation, where R is symmetric and S is skew symmetric. i.e. $R^T = R, S^T = -S$

Consider $A^T = (R+S)^T = R^T + S^T = R - S \rightarrow (2)$

$(1)-(2) \Rightarrow A - A^T = 2S \Rightarrow S = \frac{1}{2}\left(A-A^T\right) = Q$

Therefore every square matrix can be expressed as a sum of a symmetric and a skew symmetric matrix

Ex. Express the given matrix A as a sum of a symmetric and skew symmetric matrices

where A= $\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$

Solution: $A^T = \begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 3 & 11 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 4 & 10 & 12 \\ 10 & 14 & 18 \\ 12 & 18 & 22 \end{bmatrix} \Rightarrow P = \frac{1}{2}\left(A + A^T\right) = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}; P \text{ is symmetric}$$

$$A - A^T = \begin{bmatrix} 0 & -18 & 6 \\ 18 & 0 & 8 \\ -6 & -8 & 0 \end{bmatrix} \Rightarrow Q = \frac{1}{2}\left(A - A^T\right) = \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}; Q \text{ is skew-symmetric}$$

Now A =P+Q = $\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

Orthogonal Matrix: A square matrix A is said to be an Orthogonal Matrix if $AA^T = A^TA = I$, Similarly we can prove that $A = A^{-1}$; Hence A is an orthogonal matrix.

Note: 1. If A, B are orthogonal matrices, then AB and BA are orthogonal matrices.

2. Inverse and transpose of an orthogonal matrix is also an orthogonal matrix.

Result: If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

Result: The inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal

Solved Problems :

1. Show that A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **is orthogonal.**

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **is orthogonal.**

Sol: Given $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ Then $A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A.A^T = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\Rightarrow A.A^T = I$

Similarly $A^T.A = I$

Hence A is orthogonal matrix

3. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Sol: - For orthogonal matrix $AA^T = I$

So, $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$

We get $c = \pm\sqrt{2}b$ $a^2 = b^2+2b^2 = 3b^2$

$\Rightarrow a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1 \Rightarrow 4b^2+2b^2=1$ (since $c^2=2b^2$) $\Rightarrow b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3}b = \pm\frac{1}{\sqrt{2}}$; $b = \pm\frac{1}{\sqrt{6}}$; $c = \pm\sqrt{2}b = \pm\frac{1}{\sqrt{3}}$

4. Is matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ Orthogonal?

Sol:- Given $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$\Rightarrow AA^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I_3$

$AA^T \neq A^TA \neq I_3$

$\therefore$ Matrix is not orthogonal.

Complex matrix: A matrix whose elements are complex numbers is called a complex matrix.

Conjugate of a complex matrix: A matrix obtained from A on replacing its elements by the corresponding conjugate complex numbers is called conjugate of a complex matrix. It is denoted by $\overline{A}$

If $A = \left[a_{ij}\right]_{mxn}$, $\overline{A} = \left[\overline{a_{ij}}\right]_{mxn}$, where $\overline{a_{ij}}$ is the conjugate of a_{ij}.

Eg: If A=$\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

Note: If $\overline{A}$ and $\overline{B}$ be the conjugate matrices of A and B respectively, then

(i) $\overline{\left(\overline{A}\right)}$=A (ii) $\overline{A+B} = \overline{A} + \overline{B}$ (iii) $\overline{(KA)} = \overline{K}\ \overline{A}$

Transpose conjugate of a complex matrix: Transpose of conjugate of complex matrix is called transposed conjugate of complex matrix. It is denoted by A^θ or A^*.

Note: If A^θ and B^θ be the transposed conjugates of A and B respectively, then

(i) $\left(A^\theta\right)^\theta$ = A (ii) $(A \pm B)^\theta = A^\theta \pm B^\theta$

(iii) $(KA)^\theta = \overline{K}A^\theta$ (iv) $(AB)^\theta = A^\theta B^\theta$

Hermitian Matrix: A square matrix A is said to be Hermitian Matrix iff $A^\theta = A$.

Eg: A=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and A^θ=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Note: 1. In Hermitian matrix the principal diagonal elements are real.

2. The Hermitian matrix over the field of Real numbers is nothing but real symmetric matrix.

3. In Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij} = \overline{a_{ji}}\ \forall i, j$.

Skew-Hermitian Matrix: A square matrix A is said to be Skew-Hermitian Matrix iff $A^\theta = -A$.

Eg: Let A=$\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\overline{A}\right)^T = \begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore$ $\left(\overline{A}\right)^T$=-A $\therefore$ A is skew-Hermitian matrix.

Note: 1. In Skew-Hermitian matrix the principal diagonal elements are either Zero or Purely Imaginary.

2. The Skew- Hermitian matrix over the field of Real numbers is nothing but real Skew - Symmetric matrix.

3. In Skew-Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij} = -\overline{a_{ji}}\ \forall i, j$.

Unitary Matrix: A Square matrix A is said to be unitary matrix iff

$AA^{\theta} = A^{\theta}A = I \text{ or } A^{\theta} = A^{-1}$

Eg: $B = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

Theorem1: Every square matrix can be uniquely expressed as a sum of Hermitian and skew – Hermitian Matrices.

Proof: - Let A be a square matrix write

$$A = \frac{1}{2}(2A) = \frac{1}{2}(A+A) = \frac{1}{2}(A + A^{\theta} + A - A^{\theta})$$
$$A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta}) i.e A = P+Q$$

Let $P = \frac{1}{2}\left(A + A^{\theta}\right); Q = \frac{1}{2}\left(A - A^{\theta}\right)$

Consider $P^{\theta} = \left[\frac{1}{2}(A+A^{\theta})^{\theta}\right] = \frac{1}{2}(A+A^{\theta})^{\theta} = (A+A^{\theta}) = P$

I.e. $P^{\theta} = P, P$ is Hermitian matrix.

$$Q^{\theta} = \left[\frac{1}{2}(A-A^{\theta})\right]^{\theta} = \frac{1}{2}\left(A^{\theta} - A\right) = -\frac{1}{2}(A - A^{\theta}) = -Q$$

Ie $Q^{\theta} = -Q, Q$ is skew – Hermitian matrix.

Thus every square matrix can be expressed as a sum of Hermitian & Skew Hermitian matrices.

To prove such representation is unique:

Let A = R+S------- (1) be another representation of A where R is Hermitian matrix & S is skew – Hermitian matrix.

$\therefore R = R^{\theta}; S^{\theta} = -S$

Consider $A^{\theta} = (R+S)^{\theta} = R^{\theta} + S^{\theta} = R - S -$. Ie $A^{\theta} = R - A -----(2)$

$$(1)+(2) \Rightarrow A+A^{\theta}=2R \text{ ie } R=\frac{1}{2}\left(A+A^{\theta}\right)=P$$

$$(1)-(2) \Rightarrow A-A^{\theta}=2S \text{ ie } S=\frac{1}{2}\left(A-A^{\theta}\right)=Q$$

Thus every square matrix can be uniquely expressed as a sum of Hermitian & skew Hermitian matrices.

Solved Problems :

1) If A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **then show that A is Hermitian and iA is skew-Hermitian.**

Sol: Given A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A}=\begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\left(\overline{A}\right)^T=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A=\left(\overline{A}\right)^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B}=\begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T=\begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix}=(-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}=-B$$

$\therefore \left(\overline{B}\right)^T$=-B

$\therefore$ B= iA is a skew Hermitian matrix.

2). If A and B are Hermitian matrices, prove that AB-BA is a skew-Hermitian matrix.

Sol: Given A and B are Hermitan matrices

$\therefore \left(\overline{A}\right)^T=A$ And $\left(\overline{B}\right)^T=B$ ------------- (1)

Now $\overline{(AB-BA)}^T = \left(\overline{AB}-\overline{BA}\right)^T$

$$= \left(\overline{A}\,\overline{B}-\overline{B}\,\overline{A}\right)^T$$

$$= \left(\overline{A}\,\overline{B}\right)^T - \left(\overline{B}\,\overline{A}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T - \left(\overline{A}\right)^T\left(\overline{B}\right)^T$$

$$= BA - AB \text{ (By (1))}$$

$$= -(AB-BA)$$

Hence AB-BA is a skew-Hemitian matrix.

3). Show that A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$

Sol: Given A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^{\theta} = \left(\overline{A}\right)^T = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\therefore AA^{\theta} = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$\therefore AA^{\theta} = I$ if and only if $a^2+b^2+c^2+d^2=1$

4)Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I-A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1} = \frac{1}{1-\left(4i^2-1\right)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B=(I-A)(I+A)^{-1}$

$$B=\frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}=\frac{1}{6}\begin{bmatrix} 1+(1-2i)(-1-2i) & -1-2i-1-2i \\ 1-2i+1-2i & (-1-2i)(1-2i)+1 \end{bmatrix}$$

$$B=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Now $\overline{B}=\frac{1}{6}\begin{bmatrix} -4 & -2+4i \\ 2+4i & -4 \end{bmatrix}$ and $\left(\overline{B}\right)^T=\frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$B\left(\overline{B}\right)^T=\frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$$\left(\overline{B}\right)^T=B^{-1}$$

i.e. B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

5) Show that the inverse of a unitary matrix is unitary.

Sol: Let A be a unitary matrix. Then $AA^\theta=I$

i.e $\left(AA^\theta\right)^{-1}=I^{-1}$

$$\Rightarrow \left(A^{\theta}\right)^{-1} A^{-1} = I$$

$$\Rightarrow \left(A^{-1}\right)^{\theta} A^{-1} = I$$

Thus A^{-1} is unitary.

Rank of a Matrix:

Let A be mxn matrix. If A is a null matrix, we define its rank to be ‘0’. If A is a non-zero matrix, we say that ‘**r**’ is the rank of A if

i. Every $(r+1)^{th}$ order minor of A is ‘0’ (zero) &

ii. At least one r^{th} order minor of A which is not zero.

It is denoted by ρ (A) and read as rank of A.

Note: 1. Rank of a matrix is unique.

2. Every matrix will have a rank.

3. If A is a matrix of order mxn, then Rank of A $\leq$ min (m,n)

4. If ρ (A) = r then every minor of A of order r+1, or minor is zero.

5. Rank of the Identity matrix I_n is n.

6. If A is a matrix of order n and A is non-singular then ρ (A) = n

7. If A is a singular matrix of order n then ρ (A) < n

Important Note:

1. The rank of a matrix is $\leq$ r if all minors of $(r+1)^{th}$ order are zero.

2. The rank of a matrix is $\geq$ r, if there is at least one minor of order ‘r’ which is not equal to zero.

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

det A = 1(48-40)-2(36-28)+3(30-28) = 8-16+6 = -2 $\neq$ 0

We have minor of order 3 $\therefore \rho$ (A) =3

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank $\leq$ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is 1(-49)-2(-56) + 3(35-48) = -49+112-39 = 24 $\neq$ 0.

Hence rank of the given matrix is '3'.

Elementary Transformations on a Matrix:

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \rightarrow kR_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \rightarrow R_j + kR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \rightarrow k\, c_j$ $c_j \rightarrow c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalance of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.It is denoted as B~A.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Elementary Matrix or E-Matrix: A matrix is obtained from a unit matrix by a single elementary transformation is called elementary matrix or E-matrix.

Notations: We use the following notations to denote the E-Matrices.

1) E_{ij} $\rightarrow$ Matrix obtained by interchange of i^{th} and j^{th} rows (columns).

2) $E_{i(k)}$ $\rightarrow$ Matrix obtained by multiplying ith row (column) by a non- zero number k.

3) $E_{ij(k)}$ $\rightarrow$ Matrix obtained by adding k times of j^{th} row (column) to ith row (column).

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i) Zero rows, if any exists, they should be below the non-zero row.

(ii) The first non-zero entry in each non-zero row is equal to '1'.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note : 1. The number of non-zero rows in echelon form of A is the rank of 'A'.

1. The rank of the transpose of a matrix is the same as that of original matrix.

2. The condition (ii) is optional.

Eg: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

Solved Problems :

1. Find the rank of the matrix A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ **by reducing it to Echelon form.**

Sol: Given A = $\begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ Applying row transformations on A.

$R_1 \leftrightarrow R_3$ $\quad A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$

$R_2 \to R_2 - 3R_1;\ R_3 \to R_3 - 2R_1$ $\quad \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$

$R_2 \to R_2/7, R_3 \to R_3/9$ $\quad \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad \sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

This is the Echelon form of matrix A.

The rank of a matrix A= Number of non – zero rows =2

2. For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ **has rank '3'.**

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get $A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank A = 3 ⇨ det A =0

⇨ $4\begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$

⇨ 1[(8-4k) 3]-1(8-4k) (4k+27)] = 0

⇨ (8-4k) (3-4k-27) = 0

⇨ (8-4k)(-24-4k) =0

⇨ (2-k)(6+k)=0

⇨ k =2 or k = -6

3). Find the rank of the matrix using echelon form $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1\,; R_3 \to R_3 - 4R_1\,; R_4 \to R_4 - 4R_1$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_1 \to \frac{R_1}{-1}, R_2 \to \frac{R_2}{-1}, R_3 \to \frac{R_3}{-3}$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ A is in echelon form $\qquad\qquad \therefore$ Rank of A = 2

4). Find the rank of the matrix A = $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ **by reducing into echelon form.**

Sol: By applying $R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1; R_4 \to R_4 + R_1$ $\qquad$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\qquad$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 \leftrightarrow R_4$ $\qquad$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$ $\qquad$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clear it is in echelon form, rank of A = 2

Normal form/Canonical form of a Matrix:

Every non-zero Matrix can be reduced to any one of the following forms. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}; [I_r \quad 0]; \begin{bmatrix} I_r \\ 0 \end{bmatrix}; [I_r]$ Known as normal forms or canonical forms by using Elementary row or column or both transformations where I_r is the unit matrix of order 'r' and 'O' is the null matrix.

Note: 1.In this form "the rank of a matrix is equal to the order of an identity matrix.

2. Normal form another name is "canonical form"

Solved Problems :

1. By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ **into normal form, find its rank.**

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 3R_1$ $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$

$R_3 \to R_3/\text{-}2$ $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$

$R_3 \to R_3 + R_2$ $\quad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_2 \to c_2 - 2c_1,\ c_3 \to c_3 - 3c_1,\ c_4 \to c_4 - 4c_1$ $\quad A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_3 \to 3c_3 - 2c_2,\ c_4 \to 3c_4 - 5c_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

$c_2 \to c_2/\text{-}3,\ c_4 \to c_4/18$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$c_4 \leftrightarrow c_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

This is in normal form $[I_3\ 0]$, $\quad \therefore$ Hence Rank of A is ‘3’.

2). Find the rank of the matrix A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$ **by reducing into canonical form or normal form.**

Sol: Given A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

By applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 + 7R_2$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$

$R_4 \to R_4 - 6R_3$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$

$R_4 \to \dfrac{R_4}{-18}$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Apply $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_3 \to C_3 - 2C_2; C_4 \to C_4 + 5C_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_4 \to C_4 + 2C_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Clearly it is in the normal form $[I_4]$ $\therefore$ Rank of A = 4

3). Define the rank of the matrix and find the rank of the following matrix $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Let A= $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 4R_1$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - 3R_2$
$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form. So, rank of matrix = no. of non zero rows in echelon form. $\therefore Rank \quad \rho(A) = 2$

4). Reduce the matrix A to normal form and hence find its rank $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

$C_1 \to \frac{1}{2} C_1$
$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 1 & 3 & 7 & 5 \\ 1 & 5 & 11 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_1$
$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_3$
$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 4R_2$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 8 & 2 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to 4C_4 - C_3$
$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to \frac{1}{3}C_3$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

This is in normal form. Thus Rank of matrix = Order of identify matrix. $\therefore$ Rank $\rho(A) = 3$

5). Reduce the matrix A = $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **into canonical form and then find its rank.**

Sol: Apply $C_1 \leftrightarrow C_2$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$C_3 \to C_3 - 2C_1; C_4 \to C_4 + 2C_1$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_2 \to \frac{R_2}{2}$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \leftrightarrow C_3$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - 2C_2; C_4 \to C_4 - 3C_2$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which is in the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\qquad \therefore \rho$ (A) = 2

Note: .If A is an mxn matrix of rank r, there exists non-singular matrices P and Q such that

PAQ = $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

Suppose we want to find P and Q we have procedure.

Let order of matrix 'A' is '3 $\qquad$ i.e. A = I_3 A I_3

A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ A $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now we go on applying elementary row operations and column operations on the matrix A (L.H.S) until it is reduced to the normal form $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$

Every row operations will also be applied to the pre-factor of on R.H.S

Every column operation will also be applied to the post –factor of on R.H.S.

Solved Problems :

1. Find the non-singular matrices P and Q is of the normal form where A = $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$

Sol: Write $A = I_3\, A\, I_3$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2-2R_1,\ R_3 \to R_3-3R_1$
$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3-R_2,\ R_2 \to {}^{1}/_{3}\, R_2$
$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$c_3 \to c_3-2c_1$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ \quad \text{where } P = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Find the non-singular matrices P and Q such that the normal form of A is P A Q.

Where A = $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$**. Hence find its rank.**

Sol: we write $A = I_3 A\, I_4$

$$\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$ $\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\sim \begin{bmatrix} 1 & 3 & 6 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Applying $c_2 \to c_2 - 3c_1, c_3 \to c_3 - 6c_1$, and $c_4 \to c_4 + c_1$, we get.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -6 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $c_3 \to c_3 + c_2$ and $c_4 \to c_4 - 2c_2$, we get.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⇨ $\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} = PAQ$ Where $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & -3 & -9 & 7 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Here $A \sim \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$, ∴ Hence $\rho(A) = 2$

System of linear equations: In this chapter we shall apply the theory of matrices to study the existence and nature of solutions for a system of m linear equations in 'n' unknowns.

The system of m linear equations in 'n' unknowns $x_1, x_2, x_3 ----x_n$ given by

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\} \text{---------------------------} (1)$$

The above set of equations can be written in the Matrix form as A X = B

$$\begin{bmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix} \to (2)$$

A-Coefficient Matrix; X-Set of unknowns; B-Constant Matrix

Homogeneous Linear Equations: If $b_1 = b_2 = \ldots\ldots\ldots\ldots = b_m = 0$ then B = 0

Hence eqn (2) Reduces to AX = 0 which are known as homogeneous linear equations

Non-Homogeneous Linear equations:

It at least one of b_1, b_2----b_m is non zero. Then $B \neq 0$, the system Reduces to AX = B is known as Non-Homogeneous Linear equations.

Solutions: A set of numbers $x_1, x_2 --x_n$ which satisfy all the equations in the system is known as solution of the system.

Consistent: If the system possesses a solution then the system of equations is said to be consistent.

Inconsistent: If the system has no solution then the system of equations is said to be Inconsistent.

Augmented Matrix: A matrix which is obtained by attaching the elements of B as the last column in the coefficient matrix A is called Augmented Matrix. It is denoted by [A|B]

$$[A/B] = C = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & - & - & a_{2n} & : & b_2 \\ a_{m1} & a_{m2} & a_{m3} & - & - & a_{mn} & : & b_3 \end{bmatrix}$$

1. If $\rho(A/B) = \rho(\text{A})$, then the system of equations AX = B is consistent (solution exits).

a). If $\rho(A/B) = \rho(\text{A}) = r = n$ (no. of unknowns) system is consistent and have **a unique solution**

b). If $\rho(A/B) = \rho(\text{A}) = r < n$ (no. of unknowns) then the system of equations AX = B will have an **infinite no. of solutions**. In this case (n-r) variables can be assigned arbitrary values.

2. If $\rho(A/B) \neq \rho(\text{A})$ then the system of equations AX=B is **inconsistent (no solution).**

In case of homogeneous system AX = 0, the system is always consistent.

(or) $x_1 = 0$, $x_2 = 0$, --------, $x_n = 0$ is always the solution of the system known as the" zero solution ".

Non-trivial solution:

If $\rho(A/\text{B}) = \rho(\text{A}) = r < n$ (no. of unknowns) then the system of equations AX = 0 will have an infinite no. of non zero (non trivial) solutions. In this case (n-r) variables can be assigned arbitrary values.

Also we use some direct methods for solving the system of equations.

Note: The direct methods are Cramer's rule, Matrix Inversion, Gaussian Elimination, Gauss Jordan, Factorization Tridiagonal system. These methods will give a unique solution.

Procedure to solve AX = B (Non Homogeneous equations)

Let us first consider n equations in n unknowns ie. m=n then the system will be of the form

$a_{11}x_1+a_{12}x_2+a_{13}x_3+.....+a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+.....+a_{2n}x_n=b_2$

……………………………………..

$a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+.....+a_{nn}x_n=b_n$

The above system can be written as AX = B --------- (1)

Where A is an $n \times n$ matrix.

Solving AX = B using Echelon form:

Consider the system of m equations in n unknowns given by

$a_{11}x_1+a_{12}x_2+a_{13}x_3+.....+a_{1n}x_n=b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+.....+a_{2n}x_n=b_2$

……………………………………..

$a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+.....+a_{mn}x_n=b_m$

We know this system can be we write as AX = B

The augmented matrix of the above system is $[A / B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

The system AX = B is consistent if ρ (A) = ρ[A/B]

i). ρ (A) = ρ [A/B] = r < n (no. of unknowns).Then there is infinite no. of solutions.

ii). ρ (A) = ρ [A/B] = number of unknowns then the system will have unique solution.

iii). ρ (A) ≠ ρ [A/B] the system has no solution.

Solved Problems :

1). Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.

Sol: Write given equations is of the form AX = B i.e; $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

Consider the Augment matrix is [A /B] ⇨ $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \rightarrow R_2-2R_1$ and $R_3 \rightarrow R_3-R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$

Applying $R_3 \rightarrow R_3 - 2R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

$\therefore \rho(A) = 2$ and $\rho(A/B) = 3$

The given system is inconsistent as $\rho(A) \neq \rho[A/B]$.

2). Show that the equations given below are consistent and hence solve them

x-3y-8z = -10, 3x+y-4z =0, 2x+5y+6z =3

Sol: Matrix notation is $\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 3 \end{bmatrix}$

Augmented matrix [A/B] is $[A/B] = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 3 \end{bmatrix}$

$R_2 \rightarrow R_2 - 3R_1$ $R_3 \rightarrow R_3 - 2R_1$ $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 23 \end{bmatrix}$

$R_2 \rightarrow 1/10\ R_2$ $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 11 & 22 & 23 \end{bmatrix}$

$R_3 \rightarrow R_3 - 11R_2$ $\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$

This is the Echelon form of [A/B] $\therefore \rho(A) = 2$, $\rho(A/B) = 3$

$\rho(A) \neq \rho[A/B]$.

The given system is inconsistent.

3). Find whether the following equations are consistent, if so solve them. x+y+2z=4, 2x-y+3z=9, 3x-y-z=2

Sol: We write the given equations in the form AX=B i.e; $\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \to R_2-2R_1$ and $R_3 \to R_3-3R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{bmatrix}$

Applying $R_3 \to 3R_3-4R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -17 & -34 \end{bmatrix}$

this matrix is in Echelon form. $\rho(A) = 3$ and $\rho(A/B) = 3$

Since $\rho(A) = \rho[A/B]$. ∴ The system of equations is consistent.

Here the number of unknowns is 3

Since $\rho(A) = \rho[A/B]$ = number of unknowns

∴ The system of equations has a unique solution

We have $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$

⇨ $-17z = -34$ ⇨ $z = 2$

$-3y-z = 1$ ⇨ $-3y = z+1$ ⇨ $-3y = 3$ ⇨ $y = -1$

and $x+y+2z = 4$ ⇨ $x = 4-y-2z = 4+1-4 = 1$

∴ $x=1, y=-1, z=2$ is the solution.

4). Show that the equations x+y+z=6, x+2y+3z=14, x+4y+7z=30 are consistent and solve them.

Sol: We write the given equations in the form AX=B i.e. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

The Augmented matrix $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

Applying $R_2 \to R_2-R_1$ and $R_3 \to R_3-R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$

Applying $R_3 \to R_3-3R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This matrix is in Echelon form. $\rho(A) = 2$ and $\rho(A/B) = 2$

Since $\rho(A) = \rho[A/B]$.

The system of equations is consistent. Here the no. of unknowns are 3

Since rank of A is less than the no. of unknowns, the system of equations will have infinite number of solutions in terms of n-r=3-2=1 arbitrary constant.

The given system of equations reduced form is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$

⇨x+y+z=6……… (1), y+2z=8…….(2)

Let z=k, put z=k in (2) we get y=8-2k

Put z=k y=8-2k in (1), we get

x=6-y-z=6-8+2k=-2+k

∴ x=-2+k, y=8-2k, z=k is the solution, where k is an arbitrary constant.

5). Show that $x+2y-z=3$ **;** $3x-y+2z=1$**;** $2x-2y+3z=2; x-y+z=-1$ **are consistent and solve them**

Sol: The above system in matrix notation is $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}_{4\times 3}\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}_{4\times 1}$

$A \quad X \quad = B$

The Augmented matrix is $[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$$\begin{array}{l} R_2 \to R_2 \to 3R_1 \\ R_3 \to R_3 \to 2R_1 \\ R_4 \to R_4 \to R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -0 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$R_2 \to R_2 - R_3 \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$\begin{array}{l} R_3 \to R_3 \to 3R_1 \\ R_4 \to R_4 \to 3R_2 \end{array} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & +20 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{5} \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$$R_4 \to R_4 - 2R_3 \quad \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ρ (A) = 3 = ρ (A/B

∴ρ (A) =ρ (A/B) = No. of unknowns = 3

∴ The given system has unique solution.

The systems of equations equivalent to given system are

$x + 2y - z = 3 \qquad -y = -4; z = 4$

$x + 8 - 4 = 3 \qquad y = 4; z = 4$

$x = 3 - 4 = -1$

$\therefore x = -1, y = 4, z = 4.$

6). Solve $x + y + z = 3; 3x - 5y + 2z = 8; 5x - 3y + 4z = 14$

Sol: - $\begin{bmatrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 14 \end{bmatrix}$

Augmented Matrix is $\quad [AB] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -5 & 2 & 8 \\ 5 & -3 & 4 & 14 \end{bmatrix}$

$$\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{array} \quad \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & -8 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \quad \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho[A] = \rho[AB] = 2 <$ Number of unknowns (3)

∴ The system has infinite number of solutions.

$x + y + z = -3, -8y - z = -1 \quad \Rightarrow 8y + z - 1$

$$Let\ z = k \quad \Rightarrow y = \frac{1-k}{8} \quad and \quad x = 3 - \frac{(1-k)}{8} - k = \frac{24 - 1 + k - 8k}{8} = \frac{23 - 7k}{8}$$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{23}{8} - \frac{7}{8}k \\ \frac{1}{8} - \frac{k}{8} \\ 0 + k \end{bmatrix} \quad \Rightarrow \quad X = \begin{bmatrix} \frac{23}{8} \\ \frac{1}{8} \\ 1 \end{bmatrix} \qquad \text{where k is any real number.}$$

7). Find whether the following system of equations is consistent. If so solve them.

$x + 2y + 2z = 2, \quad 3x - 2y + z = 5, \quad 2x - 5y + 3z = -4, \quad x + 4y + 6z = 0.$

Sol: In Matrix form it is $\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$

$AX = B$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -5 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3 + 9R_2$
$R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 12 & -16 \end{bmatrix}$$

$R_4 \to \frac{1}{4} R_4$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 3 & -4 \end{bmatrix}$$

$R_4 \to 37R_4 + 3R_3$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 0 & 17 \end{bmatrix}$$ is in echelon form

$\rho[A] = 3 \text{ and } \rho[AB] = 4 \Rightarrow \rho[A] \neq \rho[AB]$. $\therefore$The given system is in consistent.

8). Discuss for what values of λ, μ the simultaneous equations x+y+z = 6, x+2y+3z =10, x+2y+λz = μ have

(i). No solution

(ii). A unique solution

(iii). An infinite number of solutions.

Sol: The matrix form of given system of Equations is A X = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$ = B

The augmented matrix is $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1,\ R_3 \rightarrow R_3 - R_1$ $\quad [A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{bmatrix}$

Case (i): let $\lambda \neq 3$ the rank of A = 3 and rank [A/B] = 3

Here the no. of unknowns is '3' $\quad \therefore \rho(A) = \rho(A/B) =$ No. of unknowns

The system has unique solution if $\lambda \neq 3$ and for any value of 'μ'.

Case (ii): Suppose $\lambda = 3$ and $\mu \neq 10$.

We have $\rho(A) = 2$, $\rho(A/B) = 3$

The system has no solution.

Case (iii): Let $\lambda = 3$ and $\mu = 10$.

We have $\rho(A) = 2$ $\rho(A/B) = 2$

Here $\rho(A) = \rho(A/B) \neq$ No. of unknowns =3

The system has infinitely many solutions.

9). Find the values of a and b for which the equations x+y+z=3; x+2y+2z=6;x+ay+3z=b have (i) No solution

(ii) A unique solution

(iii) Infinite no of solutions.

Sol: The above system in matrix notation is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$

Augmented matrix $\quad [AB] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{bmatrix}$

$R_2 \rightarrow R_2 - R_1$
$R_3 \rightarrow R_3 - R_1$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\sim\begin{bmatrix}1 & 1 & 1 & 3\\0 & 1 & 1 & 3\\0 & a-3 & 0 & b-9\end{bmatrix}$

• For $a = 3 \,\&\, b = 9$ $\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\0 & 1 & 1 & 3\\0 & 0 & 0 & 0\end{array}\right]$

$\therefore \rho[A] = \rho[AB] = 2 < 3 \Rightarrow$ It has infinite no of solutions.

• *For* $a \neq 3 \,\&\, b = any\ value$ $\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\0 & 1 & 1 & 3\\0 & a-3 & 0 & b-9\end{array}\right]$

$\therefore \rho[A] = \rho[AB] = 3$ $\Rightarrow$ It has a unique solution.

• *For* $a = 3 \,\&\, b \neq 9$ $\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\0 & 1 & 1 & 3\\0 & 0 & 0 & b-9\end{array}\right]$

$\therefore \rho[A] = 2 \neq \rho[AB] = 3$ $\Rightarrow$ Inconsistent $\Rightarrow$ no solution

10). Solve the following system completely. $x+y+z=1; x+2y+4x=\alpha; x+4y+10z=\alpha^2$

Sol: The above system in matrix notation is $\begin{bmatrix}1 & 1 & 1\\1 & 2 & 4\\1 & 4 & 10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\\alpha\\\alpha^2\end{bmatrix}$

$A \quad X \quad = \quad B$

Augmented Matrix is $[AB] = \begin{bmatrix}1 & 1 & 1 & 1\\1 & 2 & 4 & \alpha\\1 & 4 & 10 & \alpha^2\end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$ $\sim\begin{bmatrix}1 & 1 & 1 & 1\\0 & 1 & 3 & \alpha-1\\0 & 3 & 9 & \alpha^2-1\end{bmatrix}$

$R_3 \to R_3 - 3R_2$ $\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\0 & 1 & 3 & \alpha-1\\0 & 0 & 0 & \alpha^2-3\alpha+2\end{array}\right]$

Here $\rho[A] = 2$ *and* $\rho[AB] = 3$ $\Rightarrow$ The given system of equations is consistent if

$\alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha^2 - 2\alpha - \alpha + 2 = 0 \Rightarrow (\alpha-2)(\alpha-1) = 0 \Rightarrow \alpha = 2, \alpha = 1$

Case (i): When $\alpha = 1$

$\rho[A]=\rho[AB]=2<$ Number of unknowns.

$\therefore$ The system has infinite number of solutions.

The equivalent matrix is $\begin{bmatrix}1&1&1&1\\0&1&3&0\\0&0&0&0\end{bmatrix}$

The equivalent systems of equations are $x+y+z=1; y+3z=0$

$\Rightarrow Let\ z=k \quad \Rightarrow y=-3k \quad and \quad x+(-3k)+K=1 \Rightarrow x-2k=1 \Rightarrow x=1+2k$

$$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1+2k\\0-3k\\0+k\end{bmatrix}\Rightarrow X=\begin{bmatrix}1\\0\\0\end{bmatrix}+k\begin{bmatrix}2\\-3\\1\end{bmatrix}$$ where k is any arbitrary constant.

Case (ii): When $\alpha=2$

$\rho[A]=\rho[AB]=2<$ no. of unknowns.

$\therefore$ The system has infinite number of solutions.

The equivalent matrix is $\sim\begin{bmatrix}1&1&1&1\\0&1&3&1\\0&0&0&0\end{bmatrix}$

The system of equations equivalent to the given system is x + y + z = 1; y + 3z = 1

$Let\ z=k \quad \Rightarrow y=1-3k\ and\ x+(1-3K)+k=1 \Rightarrow x=2k$

$$\Rightarrow X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0+2k\\1-3k\\0+k\end{bmatrix}\Leftrightarrow X=\begin{bmatrix}0\\1\\0\end{bmatrix}+k\begin{bmatrix}2\\-3\\1\end{bmatrix}$$ where k is any arbitrary constant.

11). Show that the equations $3x+4y+5z=a; 4x+5y+6z=b; 5x+6y+7z=c$ **don't have a solution unless** $a+c=2b.$ **solve equations when a=b=c= -1.**

Sol: The Matrix notation is $\begin{bmatrix}3&4&5\\4&5&6\\5&6&7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$

$A \quad X \quad = \quad B$

Augment Matrix is $[AB]=\begin{bmatrix}3&4&5&a\\4&5&6&b\\5&6&7&c\end{bmatrix}$

$R_2\to 3R_2-4R_1$
$R_3\to 3R_3-5R_1$
$\sim\begin{bmatrix}3&4&5&a\\0&-1&-2&3b-4a\\0&-2&-4&3c-5a\end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\qquad \sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3a-6b+3c \end{bmatrix}$

Here $\rho[A]=2 \; and \; \rho[AB]=3$

$\therefore$ The given system of equations is consistent if $3a-6b+3c=0 \Rightarrow 3a+3c=6b \Rightarrow a+c=2b$

Thus the equations don't have a solution unless $a+c=2b, when \; a=b=c=-1$

The equivalent matrix is $\begin{bmatrix} 3 & 4 & 5 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho[A]=\rho[(AB)]=2<$ No. of unknowns.

$\therefore$ The system has infinite number of solutions. The system of equations equivalent to the given system $3x+4y+5z=-1; -y-2z=1 \Rightarrow y+2z=-1$

$Let \; z=k \Rightarrow y=-1-2k \; and \; 3x-4-8k+5k=-1 \Rightarrow x=1+k$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1+k \\ -1-2k \\ 0+k \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Linearly dependent set of vectors: A set $\{x_1, x_2, \text{--------------} x_r\}$ of r vectors is said to be a linearly dependent set, if there exist r scalars $k_1, k_2 \text{---} k_r$ not all zero, such that $k_1x_1+k_2x_2+\text{--------}+k_rx_r = 0$

Linearly independent set of vectors: A set $\{x_1, x_2, \text{-------------} x_r\}$ of r vectors is said to be a linearly independent set, if $k_1x_1+k_2x_2+\text{--------------}+k_rx_r=0$ then $k_1=0, k_2=0 \text{------} k_r=0$.

Linear combination of vectors:

A vector x which can be expressed in the form $x = k_1x_1+k_2x_2+\text{--------------}+k_nx_n$ is said to be a linear combination of $x_1, x_2 \text{------} x_n$ here $k_1, k_2 \text{-----------} k_n$ are any scalars.

Linear dependence and independence of Vectors:

Solved Problems :

1). Show that the vectors (1, 2, 3), (3,-2, 1), (1,-6,-5) from a linearly dependent set.

Sol: The Given Vector $X_1=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2=\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3=\begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$

The Vectors X_1, X_2, X_3 from a square matrix.

$$Let\ A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \quad Then\ |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{vmatrix}$$

= 1(10+6)-2(15-1) + 3(-18+2)

= 16+32-48 = 0

The given vectors are linearly dependent $\because |A| = 0$

2). Show that the Vector X_1=(2,2,1), X_2=(1,4,-1) and X_3=(4,6,-3) are linearly dependent.

Sol: Given Vectors X_1=(2,-2,1) X_2=(1,4,-1) and X_3=(4,6,-3) The Vectors X_1, X_2, X_3 form a square matrix.

$$Let\ A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix} \quad Then\ |A| = \begin{vmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{vmatrix}$$

= 2(-12+6) + 2(-3+4)+1(6-16) = -20$\neq$ 0

$\therefore$ The given vectors are linearly dependent $\because |A| \neq 0$

Consistency of system of Homogeneous linear equations:

A system of m homogeneous linear equations in n unknowns, namely

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = 0 \\ -------------------- \\ -------------------- \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mn}x_n = 0 \end{matrix}\right\} ------(1)$$

i.e. $$\begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots.a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots.a_{2n} \\ a_{m1} & a_{m2} & a_{m3}\ldots.a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ .. \\ .0 \end{bmatrix}$$ Here A is called Co –efficient matrix.

Note: 1. Here $x_1 = x_2$ = --------- x_n = 0 is called trivial solution or zero solution of AX = 0

2. A zero solution always linearly dependent.

Theorem: The number of linearly independent solutions of the linear system AX = 0 is (n-r), r being the rank of the matrix A and n being the number of variables.

Note: 1.if A is a non-singular matrix then the linear system AX = 0 has only the zero solution.

2. The system AX =0 possesses a non-zero solution if and only if A is a singular matrix.

Working rule for finding the solutions of the equation AX = 0

(i). Rank of A = No. of unknowns i.e. r = n

∴ The given system has zero solution.

(ii). Rank of A < No of unknowns (r<n) and No. of equations < No. of unknowns (m<n) then the system has **infinite no. of solutions**.

Note: If AX =0 has more unknowns than equations the system always has infinite solutions.

Solved Problems :

1). Solve the system of equations x+3y-2z = 0, 2x-y+4z = 0, x-11y+14z = 0

Sol: We write the given system is AX = 0 i.e. $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & -4 \\ 1 & -11 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - R_1$ $\quad A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\quad A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

The Rank of the A = 2 i.e. ρ (A) = 2 < No. of unknowns = 3

We have infinite No. of solution

Above matrix can we write as x+3y-2z = 0 -7y+8z = 0, 0 = 0

Let z = k then y=8/7k & x= -10/7 k

Giving different values to k, we get infinite no. of values of x,y,z.

2). Find all the non-trivial solution $2x - y + 3z = 0; 3x + 2y + z = 0; x - 4y + 5z = 0.$

Sol: In Matrix form it is $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$AX = 0$

$$\text{The Augumented matrix}[A/O] = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \qquad \sim \begin{bmatrix} 1 & -4 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix}$$

$$R_3 \to 2R_3 - R_2 \qquad \sim \begin{bmatrix} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{it is echelon form.}$$

The Rank of the A = 2 i.e. ρ(A) = 2 < No. of unknowns = 3

Hence the system has non trivial solutions. From echelon form, reduced equations are

$x - 4y + 5z = 0$ *and* $14y - 14z = 0$

Let $z = k$ *then* $y = k$ *and* $x - 4k + 5k = 0 \Rightarrow x = -k.$

Thus, the solution set is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \forall K.$

3). Show that the only real number λ for which the system x+2y+3z = λx, 3x+y+2z= λy, 2x+3y+z = λz, has non-zero solution is 6 and solve them.

Sol: Above system can we expressed as AX = 0 i.e. $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Given system of equations possess a non –zero solution i.e. ρ (A) < no. of unknowns.

⇨ For this we must have det A = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$R_1 \to R_1 + R_2 + R_3 \quad \Rightarrow \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1 \Rightarrow (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$

⇨ $(6-\lambda)[(-2-\lambda)(-1-\lambda)+1] = 0$

⇨ $(6-\lambda)(\lambda^2+3\lambda+3) = 0$

⇨ $\lambda = 6$ only real values.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2+3R_1, R_3 \to 5R_3+2R_1$ $\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3+R_2$ $\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⇨ $-5x+2y+3z = 0$ and $-19y+19z = 0$ ⇨ $y = z$

Let $z = k$ ⇨ $y = k$ and $x = k$.

∴ The solution is $x = y = z = k$.

Eigen Values and Eigen vectors:

Let $A = \left[a_{ij}\right]_{nxn}$ be a square Matrix. Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself i.e. $AX = Y = \lambda X$, Then the unknown scalar λ is known as an "Eigen value" of the Matrix A and the corresponding non-zero vector X is known as "Eigen Vector" of A. Corresponding to Eigen value λ. Thus the Eigen values (or) characteristic values (or) proper values (or) latent roots are scalars λ which satisfy the equation.

$AX = \lambda X$ for $X \neq 0$, $AX - \lambda IX = 0 \Rightarrow (A-\lambda I)X = 0$

Which represents a system of 'n' homogeneous equations in 'n' variables x_1, x_2, ----, x_n this system of equations has non-trivial solutions If the coefficient matrix ($A-\lambda I$) is singular i.e.

$$|A-\lambda I| = 0 \quad \Rightarrow \begin{vmatrix} a_{11-\lambda} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22-\lambda} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn-\lambda} \end{vmatrix} = 0$$

Expansion of the determinant is $(-1)^n\lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + ----+K_n$ is the n^{th} degree of a polynomial $P_n(\lambda)$ which is known as **"Characteristic Polynomial"**. Of A

$(-1)^n \lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + ----- + K_n = 0$ is known as **"Characteristic Equation"**. Thus the Eigen values of a square Matrix A are the roots of the characteristic equation.

Eg: Let A= $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

AX = $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.X$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

Eigen Value: The roots of the characteristic equation are called Eigen values or characteristic roots or latent roots or proper values.

Eigen Vector: Let A=$\left[a_{ij}\right]_{nxn}$ be a Matrix of order n. A non-zero vector X is said to be a characteristic vector (or) Eigen vector of A if there exists a scalar λ such that AX = λ X.

Method of finding the Eigen vectors of a matrix.

Let A=[a_{ij}] be a nxn matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition AX = λX.

⇒ AX = λIX

⇒ AX –λIX = 0

⇒ (A-λI)X = 0 ------- (1)

This is a homogeneous system of n equations in n unknowns.

Will have a non-zero solution X if and only if |A-λI| = 0

- A-λI is called characteristic matrix of A
- |A-λI| is a polynomial in λ of degree n and is called the characteristic polynomial of A
- |A-λI|=0 is called the characteristic equation
- Solving characteristic equation of A, we get the roots , $\lambda_1, \lambda_2, \lambda_3, \dots \dots . \lambda_n,$ These are called the characteristic roots or eigen values of the matrix.
- Corresponding to each one of these n eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors

Let A = $\begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} \\ a_{21} & a_{22} \dots . . & a_{2n} \\ \dots \dots . & \dots \dots . & \dots \dots . \\ a_{n1} & a_{n2} \dots \dots . & a_{nn} \end{bmatrix}$ *be a given matrix*

$Characteristic\ matrix\ of\ A\ is\ A - \lambda I$

$$i.e., A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A-\lambda I|$

$$say\,\phi(\lambda) = |A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

The characteristic equation is $|A-\lambda I| = 0$ we solve the $\emptyset(\lambda) = |A-\lambda I| = 0$, we get n roots, these are called eigen values or latent values or proper values.

Let each one of these eigen values say λ their eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$ and determining the non-trivial solution.

Solved Problems

1. **Find the eigen values and the corresponding eigen vectors of** $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol: $Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic matrix = $[A-\lambda I] = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(2-\lambda) + 8 = 0$$

$$\Rightarrow 16 + \lambda^2 - 10\lambda + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\,\lambda + 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = 6, 4 \text{ are eigen values of A}$$

Consider the system $\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

Eigen vector corresponding to $\lambda = 4$

Put $\lambda = 4$ in the above system, we get $\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Longrightarrow\ 4x_1 - 4x_2 = 0 - - - (1) \qquad 2x_1 - 2x_2 = 0 - - - (2)$$

$from\ (1) and\ (2) we\ have\ x_1 = x_2$

Let $x_1 = \alpha$

$$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $is\ a\ Eigen\ vector\ of\ matrix\ A, corresponding\ eigen\ value\ \lambda = 4$

Eigen Vector corresponding to $\boldsymbol{\lambda = 6}$

$put\ \lambda = 6$ in the above system, we get $\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Longrightarrow\ 2x_1 - 4x_2 = 0 - - - (1) \qquad 2x_1 - 4x_2 = 0 - - - (2)$$

from (1) and (2) we have $x_1 = 2x_2$

Let $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $is\ eigen\ vector\ of\ matrix\ A\ corresponding\ eigen\ value\ \lambda = 6$

2. **Find the eigen values and the corresponding eigen vectors of matrix** $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I|=0$

i.e. $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Longrightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Longrightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Longrightarrow\ \lambda - 2\,[-(\lambda-2)^2 - 1] = 0$$

$$\Longrightarrow\ \lambda - 2\ [-\lambda^2 + 4\lambda - 3] = 0$$

$$\Longrightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

⇨ $\lambda=1, 2, 3$

The eigen values of A is 1, 2, 3.

$For\ finding\ eigen\ vector\ the\ system\ is\ (A - \lambda I)X = 0$

$$\Longrightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 1}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_3 = 0$

$x_2 = 0$

$x_1 + x_3 = 0$

$x_1 = -x_3, x_2 = 0$

$Let\ x_3 = \alpha$

$\Rightarrow x_1 = -\alpha \quad x_2 = 0, \quad x_3 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ *is Eigen vector*

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 2}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here $x_1 = 0$ *and* $x_3 = 0$ *and we can take any arbitary value* x_2 *i.e* $x_2 = \alpha$ *(say)*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boldsymbol{Eigen\ vector\ corresponding\ to\ \lambda = 3}$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$

$-x_2 = 0$

$x_1 - x_3 = 0$

here by solving we get $x_1 = x_3, x_2 = 0$ *say* $x_3 = \propto$

$x_1 = \propto \ , \quad x_2 = 0 \quad , x_3 = \propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$Eigen\ vector\ is \begin{bmatrix}1\\0\\1\end{bmatrix}$

3. Find the Eigen values and Eigen vectors of the matrix is $\begin{bmatrix}3 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$

Sol: Let $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$

Consider characteristic equation is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}8-\lambda & -6 & 2\\ -6 & 7-\lambda & -4\\ 2 & -4 & 3-\lambda\end{vmatrix}=0$

$$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda)-(16)\right]+6\left[(-6)(3-\lambda)+8\right]+2\left[24-2(7-\lambda)\right]=0$$
$$\Rightarrow (8-\lambda)\left[21-7\lambda-3\lambda+\lambda^2-16\right]+6[-18+6\lambda+8]+2[24-14+2\lambda]=0$$
$$\Rightarrow (8-\lambda)\left[\lambda^2-10\lambda-5\right]+6[6\lambda-10]+2[10+2\lambda]=0$$
$$\Rightarrow 8\lambda^2-80\lambda-40-\lambda^3+10\lambda^3+5\lambda+36\lambda-60+20+4\lambda=0$$
$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$$
$$\Rightarrow \lambda\left[-\lambda^2+18\lambda-45\right]=0$$
$$\Rightarrow \lambda=0 \quad (OR) \quad -\lambda^2+18\lambda-45=0$$
$$\Rightarrow \lambda=0, \quad \lambda=3, \quad \lambda=15$$

Eigen Values $\lambda = 0, 3, 15$

Case (i): If $\lambda = 0$

$$\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}X=0$$

$$\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow 8x_1-6x_2+2x_3=0----(1)$$
$$-6x_1+7x_2-4x_3=0----(2)$$
$$2x_1-4x_2+3x_3=0----(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{array}$$

$$\Rightarrow \frac{x_1}{21-16} = \frac{-x_2}{-18+8} = \frac{x_3}{24-14} = k$$

$$\Rightarrow \frac{x_1}{5} = \frac{-x_2}{-10} = \frac{x_3}{10} = k$$

$$\Rightarrow x_1 = k, \quad x_2 = 2k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (ii): If $\lambda = 3$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0 ------(1)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 ------(2)$$

$$2x_1 - 4x_2 + 0 = 0 --------(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{array}$$

$$\Rightarrow \frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8} = k$$

$$\Rightarrow \frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16} = k$$

$$\Rightarrow \frac{x_1}{-2} = \frac{-x_2}{1} = \frac{x_3}{2} = k$$

$$\Rightarrow \frac{x_1}{-2} = k, \quad -x_2 = k, \quad x_3 = 2k$$

$$\Rightarrow x_1 = -2k, \quad x_2 = -k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix} = k\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Case (iii): If $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + (-6x_2) + 2x_3 = 0 ----(1)$$
$$-6x_1 - 8x_2 - 4x_3 = 0 ------(2)$$
$$2x_1 - 4x_2 - 12x_3 = 0 ------(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{array}$$

$$\Rightarrow \frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16} = k$$

$$\Rightarrow \frac{x_1}{80} = \frac{-x_2}{80} = \frac{x_3}{40} = k$$

$$\Rightarrow \frac{x_1}{2} = k, \quad \frac{x_2}{2} = k, \quad \frac{x_3}{1} = k$$

$$\Rightarrow x_1 = 2k, \quad x_2 = 2k, \quad x_3 = k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} k$

4. Find the Eigen values and the corresponding Eigen vectors of the matrix.

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & -0 \end{bmatrix}$$

Sol: Let $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & -0\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0$

$$\Rightarrow(-2-\lambda)\left[-\lambda(1-\lambda)-12\right]-2[-2\lambda-6]-3\left[2(-2)+(1-\lambda)\right]=0$$

$$\Rightarrow\lambda^3+\lambda^2-21\lambda-45=0$$
$$\Rightarrow(\lambda+3)(\lambda+3)(\lambda-5)=0$$
$$\Rightarrow\lambda=-3,-3,5$$

The Eigen values are -3,-3, and 5

Case (i): If $\lambda=-3$

We get $\begin{bmatrix}-2+3 & 2 & -3\\ 2 & 1+3 & -6\\ -1 & -2 & 0+3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

The augment matrix of the system is $\left[\begin{array}{ccc|c}1 & 2 & -3 & 0\\ 2 & 4 & -6 & 0\\ -1 & -2 & 3 & 0\end{array}\right]$

Performing R_2-2R_1, R_3+R_1, we get $\left[\begin{array}{ccc|c}1 & 2 & -3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]$

Hence we have $x_1+2x_2-3xy=0\Rightarrow x_1=-2x_2+3x_3$

Thus taking $x_2=k_1$ *and* $x_3=k_2$, we get $x_1=-2k_1+3k_2; x_2=k_1; x_3=k_2$

Hence $\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=k_1\begin{bmatrix}-2\\1\\0\end{bmatrix}+k_2\begin{bmatrix}3\\0\\1\end{bmatrix}$

So $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$ are the Eigen vectors corresponding to $\lambda=-3$

Case (ii): If $\lambda=5$

We get $\begin{pmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0 ----(1)$$
$$2x_1 - 4x_2 - 6x_3 = 0 ----(2)$$
$$-x_1 - 2x_2 - 5x_3 = 0 ----(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{array}$$

$$\Rightarrow \frac{x_1}{20-12} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-4} = k_3$$

$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{-16} = \frac{-x_3}{-8} = k_3$$

$$\Rightarrow \frac{x_1}{1} = \frac{-x_2}{-2} = \frac{-x_3}{-1} = k_3$$

$$Eigen\ vector\ is \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} k_3$$

5. Find the Eigen values and Eigen vectors of the matrix A and it's inverse where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of "A" is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$
$$\Rightarrow \lambda = 1,2,3 \quad i.e.\ Eigen\,Values\ are\ 1,2,3$$

Note: In upper Δ^{le} (or) Lower Δ^{lar} of a square matrix the Eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Case (i): If $\lambda = 1$

$$\therefore (A-\lambda I)x=0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2+4x_3=0;\ x_2+5x_3=0;\ 2x_3=0 \quad \Rightarrow x_1=k_1;\ x_2=0;\ x_3=0$$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}k_1$$

Case (ii): If $\lambda = 2$

$$\Rightarrow \begin{bmatrix} +1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1+3x_2+4x_3=0;\ 5x_2=0;\ x_3=0$$

$$\Rightarrow -x_1+3k+4(0)=0 \Rightarrow -x_1+3k=0 \Rightarrow x_1=3k$$

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix}=k\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 3x_2 + 4x_3 = 0; -x_2 + 5x_3 = 0; x_3 = 0$$

$$Let \quad x_3 = k$$

$$\Rightarrow -x_2 + 5x_3 = 0 \Rightarrow x_2 = 5k$$

$$and -2x_1 + 3x_2 + 4k = 0 \Rightarrow -2x_1 + 15k + 4k = 0$$

$$\Rightarrow -2x_1 + 19x = 0 \Rightarrow x_1 = \frac{19}{2}k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix} = k \begin{bmatrix} \frac{19}{2} \\ 5 \\ 1 \end{bmatrix}$$

Note: Eigen Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3} i.e 1, \frac{1}{2}, \frac{1}{3}$ and the Eigen vectors of A^{-1} are same as Eigen vectors of the matrix A

6. Determine the Eigen values and Eigen vectors of

$$B = 2A^2 - \frac{1}{2}A + 3I \text{ where } A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

Sol : $-$ *Given that* $B = 2A - \frac{1}{2}A + 3 \quad \Rightarrow A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$$\text{we have } A^2 = A.A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix}$$

$$B = 2A^2 - \frac{1}{2}A + 3I$$

$$= 2\begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$$

Characteristic equation of B is $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$$

i.e. $\Rightarrow \lambda^2 + 105\lambda - 2376 = 0$

$$\Rightarrow (\lambda-33)(\lambda-72) = 0$$

$$\Rightarrow \lambda = 33 \quad or \quad 72$$

Eigen Values of B are 33 and 72.

Case (i): If $\lambda = 33$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Case (ii): If $\lambda = 72$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 111-72 & -78 \\ 39 & -6-72 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Properties of Eigen Values:

Theorem 1: The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

Proof: Characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$ *expanding this we get*

$(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)-a_{12}(a\ polynomial\ of\ degree\ n-2)+a_{13}\ (a\ polynomial\ of\ degree\ n$ $-2) + \ldots = 0$

$$\Rightarrow (-1)^n\left[\lambda^n-(a_{11}+a_{22}+....+a_{nn})\lambda^{n-1}+a\ polynomial\ \ of\ \deg ree\,(n-2)\right]=0$$

$$(-1)^n\lambda^n+(-1)^{n+1}(Trace\ A)\lambda^{n-1}+a\ polynomial\ of\ \deg ree\,(n-2)\ in\ \lambda=0$$

$If\ \lambda_1, \lambda_2 \ldots..\lambda_n\ are\ the\ roots\ of\ this\ equation$

$sum\ of\ the\ roots = \frac{(-1)^{n+1}Tr(A)}{(-1)^n} = Tr(A)$s

$Further\ |A-\lambda I| = (-1)^n\lambda^n + \cdots . + a_0$

$put\ \lambda = 0\ then\ |A| = a_0$

$(-1)^n\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \ \ldots . + a_0 = 0$

$$Product\ of\ the\ roots = \frac{(-1)^n\, a_0}{(-1)^n} = a_0$$

$but\ a_0 = \ |A| = \det A$

Hence the result

Theorem 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Proof: Since λ is an eigen value of A corresponding to the eigen value X, we have

AX= λX -----------(1)

Pre multiply (1) by A, A(AX) = A(λX)

(AA)X = λ(AX)

$A^2X = \lambda(\lambda X)$

$A^2X = \lambda^2 X$

λ^2 is eigen value of A^2 with X itself as the corresponding eigen vector. Thus the theorm is true for n=2

Let we assume it is true for n = k

i.e. $A^K X = \lambda^K X$------------(2)

Premultiplying (2) by A, we get

$A(A^k X) = A(\lambda^K X)$

$(AA^K)X = \lambda^K(AX) = \lambda^K(\lambda X)$

$A^{K+1}X = \lambda^{K+1}X$

λ^{K+1} is eigen value of A^{K+1} with X itself as the corresponding eigen vector.

Thus, by Mathematical induction. λ^n is an eigen value of A^n.

Theorem 3: A Square matrix A and its transpose A^T have the same eigen values.

Proof: We have $(A-\lambda I)^T = A^T - \lambda I^T$

$= A^T - \lambda I$

$|(A-\lambda I)^T| = |A^T - \lambda I|$ (or)

$|A-\lambda I| = |A^T - \lambda I|$ $\left[\because |A^T| = |A|\right]$

$|A-\lambda I| = 0$ if and only if $|A^T - \lambda I| = 0$

Hence the theorem.

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values.

Proof: Given A is invertble i.e, A^{-1} exist

We know that if A and P are the square matrices of order n such that P is non-singular then A and $P^{-1}AP$ have the same eigen values.

Taking $A=BA^{-1}$ and P=A, we have

BA^{-1} and $A^{-1}(BA^{-1})A$ have the same eigen values

ie.,BA^{-1} and $(A^{-1}B)(A^{-1}A)$ have the same eigen values

ie.,BA^{-1} and $(A^{-1}B)I$ have the same eigen values

ie.,BA^{-1} and $A^{-1}B$ have the same eigen values

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$ are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, \ldots\ldots k\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

Proof: Let A be a square matrix of order n. Then $|KA-\lambda KI| = |K(A-\lambda I)| = K^n|A-\lambda I|$

Since $K\neq 0$, therefore $|KA-\lambda KI| = 0$ if and only if $|A-\lambda I| = 0$

i.e., $K\lambda$ *is an eigen value of* $KA \Leftrightarrow$ *if* λ *is an eigen value of* A

Thus $k\lambda_1, k\lambda_2 \ldots k\lambda_n$ *are the eigen values of the matrix KA if*

$\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of the matrix A

Theorem 6: If λ is an eigen values of the matrix A then $\lambda+k$ is an eigen value of the matrix A+KI

Proof: Let λ be an eigen value of A and X the corresponding eigen vector. Then by definition

$AX = \lambda X$

Now $(A+KI)X$

$= AX + IKX = \lambda X + KX$

$= (\lambda+K)\, X$

$\lambda+k$ is an eigen value of the matrix A+KI.

Theorem 7: If $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigen values of A, then $\lambda_1 - K, \lambda_2 - K, \dots \lambda_n - K$, *are the eigen values of the matrix* $(A - KI)$, *where K is a non – zero scalar*

Proof: Since $\lambda_1, \lambda_2, \dots . \lambda_n$ are the eigen values of A.

The characteristic polynomial of A is

$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$-----------------------1

Thus the characteristic polynomial of A-KI is

$|(A - KI) - \lambda I| = |A - (k+\lambda)I|$

$= [\lambda_1 - (\lambda+K)][\lambda_2 - (\lambda+K)] \dots\dots\dots [\lambda_n - (\lambda+K)]$.

$= [(\lambda_1 - K) - \lambda)][(\lambda_2 - K) - \lambda] \dots\dots\dots [(\lambda_n - K) - \lambda]$.

Which shows that the eigen values of A-KI are $\lambda_1 - K, \lambda_2 - K, \dots \dots . \lambda_n - K$

Theorem 8: If $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A - \lambda I)^2$

Proof: First we will find the eigen values of the matrix A- λI

Since $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigen values of A

The characteristics polynomial is

$|A - \lambda I| = (\lambda_1 - K)(\lambda_2 - K) \dots . (\lambda_n - K) \;\; ------(1)$ *where K is scalar*

The characteristic polynomial of the matrix (A- λI) is

$|A - \lambda I - KI| = |A - (\lambda+K)I|$

$= [\lambda_1 - (\lambda + K)]\,[\lambda_2 - (\lambda + K)] \dots [\lambda_n - (\lambda+K)]$

$= [(\lambda_1 - \lambda) - K)]\,[(\lambda_2 - \lambda) - K] \dots [(\lambda_n - \lambda) - K)]$

Which shows that eigen values of (A- λI) are $\lambda_1 - \lambda, (\lambda_2 - \lambda) \dots . \lambda_n - \lambda$

We know that if the eigen values of A are $\lambda_1, \lambda_2 \dots \lambda_n$ then the eigen values of A^2 are $\lambda_1^2, \lambda_2^2 \dots . \lambda_n^2$ Thus eigen values of $(A - \lambda I)^2$ *are* $(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, \dots . (\lambda_n - \lambda)^2$

Theorem 9: If λ is an eigen value of a non-singular matrix A corresponding to the eigen vector X, then λ^{-1} *is an eigen value of* A^{-1} *and corresponding eigen vector X itself.*

Proof: Since A is non-singular and product of the eigen values is equal to |A|, it follows that none of the eigen values of A is 0.

If λ is an eigen value of the non-singular matrix A and X is the corresponding eigen vector $\lambda \neq 0$ and AX= λX.

premultiplying this with A^{-1}, we get $A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow (A^{-1}A)X = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$

$\therefore X = \lambda A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X \ (\lambda \neq 0)$

Hence λ^{-1} is an eigen value of A^{-1}

Theorem 10: If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix AdjA.

Proof: Since λ is an eigen value of a non-singular matrix, therefore $\lambda \neq 0$

Also λ is an eigen value of A implies that there exists a non-zero vector X such that

AX = λX -----(1)

$\Rightarrow (adj\ A)AX = (Adj\ A)(\lambda X)$

$\Rightarrow [(adj\ A)A]X = \lambda(\mathrm{adj\ A})X$

$\Rightarrow |A|IX = \lambda\ (\mathrm{adj\ A})X\ \left[\because (adjA)A = |A|I\right]$

$\Rightarrow \frac{|A|}{\lambda}X = (adj\ A)\,X\ or\ (adj\ A)\,X = \frac{|A|}{\lambda}X$

Since X is a non – zero vector, therefore the relation (1)

it is clear that $\frac{|A|}{\lambda}$ is an eigen value of the matrix Adj A

Theorem 11: If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an Eigen value A

Proof: We know that if λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}

Since A is an orthogonal matrix, therefore $A^{-1} = A^{1}$

$\therefore\ \frac{1}{\lambda}$ is an eigen value of A^{1}

But the matrices A and A^{1} have the same eigen values, since the determinants |A- λI| and $|A^{1}$ - λI| are same.

Hence $\frac{1}{\lambda}$ is also an eigen value of A.

Theorem 12: If λ is eigen value of A then prove that the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Proof: If X be the eigen vector corresponding to the eigen valueλ, then AX = λX --- (1)

Premultiplying by A on both sides

$\Rightarrow A(AX) = A(\lambda X)$

$\Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) = \lambda^2X$

This shows that λ^2 is an eigen value of A^2

We have $B = a_0A^2+a_1A+a_2I$

$\therefore BX = a_0A^2+a_1A+a_2I) X$

$= a_0A^2 X+a_1AX+a_2 X$

$= a_0\lambda^2 X+a_1 \lambda X+a_2X = (a_0\lambda^2 +a_1 \lambda+a_2\ ) X$

$(a_0\lambda^2 +a_1 \lambda+a_2)$ is an eigen value of B and the corresponding eigen vector of B is X.

Theorem 13: Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

Proof: Consider the characteristic equation of $P^{-1}AP$

It is $|(P^{-1}AP)-\lambda I| = | P^{-1}AP-\lambda P^{-1}IP|$ $(\because I = P^{-1}P)$

$= | P^{-1} (A-\lambda I) P| = | P^{-1} | |A-\lambda I| |P|$

$= |A-\lambda I|$ since $|P^{-1} | |P| = 1$

Thus the characteristic polynomials of $P^{-1}AP$ and A are same. Hence the eigen values of $P^{-1}AP$ and A are same.

Corollary 1: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same eigen values.

Corollary 2: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

Theorem 14: The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A = \begin{vmatrix} a_{11} & a_{12}\dots\dots & a_{1n} \\ 0 & a_{22}\dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0\dots\dots & a_{nn} \end{vmatrix}$ be a triangular matrix of order n

The characteristic equation of A is $|A- \lambda I|=0$

i.e., $\begin{vmatrix} a_{11-\lambda} & a_{12}\dots\dots & a_{1n} \\ 0 & a_{22-\lambda}\dots\dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots \\ 0 & 0\dots\dots & a_{nn-\lambda} \end{vmatrix} = 0$

i.e, $(a_{11}- \lambda) (a_{22}- \lambda) \dots\dots (a_{nn}- \lambda)=0$

$\Longrightarrow \lambda = a_{11} , a_{22} ,\dots. a_{nn}$

Hence the eigen values of A are $a_{11} , a_{22} ,\dots. a_{nn}$ which are just the diagonal elements of A.

Note: Similarly we can show that the eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Theorem 15: The eigen values of a real symmetric matrix are always real.

Proof: Let λ be an eigen value of a real symmetric matrix A and Let X be the corresponding eigen vector then AX= $\lambda X - - - - (1)$

Take the conjugate $\bar{A}\bar{X} = \bar{\lambda}\,\bar{X}$

Taking the transpose $\bar{X}^T(\bar{A})^T = \bar{\lambda}\,\bar{X}^T$

Since $\bar{A} = A$ and $A^T = A$, we have $\bar{X}^T\ A = \bar{\lambda}\,\bar{X}^T$

Post multiplying by X, we get $\bar{X}^T\ AX = \bar{\lambda}\,\bar{X}^T X$------- (2)

Premultiplying (1) with $\bar{X}^T$, we get $\overline{X}^T AX = \lambda \overline{X}^T X$ ------ (3)

(1) – (3) gives $(\lambda - \overline{\lambda})\overline{X}^T X = 0$ but $\bar{X}^T X \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda - \overline{\lambda} \Rightarrow \lambda$ is real. Hence the result follows

Theorem 16: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Proof: Let λ_1, λ_2 be eigen values of a symmetric matrix A and let X_1, X_2 be the corresponding eigen vectors.

Let $\lambda_1 \neq \lambda_2$. We want to show that X_1 is orthogonal to X_2 (i.e., $X_1^T X_2 = 0$)

Sice X_1, X_2 are eigen values of A corresponding to the eigen values λ_1, λ_2 we have

$AX_1 = \lambda_1 X_1$ ----- (1) $\quad AX_2 = \lambda_2 X_2$ ------- (2)

Premultiply (1) by X_2^T

$\Longrightarrow\ X_2^T A X_1 = \lambda_1 X_2^T X_1$

Taking transpose to above, we have

$\Rightarrow X_2^T A^T \left(X_2^T\right)^T = \lambda_1 X_1^T \left(X_2^T\right)^T$

$i.e, X_1^T A X_2 = \lambda_1 X_1^T X_2$-------- (3) s

Premultiplying (2) by X_1^T, $we\ get\ X_1^T A X_2 = \lambda_2 X_1^T X_2 - - - - - (4)$

Hence from (3) and (4) we get

$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$

$\Longrightarrow\ X_1^T X_2 = 0$

$(\because \lambda_1 \neq \lambda_2)$

X_1 *is orthogonal to* X_2

Note: If λ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is f(λ) .

Theorem17: The Eigen values of a Hermitian matrix are real.

Proof: Let A be Hermitian matrix. If X be the Eigen vector corresponding to the eigen value λ of A, then AX = λX -------------------- (1)

Pre multiplying both sides of (1) by X^{θ} ,we get

$X^{\theta}AX = \lambda X^{\theta}X$ ------------------------ (2)

Taking conjugate transpose of both sides of (2)

We get $\left(X^{\theta}AX\right)^{\theta} = \left(\lambda X^{\theta}X\right)^{\theta}$

i.e $X^{\theta}A^{\theta}\left(X^{\theta}\right)^{\theta} = \bar{\lambda}X^{\theta}\left(X^{\theta}\right)^{\theta}\left[\because (ABC)^{\theta} = C^{\theta}B^{\theta}A^{\theta}\ and\ (KA)^{\theta} = \bar{K}A^{\theta}\right]$

(or) $X^{\theta}A^{\theta}X = \bar{\lambda}X^{\theta}X\left[\because \left(X^{\theta}\right)^{\theta} = X, \left(A^{\theta}\right)^{\theta} = A\right]$-------------------- (3)

From (2) and (3), we have

$$\lambda X^{\theta}X = \bar{\lambda}X^{\theta}X$$

i.e $\left(\lambda - \bar{\lambda}\right)X^{\theta}X = 0 \Rightarrow \lambda - \bar{\lambda} = 0$

$$\Rightarrow \lambda = \bar{\lambda}\left(\because X^{\theta}X \neq 0\right)$$

$\therefore$ Hence λ is real.

Note: The Eigen values of a real symmetric are all real

Corollary: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

Proof: Let A be the skew-Hermitian matrix

If X be the Eigen vector corresponding to the Eigen value λ of A, then

$AX = \lambda X (or) (iA)X = (i\lambda)X$

From this it follows that $i\lambda$ is an Eigen value of iA

Which is Hermitian (since A is skew-hermitian)

$\therefore A^{\theta} = -A$

Now $(iA)^{\theta} = \bar{i}A^{\theta} = -iA^{\theta} = -i(-A) = iA$

Hence $i\lambda$ is real. Therefore λ must be either

Zero or purely imaginary.

Hence the Eigen values of skew-Hermitian matrix are purely imaginary or zero

Theorem 18: The Eigen values of an unitary matrix have absolute value l.

Proof: Let A be a square unitary matrix whose Eigen value is λ with corresponding eigen vector X

$\Rightarrow \quad AX = \lambda X \rightarrow (1)$

$\Rightarrow \quad \overline{A}\overline{X} = \bar{\lambda}\overline{X} \Rightarrow \overline{X}^{T}\overline{A}^{T} = \bar{\lambda}X^{T} \rightarrow (2)$

Since A is unitary, we have $\left(\overline{A}\right)^T A = I \rightarrow (3)$

(1) and (2) given $\overline{X}^T \overline{A}^T \quad (AX) = \lambda\overline{\lambda}\overline{X}^T X$

i.e $\overline{X}^T X = \lambda\overline{\lambda}\overline{X}^T X$ From (3)

$\Rightarrow \overline{X}^T X\left(1-\lambda\overline{\lambda}\right)=0$

Since $\overline{X}^T X \neq 0$,we must have $1-\lambda\overline{\lambda}=0$

$\Rightarrow \lambda\overline{\lambda}=1$

Since $|\lambda| = \overline{|\lambda|}$

We must have $|\lambda|=1$

Note 1: From the above theorem, we have "The characteristic root of an orthogonal matrix is of unit modulus".

2. The only real eigen values of unitary matrix and orthogonal matrix can be $\pm$ 1

Theorem 19: Prove that transpose of a unitary matrix is unitary.

Proof: Let A be a unitary matrix, then $A.A^\theta = A^\theta.A = I$

where A^θ is the transposed conjugate of A.

$\therefore \left(AA^\theta\right)^T = \left(A^\theta A\right)^T = (I)^T$

$\therefore \left(AA^\theta\right)^T = \left(A^\theta A\right)^T = (I)^T$

$\Rightarrow \left(A^\theta\right)^T A^T = A^T\left(A^\theta\right)^T = I$

$\Rightarrow \left(A^T\right)^\theta A^T = A^T\left(A^T\right)^\theta = I$

Hence A^T is a unitary matrix.

Solved Problems:

1. For the matrix $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 3 & 2\\ 0 & 0 & -2\end{bmatrix}$ find the Eigen values of $3A^3+5A^2-6A+2I$

Sol: The Characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}1-\lambda & 2 & -3\\ 0 & 3-\lambda & 2\\ 0 & 0 & -2-\lambda\end{vmatrix}=0$

$\Rightarrow(1-\lambda)(3-\lambda)(-2-\lambda)=0$

$\therefore$ Eigen values are 1, 3,-2.

If λ is the Eigen value of A. and F (A) is the polynomial in A then the Eigen value of f (A) is f (λ)

Let $f(A)=3A^3+5A^2-6A+2I$

$\therefore$ Eigen Value of f (A) are f (1), f (-2), f (3)

f (1) = 3+5-6+2 = 4

f (-2) = 3(-8)+5(4)-6(-2)+2 = -24+20+12+2 = 10

f (3) = 3(27)+5(9)+6(3)+2 = 81+45-18+2 = 110

The Eigen values of f (a) are f (λ) = 4, 10,110

2. Find the eigen values and eigen vectors of the matrix A and its inverse, where

A = $\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$

Sol: Given A = $\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$\Longrightarrow \begin{vmatrix}1-\lambda & 3 & 4\\ 0 & 2-\lambda & 5\\ 0 & 0 & 3-\lambda\end{vmatrix}=0$

$\Longrightarrow (1-\lambda)[(2-\lambda)(3-\lambda)]=0$

$\Longrightarrow \lambda=1,2,3$

Charecetstic roots are 1, 2, 3.

Case (i): If $\lambda=1$

$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow 3x_2 + 4x_3 = 0$

$x_2 + 5x_3 = 0$

$2x_3 = 0$

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the solution where α is arbitrary constant

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

Case (i): If $\lambda = 2$

$For\ \lambda = 2, becomes \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow -x_1 + 3x_2 + 4x_3 = 0$

$5x_3 = 0 \Rightarrow x_3 = 0$

$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$

$Let\, x_2 = k$

$x_1 = 3k$

$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

is the solution where k is arbitrary constant

$\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

Case (iii): If $\lambda = 3$

$For\ \lambda = 3, becomes \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow\ -2x_1 + 3x_2 + 4x_3 = 0$

$-x_2 + 5x_3 = 0$

$Say\ x_3 = K \implies x_2 = 5K$

$x_1 = \frac{19}{2}K$

X = $\begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the solution, where k/2 is arbitrary constant.

$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Eigen values of A^{-1} are $1, \frac{1}{2}, \frac{1}{3}$.

We know Eigen vectors of A^{-1} are same as eigen vectors of A.

3. Find the eigen values of A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A} = -A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow A^T = \begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$$

$\Rightarrow \lambda = 4i, -2i$ are the Eigen values of A

4.Find the eigen values of $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$Now\ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$ $and\ \left(\overline{A}\right)^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$

We can see that $\overline{A}^T.A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Thus A is a unitary matrix

$\therefore$ The characterstic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} \frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda \end{vmatrix}=0$$

Which gives $\lambda=\frac{\sqrt{3}}{2}+i\frac{1}{2} \, and \, \frac{-\sqrt{3}}{2}+\frac{1}{2}i$ and

Hence above λ values are Eigen values of A.

Cayley-Hamilton Theorem: Every Square Matrix satisfies its own characteristic equation

To find Inverse of matrix: If A is non-singular Matrix, then A^{-1} exists, Pre multiplying (1) above by A^{-1} we have $a_0A^{n-1}+a_1A^{n-2}+---+a_{n-1}I+a_nA^{-1}=0$,

$$A^{-1}=\frac{1}{a_n}\left[a_0A^{n-1}+a_1A^{n-2}+---a_{n-1}I\right]$$

To find the powers of A: - Let K be a +ve integer such that $K\geq n$

Pre multiplying (1) by A^{k-n} we get $a_0A^K+a_1A^{K-1}+---+a_nA^{k-n}=0$,

$$A^K=\frac{-1}{a_0}\left[a_1A^{k-1}+a_2A^{K-1}---+a_nA^{K-n}\right]$$

Solved Problems :

1. S.T the matrix A = $\begin{bmatrix}1 & -2 & 1\\ 1 & -2 & 3\\ 0 & -1 & 2\end{bmatrix}$ **satisfies its characteristic equation and hence find A^{-1}**

Sol: Characteristic equation of A is det (A-λI) = 0

$$\Rightarrow \begin{vmatrix}1-\lambda & -2 & 2\\ 1 & -2-\lambda & 3\\ 0 & -1 & 2-\lambda\end{vmatrix}=0$$

$C_2 \rightarrow C_2+C_3$ $\begin{vmatrix}1-\lambda & 0 & 2\\ 1 & 1-\lambda & 3\\ 0 & 1-\lambda & 2-\lambda\end{vmatrix}=0$

$$(1-\lambda)\begin{vmatrix}1-\lambda & 0 & 2\\ 1 & 1 & 3\\ 0 & 1 & 2-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^3-\lambda^2+\lambda-1=0$$

By Cayley – Hamilton theorem, we have $A^3-A^2+A-I=0$

$$A=\begin{bmatrix}1 & -2 & 2\\ 1 & -2 & 3\\ 0 & -1 & 2\end{bmatrix} \quad A^2=\begin{bmatrix}-1 & 0 & 0\\ -1 & -1 & 2\\ -1 & 0 & 1\end{bmatrix} \quad A^3=\begin{bmatrix}-1 & 2 & -2\\ -2 & 2 & -1\\ -1 & 1 & 0\end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix

A $= \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$ *i.e.,* $\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$…..(1)

Multiply with A^{-1} we get

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \quad A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

multiplying (1) with A,we get,

$A^4-5A^3+7A^2-3A= 0$

$A^4 = 5A^3-7A^2+3A$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -390 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -69 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

3. *If* $A=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$ **Verify Cayley-Hamilton theorem hence find A^{-1}**

Sol: - Given that $A=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

i.e. $\begin{vmatrix}2-\lambda&1&2\\5&3-\lambda&3\\-1&0&-2-\lambda\end{vmatrix}=0$

$\Rightarrow(2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2\right]-1\left[-10-5\lambda+3\right]+2\left[0+(3-\lambda)\right]$

$\Rightarrow(2-\lambda)\left[\lambda^2-\lambda-6\right]-1\left[-5\lambda-7\right]+2\left[3-x\right]=0$

$\Rightarrow 2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda+5\lambda+7+6-2\lambda=0$

$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+1=0$

$\Rightarrow \lambda^3-3\lambda^2-7\lambda-1=0----(1)$

According to Cayley Hamilton theorem. Square matrix 'A' satisfies equation (1)

Substitute A in place of λ

Now $A=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}$

$$A^2=\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}7&5&3\\22&14&13\\0&-1&2\end{bmatrix}\begin{bmatrix}2&1&2\\5&3&3\\-1&0&-2\end{bmatrix}=\begin{bmatrix}36&22&23\\101&64&60\\-7&-3&-7\end{bmatrix}$$

Now $A^3-3A^2-7A-I=0$

$$\Rightarrow\begin{bmatrix}36&22&23\\101&64&60\\-7&3&-7\end{bmatrix}+\begin{bmatrix}21&-15&-9\\-66&-42&-39\\0&3&-6\end{bmatrix}+\begin{bmatrix}14&-7&-14\\-35&-21&-21\\7&0&14\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0$$

Cayley –Hamilton theorem is verified.

To find A^{-1}

$\Rightarrow$ $A^3-3A^2-7A-I=0$

Multiply A^{-1}, we get

$$A^{-1}\left(A^3-3A^2-7A-I\right)=0$$

$$\Rightarrow A^2-3A-7I-A^{-1}=0$$

$$\Rightarrow A^{-1}=A^2-3A-7I$$

$$\therefore A^{-1}=\begin{bmatrix}7 & 5 & 3\\ 22 & 14 & 13\\ 0 & -1 & 2\end{bmatrix}+\begin{bmatrix}-6 & -3 & -6\\ 15 & -9 & 9\\ 3 & 0 & 6\end{bmatrix}+\begin{bmatrix}-7 & 0 & 0\\ 0 & -7 & 0\\ 0 & 0 & -7\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-6 & 2 & -3\\ 7 & -2 & 4\\ 3 & -1 & 1\end{bmatrix}$$

Check $A.A^{-1}=I$

$$A.A^{-1}=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}\begin{bmatrix}-6 & 2 & -3\\ 7 & -2 & 4\\ 3 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

4. Using Cayley – Hamilton theorem, find A^8, if $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

Sol: Given $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$

Characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow(1-\lambda)(-1-\lambda)-4=0$$

$$\Rightarrow \lambda^2-5=0 ------(1)$$

Substitute A in place of λ

$$A^2-5I=0\Rightarrow A^2=5I$$

$$find\ A^8$$

$$\therefore A^8=5A^6=5(A^2)(A^2)(A^2)$$

$$=5(5I)(5I)(5I)$$

$$=625I$$

$$\Rightarrow A^8=625I$$

Diagonalization of a Matrix by similarity transformation:

Similar Matrix: A matrix A is said to be similar to the Matrix B if there Exists a non-singular matrix P such that B= $P^{-1}AP$. This transformation of A to B is known as "Similarity Transformation"

Diagonalization of a Matrix:

Let A be a square Matrix. If there exists a non-singular Matrix P and a diagonal Matrix D such that $P^{-1}AP=D$, then the Matrix A is said to be diagonalizable and D is said to be "Diagonal" form (or) canonical diagonal form of the Matrix A

Modal Matrix: The modal matrix which diagonalizes A is called the modal Matrix of A and is obtained by grouping the Eigen vectors of A into a Square Matrix.

Spectral Matrix: The resulting diagonal Matrix D is known as Spectral Matrix.

In this spectral Matrix D whose principal diagonal elements are the Eigen values of the Matrix.

Calculation of powers of a matrix:

We can obtain the power of a matrx by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2= (P^{-1}AP)\,(P^{-1}AP)$

$= P^{-1}A\,(PP^{-1})\,AP$

$= P^{-1}A^2P$ (since $PP^{-1}=I$)

Simlarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$……..(1)

To obtain A^n, Premultiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})A^n\,(PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$

Hence $A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix} P^{-1}$

Diagonalization of a matrix:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors $(X_1,X_2\ldots X_n)$ corresponding to the n eigen values $\lambda_1,\lambda_2\ldots.\lambda_n$ respectively then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Note: 1. If $X_1,X_2\ldots X_n$ are not linearly independent this result is not true.

2. Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1,\lambda_2\cdots\lambda_n$ then the corresponding eigen vectors $X_1,X_2\ldots X_n$ are pairwise orthogonal.

Hence if P = ($e_1, e_2 \ldots e_n$)

Where $e_1 = (X_1 / \|X_1\|)$, $e_2 = (X_2 / \|X_2\|) \ldots e_n = (X_n)/\|X_n\|$ then P will be an orthogonal matrix.

i.e, $P^TP = PP^T = I$

Hence $P^{-1} = P^T$

$\therefore P^{-1}AP = D$

Solved Problems :

1. Determine the modal matrix P of A= $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$**. Verify that $P^{-1}AP$ is a diagonal matrix.**

Sol: The characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

When $\lambda=5 \implies \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given eigen value $\lambda = -3$ we can have two linearly independent eigen vectors

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $and\ X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$P = [X_1\ X_2\ X_3]$

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ = modal matrix of A

Now det P=1(-1)-2(2)+3(0-1) = -8

$$P^{-1} = \frac{adj\ P}{\det P} = -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \text{diag } [5,-3,-3].$$

$\therefore\ P^{-1}AP$= diag [5,-3,-3].

2. Find a matrix P which transform the matrix A = $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate A^4.

Sol: Characteristic equation of A is given by |A-λI| = 0 i.e. $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

=> $(1-\lambda)[(2-\lambda)(3-\lambda)-2]-0-[2-2(2-\lambda)]=0$

=> $9\,\lambda-1)(\lambda-209\lambda-30=0$

=> $\lambda=1, \lambda=2, \lambda=3$

Thus the eigen values of A are 1, 2, 3.

If x_1, x_2, x_3 be the components of an eigen vector corresponding to the eigen value λ, we have

$$[A-\lambda I]\,X = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i): If $\lambda = 1$

$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ i.e, $0.x_1+0.x_2+0.x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$, $x_1=-x_2$

$x_1=1$, $x_2=-1$, $x_3=0$

Eigen vector is $[1,-1,0]^T$

Also every non-zero multiple of this vector is an eigen vector corresponding to λ=1

For λ=2, λ=3 we can obtain eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

The Matrix P is called modal matrix of A

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$Now\, P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$A^4 = PD^4P^{-1}$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

3. Determine the modal matrix P for $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence diagonalize A

Sol: Given that $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}=0$

$\Rightarrow (1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right]-1\left[(1-\lambda)-3\right]+3\left(1-3(5-\lambda)\right)=0$

$\Rightarrow (1-\lambda)\left(5-5\lambda-\lambda+\lambda^2-1\right)-(-2-\lambda)+3(1-15+3\lambda)=0$

$\Rightarrow (1-\lambda)\left(4-6\lambda+\lambda^2\right)-(-2-\lambda)+3(-14+3\lambda)=0$

$\Rightarrow 4-6\lambda+\lambda^2-4\lambda+6\lambda^2-\lambda^3+2+\lambda-42+9\lambda=0$

$\Rightarrow \lambda^3-7\lambda^2-9\lambda+9\lambda-36=0$

$\Rightarrow \lambda^3-7\lambda^2-36=0$

$\Rightarrow \lambda=-2,3,6$

The Eigen Values are -2, 3, and 6

Case (i): If $\lambda = -2$

⇨ $[A-\lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 3x_1+x_2+3x_3=0----(1)$

$x_1+7x_2+x_3=0----(2)$

$3x_1+x_2+3x_3=0----(3)$

From (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{array}$$

$$\Rightarrow \frac{x_1}{1-21} = \frac{-x_2}{3-3} = \frac{x_3}{21-1} = k$$

$$\Rightarrow \frac{x_1}{-20} = \frac{-x_2}{0} = \frac{x_3}{20} = k$$

$$\Rightarrow x_1 = -20k, \quad x_2 = 0, \quad x_3 = 20k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20k \\ 0k \\ 20k \end{bmatrix} = 20k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii): If $\lambda = 3$

⇨ $[A- \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + 3x_3 = 0 ----(1)$$

$$x_1 + 2x_2 + x_3 = 0 -----(2)$$

$$3x_1 + x_2 - 2x_3 = 0 ----(3)$$

Consider (1) & (2)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -2 & 1 & 3 \\ 1 & 2 & 1 \end{array}$$

$$\Rightarrow \frac{x_1}{1-6} = \frac{-x_2}{-2-3} = \frac{x_3}{-4-1} = k$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{-5} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5k \\ 5k \\ -5k \end{bmatrix} = -5k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii): If $\lambda = 6$

⇨ $[A- \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -11 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x_1 + x_2 + 3x_3 = 0 ----(1)$$
$$x_1 - x_2 + x_3 \quad = 0 ---(2)$$
$$3x_1 - x_2 - 5x_3 = 0 ---(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{matrix}$$

$$\Rightarrow \frac{x_1}{5-1} = \frac{-x_2}{-5-3} = \frac{x_3}{+1+3} = k$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{8} = \frac{x_3}{4} = k$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} k$$

$$\rho = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|\rho| = -1(-1-2) - 1(0-2) + 1(0+1)$$
$$|\rho| = (-1)(-3) - 1(-2) + 1 = 3 + 2 + 1 = 6$$

$$\rho = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$Adj(\rho) = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\rho^{-1} = \frac{Adj\,\rho}{|\rho|}$$

$$Cofactor\ of\ \rho = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$D = \rho^{-1} A\, \rho$$

$$D = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1+0+3 & 1-1+3 & 1+2+3 \\ -1+0+1 & 1-5+1 & 1+10+1 \\ -3+0+1 & 3-1+1 & 3+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

4. $\boldsymbol{If\ A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ Find (a) A^8 (b) A^4

Sol: Given that A= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

The Characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-4\right] - 1[0+4] + 1\left[0+4(2-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[6-2\lambda-3\lambda+\lambda^2-4\right] - 4 + 8 - 4x = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 5\lambda + 2\right] + 4 - 4\lambda = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 2 - \lambda^3 + 5\lambda^2 - 2\lambda + 4 - 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

The Eigen values are 1, 2, and 3

Case (i): If $\lambda = 1$

$$[A - \lambda I]X = 0$$

$$=> \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$=> \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+x_2=0$, $x_1+x_2=0$, $-4x_1+4x_2+2x_3=0$

Let $x_3 = k$, $x_2 + k=0$, $x_2 = -k$

$$\Rightarrow -4x_1 + 4(-k) + 2K = 0 \Rightarrow -4x_1 - 2k = 0 \Rightarrow -4x_1 = 2k \Rightarrow x_1 = \frac{-k}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-k}{2} \\ -k \\ k \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} - k = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \frac{-k}{2}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Case (ii): If $\lambda = 2$

$$\Rightarrow \quad [A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 + x_3 = 0 ----(1)$$

$$x_3 = 0 ---(2)$$

$$-4x_1 + 4x_2 + x_3 = 0 ---(3)$$

Consider (1) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -1 & 1 & 1 \\ -4 & 4 & 1 \end{matrix}$$

$$\Rightarrow \frac{x_1}{1-4} = \frac{-x_2}{-1+4} = \frac{x_3}{-4+4} = k$$

$$\Rightarrow \frac{x_1}{-3} = \frac{-x_2}{3} = \frac{x_3}{0} = k$$

$$\Rightarrow x_1 = -k; \quad x_2 = -k$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}(-k) \qquad \therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$\Rightarrow \quad [A - \lambda I]X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$-4x_1 + 4x_2 = 0$$

$Let\ x_1 = k \quad and \quad x_3 = k$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0 \Rightarrow -2k + x_2 + k = 0 \Rightarrow x_2 = k$$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} k$$

$$\therefore p = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\rho^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$D = \rho^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 3^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$$

$$(a). \quad A^8 = PD^8P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$$

$$(b). \quad D^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 16 & 81 \\ 2 & 16 & 81 \\ -2 & 0 & 81 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1+64-162 & 1-48+162 & 0-16+81 \\ -2+64-162 & 2-48+162 & 0-16+81 \\ 2+0-162 & -2-0+162 & 0-0+81 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$$

UNIT-II

FUNCTIONS OF SEVERAL VARIABLES

Introduction: We know that $y = f(x)$ is a function where 'y' is dependent variable and 'x' is independent variable. We are going to expand the idea of functions to include functions for more than one independent variable. In day to day life we deal with things which depend on two or more quantity. For example, the area of the room which is a rectangle consists of two variables: length(say a) and breadth (say b) is given by $A = ab$. Similarly the volume of the rectangular parallelepiped consists of three variables a, b, c i.e., length, breadth, height is given by V = a b c.

In this chapter we say that z is a function of two variables x , y and write $z = f(x\,,y)$ where 'z' is dependent variable and 'x' & 'y' are independent variables.

Limit of a function of two variables:

A function f(x, y) is said to tend to the limit l as (x, y) tends to (a,b) i.e., x→ a and $y \to b$ if corresponding to any given positive number ∈ ∃ a positive number δ $such\ that$ $|f(x, y) - l| <\in$ for all points (x,y) whenever $|x - a| \leq \delta$, $|y - b| \leq \delta$.

In other words the variable value (x ,y) approaches a finite fixed value l when the variable value (x, y) approaches a fixed value (a, b) i.e.,

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l \ or \lim_{(x,y) \to (a,b)} f(x, y) = l$$

Continuity of a function of two variables at a point:

A function f(x,y) is continuous at a point (a, b) if, corresponding to any given positive number ∈ ∃ a positive number δ $such\ that$ $|f(x, y) - f(a, b)| <\in$ for all points (x,y) whenever $0 < (x-a)^2+(y-b)^2 < \delta^2$

Note: Every differentiable function is always continuous, but converse need not be true.

Solved Problems:

1. Evaluate $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2y}{x^2+y^2+1}$

Sol. $$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2y}{x^2+y^2+1} = \lim_{x \to 1}\left\{\lim_{y \to 2}\left[\frac{2x^2y}{x^2+y^2+1}\right]\right\}$$
$$= \lim_{x \to 1} \frac{4x^2}{x^2+5}$$
$$= \frac{4}{6}$$
$$= \frac{2}{3}$$

(or)

$$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2y}{x^2+y^2+1} = \lim_{y \to 2}\left\{\lim_{x \to 1}\left[\frac{2x^2y}{x^2+y^2+1}\right]\right\}$$
$$= \lim_{x \to 1} \frac{2y}{y^2+2}$$

$$= \frac{4}{6}$$
$$= \frac{2}{3}$$

2. If f(x ,y) = $\frac{x-y}{2x+y}$ show that $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} \neq \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$

Sol. $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x-y}{2x+y}\right\}$

$$= \lim_{x\to 0} \frac{x}{2x}$$
$$= \frac{1}{2}$$

$\lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x-y}{2x+y}\right\}$

$$= \lim_{y\to 0} \frac{-y}{y}$$
$$= -1$$

Hence the result follows.

3. Discuss the continuity of the function

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Sol. Let us consider the limit of the function for testing the continuity along the line y = mx.

Now $\lim_{\substack{x\to o \\ y\to 0}} f(x,y) = \lim_{\substack{x\to o \\ y\to o}} \frac{2xy}{x^2+y^2}$

$$= \lim_{x\to 0} \frac{2mx^2}{x^2+m^2x^2}$$
$$= \frac{2m}{1+m^2}$$

Which is different for the different m selected.

∴ $\lim_{\substack{x\to o \\ y\to 0}} f(x,y)$ does not exist.

Consider

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{2x(0)}{x^2+0} = \lim_{x\to 0} 0 = 0 = f(0,0)$$
$$\lim_{y\to 0} f(y,0) = \lim_{y\to 0} \frac{2y(0)}{y^2+0} = \lim_{y\to 0} 0 = 0 = f(0,0)$$

∴ f(x,y) is continuous for given values of x and y but it is not continuous at (0,0)

Partial Differentiation:

Let $z = f(x,y)$ be a function of two variables x and y. Then $\lim_{x\to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$, if it exists , is said to be partial derivative or partial differential coefficient of z or f(x,y) , w.r.t.x . It is denoted by the symbol $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x .

i.e, for the partial derivative of $z = f(x,y)$ w.r.t. 'x' , 'y' is kept constant.

Similarly, the partial derivative of $z = f(x,y)$ w.r.t. 'y' , 'x' is kept constant and is defined as $\lim_{y\to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$ and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Higher order Partial Derivatives:

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y and they can be differentiated repeatedly to get higher order partial derivatives,

So $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}$,

$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3}$, $\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial y^3}$, $\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial x\,\partial y^2}$, and so on.

The chain rule of Partial Differentiation:

Let $z = f(u, v)$ where u = $\emptyset$(x,y) and v = g(x,y) . Then

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Total differential coefficient:

Let $z = f(x, y)$ where x = $\emptyset(t)$ and y = g(t)

Substituting x and y in z = f(x,y) , z becomes a function of a single variable t.

Then the derivative of z w.r.t. ‘t’ i.e, $\frac{\partial z}{\partial t}$ is called the total differential coefficient or total derivative of z.

$$\therefore \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Note: In the differential form , this result can be written as $du = \frac{\partial u}{\partial x}\,.\,dx + \frac{\partial u}{\partial y}\,.\,dy$

Here, *du* is called the total differential of u.

Solved Problems:

1. **If U = log($x^3+y^3+z^3-3xyz$) , prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U=\frac{-9}{(x+y+z)^2}$**

Sol: Given that U = log($x^3+y^3+z^3-3xyz$)

$\therefore \frac{\partial U}{\partial x}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$ (y and z are constant)

$\frac{\partial U}{\partial y}=\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$ (x and z are constant)

$\frac{\partial U}{\partial z}=\frac{3z^2-3yx}{x^3+y^3+z^3-3xyz}$ (y and x are constant)

$$\therefore \frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}=\frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}+\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}+\frac{3z^2-3yx}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Rightarrow \frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}=\frac{3}{x+y+z} \qquad \text{……..(1)}$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}\right)$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad \text{[from (1)]}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}$$

2. If $x^x y^y z^z = e$ show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$

Sol: Given that $x^x y^y z^z = e$

Taking logarithm on both sides, we get

$$x \log x + y \log y + z \log z = \log e$$

$\Longrightarrow$ $z \log z = 1 - x \log x - y \log y$

Differentiating partially w.r.t ,'x', we get

$$\left(z.\frac{1}{z} + 1.\log z\right)\frac{\partial z}{\partial x} = -\left(x.\frac{1}{x} + 1.\log x\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \qquad \text{……..(1)}$$

$$\text{Similarly } \frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)} \qquad \text{……..(2)}$$

When $x = y = z$, we have

$$\frac{\partial z}{\partial x} = -1 \text{ and } \frac{\partial z}{\partial y} = -1$$

Now differentiating (2) partially w.r.t, 'x', we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right]$$

$$= -(1+\log y)\left[-(1+\log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}\right] = \frac{1+\log y}{z(1+\log z)^2}\frac{\partial z}{\partial x} \qquad \text{…….(3)}$$

When $x = y = z$ from (3) , we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1+\log x}{x(1+\log x)^2}(-1) \quad \left(\text{since } \frac{\partial z}{\partial x} = -1\right)$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e+\log x)} \qquad (\text{since } \log e = 1)$$

$$= -\frac{1}{x\log ex} = -(x\log ex)^{-1}$$

3. If $u = f(y-z, z-x, x-y)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol: Let $r = y - z$, $s = z - x$, $t = x - y$ .Then $u = f(r,s,t)$

Now $\frac{\partial r}{\partial x} = 0$, $\frac{\partial r}{\partial y} = 1$, $\frac{\partial r}{\partial z} = -1$

$\frac{\partial s}{\partial x} = -1$, $\frac{\partial s}{\partial x} = 0$, $\frac{\partial s}{\partial x} = 1$

and $\frac{\partial t}{\partial x} = 1$, $\frac{\partial t}{\partial y} = -1$, $\frac{\partial t}{\partial z} = 0$

$\therefore$ By chain rule of partial differentiation , we have

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}(-1)+\frac{\partial u}{\partial t}(1)=-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \quad \ldots(1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$=\frac{\partial u}{\partial r}(1)+\frac{\partial u}{\partial s}(-0)+\frac{\partial u}{\partial t}(-1)=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t} \quad \ldots(2)$$

and

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$=\frac{\partial u}{\partial r}(-1)+\frac{\partial u}{\partial s}(1)+\frac{\partial u}{\partial t}(0)=-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s} \quad \ldots(3)$$

(1) + (2) + (3) gives

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\left(-\frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)+\left(\frac{\partial u}{\partial r}-\frac{\partial u}{\partial t}\right)+\left(-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}\right)=0$$

Jacobian :

Let u = u (x , y) , v = v(x , y) are two functions of the independent variables x , y.

The jacobian of (u , v) w .r .t (x , y) or the jacobian transformation is given by the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ (or) $\begin{vmatrix} u_x & u_y \\ v_x & v_y\end{vmatrix}$

The determinant value is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Similarly if u = u(x, y, z) , v = v (x, y , z) , w = w(x, y , z) ,then the Jacobian of u , v , w w.r.to x , y , z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right)=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix} u_x & u_y & u_z\\ v_x & v_y & v_z\\ w_x & w_y & w_z\end{vmatrix}$$

Properties of Jacobians

1. If $J=\frac{\partial(u,v)}{\partial(x,y)}$ $and\ J^1=\frac{\partial(x,y)}{\partial(u,v)}$ $then\ JJ^1=1$
2. If u,v are functions of r, s and r, s are functions of x ,y , then $\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\cdot\frac{\partial(r,s)}{\partial(x,y)}$

Solved Problems:

1. **If $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$ find** $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

$= 1(1-0) - 2y(0 - 4xz) + 0$

$= 1 - 2y(-4xz)$

$= 1 + 8xyz$

$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]} = \frac{1}{1+8xyz}$

2. Show that the functions $u = x + y + z$, $v = x^2 + y^2 + z^2 -2xy - 2yz -2xz$ and $w = x^3 + y^3 + z^3 -3xyz$ are functionally related.

Sol: Given $u = x + y + z$

$v = x^2 + y^2 + z^2 -2xy - 2yz -2xz$

$w = x^3 + y^3 + z^3 -3xyz$

we have

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$

$=6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$

$c_1 \to c_1 - c_2$

$c_2 \to c_2 - c_3$

$= 6\begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$

$=6[2(x - y) (y^2 + xy - xz -z^2)-2(y - z)(x^2 + xz - yz - y^2)]$

$=6[2(x - y)(y - z)(x + y + z) - 2(y - z)(x - y)(x + y + z)]$

$=0$

Hence there is a relation between u,v,w.

3. If $x + y + z = u$, $y + z = uv$, $z = uvw$ then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: $x + y + z = u$

$y + z = uv$

$z = uvw$

$y = uv - uvw = uv (1 - w)$

$x = u - uv = u (1 - v)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

= uv [u –uv +uv]

= u^2v

4. If u = $x^2 - y^2$, v =2xy where x = r cos θ , y = r sin θ S.T $\frac{\partial(u,v)}{\partial(r,\theta)}$ = $4r^3$

Sol: Given u = $x^2 - y^2$, v = 2xy

=$r^2\cos^2\theta - r^2\sin^2\theta$ = 2rcos θ r sin θ

= $r^2(\cos^2\theta - \sin^2\theta)$ = $r^2 \sin 2\theta$

= $r^2 \cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

= $4r^2 [r\cos^2 2\theta + r\sin^2 2\theta]$

=$4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$

=$4r^3$

5. If u = $\frac{yz}{x}$, v = $\frac{xz}{y}$, w = $\frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol: Given u = $\frac{yz}{x}$, v = $\frac{xz}{y}$, w = $\frac{xy}{z}$

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$u_x = yz(-1/x^2) = \frac{-yz}{x^2}$, $u_y = \frac{z}{x}$, $u_z = \frac{y}{x}$

$v_x = \frac{z}{y}$, $v_y = xz(-1/y^2) = \frac{-xz}{y^2}$, $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $w_y = \frac{x}{z}$, $w_z = xy(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1[-1(1-1) -1(-1-1) + (1+1)]$

$= 0 -1(-2) + (2)$

$=2 + 2$

$=4$

6. **If $x = e^r \sec\theta$, $y = e^r \tan\theta$ P.T $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$**

Sol: Given $x = e^r \sec\theta$, $y = e^r \tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r \sec\theta = x$, $x_\theta = e^r\sec\theta \tan\theta$

$y_r = e^r \tan\theta = y$, $y_\theta = e^r \sec^2\theta$

$x^2 - y^2 = e^{2r} (\sec^2\theta - \tan^2\theta)$

⇨ $2r = \log (x^2 - y^2)$

⇨ $r = ½ \log (x^2 - y^2)$

$$r_x = \frac{1}{2}\frac{1}{x^2-y^2}(2x) = \frac{x}{(x^2-y^2)}$$

$$r_y = \frac{1}{2}\frac{1}{x^2-y^2}(-2y) = \frac{-y}{(x^2-y^2)}$$

$$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$$

⇨ $\text{Sin}\theta = \frac{y}{x}$, $\theta = \sin^{-1}(\frac{y}{x})$

$$\theta_x = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\,y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}(1/x) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r\sec\theta\ \tan\theta \\ e^r\sec2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\, e^r \sec\theta \tan\theta$$

$= e^{2r} \sec \theta[\sec^2\theta - \tan^2\theta] = e^{2r} \sec\theta$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x^2-y^2)} & \frac{-y}{(x^2-y^2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2-y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} = \frac{1}{x\sqrt{x^2-y^2}} = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

Functional Dependence:

Two functions u and v are functionally dependent if their Jacobian i.e,

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

If the Jacobian of u, v is not equal to zero then those functions u, v are functionally independent.

Solved Problems :

1. **If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functionally dependent. Find the relation between them.**

Sol : Given $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

$\therefore \frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}$, $\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$, $\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$ and $\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$\therefore$ u and v are functionally dependent .

Now $v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$

$\therefore v = \tan^{-1}u$ is the functional relation between u and v.

2. **Determine whether the following functions are functionally dependent or not. If they are functionally dependent , find a relation between them.**

i) $\mathbf{u = e^x \sin y , v = e^x \cos y}$ **ii)** $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

Sol: i) Jacobian $= \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$

$= e^x(-\sin^2 y - \cos^2 y) = -e^x \neq 0$

$\therefore$ u,v are functionally independent .

ii) $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

$\therefore J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix} = \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2} = 0$

$\therefore$ u and v are functionally dependent ,

Now $v = \frac{x+y}{x-y} = \frac{y\left(\frac{x}{y}+1\right)}{y\left(\frac{x}{y}-1\right)} = \frac{u+1}{u-1}$

$\therefore v = \frac{u+1}{u-1}$ is the functional relation between u and v .

3. Show that the functions u= xy + yz + zx , v = $x^2 + y^2 + z^2$ and w= x + y + z are functionally related .find the relation between them.

Sol: We have

$u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad (\text{Applying } R_1 \to R_1 + R_2)$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$= 2(x + y + z)\ (0)$ (since R_1 and R_1 are identical)

$= 0$

Hence u,v and w are functionally dependent . That is , the functional relationship exists between them.

Now $w^2 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = v + 2u$

$\therefore\ w^2 = v + 2u$ is the functional relation between u ,v and w.

4. Verify if u = 2x – y + 3z , v = 2x – y – z , w = 2x – y + z are functionally dependent and if so , find the relation between them.

Sol: Given u = 2x – y + 3z , v = 2x – y – z , w = 2x – y + z

The functions u,v,w are functionally dependent if and only if $J\left(\frac{u,v,w}{xy,z}\right) = 0$

Now $J\left(\frac{u,v,w}{xy,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2(-1)\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-2)(0) = 0$

$\therefore$ u, v, w are functionally dependent

Now u + v -2w = (2x-y +3z) + (2x – y –z) – 2 (2x – y + z)

= (4x – 2y + 2z) – (4x – 2y +2z) = 0

Hence u + v -2w = 0 is the functional relationship between u, v and w.

5. Show that the functions u = x+y+z, v = $x^2+y^2+z^2$-2xy-2yz-2zx and w = $x^3+y^3+z^3$-3xyz are functionally related.

Sol. Given u = x+y+z, v = $x^2+y^2+z^2$-2xy-2yz-2zx and w = $x^3+y^3+z^3$-3xyz

Now $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-y-x) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$

$= 6\begin{vmatrix} 1 & 1 & 1 \\ (x-y-z) & (y-x-z) & (z-y-x) \\ (x^2-yz) & (y^2-xz) & (z^2-xy) \end{vmatrix}$

$= 6\begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & (z-y-x) \\ (x-y)(x+y+z) & (y-z)(x+y+z) & (z^2-xy) \end{vmatrix}$ $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$

$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 12\begin{vmatrix} x-y & y-z \\ (x-y)(x+y+z) & (y-z)(x+y+z) \end{vmatrix}$

$= 12\,(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ (x+y+z) & (x+y+z) \end{vmatrix}$

= 12 (x-y)(y-z) (0) [C_1 and C_2 are identical]

= 0

Hence the functional relationship exists between u, v, w.

6. Prove that u = $\frac{x^2-y^2}{x^2+y^2}$, v = $\frac{2xy}{x^2+y^2}$ are functionally dependent and find the relation between them.

Sol. We are given u = $\frac{x^2-y^2}{x^2+y^2}$, v = $\frac{2xy}{x^2+y^2}$

$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2).2x-(x^2-y^2).2x}{(x^2+y^2)^2} = \frac{2x(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{(x^2+y^2).(-2y)-(x^2-y^2).2y}{(x^2+y^2)^2} = \frac{(-2y)(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2} = \frac{-4yx^2}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial x} = 2y\left[\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}\right] = \frac{2y(y^2-x^2)}{(x^2+y^2)^2}$ and

$\frac{\partial v}{\partial y} = 2x\left[\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}\right] = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$

Thus $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4xy^2}{(x^2+y^2)^2} \\ \frac{2y(y^2-x^2)}{(x^2+y^2)^2} & \frac{2x(x^2-y^2)}{(x^2+y^2)^2} \end{vmatrix}$

$$= \frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4} + \frac{8x^2y^2(y^2-x^2)}{(x^2+y^2)^4}$$

$$= \frac{8x^2y^2(x^2-y^2) - 8x^2y^2(y^2-x^2)}{(x^2+y^2)^4} = 0$$

∴ u,v are functionally dependent.

$u^2 + v^2 = \frac{(x^2-y^2)}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2} = \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1$

Hence $u^2 + v^2 = 1$ is the functional relation between u and v.

Maxima & Minima for functions of two Variables:

Definition : Let f(x,y) be a function of two variables x and y.

At x = a ; y = b , f(x ,y) is said to have maximum or minimum value , if f (a ,b) > f(a +h , b +k) or f (a ,b) < f(a +h , b +k) respectively where h and k are small values.

Extremum : A function which have a maximum or minimum or both is called 'extremum'

Extreme value :- The maximum value or minimum value or both of a function is Extreme value.

Stationary points: - To get stationary points we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ i.e the pairs (a_1, b_1), (a_2, b_2) ………….. are called Stationary.

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a_1,b_1) (a_2,b_2) (a_3,b_3) ………………
2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x\, \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$
3. i) If $ln - m^2 > 0$ and $l < 0$ at (a_1,b_1) then f(x ,y) is maximum at (a_1,b_1) and maximum value is $f(a_1,b_1)$

   ii) If $ln - m^2 > 0$ and $l > 0$ at (a_1,b_1) then f(x ,y) is minimum at (a_1,b_1) and minimum value is $f(a_1,b_1)$.

iii) If $ln - m^2 < 0$ and at $(a_1, b1)$ then f(x, y) is neither maximum nor minimum at (a_1, b_1). In this case (a_1, b_1) is saddle point.

iii) If $ln - m^2 = 0$ and at (a_1, b_1), no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

Solved Problems:

1. Locate the stationary points & examine their nature of the following functions.

$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ -------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ -------------------> (2)

Adding (1) & (2),

$x^3 + y^3 = 0$

$\Rightarrow x = -y$ -------------------> (3)

(1) $\Rightarrow x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 4$ & $n = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$ and $l = 20 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At (0,0), $ln - m^2 = (0 - 4)(0 - 4) - 16 = 0$

(0,0) is not a extreme value.

2. Investigate the maxima & minima, if any, of the function $f(x) = x^3y^2(1-x-y)$.

Sol: Given $f(x) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\quad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0 \Rightarrow x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $\quad 4x + 3y - 3 = 0$

$2x + 3y - 2 = 0$

$2x = 1 \Rightarrow x = ½$

$4(½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2,\ y = (1/3)$

$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9 = -1/9$

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2,1/3)} = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$

$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$

$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$

$ln - m^2 = (-1/9)(-1/8) - (-1/12)^2 = \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$ and $l = \frac{-1}{9} < 0$

The function has a maximum value at (1/2 , 1/3)

$\therefore$ Maximum value is $f\left(\frac{1}{2},\frac{1}{3}\right) = \left(\frac{1}{8}\times\frac{1}{9}\right)\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$

3. Find three positive numbers whose sum is 100 and whose product is maximum.

Sol: Let x ,y ,z be three +ve numbers.

Then $x + y + z = 100$

$\Rightarrow z = 100 - x - y$

Let $f(x,y) = xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0 \Rightarrow y(100 - 2x - y) = 0$ ----------------> (1)

$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0 \Rightarrow x(100 - x - 2y) = 0$ ------------------> (2)

From (1) & (2)

$100 - 2x - y = 0$

$200 - 2x - 4y = 0$

$-100 + 3y = 0 \Rightarrow 3y = 100 \Rightarrow y = 100/3$

$100 - x - (200/3) = 0 \Rightarrow x = 100/3$

$l = \frac{\partial^2 f}{\partial x^2}$ =- 2y

$\left(\frac{\partial^2 f}{\partial x^2}\right)$ (100/3 , 100/3) = - 200/3

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ = 100 -2x -2y

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ (100/3 , 100/3) = 100 –(200/3) –(200/3) = -(100/3)

$n = \frac{\partial^2 f}{\partial y^2}$ = -2x

$\left(\frac{\partial^2 f}{\partial y^2}\right)$ (100/3 , 100/3) = - 200/3

ln -m^2 = (-200/3) (-200/3) - (-100/3)^2 = (100)^2 /3

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\quad$ z $= 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required numbers are x = 100/3, y = 100/3, z = 100/3

4. Find the maxima & minima of the function f(x) = 2(x^2 –y^2) –x^4 +y^4

Sol: Given f(x) = 2(x^2 –y^2) –x^4 +y^4 = 2x^2 –2y^2 –x^4 +y^4

For maxima & minima $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0

$\frac{\partial f}{\partial x}$ = 4x - 4x^3 = 0 => 4x(1-x^2) = 0 => x = 0 , x = ± 1

$\frac{\partial f}{\partial y}$ = -4y + 4y^3 = 0 => -4y (1-y^2) = 0 =>y = 0, y = ± 1

$l = \left(\frac{\partial^2 f}{\partial x^2}\right)$ = 4-12x^2

$m = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ = 0

$n = \left(\frac{\partial^2 f}{\partial y^2}\right)$ = -4 +12y^2

we have ln – m^2 = (4-12x^2)(-4 +12y^2) – 0

= -16 +48x^2 +48y^2 -144x^{2}y^2

= 48x^2 +48y^2 -144x^{2}y^2 -16

i) At (0 , ± 1)

ln – m^2 = 0 + 48 - 0 -16 =32 > 0

l = 4-0 = 4 > 0

f has minimum value at (0 , ± 1)

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$

The minimum value is '-1 '.

ii) At (± 1 ,0)

$ln - m^2 = 48 + 0 - 0 - 16 = 32 > 0$

$l = 4 - 12 = -8 < 0$

f has maximum value at (± 1 ,0)

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

The maximum value is '1 '.

iii) At (0,0) , (± 1 , ± 1)

$ln - m^2 < 0$

$l = 4 - 12x^2$

(0 , 0) & (± 1 , ± 1) are saddle points.

f has no max & min values at (0 , 0) , (± 1 , ± 1).

Lagrange's method of undetermined multipliers

Suppose it is required to find the extremum for the function f(x , y , z) subject to condition

$$\emptyset(x, y, z) = 0 \text{ ------------- } (1)$$

Step 1 : Form lagrangean function $F(x, y, z) = f(x, y, z) + \gamma\, \emptyset(x, y, z)$ where γ is called Lagrange's constant, which is determined by the following conditions.

Step 2: Obtain the equations

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0 \text{ --------------- } (2)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0 \text{ --------------- } (3)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0 \text{ --------------- } (4)$$

Step 3: Solving the equations (1) (2) (3) & (4) we get the stationary point (x, y, z).

Step 4 : Substitute the value of x , y , z so obtained in equation (1) we get the extremum.

Solved Problems:

1. **Find the minimum value of $x^2 + y^2 + z^2$, given $x + y + z = 3a$**

Sol: $u = x^2 + y^2 + z^2$

$\emptyset = x + y + z - 3a = 0$

Using Lagrange's function

$F(x, y, z) = u(x, y, z) + \gamma\, \emptyset(x, y, z)$

For maxima or minima

$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 2x + \gamma = 0$ ------------ (1)

$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 2y + \gamma = 0$ ------------ (2)

$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 2z + \gamma = 0$ ------------ (3)

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\emptyset = x + x + x - 3a = 0$

$x = a$

$x = y = z = a$

Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$

2. **Find the minimum value of $x^2 + y^2 + z^2$, given that $xyz = a^3$**

Sol: Let $u = x^2 + y^2 + z^2$ …… (1)

And $\emptyset = xyz - a^3 = 0$ …… (2)

Consider the lagrangean function $F(x,y,z) = u(x,y,z) + \lambda\, \phi(x, y, z)$

i.e , $F(x,y,z) = x^2 + y^2 + z^2 + \lambda(xyz - a^3)$ ……(3)

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x} = 2x + \lambda yz = 0$ ……(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y} = 2y + \lambda xz = 0$ ……(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} + \lambda \frac{\partial \emptyset}{\partial z} = 2z + \lambda yx = 0$ ……(6)

From (4) , (5) and (6) ,we have $\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = -\frac{\lambda}{2}$ ……(7)

From the first two members , we have $\frac{x}{yz} = \frac{y}{xz} \Rightarrow x^2 = y^2$ …(8)

From the last members , we have $\frac{y}{xz} = \frac{z}{xy} \Rightarrow y^2 = z^2$ ……(9)

From (8) and (9) , we have $x^2 = y^2 = z^2 \Rightarrow x = y = z$ ……(10)

on solving (2) and (10) , we get , x = y = z = a

$\therefore$ Minimum value of $u = a^2 + a^2 = 3a^2$

3. Find the maximum value of $u = x^2y^3z^4$ if 2x + 3y + 4z = a

Sol: Given $u = x^2y^3z^4$ …… (1)

Let $\phi(x, y, z) = 2x + 3y + 4z - a = 0$ ……(2)

Consider the lagrangean function F(x,y,z) = u(x,y,z) + λ ϕ(x, y, z)

i.e, , $F(x,y,z) = x^2y^3z^4 + \lambda(2x + 3y + 4z - a)$ ……(3)

for maxima or minima $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 2xy^3z^4 + 2\lambda = 0 \Rightarrow xy^3z^4 = -\lambda$ ……(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + 3\lambda = 0 \Rightarrow x^2y^2z^4 = -\lambda$ …….(5)

and $\frac{\partial F}{\partial z} = 0 \Rightarrow 4x^2y^3z^3 + 4\lambda = 0 \Rightarrow x^2y^3z^3 = -\lambda$ …….(6)

From (4) and (5) , we have x = y …….(7)

From (5) and (6) , we have y = z …….(8)

Hence from (7) and (8) , we get x = y = z …….(9)

On solving (2) and (9) ,we get $x = y = z = \frac{a}{9}$

$$\therefore \text{Maximum value of u} = \left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^9$$

4. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol: Let 2x ,2y, 2z are the length , breadth and height of rectangular solid

Then its volume V = 8 xyz ……..(1)

Let the sphere have a radius of 'r' so that $x^2 + y^2 + z^2 = r^2$ ……(2)

Consider the lagrangean function F(x,y,z) = u(x,y,z) + λ ϕ(x, y, z)

i.e, $F(x,y,z) = V + \lambda(x^2 + y^2 + z^2 - r^2)$

$= 8xyz + \lambda(x^2 + y^2 + z^2 - r^2)$ …….(3)

For maxima or minima $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + 2\lambda x = 0$ ……(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 8zx + 2\lambda y = 0$ ……(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xz + 2\lambda z = 0$ ……(6)

From (4) ,(5) and (6) we have $2x^2\lambda = -8xyz = -2y^2\lambda = -2z^2\lambda$

$\Rightarrow x = y = z$

Thus for a maximum value x = y = z which shows that the rectangular solid is a cube.

Taylor's series for a function of two variables:

Consider a function f(x,y) defined in a region enclosing (a,b) and having successive partial derivatives ,then taylor's series gives an expansion of f(x,y) in powers of (x-a) and (y-b) and partial derivatives of f at (a,b) and is expressed in ascending powers of (x-a) and (y-b) .

$f(x,y) = f(a,b) + (x-a)\, f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}[\,(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a.b)\big] + \frac{1}{3!}[\,(x-a)^3 f_{xxx}(a,b) + 3\,(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)\big] + \ldots\ldots$

Note: The above expansion is called the expansion of f(x,y) at (a,b) or in the neighbouhood of (a,b) or in powers of (x-a) and (y-b) .

Solved Problems:

1. **Expand $e^x \cos y$ near $(1, \frac{\pi}{4})$**

Sol: Let f(x,y) = $e^x \cos y \;\Rightarrow\; f(1,\frac{\pi}{4}) = \frac{e}{\sqrt{2}}$. Then

$f_x(x,y) = e^x \cos y \;\Rightarrow\; f(1,\frac{\pi}{4}) = \frac{e}{\sqrt{2}}$

$f_y(x,y) = -e^x \sin y \;\Rightarrow\; f(1,\frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{xx}(x,y) = -e^x \cos y \;\Rightarrow\; f(1,\frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{xy}(x,y) = -e^x \sin y \;\Rightarrow\; f(1,\frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{yy}(x,y) = -e^x \cos y \;\Rightarrow\; f(1,\frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$\text{By substituting above values in taylor's series , we get}$

$$e^x \cos y = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y-\frac{\pi}{4}\right) + \frac{(x-1)^2}{2!} - (x-1)\left(y-\frac{\pi}{4}\right) - \frac{\left(y-\frac{\pi}{4}\right)^2}{2!} + \cdots\right]$$

2. **Expand $x^2y + 3y - 2$ in powers of $(x-1)$and $(\,y+2)$ using taylor's thorem.**

Sol: Let f(x,y) = $x^2y + 3y - 2$, a = 1 , b = -2

Now f(a,b) = f(1,-2) = -10

$f_x(a,b) = 2xy \Rightarrow \; f_x(1,-2) = -10$

$f_y(a,b) = x^2 + 3 \;\Rightarrow\; f_y(1,-2) = 4$

$f_{xx}(a,b) = 2y \;\Rightarrow\; f_{xx}(1,-2) = -4$

$f_{xy}(a,b) = 2x \;\Rightarrow\; f_{xy}(1,-2) = 4$

$f_{yy}(a,b) = 0 \;\Rightarrow\; f_{yy}(1,-2) = 0$

$f_{xxx}(a,b) = 0 \;\Rightarrow\; f_{xxx}(1,-2) = 0$

$f_{xxy}(a,b) = 2 \;\Rightarrow\; f_{xxy}(1,-2) = 2$

$f_{xyy}(a,b) = 0 \;\Rightarrow\; f_{xyy}(1,-2) = 0$

$f_{yyy}(a,b) = 0 \;\Rightarrow\; f_{yyy}(1,-2) = 0$

All other partial derivatives of higher order will vanish

By substituting above values in taylor's series , we get

$x^2y + 3y - 2 = 10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0) + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$

$= 10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$

UNIT – III

ORDINARY DIFFERENTIAL EQUATIONS

I.Differential equations of first order and first degree

Definition: An equation involving variables and its differentials is called a Differential equation.

They are of two types

1. Ordinary differential equations
2. Partial differential equations

Ordinary differential equation: An equation is said to be ordinary if one or more variables are differentiated w.r.to only one independent variable.

Ex . (1) $\frac{dy}{dx} + 7xy = x^2$ **(2)** $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$

Partial Differential equation: A Differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

E. g: 1. $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$

2. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

Order of a Differential equation: It is the order of the highest derivative occurring in the Differential equation. Differential equation is said to be of order 'n' if the n^{th} derivative is the highest derivative in that equation.

E. g : (1). $\mathbf{(x^2+1) . \frac{dy}{dx} + 2xy = 4x^2}$

Order of this Differential equation is 1.

(2) $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

Order of this Differential equation is 2.

(3) $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 2y = 0$.

Order=2

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Order is 2.

Degree of a Differential equation: Degree of a differential Equation is the highest degree of the highest derivative in the equation, after the equation is made free from radicals and fractions in its derivations.

E.g : 1) $\mathbf{y = x \cdot \frac{dy}{dx}} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ on solving we get

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 + 2xy.\frac{dy}{dx} + \left(1-y^2\right) = 0. \text{ Degree} = 2$$

2) a. $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ on solving . we get

$$a^2.\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3. \text{ Degree} = 2$$

The general form of first order, first degree differential equation is $\frac{dy}{dx} = f(x,y)$ or $f(x,y,y') = 0$ [i.e $Mdx + Ndy = 0$ Where M and N are functions of x and y]. There is no general method to solve any first order differential equation The equation which belong to one of the following types can be easily solved.

In general the first order first degree differential equation can be classified as:

(1) Exact equations

(2) Non exact equations (reducible to exact equations).

Exact Differential Equations

Def: Let M(x, y) dx +N(x, y) dy =0 be a first order and first degree Differential Equation where M & N are real valued functions of x, y . Then the equation M dx + N dy =0 is said to be an exact Differential equation if ∃ a function f ∋ .

$$d\left[f(x,y)\right] = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$Eg: d(x^2y) = 2xydx + x^2dy$$

Condition for Exactness: If $M(x,y)$ & $N(x,y)$ are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation $M\,dx + N\,dy = 0$ is to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence solution of the exact equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is

$$\int Mdx \quad + \quad \int Ndy = c\,.$$

(y is taken as constant) (terms free from x are taken).

Solved Problems :

1. Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$.

Sol : Given equation can be written as

$$(y\cos x+\sin y+y)dx+(\sin x+x\cos y+x)dy=0\ldots(1)$$

It is of the form $Mdx+Ndy=0$.

Here

$M=y\cos x+\sin y+y$

$N=\sin x+x\cos y+x$

$\frac{\partial M}{\partial y}=\cos x+\cos y+1$

$\frac{\partial N}{\partial x}=\cos x+\cos y+1$

Clearly $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

$\Rightarrow$Equation is exact.

The general solution is given by $\int Mdx+\int Ndy=c$

(y constant) (terms independent of x)

$$\Rightarrow \int(y\cos x+\sin y+y)dx+\int(0)dy=c$$

$$\Rightarrow y\sin x+(\sin y+y)x=c\,.$$

2. Solve $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

Sol : Here $M=1+e^{\frac{x}{y}}$ & $N=e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y}=e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \;\&\; \frac{\partial N}{\partial x}=e^{\frac{x}{y}}\left(\frac{-1}{y}\right)+\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right)$$

$$\Rightarrow \frac{\partial M}{\partial y}=e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \;\&\; \frac{\partial N}{\partial x}=e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ equation is exact

General solution is

$$\int Mdx \quad + \quad \int Ndy \quad = c.$$

(y constant) (terms free from x)

$$\Rightarrow \int \left(1+e^{\frac{x}{y}}\right) dx + \int 0\ dy = c.$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\Rightarrow x + ye^{\frac{x}{y}} = c$$

3. Solve the D.E (x+y-1) dy-(x-y+2) dx=0

Sol : Here $M = -(x-y+2)$;

$$N = (x+y-1)$$

$$\frac{\partial M}{\partial y} = 1; \frac{\partial N}{\partial x} = 1$$

$$\text{Clearly}\ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus the equation is exact.

General solution is $\int Mdx \quad + \quad \int Ndy \quad = c$.

(y constant) (terms free from x)

$$\Rightarrow \int -(x-y+2)dx + \int (y-1)dy = c$$

$$\Rightarrow -\frac{x^2}{2} + xy - 2x + \frac{y^2}{2} - y = c$$

$$\Rightarrow x^2 - y^2 - 2xy + 4x + 2y = c_1$$

4. Solve $(e^y+1).\cos x dx + e^y \sin x dx = 0$.

Sol. $M = (e^y+1)\cos, N = e^y \sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x; \frac{\partial N}{\partial y} = e^y \cos x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^y \cos x$$

Equation is exact.

Gen Sol. is $\int Mdx \quad + \quad \int Ndy \quad = c$.

(y constant) (terms free from x)

$$\int (e^y+1)\cos x\,dx+\int 0\,dy=c$$

$$\Rightarrow e^y\sin x=c$$

5. Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left[x+\log x-x\sin y\right]dy=0$.

Sol : Here $M=y\left(1+\frac{1}{x}\right)+\cos y,N=x+\log x-x\sin y$

$$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y\ \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$ so the equation is exact

General sol $\int Mdx \quad + \quad \int Ndy \quad =c$.

(y constant) (terms free from x)

$$\Rightarrow \int\left[y+\frac{y}{x}+\cos y\right]dx+\int 0.dy=c.$$

$$\Rightarrow y(x+\log x)+x\cos y=c.$$

6. Solve (cosx-xcosy)dy – (siny+ysinx)dx =0

Sol : $N=\cos x-x\cos y\ \&\ M=-\sin y-y\sin x$

$$\frac{\partial N}{\partial x}=-\sin x-\cos y \qquad \frac{\partial M}{\partial y}=-\cos y-\sin x$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow$$ the equation is exact.

General sol $\int Mdx \quad + \quad \int Ndy \quad =c$.

(y constant) (terms free from x)

$$\Rightarrow \int(-\sin y-y\sin x).dx \quad +\int 0.dy=c$$

$$\Rightarrow -x\sin y+y\cos x=c$$

$$\Rightarrow \text{ycosx - xsiny}=c$$

7. Solve $(r+\sin\theta-\cos\theta)dr+r(\sin\theta+\cos\theta)d\theta=0$

Sol : Given equation is $(r+\sin\theta-\cos\theta)dr+r(\sin\theta+\cos\theta)d\theta=0$............(1)

This is of the form $Md\theta+Ndr=0$

Where $M=r(\sin\theta+\cos\theta);N=r+\sin\theta-\cos\theta$

We have $\frac{\partial M}{\partial r} = \sin\theta + \cos\theta; \frac{\partial N}{\partial \theta} = \cos\theta + \sin\theta$

Clearly $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$

$\therefore$ The given equation is exact.

The general solution is given by $\int M d\theta + \int N dr = c$

(r constant) (terms independent of θ)

$$\Rightarrow \int r(\sin\theta + \cos\theta) d\theta + \int r dr = c$$

$$\Rightarrow r(\sin\theta - \cos\theta) + \frac{r^2}{2} = c$$

The general solution is $r^2 + 2r\left(\sin\theta - \cos\right) = c_1$.

REDUCTION OF NON-EXACT DIFFERENTIAL EQUATIONS TO EXACT FORM USING INTEGRATING FACTORS

Definition: If the Differential Equation M(x,y) dx + N (x,y) dy = 0 be not an exact differential equation it can be made exact by multiplying with a suitable function u (x,y)≠0 called an Integrating factor(I.F).

Note: There may exits several integrating factors.

<u>**Some methods to find an I.F to a non-exact Differential Equation Mdx+N dy =0**</u>

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1 . d (xy) = xdy +y dx

2. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

3. $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{x^2}$

4. $d\left(\frac{x^2 + y^2}{2}\right)$ = x dx + y dy

5. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{xdy - ydx}{xy}$

6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{xy}$

7. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx - xdy}{x^2 + y^2}$

8. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{xdy - ydx}{x^2 + y^2}$

9. $d\left(\log(xy)\right) = \frac{xdy + ydx}{xy}$

10. $d\left(\log\left(x^2 + y^2\right)\right) = \frac{2\left(xdx + ydy\right)}{x^2 + y^2}$

11. $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

Solved Problems

1) Solve $\frac{y(xy + e^x)dx - e^x dy}{y^2} = 0$.

Sol : It can be written as $\frac{(xy^2 + ye^x)dx - e^x dy}{y^2} = 0$.

$$\Rightarrow \frac{xy^2}{y^2}dx + \frac{e^x ydx - e^x dy}{y^2} = 0$$

$$\Rightarrow xdx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$\Rightarrow xdx + d\left(\frac{e^x}{y}\right) = 0$$

On integrating, we get $\frac{x^2}{2} + \frac{e^x}{y} = c$

This is the required solution.

2) Solve the differential equation $\left(y - x^2\right)dx + \left(x^2 \cot y - x\right)dy = 0$

Sol : Given equation can be written as $ydx - xdy = x^2 dx - x^2 \cot y\, dy$

Dividing with x^2, we get

$$\frac{ydx - xdy}{x^2} = dx - \cot y\, dy \,(or)\, -\left(\frac{xdy - ydx}{x^2}\right) = dx - \cot y\, dy$$

i.e , $-d\left(\frac{y}{x}\right) = dx - \cot y\, dy$

Integrating, we get $-\frac{y}{x} = x + \cos ec^2 y + \cos y = -x\left(x + \cos ec^2 y\right) + cx$

which is the required solution.

3) Solve $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$

Sol : Given equation is $xdx + ydy + \frac{xdy - ydx}{x^2 + y^2} = 0$

$d\left(\frac{x^2 + y^2}{2}\right) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$ on Integrating we get

$\frac{x^2 + y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right) = c$.

4) Solve $y\left(x^3 . e^{xy} - y\right)dx + x\left(y + x^3 . e^{xy}\right)dy = 0$ **.**

Sol : Given equation is on regrouping

We get $yx^3 e^{xy} dx - y^2 dx + xydy + x^4 e^{xy} dy = 0$

$x^3 e^{xy}\left(ydx + xdy\right) + y\left(xdx - ydx\right) = 0$

Dividing by x^3

$e^{xy}\left(ydx + xdy\right) + \left(\frac{y}{x}\right) . \left(\frac{xdy - ydx}{x^2}\right) = 0$

$d\left(e^{xy}\right) + \left(\frac{y}{x}\right) . d\left(\frac{y}{x}\right) = 0$

on Integrating

$e^{xy} + \frac{1}{2}\left(\frac{y}{x}\right)^2 = c$ is required G.S.

5) Solve (1+xy) x dy + (1- yx) y dx = 0

Sol: Given equation is (1+xy) xdy + (1-yx) y dx =0.

$(xdy + ydx) + xy (xdy - ydx) = 0.$

Divided by $x^2y^2 \Rightarrow \left(\frac{xdy + ydx}{x^2y^2}\right) + \left(\frac{xdy - ydx}{xy}\right) = 0$

$\Rightarrow d\left(-\frac{1}{xy}\right) + \frac{1}{y}dy - \frac{1}{x}dx = 0.$

On integrating we get $-\frac{1}{xy} + \log y - \log x = \log c$

$$-\frac{1}{xy}-\log x+\log y=\log c\,.$$

6) Solve : $ydx-xdy=a\left(x^2+y^2\right)dx$

Sol : Given equation is ydx $ydx-xdy=a\left(x^2+y^2\right)dx$

$$\Rightarrow \frac{ydx-xdy}{\left(x^2+y^2\right)}=a\,dx$$

$$\Rightarrow d\left(\tan^{-1}\frac{x}{y}=a\,dx\right)$$

On Integrating $\tan^{-1}\frac{x}{y}=ax+c$ where c is an arbitrary constant.

Method -2: If $M(x,y)dx+N(x,y)dy=0$ is a homogeneous differential equation and $Mx+Ny\neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor of M dx+ N dy =0.

Solved Problems :

1 . Solve $x^2ydx-\left(x^3+y^3\right)dy=0$

Sol : Given equation is $x^2ydx-\left(x^3+y^3\right)dy=0$ -----------------(1)

Where $M=x^2y$ & $N=\left(-x^3-y^3\right)$

Consider $\frac{\partial M}{\partial y}=x^2$ & $\frac{\partial N}{\partial x}=-3x^2$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

equation is not exact .

But given equation (1) is homogeneous differential equation then

$$Mx+Ny=x\left(x^2y\right)-y\left(x^3+y^3\right)=-y^4\neq 0.$$

$$\text{I.F}=\frac{1}{Mx+Ny}=-\frac{1}{y^4}$$

Multiplying equation (1) by $-\frac{1}{y^4}$

$$\Rightarrow \frac{x^2y}{-y^4}dx-\frac{x^3+y^3}{-y^4}dy=0 \text{ ---------------------- (2)}$$

$$\Rightarrow -\frac{x^2}{y^3}dx-\frac{x^3+y^3}{-y^4}dy=0$$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x^2}{y^3}$ & $N_1 = \frac{x^3+y^3}{y^4}$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4} \;\&\; \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}$$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ equation (2) is an exact D.E.

General sol $\int M_1dx + \int N_1dy = c$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \frac{-x^2}{y^3}dx + \int \frac{1}{y}dy = c.$$

$$\Rightarrow \frac{-x^3}{3y^3} + \log|y| = c$$

2. Solve $y\left(y^2-2x^2\right)dx + x\left(2y^2-x^2\right)dy = 0$

Sol : Given equation is $y\left(y^2-2x^2\right)dx + x\left(2y^2-x^2\right)dy = 0$ -----------(1)

It is the form Mdx + Ndy = 0

Where $M = y\left(y^2-2x^2\right), N = x\left(2y^2-x^2\right)$

Consider $\frac{\partial M}{\partial y} = 3y^2-2x^2$ & $\frac{\partial N}{\partial x} = 2y^2-3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation(1) is Homogeneous differential equation then

Consider $Mx + Ny = x\left[y\left(y^2-2x^2\right)\right] + y\left[x\left(2y^2-x^2\right)\right]$

$$= 3xy\left(y^2-x^2\right) \neq 0 .$$

$$\Rightarrow \text{I.F.} = \frac{1}{3xy\left(y^2-x^2\right)}$$

Multiplying equation (1) by $\frac{1}{3xy\left(y^2-x^2\right)}$ we get

$$\Rightarrow \frac{y\left(y^2-2x^2\right)}{3xy\left(y^2-x^2\right)}dx + \frac{x\left(2y^2-x^2\right)}{3xy\left(y^2-x^2\right)}dy = 0$$

$$M_1 = \frac{y(y^2 - 2x^2)}{3xy(y^2 - x^2)}; N_1 = \frac{x(2y^2 - x^2)}{3xy(y^2 - x^2)}$$

$$\frac{\partial M_1}{\partial y} = \frac{2xy}{3(y^2 - x^2)^2} = \frac{\partial N_1}{\partial x}$$

Now it is exact

$$\frac{(y^2 - x^2) - x^2}{3x(y^2 - x^2)}dx + \frac{y^2 + x(y^2 - x^2)}{3y(y^2 - x^2)}dy = 0$$

$$\frac{dx}{x} - \frac{xdx}{y^2 - x^2} + \frac{ydy}{y^2 - x^2} + \frac{dy}{y} = 0.$$

$$\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{2ydy}{2(y^2 - x^2)} - \frac{2xdx}{2(y^2 - x^2)} = 0$$

On integrating we get

$$\log x + \log y + \frac{1}{2}\log(y^2 - x^2) - \frac{1}{2}\log(y^2 - x^2) = \log c \Rightarrow xy = c$$

Method- 3: If the equation $Mdx + Ndy = 0$ **is of the form**

$y.f(xy)dx + x.g(xy)dy = 0$ **&** $Mx - Ny \neq 0$ **then** $\dfrac{1}{Mx - Ny}$ **is an integrating factor of**

Mdx+ Ndy =0.

Problems :

1 . Solve $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$**.**

Sol : Given equation $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$-------(1).

Equation (1) is of the form y. $f(xy)dx + x.g(xy)dy = 0$.

Where $M = (xy\sin xy + \cos xy)y$

$N = (xy\sin xy - \cos xy)x$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Longrightarrow$ equation (1) is not exact

Now consider Mx-Ny

Mx-Ny =2xycosxy $\neq 0$

Integrating factor $= \dfrac{1}{2xy\cos xy}$

So equation (1) x I.F gives

$$\Rightarrow \frac{(xy\sin xy)y}{2xy\cos xy}dx + \frac{(xy\sin xy - \cos xy)x}{2xy\cos xy}dy = 0$$

$$\Rightarrow \left(y\tan xy + \frac{1}{x}\right)dx + \left(y\tan xy - \frac{1}{y}\right)dy = 0$$

$$\Rightarrow M_1dx + N_1dy = 0$$

$$\frac{\partial M_1}{\partial y} = \tan xy + xy\sec^2 xy = \frac{\partial N_1}{\partial x}$$

Now the equation is exact.

$\therefore$ General solution is $\int M_1 dx + \int N_1 dy = c.$.

(y constant) (terms free from x in N_1)

$$\Rightarrow \int\left(y\tan xy + \frac{1}{x}\right)dx + \int \frac{-1}{y}dy = c.$$

$$\Rightarrow \frac{y.\log|\sec xy|}{y} + \log x + (-\log y) = \log c$$

$$\Rightarrow \log|\sec(xy)| + \log\frac{x}{y} = \log c.$$

$$\Rightarrow \frac{x}{y}.\sec xy = c.$$

2. Solve $(1+xy)ydx + (1-xy)xdy = 0$

Sol : Here $M = (1+xy)y : N = (1-xy)x$

$$\frac{\partial M}{\partial y} = 1 + 2xy; \frac{\partial N}{\partial x} = 1\text{-}2xy$$

Hence, the equation is not exact

Also Mx - Ny= $2x^2y^2 \neq 0$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \neq 0$$

Multiply the given equation by I.F, we get

$$\Rightarrow \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \frac{-1}{2y}dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{2x^2y^2} = \frac{\partial N_1}{\partial x}$$

$\Rightarrow$Equation is exact.

On integrating, we get

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\frac{-1}{2y}dy=c$$

$$\Rightarrow\frac{-1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c.$$

$$\Rightarrow\frac{-1}{xy}+\log\left(\frac{x}{y}\right)=c_1 \quad \text{where } c_1=2c.$$

Method 4: If there exists a continuous single variable function $f(x)$ such that $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$, then I.F. of $Mdx+Ndy=0$ is $e^{\int f(x)dx}$

Solved Problems :

1 . Solve $(3xy-2ay^2)dx+(x^2-2axy)dy=0$

Sol : Given equation is $(3xy-2ay^2)dx+(x^2-2axy)dy=0$

This is of the form Mdx+ Ndy = 0

$$\Rightarrow M=3xy-2ay^2 \quad \& \quad N=x^2-2axy$$

$$\frac{\partial M}{\partial y}=3x-4ay \quad \& \quad \frac{\partial N}{\partial x}=2x-2ay$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \quad \text{equation is not exact .}$$

Now consider $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\dfrac{(3x-4ay)-(2x-2ay)}{x(x-2ay)}$

$$\Rightarrow\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1}{x}=f(x).$$

$$\Rightarrow e^{\int\frac{1}{x}dx}=x \text{ is an Integrating factor of (1)}$$

Multiplying equation (1) with I.F we get

$$\Rightarrow\frac{(3xy-2ay^2)}{1}xdx+\frac{(x^2-2axy)}{1}xdy=0$$

$$(3x^2y-2ay^2x)dx+(x^3-2ax^2y)dy=0$$

It is the form $M_1dx + N_1dy = 0$

$$M_1 = 3x^2y - 2ay^2x, \quad N_1 = x3 - 2ax^2y$$

$$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy, \quad \frac{\partial N_1}{\partial x} = 3x^2 - 4axy$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \qquad \therefore \text{ Equation is exact}$$

General sol $\int M_1 dx + \int N_1 dy = c.$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \left(3x^2y - 2ay^2x\right)dx + \int 0dy = c$$

$$\Rightarrow x^3y - ax^2y^2 = c \,..$$

2. Solve $ydx + xdy + \left(1 + x^2\right)dx + x^2 \sin ydy = 0$

Sol: Given equation is $\left(y + 1 + x^2\right)dx + \left(x^2 \sin y - x\right)dy = 0.$

$$M = y + 1 + x^2 \quad \& \quad N = x^2 \sin y - x$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2x \sin y - 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{the equation is not exact.}$$

So consider $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{\left(1 - 2x \sin y + 1\right)}{x^2 \sin y - x} = \dfrac{-2x \sin y + 2}{x^2 \sin y - x} = \dfrac{-2\left(x \sin y - 1\right)}{x\left(x \sin y - 1\right)} = \dfrac{-2}{x}$

$$\text{I.F} = e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

Equation (1) x I.F gives $\Rightarrow \dfrac{y + 1 + x^2}{x^2}dx + \dfrac{x^2 \sin y - x}{x^2}dy = 0$

It is the form of $M_1dx + N_1dy = 0$.

$$\frac{\partial M_1}{\partial y} = \frac{1}{x^2} = \frac{\partial N_1}{\partial x}$$

$\therefore$ Equation is exact

Gen. sol. is thus $\Rightarrow \int \left(\dfrac{y}{x^2} + \dfrac{1}{x^2} + 1\right)dx + \int \sin ydy = 0$

$$\Rightarrow \frac{-y}{x} - \frac{1}{x} + x - \cos y = c\,.$$

$$\Rightarrow x^2 - y - 1 - x\cos y = cx.$$

3. Solve 2xy dy – (x^2+y^2+1) dx = 0

Sol. Given equation is 2xy dy – (x^2+y^2+1) dx = 0 ………(1)
This is of the form M dx + N dy = 0, where N = 2xy, M = $-x^2 - y^2 - 1$
We have $\frac{\partial M}{\partial y}$ = -2y and $\frac{\partial N}{\partial x}$ = 2y, so that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$
∴ The given equation is not exact.
We have $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}$ (-2y-2y) = $\frac{-2}{x}$ = f(x)
∴ I.F. = $e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$
Multiplying (1) with $\frac{1}{x^2}$, we get $\frac{2y}{x}\,dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx$ ….(2)
This is of the form M_1 dx + N_1 dy = 0, where $M_1 = -1 - \frac{y^2}{x^2} - \frac{1}{x^2}$ and $N_1 = \frac{2y}{x}$
We have $\frac{\partial M1}{\partial y} = \frac{-2y}{x^2}$ and $\frac{\partial N1}{\partial x} = \frac{-2y}{x^2}$
Since $\frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$, ∴ (2) is exact.
General solution is given by

$$\int M_1 dx + \int N_1 dy = c$$

(y constant) (terms free from x in N_1)

$\Rightarrow \int (-1 - \frac{y^2}{x^2} - \frac{1}{x^2})\,dx + \int 0\,dy = c$

(y constant)
$\Rightarrow -x + \frac{y^2}{x} + \frac{1}{x} = c$
This is the general solution o (2) and hence of (1)

Method -5: For the equation Mdx + Ndy = 0 if $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$ (is a function of y alone)

then $e^{\int -g(y)dy}$ is an integrating factor of M dx + N dy =0.

Solved Problems :

1 . Solve ($3x^2y^4+2xy$)dx +($2x^3y^3-x^2$) dy = 0

Sol : Given equation ($3x^2y^4+2xy$)dx +($2x^3y^3-x^2$) dy =0 -----------------(1).

Equation is of the form Mdx + Ndy =0.

where M =$3x^2y^4+2xy$ & N = $2x^3y^3-x^2$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x \;;\; \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow \text{equation(1) is not exact.}$$

So consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y) = \frac{2}{y} = g(y)$

$$I.F = e^{\int -g(y)dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = \frac{1}{y^2}.$$

Equation (1) x I.F $\left(\dfrac{3x^2y^4 + 2xy}{y^2}\right)dx + \left(\dfrac{2x^3y^3 - x^2}{y^2}\right)dy = 0$

$$\Rightarrow \left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

It is the form $M_1dx + N_1dy = 0$

$$\frac{\partial M_1}{\partial y} = 6x^2y - 2\frac{x}{y^2} = \frac{\partial N_1}{\partial x}$$

$\therefore$ Equation is exact

General sol. is $\int M_1dx + \int N_1dy = c$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \left(3x^2y^2 + \frac{2x}{y}\right)dx + \int 0dy = c.$$

$$\Rightarrow \frac{3x^3y^2}{3} + \frac{2x^2}{2y} = c.$$

$$\Rightarrow x^3y^2 + \frac{x^2}{y} = c.$$

2 . Solve $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$

Sol : Here $M = xy^3\ y$; $N = 2\left(x^2y^2 + x + y^4\right)$

$$\frac{\partial M}{\partial y} = 3xy^2 + 1; \quad \frac{\partial N}{\partial x} = 4xy^2 + 2$$

We see equation is not exact.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Also $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = g(y)$

$$\frac{-xy^2 - 1}{y\left(xy^2 + 1\right)} = -\frac{1}{y} = g(y)$$

Thus $I.F = e^{\int -g(y)dy} = e^{\int \frac{1}{y}dy} = y.$

$$\frac{\partial M_1}{\partial y}=4xy^3+2y=\frac{\partial N_1}{\partial x} \quad \text{where } M_1=xy^4+y^2; N_1=2x^2y^3+2xy+2y^5$$

Gen Sol: $\int\left(xy^4+y^2\right)dx+\int\left(2y^5\right)dy=c$

$$\frac{x^2y^4}{2}+y^2x+\frac{2y^6}{6}=c.$$

3. Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

Sol: The given equation is not exact.

Also $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y)=\left(\dfrac{\left(y^3-4\right)-\left(4y^3+2\right)}{y^4+2y}\right)=\dfrac{-3}{y}=g\left(y\right).$

$$I.F=e^{\int g\left(y\right)dy}=e^{-3\int\frac{1}{y}dy}=\frac{1}{y^3}$$

Here $M_1=y+\dfrac{2}{y^2}; N_1=x+2y-\dfrac{4x}{y^3}$

$$\frac{\partial M_1}{\partial y}=1-\frac{2}{y^3}=\frac{\partial N_1}{\partial x}$$

∴Equation is exact.

Gen sol is $\int\left(y+\dfrac{2}{y^2}\right)dx+\int 2y\,dy=c.$

$$\left(y+\frac{2}{y^2}\right)x+y^2=c.$$

APPLICATIONS OF DIFFERENTIAL EQUATIONS OF FIRST ORDER FIRST DEGREE

ORTHOGONAL TRAJECTORIES (O.T)

Def: A curve which cuts every member of a given family of curves at a right angle is an orthogonal trajectory of the given family.

Orthogonal trajectories in Cartesian co-ordinates:

Working rule: To find the family of O.T in Cartesian form : Let f(x,y,c) =0 ……(1) be the given equation of family of curves in Cartesian form.

1) Differentiate (1) with respect to 'x' and obtain F(x, y, y') = 0 ----------(2) of the given family of curves.

2) Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (2)

Then the Differential Equation of family of O.T is

F(x, y, $\frac{-dx}{dy}$) =0 ---------------- (3).

3) Solve equation (3) to get the equation of family of O.T's of equation(1).

Solved Problems

1 . Find the O.T's of family of semi-cubical parabolas $ay^2=x^3$ where a is a parameter.

Sol : The given family of semi-cubical parabola is $ay^2=x^3$ ---------------(1)

Differentiating with respect to 'x' $\Rightarrow a\,2y\frac{dy}{dx} = 3x^2$ -------------(2)

Eliminating 'a' from (1) and (2) $\Rightarrow \frac{x^3}{y^2}2y\frac{dy}{dx} = 3x^2$

$$\Rightarrow \frac{2x^3}{y}\frac{dy}{dx} = 3x^2$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy} \Rightarrow \frac{2x^3}{y}\left(-\frac{dx}{dy}\right) = 3x^2$

$$\Rightarrow \frac{-2}{3}x\frac{dx}{dy} = y$$

$$\Rightarrow \int\frac{-2}{3}xdx - \int ydy = c$$

$$\Rightarrow \frac{-x^2}{3} - \frac{y^2}{2} = c$$

$$\Rightarrow \frac{x^2}{3c} + \frac{y^2}{2c} = 1$$

2. Find the O.T of the family of circles $x^2+y^2+2gx+c=0$, Where g is the parameter

Sol : $x^2+y^2+2gx+c=0$. --------- (1)

represents a system of co- axial circles with g as parameter.

Differentiating with respect to 'x' $\Rightarrow 2x+2y\frac{dy}{dx}+2g=0$--------------(2)

Substituting equation from (2) in (1)

$$\Rightarrow x^2+y^2 -(2x+2y\frac{dy}{dx})x + c = 0.$$

$$\Rightarrow y^2 - x^2 - 2xy\frac{dy}{dx} + c = 0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow y^2 - x^2 - 2xy\left(-\frac{dx}{dy}\right) + c = 0$$

$$\Rightarrow y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0$$

This can be written as

$$2x\frac{dx}{dy} - \frac{1}{y}x^2 = \frac{-(c+y^2)}{y}$$

This is a Bernoulli's equation in x

So put $x^2 = u \Rightarrow 2x\frac{dx}{dy} = \frac{du}{dy}$

$$\Rightarrow \frac{du}{dy} - \frac{1}{y}u = \frac{-(c+y^2)}{y}$$

which is a linear equation in 'u'

$$\Rightarrow \text{ I.F} = e^{\int \frac{-1}{y}dy} = e^{-\log y} = \frac{1}{y}$$

General solution is $u(\text{I.F}) = \int Q(y).\text{I.F}dy + k$

$$\Rightarrow x^2\frac{1}{y} = \int \frac{-(c+y^2)}{y}\frac{1}{y}dy + k$$

$$= -c\left(\frac{-2}{y}\right) - y + k$$

$$\Rightarrow \frac{x^2}{y} = \frac{c}{y} - y + k$$

3. Find orthogonal trajectories of the family of curves $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Sol : The equation of the given family of curves is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ …(1)

Differentiating (1) w.r.t x, we get $\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y^{\frac{-1}{3}}}{x^{\frac{-1}{3}}}$ …(2)

This is the differential equation of the given family of curves.

Changing $\frac{dy}{dx}$ to $-\frac{dx}{dy}$ in (2), we get $-\frac{dx}{dy}=-\frac{y^{\frac{-1}{3}}}{x^{\frac{-1}{3}}} \Rightarrow \frac{dx}{dy}=\frac{y^{\frac{-1}{3}}}{x^{\frac{-1}{3}}}$ …(3)

Separating the variables in (3), we get $x^{\frac{1}{3}}dx = y^{\frac{1}{3}}dy$

Integrating, we get $\int x^{\frac{1}{3}}dx = \int y^{\frac{1}{3}}dy + c$

$\Rightarrow \frac{3}{4}x^{\frac{4}{3}} = \frac{3}{4}y^{\frac{4}{3}} + c$ or $x^{\frac{4}{3}} - y^{\frac{4}{3}} = c_1$

This is the orthogonal equation.

4.Find the O.T's of the family of parabolas through origin and foci on y –axis.

Sol : The equation of the family of parabolas through the origin and foci on y-axis is x^2=4ay where 'a' is parameter.

Differentiating with respective 'x' $\Rightarrow$ $2x = 4a\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

O.T $\Rightarrow -\frac{dx}{dy} = \frac{x}{2a}$

$\Rightarrow -\frac{dx}{x} = \frac{dy}{2a}$

On integrating, we get $-\log x = \frac{y}{2a} + c$

$\Rightarrow y + 2a\log x = c_1$ is the equation of required O.T.

5.Find the O.T of the one parameter family of curves $e^x + e^{-y}$ =c.

Sol : Given equation is $e^x + e^{-y}$ =c.

Differentiating with respective 'x' $\Rightarrow$ $e^x + e^{-y}(-\frac{dy}{dx}) = 0$

Its O.T $\Rightarrow e^x + e^{-y}(\frac{dx}{dy}) = 0 \Rightarrow e^{-x}dx + e^y dy = 0$

On integrating, we get, $-e^{-x} + e^y = c$ which is the required O.T

ORTHOGONAL TRAJECTORIES IN POLAR FORM

Working Rule: To find the O.T of a given family of curves in polar-co ordinates.

Let f(r, θ, c)=0-----(1) be the given family of curves in polar form.

1.) Differentiating (1) with respect to θ *and* obtain F $[r, \theta, \frac{dr}{d\theta}]=0$ by eliminating the parameter c.

2.) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ then the Differential Equation of family of O.T is

$$F\left[r, \theta, -r^2\frac{d\theta}{dr}\right]=0$$

3.) Solve the above equation to get the equation of O.T of (1)

Self-orthogonal

A family of curves is said to be self orthogonal when the differential equation of the family of O.T is same as that of the original family.

Solved Problems

1. Find the orthogonal trajectories of the family of curves $r^n\cos\theta = a^n$.

Sol : Given family of curves is $r^n\cos\theta = a^n$

Taking logarithms on both sides,

$n\log r + \log\cos\theta = n\log a$

Differentiating w.r.t θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} + \frac{1}{\cos\theta}(-\sin\theta) = 0$$

$$\frac{n}{r}\frac{dr}{d\theta} - \tan\theta = 0 \qquad \ldots(2)$$

is the differential equation of family of curves.

To get the differential equation of orthogonal trajectories, replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$

Now, we have $\frac{n}{r}\left(-r^2\frac{d\theta}{dr}\right) - \tan\theta = 0$

$$\Rightarrow nr\frac{d\theta}{dr} + \tan\theta = 0 \qquad \ldots(3)$$

This is differential equation of orthogonal trajectories.

Separating variables, we get

$$n\cot\theta d\theta + \frac{dr}{r} = 0$$

Integrating, we get

$n\log\sin\theta + \log r = \log \Rightarrow \log\sin^n\theta + \log r = \log c$

or $\log(r\sin^n\theta) = \log c$

$\Rightarrow c = r\sin^n\theta$ which is the required O.T.

2. Find the Orthogonal trajectories of the family of cardioids $r = a(1-\cos\theta)$ where 'a' is the parameter.

Sol : Given $r = a(1-\cos\theta) \qquad \ldots(1)$

Differentiating the given equation w.r.t 'θ', we get

$$\frac{dr}{d\theta} = a\sin\theta \Rightarrow a = \frac{1}{\sin\theta}\frac{dr}{d\theta} \qquad (2)$$

Eliminating 'a' from (1) and (2) we get $r = \frac{1-\cos\theta}{\sin\theta}\frac{dr}{d\theta}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r\sin\theta}{1-\cos\theta} = \frac{2r\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \frac{dr}{d\theta} = r\cot\left(\frac{\theta}{2}\right) \qquad \ldots(3)$$

The differential equation of required O.T is obtained by replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ in (3)

$$\therefore -r^2\frac{d\theta}{dr} = r\cot\left(\frac{\theta}{2}\right) Þ \frac{dr}{r} = -\tan\left(\frac{\theta}{2}\right)d\theta$$

On integrating, we get

$$\log r = 2\log|\cos(\theta/2)| + \log(2c) = \log(2c\cos^2(\theta/2))$$

or $r = 2c\cos^2(\theta/2) \Rightarrow r = c(1+\cos\theta)$

This is the equation of family of orthogonal trajectories.

3. Prove that the system of parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Sol : Given parabola is

$$y^2 = 4a(x+a)$$

$$\Rightarrow y' = 4ax + 4a^2 \qquad \ldots(1)$$

Differentiating (1) with respect to x,we get $2yy_1 = 4a \Rightarrow a = \frac{yy_1}{2}$ …(2)

Substituting (2) in (1), $y^2 = 4\frac{yy_1}{2}x + 4\frac{y^2y_1^2}{4}$

or $y^2 = 2xyy_1 + y^2y_1^2$ ….. (3)

Equation (3) is the differential equation of the given system of parabolas. Replacing y_1 with $\frac{-1}{y_1}$, we get equation of the orthogonal trajectories as

$$y^2 = 2xy\left[\frac{-1}{y_1}\right] + y^2\left[\frac{-1}{y_1}\right]^2 \Rightarrow y^2 = \frac{2xy}{y_1} + \frac{y^2}{y_1^2}$$

$\therefore y^2y_1^2 = -2xyy_1 + y^2$ or $y^2 = 2xyy_1 + y^2y_1^2$ …(4)

which is differential equation of the orthogonal trajectories of the given family.

Equations (3) and (4) are same, hence the given system is self orthogonal.

4. Show that the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where λ is a parameter,
is self orthogonal.

Sol : Given equation of family of confocal conics is $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ …(1)

Differentiating (1) w.r.t x, we get

$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$. For convenience, we write $\frac{dy}{dx}=p$

$$\Rightarrow \frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}p=0 \Rightarrow x(b^2+\lambda)+py(a^2+\lambda)=0$$

$$\Rightarrow \lambda=\frac{-(b^2x+a^2yp)}{x+yp}$$

$$\therefore a^2+\lambda=\frac{(a^2-b^2)x}{x+yp} \text{ and } b^2+\lambda=\frac{-(a^2-b^2)yp}{x+yp} \quad \ldots(2)$$

Eliminating λ from (1) and (2), we get

$$\frac{x(x+yp)}{a^2-b^2}-\frac{y(x+yp)}{(a^2-b^2)p}=1 \Rightarrow \left(\frac{x+yp}{a^2-b^2}\right)\left(x-\frac{y}{p}\right)=1$$

$$\Rightarrow (x+yp)\left(x-\frac{y}{p}\right)=a^2-b^2 \quad \ldots(3)$$

This is the differential equation of the family of curves (1). We get the differential equation of the family of orthogonal trajectories by replacing

$$p=\frac{dy}{dx} \text{ with } -\frac{dx}{dy}=-\frac{1}{\left(\frac{dy}{dx}\right)}=-\frac{1}{p}.$$

Hence the differential equation of orthogonal trajectories is

$$\left(x-\frac{y}{p}\right)(x+py)=a^2-b^2 \quad \ldots(4)$$

which is same as (3). Thus we see that the differential equation of the family of orthogonal trajectories is same as that of the original family. Hence the given family of curves is orthogonal to itself. Hence it is a self orthogonal family of curves.

NEWTON'S LAW OF COOLING

STATEMENT: The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let 'θ' be the temperature of the body at time 't' and θ_0 be the temperature of its surrounding medium(usually air). By the Newton's law of cooling , we have

$$\frac{d\theta}{dt}\alpha(\theta-\theta_0) \Rightarrow \frac{d\theta}{dt}=-k(\theta-\theta_0) \text{ k is +ve constant}$$

$$\Rightarrow \int \frac{d\theta}{(\theta - \theta_0)} = -k \int dt$$

$$\Rightarrow \log(\theta - \theta_0) = -kt + c.$$

If initially $\theta = \theta_1$ is the temperature of the body at time t=0 then

$$c = \log(\theta_1 - \theta_0)$$

$$\Rightarrow \log(\theta - \theta_0) = -kt + \log(\theta_1 - \theta_0)$$

$$\Rightarrow \log\left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = -kt.$$

$$\Rightarrow \left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = e^{-kt}$$

$$\theta = \theta_0 + (\theta_1 - \theta_0) = e^{-kt}$$

which gives the temperature of the body at time't'.

Solved Problems

1. A pot of boiling water 100^0C is removed from the fire and allowed to cool at 30^0C room temperature. 2 mins later, the temperature of the water in the pot is 90^0C. What will be the temperature of water after 5 mins?

Sol : We have $\theta - 30 = ce^{-kt}$ …(1)

When $t = 0, \theta = 100$

from (1), we get $c = 70$

$\therefore \theta - 30^0 = 70e^{-kt}$ …(2)

When $t = 2, \theta = 90$

From (2), $90 - 30 = 70e^{-2k}$

$\Rightarrow 60 = 70e^{-2k}$

$\Rightarrow -2k = \log\left(\frac{6}{7}\right)$

$= 0.1542$

$\Rightarrow k = 0.0771$

When $t = 5, \theta - 30 = 70e^{-5k}$

$\Rightarrow \theta = 77.46^0$

2. **A body is originally at 80^0C and cools down to 60^0C in 20 min . If the temperature of the air is 40^0C find the temperature of body after 40 min.**

Sol : By Newton's law of cooling we have

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \text{ where } \theta_0 \text{ is the temperature of the air.}$$

$$\Rightarrow \int \frac{d\theta}{(\theta - \theta_0)} = -k\int dt \Rightarrow \log(\theta - \theta_0) = -kt + \log c$$

Here $\theta_0 = 40^0 C$

$$\Rightarrow \log(\theta - 40) = -kt + \log c$$

$$\Rightarrow \log\left(\frac{\theta - 40}{c}\right) = -kt$$

$$\Rightarrow \left(\frac{\theta - 40}{c}\right) = e^{-kt}$$

$$\Rightarrow \theta = 40 + ce^{-kt} \text{ --------------(1)}$$

When t=0, $\theta = 80^0 C$

$$\Rightarrow 80 = 40 + c \Rightarrow c = 40 \text{ ------------(2).}$$

When t=20, $\theta = 60^0 C \Rightarrow 60 = 40 + ce^{-20k}$ ------------(3).

Solving (2) & (3) $\Rightarrow ce^{-20k} = 20$

$$\Rightarrow 40e^{-2k} = 20$$

$$\Rightarrow k = -20\log 2$$

When $t = 40^0 C$ then equation (1) is $\theta = 40 + 40e^{-\left(\frac{1}{20}\log 2\right)40}$

$$= 40 + 40\, e^{-2\log 2}$$

$$= 40 + \left(40X\frac{1}{4}\right)$$

$$\Rightarrow \theta = 50^0 C$$

3. **An object whose temperature is 75^0C cools in an atmosphere of constant temperature C, at the rate of $k\theta$, being the excess temperature of the body over that of the temperature. If after 10min, the temperature of the object falls to 65^0 C , find its temperature after 20 min. Also find the time required to cool down to 55^0C.**

Sol : We will take one minute as unit of time.

It is given that $\frac{d\theta}{dt} = -kt$

$\Rightarrow \theta = ce^{-kt}$ ------------(1).

Initially when t=0 $\Rightarrow \theta = 75^0 - 25^0 = 50^0$

$\Rightarrow$ c= 50^0

Hence $c = 50 \Rightarrow \theta = 50e^{-kt}$ -----------------(2)

When t= 10 min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$\Rightarrow 40 = 50\,e^{-10k}$

$\Rightarrow e^{-10k} = \frac{4}{5}$ ---------------------(3).

The value of θ when t=20 $\Rightarrow \theta = ce^{-kt}$

$\theta = 50e^{-k}$

$\theta = 50(e^{-10k})^2$

$\theta = 50\left(\frac{4}{5}\right)^2$

When t=20 $\Rightarrow \theta = 32^0C$.

Hence the temperature after 20min $= 32^0C + 25^0C = 57^0C$

When the temperature of the object = 55^0C

$\theta = 55^0C - 25^0C = 30^0C$

Let t, be the corresponding time from equation (2)

$30 = 50e^{-kt}$ -----------(4)

From equation (3) $e^{(-k)^{10}} = \frac{4}{5}$ i.e $e^{-k} = \left(\frac{4}{5}\right)^{\frac{1}{10}}$

From equation (4), we get $30 = 50\left(\frac{4}{5}\right)^{\frac{t_1}{10}} \Rightarrow \frac{t_1}{10}\log\frac{4}{5} = \log\frac{3}{5}$

$$\Rightarrow t_1 = 10\left[\frac{\log\left(\frac{3}{5}\right)}{\log\left(\frac{4}{5}\right)}\right] = 22.9\text{min}$$

4. A body kept in air with temperature 25^0C cools from 140^0C to 80^0C in 20 min. Find when the body cools down in 35^0C.

Sol : By Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0) \Rightarrow \frac{d\theta}{\theta - \theta_0} = -kdt$

$\Rightarrow \log(\theta - \theta_0) = kt + c$. Here, $\theta_0 = 25^0C$

$\Rightarrow \log(\theta - 25) = kt + c$ -------------(1).

When t=0, $\theta = 140^0C$

$\Rightarrow$ log (115) =c

$\Rightarrow$ c =log (115).

$\Rightarrow kt = -\log(\theta - 25) + \log 115$--------(2)

When $t=20, \theta = 80^0C$

$\Rightarrow$ log (80-25)= -20k + log 115

$\Rightarrow$ 20 k =log (115) - log(55) ---------(3)

Divide equation (2) by (3), we get $\frac{kt}{20k} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

$$\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$$

When $\theta = 35^0C$ $\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$$\Rightarrow \frac{t}{20} = \frac{\log(11.5)}{\log\left(\frac{28}{11}\right)} = 3.31$$

$\Rightarrow$ temperature $= 20 \times 3.31 = 66.2$

The temp will be 35^0C after 66.2 min.

5. **The temperature of the body drops from 100^0C to 75^0C in 10 min. When the surrounding air is at 20^0C temperature. What will be its temp after half an hour. When will the temperature be 25^0C .**

Sol : $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\log(\theta - 20) = -kt + \log c$

when $t=0, \theta = 100^0 \Rightarrow c=80$

when $t=10, \theta = 75^0 \Rightarrow e^{-10k} = \frac{11}{16}$.

when t =30min$\Rightarrow \theta = 20 + 80\left(\frac{1331}{4096}\right) = 46^0C$

$$\text{when } \theta = 25^0\text{C} \Rightarrow t = 10\frac{\log 5 - \log 80}{(\log 11 - \log 18)} = 74.86\text{min}$$

6. If the air is maintained at 15^0C and the temperature of the body drops from 70^0C to 40^0C in 10 minutes. What will be its temperature after 30 minutes?

Sol : If θ be the temperature of the body at time t, then $\frac{d\theta}{dt} = -k(\theta - 15)$, where k is constant

Integrating, $\int \frac{d\theta}{\theta - 15} = -k\int dt + \log c$

i.e $\log(\theta - 15) = -kt + \log c$ i.e , $\theta - 15 = ce^{-kt}$ …(1)

When $t = 0, \theta = 70^0 C$ and when $t = 10, \theta = 40^0 C$

$\therefore 70 - 15 = ce^{-0} \Rightarrow c = 55$ and $40 - 15 = ce^{-10k}$

$\Rightarrow \frac{25}{55} = e^{-10k}$ or $e^{-10k} = \frac{5}{11}$ …(2)

Then (1) becomes $\theta - 15 = 55e^{-kt}$

When t=30 min, $\theta = 15 + 55e^{-30k}$

$\therefore \theta = 15 + 55(e^{-10k})^3 = 15 + 55\left(\frac{5}{11}\right)^3$ using (2)

$$= 15 + \frac{625}{121} = \frac{2441}{121} = 20.16^0\text{C}.$$

7. In a pot of boiling water 100^0C is removed from the fire and allowed to cool at 30^0C room temperature. Two minutes later, the temperature of the water in the pot is 90^0C. What will be the temperature of the water after 5 minutes?

Sol : We have $\theta - 30^0\text{C} = ce^{-kt}$ …(1)

when $t = 0, \theta = 100 \Rightarrow c = 70$

$\therefore \theta - 30^0\text{C} = 70e^{-kt}$ …(2)

when, $t = 2, \theta = 90^0$

from (2), $60 = 70e^{-2k}$

$\Rightarrow -2k = \log\left(\frac{6}{7}\right) = -0.1542$

$\Rightarrow k = 0.0771$

when $t = 5, \theta - 30^0 = 70e^{-5k}$

$\Rightarrow \theta = 77.46^0$

8. The temperature of a cup of coffee is 92^0C when freshly poured, the room temperature being 24^0C. In one min it was cooled to 80^0C. How long a period must elapse, before the temperature of the cup becomes 65^0C.

Sol : By Newton's Law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)\ ;k>0$$

$\theta_0 = 24^0 C \Rightarrow \log(\theta - 24) = -kt + \log c$--------------(1).

When t=0; $\theta = 92 \Rightarrow$ c =68

When t =1; $\theta = 80^0 C \Rightarrow e^{-k} = \frac{68}{56}$

$\Rightarrow k = \log\frac{56}{68}$.

When $\theta = 65^0 C, t = \frac{65x41}{68^2} = 0.576\text{min}$

<u>LAW OF NATURAL GROWTH OR DECAY</u>

Statement : Let x(t) or x be the amount of a substance at time ‘ t’ and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount x(t) of a chemically changed substance is proportional to the amount of the substance available at that time

$$\frac{dx}{dt}\alpha x$$

Note: a) In case of Natural growth we take

$$\frac{dx}{dt} = kx \quad (k > 0)$$

b) In case of Natural decay, we take $\frac{dx}{dt} = -kx$ (k>0)

where k is a constant of proportionality

RATE OF DECAY OF RADIO ACTIVE MATERIALS

Statement : The disintegration at any instant is proportional to the amount of material present in it.

If u is the amount of the material at any time ‘t’ , then $\frac{du}{dt} = -\ ku$, where k is any constant (k >0). i.e Law of Natural Decay is applied.

Solved Problems

1. **The number N of bacteria in a culture grew at a rate proportional to N . The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hrs .**

Sol : The differential equation to be solved is $\frac{dN}{dt} = kN$

$$\Rightarrow \frac{dN}{N} = kdt$$

$$\Rightarrow \int \frac{dN}{N} = \int kdt$$

$$\Rightarrow \log N = kt + \log c$$

$$\Rightarrow N = c\,e^{-kt} \text{ ------------(1).}$$

When t= 0sec, N =100 $\Rightarrow$ 100 = c $\Rightarrow$ c =100

When t =3600sec, N =332 $\Rightarrow$ 332 =100 e^{3600k}

$$\Rightarrow e^{3600k} = \frac{332}{100}$$

Now when $t = \frac{3}{2}$ hours = 5400 sec then N=100 e^{5400k}

$$\Rightarrow N = 100\left[e^{3600k}\right]^{\frac{3}{2}}$$

$$\Rightarrow N = 100\left[\frac{332}{100}\right]^{\frac{3}{2}} = 605.$$

$$\Rightarrow N = 605.$$

2. **A bacterial culture, growing exponentially, increases from 100 to 400 gms in 10 hrs. How much was present after 3 hrs, from the initial instant?**

Sol : Let N be the weight of bacteria culture at any $t > 0$.

Then $N = ce^{-kt}$ …(1)

By data, when $t = 0, N = 100g$

$\therefore 100 = c$

Substituting in (1), we get $N = 100e^{-kt}$ …(2)

When $t = 100, N = 400g$

from (2), $400 = 100e^{-10k}$

$\Rightarrow 4 = e^{-10k}$

$\Rightarrow -10k = \log 4$

$\Rightarrow k = -\frac{1}{10}\log 2^2 = -\frac{1}{5}\log(2)$ …(3)

When $t = 3, N = 100e^{-3k}$

$$= 100e^{-3\left(-\frac{1}{5}\log 2\right)} = 100e^{(\log 2)^{\frac{3}{5}}}$$

$$= 100X(2)^{\frac{3}{5}} = 100X8^{\frac{1}{5}} = 100X1.414$$

$$= 141.4gms$$

3.If a radioactive Carbon-14 has a half life of 5750 years, what will remain of one gram after 3000years?

Sol : Let mass of radioactive Carbon-14 at any time be denoted by $x(t)$.

Then it is known that $\frac{dx}{dt} = -kt$ where k is a constant

$\Rightarrow x = Ae^{-kt}$ where A is also a constant.

It is known that at t=0, we have 1gm of Carbon-14

$\therefore 1 = Ae^{0} \Rightarrow A = 1$

$\therefore x = e^{-kt}$

However when t=5750 years, we have 1/2gm of Carbon-14.

$$\therefore \frac{1}{2} = e^{-k(5750)} \Rightarrow k = \frac{1}{5750}\log 2$$

Suppose t=3000years, we have to find x.

$$\therefore x = e^{-kt} = e^{-3000k} = e^{-\frac{3000}{5750}\log 2}$$

$$\Rightarrow x = (2)^{-\frac{3000}{5750}gms}$$

4. If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear?

Sol : The differential equation of the diffusing radioactive material is,

$$\frac{dm}{dt} = -km \qquad \ldots(1)$$

Separating the variables and integrating, we get

$$m = ce^{-kt} \qquad \ldots(2)$$

When $t = 0$, let $m = m_1$

$$\Rightarrow m_1 = c \qquad \ldots(3)$$

By data, when $t = 10, m = \frac{70m}{100}$

$$\Rightarrow \frac{70m_1}{100} = ce^{-10k} = m_1e^{-10k}$$

$$\Rightarrow e^{-10k} = \frac{7}{10} \Rightarrow k = -\frac{1}{10}\log\left(\frac{7}{10}\right)$$

$$\therefore k = \frac{1}{10}\log\left(\frac{10}{7}\right) \qquad \ldots(4)$$

Required time at t is

$$\frac{10m_1}{100} = ce^{-kt} = m_1 e^{-kt} \Rightarrow \frac{1}{10} = e^{-kt}$$

$$\Rightarrow t = \frac{1}{k}\log(10)$$

$$= \frac{10\log(10)}{\log 10 - \log 7} = 64.5 \text{ days.}$$

II .LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

Definition: An equation of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n(x)y = Q(x)$

where $P_1(x), P_2(x), P_3(x) \ldots P_n(x)$ and $Q(x)$ (functions of x) are continuous is called a linear differential equation of order n.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Def: An equation of the form $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_n y = Q(x)$ where $P_1, P_2, P_3 \ldots P_n$, are real constants and Q(x) is a continuous function of x is called an linear differential equation of order ‘ n’ with constant coefficients.

Note:

1. Operator $D = \frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$; …………………… $D^n = \frac{d^n}{dx^n}$

$$D\,y = \frac{dy}{dx};\ D^2 y = \frac{d^2 y}{dx^2}; \ldots\ldots\ldots\ldots\ D^n y = \frac{d^n y}{dx^n}$$

2. Operator $\frac{1}{D}Q = \int Q dx$ i e $D^{-1}Q$ is called the integral of Q.

<u>To find the general solution of f(D).y = 0</u> :

Here $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n$ is a polynomial in D.

Now consider the auxiliary equation: f(m) = 0

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0$

where $P_1, P_2, P_3 \ldots P_n$ are real constants.

Let the roots of f(m) =0 be $m_1, m_2, m_3 \ldots m_n$.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E $f(m)=0$	Complementary function(C.F)
1.	$m_1, m_2, ..m_n$ are real and distinct.	$y_c = c_1e^{m_1x} + c_2e^{m_2x} + \ldots c_ne^{m_nx}$
2.	$m_1, m_2, ..m_n$ and two roots are equal i.e., m_1, m_2 are equal and real (i.e repeated twice) &the rest are real and different.	$y_c = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \ldots c_ne^{m_nx}$
3.	$m_1, m_2, ..m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real (i.e repeated thrice) &the rest are real and different.	$y_c = (c_1 + c_2x + c_3x^2)e^{m_1x} + c_4e^{m_4x} + \ldots c_ne^{m_nx}$
4.	Two roots of A.E are complex say α+i β α -iβ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \ldots c_ne^{m_nx}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x\right] + c_5e^{m_5x} + \ldots c_ne^{m_nx}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2x + c_3x^2)\cos\beta x + (c_4 + c_5x + c_6x^2)\sin\beta x\right]$ $+c_7e^{m_7x} + \ldots c_ne^{m_nx}$
7.	If roots of A.E.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$

	irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	

Solved Problems

1. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol : Given equation is of the form f(D).y = 0

Where $f(D) = (D^3 - 3D + 2)y = 0$

Now consider the auxiliary equation f (m) = 0

$f(m) = (m^3 - 3m + 2)y = 0 \Rightarrow$ (m-1) (m-1) (m+2) = 0

$\Rightarrow$ m =1, 1, -2

Since m_1 and m_2 are equal and m_3 is -2

We have $y_c = (c_1 + c_2)e^x + c_3e^{-2x}$

2. Solve $(D^4-2D^3-3D^2+4D+4)y = 0$

Sol : Given f(D) $= (D^4 - 2D^3 - 3D^2 + 4D + 4)\, y = 0$ …(1)

Auxiliary equation is f(m)=0

$\Rightarrow m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$ …(2)

By inspection m+1 is its factor.

$(m+1)(m^3 - 3m^2 + 4) = 0$ …(3)

By inspection m+1 is factor of $(m^3 - 3m^2 + 4)$.

$\therefore (3)$ is $(m+1)(m+1)(m^2 - 4m + 4) = 0$

$\Rightarrow (m+1)^2(m-2)^2 = 0$

$\Rightarrow m = -1, -1, 2, 2$

Hence general solution of (1) is

$y = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^{2x}$

3.Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol : Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8m^2 + 16) = 0$

⇨ $(m^2 + 4)^2 = 0$

⇨ $(m+2i)^2\ (m+2i)^2 = 0$

⇨ m= 2i ,2i , -2i , -2i

Here roots are complex and repeated

Hence general solution is

$y_c = [(c_1+c_2x)\cos 2x + (c_3+c_4x)\sin 2x)]$

4.Solve $y^{11}+6y^1+9y = 0$; $y(0) = -4$, $y^1(0) = 14$

Sol : Given equation is $y^{11}+6y^1+9y = 0$

Auxiliary equation f(D) y = 0 ⇒ $(D^2 +6D +9)\ y = 0$

A.equation f(m) = 0 ⇒ $(m^2 +6m +9) = 0$

⇒ m = -3 ,-3

$y_c = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to x ⇒ $y^1 =(c_1+c_2x)(-3e^{-3x}) + c_2(e^{-3x})$

Given $y_1\ (0) =14$ ⇒ $c_1 = -4$ & $c_2 =2$

Hence we get $y =(-4 + 2x)\ (e^{-3x})$

5.Solve $4y^{111} + 4y^{11} +y^1 = 0$

Sol : Given equation is $4y^{111} + 4y^{11} +y^1 = 0$

That is $(4D^3+4D^2+D)y=0$

Auxiliary equation f(m) = 0

$4m^3 +4m^2 + m = 0$

$m(4m^2 +4m + 1) = 0$

$m\,(2m+1)^2 = 0$

m = 0 , -1/2 ,-1/2

$y =c_1 + (c_2 + c_3x)\ e^{-x/2}$

6.Solve $(D^2 - 3D +4)\ y = 0$

Sol : Given equation $(D^2 - 3D +4)\ y = 0$

A.E. f(m) = 0

$m^2-3m + 4 = 0$

$$m = \frac{3\pm\sqrt{9-16}}{2} = \frac{3\pm i\sqrt{7}}{2}$$

$$\alpha \pm i\beta = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}}\ (c_1\cos\frac{\sqrt{7}}{2}x + c_2\sin\frac{\sqrt{7}}{2}x)$$

<u>**To Find General solution of f(D) y = Q(x)**</u>

It is given by $y = y_c + y_p$

i.e. y = C.F+P.I

Where the P.I consists of no arbitrary constants and P.I of f (D) y = Q(x)

Is evaluated as $\quad P.I = \frac{1}{f(D)} Q(x)$

Depending on the type of function of Q(x) , P.I is evaluated .

1. Find $\frac{1}{D}(x^2)$

Sol : $\frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3}$

2. **Find Particular value of** $\frac{1}{D+1}(x)$

Sol : $\frac{1}{D+1}(x) = e^{-x}\int xe^{x}dx$ (By definition)

$= e^{-x}(xe^{x} - e^{x})$

$= x-1$

General methods of finding Particular integral :

P.I of f(D)y=Q(x), when $\frac{1}{f(D)}$ is expressed as partial fractions**.**

Q. Solve $(D^2 + a^2)y = \sec ax$

Sol : Given equation is …(1)

Let $f(D) = D^2 + a^2$

The AE is f(m) = 0 i.e $m^2 + a^2 = 0$ …(2)

The roots are m= -ai , -ai

$y_c = c_1\cos ax + c_2\sin ax$

$$y_p = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\sec ax \quad \ldots(3)$$

$$\frac{1}{D-ai}\sec ax = e^{iax}\int \sec ax\,dx = e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx$$

$$= e^{iax}\int (1 - i\tan ax)dx = e^{iax}\left[x + \frac{i}{a}\log\cos ax\right] \ldots(4)$$

Similarly we get $\frac{1}{D+ai}\sec ax = e^{-iax}\left[x - \frac{i}{a}\log\cos ax\right]$ ...(5)

From (3),(4) and (5), we get

$$y_p = \frac{1}{2ai}\left[e^{iax}\left\{x+\frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x-\frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{x(e^{iax}-e^{-iax})}{2ai} + \frac{1}{a^2}(\log\cos ax)\frac{(e^{iax}+e^{-iax})}{2}$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

RULES FOR FINDING P.I IN SOME SPECIAL CASES;

Type 1. P.I of f (D)y=Q(x) where Q(x) $=e^{ax}$, where ‘a’ is constant.

Case1. $P.I = \frac{1}{f(D)}.Q(x) = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

provided $f(a) \neq 0$

i.e In f(D), put D=a and Particular integral will be calculated.

Case 2: If f(a)=0 then the above method fails. Then if $f(D) = (D-a)^k \phi(D)$ (i.e ‘a’ is repeated root k times).

Then $P.I = \frac{1}{\phi(a)}e^{ax}.\frac{1}{k!}x^k$ provided $\phi(a) \neq 0$

Type 2. P.I of $f(D)y = Q(x)$ where Q(x) = sinax or Q(x) = cosax where ‘a’ is constant then P.I= $\frac{1}{f(D)}Q(x)$.

Case 1: In f(D) put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then $P.I = \frac{\sin ax}{f(-a)^2}$

Case 2: If $f(-a^2)=0$ then $D^2 + a^2$ is a factor of $\phi(D^2)$ and hence it is a factor of f(D). Then let $f(D)=(D^2 + a^2)$ $f(D) = (D^2+a^2)\phi(D^2)$.

Then $\frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\sin ax}{D^2+a^2} = \frac{1}{\phi(-a^2)}\frac{-x\cos ax}{2a}$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\cos ax}{D^2+a^2} = \frac{1}{\phi(-a^2)}\frac{x\sin ax}{2a}$$

Type 3.P.I for f(D)y=Q(x) where $Q(x)=x^k$ where k is a positive integer .f(D) can be expressed as $f(D)=[1\pm\phi(D)]$

Express $\frac{1}{f(D)}=\frac{1}{[1\pm\phi(D)]}=[1\pm\phi(D)]^{-1}$

Hence $P.I=\frac{1}{[1\pm\phi(D)]}Q(x)$

$=[1\pm\phi(D)]^{-1}x^k$

Type 4.P.I of f(D)y=Q(x) when $Q(x)=e^{ax}V$ where 'a' is a constant and V is function of x. where V =sinax or cosax or x^k

Then $P.I=\frac{1}{f(D)}Q(x)$

$=\frac{1}{f(D)}e^{ax}V$

$=e^{ax}\left[\frac{1}{f(D+a)}V\right]$ & $\frac{1}{f(D+a)}V$ is evaluated depending on V.

Type 5. P.I of f(D)y=Q(x) when Q(x)=xV where V is a function of x.

Then $P.I=\frac{1}{f(D)}Q(x)$

$=\frac{1}{f(D)}V$

$=\left[x-\frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V$

Type 6. P.I. of f(D)y = Q(x) where $Q(x)=x^m v$ where v is a function of x.

When $P.I.=\frac{1}{f(D)}\times Q(x)=\frac{1}{f(D)}x^m v$, where v = cosax or sinax

i. $P.I.=\frac{1}{f(D)}x^m\sin ax=I.P.\text{of}\ \frac{1}{f(D)}x^m e^{iax}$

ii. $P.I.=\frac{1}{f(D)}x^m\cos ax=R.P.\text{of}\ \frac{1}{f(D)}x^m e^{iax}$

Formulae

1. $\frac{1}{1-D}=(1-D)^{-1}=1+D+D^2+D^3+$ ------------------
2. $\frac{1}{1+D}=(1+D)^{-1}=1-D+D^2-D^3+$ ------------------

3. $\frac{1}{(1-D)^2} = (1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 +$ ------------------

4. $\frac{1}{(1+D)^2} = (1 + D)^{-2} = 1 - 2D + 3D^2 - 4D^3 +$ ------------------

5. $\frac{1}{(1-D)^3} = (1 - D)^{-3} = 1 + 3D + 6D^2 + 10D^3 +$ ------------------

6. $\frac{1}{(1+D)^3} = (1 + D)^{-3} = 1 - 3D + 6D^2 - 10D^3 +$ ------------------

Solved Problems

1. Solve $(4D^2 - 4D + 1)y = 100$

Sol : A.E is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{-1}{2}$

$$C.F = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$\text{Now P.I} = \frac{100}{4D^2 - 4D + 1} = \frac{100e^{0x}}{(2D-1)^2} = \frac{100}{(0-1)^2} = 100 \ \{ \text{since } 100e^{0x} = 100 \}$$

Hence the general solution is $y = C.F + P.F = (c_1 + c_2 x)e^{\frac{x}{2}} + 100$

2. Solve the differential equation $(D^2 + 4)y = \sinh 2x + 7$.

Sol : Auxillary equation is $m^2 + 4 = 0$

$$\Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$$

$\therefore$ C.F is $y_c = c_1\cos 2x + c_2\sin 2x$ ...(1)

To find P.I :

$$y_p = \frac{1}{D^2+4}(\sinh 2x + 7)$$

$$= \frac{1}{D^2+4}\left(\frac{e^{2x}+e^{-2x}}{2} + 7e^0\right)$$

$$= \frac{1}{2}\cdot\frac{e^{2x}}{D^2+4} + \frac{1}{2}\frac{e^{-2x}}{D^2+4} + 7\frac{e^0}{(D^2+4)}$$

$$= \frac{e^{2x}}{2(4+4)} + \frac{e^{-2x}}{2(4+4)} + \frac{7}{(0+4)}$$

$$= \frac{e^{2x}+e^{-2x}}{16} + \frac{7}{4} = \frac{1}{8}\sinh 2x + \frac{7}{4} \qquad \ldots(2)$$

$$y = y_c + y_p$$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{8}\sinh 2x + \frac{7}{4}$$

3. Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol : The given equation is

$$(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x \qquad \ldots(1)$$

This is of the form $f(D)y = e^{-2x} + 2\sinh x$

A.E is $f(m) = 0 \Rightarrow (m+2)(m-1)^2 = 0 \therefore m = -2,1,1$

The roots are real and one root is repeated twice.

$\therefore$ C.F is $y_c = c_1e^{-2x} + (c_2 + c_3x)e^x$.

$$P.I = \frac{e^{-2x} + 2\sinh x}{(D+2)(D-1)^2} = \frac{e^{-2x} + e^x - e^{-x}}{(D+2)(D-1)^2} = y_{p_1} + y_{p_2} + y_{p_3}$$

Now $y_{p_1} = \dfrac{e^{-2x}}{(D+2)(D-1)^2}$

Hence $f(-2) = 0$. Let $f(D) = (D-1)^2$. Then $\phi(2) \neq 0$ and $m = 1$

$$\therefore y_{p_1} = \frac{e^{-2x}x}{9} = \frac{xe^{-2x}}{9}$$

and $y_{p_2} = \dfrac{e^x}{(D+2)(D-1)^2}$. Here f(1)=0

$$= \frac{e^x x^2}{(3)2!} = \frac{x^2e^x}{6}$$

and $y_{p_3} = \dfrac{e^{-x}}{(D+2)(D-1)^2}$

Putting D=-1, we get $y_{p_3} = \dfrac{e^{-x}}{(1)(-2)^2} = \dfrac{e^{-x}}{4}$

$\therefore$ The general solution is $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

i.e $y = c_1e^{-2x} + (c_c + c_3x)e^x + \dfrac{xe^{-2x}}{9} + \dfrac{x^2e^x}{6} - \dfrac{e^{-x}}{4}$

4. Solve the differential equation $(D^2 + D + 1)y = \sin 2x$.

Sol : A.E is $m^2 + m + 1 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore y_c = e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2} + c_2\sin\frac{x\sqrt{3}}{2}\right) \quad \ldots(1)$$

To find P.I :

$$y_p = \frac{\sin 2x}{D^2 + D + 1} = \frac{\sin 2x}{-4 + D + 1}$$

$$= \frac{\sin 2x}{D-3} = \frac{(D+3)\sin 2x}{D^2 - 9} = \frac{(D+3)\sin 2x}{-4-9}$$

$$= \frac{D\sin 2x + 3\sin 2x}{-13} = \frac{2\cos 2x + 3\sin 2x}{-13}$$

$$\therefore y = y_c + y_p = e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2} + c_2\sin\frac{x\sqrt{3}}{2}\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$$

5. Solve $(D^2 - 4)y = 2\cos^2 x$

Sol : Given equation is $(D^2 - 4)y = 2\cos^2 x$ …(1)

Let $f(D) = D^2 - 4$ A.E is $f(m) = 0$ i.e $m^2 - 4 = 0$

The roots are m=2,-2. The roots are real and different.

$\therefore C.F = y_c = c_1e^{2x} + c_2e^{-2x}$

$$P.I = y_p = \frac{1}{D^2 - 4}(2\cos^2 x) = \frac{1}{D^2 - 4}(1 + \cos 2x)$$

$$= \frac{e^{0x}}{D^2 - 4} + \frac{\cos 2x}{D^2 - 4} = P.I_1 + P.I_2$$

$$P.I_1 = y_{p_1} = \frac{e^{0x}}{D^2 - 4} \text{ [Put D=0] } = \frac{e^{0x}}{-4} = -\frac{1}{4}$$

$$P.I_2 = y_{p_2} = \frac{\cos 2x}{D^2 - 4} = \frac{\cos 2x}{-8} \text{ [Put } D^2 = -2^2 = -4 \text{]}$$

$\therefore$ The general solution of (1) is $y = y_c + y_{p_1} + y_{p_2}$

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{\cos 2x}{8}$$

6. Solve $(D^2 + 1)y = \sin x \sin 2x$

Sol : Given D.E is $(D^2 + 1)y = \sin x \sin 2x$

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

The roots are complex conjugate numbers.

C.F is $y_c = c_1\cos x + c_2\sin x$

w.k.t 2sinAsinB=cos(A-B)–cos(A+B)

$$P.I = \frac{\sin x \sin 2x}{(D^2 + 1)} = \frac{1}{2}\frac{\cos x - \cos 3x}{(D^2 + 1)} = P.I_1 + P.I_2$$

Now $P.I_1 = \frac{1}{2}\frac{\cos x}{D^2 + 1}$

Put $D^2 = -1$ we get $D^2 + 1 = 0$

$$\therefore P.I_1 = \frac{1}{2}\frac{x\sin x}{2} = \frac{x\sin x}{4} \qquad \left[\because \text{Case of failure} : \frac{\cos ax}{D^2 + a} = \frac{x}{2a}\sin ax\right]$$

and $P.I_2 = -\frac{1}{2}\frac{\cos 3x}{D^2 + 1}$

Put $D^2 = -9$, we get

$$P.I_2 = -\frac{1}{2}\frac{\cos 3x}{-9 + 1} = \frac{\cos 3x}{16}$$

General solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1\cos x + c_2\sin x + \frac{x\sin x}{4} + \frac{\cos 3x}{16}$$

7. Solve the differential equation $(D^3 - 3D^2 - 10D + 24)y = x + 3$.

Sol : The given D.E is $(D^3 - 3D^2 - 10D + 24)y = x + 3$

A.E is $m^3 - 3m^2 - 10m + 24 = 0$

$\Rightarrow$m=2 is a root.

The other two roots are given by $m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m=2$ (or) $m = -1$

One root is real and repeated, other root is real.

C.F is $y_c = e^{2x}(c_1 + c_2 x) + c_3 e^{-x}$

$$y_p = \frac{x+3}{(D^3 - 3D^2 - 10D + 24)} = \frac{1}{24}\frac{x^3+3}{1+\left(\dfrac{D^3-3D^2-10D}{24}\right)}$$

$$= \frac{1}{24}\left[\frac{1+D^3-3D^2-10D}{24}\right]^{-1}(x+3)$$

$$= \frac{1}{24}\left[1-\left(\frac{D^3-3D^2-10D}{24}\right)\right](x+3)$$

$$= \frac{1}{24}\left[x+3+\frac{10}{24}\right] = \frac{24x+82}{576}$$

General solution is $y = y_c + y_p$

$$\Rightarrow y = e^{2x}(c_1 + c_2 x) + c_3 e^{-x} + \frac{24x+82}{576}$$

8. Solve the differential equation $(D^2 - 4D + 4)y = e^{2x} + x^2 + \sin 3x$.

Sol : The A.E is $(m^2 - 4m + 4) = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore y_c = (c_1 + c_2 x)e^{2x}$ …(1)

To find y_p : $y_p = \dfrac{1}{D^2 - 4D + 4}(e^{2x} + x^2 + \sin 3x)$

$$= \frac{e^{2x}}{(D-2)^2} + \frac{x^2}{(D-2)^2} + \frac{\sin 3x}{D^2 - 4D + 4}$$

$$= \frac{x^2}{2!}e^{2x} + \frac{x^2}{4\left(1-\dfrac{D}{2}\right)^2} + \frac{\sin 3x}{-9-4D+4}$$

$$= \frac{x^2}{2}e^{2x} + \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2 - \frac{(4D-5)\sin 3x}{(5+4D)}$$

$$= \frac{x^2}{2}e^{2x} + \frac{1}{4}\left(1+\frac{2D}{2}+\frac{3D^2}{4}\right)x^2 - \frac{(4D-5)\sin 3x}{16D^2-25}$$

$$= \frac{x^2}{2}e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} - \frac{(12\cos 3x - 5\sin 3x)}{-144-25}$$

$$= \frac{x^2}{2}e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} + \frac{(12\cos 3x - 5\sin 3x)}{169} \qquad \ldots(2)$$

$$y = y_c + y_p = (c_1 + c_2 x)e^{2x} + = \frac{x^2}{2}e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} + \frac{(12\cos 3x - 5\sin 3x)}{169}$$

9. Solve the differential equation $(D^2 + 4)y = x\sin x$.

Sol : Auxiliary equation is $m^2+4-0 \Rightarrow m^2=(2i)^2$

$\therefore m=\pm 2i.$ The roots are complex and conjugate.

Hence Complementary Function, $y_c=c_1\cos 2x+c_2\sin 2x$

Particular integral, $y_p=\dfrac{1}{D^2+4}x\sin x$

$$=\text{I.P of }\frac{1}{D^2+4}xe^{ix}$$

$$=\text{I.P of}e^{ix}\frac{1}{(D+i)^2+4}x=\text{I.P of }e^{ix}\frac{1}{D^2+2Di+3}x$$

$$=\text{I.P of }\frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1}x$$

$$=\text{I.P of }\frac{e^{ix}}{3}\left(1-\frac{D^2+2Di}{3}+\ldots\right)x$$

$$=\text{I.P of }\frac{e^{ix}}{3}\left(1-\frac{2}{3}Di\right)x\left[D^2(x)=0,\text{etc}\right]$$

$$=\text{I.P of }\frac{1}{3}(\cos x+i\sin x)\left(x-i\frac{2}{3}\right)$$

$$=\frac{1}{3}\left(-\frac{2}{3}\cos x+x\sin x\right)$$

Hence the general solution is

$$y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}\left(x\sin x-\frac{2}{3}\cos x\right)$$

where c_1 and c_2 are constants.

Other Method (using type 5): $y_p=\dfrac{1}{D^2+4}x\sin x$

$$=\left\{x-\frac{2D}{D^2+4}\right\}\frac{\sin x}{D^2+4}$$

$$=\frac{x\sin x}{3}-\frac{2(D\sin x)}{3(D^2+4)}$$

$$=\frac{x\sin x}{3}-\frac{2\cos x}{9}$$

Hence the general solution is

$$y=y_c+y_p=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}\left(x\sin x-\frac{2}{3}\cos x\right)$$

10. Solve the Differential equation$(D^2+5D+6)y=e^x$

Sol : Given equation is $(D^2+5D+6)y=e^x$

Here $Q(x)=e^{x}$

Auxiliary equation is $f(m)=m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

m=-2 or m=-3

The roots are real and distinct

C.F=y_c=$c_1e^{-2x}+c_2e^{-3x}$

Particular Integral=y_p=$\frac{1}{f(D)}Q(x)$

$$=\frac{1}{D^2+5D+6}e^x=\frac{1}{(D+2)(D+3)}e^x$$

Put D = 1 in f(D)

P.I=$\frac{1}{(3)(4)}e^x$

Particular Integral = y_p = $\frac{1}{12}e^x$

General solution is $y=y_c+y_p$

$$y=c_1e^{-2x}+c_2e^{-3x}+\frac{e^x}{12}$$

11. Solve $y''-4y'+3y=4e^{3x}, y(0)=-1, y'(0)=3$

Sol : Given equation is $y''-4y'+3y=4e^{3x}$

i.e $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=4e^{3x}$ it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x)=4e^{3x}$; $f(D)=D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3=0$

$m^2-3m-m+3=0$

$m(m-3)-1(m-3)=0 \Longrightarrow m=3$ or 1

The roots are real and distinct.

C.F= $y_c=c_1e^{3x}+c_2e^x$

P.I= $y_p=\frac{1}{f(D)}Q(x)$

$$=\frac{1}{D^2-4D+3}4e^{3x}$$

$$=\frac{1}{(D-1)(D-3)}4e^{3x}$$

Put D=3

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x'}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y = y_c+y_p$

$$y = c_1e^{3x} + c_2e^{x} + 2xe^{3x} \qquad \ldots(3)$$

Equation (3) differentiating with respect to 'x'

$$y' = 3c_1e^{3x} + c_2e^{x} + 2e^{3x} + 6xe^{3x} \qquad \ldots(4)$$

By data, $y(0) = -1$, $y^1(0)=3$

From (3), $-1=c_1+c_2$ …(5)

From (4), $3=3c_1+c_2+2$

$3c_1+c_2=1$ …(6)

Solving (5) and (6) we get $c_1=1$ and $c_2 = -2$

$y=-2e^x + (1+2x)\, e^{3x}$

12. **Solve $y''+4y'+4y = 4\cos x + 3\sin x, y(0) = 0, y'(0) = 0$**

Sol : Given differential equation in operator for

$$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$$

A.E is $m^2+4m+4 = 0$

$(m+2)^2=0$ then $m=-2, -2$

$\therefore$ C.F is $y_c = (c_1 + c_2x)e^{-2x}$

P.I is= $y_p = \frac{4\cos x + 3\sin x}{(D^2 + 4D + 4)}$ put $D^2 = -1$

$$y_p = \frac{4\cos x + 3\sin x}{(4D+3)} = \frac{(4D-3)(4\cos x + 3\sin x)}{(4D-3)(4D+3)}$$

$$= \frac{(4D-3)(4\cos x + 3\sin x)}{16D^2 - 9}$$

$$y_p = \frac{(4D-3)(4\cos x + 3\sin x)}{-16-9}$$

$$= \frac{-16\sin x + 12\cos x - 12\cos x - 9\sin x}{-25} = \frac{-25\sin x}{-25} = \sin x$$

$\therefore$ General equation is $y=y_c+y_p$

$$y = (c_1 + c_2x)e^{-2x} + \sin x \qquad \ldots(1)$$

By given data $y(0) = 0$, $c_1 = 0$ and

Differentiating (1) w.r.t 'x', $y' = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x$ …(2)

given $y'(0) = 0$

Substitute in (2) $\Rightarrow$ $-2c_1 + c_2+1=0$ $\therefore c_2 = -1$

$\therefore$ Required solution is $y = -xe^{-2x} + \sin x$

13. Solve $(D^2+9)y = \cos 3x$

Sol : Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore m = \pm 3i$

$y_c = C.F = c_1\cos 3x + c_2\sin 3x$

$$y_p = P.I = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$$

$$= \frac{x}{2(3)}\sin 3x = \frac{x}{6}\sin 3x$$

General equation is $y=y_c+y_p$

$$y = c_1\cos 3x + c_2\sin 3x + \frac{x}{6}\sin 3x$$

14. Solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$

Sol : Given equation can be written as

$(D^3 + 2D^2 - D - 2)y = 1 - 4x^3$

A.E is $m^3 + 2m^2 - m - 2 = 0$

$(m^2 - 1)(m+2) = 0$

$m^2 = 1$ or $m = -2$

$m = 1, -1, -2$

C.F$= c_1e^x + c_2e^{-x} + c_3e^{-2x}$

$$P.I = \frac{1}{(D^3+2D^2-D-2)}(1-4x^3) = \frac{-1}{2\left[1-\frac{(D^3+2D^2-D)}{2}\right]}(1-4x^3)$$

$$= \frac{-1}{2}\left[1-\frac{(D^3+2D^2-D)}{2}\right]^{-1}(1-4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+\ldots\right](1-4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{1}{2}\left(D^3+2D^2-D\right)+\frac{1}{4}\left(D^2-4D^3\right)+\frac{1}{8}\left(-D^3\right)\right]\left(1-4x^3\right)$$

$$=\frac{-1}{2}\left[1-\frac{5}{8}D^3+\frac{5}{4}D^2-\frac{1}{2}D\right](1-4x^3)$$

$$=\frac{-1}{2}\left[(1-4x^3)-\frac{5}{8}(-24)+\frac{5}{4}(-24x)-\frac{1}{2}(-12x^2)\right]$$

$$=\frac{-1}{2}\left[-4x^3+6x^2-30x+16\right]$$

$$=[2x^3-3x^2+15x-8]$$

The general solution is

y= C.F + P.I

$$y=c_1e^x+c_2e^{-x}+c_3e^{-2x}+[2x^3-3x^2+15x-8]$$

15. Solve $(D^3-7D^2+14D-8)y=e^x\cos 2x$

Sol : Given equation is

$$(D^3-7D^2+14D-8)y=e^x\cos 2x$$

A.E is $(m^3-7m^2+14m-8)=0$

(m-1)(m-2)(m-4)=0

Then m=1,2,4

C.F= $c_1e^x+c_2e^{2x}+c_3e^{4x}$

$$\text{P.I}=\frac{e^x\cos 2x}{(D^3-7D^2+14D-8)}$$

$$=e^x\frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}\cos 2x$$

$$\left[\because \text{P.I}=\frac{1}{f(D)}e^{ax}v=e^{ax}\frac{1}{f(D+a)}v\right]$$

$$=e^x\frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$=e^x\frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$=e^x\frac{1}{(-4D+3D+16)}\cos 2x \text{ (Replacing } D^2 \text{ with } -2^2)$$

$$=e^x\frac{1}{(16-D)}\cos 2x$$

$$=e^x\frac{16+D}{(16-D)(16+D)}\cos 2x$$

$$= e^x \frac{16+D}{256-D^2}\cos 2x$$

$$= e^x \frac{16+D}{256-(-4)^2}\cos 2x$$

$$= \frac{e^x}{260}(16\cos 2x - 2\sin 2x)$$

$$= \frac{2e^x}{260}\left(8\cos 2x - \sin 2x\right)$$

$$= \frac{e^x}{130}\left(8\cos 2x - \sin 2x\right)$$

General solution is $y = y_c + y_p$

$$y = c_1e^x + c_2e^{2x} + c_3e^{4x} + \frac{e^x}{130}\left(8\cos 2x - \sin 2x\right)$$

16. Solve $(D^2 - 4D + 4)y = x^2\sin x + e^{2x} + 3$

Sol : Given $(D^2 - 4D + 4)y = x^2\sin x + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2$=0 then m=2,2

C.F= $(c_1 + c_2x)e^{2x}$

$$\text{P.I}= \frac{x^2\sin x + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2\sin x) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$$

$$\text{Now } \frac{1}{(D-2)^2}(x^2\sin x) = \frac{1}{(D-2)^2}(x^2) \quad (\text{I.P of } e^{ix})$$

$$= \text{I.P of } \frac{1}{(D-2)^2}(x^2)e^{ix}$$

$$= \text{I.P of } (e^{ix})\frac{1}{(D+i-2)^2}(x^2)$$

$$\text{I.P of } (e^{ix})\frac{1}{(D+i-2)^2}(x^2)$$

On simplification, we get

$$\frac{1}{(D+i-2)^2}(x^2\sin x) = \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right]$$

$$\text{and } \frac{1}{(D-2)^2}e^{2x} = \frac{x^2}{2}e^{2x},$$

$$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$$

$$\text{P.I}= \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$$

$y = y_c + y_p$

$$y = (c_1 + c_2x)e^{2x} + \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$$

Linear equations of second order with variable coefficients

An equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, where P(x), Q(x), R(x) are real valued functions of 'x'is called linear equation of second order with variable coefficients.

Variation of Parameters :

This method is applied when P,Q in above equation are either functions of 'x' or real constants but R is a function of 'x'.

Working Rule :

1. Find C.F. Let C.F= $y_c=c_1u(x)+c_2u(x)$
2. Take P.I= y_p=Au+Bv where A= $-\int\frac{vRdx}{uv'-vu'}$ and $B=\int\frac{uRdx}{uv'-vu'}$
3. Write the G.S. of the given equation $y=y_c+y_p$

1. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+y=\text{cosec}x$

Sol : Given equation in the operator form is $(D^2+1)y=\text{cosec}x$ …(1)

A.E is $(m^2+1)=0$

$\therefore m=\pm i$

The roots are complex conjugate numbers.

C.F is $y_c=c_1\cos x+c_2\sin x$

Let y_p= A cosx+ B sinx be P.I. of (1)

$u\frac{dv}{dx}-v\frac{du}{dx}=\cos^2x+\sin^2x=1$

A and B are given by

$$A=-\int\frac{vRdx}{uv'-vu'}=-\int\frac{\sin x\,\text{cosec}x}{1}dx=-\int dx=-x$$

$$B=\int\frac{uRdx}{uv^1-vu^1}=\int\cos x.\text{cosec}x dx=\int\cot x dx=\log(\sin x)$$

$\therefore y_p=-x\cos x+\sin x.\log(\sin x)$

$\therefore$ General solution is y= y_c+y_p.

$y=c_1\cos x+c_2\sin x-x\cos x+\sin x.\log(\sin x)$

2. Solve $(D^2-2D+2)y=e^x\tan x$ by method of variation of parameters.

Sol : A.E is $m^2-2m+2=0$

$$\therefore m=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm i2}{2}=1\pm i$$

We have $y_c = e^x(c_1\cos x + c_2\sin x) = c_1e^x\cos x + c_2e^x\sin x$
$= c_1(u) + c_2(u)$

where $u = e^x\cos x, v = e^x\sin x$

$$\frac{du}{dx} = e^x(-\sin x) + e^x\cos x, \frac{dv}{dx} = e^x\cos x + e^x\sin x$$

$$u\frac{dv}{dx} - v\frac{du}{dx} = e^x\cos x(e^x\cos x + e^x\sin x) - e^x\sin x(e^x\cos x - e^x\sin x)$$

$$= e^{2x}(\cos^2 x + \cos x\sin x - \sin x\cos x + \sin^2 x) = e^{2x}$$

Using variation of parameters,

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} = -\int \frac{e^x\tan x}{e^{2x}}(e^x\sin x)dx$$

$$= -\int \tan x\sin x dx = \int\left(\frac{\sin^2 x}{\cos x}dx\right) = \int\frac{(1-\cos^2 x)}{\cos x}dx$$

$$= \int(\sec x - \cos x)dx = \log(\sec x + \tan x) - \sin x$$

$$B = \int\frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$$

$$= \int\frac{e^x\cos x.e^x\tan x}{e^{2x}}dx = \int \sin x dx = -\cos x$$

General solution is given by $y = y_c + Au + Bv$

i.e $y = c_1e^x\cos x + c_2e^x\sin x + [\log(\sec x + \tan x) - \sin x]e^x\cos x - e^x\cos x\sin x$

or $y = c_1e^x\cos x + c_2e^x\sin x + [\log(\sec x + \tan x) - 2\sin x]e^x\cos x$

3. Solve the differential equation (D^2 + 4) y = sec2x by the method of variation of parameters.

Sol. Given equation is $(D^2 + 4)\, y = \sec 2x$ …..(1)

∴ A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

The roots are complex conjugate numbers.

∴ $y_c = C.F = c_1 \cos 2x + c_2 \sin 2x$

Let $y_p = P.I = A \cos 2x + B \sin 2x$

Here $u = \cos 2x$, $v = \sin 2x$ and $R = \sec 2x$.

∴ $\frac{du}{dx} = -2 \sin 2x$ and $\frac{dv}{dx} = 2 \cos 2x$

∴ $u\frac{dv}{dx} - v\frac{du}{dx} = (\cos 2x)(2 \cos 2x) - (\sin 2x)(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x = 2(\cos^2 2x + \sin^2 2x) = 2$

A and B are given by :

$$A = -\int\frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx = -\int\frac{\sin 2x \sec 2x}{2}dx = -\frac{1}{2}\int \tan 2x\, dx = \frac{1}{2}\frac{\log|\cos 2x|}{2}$$

$$\Rightarrow A = \frac{\log|\cos 2x|}{4}$$

$$B = \int\frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx = \int\frac{\cos 2x \sec 2x}{2}dx = \frac{1}{2}\int dx = \frac{x}{2}$$

∴ $y_p = P.I = \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

∴ The general solution is given by :

$y = y_c + y_p = C.F. + P.I$

i.e., $y = c_1 \cos 2x + c_2 \sin 2x + \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

UNIT-IV

PARTIAL DIFFERENTIAL EQUATIONS

Definition:

A Differential equation involves a dependent variable and its derivatives with respect to two or more independent variables is called partial differential equation.

Ex: $x\frac{\partial z}{\partial y} + 4y\frac{\partial z}{\partial x} = 2z + 3xy$

Linear & non linear p.d.e:

If the partial derivatives of the dependent variable occur in first degree only and separately, Such a P.D.E is said to the linear P.D.E, otherwise it is said as non –linear P.D.E

Homogeneous & non homogeneous p.d.e:

A P.D.E is said to the Homogeneous if each term of the equation contains either the dependent variable or one of its derivatives. Otherwise it is said to be Non - Homogeneous

Formation & partial differential equations:

Partial Differential equations can be formed by two methods

1.By the elimination of arbitrary constants

2.By the elimination of arbitrary functions

1.By elimination of arbitrary constants

Let the given function be $f(x, y, z, a, b) = 0$ ………(1) where a and b are arbitrary constants.

To eliminate a and b, differentiating (1) partially w.r.t. *'x' and 'y'*

$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.p=0$ ………………(2) and

$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.q=0$ ………………(3)

Now eliminate the constants a and b from (1), (2) and (3). We get a partial differential equation of the first order of the form. $\phi(x, y, z, p, q)=0$

Note : 1. If the number of arbitrary constants is equal to the number of variables, a partial differential equation of first order can be obtained.

2.If the number of arbitrary constants is greater than the number of variables, a partial differential equation of order higher than one can be obtained.

Solved Problems

1. **Form the partial differential equation by eliminating the arbitrary constants $\boldsymbol{a}$ and $\boldsymbol{b}$ from (i) $\boldsymbol{z = ax + by + ab}$**

Sol: we have $z = ax + by + ab \ldots\ldots.(1)$

Differentiating (1) partially w.r.t. $'x'$ and $'y'$, we get

$\frac{\partial z}{\partial x} = a \Rightarrow p = a \ldots\ldots\ldots$ (2) and $\frac{\partial z}{\partial y} = b \Rightarrow q = b \ldots\ldots\ldots.(3)$

Putting the values of a and b from equation (2) and (3) in (1), we get

$$\boxed{\boldsymbol{z = px + qy + pq}}$$

This is the required partial differential equation

2. **Form the partial differential equation by eliminating the arbitrary constants a and b from (a)** $z = ax + by + a^2 + b^2$ (b) $z = ax + by + \frac{a}{b} - b$

Sol. (a) we have $z = ax + by + a^2 + b^2$ .... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\frac{\partial z}{\partial x} = a \Rightarrow p = a \ldots\ldots\ldots$ (2) and $\frac{\partial z}{\partial y} = b \Rightarrow q = b \ldots\ldots\ldots.(3)$

Putting the values of a and b from equation (2) and (3) in (1), we get

$$\boxed{z = px + qy + p^2 + q^2}$$

This is the required partial differential equations

(a) We have $z = ax + by + \frac{a}{b} - b \ldots\ldots\ldots\ldots.(1)$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\frac{\partial z}{\partial x} = a \Rightarrow p = a$ ..... (2) and $\frac{\partial z}{\partial y} = b \Rightarrow q = b \ldots\ldots\ldots\ldots$ (3)

Putting the values of a and b from equation (2) and (3) in (1), we get

$$\boxed{z = px + qy + \frac{p}{q} - q}$$

This is the required partial differential equation.

3. **Form the partial differential equation by eliminating the arbitrary constants from** $(x-a)^2 + (y-b)^2 + z^2 = r^2$

(OR)

Find the differential equation of all spheres of fixed radius having their centre on the xy-plane.

Sol. The equation of sphere of radius r having their centers on xy-plane is

$(x-a)^2+(y-b)^2+z^2=r^2$ ……….(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get.

$2(x-a)+2z.\frac{\partial z}{\partial x}=0 \Rightarrow (x-a)+zp=0\, or\, x-a=-zp \rightarrow$ (2)

And $2(y-b)+2z.\frac{\partial z}{2y}=0\, or (y-b)+zq=0\, or\, y-b=-zq \rightarrow$ (3)

Putting the values of (x-a) and (y-b) from (2) and (3) in (1), we get

$$\boxed{(-zp)^2+(-zq)^2+z^2=r^2}$$

is the required partial differential equation.

4. **Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x+a)(y+b)$**

Sol:-The given equation $z = (x+a)(y+b)$ −−−−−−−−−−−−(1)

Differentiating (1) w.r.t., x

$\mathrm{P}=\frac{\partial z}{\partial x}=1.(y+b)$ −−−−−−−−−−−−(2)

Differentiating (1) w.r.t., x

$\mathrm{q}=\frac{\partial z}{\partial y}=1.(x+a)$ −−−−−−−−−−− (3)

from (2) $P = (y+b)$

$from\ (3)\ q = (x+a)$

Substituting in (1) we get

$$\boxed{z = p.q}$$

This is the required partial differential equations

5. **Form the partial differential by eliminating the arbitrary constants from** $\log(az-1)=x+ay+b$

Sol. We have $\log(az-1)=x+ay+b$ ………….(1)

Differentiating (1) partially $w.r.t.\ 'x'\ and\ 'y'$, we get

$\frac{1}{(az-1)}.a.\frac{\partial z}{\partial x}=1\, or\, \frac{1}{(az-1)}ap=1\, or\, ap=az-1$ …………….(2)

and $\frac{1}{(az-1)}a.\frac{\partial z}{\partial y}=a \Rightarrow aq=(az-1)a$ …………(3)

$(3)\div(2), gives\, \frac{q}{p}=a \Rightarrow ap=q$ …………..(4)

Putting (4) in (2), we get

$$\boxed{q = \frac{q}{p}z - 1 \, or \, pq = qz - p \, or \, p(q+1) = q^2}$$

is the required partial differential equation.

6.**Form the differential equation by eliminating a and b from** $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$

Sol: We have $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$ …………..(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have,

$$2\frac{\partial z}{\partial x} = 2p = \frac{1}{2\sqrt{x+a}} \Rightarrow \frac{1}{\sqrt{x+a}} = 4p$$

or $\sqrt{x+a} = \dfrac{1}{4p}$

or $x + a = \dfrac{1}{16p^2} \to$ (2)

And $2\dfrac{\partial z}{\partial y} = \dfrac{1}{2\sqrt{y-a}} \, or \, 2q = \dfrac{1}{2\sqrt{y-a}} \, or \, \sqrt{y-a} = \dfrac{1}{4q}$

$\therefore y - q = \dfrac{1}{16q^2} \to$ ………… (3)

Adding (2) and (3), we get

$$x + y = \frac{1}{16}\left(\frac{1}{p^2} + \frac{1}{q^2}\right)$$

$$\boxed{or \, 16(x+y)p^2q^2 = p^2 + q^2}$$

is the required partial differential equation.

7.Form the partial differential equation by eliminating the arbitrary constants a and b from $z = ax^3 + by^3$

Sol. We have $z = ax^3 + by^3 \to$ (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial z}{\partial x} = 3ax^2 \, or \, p = 3ax^2 \Rightarrow a = \frac{p}{3x^2} \to (2)$$

And $\dfrac{\partial z}{\partial y} = 3by^2 \, or \, q = 3by^2 \Rightarrow b = \dfrac{q}{3y^2} \to$ (3)

Putting the values of 'a' and 'b' from (2) and (3) in (1), we get

$$z = \frac{p}{3}x + \frac{q}{3}y$$

Or

$$\boxed{3z = px + qy}$$

8.Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x^2+a)(y^2+b)$

Sol:-The given equation $z = (x^2+a)(y^2+b)$ ———————————(1)

Differentiating (1) w.r.t., x

P = $\frac{\partial z}{\partial x}$ = $2x(y^2+b)$ ———————————(2)

$$\therefore (y^2+b) = \frac{p}{2x}$$

Differentiating (1) w.r.t., y,we get

q = $\frac{\partial z}{\partial y}$ = 2y (x^2+a) —————————— (3)

$$\therefore (x^2+a) = \frac{q}{2y}$$

Substituting in (1) we get $z = \frac{pq}{4xy}$ implies that

$$\boxed{pq - 4xyz = 0}$$

is the required partial differential equation.

9.Form the partial differential equation by eliminating the arbitrary constants from

$$(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$$

Sol:Given

$(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$……..(1)

Differentiating (1) w.r.t., x

$(x-a) = z\,\mathbf{p}\cot^2\alpha$

Differentiating (1) w.r.t., y

$(y-b) = z\,\mathbf{q}\cot^2\alpha$

Substituting (2),(3) in (1),we get

$(z\,\mathbf{p}\cot^2\alpha)^2 + (z\,\mathbf{q}\cot^2\alpha)^2 = z^2\cot^2\alpha$

∴The required Partial differential equation is

$$\boxed{p^2+q^2 = \tan^2\alpha}$$

Formation of the partial differential equation by the elimination of arbitrary functions:

Derive a p.diff.eqn by the elimination of the arbitrary function ϕ from $\phi(u,v)=0$ where u, v are functions of x, y and z.

$\phi(u,v)=0$ …. (1)

Differentaite partially equation (1) w.r.to. x, y

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right)=0$$

i.e., $\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \ldots\ldots(2)$

and $\frac{\partial\phi}{\partial 4}\left(\frac{\partial 4}{\partial y}+q\frac{\partial 4}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \ldots\ldots(3)$

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from (2) and (3)

$$\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)$$

i.e. $\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

is the P.D.E after the elimination of ϕ from $\phi(u,v)=0.$ Written in a simpler form

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

Above equation is generally written as pP+qQ=R where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\ Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\ \ and\ \ R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Solved Problems

1.Form the partial differential equation by eliminating a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol. Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ….. (1)

Differentiating (1) partially w.r.t. ‘x’ and ‘y’.

$$\frac{2x}{a^2}+\frac{2z}{c^2}.p=0\ or\ \frac{x}{a^2}+\frac{z}{c^2}.p=0\rightarrow (2)$$

And $\frac{2y}{b^2}+\frac{2z}{c^2}.q=0\ or\ \frac{y}{b^2}+\frac{z}{c^2}.q=0\rightarrow (3)$

Since it is not possible to eliminate a, b,c from eqn (1), (2) and (3). We require one more relation.

Differentiating (2), partially w.r.t. 'x', we get

$$\frac{1}{a^2}+\frac{1}{c^2}\left(z.\frac{\partial p}{\partial x}+p.\frac{\partial z}{\partial x}\right)=0 \ or \ \frac{1}{a^2}+\frac{1}{c^2}.z.\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}.p$$

$$\therefore \frac{1}{a^2}+\frac{1}{c^2}.zr+\frac{p^2}{c^2}=0\rightarrow (4)$$

Multiplying (4) by 'x' and then subtracting (2) from it, we get

$$\frac{xz}{c^2}.r+\frac{xp^2}{c^2}-\frac{z}{c^2}.p=0 \ or \ \frac{1}{c^2}\left(xzr+xp^2-zp\right)=0$$

$$\boxed{\therefore pz=xp^2+xzr}$$

is the required partial differential equation.

2. Form a partial differential equation by eliminating the arbitrary function$\boldsymbol{\varphi(x^2+y^2,z-xy)=0}$

Sol:-Given $\varphi(x^2+y^2,z-xy)=0$

This can be written as $z-xy=$ f(x^2+y^2)--------------(1)

Now we have to eliminate f from (1)

Differentiating (1) w.r.t., x

$$\frac{\partial z}{\partial x}-y=f'(x^2+y^2)(2x)$$

$$p-y=f'(x^2+y^2)(2x)\text{--------}(2)$$

Differentiating (2) w.r.t., y

$$q-x=f'(x^2+y^2)(2y)\text{--------}(3)$$

Dividing (2) by (3)

$$\boxed{p\,y-q\,x=y^2-x^2}$$

is the required partial differential equation.

3. Form a partial differential equation by eliminating the arbitrary function from $z=f\left(x^2-y^2\right)$

Sol. We have $z=f\left(x^2-y^2\right)\rightarrow(1)$

Put $u=x^2-y^2$, we have $z=f\left(u\right)\rightarrow(2)$

Differentiating (2) partially w.r.t. 'x' and 'y',

$$\frac{\partial z}{\partial x}=f^1\left(u\right).\frac{\partial u}{\partial x}=f^1\left(u\right).2x$$

$$\therefore p=f^1\left(u\right)2x\rightarrow(3)$$

Similarly we get

$q=-f^1$ (u)$2y$(4)

$\therefore (3)\div(4)$, gives $\frac{p}{q} = \frac{x}{-y}$

$$\boxed{\therefore px + qy = 0}$$

is the required partial differential equation.

4. Form the partial differential equation by eliminating the arbitrary functions from

$xyz = f\left(x^2 + y^2 + z^2\right)$

Sol. We have $xyz = f\left(x^2 + y^2 + z^2\right) \rightarrow$ (1)

Differentiating (1) partially w.r.t. x and y

$$yz + xy.p = f^1\left(x^2 + y^2 + z^2\right).\left(2x + 2z.\frac{\partial z}{\partial x}\right)$$

(or) $yz + xyp = f^1\left(x^2 + y^2 + z^2\right).(2x + 2zp) \rightarrow$ (2)

And $xz + xy.q = f^1\left(x^2 + y^2 + z^2\right).(2y + 2z.q) \rightarrow$ (3)

$\therefore (2)\div(3)$, gives

$$\frac{yz + xyp}{xz + xyq} = \frac{2x + 2zp}{2y + 2zq}$$

$$(yz + xyp)(y + zq) = (xz + xyq)(x + zp)$$

$$y^2z + z^2yq + xy^2p + xyzpq = x^2z + x^2zp + x^2yq + xyzpq$$

$$\boxed{x\left(y^2 - z^2\right)p + y\left(z^2 - x^2\right)q = \left(x^2 - y^2\right)z}$$

is the required partial differential equation.

5. Form the partial differential equation by eliminating the arbitrary functions

From $xyz = f(x + y + z)$

Sol: Given equations $xyz = f(x + y + z)$---------(1)

Differentiating (1) partially w.r.t. 'x'

y(xp+z) = $f'(x + y + z)(1 + p)$----------(2)

Differentiating (1) partially w.r.t. 'x'

x(y q + z) = $f'(x + y + z)(1 + q)$------------(3)

Dividing (2) by (3) $\frac{y(xp+z)}{x(yq+z)} = \frac{1+p}{1+q}$

Y(xp + z)(1 + q) = x(yq + z)(1 + p)

$$(xy - zx)p + (yz - xy)q = zx - yz$$

$$\boxed{x(y - z)p + y(z - x)q = z(x - y)}$$

This is the required partial differential equation.

6. Form the partial differential equation by eliminating the arbitrary function from $xy + yz + zx = f\left(\frac{z}{x+y}\right)$

Sol. We have $xy + yz + zx = f\left(\frac{z}{x+y}\right) \rightarrow$ (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$y + y.p + z + x.p = f^1\left(\frac{z}{x+y}\right)\frac{\left[(x+y).p - z\right]}{(x+y)^2} \rightarrow (2)$$

And $x + z + yq + xq = f^1\left(\frac{z}{x+y}\right)\frac{\left[(x+y)q - z\right]}{(x+y)^2} \rightarrow$ (3)

Dividing (2) by (3), we get

$$\boxed{\frac{(x+y)p + y + z}{(x+y)q + x + z} = \frac{(x+y)p - z}{(x+y)q - z}}$$

is the required partial differential equation.

7. **Form the partial differential equation by eliminating the arbitrary function from** $z = f(x) + e^y.g(x)$

Sol. We have $z = f(x) + e^y.g(x) \rightarrow$ (1)

Differentiating (1) partially w.r.t. 'x' and y, we get

$$\frac{\partial z}{\partial x} = f^1(x) + e^y.g^1(x) \text{ or } p = f^1(x) + e^y.g^1(x) \rightarrow (2)$$

And $q = e^y.g(x)$ or $\frac{\partial z}{\partial y} = e^y.g(x) \rightarrow$ (3)

Differentiating (3), partially w.r.t. 'y', we get

$$\frac{\partial^2 z}{\partial y^2} = e^y.g(x) = \frac{\partial z}{\partial y} \text{ [using (3)]}$$

$$\therefore \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0$$

$$\boxed{\therefore t - q = 0}$$

Which is the required P.D.E.

8.Form a partial differential equation by eliminating the arbitrary function

$$\boldsymbol{z = f(x^2 + y^2)}$$

Sol. We have

$$\boldsymbol{z = f(x^2 + y^2) \ldots\ldots (1)}$$

Put $u = x^2 - y^2$, we have $z = f(u) \to (2)$

Differentiating (2) partially w.r.t. 'x' and 'y',

$$\frac{\partial z}{\partial x} = f^1(u).\frac{\partial u}{\partial x} = f^1(u).2x$$

$$\therefore p = f^1(u)2x \to (3)$$

And $\dfrac{\partial z}{\partial y} = f^1(u).\dfrac{\partial u}{\partial y} = f^1(u).2y$

$$\therefore q = f^1(u)2y \to (4)$$

$\therefore (3) \div (4)$, gives $\dfrac{p}{q} = \dfrac{f^1(u).2x}{f^1(u)2y} = \dfrac{x}{y}$

$$\boxed{\therefore py - qx = 0}$$

This is the required partial differential equation.

9.Form a partial differential equation by eliminating the arbitrary function $\boldsymbol{\varphi(x^2 + y^2 + z^2, ax + by + cz) = 0}$

Sol: Given function can be written as

$$\boldsymbol{x^2 + y^2 + z^2 = f(ax + by + cz) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)}$$

Differentiating (1) partially w.r.t. 'x' and 'y',we get

$$2x + 2zp = \boldsymbol{(a + cp)f^1(ax + by + cz) \ldots . (2)}$$

$$and$$

$$2y + 2zq = \boldsymbol{(b + cq)f^1(ax + by + cz) \ldots . (3)}$$

$\boldsymbol{\frac{(2)}{(3)}}$ **implies**

$$\frac{x+zp}{y+zq} = \frac{(a+cp}{(b+cq)}$$

Which is the required complete solution of given Partial differential equation.

SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS :

Complete integral:

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.

Particular integral :

A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral or particular solution.

Singular integral:

$Let\ f(x, y, z, p, q) = 0 \to$ (1) be the partial differential equation.

$Let\ \phi(x, y, z, a, b) = 0 \to (2)$

Be the complete integral of (1). Where a and b are arbitrary constants.

Now find $\frac{\partial \phi}{\partial a} = 0 \to$ (3) $\frac{\partial \phi}{\partial b} = 0 \to$ (4)

Eliminate a and b between the equations(2), (3) & (4) When it exists is called the singular integral of (1).

General integral : In the complete integral (2). Assume that one of the constant is a function of the other i.e. b=f(a) Then (2), becomes $\phi(x, y, z, a, f(a)) = 0 \to$ (5)

Differentiating (5) partially w.r.t. ‘a’, we get $\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial f} . f^1(a) = 0 \to$ (6)

Eliminate ‘a’ between (5) and (6), when it exists is called the general integral or general solution of (1).

LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER:

A differential equation involving partial derivatives p and q only and no higher order derivatives is called a first order equation. If p and q occur in the first degree, it is called a linear partial differential equation of first order; otherwise it is called a non-linear partial differential equation of the first order.

For example: $px + qy^2 = z$ is a linear p.d.e of first order and $p^2 + q^2 = 1$ is non-linear

Lagrange's linear equation:

A linear partial differential equation of order one involving a dependent variable z and two independent variables x and y of the form $Pp+Qq=R$

Where P, Q, R are functions of x, y, z is called Lagrange's linear equation.

Lagrange's auxiliary equations or Lagrange's subsidiary equations

The equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ are called Lagrange's auxiliary equations.

WORKING RULE TO SOLVE LAGRANGE'S LINEAR EQUATION $\boldsymbol{Pp+Qq=R}$

Step 1: Write down the auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Step 2 : Solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions u=a and v=b where a,b are arbitrary constants

Step 3: Then Q(u,v) =0 or u=f(v) is the general solution of the equation $Pp+Qq=R$

To solve $\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}$ …… (1)

(i) **Method of grouping** : In some problems, it is possible that two of the equations $\frac{dx}{P}=\frac{dy}{Q}$ or $\frac{dy}{Q}=\frac{dz}{R}$ or $\frac{dx}{P}=\frac{dz}{R}$ are directly solvable to get solutions u(x,y) = constant or v(y,z)=constant or w(x,z)=constant. These give the complete solutions of (1)

Sometimes one of them, say $\frac{dx}{P}=\frac{dy}{Q}$ may give rise to solution u(x,y)=c_1

From this we may express y, as a function of x. Using this in $\frac{dy}{Q}=\frac{dz}{R}$ and integrating we get $v(y,z)=c2.$ *These two relations* $u= c1, v=c2$ *give the complete solution of* (1)

2. **Method of multipliers**: This is base on the following elementary result.

If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=......=\frac{a_n}{b_n}$ then each ratio is equal to $\frac{l_1a_1+l_2a_2+....+l_na_n}{l_1b_1+l_2b_2+....+l_nb_n}$

Consider $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

If possible identity multipliers l, m, n, not necessarily constant, so that each ratio

$=\frac{ldx+mdy+ndz}{lP+mQ+nR}$

Where $L\,P + m\,Q + n\,R = 0$ Then $l\,dx + m\,dy + n\,dz = 0$

Integrating this we get $u(x,y,z) = c1.$

Similarly we get another solution $v(x,y,z) = c2$ independent of the earlier one.

We have the complete solution of (1) constituted $by\ u = c1\ and\ v = c2$

Linear Partial P.D.E:

Solved Problems

1. **Solve $p\tan x + q\tan y = \tan z$**

Sol. The given equations can be written as $\tan xp + \tan yq = \tan z \rightarrow$ (1)

Comparing with $Pp + Qq = R$, we have $P = \tan x, Q = \tan y, R = \tan z$

$\therefore$ The auxiliary equations are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

Taking the first two members, we have $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y}$

Integrating $\log\sin x = log\sin y + l\log c_1$

or $\log\dfrac{\sin x}{\sin y} = \log c_1$ or $\dfrac{\sin x}{\sin y} = c_1 \rightarrow$ (2)

Taking the last two members, we have $\dfrac{dy}{\tan y} = \dfrac{dt}{\tan z}$

Integrating, $log\ siny = log\ sinz + logc2$

or $\log\dfrac{\sin y}{\sin z} = \log c_2$ or $\dfrac{\sin y}{\sin z} = c_2 \rightarrow$ (3)

From (2) and (3). The general solution of (1) is

$\phi(c_1, c_2) = 0$

$$\boxed{i.e.\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0}$$

is the required Complete Solution.

2. **Find the general solution of** $y^2zp + x^2zq = y^2x$

Sol. We have $y^2zp + x^2zq = y^2x \rightarrow$ (1)

Comparing $with\ Pp + Qq = R, we\ have$

$P = y^2z, Q = x^2z,\ R = y^2x$

$\therefore$ The auxiliary equations are $\dfrac{dx}{y^2z} = \dfrac{dy}{x^2z} = \dfrac{dz}{y^2x}$

Taking the first two members, we have

$\dfrac{dx}{y^2z} = \dfrac{dy}{x^2z} \Rightarrow \dfrac{dx}{y^2} = \dfrac{dy}{x^2}$ $or\ x^2dx = y^2dy$

Integrating, $\dfrac{x^3}{3} = y\,{}^3\!/_3 + c_1$ or $\dfrac{x^3}{3} - \dfrac{y^3}{3} = c_1 \rightarrow$ (2)

Taking the first and last two members, we have

$$\frac{dx}{y^2z}=\frac{dz}{y^2x} \text{ or } xdx=zdz$$

Integrating $\frac{x^2}{2}=\frac{z^2}{2}+c_2$ or $\frac{x^2}{2}-\frac{z^2}{2}=c^2 \rightarrow$ (3)

From (2) and (3) The general solution of (1) is

$\phi(c_1,c_2)=0$ i.e.

$$\boxed{\phi\left(\frac{x^3}{3}-\frac{y^3}{3},\frac{x^2}{2}-\frac{z^2}{2}\right)=0}$$

is the required Complete Solution.

3. **Solve** $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

Sol. The given equation can be written as

$\sqrt{x}p+\sqrt{y}q=\sqrt{z}\rightarrow(1)$

Comparing with Pp+Qq=R, we have

$P=\sqrt{x},Q=\sqrt{y},R=\sqrt{z}$

$\therefore$ The auxiliary equations are $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

From the first two members, we have $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}$

Integrating, $2\sqrt{x}=2\sqrt{y}+c_1$ or $2\sqrt{x}-2\sqrt{y}=c_1$ or $\sqrt{x}-\sqrt{y}=a\rightarrow$ (2)

From the last two members, we have $\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

Integrating, $2\sqrt{y}=2\sqrt{z}+c_2$ or $2\sqrt{y}-2\sqrt{z}=c_2$

or $\sqrt{y}-\sqrt{z}=b\rightarrow$ (3)

From (2) and (3). The general solution of (1) is

$\phi(a,b)=0$ i.e.,

$$\boxed{\phi\left(\sqrt{x}-\sqrt{y},\sqrt{y}-\sqrt{z}\right)=0}$$

is the required Complete Solution.

4. **Solve** $x(y-z)p+y(z-x)q=z(x-y)$

Sol. We have $x(y-z)p+y(z-x)q=z(x-y)\rightarrow$ (1)

Comparing with $Pp+Qq=R$, we have

$P = x(y-z), Q = y(z-x), R = z(x-y)$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

Using l=1,m=1, n=1 as multipliers, we get

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{0} \qquad \left[\because x(y-z)+y(z-x)+z(x-y)=0\right]$$

$\therefore dx + dy + dz = 0$

Integrating, $x + y + z = a \to (2)$

Again using $l = \dfrac{1}{x}, m = \dfrac{1}{y}, n = \dfrac{1}{z}$ as multipliers, we get

Each fraction $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$ =k(say)

$\therefore \dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0$

Integrating, logx+logy+logz=logb. or xyz=b……. (3)

From (2) and (3). The general solution of (1) is

$\phi(a,b) = 0$ *i.e.,*

$$\boxed{\phi(x+y+z, xyz) = 0}$$

is the required Complete Solution.

5. **Solve** $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol. Given $x^2(y-z)p + y^2(z-x)q = z^2(x-y) \to (1)$

Comparing with $Pp + Qq = R$, we have

$P = x^2(y-z), Q = y^2(z-x), R = z^2(x-y)$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$

Using $l = \dfrac{1}{x^2}, m = \dfrac{1}{y^2}, n = \dfrac{1}{z^2}$ as multipliers, we get

Each fraction $= \dfrac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$ =k(say)

$\therefore \dfrac{1}{x^2}dx + \dfrac{1}{y^2}dy + \dfrac{1}{z^2}dz = 0$

Integrating , $-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=a \quad or \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1 \rightarrow (2)$

Again using $l=\frac{1}{x}, m=\frac{1}{y}, n=\frac{1}{z}$ as multipliers, we get

Each fraction $=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$ =k(say)

$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating $\log x+\log y+\log z=\log c_2$

$or\ xyz=c_2 \rightarrow$ (3)

From (2) and (3), The general solution of (1) is .

$$\boxed{\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right)=0}$$

is the required Complete Solution.

6. **Solve** $(mz-ny)p+(nx-lz)q=ly-mx$

Sol. Given eqn is $(mz-ny)p+(nx-lz)q=ly-mx \rightarrow (1)$

Comparing with $Pp+Qq=R$, we have

$P=mz-ny, Q=nx-lz, R=ly-mx$

$\therefore$ The auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using l=x, m=y, n=z as multipliers, we get

Each fraction $=\frac{xdx+ydy+zdz}{0}$

$\therefore xdx+ydy+zdz=0$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=a\ or\ x^2+y^2+z^2=c_1 \rightarrow (2)$

Again using l, m, n as multipliers, we get

Each fraction $=\frac{ldx+mdy+nz}{0}$ =k(say)

$\therefore ldx$+mdy+ndz=0

Integrating, $lx+my+nz=c_2 \rightarrow$ (3)

From (2) and (3), the general solution of (1) is

$$\boxed{\phi\left(x^2+y^2+z^2, lx+my+nz\right)=0}$$

is the required Complete Solution.

7. Solve $xp - yq = y^2 - x^2$

Sol. Here $P = x, Q = y, R = y^2 - x^2$

$\therefore$ The auxiliary eqn's are $\dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{dz}{y^2 - x^2}$

From the first two members, $\dfrac{dx}{x} = \dfrac{dy}{-y}$

Integrating, $\log x + \log y = \log c_1 \; or \; xy = c_1 \rightarrow (1)$

Using l=x, m=y, n=1 as multipliers, we get

Each fraction $= \dfrac{xdx + ydy + dz}{0}$

$\therefore xdx + ydy + dz = 0$

Integrating, $\dfrac{1}{2}x^2 + \dfrac{1}{2}y^2 + z = c \; or \; x^2 + y^2 + 2z = c_2 \rightarrow$ (2)

From (1) and (2), The general solution is

$$\boxed{\phi\left(xy, x^2+y^2+2z\right)=0}$$

is the required Complete Solution.

8. Find the integral surface of $x\left(y^2+z\right)p - y\left(x^2+z\right)q = \left(x^2-y^2\right)z$

Which contains the straight line x+y=0, z=1

Sol. Given that $x\left(y^2+z\right)p - y\left(x^2+z\right)q = \left(x^2-y^2\right)z$ ....(1)

Comparing with $Pp + Qq = R$, we have

$P = x\left(y^2+z\right), Q = -y\left(x^2+z\right), R = \left(x^2-y^2\right)z$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x\left(y^2+z\right)} = \frac{dy}{-y\left(x^2+z\right)} = \frac{dz}{\left(x^2-y^2\right)z}$$

Using $l = \dfrac{1}{x}, m = \dfrac{1}{y}, n = \dfrac{1}{z}$ as multipliers, we get

Each fraction $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, $\log x + \log y + \log z = \log a$

or $xyz = a \rightarrow$ (2)

Again using l=x, m=y, n=-1 as multipliers, we get

$\therefore$ Each fraction $= \dfrac{xdx + ydy - dz}{0}$ =k(say)

$$\therefore xdx + ydy - dz = 0$$

Integrating, $\dfrac{x^2}{2} + \dfrac{y^2}{2} - z = c \, or \, x^2 + y^2 - 2z = b \rightarrow$ (3)

Given that z=1, using this (2) and (3), we get

xy=a and $x^2 + y^2 - 2 = b$

Now b+2a=$x^2 + y^2 - 2 + 2xy = (x+y)^2 - 2$=0-2 $[\therefore x + y = 0]$=-2

$$\therefore 2a + b + 2 = 0$$

Hence the required surface is

$$\boxed{x^2 + y^2 - 2z + 2xyz + 2 = 0}$$

is the required Complete Solution.

9.Solve $px + qy = z$

Sol:Given

$px + qy = z$ is a Lagrange's linear equation

The Auxillary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

By Consider first group, we get

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c1$$

$c_1 = \frac{x}{y}$.....(1)

By Consider second group, we get

$$\int \frac{dz}{z} = \int \frac{dy}{y}$$

$$\log y = \log z + \log c2$$

$c_2 = \frac{y}{z}$.....(2)

$$\boxed{\therefore f\left(\frac{x}{y}, \frac{y}{z}\right) = 0}$$

is the required solution.

10. Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - xz)q = z(x - y)$

Sol: The auxiliary equations are

$$\frac{dx}{(x^2 - y^2 - yz)} = \frac{dy}{(x^2 - y^2 - xz)} = \frac{dz}{z(x - y)}$$

Taking 1,-1-1 multipliers,we get

$$\frac{dx - dy - dz}{(x^2 - y^2 - yz - x^2 + y^2 + xz - xz + yz)} = \frac{dx}{(x^2 - y^2 - yz)}$$

$$dx - dy - dz = 0$$

Integrating,we get

$x - y - z = c_1$.........(1)

Taking $x, -y, 0$ as multipliers,we get

$$\frac{xdx - ydy}{(x^3 - xy^2 - xyz - yx^2 + y^3 + xyz)} = \frac{dz}{z(x - y)}$$

$$\frac{xdx - ydy}{(x^2 - y^2)(x - y)} = \frac{dz}{z(x - y)}$$

$$\frac{1}{2}\log(x^2 - y^2) = \log z$$

$\frac{x^2 - y^2}{z^2} = c_2$ ... ... (2)

$$\therefore \textit{Complete solution of given pde is } \varphi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$$

11. Solve $x(y^2 - z^2)p - y(x^2 + z^2)q = z(x^2 + y^2)$

Sol: The auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(x^2 + z^2)} = \frac{dz}{z(x^2 + y^2)}$$

Taking x, y, z, multipliers, we get

$$\frac{xdx+ydy+zdz}{(x^2y^2-x^2z^2-y^2x^2-z^2y^2+x^2z^2+y^2z^2)}=\frac{dx}{x(y^2-z^2)}$$

$$xdx + ydy + zdz = 0$$

$$x^2 + y^2 + z^2 = c_1 \dots\dots\dots\dots\dots\dots\dots.(1)$$

Taking $-\frac{1}{x}, \frac{1}{y}\frac{1}{z}$, multipliers, we get

$$-\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, we get

$$\frac{yz}{x} = c_2 \dots\dots(2)$$

From (1),(2),

$$\therefore \textit{Complete solution of given pde is } \varphi\left(\frac{yz}{x}, x^2 + y^2 + z^2\right) = 0$$

12.Solve $(y^2)p - xyq = x(z - 2y)$

Sol:Comparing with Pp+Qq=R, we have

The auxiliary equations are

$$\therefore \frac{dx}{y^2} = \frac{dy}{-yx} = \frac{dz}{x(z-2y)}$$

From the first two members, we have Type equation here.

$$\frac{dx}{y} = \frac{dy}{-x}$$

Integrating,we get

$$x^2 + y^2 = c_1 \dots..(2)$$

From the last two members, we have

$$\frac{dy}{-y} = \frac{dz}{(z-2y)}$$

$$-ydz = zdy - 2ydy$$

$$d(yz) - 2ydy = 0$$

$yz - y^2 = c_2$ **...........(3)**

From (2) and (3). The general solution of (1) is *i.e.*, $\emptyset(yz-\boldsymbol{y^2},\boldsymbol{x^2}+\boldsymbol{y^2})$=0

12. Solve $(\boldsymbol{y}+\boldsymbol{z})\boldsymbol{p}+(\boldsymbol{z}+\boldsymbol{x})\boldsymbol{q}=(\boldsymbol{x}+\boldsymbol{y})$

Sol:Comparing with Pp+Qq=R, we have

The auxiliary equations are

$\therefore \ \frac{dx}{(y+z)}=\frac{dy}{(z+x)}=\frac{dz}{(x+y)}$

Taking 1,1,1 and 1,-1,0 and 0,1,-1 as multipliers , we have $\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{(y-x)}=\frac{dy-dz}{(z-y)}$

From the last two members, we have

$$\frac{dx-dy}{(y-x)}=\frac{dy-dz}{(z-y)}$$

Integrating,we get

$$\log\frac{(y-x)}{(z-y)}=\log C_2$$

$$\frac{(y-x)}{(z-y)}=c_2\text{.....(1)}$$

From the first two members, we have

$$\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{(y-x)}$$

Integrating, we get

$$\frac{1}{2}\log(x+y+z)=\log(y-x)+\log c_1$$

$(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{x})^2=\boldsymbol{C_1}$**..........(2)**

From (2) and (1). The general solution of given pde is

$$\boxed{i.e.,\ \emptyset\left(\frac{(y-x)}{(z-y)},(x+y+z)(y-x)^2\right)=0}$$

13. Solve $\boldsymbol{x^2p}-\boldsymbol{y^2q}=\boldsymbol{z}(\boldsymbol{x}-\boldsymbol{y})$

Sol:Comparing with $Pp+Qq=R$, we have

The auxiliary equations are

$\therefore \ \frac{dx}{x^2}=\frac{dy}{-y^2}=\frac{dz}{(z(x-y)}$

From the first two members, we have

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating,we get

$$\frac{1}{x} + \frac{1}{y} = c_1.....(1)$$

Taking 1,1,0 as multipliers,we get

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)} = \frac{dz}{z}$$

Integrating,we get

$$\frac{x+y}{z} = c_2 \quad ....(2)$$

From (2) and (1). The general solution is

$$\boxed{\text{i.e., } \emptyset\left(\frac{x+y}{z}, \frac{1}{x} + \frac{1}{y}\right) = 0}$$

14. Solve $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$

Sol: The auxiliary equations are

$$\frac{dx}{(x^2-yz)} = \frac{dy}{(y^2-xz)} = \frac{dz}{(z^2-xy)}$$

Taking 1,-1,0 and 0,-1,-1 as multipliers,we get

$\frac{dx-dy}{(x^2-yz)-(y^2-xz)}$ and also $\frac{dy-dz}{\left((-z^2+yx)+(y^2-xz)\right)}$

$$\therefore \frac{dx-dy}{(x^2-yz)-(y^2-xz)} = \frac{dy-dz}{\left((-z^2+yx)+(y^2-xz)\right)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z))}$$

solving it,we get

$$\frac{(x-y)}{y-z} = c_1.....(1)$$

Taking x, y, z *and* 1,1,1 *as* multipliers,we get

$$\frac{(xdx+ydy+zdz)}{x^3+y^3+z^3-3xyz} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{(xdx+ydy+zdz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$$

$$(x+y+z)(dx+dy+dz) = (xdx+ydy+zdz)$$

$$(x+y+z)d(x+y+z) = (xdx+ydy+zdz)$$

Integrating,we get

$$\frac{(x+y+z)^2}{2} = \frac{x^2+y^2+z^2}{2} + c$$

$$\therefore (x+y+z)^2 = x^2+y^2+z^2+c_2$$

$xy+yz+zx = c_2$.....(2)

$$\therefore \textit{Complete solution of given pde is } \varphi\left(xy+yz+zx, \frac{(x-y)}{y-z}\right) = 0$$

NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

A partial differential equation which involves first order partial derivatives p *and* q with degree higher than one and the products *of* p *and* q *is* called a non-linear partial differential equations.

CHARPIT'S METHOD

In this method give D.E of the form $f(x,y,z,p,q) = 0$ to find another relation of the form $\varphi(x,y,z,p,q) = 0$ which is compatible with *the* $f(x,y,z,p,q) = 0$ the we solve for p, q and substitute these values in the relation $dz = p\,dx + q\,dy$.

Which on integration gives the required solution of $f(x,y,z,p,q) = 0$

Charpit's equation :

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Solved Problems

1.Solve $\boldsymbol{px+qy=pq}$

Sol: Given $\boldsymbol{f = px+qy-pq}$

The auxillary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Here $f_p = x;\ f_q = y;\ f_x = p\ ;\ f_y = q\ ;\ f_z = 0$

Substituting them in above ,we get

$$\frac{dx}{-x} = \frac{dy}{-y} = \frac{dz}{px+qy} = \frac{dp}{p} = \frac{dq}{q}$$

By considering first and last groups,

$$\frac{dx}{-x} = \frac{dy}{-y}$$

Integrating,we get $\log x = \log y + \log c$

$$\therefore y = cx \,\ldots\ldots(1)$$

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating,we get $\log p = \log q + \log a$

$$\therefore p = aq \,\ldots\ldots(2)$$

Substitute (1),(2) in given pde ,we get

$$p = ax + y\,;\, q = \frac{ax + y}{a}$$

But we have

$$dz = p\,dx + q\,dy$$

$$dz = (ax + y)\,dx + \frac{ax + y}{a}\,dy$$

Integrating it we get required solution as

$$\boxed{z = \frac{1}{a}(ax + y)^2 + c}$$

2.Solve $\boldsymbol{z^2 = pqxy}$

Sol: Given $f = \boldsymbol{z^2 - pqxy}$

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Here $f_p = -qxy;\ f_q = -pxy;\ f_x = -qxy\,;\ f_y = -pxy\,;\ f_z = 2z$

Substituting them in above ,we get

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz}$$

Considering

$\frac{dx}{qxy} = \frac{dp}{-pqy+2pz}$ and $\frac{dy}{pxy} = \frac{dq}{-pqx+2qz}$

Taking

p, x *as multipliers for first and* q, y *for second group, then equating them we get*

$$\frac{xdp+pdx}{-xpqy+2xpz+pqxy} = \frac{ydq+qdy}{-ypqx+2qz+pqxy}$$

Solving ,

$$\frac{d(px)}{px} = \frac{d(qy)}{qy}$$

Integrating,we get

$$\log px = \log qy + \log c$$

$$px = qyc$$

Substituting in given pde,we get

$$z^2 = q^2y^2c$$

$\therefore q = \frac{z}{y\sqrt{c}}$; $p = \frac{z}{x}\sqrt{c}$

But we have

$$dz = p\,dx + q\,dy$$

$$dz = \frac{z}{x}\sqrt{c}dx + \frac{z}{y\sqrt{c}}\,dy$$

$$\int \frac{dz}{z} = \int \frac{1}{x}\sqrt{c}dx + \int \frac{1}{y\sqrt{c}}\,dy$$

Integrating,we get,

$\therefore$ Required solution is

$$\boxed{z = x^{\sqrt{c}} y^{\frac{1}{\sqrt{c}}} a}$$

Now let us start solving some standard forms of first order partial differential equations by using Charpit's method

STANDARD FORM I:

<u>Equation of the form f(p,q)=0</u>

i.e., equations containing p and q only.

Given partial differential equation is $\boldsymbol{f(p,q) = 0} \ldots \ldots (\mathbf{1})$

The auxillary equations are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{[f_x + pf_z]} = \frac{dq}{[f_y + qf_z]}$$

Here $f_x = 0$; $f_y = 0$; $f_z = 0$

Substituting above and considering last group, we get

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$$\therefore \boldsymbol{dp = 0}, \textit{integrating we get } \boldsymbol{p = a}$$

Put $p = a$ in (1), *then we get q value in terms of a*, *say* $\varphi(a)$.

But we have

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

$$dz = p\, dx + q\, dy$$

$$dz = a\, dx + \varphi(a) dy \ldots\ldots(2)$$

Integrating (2),we get required complete solution of (1) is

$$\boxed{z = ax + \phi(a)y + c}$$

Which contains two arbitrary constants a and c.

PROCEDURE:

Given partial differential equation is $f(p, q) = 0 \ldots \ldots (1)$

STEP1: Put $p = a$ *in* (1), *then we get q value in terms of a* ..then we can obtain 'p' value.

$STEP2$: *Sub p, q values in* $dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$

i.e $dz = p\, dx + q\, dy$

STEP3:Integrating it ,we get required complete solution of (1) .

Solved Problems

1.Solve $pq = k$, where k is a constant.

Sol. Given that $pq = k \ldots.(1)$

Since (1) is of the form $f(p, q) = 0$

Put **p=a** in (1),we get $q = \frac{k}{a}$

But we have

$$dz = p\, dx + q\, dy$$

$$dz = a\, dx + \frac{k}{a}\, dy$$

Integrating,we get ,

$$\boxed{z = ax + \frac{k}{a}y + c}$$

which contains two arbitrary constants a and c.

2.Solve $p^2 + q^2 = npq$

Sol : Given that $p^2 + q^2 = npq \ldots\ldots\ldots\ldots(1)$

Since (1) is of the form $f(p,q)=0$

Put p=a in (1),then we get $q=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]$

But we have

$\boldsymbol{dz = p\,dx + q\,dy}$

$dz = a\,dx + \frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]dy$

Integrating,we get , $dz = a\int dx + \frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]\int dy$

$$\boxed{z = ax + \frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]y + c}$$

This is the complete integral of (1), which contains. two arbitrary constants a and c.

3.Find the complete integral of $\boldsymbol{p^2 + q^2 = m^2}$

Sol. Given that $\boldsymbol{p^2 + q^2 = m^2}$ ……….(1)

Since (1) is of the form $f(p,q)=0$

Put $p = a\ in\ (1), we\ get\ q = \sqrt{m^2-a^2}$

But we have

$\boldsymbol{dz = p\,dx + q\,dy}$ …….(2)

Put the values of p,q in (2), we get

$$\boxed{z = ax + \left(\sqrt{m^2-a^2}\right)y + c}$$

Which is the complete integral of (1)

STANDARD FORM II :

<u>Equation of the form $f(p,q,z)=0$(i.e., not containing x and y)</u>

PROCEDURE

Given partial differential equation is $\boldsymbol{f(p,q,z)=0}$ … … $\boldsymbol{(1)}$

STEP1: Put $\boldsymbol{p = aq}$ $in\ (1), then\ we\ get\ q\ value\ in\ terms\ of\ a\ , z$ .then

$\boldsymbol{STEP2}$: $Sub\ p,q\ values\ in\ \boldsymbol{dz} = \frac{\partial z}{\partial x}\boldsymbol{dx} + \frac{\partial z}{\partial y}\boldsymbol{dy}$

$\boldsymbol{i.e\ dz = p\,dx + q\,dy}$

STEP3:Integrating it ,we get required complete solution of (1) .

Solved Problems :

Solve the following partial differential equations

1. $\boldsymbol{z = p^2 + q^2}$ **2.** $\boldsymbol{p^2z^2 + q^2 = p^2q}$ **3.** $\boldsymbol{zpq = p + q}$

Sol. **1.** We have $z = p2 + q2 \ldots\ldots(1)$

Since (1) is of the form $f(z, p, q) = 0$

Put $\boldsymbol{p} = \boldsymbol{aq}\ in\ (1), then\ we\ get\ q = \sqrt{\frac{z}{1+a^2}}$

$$\therefore p = a\sqrt{\frac{z}{1+a^2}}$$

Putting the values $of\ p\ and\ q\ in\ \boldsymbol{dz} = \boldsymbol{p}\ \boldsymbol{dx} + \boldsymbol{q}\ \boldsymbol{dy}$, we get

$\frac{\mathbf{1}}{\sqrt{\boldsymbol{z}}}\boldsymbol{dz} = \frac{\mathbf{1}}{\sqrt{\mathbf{1}+\boldsymbol{a}^{\mathbf{2}}}}(\boldsymbol{a}\ \boldsymbol{dx} + \boldsymbol{dy})$,

Integrating ,we get

$$\int \frac{\mathbf{1}}{\sqrt{\boldsymbol{z}}}\boldsymbol{dz} = \frac{\mathbf{1}}{\sqrt{\mathbf{1}+\boldsymbol{a}^{\mathbf{2}}}}\int(\boldsymbol{a}\ \boldsymbol{dx} + \boldsymbol{dy})$$

$$\boxed{\therefore \mathbf{2}\sqrt{\boldsymbol{z}} = \frac{\mathbf{1}}{\sqrt{\mathbf{1}+\boldsymbol{a}^{\mathbf{2}}}}(\boldsymbol{ax} + \boldsymbol{y})}$$

This is the required solution of (1)

2. **Given that** $p^2z^2 + q^2 = p^2q \rightarrow\ (1)$

Since (1) is of the form $f(z, p, q) = 0$

Put $\boldsymbol{p} = \boldsymbol{aq}\ in\ (1), then\ we\ get\ q = \frac{(a^2z^2+1)}{a^2}$

$$\therefore p = \frac{(a^2z^2+1)}{a}$$

Putting the values of p and q in $\boldsymbol{dz} = \boldsymbol{p}\ \boldsymbol{dx} + \boldsymbol{q}\ \boldsymbol{dy}$,we get

$\frac{\boldsymbol{dz}}{(a^2z^2+1)} = \frac{\mathbf{1}}{\boldsymbol{a}^{\mathbf{2}}}(\boldsymbol{adx} + \boldsymbol{dy})$

Integrating,we get

$\int \frac{\boldsymbol{dz}}{(a^2z^2+1)} = \frac{\mathbf{1}}{\boldsymbol{a}^{\mathbf{2}}}\int(\boldsymbol{adx} + \boldsymbol{dy})$

$$\therefore \mathrm{a}\tan^{-1}(az) = ax + y + c$$

which is the required complete solution of (1)

3. **Given** that $zpq = p + q\ldots..(1)$

Since (1) is of the form $f(z, p, q) = 0$

Put $\boldsymbol{p} = \boldsymbol{aq}\ in\ (1), then\ we\ get\ q = \frac{a+1}{az}$

$$\therefore p = \frac{a+1}{z}$$

Putting the values $of\ p\ and\ q\ in\ \boldsymbol{dz} = \boldsymbol{p}\ \boldsymbol{dx} + \boldsymbol{q}\ \boldsymbol{dy}$, we get

$\boldsymbol{zdz} = \frac{\boldsymbol{a}+\mathbf{1}}{\boldsymbol{a}}(\boldsymbol{a}\ \boldsymbol{dx} + \boldsymbol{dy})$, Integrating ,we get

$$\int z\,dz = \frac{a+1}{a}\int (a\,dx + dy)$$

$$\therefore \frac{aZ^2}{2(a+1)} = ax + y + c$$

This is the required solution of (1)

STANDARD FORM III :

Equation of the form $f_1(x,p) = f_2(y,q)$ i.e. Equations not involving z and the terms containing x and p can be separated from those containing y and q.

We assume that these two functions should be equal to a constant say k.

$\therefore f_1(x,p) = f_2(y,q) = k$

Solve for p and q from the resulting equations

$\therefore f_1(x,p) = k \; and \; f_2(y,q) = k$

Solve for p and q, we obtain

$p = F_1(x,k) \; and \; q = F_2(y,k)$

Since z is a function of x and y

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ [By total differentiation]

$dz = p\,dx + q\,dy$

$\therefore dz = F_1(x,k)dx + F_2(y,k)dy$

Integrating on both sides

$z = \int F_1(x,k)dx + \int F_2(y,k)dy + c$

Which is the complete solution of given equation

Solved Problems:

1. Solve $p^2 + q^2 = x + y$

Sol :. Given that $p^2 + q^2 = x + y$ ....(1)

Separating $p \; and \; x \; from \; q \; and \; y$, the given equation can be written as

$$p^2 - x = -q^2 + y$$

Let $p^2 - x = -q^2 + y = k$ (constant)

$\therefore p^2 - x = k \; and \; -q^2 + y = k$

$\Rightarrow p^2 = k + x \; and \; q^2 = y - k$

$$\therefore p=\sqrt{k+x} \ and \ q=\sqrt{y-k}$$

Since $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=\sqrt{k+x}\,dx+\sqrt{y-k}\,dy$$

Integrating on both sides

$$z=\int(k+x)^{\frac{1}{2}}dx+\int(y-k)^{\frac{1}{2}}dy+c$$

$$\boxed{\therefore z=\frac{2}{3}(k+x)^{\frac{3}{2}}+\frac{2}{3}(y-k)^{\frac{3}{2}}+c}$$

Which is the complet solution of (1)

2. Solve $xp-yq=y^2-x^2$

Sol: Given that $xp-yq=y^2-x^2 \rightarrow (1)$

Separating p and x from q and y. The given equation can be written as.

$$xp+x^2=yq+y^2$$

Let $xp+x^2=yq+y^2=k$ (arbitrary constant)

$$\therefore xp+x^2=k \ and \ yq+y^2=k$$

$$\Rightarrow p=\frac{k-x^2}{x} and \ q=\frac{k-y^2}{y}$$

We have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=\left(\frac{k}{x}-x\right)dx+\left(\frac{k}{y}-y\right)dy$$

Integrating on both sides

$$z=\int\left(\frac{k}{x}-x\right)dx+\int\left(\frac{k}{y}-y\right)dy+c$$

$$=k\log x-\frac{x^2}{2}+k\log y-\frac{y^2}{2}+c$$

$$\boxed{\therefore z=k\log(xy)-\frac{1}{2}\left(x^2+y^2\right)+c}$$

Which is the complete integral of (1)

3. Solve $\left(\frac{p}{2}+x\right)^2+\left(\frac{q}{2}+y\right)^2=1$

Sol: Separating p *and* x *from* q *and* y, the given equation can be written as.

$$\left(\frac{p}{2}+x\right)^2=1-\left(\frac{q}{2}+y\right)^2$$

Let $\left(\frac{p}{2}+x\right)^2=1-\left(\frac{q}{2}+y\right)^2=k^2$ (arbitrary constant)

$$\therefore\left(\frac{p}{2}+x\right)^2=k^2 \text{ and } 1-\left(\frac{q}{2}+y\right)^2=k^2$$

$$\Rightarrow \frac{p}{2}+x=k \text{ and } \left(\frac{q}{2}+y\right)^2=1-k^2 \text{ or } \frac{q}{2}+y=\sqrt{1-k^2}$$

$$\Rightarrow p=2(k-x) \text{ and } q=2\left[\sqrt{1-k^2}-y\right]$$

We have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=2(k-x)dx+2\left[\sqrt{1-k^2}-y\right]dy$$

Integrating on both sides

$$z=2\int(k-x)dx+2\int\left[\sqrt{1-k^2}-y\right]dy+c$$

$$z=2(kx-\frac{x^2}{2})+2\left[\left(\sqrt{1-k^2}\right)y-\frac{y^2}{2}\right]+c$$

$$\boxed{\therefore z=2kx-x^2+2\left(\sqrt{1-k^2}\right)y-y^2+c}$$

This is the complete solution of (1)

4.Solve $\boldsymbol{p-x^2=q+y^2}$

Sol:Let $p-x^2=q+y^2=k^2(say)$

Then $p-x^2=k^2$ and $q+y^2=k^2$

$$\therefore p=k^2+x^2 \text{ and } q=k^2+y^2$$

But we have

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$$

Integrating,we get

$$\boxed{z = \frac{x^3}{3} + k^2x + k^2y + \frac{y^3}{3} + c}$$

is the required complete solution.

5. Solve $q^2 - p = y - x$

Sol: Let $p - x = q^2 - y = k(say)$

Then $p = k + x \; and \; q = \sqrt{k + y}$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

Integrating,we get

$$\boxed{z = \frac{x^2}{2} + kx + \frac{2}{3}(k + y)^{\frac{3}{2}} + C}$$

is the required complete solution.

6.Solve $q = px + p^2$

Sol: Let $q = px + p^2 = k(say)$

Then we get

$$p^2 + px - k = 0 \; and \; q = k$$

Solving,we get

$p = \frac{-x \pm \sqrt{(x^2+4k}}{2}$ and $q = k$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

Integrating,we get

$$\boxed{z = -\frac{x^2}{4} + \frac{1}{2}[\frac{x}{2}\sqrt{(x^2 + 4k} + 2k \sinh^{-1}(\frac{x}{2\sqrt{k}}) + ky + C}$$

is the required complete solution.

STANDARD FORM IV: $Z = px + qy + f(p,q)$

An equation analogous to the clairaut's equation it is complete solution is $Z = ax + by + f(a,b)$ which is obtained by writing $a \; for \; p \; and \; b \; for \; q$ The differential equation which satisfies some specified conditions known as the boundary conditions.The differential equation together with these boundary conditions, constitute a boundary value problem

Solved Problems:

1. **Solve $z = px + qy + pq$**

Sol : The given PDE is form IV

Therefore complete solution is given by

$$z = ax + by + ab$$

2. Find the solution of $(p+q)(z-px-qy)=1$

Sol. The given equation can be written as

$$z-px-qy=\frac{1}{p+q}$$

$$\therefore z = px+qy+\frac{1}{p+q} \rightarrow (1)$$

Hence the complete solution of (1) is given by

$$\boxed{z = ax+by+\frac{1}{a+b}}$$

3. Solve $pqz = p^2\left(qx+p^2\right)+q^2\left(py+q^2\right)$

Sol. The given equation can be written as

$$pqz = p^2q\left(x+\frac{p^2}{q}\right)+q^2p\left(y+\frac{q^2}{p}\right)$$

$$\therefore z = p\left(x+\frac{p^2}{q}\right)+q\left(y+\frac{q^2}{p}\right)$$

$$\therefore z = px+qy+\left(\frac{p^3}{q}+\frac{q^3}{p}\right) \rightarrow (1)$$

Since it is in the form $z = px+qy+f(p,q)$

Hence the complete solution of (1) is given by

$$\boxed{z = ax+by+\frac{a^3}{b}+\frac{b^3}{a}}$$

4. Solve $z = px + qy + pq + q^2$

Sol. We have $z = px + qy + pq + q^2 \ldots\ldots\ldots\ldots\ldots(1)$

Since (1) is of the form $z = px + qy + f(p, q)$.

Hence the complete solution of (1) is given by

$$z = ax + by + ab + b^2 \ldots\ldots(2)$$

For singular solution, differentiating (2) partially w.r.t. a and b, we get

$$\frac{\partial z}{\partial a}=0, \frac{\partial z}{\partial b}=0,$$

Implies that

$0=x+b \ldots .(3)$ *and* $0=y+a+2b \ldots \ldots \ldots \ldots(4)$

Eliminating a, b between (2), (3) and (4), we get

$$z=x(2x-y)-xy-(2x-y)x+x^2$$

$\boxed{\therefore z=x^2}$

is the singular solution

EQUATIONS REDUCIBLE TO STANDARD FORMS:

EQUATIONS OF THE FORM$F(x^m p, y^n q)=0$ *where m and n are constants*

The above form of the equation of the type can be transformed to an equation of the form f(p,q)=0 By substitutions given below.

Case (i):- when $m \neq 1$ *and* $n \neq 1$

Put $X=x^{1-m}$ *and* $Y=y^{1-n}$ *then* $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}=P(1-m)x^{-m}$ *where* $P=\frac{\partial z}{\partial x}$

$$x^m p=P(1-m) \quad and\ q=\frac{\partial z}{\partial y}=\frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y}= Q(1-n)y^{-n}\, whereQ=\frac{\partial Z}{\partial Y} \rightarrow y^n q=Q(1-n)$$

Now the given equation reduces to $f[(1-m)P,(1-n)Q]=0$ *which is of the form* $f(P,Q)=0$

Case(ii):- when $m=1, n=1$

Put $X=logx$ *and* $Y=logy$ *then*

p=$\frac{\partial z}{\partial x}=\frac{\partial Z}{\partial X}\frac{\partial X}{\partial x}=\frac{\partial Z}{\partial X}\frac{1}{x}$ *implies* $px=P=$ *where* $P=\frac{\partial z}{\partial X}$

$$similarly\quad qy=Q \quad whereQ=\frac{\partial Z}{\partial Y}$$

now the given equation reduces to the form $f(P,Q)=0$

EQUATIONS OF THE FORM$F(x^m p, y^n q, z)=0$ *where m and n are constants*:

This can be reduced to an equation of the form $f(P,Q,z)=0$ by the substitutions given for the equation $F(x^m p, y^n q, z)=0$ as above.

Solved Problems:

1. **Solve the partial differential equation** $\frac{x^2}{p}+\frac{y^2}{q}=z$

Sol. Given equation can be written as

$x^2p^{-1}+y^2q^{-1}=z \ or \left(x^{-2}p\right)^{-1}+\left(y^{-2}q\right)^{-1}=z \rightarrow \ (1)$

This is of the form $f\left(x^m p, y^n q, z\right)=0$ with $m=-2, and\ n=-2.$

Put $X=x^{1-m}=x^{1-(-2)}=x^3$ and $Y=y^{1-n}=y^{1+2}=y^3$

Then $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}.\frac{\partial X}{\partial x}=P.3x^2$ where $P=\frac{\partial z}{\partial X}$

$\therefore x^{-2}p=3P$

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y}=Q.3y^2$ where $Q=\frac{\partial z}{\partial Y}$

$\therefore y^{-2}q = 3Q$

Now equation (1), becomes.

$(3P)^{-1} + (3Q)^{-1} = z \rightarrow$ (2)

Since (2) is of the form $f(P,Q,z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}$ *in* (1), *then we get* $Q = \frac{(a+1)}{3az}$

$$\therefore P = \frac{(a+1)}{3z}$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P}\,\boldsymbol{dX} + \boldsymbol{Q}\,\boldsymbol{dY}$,we get

$\frac{3az}{a+1}dz = (\boldsymbol{adX} + \boldsymbol{dY})$

Integrating,we get

$\int \frac{3az}{a+1}dz = (\boldsymbol{a}\int \boldsymbol{dX} + \int \boldsymbol{dY})$

$$\frac{3az^2}{2(a+1)} = (aX + Y) + c$$

$$\boxed{\therefore 3z^2 = 2\left(\frac{a+1}{a}\right)\left(x^3 + ay^3\right) + c_1}$$

,taking $c_1 = 2(\frac{a+1}{a})c$

Which is the required solution of (1)

2. **Solve the partial differential equation** $\frac{p}{x^2} + \frac{q}{y^2} = z$

Sol. The given equation can be written as

$px^{-2} + qy^{-2} = z \rightarrow (1)$

Since (1) is of the form $f(x^m p, y^n q, z) = 0$ With $m = -2, and\ n = -2$

Put $X = x^{1-m} = x^3$, and $Y = y^{1-n} = y^3$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2$ where $P = \frac{\partial z}{\partial X}$

$\therefore x^{-2}p = 3P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q3y^2$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^{-2}q = 3Q$

Equation (1) becomes, $3P + 3Q = z \rightarrow (2)$

Since (2) is of the form $f(P,Q,z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}$ *in* (1), *then we get* $Q = \frac{z}{3(a+1)}$

$$\therefore P = \frac{az}{3(a+1)}$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P}\,\boldsymbol{dX} + \boldsymbol{Q}\,\boldsymbol{dY}$,we get

$\frac{dz}{z} = \frac{1}{3(a+1)}(\boldsymbol{adX} + \boldsymbol{dY})$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{3(a+1)}(\boldsymbol{a} \int \boldsymbol{dX} + \int \boldsymbol{dY})$

$$\log z = \frac{1}{3(a+1)}(aX + Y) + C$$

$$\boxed{\Rightarrow \log z = \frac{1}{3(1+a)}\left(x^3 + ay^3\right) + c}$$

This is the complete solution of (1)

3. **Solve** $q^2y^2 = z(z - px)$

Sol. Given equation can be written as

$q^2y^2 = z^2 - zpx \, or \, (xp)z + (qy)^2 = z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = 1 \; and \; n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = P \cdot \frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = Q \cdot \frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes, $Pz + Q^2 = z^2 \rightarrow (2)$

Since (2) is of the form $f(P, Q, z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}$ in $(1), then \; we \; get$ $Q = \frac{z}{2}[-\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}]$

$$\therefore P = \frac{aZ}{2}[-\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}]$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P}\,\boldsymbol{dX} + \boldsymbol{Q}\,\boldsymbol{dY}$,we get

$\frac{dz}{z} = \frac{1}{2}[-\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}](\boldsymbol{adX} + \boldsymbol{dY})$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{2}[-\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}](\boldsymbol{a} \int \boldsymbol{dX} + \int \boldsymbol{dY})$

$\log z = \frac{1}{2}[-\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}](\boldsymbol{aX} + \boldsymbol{Y}) + \boldsymbol{c}$

$$\boxed{\therefore \log z = \frac{1}{2}[-\mathrm{a} \pm \sqrt{\mathrm{a}^2 + 4}](\boldsymbol{ax^3} + \boldsymbol{y^3}) + \boldsymbol{c}}$$

is the complete integeral of (1)

4. **Solve the partial differential equation** $p^2x^4 + y^2zq = 2z^2$

Sol. Given that $p^2x^4 + y^2zq = 2z^2$

Then given equation can be written as

$\left(px^2\right)^2 + \left(qy^2\right)z = 2z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with m=2 and n=2

Put $X = x^{1-m} = x^{1-2} = x^{-1} = \frac{1}{x}$ and $Y = y^{-1} = \frac{1}{x}$

Now $P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\left(\frac{-1}{x^2}\right)$, where $P = \frac{\partial z}{\partial X}$

$\therefore x^2 p = -P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\left(\frac{-1}{y^2}\right)$, where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^2 q = -Q$

Now equation (1) becomes, $P^2 - Qz = 2z^2$ or $P^2 - Qz = 2z^2 \rightarrow (2)$

Since (2) is of the form $f(P, Q, z) = 0$

Put $\boldsymbol{P} = \boldsymbol{aQ}$ in $(1), then\ we\ get$ $Q = \frac{z}{2a^2}[1 \pm \sqrt{8a^2+1}]$

$$\therefore P = \frac{Z}{2a}[1 \pm \sqrt{8a^2+1}]$$

Putting the values of Pand Q in $\boldsymbol{dz} = \boldsymbol{P\,dX} + \boldsymbol{Q\,dY}$,we get

$\frac{dz}{z} = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](\boldsymbol{adX} + \boldsymbol{dY})$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](\boldsymbol{a} \int \boldsymbol{dX} + \int \boldsymbol{dY})$

$\log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}]\ (\boldsymbol{aX} + \boldsymbol{Y}) + \boldsymbol{c}$

$\boxed{\therefore \log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](\boldsymbol{ax^3} + \boldsymbol{y^3}) + \boldsymbol{c}}$

Which is the complete integral of (1).

5. **Solve** $x^2p^2 + xpq = z^2$

Sol. The given equation can be written as

$\left(xp\right)^2 + \left(xp\right)q = z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p.y^n q, z\right) = 0$ with m=1 and n=0

Put $X = \log x$

Now $P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P\frac{1}{x}$, where

$P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

Equation (1) becomes, $P^2 + Pq = z^2 \rightarrow$ (2)

Since (2) is of the form $f(P, q, z) = 0$

Put $P = aq\ in\ (2), we\ get$

$q = \frac{z}{\sqrt{a(a+1)}}$, P=a$\frac{z}{\sqrt{a(a+1)}}$

But we have

$\boldsymbol{dz = P\,dX + q\,dy}$

Substuting P,q ,we get

$\frac{\boldsymbol{dz}}{\boldsymbol{z}} = \frac{1}{\sqrt{a(a+1)}}(\boldsymbol{a\,dX + dy})$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a(a+1)}}\left(\boldsymbol{a}\int \boldsymbol{dX + d\,y}\right)$$

$$\boxed{\sqrt{a(a+1)}\log z = (aX + y) + C}$$

be the complete integral of (1)

6.Solve $z = p^2x + q^2y$

Sol. Given that $z = p^2x + q^2y$

The given equation can be written as

$$\left(p\sqrt{x}\right)^2 + \left(q\sqrt{y}\right)^2 = z \; or \left(px^{\frac{1}{2}}\right)^2 + \left(qy^{\frac{1}{2}}\right)^2 = z \rightarrow (1)$$

This is of the form $f\left(x^m p, y^n q, z\right) = 0 \; with \, m = n = \frac{1}{2}$

Put $X = x^{1-m} = x^{1-\frac{1}{2}} = x^{\frac{1}{2}} \; and \; Y = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P\left(\frac{1}{2}x^{\frac{-1}{2}}\right), where \; P = \frac{\partial z}{\partial X}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x} = Q\left(\frac{1}{2}y^{\frac{-1}{2}}\right), where \, Q = \frac{\partial z}{\partial Y}$

$\therefore px^{\frac{1}{2}} = \frac{P}{2} and \; qy^{\frac{1}{2}} = \frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = z \; i.e. P^2 + Q^2 = 4z \rightarrow (2)$

This if of the form $f\left(P, Q, z\right) = 0$

Put $P = aQ \; in \; (2), we \; get$

$$a^2Q^2 + Q^2 = 4z$$

$Q = \sqrt{\frac{4z}{a^2+1}}$, P=a$\sqrt{\frac{4z}{a^2+1}}$,

But we have

$\boldsymbol{dz = P\,dX + Q\,dY}$

Substuting P,Q ,we get

$$dz = \sqrt{\frac{4z}{a^2+1}}(a\,dX + dY)$$

$$\frac{dz}{\sqrt{z}} = \frac{2}{\sqrt{a^2+1}}(a\,dX + dY)$$

Integrating on both sides

$$\int dz/\sqrt{z} = \frac{2}{\sqrt{a^2+1}}\left(a\int dX + \int dY\right)$$

$$\sqrt{(a^2+1)}\sqrt{z} = (aX + Y) + C$$

$$\boxed{\sqrt{(a^2+1)}\sqrt{z} = \left(a\sqrt{x} + \sqrt{y}\right) + C}$$

Which is the complete integral of (1)

7.Solve $x^2p^2 + y^2q^2 = z^2$

Sol: Given $x^2p^2 + y^2q^2 = z^2 \ldots\ldots(1)$

$$(xp)^2 + (yq)^2 = z^2$$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$$P^2 + Q^2 = z^2 \ldots\ldots(2)$$

$$Put\ P = aQ\ in\ (2), we\ get\ Q = \frac{z}{\sqrt{a^2+1}}; P = \frac{az}{\sqrt{a^2+1}}$$

But we have

$$dz = P\,dX + Q\,dY$$

Substuting P,Q ,we get

$$dz = \frac{z}{\sqrt{a^2+1}}(a\,dX + dY)$$

$$\frac{dz}{z} = \frac{1}{\sqrt{a^2+1}}(a\,dX + dY)$$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a^2+1}}\left(a\int dX + \int dY\right)$$

$$\sqrt{(a^2+1)}\log z = (aX + Y) + C$$

$$\boxed{\sqrt{(a^2+1)}\log z = (a\log x + \log y) + C}$$

is the Complete solution of (1)

8. **Solve $x^2p^2 + y^2q^2 = 1$**

Sol: Given $\boldsymbol{x^2p^2 + y^2q^2 = 1} \dots\dots (\mathbf{1})$

$$(xp)^2 + (yq)^2 = 1$$

Since (1) is of the form $f(x^m p, y^n q) = 0$ with $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X}.\dfrac{\partial X}{\partial x} = P.\dfrac{1}{x}$ where $P = \dfrac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial Y}.\dfrac{\partial Y}{\partial y} = Q.\dfrac{1}{y}$ where $Q = \dfrac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$P^2 + Q^2 = 1$......(2)

$$Put\ P = a\ in\ (2), we\ get\ Q = \sqrt{1 - a^2}$$

But we have

$\boldsymbol{dz = P\,dX + Q\,dY}$

Substuting P,Q ,we get

$\boldsymbol{dz = (a\,dX + \sqrt{1 - a^2}\,dY)}$

Integrating on both sides

$$\int \boldsymbol{dz} = (\boldsymbol{a} \int \boldsymbol{dX} + \sqrt{1 - a^2} \int \boldsymbol{dY})$$

z $= \left(aX + \sqrt{1 - a^2}Y\right) + C$

$$\boxed{z = \left(a \log x + \sqrt{1 - a^2} \log y\right) + C}$$

is the Complete solution of (1)

EQUATIONS OF THE FORM$\boldsymbol{F(z^n p, z^n q) = 0}$ *where n is a constant*:

Use the following substitution to reduce the above form to an equation of the form f(P,Q)=0

$$put \quad Z = \begin{cases} z^{n+1} & if\ n \neq -1 \\ \log z, & if\ n = -1 \end{cases}$$

EQUATIONS OF THE FORM$\boldsymbol{f(x, z^n p) = g(y, z^n q)}$ *where n is a constant*:

An equation of the above form can be reduced to an equation of the form f(P,Q)=0 by the substitutions given for the equation $F(z^n p, z^n q) = 0$ as above

Solved Problems :

1. **Solve** $z^2\left(p^2 + q^2\right) = x^2 + y^2$

Sol. Given that $z^2\left(p^2 + q^2\right) = x^2 + y^2$

The given equation can be written as

$z^2p^2+z^2q^2=x^2+y^2 \; or \; z^2p^2-x^2=y^2-z^2q^2$

Or $(zp)^2-x^2=y^2-(zq)^2 \rightarrow (1)$

Since (1) is the of the form $f\left(x,pz^n\right)=g\left(y,qz^n\right)$. with n=1

$\therefore put\; Z=z^{n+1}=z^{1+1}=z^2$

Then $\frac{\partial Z}{\partial x}=2z.\frac{\partial z}{\partial x} \Rightarrow P=2zp$ where $P=\frac{\partial z}{\partial x}$

$\therefore pz=\frac{P}{2}$

and $\frac{\partial Z}{\partial y}=2z.\frac{\partial z}{\partial y} \Rightarrow Q=2zq$ where $Q=\frac{\partial z}{\partial y} \therefore qz=\frac{Q}{2}$

$\therefore$ Equation (1) becomes, $\frac{P^2}{4}-x^2=y^2-\frac{Q^2}{4}$

$i.e., P^2-4x^2=4y^2-Q^2 \rightarrow (2)$

This is of the form $f_1(x,P)=f_2(y,Q)$

Let $P^2-4x^2=4y^2-Q^2=4k^2$ (say)

$\therefore P^2-4x^2=4k^2 \; and \; 4y^2-Q^2=4k^2$

$\Rightarrow P^2=4x^2+4k^2 \; and \; Q^2=4y^2-4k^2$

$\therefore P=2\sqrt{x^2+k^2} \; and \; Q=2\sqrt{y^2-k^2}$

We have $dZ=\frac{\partial Z}{\partial x}.dx+\frac{\partial Z}{\partial y}dy$

$=Pdx+Qdy$ [By total differentiation]

$\therefore dZ=2\sqrt{x^2+k^2}\,dx+2\sqrt{y^2-k^2}\,dy$

Integrating on both sides

$Z=2\int\sqrt{x^2+k^2}\,dx+2\int\sqrt{y^2-k^2}\,dy$

$=2\left[\frac{x}{2}\sqrt{x^2+k^2}+\frac{k^2}{2}\sinh^{-1}\left(\frac{x}{k}\right)\right]+2\left[\frac{x}{2}\sqrt{y^2-k^2}-\frac{k^2}{2}\cosh^{-1}\left(\frac{y}{k}\right)\right]+c$

$=x\sqrt{x^2+k^2}+k^2\sinh^{-1}\left(\frac{x}{k}\right)+x\sqrt{y^2-k^2}+k^2\cosh^{-1}\left(\frac{y}{k}\right)+c$

$or\; z^2=x\sqrt{x^2+k^2}+y\sqrt{y^2-k^2}+k^2\left[\sinh^{-1}\left(\frac{x}{k}\right)-\cosh^{-1}\left(\frac{y}{k}\right)\right]+c$

$$\boxed{or\; z^2=x\sqrt{x^2+k^2}+y\sqrt{y^2-k^2}+k^2\log\left(\frac{x+\sqrt{x^2+k^2}}{y+\sqrt{y^2-k^2}}\right)+c}$$

This is the complete solution of (1)

2. **Solve the partial differential equation.** $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

Sol. Given that $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

The given equation can be written as

$(pz)^2\sin^2 x+(qz)^2\cos^2 y=1 \, or \, (pz)^2\sin^2 x=1-(qz)^2\cos^2 y\rightarrow(1)$

Since (1) is of the form $f\left(x,pz^n\right)=g\left(y,qz^n\right)$ with n=1.

Put $Z=z^{n+1}=z^2$

Now $\frac{\partial Z}{\partial x}=2z.\frac{\partial z}{\partial x}\Rightarrow P=2zp \, or \, pz=\frac{P}{2}$ where $P=\frac{\partial Z}{\partial x}; Q=\frac{\partial Z}{\partial y}$

and $\frac{\partial Z}{\partial y}=2z.\frac{\partial z}{\partial y}\Rightarrow Q=2zq \, or \, qz=\frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2\sin^2 x=1-\left(\frac{Q}{2}\right)^2\cos^2 y$

$i.e.\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y\rightarrow(2)$

This is of the form $f_1(x,p)=f_2(y,q)$

Let $\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y=k^2$ (constant)

$\therefore\frac{P^2}{4}\sin^2 x=k^2 \, and \, 1-\frac{Q^2}{4}\cos^2 y=k^2$

$\Rightarrow P^2\sin^2 x=4k^2 \, and \, Q^2\cos^2 y=4\left(1-k^2\right)$

$\Rightarrow P=\frac{2k}{\sin x} \, and \, Q=\frac{2\sqrt{1-k^2}}{\cos y}$

We have $dZ=\frac{\partial Z}{\partial x}dx+\frac{\partial Z}{\partial y}dy$ [By total differential]

$\therefore dZ=Pdx+Qdy$

$dZ=\frac{2k}{\sin x}dx+\frac{2\sqrt{1-k^2}}{\cos y}dy$

Integrating on both sides

$$z=2k\int\csc x\,dx+2\sqrt{1-k^2}\int\sec y\,dy$$

$=2k\log\left(\operatorname{cosec} x-\cot x\right)+2\sqrt{1-k^2}\log\left(\sec y+\tan y\right)+c$

$$\boxed{\therefore z^2=2k\log\left(\operatorname{cosec} x-\cot x\right)+2\sqrt{1-k^2}\log\left(\sec y+\tan y\right)+c}$$

This is the required complete solution of (1)

3.Solve $(x+pz)^2+(y+qz)^2=1$

Sol:Given $(x+pz)^2+(y+qz)^2=1$…….(1)

$since\ (1)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y) = 0 \quad n = 1$

Put $Z = z^{n+1} = z^2$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z} = 2z\ implies\ that\ \frac{\partial z}{\partial Z} = \frac{1}{2z}$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = \frac{P}{2z}$ implies $\frac{\partial Z}{\partial x} = \frac{P}{2} = zp; Similarly\ we\ get\ qz = \frac{Q}{2}$

Substitute in (1),we get

$(x + \frac{P}{2})^2 + (y + \frac{Q}{2})^2 = 1$

Separating $P\ and\ x\ from\ Q\ and\ y$, the given equation can be written as.

$$\left(x + \frac{P}{2}\right)^2 = 1 - \left(y + \frac{Q}{2}\right)^2 = K^2$$

$$\left(x + \frac{P}{2}\right)^2 = K^2\ AND\ 1 - \left(y + \frac{Q}{2}\right)^2 = K^2$$

$$\left(x + \frac{P}{2}\right) = K$$

Implies that

$Q = 2(\sqrt{(1 - K^2)} - y)$

$P = 2(K - x)$

We have $$dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$$

$$\therefore dz = 2(k - x)dx + 2\left[\sqrt{1 - k^2} - y\right]dy$$

Integrating on both sides

$$z = 2\int (k - x)dx + 2\int \left[\sqrt{1 - k^2} - y\right]dy + c$$

$$z = 2(kx - \frac{x^2}{2}) + 2\left[\left(\sqrt{1 - k^2}\right)y - \frac{y^2}{2}\right] + c$$

$$\boxed{\therefore z = 2kx - x^2 + 2\left(\sqrt{1 - k^2}\right)y - y^2 + c}$$

This is the complete solution of (1).

4. Solve $\boldsymbol{z(p^2 - q^2) = x - y}$

Sol: Given

$$\boldsymbol{z(p^2 - q^2) = x - y} \ldots\ldots\ldots\ldots (\mathbf{1})$$

$$\boldsymbol{(z^{\frac{1}{2}}p)^2 - (z^{\frac{1}{2}}q)^2 = x - y} \ldots..(\mathbf{2})$$

$$since\ (2)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y) = 0 \quad n = \frac{1}{2}$$

Put $Z = z^{n+1} = z^{\frac{3}{2}}$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z} = \frac{3}{2}z^{\frac{1}{2}}$

$$implies\ that\ \frac{\partial z}{\partial Z} = \frac{2}{3z^{\frac{1}{2}}}$$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$=Pimplies $\frac{2}{3}P = z^{\frac{1}{2}}p; Similarly\ we\ get \frac{2}{3}Q = z^{\frac{1}{2}}q$ Substitute in (2),we get

$$(\frac{2}{3}P)^2 - (\frac{2}{3}Q)^2 = \boldsymbol{x - y}$$

Separating $P\ and\ x\ from\ Q\ and\ y$, the given equation can be written as.

$$(\frac{2}{3}P)^2 - \boldsymbol{x} = -\boldsymbol{y} + (\frac{2}{3}Q)^2 = \boldsymbol{k}$$

Solving, we get

$\boldsymbol{P} = \frac{3}{2}\sqrt{\boldsymbol{k+x}}$ **and** $\boldsymbol{Q} = \frac{3}{2}\sqrt{\boldsymbol{k+y}}$

We have $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$ [By total differential]

$\therefore dZ = Pdx + Qdy$

$dZ = \frac{3}{2}[\sqrt{\boldsymbol{k+x}}\boldsymbol{dx} + \sqrt{\boldsymbol{k+y}}\boldsymbol{dy}]$

Integrating on both sides

$$Z = \frac{3}{2}[\int \sqrt{\boldsymbol{k+x}}\boldsymbol{dx} + \int \sqrt{\boldsymbol{k+y}}\ dy$$

$$z^{\frac{3}{2}} = (\ k + x\ )^{\frac{3}{2}} + (k+y)^{\frac{3}{2}} + c$$

This is the required complete solution of (1)

Methods Of Separation Of Variables:

This method is used to reduce one partial differential equation to two or more ordinary differential equations,each one involving one of the independent variables.This will be done by separating these variables from the beginning. This method is explained through following examples.

1. **Solve by the method of separation of variables $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$ where U(x,0)=6e^{-3x}**

Solution:- Given equation is $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$----------(1)

Let U(x,t)= X(x) T(t) =XT ---------------(2)

be a solution of (1)

Differentiating (2) partially w.r.t x ant t

$$\frac{\partial U}{\partial x} = X'T \quad , \quad \frac{\partial U}{\partial t} = T'X$$

Put these values in equation (1), we have

$X'T = 2\,T'X\ + XT$ Dividing by XT

$\frac{X'}{X} = 2\frac{T'}{T} + 1$--------(3)

Since L.H.S is a function of 'x' and the R.H.S is a function of 't' where x and t are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 't' if and only if both sides are equal to a constant.

Therefore $\frac{X'}{X} = 2\frac{T'}{T} + 1$=k--------(4) where k is a constant

Now from (4) $\frac{X'}{X}$ = k----(5) and $2\frac{T'}{T} + 1$=k-------(6)

Now consider (5) $\frac{X'}{X}$ = k $\Rightarrow$ $X' - kX = 0$ $\Rightarrow X = C_1 e^{kx}$

Now consider (6) $2\frac{T'}{T} + 1$=k $\Rightarrow$ $T' - \left(\frac{k-1}{2}\right)T = 0$ $\Rightarrow T = C_2 e^{\left(\frac{k-1}{2}\right)t}$ ---(8)

Substituting the values of X and T in (2) we get

$$U(x,t) = X = C_1 e^{kx} C_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$U(x,t) = X = Ae^{kx} e^{\left(\frac{k-1}{2}\right)t} \quad (where\ A=C_1 C_2)$$

Put t=0 in the above equation ,we have U(x,0) = A e^{kx} -----(9)

but given that U(x,0)=$6e^{-3x}$--------(10)

from (9) and (10) we have A e^{kx} =$6e^{-3x}$

A=6 and k=-3 the solution of the given equation becomes

$$\boxed{U(x,t) = X = 6e^{-3x} e^{(-2)t} = 6e^{-(3x+2t)}}$$

2. Solve the equation by the method of separation of variables$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$

Sol: Given equation is $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$ ----------(1)

Let U(x,y)= X(x) Y(y) =X Y ---------------(2)

be a solution of (1)

Differentiating (2) partially w.r.t x ant y

$$\frac{\partial U}{\partial x} = X'Y \quad , \quad \frac{\partial U}{\partial y} = Y'X \quad \frac{\partial^2 U}{\partial x^2} = X''Y$$

Put these values in equation (1), we have

$$X''Y = Y'X + 2XY$$

Dividing by XY on both sides we have $\frac{X''}{X} = \frac{Y''}{Y}$+2

$$\frac{X''}{X} - 2 = \frac{Y''}{Y}\text{-----------------(3)}$$

Since L.H.S is a function of 'x' and the R.H.S is a function of 'y' where x and y are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 'y' if and only if both sides are equal to a constant.

$\frac{X''}{X} - 2 = \frac{Y''}{Y} = k$--------(4)

Now from (4)

$$\frac{X''}{X} - 2 = k - - - - - - -(5)$$

And $$\frac{Y''}{Y} = k - - - - - - -(6)$$

From (5) $X'' - 2X = kX$ $X'' - (2 + k)X = 0$

Which is second order differential equation

Auxiliary equation is $m^2 - (2 + k) = 0$ $\rightarrow$ m=$\pm\sqrt{(2 + k)}$

Solution of the given equation (5) is $X = C_1 e^{\sqrt{(2+k)}}$

Now consider equation (6) $Y' = kY$ $\rightarrow \frac{Y'}{Y} = k$

Integrating on both sides we get logy = ky + logC_3

$$\Rightarrow \log\left(\frac{Y}{C_3}\right) = ky \quad \Rightarrow Y = C_3 e^{ky} - - -(8)$$

Substituting the values of X and Y in (2) we have

$$U = \left[C_1 e^{\sqrt{(2+k)}x} + C_2 e^{-\sqrt{(2+k)}x}\right] C_3 e^{ky}$$

$$U = \left[Ae^{\sqrt{(2+k)}x} + Be^{-\sqrt{(2+k)}x}\right] e^{ky}$$

Where $A = C_1 C_3 \quad and \quad B = C_1 C_2$

UNIT V

LAPLACE TRANSFORMS

INTRODUCTION

Laplace Transformations were introduced by Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician known as a Newton of French. Laplace Transformations is a powerful technique, it replaces operations of calculus by operations of algebra. An Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation by this method.

USES

- Particular Solution is obtained without first determining the general solution.
- Non-Homogeneous Equations are solved without obtaining the complementary integral.
- L.T is applicable not only to continuous functions but also to piecewise continuous functions, complicated periodic functions, step functions and impulse functions.

APPLICATIONS:

- L.T is very useful in obtaining solution of linear differential equations, both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations, solution of linear difference equations and in the evaluation of definite integrals.

DEFINITION:

Let f (t) be a function of'tʼ defined for all positive values of t. Then Laplace transforms of f (t) is denoted by L {f (t)} is defined by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \bar{f}(s) \rightarrow (1)$$

provided that the integral exists. Here the parameter's' is a real (or) complex number.

The relation (1) can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$

In such a case the function f(t) is called the inverse Laplace transform of $\bar{f}(s)$ .The symbol 'L' which transform f(t) into $\bar{f}(s)$ is called the Laplace transform operator. The symbol 'L^{-1}' which transforms $\bar{f}$(s) to f (t) can be called the inverse Laplace transform operator.

Conditions for Laplace Transforms

Exponential order: A function f (t) is said to be of exponential order 'a' If $\underset{t\to\infty}{lt}\, e^{-st} f(t) =$ a finite quantity.

Ex: (i). The function t^2 is of exponential order

(ii). The function e^{t^3} is not of exponential order (which is not finite quantity)

Piece – wise Continuous function: A function f (t) is said to be piece-wise continuous over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided into a finite number of sub intervals, in each of which f (t) is continuous and has both right and left hand limits at every end point of the subinterval.

Sufficient conditions for the existence of the Laplace transform of a function:

The function f (t) must satisfy the following conditions for the existence of the L.T.

(i).The function f(t) must be piece-wise continuous (or sectionally continuous) in any limited interval $0 < a \le t \le b$.

(ii).The function f (t) is of exponential order.

Laplace Transforms of standard functions:

1. Prove that $L\{1\} = \frac{1}{s}$

Proof: By definition

$$L\{1\} = \int_0^\infty e^{-st}.1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s}\; if\; s > 0$$

$$L\{1\} = \frac{1}{s}\; \left(\therefore e^{-\infty} = 0\right)$$

2. Prove that $L\{t\} = \frac{1}{s^2}$

Proof: By definition

$$L\{t\} = \int_0^\infty e^{-st}.tdt = \left[t.\left(\frac{e^{-st}}{-s}\right) - \int 1.\frac{e^{-st}}{-s}dt\right]_0^\infty$$

$$= \left[t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right]_0^\infty = \frac{1}{s^2}$$

3. Prove that $L\{t^n\} = \frac{n!}{s^{n+1}}$ **where n is a +ve integer**

Proof: By definition $L\{t^n\} = \int_0^\infty e^{-st}.t^n dt = \left[t^n.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n.t^{n-1}.\frac{e^{-st}}{-s}dt$

$$= 0 - 0 + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} dt$$

$$= \frac{n}{s} L\left\{t^{n-1}\right\}$$

Similarly $L\left\{t^{n-1}\right\} = \frac{n-1}{s} L\left\{t^{n-2}\right\}$

$$L\left\{t^{n-2}\right\} = \frac{n-2}{s} L\left\{t^{n-3}\right\}$$

By repeatedly applying this, we get

$$L\left\{t^n\right\} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.....\frac{2}{s}.\frac{1}{s}L\left\{t^{n-n}\right\}$$

$$= \frac{n!}{s^n} L\{1\} = \frac{n!}{s^n}.\frac{1}{s} = \frac{n!}{s^{n+1}}$$

Note: $L\left\{t^n\right\}$ can also be expressed in terms of Gamma function.

$$i.e., \mathrm{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}} \left(\because \Gamma(n+1) = n!\right)$$

Def: If n>0 then Gamma function is defined by $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

We have $L\left\{t^n\right\} = \int_0^\infty e^{-st}.t^n dt$

Putting x=st on R.H.S, we get

$$L\left\{t^n\right\}t = \int_0^\infty e^{-x}.\frac{x^n}{s^n}.\frac{1}{s}dx \qquad \begin{pmatrix} x = st \\ \frac{1}{s}dx = dt \end{pmatrix}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x}.x^n dx \qquad \begin{pmatrix} When & t = 0, x = 0 \\ When & t = \infty, x = \infty \end{pmatrix}$$

$$L\left\{t^n\right\} = \frac{1}{s^{n+1}}.\Gamma(n+1)$$

If 'n' is a +ve integer then $\Gamma(n+1) = n!$

$$\therefore L\left\{t^n\right\} = n!/s^{n+1}$$

Note: The following are some important properties of the Gamma function.

1. $\Gamma(n+1) = n.\Gamma(n)$ *if* $n > 0$
2. $\Gamma(n+1) = n!$ if n is a +ve integer
3. $\Gamma(1) = 1, \Gamma\left(1/2\right) = \sqrt{\pi}$

Note: Value of $\Gamma(n)$ in terms of factorial

$$\Gamma(2) = 1\times\Gamma(1) = 1!$$
$$\Gamma(3) = 2\times\Gamma(2) = 2!$$
$$\Gamma(4) = 3\times\Gamma(3) = 3! \text{ and so on.}$$

In general $\Gamma(n+1) = n!$ provided 'n' is a +ve integer.

Taking n=0, it defined 0! = $\Gamma(1) = 1$

4. **Prove that** $L\left\{e^{at}\right\} = \dfrac{1}{s-a}$

Proof: By definition,

$$L\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$
$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$
$$= \frac{-e^{-\infty}}{s-a} + \frac{e^0}{s-a} = \frac{1}{s-a}\, if\ s > a$$

Similarly $L\left\{e^{-at}\right\} = \dfrac{1}{s+a}\, if\ s > -a$

5. **Prove that** $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$

Proof: $L\{\sinh at\} = L\left\{\dfrac{e^{at} - e^{-at}}{2}\right\} = \dfrac{1}{2}\left[L\left\{e^{at}\right\} - L\left\{e^{-at}\right\}\right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] = \frac{2a}{2\left(s^2-a^2\right)} = \frac{a}{s^2-a^2}$$

6. **Prove that** $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$

Proof: $L\{\cosh at\} = L\left\{\dfrac{e^{at} + e^{-at}}{2}\right\}$

$$= \frac{1}{2}\left[L\left\{e^{at}\right\} + L\left\{e^{-at}\right\}\right] = \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$
$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right] = \frac{2s}{2\left(s^2-a^2\right)} = \frac{s}{s^2-a^2}$$

7. **Prove that** $L\{\sin at\} = \dfrac{a}{s^2 + a^2}$

Proof: By definition, $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$

$$=\left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at-a\cos at\right)\right]_0^\infty$$

$$\left[\because\int e^{ax}\sin bx dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)\right]$$

$$=\frac{a}{s^2+a^2}$$

8. Prove that $L\{\cos at\}=\dfrac{s}{s^2+a^2}$

Proof: We know that $L\{e^{at}\}=\dfrac{1}{s-a}$

Replace 'a' by 'ia' we get

$$L\{e^{iat}\}=\frac{1}{s-ia}=\frac{s+ia}{(s-ia)(s+ia)}$$

$$i.e., \mathrm{L}\{cosat+i\sin at\}=\frac{s+ia}{s^2+a^2}$$

Equating the real and imaginary parts on both sides, we have

$$L\{\cos at\}=\frac{s}{s^2+a^2}\ and\ L\{\sin at\}=\frac{a}{s^2+a^2}$$

Solved Problems :

1. Find the Laplace transforms of $(t^2+1)^2$

Sol: Here $\mathrm{f(t)}=(t^2+1)^2=t^4+2t^2+1$

$$L\{(t^2+1)^2\}=L\{t^4+2t^2+1\}=L\{t^4\}+2L\{t^2\}+L\{1\}$$

$$=\frac{4!}{s^{4+1}}+2.\frac{2!}{s^3}+\frac{1}{s}=\frac{4!}{s^5}+2.\frac{2!}{s^3}+\frac{1}{s}$$

$$=\frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}=\frac{1}{s^5}(24+4s^2+s^4)$$

2. Find the Laplace transform of $L\left\{\dfrac{e^{-at}-1}{a}\right\}$

Sol: $$L\left\{\frac{e^{-at}-1}{a}\right\}=\frac{1}{a}L\{e^{-at}-1\}=\frac{1}{a}\left[L\{e^{-at}\}-L\{1\}\right]$$

$$=\frac{1}{a}\left[\frac{1}{s+a}-\frac{1}{s}\right]=-\frac{1}{s(s+a)}$$

3. Find the Laplace transform of Sin2tcost

Sol: W.K.T $\sin 2t\cos t=\dfrac{1}{2}[2\sin 2t\cos t]=\dfrac{1}{2}[\sin 3t+\sin t]$

$$\therefore L\{\sin 2t\cos t\} = L\left\{\frac{1}{2}[\sin 3t+\sin t]\right\} = \frac{1}{2}\left[\mathrm{L}\{\sin 3t\}+L\{\sin t\}\right]$$

$$=\frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]=\frac{2(s^2+3)}{(s^2+1)(s^2+9)}$$

4. Find the Laplace transform of Cosh^{2}2t

Sol: w.k.t $\cosh^2 2t=\frac{1}{2}[1+\cosh 4t]$

$$L\left\{\cosh^2 2t\right\}=\frac{1}{2}[L(1)+L\{\cosh 4t\}]$$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]=\frac{s^2-8}{s(s^2-16)}$$

5. Find the Laplace transform of Cos^{3}3t

Sol: Since cos9t=cos3(3t)

cos9t=4cos^{3}3t-3cos3t (or) $\cos^3 3t=\frac{1}{4}[\cos 9t+3\cos 3t]$

$$L\{\cos^3 3t\}=\frac{1}{4}L\{\cos 9t\}+\frac{3}{4}L\{\cos 3t\}$$

$$\therefore \qquad =\frac{1}{4}.\frac{s}{s^2+81}+\frac{3}{4}.\frac{s}{s^2+9}$$

$$=\frac{\mathrm{s}}{4}\left[\frac{1}{s^2+81}+\frac{3}{s^2+9}\right]=\frac{s\left(s^2+63\right)}{\left(s^2+9\right)\left(s^2+81\right)}$$

6. Find the Laplace transforms of $\left(\sin t+\cos t\right)^2$

Sol: Since $\left(\sin t+\cos t\right)^2=\sin^2 t+\cos^2 t+2\sin t\cos t \;=1+\sin 2t$

$$L\{(\sin t+\cos t)^2\}=L\{1+\sin 2t\}$$

$$=L\{1\}+L\{\sin 2t\}$$

$$=\frac{1}{s}+\frac{2}{s^2+4}=\frac{s^2+2s+4}{s\left(s^2+4\right)}$$

7. Find the Laplace transforms of cost cos2t cos3t

Sol: $\cos t\cos 2t\cos 3t=\frac{1}{2}.\cos t\left[2.\cos 2t.\cos 3t\right]$

$$=\frac{1}{2}\cos t[\cos 5t+\cos t]=\frac{1}{2}[\cos t\cos 5t+\cos^2 t]$$

$$=\frac{1}{4}\left[2\cos t\cos 5t+2\cos^2 t\right]=\frac{1}{4}\left[\left(\cos 6t+\cos 4t\right)+\left(1+\cos 2t\right)\right]$$

$$= \frac{1}{4}[1+\cos 2t+\cos 4t+\cos 6t]$$

$$\therefore L\{\cos t\cos 2t\cos 3t\} = \frac{1}{4}L\{1+\cos 2t+\cos 4t+\cos 6t\}$$

$$= \frac{1}{4}[L\{1\}+L\{\cos 2t\}+L\{\cos 4t\}+L\{\cos 6t\}]$$

$$= \frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+4}+\frac{s}{s^2+16}+\frac{s}{s^2+36}\right]$$

8. Find L.T. of Sin^2t

Sol: $$L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$$

$$= \frac{1}{2}[\text{L}\{1\}-L\{\cos 2t\}] = \frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

9. Find $\mathbf{L}(\sqrt{t})$

Sol: $$L\left\{\sqrt{t}\right\} = L\left[t^{1/2}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$$ where n is not an integer

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \quad \because \Gamma(n+1) = n.\Gamma(n)$$

10. Find $L\ \{\sin(\omega t + \alpha)\}$, where α a constant is

Sol: $$L\{\sin(\omega t+\alpha)\} = L\{\sin\omega t\cos\alpha + \cos\omega t\sin\alpha\}$$

$$= \cos\alpha\ L\{\sin\omega t\} + \sin\alpha\ L\{\cos\omega t\}$$

$$= \cos\alpha \frac{\omega}{s^2+\omega^2} + \sin\alpha \frac{\omega}{s^2+\omega^2}$$

Properties of Laplace transform:

Linearity Property:

Theorem1: The Laplace transform operator is a Linear operator.

i.e. $(i).L\{cf(t)\} = c.\text{L}\{f(t)\}$ $\quad (ii).L\{f(t)+g(t)\} = L\{f(t)\}+L\{g(t)\}$ Where 'c' is constant

Proof: (i) By definition

$$L\{cf(t)\} = \int_0^\infty e^{-st}cf(t)\,dt = c\int_0^\infty e^{-st}f(t)\,dt = cL\{f(t)\}$$

(ii) By definition

$$L\{f(t)+g(t)\}=\int_0^\infty e^{-st}\{f(t)+g(t)\}dt$$

$$=\int_0^\infty e^{-st}f(t)dt+\int_0^\infty e^{-st}g(t)dt=L\{f(t)\}+L\{g(t)\}$$

Similarly the inverse transforms of the sum of two or more functions of ‘s’ is the sum of the inverse transforms of the separate functions.

Thus, $L^{-1}\{\overline{f}(s)+\overline{g}(s)\}=L^{-1}\{\overline{f}(s)\}+L^{-1}\{\overline{g}(s)\}=f(t)+g(t)$

Corollary: $L\{c_1f(t)+c_2g(t)\}=c_1L\{f(t)\}+c_2L\{g(t)\}$, where c_1, c_2 are constants

Theorem2: If a, b, c be any constants and f, g, h any functions of t, then

$$L\{af(t)+bg(t)-ch(t)\}=a.L\{f(t)\}+b.L\{g(t)\}-cL\{h(t)\}$$

Proof: By the definition

$$L\{af(t)+bg(t)-ch(t)\}=\int_0^\infty e^{-st}\{af(t)+bg(t)-ch(t)\}dt$$

$$=a.\int_0^\infty e^{-st}f(t)dt+b\int_0^\infty e^{-st}g(t)dt-c\int_0^\infty e^{-st}h(t)dt$$

$$=a.L\{f(t)\}+bL\{g(t)\}-cL\{h(t)\}$$

Change of Scale Property:

If $L\{f(t)\}=\overline{f}(s)$ $then$ $L\{f(at)\}=\frac{1}{a}.\overline{f}\left(\frac{s}{a}\right)$

Proof: By the definition we have

$$L\{f(at)\}=\int_0^\infty e^{-st}f(at)dt$$

$$Put\quad at=u\Rightarrow dt=\frac{du}{a}$$

$$when\ \ t\to\infty\quad then\ \ u\to\infty\ \ and\ t=0\ \ then\ u=0$$

$$\therefore L\{f(at)\}=\int_0^\infty e^{-\frac{su}{a}}f(u)\frac{du}{a}=\frac{1}{a}.\int_0^\infty e^{-\left(\frac{s}{a}\right).u}f(u)du=\frac{1}{a}.\overline{f}\left(s/a\right)$$

Solved Problems :

1. Find $L\{\sinh 3t\}$

Sol: $L\{\sinh t\}=\frac{1}{s^2-1}=\overline{f}(s)$

$\therefore L\{\sinh 3t\}=\frac{1}{3}\overline{f}(s/3)$(Change of scale property)

$$=\frac{1}{3}\frac{1}{\left(s/3\right)^2-1}=\frac{3}{s^2-9}$$

2. Find $L\{\cos 7t\}$

Sol: $L\{\cos t\}=\frac{s}{s^2+1}=\overline{f}(s)\ (say)$

$L\{\cos 7t\}=\frac{1}{7}\overline{f}\left(s/7\right)$ (Change of scale property)

$L\{\cos 7t\}=\frac{1}{7}\frac{s/7}{\left(s/7\right)^2+1}=\frac{s}{s^2+49}$

First shifting property:

If $L\{f(t)\}=\overline{f}(s)$ then $L\{e^{at}f(t)\}=\overline{f}(s-a)$

Proof: By the definition

$$L\{e^{at}f(t)\}=\int_0^\infty e^{-st}e^{at}f(t)dt$$

$$=\int_0^\infty e^{-(s-a)t}f(t)dt$$

$$=\int_0^\infty e^{-ut}f(t)dt \text{ where } u=s-a$$

$$=\overline{f}(u)=\overline{f}(s-a)$$

Note: Using the above property, we have $L\{e^{-at}f(t)\}=\overline{f}(s+a)$

Applications of this property, we obtain the following results

1. $L\{e^{at}t^n\}=\frac{n!}{(s-a)^{n+1}}\left[\because L(t^n)=\frac{n!}{s^{n+1}}\right]$

2. $L\{e^{at}\sin bt\}=\frac{b}{(s-a)^2+b^2}\left[\because L(\sin bt)=\frac{b}{s^2+b^2}\right]$

3. $L\{e^{at}\cos bt\}=\frac{s-a}{(s-a)^2+b^2}\left[\because L(\cos bt)=\frac{s}{s^2+b^2}\right]$

4. $L\{e^{at}\sinh bt\}=\frac{b}{(s-a)^2-b^2}\left[\because L(\sinh bt)=\frac{b}{s^2-b^2}\right]$

5. $L\{e^{at}\cosh bt\}=\frac{s-a}{(s-a)^2-b^2}\left[\because L(\cosh bt)=\frac{s}{s^2-b^2}\right]$

Solved Problems :

1. Find the Laplace Transforms of t^3e^{-3t}

Sol: Since $L\{t^3\}=\frac{3!}{s^4}$

Now applying first shifting theorem, we get

$$L\{t^3 e^{-3t}\} = \frac{3!}{(s+3)^4}$$

2. Find the L.T. of $e^{-t}\cos 2t$

Sol: Since $L\{\cos 2t\} = \frac{s}{s^2+4}$

Now applying first shifting theorem, we get

$$L\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Find L.T of $e^{2t}cos^2t$

Sol: - $L\,[e^{2t}cos^2t] = L\,[e^{2t}(\frac{1+cos2t}{2})]$

$$=\frac{1}{2}\{L[e^{2t}] + L[e^{2t}cos2t]\}$$

$$=\frac{1}{2}\left(\frac{1}{s-2}\right)+\frac{1}{2}\{L[cos2t]\}_{s\to s-2}$$

$$=\frac{1}{2}\left(\frac{1}{s-2}\right)+\frac{1}{2}\frac{s-2}{(s-2)^2+2^2}$$

$$=\frac{1}{2}\left(\frac{1}{s-2}\right)+\frac{1}{2}\frac{s-2}{(s^2-4s+8)}$$

Second translation (or) second Shifting theorem:

If $L\{f(t)\} = \overline{f}(s) and\ g(t) = \begin{cases} f(t-a), & t>a \\ 0 & t<a \end{cases} then\ L\{g(t)\} = e^{-as}\overline{f}(s)$

Proof: By the definition

$$L\{g(t)\} = \int_0^\infty e^{-st}\, g(t)dt = \int_0^a e^{-st}\, g(t)dt + \int_a^\infty e^{-st}\, g(t)dt$$

$$= \int_0^\infty e^{-st}.odt + \int_a^\infty e^{-st}\, f(t-a)dt = \int_a^\infty e^{-st}\, f(t-a)dt$$

Let t-a = u so that dt = du And also u = 0 when t = a and u $\to\infty$ when t $\to\infty$

$$\therefore L\{g(t)\} = \int_0^\infty e^{-s(u+a)}\, f(u)du = e^{-as}\int_0^\infty e^{-su}\, f(u)du = e^{-as}\int_a^\infty e^{-st}\, f(t)dt$$

$$= e^{-as}L\{f(t)\} = e^{-as}\overline{f}(s)$$

Another Form of second shifting theorem:

If $L\{f(t)\} = \overline{f}(s)\ and\ a > 0\ then\ L\{F(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$

where H (t) = $\begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$ and H(t) is called Heaviside unit step function.

Proof: By the definition

$$L\{F(t-a)H(t-a)\} = \int_0^\infty e^{-st}\, F(t-a)H(t-a)dt \to (1)$$

Put t-a=u so that dt= du and also when t=0, u=-a when t $\to\infty$, u$\to\infty$

Then $L\{F(t-a)H(t-a)\} = \int_a^\infty e^{-s(u+a)}\, F(u)H(u)du. \quad [by\ eq(1)]$

$$= \int_{-a}^{0} e^{-s(u+a)} F(u) H(u) du + \int_{0}^{\infty} e^{-s(u+a)} F(u) H(u) du$$

$$= \int_{-a}^{0} e^{-s(u+a)} F(u).0du + \int_{0}^{\infty} e^{-s(u+a)} F(u).1du$$

[Since By the definition of H (t)]

$$= \int_{0}^{\infty} e^{-s(u+a)} F(u)du = e^{-as} \int_{a}^{\infty} e^{-su} F(u)du$$

$= e^{-sa} \int_{0}^{\infty} e^{-st} F(t) dt$ by property of Definite Integrals

$$= e^{-as} L\{F(t)\} = e^{-as} \overline{f}(s)$$

Note: $H(t-a)$ *is also denoted by* $u(t-a)$

Solved Problems

1. Find the L.T. of g (t) when $g(t) = \begin{cases} \cos\left(t - \pi/3\right) & if\ t > \pi/3 \\ 0 & if\ t < \pi/3 \end{cases}$

Sol. Let $f(t) = \cos t$

$$\therefore L\{F(t)\} = L\{cost\} = \frac{s}{s^2+1} = \overline{f}(s)$$

$$g(t) = \begin{cases} f(t - \pi/3) = cos(t - \pi/3), & if\ t > \pi/3 \\ 0 & , if\ t < \pi/3 \end{cases}$$

Now applying second shifting theorem, then we get

$$L\{g(t)\} = e^{\frac{-\pi s}{3}} \left(\frac{s}{s^2+1}\right) = \frac{s.e^{\frac{-\pi s}{3}}}{s^2+1}$$

2. Find the L.T. of (ii)$(t-2)^3 u(t-2)$ (ii) $e^{-3t} u(t-2)$

Sol: (i). Comparing the given function with f(t-a) u(t-a), we have a=2 and f(t)=t^3

$$\therefore L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = \overline{f}(s)$$

Now applying second shifting theorem, then we get

$$L\{(t-2)^3 u(t-2)\} = e^{-2s} \frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$$

(ii). $L\{e^{-st} u(t-2)\} = L\{e^{-s(t-2)}.e^{-6} u(t-2)\} = e^{-6} L\{e^{-3(t-2)} u(t-2)\}$

$f(t) = e^{-3t}$ then $\overline{f}(s) = \frac{1}{s+3}$

Now applying second shifting theorem then, we get

$$L\{e^{-3t} u(t-2)\} = e^{-6}.e^{-2s} \frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Multiplication by't':

Theorem: If $L\{f(t)\} = \bar{f}(s)$ *then* $L\{tf(t)\} = \frac{-d}{ds}\bar{f}(s)$

Proof: By the definition $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds}\{\bar{f}(s)\} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

By Leibnitz's rule for differentiating under the integral sign,

$$\frac{d}{ds}\bar{f}(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\{tf(t)\}dt = -L\{tf(t)\}$$

Thus $L\{tf(t)\} = \frac{-d}{ds}\bar{f}(s)$

$\therefore L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} = \bar{f}(s)$

Note: Leibnitz's Rule

If $f(x,\alpha)$ *and* $\frac{\partial}{\partial\alpha} f(x,\alpha)$ be continuous functions of x and α then

$\frac{d}{d\alpha}\left\{\int_a^b f(x,\alpha)dx\right\} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$

Where a, b are constants independent of α

Solved Problems:

1. Find L.T of tcosat

Sol: Since $L\{t\cos at\} = \frac{s}{s^2+a^2}$

$$L\{t\cos at\} = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{-s^2+a^2-s.2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2. Find t^2sin at

Sol: Since $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L\{t^2.\sin at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$=\frac{d}{ds}\left(\frac{-2as}{\left(s^2+a^2\right)^2}\right)=\frac{2a\left(3s^2-a^2\right)}{\left(s^2+a^2\right)^3}$$

3. Find $L.T\ of\ te^{-t}\sin 3t$

Sol: Since $L\{sin\ 3t\}=\frac{3}{s^2+3^2}$

$\therefore L\{tsin\ 3t\}=\frac{-d}{ds}\left[\frac{3}{s^2+3^2}\right]=\frac{6s}{(s^2+9)^2}$ Now using the shifting property, we get

$$L\{te^{-t}sin\ 3t\}=\frac{6(s+1)}{((s+1)^2+9)^2}=\frac{6(s+1)}{(s^2+2s+10)^2}$$

4. Find $L\{te^{2t}sin\ 3t\}$

Sol: Since $L\{sin\ 3t\}=\frac{3}{s^2+9}$

$$\therefore \text{L}\left\{e^{2t}\sin 3t\right\}=\frac{3}{\left(s-2\right)^2+9}=\frac{3}{s^2-4s+13}$$

$$L\{te^{2t}sin\ 3t\}=(-1)\frac{d}{ds}\left[\frac{3}{s^2-4s+13}\right]=(-1)\left[\frac{0-3(2s-4)}{(s^2-4s+13)^2}\right]$$

$$=\frac{3(2s-4)}{\left(s^2-4s+13\right)^2}=\frac{6(s-2)}{\left(s^2-4s+13\right)^2}$$

5. Find the L.T. of $\left(1+te^{-t}\right)^2$

Sol: Since $(1+te^{-t})^2=1+2te^{-t}+t^2e^{-2t}$

$$\therefore \text{L}\left(1+te^{-t}\right)^2=\text{L}\{1\}+2\text{L}\left\{te^{-t}\right\}+\text{L}\left\{t^2e^{-2t}\right\}$$

$$=\frac{1}{s}+2(-1)\frac{d}{ds}\left(\frac{1}{s+1}\right)+(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$=\frac{1}{s}+\frac{2}{\left(s+1\right)^2}+\frac{d}{ds}\left(\frac{-1}{\left(s+2\right)^2}\right)$$

$$=\frac{1}{s}+\frac{2}{\left(s+1\right)^2}+\frac{2}{\left(s+2\right)^3}$$

6. Find the L.T of t^3e^{-3t} (already we have solved by another method)

Sol: $L\{t^3e^{-3t}\}=(-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$

$$=-\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)=\frac{-3!(-1)^3}{(s+3)^4}$$

$$=\frac{3!}{(s+3)^4}$$

7. Find $L\{\cosh at\sin at\}$

Sol. $L\{\cosh at\sin at\}=L\left\{\frac{e^{at}+e^{-at}}{2}.\sin at\right\}$

$$=\frac{1}{2}[L\{e^{at}\sin at\}+L\{e^{-at}\sin at\}]$$

$$=\frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right]$$

8. Find the L.T of the function $f(t)=(t-1)^2, \quad t>1$
$\quad =0 \quad 0<t<1$

Sol: By the definition

$$L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt=\int_0^{1}e^{-st}f(t)dt+\int_1^{\infty}e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}0dt+\int_1^{\infty}e^{-st}(t-1)^2dt$$

$$=\int_1^{\infty}e^{-st}(t-1)^2dt=\left[(t-1)^2\frac{e^{-st}}{-s}\right]_1^{\infty}-\int_1^{\infty}2(t-1)\frac{e^{-st}}{-s}dt$$

$$=0+\frac{2}{s}\int_1^{\infty}e^{-st}(t-1)dt$$

$$=\frac{2}{s}\left[\left\{(t-1)\left(\frac{e^{-st}}{-s}\right)\right\}_1^{\infty}-\int_1^{\infty}\frac{e^{-st}}{-s}dt\right]$$

$$=\frac{2}{s}\left[0+\frac{1}{s}\int_1^{\infty}e^{-st}dt\right]=\frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_1^{\infty}=\frac{-2}{s^3}\left(e^{-st}\right)_1^{\infty}$$

$$=\frac{-2}{s^3}\left(0-e^{-s}\right)=\frac{2}{s^3}e^{-s}$$

9. Find the L.T of f (t) defined as $f(t)=3$**, t >2**
$\quad =0$**, 0<t<2**

Sol: $L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt$

$$=\int_0^2 e^{-st}f(t)dt+\int_2^{\infty}e^{-st}f(t)dt$$

$$=\int_0^2 e^{-st}.0dt+\int_2^{\infty}e^{-st}3dt$$

$$=0+\int_2^{\infty}e^{-st}3dt=\frac{-3}{s}\left(e^{-st}\right)_2^{\infty}=\frac{-3}{s}\left(0-e^{-2s}\right)$$

$$=\frac{3}{s}e^{-2s}$$

10. Find $L\{t\cos(at+b)\}$

Sol: $L\{\cos(at+b)\}=L\{\cos at\cos b-\sin at\sin b\}$

$$=\cos b.L\{\cos at\}-\sin b\,L\{\sin at\}$$

$$=\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}$$

$$L\{t.cos(at+b)\} = \frac{-d}{ds}\left[\cos b.\frac{s}{s^2+a^2} - \sin b.\frac{a}{s^2+a^2}\right]$$

$$= -\cos b.\left(\frac{s^2+a^2.1-s.2s}{\left(s^2+a^2\right)^2}\right)+\sin b\left(\frac{\left(s^2+a^2\right).0-a.2s}{\left(s^2+a^2\right)^2}\right)$$

$$= \frac{1}{\left(s^2+a^2\right)^2}\left[\left(s^2-a^2\right)^2\cos b - 2as\sin b\right]$$

11. Find L.T of L $[\mathbf{te}^{t}sint]$

Sol: - We know that L[sint] = $\frac{1}{s^2+1}$

$$\text{L[tsint]} = (-1)\frac{d}{ds}\text{L[sint]} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\frac{(-1)2s}{(s^2+1)^2}$$

$$= \frac{2s}{(s^2+1)^2}$$

By First Shifting Theorem

$$\text{L}\,[te^{t}sint] = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s-1} = \frac{2(s-1)}{((s-1)^2+1)^2} = \frac{2(s-1)}{(s^2-2s+2)^2}$$

Division by't':

Theorem: If $L\{f(t)\} = \overline{f}(s)$ then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \overline{f}(s)ds$

Proof: We have $\overline{f}(s) = \int_0^\infty e^{-st}f(t)dt$

Now integrating both sides w.r.t s from s to ∞, we have

$$\int_0^\infty \overline{f}(s)ds = \int_s^\infty\left[\int_0^\infty e^{-st}f(t)dt\right]ds$$

$$= \int_0^\infty\int_s^\infty f(t)e^{-st}dsdt \text{ (Change the order of integration)}$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt \text{ (}\because\text{ t is independent of's')}$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}dt (or) L\left\{\frac{1}{t}f(t)\right\}$$

Solved Problems:

1. Find $L\left\{\frac{\sin t}{t}\right\}$

Sol: Since $L\{sint\} = \frac{1}{s^2+1} = \overline{f}(s)$

Division by't', we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= [Tan^{-1}s]_s^{\infty} = Tan^{-1}\infty - Tan^{-1}s$$

$$= \pi/2 - Tan^{-1}s = \cot^{-1} s$$

2. **Find the L.T of** $\frac{\sin at}{t}$

Sol: Since $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Division by t, we have

$$L\left\{\frac{\sin a t}{t}\right\} = \int_s^{\infty} \overline{f}(s)ds = \int_s^{\infty} \frac{a}{s^2+a^2} ds$$

$$= a.\frac{1}{a}\left[Tan^{-1} s/a\right]_s^{\infty} = Tan^{-1}\infty - Tan^{-1} s/a$$

$$= \pi/2 - Tan^{-1}\left(s/a\right) = \cot^{-1} s/a$$

3. **Evaluate** $\boldsymbol{L\left\{\frac{1-\cos a t}{t}\right\}}$

Sol: Since $L\{1 - \cos a t\} = L\{1\} - L\{\cos at\} = \frac{1}{s} - \frac{s}{s^2+a^2}$

$$L\left\{\frac{1-\cos a t}{t}\right\} = \int_s^{\infty} \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[2\log s - \log\left(s^2+a^2\right)\right]_s^{\infty} = \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[l\,og\left(\frac{1}{1+a^2/s^2}\right)\right]_s^{\infty} = \frac{1}{2}\left[\log 1 - \log\frac{s^2}{s^2+a^2}\right]$$

$$= -\frac{1}{2}l\,og\left(\frac{s^2}{s^2+a^2}\right) = \log\left(\frac{s^2}{s^2+a^2}\right)^{\frac{-1}{2}} = \log\sqrt{\frac{s^2+a^2}{s^2}}$$

Note: $\boldsymbol{L\left\{\frac{1-\cos t}{t}\right\}} = \log\sqrt{\frac{s^2+1}{s}}$ (Putting a=1 in the above problem)

4. **Find** $\boldsymbol{L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}}$

Sol: $$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^{\infty} \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^{\infty} = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^{\infty}$$

$$= \underset{s\to\infty}{lt}\left\{\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right\} - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log 1 - \log(s+a) + \log(s+b) = \log\left(\frac{s+b}{s+a}\right)$$

5. **Find** $L\left\{\frac{1-\cos t}{t^2}\right\}$

Sol: $L\left\{\frac{1-\cos t}{t^2}\right\} = L\left\{\frac{1}{t}.\frac{1-\cos t}{t}\right\}$.....(1)

Now $L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds = \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty = \frac{-1}{2}\left[\log\frac{s^2}{s^2+1}\right] = \frac{1}{2}\log\frac{s^2+1}{s^2}$$

$$\therefore L\left[\frac{1-\cos t}{t^2}\right] = \int_s^\infty \frac{1}{2}\log\frac{s^2+1}{s^2}ds$$

$$= \frac{1}{2}\left[\left\{\log\left(\frac{s^2+1}{s^2}\right)\right\}.s\right]_s^\infty - \int_s^\infty \frac{s^2}{s^2+1}\left(\frac{-2}{s^3}\right).sds$$

$$= \frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s.\log\left(1+\frac{1}{s^2}\right)\right\} - s\log\left(\frac{s^2+1}{s^2}\right) + 2\int_s^\infty \frac{ds}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s\left(\frac{1}{s^2} - \frac{1}{2s^4} + \frac{1}{3s^6} + ...\right) - s\log\frac{s^2+1}{s^2}\right\} + 2Tan^{-1}s\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{0 - s\log\left(1+\frac{1}{s^2}\right) + 2\left(\frac{\pi}{2} - Tan^{-1}s\right)\right\}\right] \quad \because \left(\log\left(1+x\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\right)$$

$$= \cot^{-1}s - \frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)$$

6.Find L.T of $\frac{e^{-at}-e^{-bt}}{t}$

Sol: W.K.T $L\left[e^{-at}\right] = \frac{1}{s+a}$, $L\left[e^{-bt}\right] = \frac{1}{s+b}$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)ds$$

$$\therefore \quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds$$

$$= [\log(s+a) - \log(s+b)]_s^\infty$$

$$= \log\left(\frac{s+a}{s+b}\right)_s^\infty$$

$$= \log(\frac{1+\frac{a}{s}}{1+\frac{b}{s}})_s^\infty$$

$$= \log(1) - \log(\frac{s+a}{s+b})$$

$$= 0 - \log(\frac{s+a}{s+b}) = \log(\frac{s+b}{s+a})$$

Laplace transforms of Derivatives:

If $f^1(t)$ be continuous and $L\{f(t)\} = \overline{f}(s)$ *then* $L\{f^1(t)\} = s\overline{f}(s) - f(0)$

Proof: By the definition

$$L\{f^1(t)\} = \int_0^\infty e^{-st} f^1(t)dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t) dt \qquad \text{(Integrating by parts)}$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t) dt$$

$$= \underset{t\to\infty}{\mathrm{lt}}\, e^{-st} f(t) - f(0) + s.L\{f(t)\}$$

Since f (t) is exponential order

$$\therefore \underset{t\to\infty}{\mathrm{lt}}\, e^{-st} f(t) = 0$$

$$\therefore L\{f^1(t)\} = 0 - f(0) + sL\{f(t)\}$$

$$= s\overline{f}(s) - f(0)$$

The Laplace Transform of the second derivative $f^{11}(t)$ is similarly obtained.

$$\therefore L\{f^{11}(t)\} = s.L\{f^1(t)\} - f^1(0)$$

$$= s.\left[s\overline{f}(s) - f(0)\right] - f^1(0)$$

$$= s^2\overline{f}(s) - sf(0) - f^1(0)$$

$$\therefore L\{f^{111}(t)\} = s.L\{f^{11}(t)\} - f^{11}(0)$$

$$= s[s^2L\{f(t)\} - sf(0) - f^1(0)] - f^{11}(0)$$

$$= s^3L\{f(t)\} - s^2f(0) - sf^1(0) - f^{11}(0)$$

Proceeding similarly, we have

$$L\{f^n(t)\} = s^nL\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^1(0) \dots \dots f^{n-1}(0)$$

Note 1: $L\{f^n(t)\} = s^n\overline{f}(s)$ *if* $f(0) = 0$ *and* $f^1(0) = 0, f^{11}(0) = 0 \dots\ f^{n-1}(0) = 0$

Note 2: Now $|f(t)| \le M.e^{at}$ *for all* $t \ge 0$ *and for some constants a amd M.*

We have $|e^{-st}f(t)| = e^{-st}|f(t)| \le e^{at}.Me^{at}$

$= M.\, e^{-(s-a)t} \to 0$ *as* $t \to \infty$ if s>a

$$\therefore \underset{t\to\infty}{lt}\, e^{-st} f(t) = 0 \; for \;\; s > a$$

Solved Problems:

Using the theorem on transforms of derivatives, find the Laplace Transform of the following functions.

(i). e^{at} (ii). cosat (iii). t sin at

(i). $Let\, f(t) = e^{at}\; Then\; f^1(t) = a.e^{at}\; and\; f(0) = 1$

$$Now\; L\{f^1(t)\} = s.L\{f(t)\} - f(0)$$

$$i.e., L\{ae^{at}\} = s.L\{e^{at}\} - 1$$

$$i.e., L\{e^{at}\} - s.L\{e^{at}\} = -1$$

$$i.e., (a-s)L\{e^{at}\} = -1$$

$$\therefore L\{e^{at}\} = \frac{1}{s-a}$$

(ii). $Let\; f(t) = cosat\; then\; f^1(t) = -asinat\; and\; f^{11}(t) = -a^2 cosat$

$$\therefore L\{f^{11}(t)\} = s^2 L\{f(t)\} - s.f(0) - f^1(0)$$

$$Now\, f(0) = \cos 0 = 1\, and\;\; f^1(0) = -a\sin 0 = 0$$

$$Then\; L\{-a^2\cos at\} = s^2 L\{\cos at\} - s.1 - 0$$

$$\Longrightarrow -a^2 L\{\cos at\} - s^2 L\{\cos at\} = -s$$

$$\Longrightarrow -(s^2+a^2)L\{\cos at\} = -s \Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$$

(iii). $Let\; f(t) = t\sin at\; then\; f^1(t) = \sin at + at\cos at$

$$f^{11}(t) = a\cos at + a[\cos at - at\sin at] = 2a\cos at - a^2 t\sin at$$

Also $f(0) = 0$ and $f^1(0) = 0$

$$Now\; L\{f^{11}(t)\} = s^2 L\{f(t)\} - sf(0) - f^1(0)$$

$$i.e., L\{2a\cos at - a^2 t\sin at\} = s^2 L\{t\sin at\} - 0 - 0$$

$$i.e., 2a\, L\{\cos at\} - a^2 L\{t\sin at\} - s^2 L\{t\sin at\} = 0$$

$$i.e., -(s^2+a^2)L\{t\sin at\} = \frac{-2as}{s^2+a^2} \Rightarrow L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Laplace Transform of Integrals:

If $L\{f(t)\} = \overline{f}(s)\; then\; L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Proof: $Let\; g(t) = \int_0^t f(x)\,dx$

Then $g^1(t) = \frac{d}{dt}\left[\int_0^t f(x)\,dx\right] = f(t)\; and\; g(0) = 0$

Taking Laplace Transform on both sides

$L\{g^1(t)\} = L\{f(t)\}$

But $L\{g^1(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\} - 0$ $[Since\ g(0) = 0]$

$\therefore L\{g^1(t)\} = L\{f(t)\}$

$\Longrightarrow sL\{g(t)\} = L\{f(t)\} \Rightarrow L\{g(t)\} = \frac{1}{s}L\{f(t)\}$

$But\ g(t) = \int_0^t f(x)\,dx$

$\therefore L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Solved Problems:

1. Find the L.T of $\int_0^t \sin at dt$

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Using the theorem of Laplace transform of the integral, we have

$L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

$\therefore L\left\{\int_0^t \sin at\right\} = \frac{a}{s(s^2+a^2)}$

2. Find the L.T of $\int_0^t \frac{\sin t}{t} dt$

Sol: $L\{\sin t\} = \frac{1}{s^2+1}\ also\ \underset{t\to 0}{lt}\frac{\sin t}{t} = 1\ exists$

$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}ds = \int_s^\infty \frac{1}{s^2+1}ds$

$= \left[Tan^{-1}s\right]_s^\infty = Tan^{-1}\infty - Tan^{-1}s = \pi/2 - Tan^{-1}s = \cot^{-1}s\,(or)\,Tan^{-1}\left(1/s\right)$

$i.e., L\left\{\frac{\sin t}{t}\right\} = Tan^{-1}(1/s)(or)cot^{-1}s$

$\therefore L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}Tan^{-1}(1/s)\ (or)\ \frac{1}{s}cot^{-1}s$

3. Find L.T of $\boldsymbol{e^{-t}\int_0^t \frac{sint}{t}dt}$

Sol: L $[\ e^{-t}\int_0^t \frac{sint}{t}dt]$

We know that

L $\{sint\} = \frac{1}{s^2+1} = \bar{f}(s)$

L $\{\frac{sint}{t}\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}\ ds$

$=(tan^{-1}s)_s^\infty$

$=tan^{-1}\infty - tan^{-1}s = \frac{\pi}{2} - tan^{-1}s\ = cot^{-1}s$

$\therefore$ L $\left\{\frac{sint}{t}\right\} = cot^{-1}s$

Hence $\quad$ L $\{\int_0^t \frac{sint}{t} dt\} = \frac{1}{s} cot^{-1}s$

By First Shifting Theorem

$$\text{L}\,[\, e^{-t}\int_0^t \frac{sint}{t} dt] = \bar{f}(s+1) = (\frac{cot^{-1}s}{s})_{s\to s+1}$$

$$\therefore \text{L}\left[e^{-t}\int_0^t \frac{sint}{t} dt\right] = \frac{1}{s+1} cot^{-1}(s+1)$$

Laplace transform of Periodic functions:

If f (t) is a periodic function with period ‘a’. i.e, $f(t+a)=f(t)$ then

$$\text{L}\,\{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)\,dt$$

Eg: sin x is a periodic function with period 2π

$i.e.,\ \sin x = \sin(2\pi + x) = \sin(4\pi + x)$............

Solved Problems:

1. A function f (t) is periodic in (0,2b) and is defined as $f(t) \begin{array}{l} =1 \ if\ \ 0<t<b \\ =-1 \ if\ \ b<t<2b \end{array}$

Find its Laplace Transform.

Sol: $\quad \text{L}\{f(t)\} = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^b e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)}\left[-\left(e^{-sb}-1\right)+\left(e^{-2bs}-e^{-sb}\right)\right]$$

$$\text{L}\{f(t)\} = \frac{1}{s\left(1-e^{-2bs}\right)}\left[1-2e^{-sb}+e^{-2bs}\right]$$

2. Find the L.T of the function $f(t) = \sin \omega t \ if\ 0<t<\frac{\pi}{\omega}$

$$=0\ if\ \frac{\pi}{\omega}<t<\frac{2\pi}{\omega}\ where\ f(t)\ has\ period\ \frac{2\pi}{\omega}$$

Sol: Since f (t) is a periodic function with period $\frac{2\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-s2\pi/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s\pi/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0dt\right]$$

$$= \frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$\because \int_a^b e^{at}\sin bt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$$

$$= \frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{1}{s^2+\omega^2}\left(e^{-s\pi/\omega}.\omega+\omega\right)\right]$$

Laplace Transform of Some special functions:

1. The Unit step function or Heaviside's Unit functions:

It is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Laplace Transform of unit step function:

To prove that $L\{u(t-a)\} = \frac{e^{-as}}{s}$

Proof: Unit step function is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Then $L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)\,dt$

$$= \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt$$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt$$

$$= \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}.\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}$$

$\therefore L\{u(t-a)\} = \frac{e^{-as}}{s}$

Laplace Transforms of Dirac Delta Function:

The Dirac delta function or Unit impulse function $f_\in(t) = \begin{cases} 1/\in & 0 \le t \le \in \\ 0 & t > \in \end{cases}$

2. **Prove that $L\{f_\in(t)\} = \frac{1-e^{-s\in}}{s\in}$ hence show that $L\{\delta(t)\} = 1$**

Proof: By the definition $f_\epsilon(t) = \begin{cases} 1/\epsilon & 0 \le t \le \epsilon \\ 0 & t > \epsilon \end{cases}$

And Hence $L\{f_\epsilon(t)\} = \int_0^\infty e^{-st} f_\epsilon(t)\, dt$

$$= \int_0^\epsilon e^{-st} f_\epsilon(t)\, dt + \int_\epsilon^\infty e^{-st} f_\epsilon(t)\, dt$$

$$= \int_0^\epsilon e^{-st} \frac{1}{\epsilon} dt + \int_\epsilon^\infty e^{-st}.0\, dt$$

$$= \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_0^\epsilon = -{}^1/_{\epsilon\, s}\left[e^{-s\epsilon} - e^0\right] = \frac{1-e^{-s\epsilon}}{s\epsilon}$$

$$\therefore L\{f_\epsilon(t)\} = \frac{1-e^{-s\epsilon}}{s\epsilon}$$

Now $L\{\delta(t)\} = \underset{\epsilon\to 0}{lt} L\{f_\epsilon(t)\} = \underset{\epsilon\to 0}{lt} \frac{1-e^{-s\epsilon}}{s\epsilon}$

$\therefore L\{\delta(t)\} = 1$ *using* L-Hospital rule.

Properties of Dirac Delta Function:

1. $\int_0^\infty \delta(t)\, dt = 0$
2. $\int_0^\infty \delta(t) G(t)\, dt = G(0)$ where G(t) is some continuous function.
3. $\int_0^\infty \delta(t-a) G(t)\, dt = G(a)$ where G(t) is some continuous function.
4. $\int_0^\infty G(t)\delta^1(t-a) = -G^1(a)$

Solved Problems:

1. Prove that $L\{\delta(t-a)\} = e^{-as}$

Sol: By Translation theorem

$L\{\delta(t-a)\} = e^{-as} L\{\delta(t)\}$

$= e^{-as}$ [$\sin ce\; L\{\delta(t)\} = 1$]

2. Evaluate $\int_0^\infty \cos 2t\, \delta\left(t - {}^\pi/_3\right) dt$

Sol: By using property (3) then we get

$\int_0^\infty \delta(t-a) G(t) dt = G(a)$

Here $a = {}^\pi/_3, G(t) = \cos 2t$

$\therefore G(a) = G\left({}^\pi/_3\right) = \cos 2\, {}^\pi/_3 = -{}^1/_2$

$\therefore \int_0^\infty \cos 2at\, \delta\left(t - {}^\pi/_3\right) dt = \cos 2\, {}^\pi/_3 = {}^{-\pi}/_2$

3. Evaluate $\int_0^\infty e^{-4t}\delta^1(t-2)\, dt$

Sol: By the 4th Property then we get

$$\int_0^\infty \delta^1(t-a)G(t)\,dt = -G^1(a)$$

$$G(t) = e^{-4t} \text{ and } a = 2$$

$$G^1(t) = -4.e^{-4t}$$

$$\therefore G^1(a) = G^1(2) = -4.e^{-8}$$

$$\therefore \int_0^\infty e^{-4t}\delta^1(t-2)\,dt = -G^1(a) = 4.e^{-8}$$

Inverse Laplace Transforms:

If $\overline{f}(s)$ is the Laplace transforms of a function of f (t) i.e. $L\{f(t)\} = \overline{f}(s)$ then f (t) is called the inverse Laplace transform of $\overline{f}(s)$ and is written as $f(t) = L^{-1}\{\overline{f}(s)\}$

$\therefore L^{-1}$ is called the inverse L.T operator.

Table of Laplace Transforms and Inverse Laplace Transforms

S.No.	$L\{f(t)\} = \overline{f}(s)$	$L^{-1}\{\overline{f}(s)\} = f(t)$
1.	$L\{1\} = {}^1/_s$	$L^{-1}\{{}^1/_s\} = 1$
2.	$L\{e^{at}\} = \dfrac{1}{s-a}$	$L^{-1}\{{}^1/_{s-a}\} = e^{at}$
3.	$L\{e^{-at}\} = \dfrac{1}{s+a}$	$L^{-1}\{{}^1/_{s+a}\} = e^{at}$
4.	$L\{t^n\} = \dfrac{n!}{s^{n+1}}\, n \text{ is a } + ve \text{ integer}$	$L^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$
5.	$L\{t^{n-1}\} = \dfrac{(n-1)!}{s^n}$	$L^{-1}\{{}^1/_{s^n}\} = \dfrac{t^{n-1}}{(n-1)!}, n = 1,2,3\ldots$
6.	$L\{\sin at\} = \dfrac{a}{s^2+a^2}$	$L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}.\sin at$
7.	$L\{\cos at\} = \dfrac{s}{s^2+a^2}$	$L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$
8.	$L\{\sinh at\} = \dfrac{a}{s^2-a^2}$	$L^{-1}\left\{\dfrac{1}{s^2-a^2}\right\} = \dfrac{1}{a}\sinh at$
9.	$L\{\cosh at\} = \dfrac{s}{s^2-a^2}$	$L^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$
10.	$L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}$	$L^{-1}\left\{\dfrac{1}{(s-a)^2+b^2}\right\} = \dfrac{1}{b}.e^{at}\sin bt$
11.	$L\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$	$L^{-1}\left\{\dfrac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

12.	$L\{e^{at}\sinh bt\} = \dfrac{b}{(s-a)^2-b^2}$	$L^{-1}\left\{\dfrac{1}{(s-a)^2-b^2}\right\} = \dfrac{1}{b}.e^{at}\sinh bt$
13.	$L\{e^{at}\cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}$	$L^{-1}\left\{\dfrac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
14.	$L\{e^{-at}\sin bt\} = \dfrac{b}{(s+a)^2+b^2}$	$L^{-1}\left\{\dfrac{1}{(s+a)^2+b^2}\right\} = \dfrac{1}{b}.e^{-at}\sin bt$
15.	$L\{e^{-at}\cos bt\} = \dfrac{s+a}{(s+a)^2+b^2}$	$L^{-1}\left\{\dfrac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$
16.	$L\{e^{at}f(t)\} = \overline{f}(s-a)$	$L^{-1}\{\overline{f}(s-a)\} = e^{at}L^{-1}\{\overline{f}(s)\}$
17.	$L\{e^{-at}f(t)\} = \overline{f}(s+a)$	$L^{-1}\{\overline{f}(s+a)\} = e^{-at}f(t)e^{-at}L^{-1}\{\overline{f}(s)\}$

Solved Problems :

1. Find the Inverse Laplace Transform of $\dfrac{s^2-3s+4}{s^3}$

Sol: $L^{-1}\left\{\dfrac{s^3-3s+4}{s^3}\right\} = L^{-1}\left\{{}^{1}/_{s} - 3.{}^{1}/_{s^2} + {}^{4}/_{s^3}\right\}$

$$= L^{-1}\left\{{}^{1}/_{s}\right\} - 3L^{-1}\left\{{}^{1}/_{s^2}\right\} + L^{-1}\left\{{}^{4}/_{s^3}\right\}$$

$$= 1 - 3t + 4.\frac{t^2}{2!} = 1 - 3t + 2t^2$$

2. Find the Inverse Laplace Transform of $\dfrac{s+2}{s^2-4s+13}$

Sol: $L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\dfrac{s+2}{(s-2)^2+9}\right\} = L^{-1}\left\{\dfrac{s-2+4}{(s-2)^2+3^2}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + 4.L^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

3. Find the Inverse Laplace Transform of $\dfrac{2s-5}{s^2-4}$

Sol: $L^{-1}\left\{\dfrac{2s-5}{s^2-4}\right\} = L^{-1}\left\{\dfrac{2s}{s^2-4} - \dfrac{5}{s^2-4}\right\}$

$$= 2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= 2.\cosh 2t - 5.\frac{1}{2}\sinh 2t$$

4. Find $L^{-1}\left\{\dfrac{2s+1}{s(s+1)}\right\}$

Sol: $L^{-1}\left\{\frac{s+s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s+1}+\frac{1}{s}\right\}$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s}\right\} = e^{-t}+1$$

5. **Find** $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

Sol: $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\} = L^{-1}\left\{\frac{3s}{4s^2+25}\right\} - 8L^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$$= \frac{3}{4}L^{-1}\left\{\frac{s}{s^2+(5/2)^2}\right\} - \frac{8}{4}L^{-1}\left\{\frac{1}{s^2+(5/2)^2}\right\}$$

$$= \frac{3}{4}.\cos\frac{5}{2}t - \frac{8}{4}.\frac{2}{5}\sin\frac{5}{2}t$$

$$= \frac{3}{4}\cos\frac{5}{2}t - \frac{4}{5}\sin\frac{5}{2}t$$

6. **Find the Inverse Laplace Transform of** $\frac{s}{(s+a)^2}$

Sol: $L^{-1}\left\{\frac{s}{(s+a)^2}\right\} = L^{-1}\left\{\frac{s+a-a}{(s+a)^2}\right\} = e^{-at}L^{-1}\left\{\frac{s-a}{s^2}\right\}$

$$= e^{-at}L^{-1}\left\{\frac{1}{s}-\frac{a}{s^2}\right\}$$

$$= e^{-at}\left[L^{-1}\left\{\frac{1}{s}\right\} - a.L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= e^{-at}\left[1-at\right]$$

7. **Find** $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Sol: $Let\ \frac{3s+7}{s^2-2s-3} = \frac{A}{s+1}+\frac{B}{s-3}$

$A(s-3)+B(s+1) = 3s+7$

$put\ s = 3, 4B = 16 \Rightarrow B = 4$

$put\ s = -1, -4A = 4 \Rightarrow A = -1$

$\therefore \frac{3s+7}{s^2-2s-3} = \frac{-1}{s+1}+\frac{4}{s-3}$

$$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{-1}{s+1}+\frac{4}{s-3}\right\} = -1L^{-1}\left\{\frac{1}{s+1}\right\} + 4L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -e^{-t}+4.e^{3t}$$

8. **Find** $L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$

Sol: $\frac{s}{(s+1)^2(s^2+1)} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+1}$

$A(s+1)(s^2+1)+B(s^2+1)+(Cs+D)(s+1)^2=s$

Equating Co-efficient of s^3, A+C=0……..(1)

Equating Co-efficient of s^2, A+B+2C+D=0…….(2)

Equating Co-efficient of s, A+C+2D=1…….(3)

$put\ s=-1, 2B=-1 \Rightarrow B=-\frac{1}{2}$

Substituting (1) in (3) $2D=1 \Rightarrow D=\frac{1}{2}$

Substituting the values of B and D in (2)

i.e. $A-\frac{1}{2}+2C+\frac{1}{2}=0 \Rightarrow A+2C=0, also\ A+C=0 \Rightarrow A=0, C=0$

$$\therefore \frac{s}{(s+1)^2(s^2+1)}=\frac{\frac{-1}{2}}{(s+1)^2}+\frac{\frac{1}{2}}{s^2+1}$$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}=\frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\}-L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right]$$

$$=\frac{1}{2}\left[\sin t - e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$=\frac{1}{2}\left[\sin t - te^{-t}\right]$$

9. Find $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol: Since $s^4+4a^4=\left(s^2+2a^2\right)^2-(2as)^2$

$$=(s^2+2as+2a^2)(s^2-2as+2a^2)$$

$$\therefore Let\ \frac{s}{s^4+4a^4}=\frac{As+B}{s^2+2as+2a^2}+\frac{Cs+D}{s^2-2as+2a^2}$$

$(As+B)(s^2-2as+2a^2)+(Cs+D)(s^2+2as+2a^2)=s$

Solving we get $A=0, C=0, B=\frac{-1}{4a}, D=\frac{1}{4a}$

$$L\left\{\frac{s}{s^4+4a^4}\right\}=L^{-1}\left\{\frac{-\frac{1}{4a}}{s^2+2as+2a^2}\right\}+L^{-1}\left\{\frac{\frac{1}{4a}}{s^2-2as+2a^2}\right\}$$

$$=\frac{-1}{4}a.L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}+\frac{1}{4a}..L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$=\frac{-1}{4a}.\frac{1}{a}.e^{-at}\sin at+\frac{1}{4a}.\frac{1}{a}e^{at}\sin at$$

$$=\frac{1}{4a^2}\sin at\left(e^{at}-e^{-at}\right)=\frac{1}{4a^2}.\sin at.2\sinh at=\frac{1}{2a^2}\sin at\sinh at$$

10. Find i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$ ii. $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$

Sol:

i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{s^2}{s^3}-\frac{3s}{s^3}+\frac{4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}\right\}$

$$= L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3t + 4\frac{t^2}{2!} = 1 - 3t + 2t^2$$

ii. $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{(s^2-2)^2}{s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right\} + \frac{3}{2}\left\{L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{\frac{1}{s^3}\right\} + 4L^{-1}\left\{\frac{1}{s^5}\right\}\right\}$$

$$= \frac{3}{2}\left[1-4\frac{t^2}{2!}+\frac{4t^4}{4!}\right] = \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right] = \frac{1}{4}\left[t^4-6t^2+6\right]$$

11. Find $L^{-1}\left[\frac{s}{s^2-a^2}\right]$

Sol:

$$L^{-1}\left[\frac{s}{s^2-a^2}\right] = L^{-1}\left[\frac{2s}{2(s^2-a^2)}\right] = \frac{1}{2}L^{-1}\left[\frac{2s}{(s-a)(s+a)}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[e^{at}+e^{-at}\right] = \cosh at$$

12. Find $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]$

Sol: $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right] = 4L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = 4\,L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right] = 4[\mathrm{e}^{-t}-\mathrm{e}^{-2t}]$

13. Find $L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}$

Sol: $\frac{1}{(s+1)^2(s^2+4)} = \frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4}$

$$A=\frac{2}{25}, B=\frac{1}{5}, C=\frac{-2}{25}, D=\frac{-3}{25}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\} = \frac{2}{25}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{2}{25}L^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{3}{25}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{2}{25}e^{-t}\,L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{5}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{2}{25}\cos 2t-\frac{3}{25}.\frac{1}{2}\sin 2t$$

$$=\frac{2}{25}e^{-t}+\frac{1}{5}e^{-t}.t-\frac{2}{25}\cos 2t-\frac{3}{50}\sin 2t$$

14. Find $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(\mathrm{s}-2)}\right]$

Sol: $\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$

Comparing with s^2, s, constants, we get

$A=1/3, B=4/15, C=2/5$

$$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(\mathrm{s}-2)}\right]=L^{-1}\left[\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}\right]$$

$$=L^{-1}\left[\frac{1}{3s}\right]+L^{-1}\left[\frac{4}{15(s+3)}\right]+L^{-1}\left[\frac{2}{5(s-2)}\right]$$

$$=\frac{1}{3}+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

15. Find $L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(\mathrm{s}-5)}\right]$

Sol: $\frac{s^2+2s-4}{(s^2+9)(\mathrm{s}-5)}=\frac{A}{s-5}+\frac{Bs+C}{s^2+9}$

Comparing with s^2, s, constants, we get

$A=31/34, B=3/34, C=83/34$

$$L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(\mathrm{s}-5)}\right]=L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(\mathrm{s}-5)}\right]$$

$$=L^{-1}\left[\frac{31}{34(\mathrm{s}-5)}\right]+L^{-1}\left[\frac{3}{34(s^2+9)}\right]+L^{-1}\left[\frac{83}{34(s^2+9)}\right]$$

$$=\frac{31}{34}e^{5t}+1/34\left[3\cos 3\mathrm{t}+\frac{83}{3}\sin 3t\right]$$

First Shifting Theorem:

If $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, then $\mathrm{L}^{-1}\left\{\overline{f}(s-a)\right\}=e^{at}f(t)$

Proof: We have seen that $L\left\{e^{at}f(t)\right\}=\overline{f}(s-a)\ \therefore L^{-1}\left\{\overline{f}(s-a)\right\}=e^{at}f(t)=e^{at}L^{-1}\left\{\overline{f}(s)\right\}$

Solved Problems :

1. **Find** $L^{-1}\left\{\dfrac{1}{(s+2)^2+16}\right\} = L^{-1}\left\{\overline{f}(s+2)\right\}$

Sol: $L^{-1}\left\{\dfrac{1}{(s+2)^2+16}\right\} = e^{-2t}L^{-1}\left\{\dfrac{1}{s^2+16}\right\}$

$$= e^{-2t}.\frac{1}{4}\sin 4t = \frac{e^{-2t}\sin 4t}{4}$$

2. **Find** $L^{-1}\left\{\dfrac{3s-2}{s^2-4s+20}\right\}$

Sol: $L^{-1}\left\{\dfrac{3s-2}{s^2-4s+20}\right\} = L^{-1}\left\{\dfrac{3s-2}{(s-2)^2+16}\right\} = L^{-1}\left\{\dfrac{3(s-2)+4}{(s-2)^2+4^2}\right\}$

$$= 3L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + 4e^{2t}\frac{1}{4}\sin 4t$$

3. **Find** $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\}$

Sol: $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\} = L^{-1}\left\{\dfrac{s+3}{(s-5)^2+2^2}\right\} = L^{-1}\left\{\dfrac{s-5+8}{(s-5)^2+2^2}\right\}$

$$= e^{5t}\mathrm{L}^{-1}\left\{\frac{s+8}{s^2+2^2}\right\} = e^{5t}\left\{\cos 2t + 8.\frac{1}{2}\sin 2t\right\}$$

Second shifting theorem:

If $L^{-1}\left\{\overline{f}(s)\right\} = f(t), then \quad L^{-1}\{e^{-as}\overline{f}(s)\} = G(\mathrm{t})$, where $\mathrm{G}(t) = \begin{cases} f\{t-a\} & if\ t>a \\ 0 & if\ t<a \end{cases}$

Proof: We have seen that $\mathrm{G}(t) = \begin{cases} f\{t-a\} & if\ t>a \\ 0 & if\ t<a \end{cases}$

$$then \quad L\{G(t)\} = e^{-as}.\overline{f}(s)$$

$$\therefore L^{-1}\left\{e^{-as}\overline{f}(s)\right\} = G(t)$$

Solved Problems :

1. **Evaluate (i)** $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$ **(ii)** $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Sol: (i) $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$, say

$\therefore$ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi) & , if\ t > \pi \\ 0 & , if\ t < \pi \end{cases}$

or $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$=sin(t-π)H(t-π)= -sint. H(t-π)

Hence $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$=sint-sint. H (t-π) =sint [1- H (t-π)]

Where H (t-π) is the Heaviside unit step function

(ii) Since $L^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}L^{-1}\left\{\frac{1}{s^2}\right\}$

$= e^{4t}.t = f(t)$, say

$\therefore$ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = \begin{cases} e^{4(t-3)}.(t-3) & , if\ t > 3 \\ 0 & , if\ t < 3 \end{cases}$

or $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = e^{4(t-3)}.(t-3)$ H(t-3)

Where H (t-3) is the Heaviside unit step function

Change of scale property:

If $L\{f(t)\} = \overline{f}(s)$, Then $L^{-1}\{\overline{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0$

Proof: We have seen that $L\{f(t)\} = \overline{f}(s)$

$$Then\quad \overline{f}(as) = \frac{1}{a} L\left\{f\left(\frac{t}{a}\right)\right\}, a > 0$$

$$\therefore L^{-1}\{\overline{f}(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0$$

Solved Problems :

1. **If** $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$, **find** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol: We have $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$,

Writing as for s,

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\}=\frac{1}{2}.\frac{1}{a}.\frac{t}{a}\sin\frac{t}{a}=\frac{t}{2a^2}.\sin\frac{t}{a}$$, by change of scale property.

Putting a=2, we get

$$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\}=\frac{t}{8}\sin\frac{t}{2} \textbf{ or } L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}=\frac{1}{2}\sin\frac{t}{2}$$

Inverse Laplace Transform of derivatives:

Theorem: $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, then $L^{-1}\left\{\overline{f}^{n}(s)\right\}=(-1)^n t^n f(t)$ where $\overline{f}^{n}(s)=\frac{d^n}{ds^n}\left[\overline{f}(s)\right]$

Proof: We have seen that $L\left\{t^n f(t)\right\}=(-1)^n\frac{d^n}{ds^n}\overline{f}(s)$

$$\therefore L^{-1}\left\{\overline{f}^{n}(s)\right\}=(-1)^n t^n f(t)$$

Solved Problems :

1. **Find** $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Sol: Let $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}=f(t)$

$$L\{f(t)\}=\log\frac{s+1}{s-1}$$

$$L\{tf(t)\}=\frac{-d}{ds}\left\{\log\frac{s+1}{s-1}\right\}$$

$$L\{tf(t)\}=\frac{-1}{s+1}+\frac{1}{s-1}$$

$$tf(t)=L^{-1}\left\{\frac{-1}{s+1}+\frac{1}{s-1}\right\}$$

$$tf(t)=-1.\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$=e^{-t}+e^{t}$$

$$t\,f(t) = 2\sinh t \Rightarrow f(t) = \frac{2\sinh t}{t}$$

$$\therefore L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = \frac{2\sinh t}{t}$$

Note: $L^{-1}\left\{\log\frac{1+s}{s}\right\} = \frac{1-e^{-t}}{t}$

2. **Find** $L^{-1}\left\{\cot^{-1}(s)\right\}$

Sol: Let $L^{-1}\left\{\cot^{-1}(s)\right\} = f(t)$

$$L\{f(t)\} = \cot^{-1}(s)$$

$$L\{tf(t)\} = \frac{-d}{ds}[\cot^{-1}(s) = -\left[\frac{-1}{1+s^2}\right] = \frac{1}{1+s^2}$$

$$tf(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

$$\therefore L^{-1}\left\{\cot^{-1}(s)\right\} = \frac{1}{\mathrm{t}}\sin t$$

Inverse Laplace Transform of integrals:

Theorem: $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, then $L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$

Proof: we have seen that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)ds$

$$\therefore L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$$

Solved Problems :

1. **Find** $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$

Sol: **Let** $\overline{f}(s) = \frac{s+1}{(s^2+2s+2)^2}$

Then $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\int_s^\infty \frac{s+1}{(s^2+2s+2)^2}ds\right\}$

$$= L^{-1}\left\{\frac{s+1}{[(s+1)^2+1]^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}, \text{ by First Shifting Theorem}$$

$$= e^{-t}\frac{t}{2}\sin t = \frac{t}{2}e^{-t}\sin t \because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

Multiplication by power of's':

Theorem: $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, and f (0),then $L^{-1}\left\{s\overline{f}(s)\right\} = f^1(t)$

Proof: we have seen that $L\left\{f^1(t)\right\} = s\overline{f}(s) - f(0)$

$$\therefore L\left\{f^1(t)\right\} = s\overline{f}(s) \quad [\because f(0)=0] \text{ or}$$

$$L^{-1}\left\{s\overline{f}(s)\right\} = f^1(t)$$

Note: $L^{-1}\left\{s^n\overline{f}(s)\right\} = f^n(t), \text{if } f^n(0) = 0 \text{ for } n = 1,2,3.........n-1$

Solved Problems :

1. **Find (i)** $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$ **(ii)** $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol: Let $\overline{f}(s) = \frac{1}{(s+2)^2}$ **Then**

$$L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} = e^{-2t}.t = f(t),$$

Clearly f (0) =0

Thus $L^{-1}\left\{\frac{s}{(s+2)^2}\right\} = L^{-1}\left\{s.\frac{s}{(s+2)^2}\right\} = L^{-1}\left\{s.\overline{f}(s)\right\} = f^1(t)$

$$= \frac{d}{dt}(te^{-2t}) = t(-2e^{-2t}) + e^{-2t}.1 = e^{-2t}(1-2t)$$

Note: in the above problem put 2=3, then $L^{-1}\left\{\frac{s}{(s+3)^2}\right\} = e^{-3t}(1-3t)$

Division by S:

Theorem: If $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, Then $L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\,du$

Proof: We have seen that $L\left\{\int_0^t f(u)\,du\right\} = \frac{\overline{f}(s)}{s}$

$$\therefore L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\} = \int_0^t f(u)\,du$$

Note: If $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, then $L^{-1}\left\{\frac{\overline{f}(s)}{s^2}\right\} = \int_0^t\int_0^t f(u)\,du.du$

Solved Problems :

1. Find the inverse Laplace Transform of $\frac{1}{s^2(s^2+a^2)}$

Sol: Since $L^{-1}\left[\frac{1}{(s^2+a^2)}\right] = \frac{1}{a}\text{sinat}$, we have

$$L^{-1}\left[\frac{1}{\text{s}(s^2+a^2)}\right] = \int_0^t \frac{1}{a}\sin at\,dt$$

$$= \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t = -\frac{1}{a^2}(\text{cosat}-1) = \frac{1}{a^2}(1-\text{cosat})$$

Then $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \int_0^t \frac{1}{a^2}(1-\cos at)\,dtdt$

$$= \frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)_0^t = \frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$$

Convolution Definition:

If f (t) and g (t) are two functions defined for $t \ge 0$ then the convolution of f (t) and g (t) is defined as $f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$

$f(t)*g(t)\,can\,also\,be\,written\,as\,(f*g)(t)$

Properties:

The convolution operation * has the following properties

1. Commutative i.e. $(f*g)(t)=(g*f)(t)$

2. Associative $[f*(g*h)](t)=[(f*g)*h](t)$

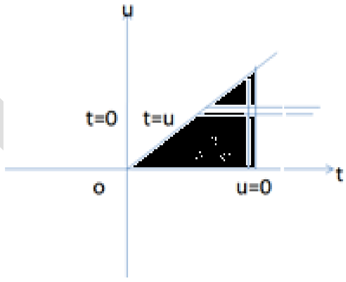
3. Distributive $[f*(g+h)](t)=(f*g)(t)+(f*h)(t)\ for\ t\geq 0$

Convolution Theorem: If $f(t)$ and $g(t)$ are functions defined for $t\geq 0$ then

$$\text{L}\{f(t)*g(t)\}=\text{L}\{f(t)\}\text{L}\{g(t)\}=\bar{f}(s).\bar{g}(s)$$

i.e., The L.T of convolution of f(t) and g(t) is equal to the product of the L.T of f(t) and g(t)

Proof: WKT $\text{L}\{\phi(t)\}=\int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt$



$$=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\,dt$$

The double integral is considered within the region enclosed by the line u=0 and u=t

On changing the order of integration, we get

$$\text{L}\ \{\phi(t)\}=\int_0^\infty\int_u^\infty e^{-st}f(u)g(t-u)dt\,du$$

$$=\int_0^\infty e^{-su}f(u)\left\{\int_u^\infty e^{-s(t-u)}g(t-u)dt\right\}du$$

$$=\int_0^\infty e^{-su}f(u)\left\{\int_0^\infty e^{-sv}g(v)dv\right\}du \qquad put\ \ t-u=v$$

$$=\int_0^\infty e^{-su}f(u)\{\bar{g}(s)\}du=\bar{g}(s)\int_0^\infty e^{-su}f(u)du=\bar{g}(s).\bar{f}(s)$$

$$\text{L}\{f(t)*g(t)\}=\text{L}\{f(t)\}\text{L}\{g(t)\}=\bar{f}(s).\bar{g}(s)$$

Solved Problems :

1. Using the convolution theorem find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol: $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}$

$Let\ \bar{f}(s)=\frac{s}{s^2+a^2}\ and\ \bar{g}(s)=\frac{1}{s^2+a^2}$

So that $\text{L}^{-1}\{\bar{f}(s)\}=\text{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at=f(t)-say$

$$L^{-1}\left\{\bar{g}(s)\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t) \to say$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\frac{1}{a}.\sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t [\sin(\text{au}+at-au)-\sin(au-at+au)]du$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\sin at.u + \frac{1}{2a}.\cos(2au-at)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos\left(2at-at\right) - \frac{1}{2a}\cos\left(-at\right)\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right]$$

$$= \frac{t}{2a}\sin at$$

2. Use convolution theorem to evaluate $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol: $$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\}$$

$$Let\ \bar{f}(s) = \frac{s}{s^2+a^2}\ and\ \bar{g}(s) = \frac{s}{s^2+b^2}$$

So that $L^{-1}\left\{\bar{f}(s)\right\} = L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at = f(t) \to say$

$$L^{-1}\left\{\bar{g}(s)\right\} = L^{-1}\left\{\frac{s}{(s^2+b^2)}\right\} = \cos bt = g(t) \to say$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\} = \int_0^t \cos au.\cos b(t-u)du$$

$$= \frac{1}{2}\int_0^t [\cos(au-bu+bt)+\cos(au+bu-bt)]du$$

$$=\frac{1}{2}\left[\frac{\sin(au-bu+bt)}{a-b}+\frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right]=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

3. Use convolution theorem to evaluate $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}=L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}$

$Let\ \overline{f}(s)=\frac{1}{s^2}\ and\ \overline{g}(s)=\frac{s}{\left(s^2+4\right)^2}$

So that $L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s^2}\right\}=t=g(t)\rightarrow say$

$$L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}=\frac{t.\sin 2t}{4}=f(t)-say\left[\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{ts\text{in}\,2t}{2a}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}=\int_0^t\frac{u}{4}\sin 2u(t-u)du$$

$$=t/4\int_0^t u\sin 2udu-\frac{1}{4}\int_0^t u^2\sin 2udu$$

$$=t/4\left(-\frac{u}{2}\cos 2u+\frac{1}{4}\sin 2u\right)_0^t$$

$$=-\frac{1}{4}\left[\frac{-u^2}{2}\cos 2u+\frac{u}{2}\sin 2u+\frac{1}{4}\cos 2u\right]_0^t$$

$$=\frac{1}{16}[1-t\sin 2t-\cos 2t]$$

4. Find $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]$

Sol: $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]=L^{-1}\left[\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right]$

$Let\ \overline{f}(s)=\frac{1}{s-2}\ and\ \overline{g}(s)=\frac{1}{s^2+1}$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=f(t)\rightarrow say$

$$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = g(t) \rightarrow say$$

$$\therefore L^{-1}\left\{\frac{1}{s-2}.\frac{1}{s^2+1}\right\} = \int_0^t f(u).\mathrm{g}(\mathrm{t}-u)\,\mathrm{du} \qquad \text{(By Convolution theorem)}$$

$$= \int_0^t e^{2u}\sin(\mathrm{t}-\mathrm{u})\,\mathrm{du} \ \ \text{(or)} \ \int_0^t \mathrm{sinu}.e^{2(t-u)}\,\mathrm{du}$$

$$= e^{2t}\int_0^t \sin u e^{-2u}du$$

$$= e^{2t}\left[\frac{e^{-2u}}{2^2+1}\left[-2\sin u - \cos u\right]\right]_0^t$$

$$= e^{2t}\left[\frac{1}{5}e^{-2t}\left(-2\sin t - \cos t\right) - \frac{1}{5}(-1)\right]$$

$$= \frac{1}{5}\left(e^{2t} - 2\sin t - \cos t\right)$$

5. **Find** $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = L^{-1}\left\{\frac{1}{s+1}.\frac{1}{s-2}\right\}$

$$Let\ \overline{f}(s) = \frac{1}{s+1}\ and\ \overline{g}(s) = \frac{1}{s-2}$$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} = f(t) \rightarrow say$

$$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = g(t) \rightarrow say$$

$\therefore$ By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(s+1)(\mathrm{s}-2)}\right\} = \int_0^t e^{-u}e^{2(t-u)}du$$

$$= \int_0^t e^{2t}e^{-3u}du = e^{2t}\int_0^t e^{-3u}du = e^{2t}\left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{1}{3}\left[e^{2t} - e^{-t}\right]$$

6. **Find** $\mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Sol: $\mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \mathrm{L}^{-1}\left\{\frac{1}{s^2}.\frac{1}{s^2-a^2}\right\}$

$Let\,\bar{f}(s)=\frac{1}{s^2}\,and\,\bar{g}(s)=\frac{1}{s^2-a^2}$

So that $\mathrm{L}^{-1}\left\{\bar{f}(s)\right\}=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}=t=f(t)-say$

$\mathrm{L}^{-1}\left\{\bar{g}(s)\right\}=\mathrm{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\}=\frac{1}{a}\sinh at=g(t)-say$

By using convolution theorem, we have

$$\mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}=\int_0^t u.\frac{1}{a}\sinh a(t-u)du$$

$$=\frac{1}{a}\int_0^t u\sinh(at-au)du$$

$$=\frac{1}{a}\left[\frac{-u}{a}\cosh(at-au)-\frac{\sin(at-au)}{a^2}\right]_0^t$$

$$=\frac{1}{a}\left[\frac{-t}{a}\cosh(at-at)-0-\frac{1}{a^2}[0-\sinh at]\right]$$

$$=\frac{1}{a}\left[\frac{-t}{a}+\frac{1}{a^2}\sinh at\right]$$

$$=\frac{1}{a^3}\left[-at+\sinh at\right]$$

3. Using Convolution theorem, evaluate $L^{-1}\{\frac{s}{(s+2)(s^2+9)}\}$

Sol: $L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\}=L^{-1}\{\frac{1}{s+2}.\frac{s}{s^2+3^2}\}=L^{-1}\{\bar{f}(s).\bar{g}\,(s)\}$

$\bar{f}(s)=\frac{1}{s+2}=L\{f(t)\}\Rightarrow f(t)=L^{-1}\left\{\frac{1}{s+2}\right\}=e^{-2t}$----------------- (1)

$\bar{g}(s)=\frac{s}{s^2+3^2}=L\{g(t)\}\Rightarrow g(t)=L^{-1}\left\{\frac{s}{s^2+3^2}\right\}=cos3t$----------------- (2)

By Convolution theorem we have

$$L^{-1}\{\bar{f}(s).\bar{g}\,(s)\}=f(t)*g(t)$$

Where $f(t)*g(t)=\int_0^t g(u)f(t-u)du$

$\therefore\quad L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\}=\int_0^t e^{-2(t-u)}cos3udu$

$$=e^{-2t}\int_0^t e^{2u}cos3udu$$

$$=e^{-2t}.\frac{1}{2^2+3^2}[2cos3u-3sin3u]_0^t$$

$$=\frac{e^{-2t}}{13}[2cos3t-2-3sin3t]$$

$$=\frac{1}{13}[e^{-2t}(2cos3t-3sin3t)]-\frac{2e^{-2t}}{13}$$

Application of L.T to ordinary differential equations:

(Solutions of ordinary DE with constant coefficient):

1. **Step1:** Take the Laplace Transform on both the sides of the DE and then by using the formula $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-1} f^1(0) - s^{n-2} f^2(0) - ....f^{n-1}(0)$ and apply given initial conditions. This gives an algebraic equation.
2. **Step2:** replace f (0), $f^1(0)$, $f^2(0)$,……… $f^{n-1}(0)$ with the given initial conditions.

   Where $f^1(0) = s\overline{f}(0) - f(0)$

   $f^2(0) = s^2\overline{f}(s) - s f(0) - f^1(0)$, and so on
3. **Step3:** solve the algebraic equation to get derivatives in terms of s.
4. **Step4:** take the inverse Laplace transform on both sides this gives f as a function of t which gives the solution of the given DE

Solved Problems :

1. **Solve** $y^{111} + 2y^{11} - y^1 - 2y = 0$ **using Laplace Transformation given that** $y(0) = y^1(0) = 0$ **and** $y^{11}(0) = 6$

Sol: Given that $y^{111} + 2y^{11} - y^1 - 2y = 0$

Taking the Laplace transform on both sides, we get

$$L\left\{y^{111}(t)\right\} + 2L\left\{y^{11}(t)\right\} - L\left\{y^1\right\} - 2L\left\{y\right\} = 0$$

$$\Rightarrow s^3L\left\{y(t)\right\} - s^2y(0) - sy^1(0) - y^{11}(0) + 2\left\{s^2L\left\{y(t)\right\} - sy(0) - y^1(0)\right\} - \left\{sL\left\{y(t)\right\} - y(0)\right\} - 2L\left\{y(t)\right\} = 0$$

$$\Rightarrow \left\{s^3 + 2s^2 - s - 2\right\}L\left\{y(t)\right\} = s^2y(0) + sy^1(0) + y^{11}(0) + 2sy(0) + 2y^1(0) - y(0)$$

$$= 0 + 0 + 6 + 2.0 + 2.0 - 0$$

$$\Rightarrow \left\{s^3 + 2s^2 - s - 2\right\}L\left\{y(t)\right\} = 6$$

$$L\left\{y(t)\right\} = \frac{6}{s^3 + 2s^2 - s - 2} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) = 6$$

$$\Rightarrow A(s^2 + 3s + 2) + B(s^2 - s - 2) + C(s^2 - 1) = 6$$

Comparing both sides s^2,s,constants,we have

$\Rightarrow A+B+C=0, 3A-B=0, 2A-2B-C=6$

$$\begin{aligned} A+B+C&=0\\ 2A-2B-C&=6 \end{aligned}$$

$$\begin{aligned} 3A-B&=6\\ 3A+B&=0 \end{aligned}$$

$6A=6 \Rightarrow A=1$

$3A+B=0 \Rightarrow B=-3A \Rightarrow B=-3$

$\therefore A+B+C=0 \Rightarrow C=-A-B=-1+3=2$

$$\therefore L\{y(t)\}=\frac{1}{s-1}-\frac{3}{s+1}+\frac{2}{s+2}$$

$$y(t)=L^{-1}\left\{\frac{1}{s-1}\right\}-3.L^{-1}\left\{\frac{1}{s+1}\right\}+2.L^{-1}\left\{\frac{1}{s+2}\right\}=e^{t}-3e^{-t}+2.e^{-2t}$$

Which is the required solution

2. Solve $y^{11}-3y^{1}+2y=4t+e^{3t}$ using Laplace Transformation given that $y(0)=1 \, and \, y^{1}(0)=-1$

Sol: Given that $y^{11}-3y^{1}+2y=4t+e^{3t}$

Taking the Laplace transform on both sides, we get

$$L\{y^{11}(t)\}-3L\{y^{1}(t)\}+2L\{y(t)\}=4L\{t\}+L\{e^{3t}\}$$

$$\Rightarrow s^2L\{y(t)\}-sy(0)-y^{1}(0)-3\left[sL\{y(t)\}-y(0)\right]+2L\{y(t)\}=\frac{4}{s^2}+\frac{1}{s-3}$$

$$\Rightarrow (s^2-3s+2)L\{y(t)\}=\frac{4}{s^2}+\frac{1}{s-3}+s-4$$

$$\Rightarrow (s^2-3s+2)L\{y(t)\}=\frac{4s-12+s^4+s^2-3s^3-4s^3+12s^2}{s^2(s-3)}$$

$$\Rightarrow L\{y(t)\}=\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s^2-3s+2)}$$

$$\Rightarrow L\{y(t)\}=\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)}=\frac{As+B}{s^2}+\frac{C}{s-3}+\frac{D}{s-1}+\frac{E}{s-2}$$

$$=\frac{(As+B)(s-1)(s-2)(s-3)+C(s^2)(s-1)(s-2)+D(s^2)(s-2)(s-3)+E(s^2)(s-1)(s-3)}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow s^4-7s^3+13s^2+4s-12=(As+B)(s^3-6s^2+11s-6)+$$
$$C(s^2)(s^2-3s+2)+D(s^2)(s^2-5s+6)+E.s^2(s^2-4s+3)$$

Comparing both sides s^4,s^3,we have

$A+C+D+E=1$......................(1)

$-6A+B-3C-5D-4E=-7$......................(2)

$$put\ s=1, 2D=-1\Rightarrow D=\frac{-1}{2}$$

$$put\ s=2, -4E=8\Rightarrow E=-2$$

$$put\ \ s=3, 18C=9\Rightarrow C=\frac{1}{2}$$

$$from\,eq.(1)\ A=1-\frac{1}{2}+\frac{1}{2}+2\Rightarrow A=3$$

from eq.(2) B= -7+18+$\frac{3}{2}-\frac{5}{2}-8=3-1=2$

$$y(t)=L^{-1}\left\{\frac{3}{s}+\frac{2}{s^2}+\frac{1}{2(s-3)}-\frac{1}{2(s-1)}-\frac{2}{s-2}\right\}$$

$$y(t)=3+2t+\frac{1}{2}e^{3t}-\frac{1}{2}e^{t}-2.e^{2t}$$

3. Using Laplace Transform Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t,$ **given that** $y=\frac{dy}{dt}=0$ **when t=0**

Sol: Given equation is $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t.$

$$L\{y^{11}(t)\}+2L\{y^{1}(t)\}-3L\{y(t)\}=L\{\sin t\}$$

$$s^2L\{y(t)\}-sy(0)-y^{1}(0)+2\left[sL\{y(t)\}-y(0)\right]-3.L\{y(t)\}=\frac{1}{s^2+1}$$

$$\Rightarrow(s^2+2s-3)L\{y(t)\}=\frac{1}{s^2+1}$$

$$\Rightarrow L\{y(t)\}=\left(\frac{1}{(s^2+1)(s^2+2s-3)}\right)$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{1}{(s-1)(s+3)(s^2+1)}\right)$$

Now consider

$$\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}$$

$$A(s+3)(s^2+1)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)=1$$

Comparing both sides s^3,we have

$$put\ s=1, 8A=1 \Rightarrow A=\frac{1}{8}$$

$$put\ s=-3,-40B=1 \Rightarrow B=\frac{-1}{40}$$

$$A+B+C=0 \Rightarrow C=0-\frac{1}{8}+\frac{1}{40}$$

$$C=\frac{-5+1}{40}=\frac{-4}{40}=\frac{-1}{10}$$

$$3A-B+2C+D=0 \Rightarrow D=-\frac{3}{8}-\frac{1}{40}+\frac{1}{5}$$

$$D=\frac{-15-1+8}{40}=\frac{-8}{40}=\frac{-1}{5}$$

$$\therefore y(t)=\mathrm{L}^{-1}\left\{\frac{\frac{1}{8}}{s-1}+\frac{\frac{-1}{40}}{s+3}+\frac{\frac{-1}{10}s-\frac{1}{5}}{s^2+1}\right\}$$

$$=\frac{1}{8}\mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{40}\mathrm{L}^{-1}\left\{\frac{1}{s+3}\right\}-\frac{1}{10}\mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{5}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\therefore y(t)=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\cos t-\frac{1}{5}\sin t$$

4. **Solve** $\frac{dx}{dt}+x=\sin\omega \mathrm{t}, x(0)=2$

Sol: Given equation is $\frac{dx}{dt}+x=\sin\omega \mathrm{t}$

$$\mathrm{L}\{x^1(t)\}+\mathrm{L}\{x(t)\}=\mathrm{L}\{\sin\omega \mathrm{t}\}$$

$$\Rightarrow \mathrm{s.L}\{x(t)\}-x(0)+\mathrm{L}\{x(t)\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow \mathrm{s.L}\{x(t)\}-2+\mathrm{L}\{x(t)\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow (s+1)\mathrm{L}\{x(t)\}=\frac{\omega}{s^2+\omega^2}+2$$

$$\Rightarrow x(t) = L^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)} + \frac{2}{s+1}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}\right\}$$ (By using partial fractions)

$$= 2e^{-t} + L^{-1}\left\{\frac{\frac{\omega}{\omega^2+1}}{s+1} - \frac{\frac{s\omega}{1+\omega^2}}{s^2+\omega^2} + \frac{\frac{\omega}{1+\omega^2}}{s^2+\omega^2}\right\}$$

$$= 2e^{-t} + \frac{\omega}{\omega^2+1}e^{-t} - \frac{\omega}{1+\omega^2}\cos\omega t + \frac{\omega}{1+\omega^2}.\frac{1}{\omega}\sin\omega t$$

5. **Solve** $\left(D^2+n^2\right)x = a\sin(nt+\alpha)$ **given that x=Dx=0, when t=0**

Sol: Given equation is $\left(D^2+n^2\right)x = a\sin(nt+\alpha)$

$$x^{11}(t) + n^2x(t) = a\sin(nt+\alpha)$$

$$L\left\{x^{11}(t)\right\} + n^2L\left\{x(t)\right\} = L\left\{a\sin nt\cos\alpha + a\cos nt\sin\alpha\right\}$$

$$\Rightarrow s^2L\left\{x(t)\right\} - sx(0) - x^1(0) + n^2L\left\{x(t)\right\} = a\cos\alpha L\left\{\sin nt\right\} + a\sin\alpha L\left\{\cos nt\right\}$$

$$\Rightarrow \left(s^2+n^2\right)L\left\{x(t)\right\} = a\cos\alpha\frac{n}{s^2+n^2} + a\sin\alpha.\frac{s}{s^2+n^2}$$

$$\Rightarrow L\left\{x(t)\right\} = a\cos\alpha\frac{n}{\left(s^2+n^2\right)^2} + a\sin\alpha\frac{s}{\left(s^2+n^2\right)^2}$$

(By using convolution theorem I –part, partial fraction in II-part)

$$= na\cos\alpha\int_0^t \frac{1}{n}.\sin nx.\frac{1}{n}\sin n(t-x)\,dx - \frac{a\sin\alpha}{2}L^{-1}\left\{\frac{d}{ds}\frac{1}{\left(s^2+n^2\right)}\right\}$$

$$= \frac{a\cos\alpha}{2n}\int_0^t\left\{\cos(nt-2nx) - \cos nt\right\}dx + \frac{a\sin\alpha}{2}t\frac{1}{n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\int_0^t\left\{\cos n(t-2x) - \cos nt\right\}dx + \frac{a}{2n}\sin\alpha t\sin nt\right]$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{-1}{2n}.\sin n(t-2x) - x\cos nt\right]_0^t + \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{\sin nt}{2n} - t\cos nt\right] + \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}\left[\cos\alpha\cos nt - \sin\alpha\sin nt\right]$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}\cos(\alpha + nt)$$

6. Solve $y^{11} - 4y^1 + 3y = e^{-t}$ using L.T given that y (0) = y¹ (0) = 1.

Sol: Given equation is $y^{11} - 4y^1 + 3y = e^{-t}$

Applying L.T on both sides we get $L(y^{11}) - 4L(y^1) + 3L(y) = L(e^{-t})$

$\Rightarrow \{s^2L[y] - s\,y(0) - y^1(0)\} - 4\{s\,L[y] - y(0)\} + 3L\{y\} = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} - s - 1 - 4 = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} = \frac{1}{s+1} + s + 5$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} = \frac{1}{s+1} + s + 5$

$$L\{y\} = \frac{1}{(s+1)(s^2+4s+3)} + \frac{s+5}{(s^2+4s+3)}$$

$$y = L^{-1}[\frac{1}{(s+1)(s^2+4s+3)}] + L^{-1}[\frac{s+5}{(s^2+4s+3)}]$$

Let us consider

$$L^{-1}[\frac{1}{(s+1)(s^2+4s+3)}] = L^{-1}[\frac{1}{(s+1)^2(s+3)}]$$

$$\frac{1}{(s+1)(s^2+4s+3)} = \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{(S+1)^2} + \frac{C}{S+3}$$

$$= \frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}$$

$$= L^{-1}[\frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}]$$

$$= L^{-1}[\frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}]$$

$$= -\frac{1}{4}L^{-1}[\frac{1}{s+1}] + \frac{1}{2}L^{-1}[\frac{1}{(S+1)^2}] + \frac{1}{4}L^{-1}[\frac{1}{S+3}]$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] = -\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t} ----\to (1)$$

$$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right] = L^{-1}[\frac{s+2}{((s+2)^2-1)}] + L^{-1}[\frac{3}{((s+2)^2-1)}]$$

$$= \boldsymbol{e^{-2t}}L^{-1}[\frac{s}{(s^2-1)}] + L^{-1} + \boldsymbol{3e^{-2t}}L^{-1}[\frac{1}{(s^2-1)}]$$

$$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right] = \boldsymbol{cost} + \boldsymbol{3e^{-2t}sint} ---\to (2)$$

From (1) & (2)

$$\therefore y = -\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t} + \boldsymbol{e^{-2t}cost} + \boldsymbol{3e^{-2t}sint}$$

7. Solve $\frac{d^2x}{d^2t} + 9x = cos2t$ using L.T. given x (0) =1, x ($\frac{\pi}{2}$) = -1.

Sol: Given $x^{11} + 9x = cos2t$

L $[x^{11}] + 9[x] = L[cos2t]$

$\Rightarrow s^2L[x] - sx(0) - x^1(0) + 9L[x\} = \frac{s}{s^2+4}$

$\Rightarrow (s^2+9)L[x] - s - a = \frac{s}{s^2+4}$

$\Rightarrow (s^2+9)L[x] = \frac{s}{s^2+4} + (s+a)$

$L[x] = \frac{s}{(s^2+4((s^2+9)} + \frac{s}{(s^2+9)} + \frac{a}{(s^2+9)}$

$X = L^{-1}[\frac{s}{(s^2+4((s^2+9)}] + L^{-1}[\frac{s}{(s^2+9)}] + L^{-1}[\frac{a}{(s^2+9)}]$

$= \frac{1}{5}L^{-1}[\frac{s}{s^2+4} - \frac{s}{s^2+9}] + cos3t + \frac{a}{3}sin3t$

$= \frac{1}{5}L^{-1}[\frac{s}{s^2+4}] - \frac{1}{5}L^{-1}[\frac{s}{s^2+9}] + cos3t + \frac{a}{3}sin3t$

$= \frac{1}{5}cos2t - \frac{1}{5}cos3t + cos3t + \frac{a}{3}sin3t$ ------------------→ (1)

Given x ($\frac{\pi}{2}$) = -1.

$\therefore\ -1 = \frac{1}{5}cos2\left(\frac{\pi}{2}\right) - \frac{1}{5}cos\frac{3\pi}{2} + cos\frac{3\pi}{2} + cos\frac{3\pi}{2} + \frac{a}{3}sin\frac{3\pi}{2}$

$\Rightarrow\ -1 = -\frac{1}{5} - 0 + 0 - \frac{a}{3}$

$\frac{a}{3} = -\frac{1}{5} + 1$

$\frac{a}{3} = \frac{4}{5}$

$\therefore\ x = \frac{1}{5}cos2t + \frac{4}{5}cos3t + \frac{4}{5}sin3t$ From (1)

8. $Solve(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ Using L.T given y (0) =1, $y^1 = 0, y^{11}(0) = -2$

Sol: Given $y^{111} - 3y^{11} + 3y^1 - y = t^2e^t$

$L[y^{111}] - 3L[y^{11}] + 3L[y^1] - L[y] = L[t^2e^t]$

$\Rightarrow \{s^3L[y] - s^2y(0) - sy^1(0) - y^{11}(0)\} - 3\{s^2L[y] - sy^1(0) - y(0)\} +$
$3\{sL[y] - y(0)\} - L[y] = L[t^2e^t]$

$\Rightarrow (s^3 - 3s^2 + 3s - 1)L[y] - s^2 - 0 + 2 + 0 + 3(1) - 3(1) = (-1)^2\frac{d^2}{ds^2}L[e^t]$

$\Rightarrow (s-1)^3L[y] - s^2 + 2 = \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$

$= \frac{2}{(s-1)^3}$

$$\Rightarrow (s-1)^3 \mathrm{L}[\mathrm{y}] = \frac{2}{(s-1)^3} + s^2 - 2$$

$$L[y] = \frac{2}{(s-1)^6} + \frac{s^2}{(s-1)^3} - \frac{2}{(s-1)^3}$$

$$y = L^{-1}[\frac{2}{(s-1)^6}] + L^{-1}[\frac{s^2}{(s-1)^3}] - L^{-1}[\frac{2}{(s-1)^3}]$$

$$= 2L^{-1}[\frac{1}{(s-1)^6}] + L^{-1}[\frac{s^2}{(s-1)^3}] - 2L^{-1}[\frac{1}{(s-1)^3}]$$

$$=2e^t L^{-1}\left[\frac{1}{(s)^6}\right] + L^{-1}\frac{s^2}{(s-1)^3} - 2e^t L^{-1}\left[\frac{1}{s^3}\right]$$

$$= 2e^t\frac{t^5}{5!} - 2e^t\frac{t^2}{2!} + L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$$

Consider $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

W.K.T $L^{-1}\left[\frac{1}{(s-1)^3}\right] = e^t L^{-1}\left[\frac{1}{s^3}\right] = e^t\frac{t^2}{2!} = \frac{e^t t^2}{2}$

$$L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = \frac{d^2}{ds^2}\left(\frac{e^t t^2}{2}\right) = \frac{1}{2}\frac{d}{dt}(2te^t + t^2e^t) = \frac{1}{2}(2e^t + 2te^t + 2te^t + t^2e^t)$$

$$= \frac{1}{2}(2e^t + 4te^t + t^2e^t)$$

$$\therefore \quad y = 2e^t\frac{t^5}{5!} - 2e^t\frac{t^2}{2!} - \frac{1}{2}(2e^t + 4te^t + t^2e^t)$$