Indian Institute of Technology Roorkee
MAN-001(Mathematics-1): B. Tech. I Year
Autumn Semester: 2018-19
Tutorial Sheet-1: Matrix Algebra

(1) Reduce each of the following matrices into row echelon form and then find their ranks:

(a) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 7 & 8 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 4 & 6 \\ -1 & 3 & 2 \\ 1 & 4 & 6 \\ 2 & 8 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 2 \\ 4 & 6 & 0 \\ 3 & 1 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 2 & 0 & -4 \\ 1 & 2 & 0 & -4 & 1 \\ 2 & 0 & -4 & 1 & 1 \\ 0 & -4 & 1 & 1 & 2 \\ -4 & 1 & 1 & 2 & 0 \end{bmatrix}$

(2) Examine the following set of vectors over $\mathbb{R}$ for linear dependence:

(a) $\{(1,2,3),\ (2,1,6),\ (3,3,9)\}$
(b) $\{(1,-1,1),\ (2,1,1),\ (8,1,5)\}$
(c) $\{(1,1,-1,1),\ (1,-1,2,-1),\ (3,1,0,1)\}$
(d) $\{(1,2,-2),\ (-1,3,0),\ (0,-2,1)\}$

(3) (a) Find the conditions/values of α and β for which the matrix

$$\begin{pmatrix} \alpha & 1 & 2 \\ 0 & 2 & \beta \\ 1 & 3 & 6 \end{pmatrix}$$

has (i) rank $= 1$ (ii) rank $= 2$ (iii) rank $= 3$.

(b) For what values of α and β is the following system consistent?

$$\begin{aligned} 2x + 4y + (\alpha+3)z &= 2 \\ x + 3y + z &= 2 \\ (\alpha-2)x + 2y + 3z &= \beta \end{aligned}$$

(4) Solve the following system of linear equations by Gauss elimination method:

(a) $\begin{array}{ccccccc} x & + & 4y & - & z & = & 4 \\ x & + & y & - & 6z & = & -4 \\ 3x & - & y & - & z & = & 1 \end{array}$ (b) $\begin{array}{ccccccc} x & + & y & + & z & = & 4 \\ 2x & + & 5y & - & 2z & = & 3 \\ x & + & 7y & - & 7z & = & 5 \end{array}$ (c) $\begin{array}{ccccccc} x & + & 2y & + & z & = & 2 \\ 3x & + & y & - & 2z & = & 1 \\ 4x & - & 3y & - & z & = & 3 \\ 2x & + & 4y & + & 2z & = & 4 \end{array}$

(5) Consider the following systems of linear equations:

(a) $\begin{array}{ccccccc} -2x & + & y & + & z & = & a \\ x & - & 2y & + & z & = & b \\ x & + & y & - & 2z & = & c \end{array}$ (b) $\begin{array}{ccccccc} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & \lambda z & = & 3 \\ x & + & \lambda y & + & 3z & = & 2 \end{array}$ (c) $\begin{array}{ccccccc} \lambda x & + & y & + & z & = & p \\ x & + & \lambda y & + & z & = & q \\ x & + & y & + & \lambda z & = & r \end{array}$

Find the values of unknown constant(s) such that each of the above systems has
(i) no solution (ii) a unique solution (iii) infinitely many solutions .

(6) Use Gauss elimination method to show that following system has no solution:

$$\begin{array}{ccccccc} 2\sin x & - & \cos y & + & 3\tan z & = & 3 \\ 4\sin x & + & 2\cos y & - & 2\tan z & = & 10 \\ 6\sin x & - & 3\cos y & + & \tan z & = & 9 \end{array}$$

(7) Let P_2 be the set of all polynomials of degree 2 or less. Use Gauss elimination method to find all polynomials $f \in P_2 : f(1) = 2$ and $f(-1) = 6$.

(8) Find the values of k for which the following system of equations has
(i) trivial solution (ii) non-trivial solution.

$$\text{(a)} \quad \begin{aligned} (3k-8)x + 3y + 3z &= 0 \\ 3x + (3k-8)y + 3z &= 0 \\ 3x + 3y + (3k-8)z &= 0 \end{aligned} \qquad \text{(b)} \quad \begin{aligned} (k-1)x + (3k+1)y + 2kz &= 0 \\ (k-1)x + (4k-2)y + (k+3)z &= 0 \\ 2x + (3k+1)y + 3(k-1)z &= 0 \end{aligned}$$

(9) By employing elementary row operations, find the inverse of the following matrices:

$$(a) \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \quad (b) \begin{pmatrix} 2 & 4 & 5 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{pmatrix} \quad (c) \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \quad (d) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 1 & 3 \end{pmatrix}$$

(10) If p is a solution of a non-homogeneous system $AX = Y$, then show that $v + p$ is also a solution of $AX = Y$, for every solution v of the homogeneous system $AX = 0$.

(11) (a) Let A be an $n \times n$ matrix. Prove the following two statements:
(i) If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.
(ii) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

(b) If $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ -2 & 1 & 0 & 1 \\ 0 & 5 & -2 & 7 \\ -1 & 3 & -1 & 4 \end{bmatrix}$, find a 4×4 matrix $B \neq 0$ such that $AB = 0$.

(12) Consider a 4×5 matrix $A = \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 3 & 21 & 0 & 9 & 0 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{bmatrix}$.

(a) Find the row-reduced echelon form of A.

(b) Find an invertible matrix P such that $PA = \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 3 & 15 & 3 \\ 0 & 0 & 2 & 10 & 3 \\ 0 & 0 & 5 & 25 & 6 \end{bmatrix}$.

(c) Find the equation of a plane $ay_1 + by_2 + cy_3 = d$ such that for each column vector $Y = (y_1, y_2, y_3, d)^T$, the equation $AX = Y$ has a solution.

(d) If $X = (x_1, x_2, x_3, x_4, x_5)^T$, then find the conditions on x_1, x_2, x_3, x_4, x_5 such that $AX = 0$.

ANSWERS

(1) (a) 3 (b) 3 (c) 3 (d) 4

(2) (a) LD (b) LD (c) LD (d) LI

(3) (a) (i) Not possible (ii) $\alpha = \frac{1}{3}$ or $\beta = 4$ (iii) $\alpha \neq \frac{1}{3},\ \beta \neq 4$
(b) $\alpha = 3$ and $\beta = 1$; or $\alpha = -2$ and $\beta = 6$; or $\alpha \neq 3, -2$.

(4) (a) (1,1,1) (b) No solution (c) (1,0,1)

(5) (a) (i) $a + b + c \neq 0$ (ii) Not possible (iii) $a + b + c = 0$
(b) (i) $\lambda = -3$ (ii) $\lambda \neq -3, 2$ (iii) $\lambda = 2$
(c) (i) $\lambda = 1$ and $p + q - 2r \neq 0$ OR $\lambda = 1$ and $q \neq r$ OR $\lambda = -2$ and $p + q + r \neq 0$
(ii) $\lambda \neq 1, -2$
(iii) $\lambda = 1$ and $p = q = r$ OR $\lambda = -2$ and $p + q + r = 0$

(7) $f = (4 - k)x^2 - 2x + k,\ k \in \mathbb{R}$

(8) (a) (i) $k \neq \frac{2}{3}, \frac{11}{3}$ (ii) $k = \frac{2}{3}$ or $\frac{11}{3}$ (b) (i) $k \neq 0,\ 3$ (ii) $k = 0$ or 3

(9) (a) $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 1 & 2 \\ 3 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$ (c) $\frac{1}{13}\begin{bmatrix} 1 & 9 & -2 \\ -2 & -5 & 4 \\ 5 & -7 & 3 \end{bmatrix}$ (d) $\frac{1}{4}\begin{bmatrix} -16 & 4 & -4 & 12 \\ 5 & -1 & -1 & 0 \\ 9 & -1 & 3 & -8 \\ 6 & -2 & 2 & -4 \end{bmatrix}$

(11) (b) $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -7 & 2 & -5 \\ 5 & 0 & 5 & 5 \\ 0 & 5 & 0 & 5 \end{bmatrix}$ (This is just one solution. The matrix B is not unique).

(12) (a) $\begin{bmatrix} 1 & 7 & 0 & 3 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \end{bmatrix}$

(c) $y_1 + y_2 + y_3 = d$

(d) $x_1 + 7x_2 + 3x_4 = 0, \quad x_3 + 5x_4 = 0, \quad x_5 = 0$.