Indian Institute of Technology Roorkee
MAN-001(Mathematics-1): B. Tech. I Year
Autumn Semester: 2018-19
Tutorial Sheet-7: Gamma and Beta Functions

1. Evaluate: (i) $\Gamma(7)$, (ii) $\Gamma(\frac{7}{2})$.
2. Show that (i) $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$; (ii) $\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma(2m+1)}{2^{2m}\Gamma(m+1)}$;

   (iii) $2^{2m-1}\Gamma(\mathrm{m})\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2\mathrm{m})$, m is an integer in both (ii) and(iii).

3. For $s>0, p>0$, show that (i) $\int_0^\infty x^{p-1}e^{-sx}dx = \Gamma(p)/s^p$ (ii) $\int_0^\infty e^{-s^2x^2}dx = \sqrt{\pi}/2s$.
4. Show That $\Gamma(p) = \int_0^1(\log(1/y))^{p-1}dy$, $p>0$; using this evaluate $\int_0^1(\log(1/y))^{-1/2}dy$.
5. Show that for integer $\mathrm{m}>-1, \mathrm{n}>0$,

$$\int_0^1 x^m(\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

6. Show that for $\mathrm{c}>1$,

$$\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}.$$

7. Show that for $\mathrm{r}>-1$,

$$\int_0^\infty x^r e^{-s^2x^2}dx = \frac{1}{2\,s^{r+1}}\Gamma(\frac{r+1}{2}).$$

8. Using reflection property show that

$$\int_0^{\pi/2}\tan^n\theta\,d\theta = \frac{\pi}{2}\sec\frac{n\pi}{2}.$$

9. Prove the following:

   (i) $\mathrm{B}(x,y) = 2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta$, (ii) $\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$,

   (iii) $\mathrm{B}(x,y) = \mathrm{B}(x+1,y) + \mathrm{B}(x,y+1)$, (iv) $\frac{1}{x+y}\mathrm{B}(x,y) = \frac{1}{x}\mathrm{B}(x+1,y) = \frac{1}{y}\mathrm{B}(x,y+1)$,

   (v) $\int_0^1 t^{m-1}(1-t^2)^{n-1}dt = \frac{1}{2}\mathrm{B}\left(\frac{m}{2},n\right)$, (vi) $\int_0^1(1-t^6)^{-1/6}dt = \frac{\pi}{3}$.

10. Show that, for any positive integer m

$$\mathrm{B}(m,m) = \frac{\sqrt{\pi}\Gamma(\mathrm{m})}{2^{2m-1}\Gamma(\mathrm{m}+1/2)}.$$

11. Evaluate following integrals in terms of Gamma or Beta functions;

    (i) $\int_0^\infty e^{-x^4}dx$, (ii) $\int_0^\infty x^{-7/4}e^{-\sqrt{x}}dx$, (iii) $\int_0^1 x^5\,(1-x^3)^{10}dx$,

    (iv) $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx$, (v) $\int_0^a x^9\sqrt[3]{(a^6-x^6)}dx$, (vi) $\int_0^a x^3\,(a^5-x^5)^3dx$.

ANSWERS:

1. (i) 720, (ii) $\frac{15}{8}\sqrt{\pi}$ 4. $\sqrt{\pi}$

11. (i) $\Gamma(\frac{5}{4})$,

(ii) $\frac{8}{3}\sqrt{\pi}$,

(iii) $\frac{1}{3}\ \mathrm{B}(2,11) = \frac{1}{396}$,

(iv) $\frac{1}{4.2^{1/4}}\ \mathrm{B}\left(\frac{1}{4},\frac{7}{4}\right)$,

(v) $\frac{a^6}{6}\ \mathrm{B}\left(\frac{5}{3},\frac{4}{3}\right)$

(vi) $\frac{a^{19}}{65}\ \mathrm{B}\left(\frac{4}{5},4\right)$.