

**Indian Institute of Technology Roorkee**  
**MAN-001(Mathematics-1): B. Tech. I Year**  
**Autumn Semester: 2018-19**

**Assignment Sheet-3: Differential Calculus (Limit, Continuity and Differentiability)**

1. Find the following limits, if they exist

- (a)  $\lim_{(x,y) \rightarrow (-1,2)} \frac{xy^3}{x+y}$     (b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y}{x^6+y^2}$     (c)  $\lim_{(x,y) \rightarrow (0,1)} \tan^{-1} \left( \frac{y}{x} \right)$   
 (d)  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}}$     (e)  $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin^2(x+y)}{|x|+|y|}$     (f)  $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4+y^2}$   
 (g)  $\lim_{(x,y) \rightarrow (1,1)} f(x,y)$  where  $f(x,y) = \begin{cases} 1, & \text{if } x+y \geq 2 \\ -1, & \text{if } x+y < 2 \end{cases}$   
 (h)  $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xy+yz+zx}{\sqrt{x^2+y^2+z^2}}$     (i)  $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^2+y^2}$

2. (a) Consider the function  $f(x,y) = \frac{x+y}{x-y}$  for  $(x,y) \in \mathbb{R}^2$  with  $x-y \neq 0$ . Show that  $\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x,y)] = 1$ , but  $\lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x,y)] = -1$ . What can you say about the existence of  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ ?

(b) Let  $f(x,y) = 0$  if  $y = 0$ , and  $f(x,y) = x \sin \left( \frac{1}{y} \right)$ , if  $y \neq 0$ . Compute  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  and iterated limits  $\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x,y)]$  and  $\lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x,y)]$  if they exist.

(c) Let  $f(x,y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}$  if  $x^2y^2 + (x-y)^2 \neq 0$ . Show that  $\lim_{x \rightarrow 0} [\lim_{y \rightarrow 0} f(x,y)]$  and  $\lim_{y \rightarrow 0} [\lim_{x \rightarrow 0} f(x,y)] = 0$ . But  $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$  does not exist.

3. Let

$$f(x,y) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}.$$

Show that for any point  $(a,b)$ ,  $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$  does not exist.

4. Examine the continuity of the function  $f(x, y)$  at  $(0, 0)$  in each of the following cases:

$$\begin{aligned}
 \text{(a) } f(x, y) &= \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x, y) \neq (0, 0) \\ \frac{1}{2}, & (x, y) = (0, 0) \end{cases} \\
 \text{(b) } f(x, y) &= \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ 0, & xy = 0 \end{cases} \\
 \text{(c) } f(x, y) &= \begin{cases} xy \log(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \\
 \text{(d) } f(x, y) &= \begin{cases} \frac{x^2 y^2}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \quad \text{(e) } f(x, y) = \sqrt{|xy|}.
 \end{aligned}$$

5. For the function  $f(x, y) = \begin{cases} \frac{y(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ .

Compute  $f_x(0, y)$ ,  $f_y(x, 0)$ ,  $f_x(0, 0)$  and  $f_y(0, 0)$ , if they exist.

6. Show that for the function  $f(x, y) = \begin{cases} -xy, & |y| \geq |x| \\ xy, & |y| < |x| \end{cases}$ ,  $f_{xy}(0, 0)$  and  $f_{yx}(0, 0)$  both exist and are unequal.

7. Prove that  $|x| + |y|$  is continuous, but not differentiable at  $(0, 0)$ .

8. Prove that  $f(x, y) = \begin{cases} \frac{(x+y)\{\sqrt{(x^2+y^2)}+xy\}}{\sqrt{(x^2+y^2)}}, & \text{when } x^2 + y^2 \neq 0 \\ 0, & \text{when } x = y = 0, \end{cases}$  is differentiable at  $(0, 0)$ . Hence, deduce that  $f_x(0, 0) = f_y(0, 0) = 1$ .

9. Show that the function

$$f(x, y) = \begin{cases} x^3 \sin \frac{1}{x^2} + y^3 \sin \frac{1}{y^2}, & \text{when } xy \neq 0 \\ x^3 \sin \frac{1}{x^2}, & \text{when } x \neq 0 \text{ and } y = 0 \\ y^3 \sin \frac{1}{y^2}, & \text{when } x = 0 \text{ and } y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases} \text{ is differentiable at } (0, 0), \text{ whereas none of } f_x, f_y \text{ is continuous at } (0, 0).$$

10. Show that the function

$$f(x, y) = \begin{cases} (x^2 + y^2) \cos \left( \frac{1}{\sqrt{x^2 + y^2}} \right), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \text{ is differentiable at } (0, 0) \text{ and that } f_x, f_y \text{ are not continuous at } (0, 0).$$

**Answers:**

1. (a) -8    (b) does not exist    (c) does not exist    (d) 0    (e) 0    (f) does not exist    (g) does not exist    (h) 0    (i) 0
2. (a) does not exist    (b) 0, does not exist, 0
4. (a) continuous    (b) continuous    (c) continuous    (d) discontinuous    (e) continuous
5. 0, 1, 0, -1.