Marble Descent Through a Viscous Fluid

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1 Introduction

In this project, we will attempt to predict the terminal velocity of a marble falling through a viscous fluid. In order to do this, we will first go through the process of variable selection to determine what variables will have an effect on the terminal velocity. Then, we will use dimensional analysis with these variables to produce a complete set of dimensionless products. We will use these products in conjunction with Buckingham's Theorem, the Implicit Function Theorem, and the given experimental data to obtain a function ϕ that gives the value of the dimensionless product containing the dependent variable in terms of the other dimensionless products. Once we have determined ϕ, we will be able to solve for terminal velocity, which will give us a model for terminal velocity in terms of the other independent variables. We will then analyze the accuracy of this model by computing the SSR as well as the standard deviation of the model. Finally, we will create a Matlab file that takes in information about the marble and the viscous fluid and returns the terminal velocity of the marble in the fluid.

We will ultimately find that after all dimensional analysis and solving for terminal velocity, that we will obtain the following relation between velocity and the other dependent variables:

$$v_t = \frac{\mu \cdot r^2 \cdot (-0.5344 + 5.631 \cdot \frac{r^3 \cdot \rho}{m} + 0.0123 \cdot \frac{m^2 \cdot g}{r^3 \cdot \mu^2})}{m} \tag{1}$$

with an SSR of:

$$0.0018 \tag{2}$$

and a standard deviation of:

$$0.0162 \tag{3}$$

2 Dimensional Analysis

In order to select variables, we select all of the variables in the data given to use and also include gravity. The variables used in our model will therefore be:

- Mass of the marble, m, measured in kg
- Radius of the marble, r, measured in m
- Density of the fluid, ρ, measured in kg/m^3
- Viscosity of the fluid, μ, measured in $kg/m/s$
- Gravitational acceleration, g, measured in m/s^2
- Terminal velocity, v_t, measured in m/s

Now, in order to identify our dimensionless products, we first write:

$$\pi = \pi(m, r, \rho, \mu, g, v_t) = m^a r^b \rho^c \mu^d g^e v_t^f \tag{4}$$

Equating dimensions on two sides of the relation, we obtain:

$$1 = M^a L^b M^c L^{-3c} M^d L^{-d} T^{-d} L^e T^{-2e} L^f T^{-f} \tag{5}$$

which lead to the dimensions equations:

$$M : 0 = a + c + d \tag{6}$$
$$L : 0 = b - 3c - d + e + f \tag{7}$$
$$T : 0 = -d - 2e - f \tag{8}$$

Notice that we can express this system in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

Next, we reduce this matrix to row reduced echelon form using the rref function in matlab:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & -2 & -1 \\ 0 & 1 & -3 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Denote the leftmost matrix above matrix A and notice that $A \in \mathbb{R}^{3\times 6}$. By the rank-nullity theorem, we have that

$$rankA + nullityA = n \tag{11}$$

where n in this case is the number of columns, 6, and rank A is the number of linearly independent rows, 3. Therefore,

$$3 + nullityA = 6 \implies nullityA = 3 \tag{12}$$

which tells us that we are looking for 3 dimensionless products, say π_1, π_2, π_3. Notice that we can express any element $\vec{v}$ of the null space of A in the form:

$$\vec{v} = \begin{pmatrix} -c+2e+f \\ 3c-3e-2f \\ c \\ -2e-f \\ e \\ f \end{pmatrix} = c\begin{pmatrix} -1 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + e\begin{pmatrix} 2 \\ -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + f\begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{13}$$

which correspond to the following dimensionless products:

π_1	$m^{-1}r^3p$	$\frac{r^3p}{m}$
π_2	$m^2r^{-3}\mu^{-2}g$	$\frac{m^2g}{r^3\mu^2}$
π_3	$mr^{-2}\mu^{-1}v_t$	$\frac{v_tm}{r^2\mu}$

According to Buckingham's Theorem, there must exist a function f such that

$$f(\pi_1, \pi_2, \pi_3) = 0 \tag{14}$$

and according to the Implicit Function Theorem, we will be able to find a function ϕ s.t.

$$\pi_3 = \phi(\pi_1, \pi_2) \tag{15}$$

Since we know π_1, π_2, π_3 in terms of m, r, ρ, μ, g, v_t, we can plug these values into equation 18 to obtain:

$$\pi_3 = \phi(\pi_1, \pi_2) \tag{16}$$

$$\frac{v_tm}{r^2\mu} = \phi(\frac{r^3p}{m}, \frac{m^2g}{r^3\mu^2}) \tag{17}$$

$$v_t = \frac{\mu r^2\phi(\frac{r^3p}{m}, \frac{m^2g}{r^3\mu^2})}{m} \tag{18}$$

Notice that if we knew what the function ϕ was, then we would have a way to predict v_t based on the other variables.

3 Finding Linear Regression

To find the value of ϕ, we will perform a linear regression on the equation:

$$\phi_3 = a + b\phi_1 + c\phi_2 \tag{19}$$

in matlab. We begin by defining our experimental values:

```
% graviational acceleration
g = 9.81;
% densities with corn syrup first and shampoo second
p = [1427.1 1427.1 1427.1 1427.1 1427.1 1018.0 1018.0 1018.0 1018.0 1018.0];
% viscosities with shampoo first and corn syrup second
mu = [5 5 5 5 5 20 20 20 20 20];
% masses of marbles 1-5 twice
m = [0.003 0.005 0.013 0.022 0.054 0.003 0.005 0.013 0.022 0.054];
% radii of marbles 1-5 twice
r = [0.007 0.008 0.011 0.013 0.0175 0.007 0.008 0.011 0.013 0.0175];
%velocities with corn syrup first and shampoo second
v = [0.026 0.030 0.046 0.070 0.076 0.009 0.010 0.016 0.019 0.021];
```

Next, we compute the values of π_1, π_2, and π_3 using these experimental values above:

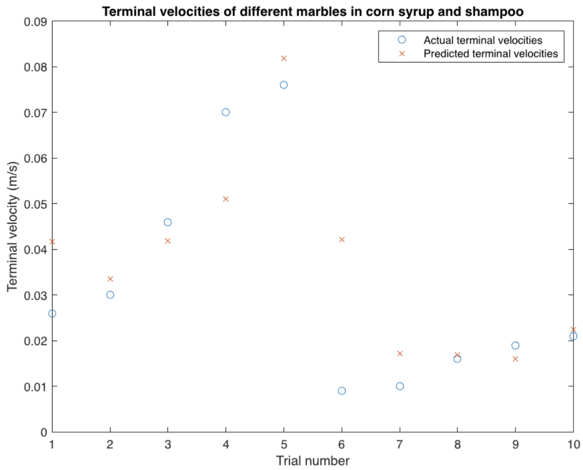
```
% pi1, pi2, pi3 values
pi1 = (r.^3 .* p) ./ m

pi1 = 1×10
    0.1632    0.1461    0.1461    0.1425    0.1416    0.1164    0.1042    0.1042    0.1017    0.1010

pi2 = (m.^2 .* g) ./ (r.^3 .* mu.^2)

pi2 = 1×10
   10.2962   19.1602   49.8239   86.4459  213.5022    0.6435    1.1975    3.1140    5.4029   13.3439

pi3 = (v .* m) ./ (r.^2 .* mu)

pi3 = 1×10
    0.3184    0.4688    0.9884    1.8225    2.6802    0.0276    0.0391    0.0860    0.1237    0.1851
```

Then, we compute our design matrix for equation 22 and find regression values for a, b, and c in equation 22:

```
% Fit
F=[ones(size(pi3))' pi1' pi2']

F = 10×3
    1.0000    0.1632   10.2962
    1.0000    0.1461   19.1602
    1.0000    0.1461   49.8239
    1.0000    0.1425   86.4459
    1.0000    0.1416  213.5022
    1.0000    0.1164    0.6435
    1.0000    0.1042    1.1975
    1.0000    0.1042    3.1140
    1.0000    0.1017    5.4029
    1.0000    0.1010   13.3439

phi_vals=F\pi3'

phi_vals = 3×1
   -0.5344
    5.6310
    0.0123
```

Next, for each of the ten trials, we compute the value of π_3 using the regression parameters obtained above and plug π_3 into equation 21 to compute terminal velocities for each trial. Finally, we plot each of the predicted terminal velocities versus the actual experimental terminal velocities:

```
x = 1:1:10;
pi3 = phi_vals(1) + phi_vals(2) .* pi1 + phi_vals(3) .* pi2;
v_t = (mu .* r.^2 .* pi3) ./ m;
% plotting exact values of v
plot(x, v, 'o')
hold on
% plotting predicted values of v
plot(x, v_t, 'x')
hold off
title("Terminal velocities of different marbles in corn syrup and shampoo");
ylabel("Terminal velocity (m/s)");
xlabel("Trial number");
legend('Actual terminal velocities', 'Predicted terminal velocities');
```



Lastly, in order to analyze the accuracy of our model, we compute its SSR (E) and standard deviation (s):

```
% SSR of fit
E = sum((v - v_t).^2)
```

```
E = 0.0018
```

```
s = sqrt(E ./ (10 - 3))
```

```
s = 0.0162
```

4 Summary

Using dimensional analysis, we have simplified the problem of predicting terminal velocity of a marble in a viscous liquid as a complex function of the mass of the marble, radius of the marble, density of the liquid, viscosity of the liquid, and gravitation acceleration into a linear regression problem where the independent variables are functions of those variables. Let's take another look at our model:

$$v_t = \frac{\mu r^2(-0.5433 + 5.631\frac{r^3 p}{m} + 0.0123\frac{m^2 g}{r^3\mu^2})}{m} \tag{20}$$

$$v_t = \frac{-0.5433\mu r^2 + 5.631\frac{\mu r^5 p}{m} + 0.0123\frac{m^2 g}{r\mu}}{m} \tag{21}$$

$$v_t = -0.5433\frac{\mu r^2}{m} + 5.631\frac{\mu r^5 p}{m^2} + 0.0123\frac{mg}{r\mu} \tag{22}$$

First, consider the mass of the marble, represented by the variable m. In the first term of equation 25, the bigger m gets, the higher the terminal velocity gets since the negative term gets smaller. The m in the third term also increases the terminal velocity. However, these are both dominated by the m^2 term in the denominator of the second term which makes the terminal velocity smaller. Therefore, according to our model, as mass of the marble increases, the terminal velocity decreases. This makes sense physically.

Next, lets consider the radius of the marble, represented by the variable r. In the first term of equation 25, the bigger r gets, the lower the terminal velocity. This is also the case for the third term. However, for the second term, a bigger radius leads to a much higher terminal velocity, which suggests that a bigger radius leads to a higher terminal velocity overall, which does not make sense physically. This suggest that the model may need some tweaking to the effect of the radius of the marble.

Next, lets consider the density of the liquid, represented by the variable ρ. This term only shows up in one place in our model, which is in the second term. The higher the density of the liquid, the higher the terminal velocity in our model, which also does not make sense physically.

Next, lets consider the viscosity of the liquid, represented by the variable μ. The effect of this variable is again determined by the second term. If the viscosity of the liquid goes up, so does the terminal velocity in our model, which does not make sense physically.

Finally, lets consider the effect of gravitational acceleration, represented by the variable g. The higher the value of g, the higher the terminal velocity in our model, which makes sense physically.

The graph of the predicted values and the actual values shows that the model certainly has some utility.However, the predicted values of trials 4 and 6 are well outside of 1 standard deviation, which leads us to believe that there may be a better model.

One way to create a better model would be to use different dimensional products. One easy way to create new dimensionless products would be to invert them noticing that this would still produce products that are dimensionless. First, we attempt to invert π_1 as follows:

$$\pi_1 = \frac{m}{r^3 p} \tag{23}$$

$$\pi_2 = \frac{m^2 g}{r^3\mu^2} \tag{24}$$

$$\pi_3 = \frac{v_t m}{r^2\mu} \tag{25}$$

After calculating our fit in Matlab, we see that our new SSR and standard deviation values are:

$$SSR = 0.0023 \tag{26}$$

$$s = 0.0181 \tag{27}$$

However, these values suggest that this model is worse than our original model. Next, we attempt to invert π_2 as follows:

$$\pi_1 = \frac{r^3 p}{m} \tag{28}$$

$$\pi_2 = \frac{r^3\mu^2}{m^2 g} \tag{29}$$

$$\pi_3 = \frac{v_t m}{r^2\mu} \tag{30}$$

After calculating our fit in Matlab, we see that our new SSR and standard deviation values are:

$$SSR = 0.0156 \tag{31}$$
$$s = 0.0472 \tag{32}$$

However, these values suggest that this model is worse than our original model. Next, we attempt to invert π_3 as follows:

$$\pi_1 = \frac{r^3 p}{m} \tag{33}$$
$$\pi_2 = \frac{m^2 g}{r^3 \mu^2} \tag{34}$$
$$\pi_3 = \frac{r^2 \mu}{v_t m} \tag{35}$$

After calculating our fit in Matlab, we see that our new SSR and standard deviation values are:

$$SSR = 0.0156 \tag{36}$$
$$s = 0.0472 \tag{37}$$

However, these values suggest that this model is worse than our original model.

Our next approach will be to find new dimensionless products. We will start by finding a dimensionless product involving v_t^2. One such product is:

$$\frac{v_t^2}{rg} \tag{38}$$

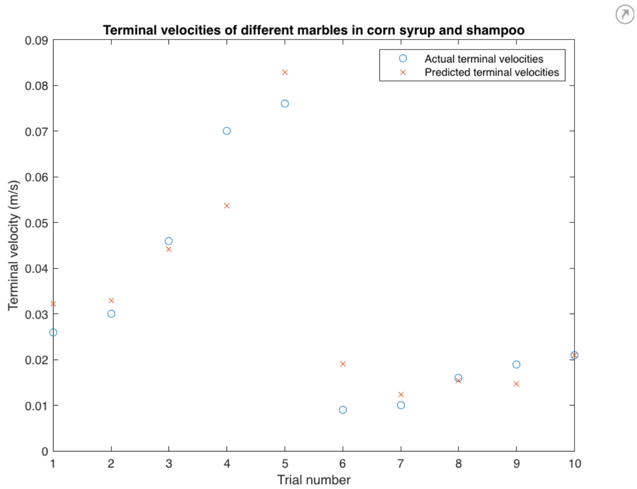
Since this dimensionless product involves v_t, it must be substituted only for π_3:

$$\pi_1 = \frac{r^3 p}{m} \tag{39}$$
$$\pi_2 = \frac{m^2 g}{r^3 \mu^2} \tag{40}$$
$$\pi_3 = \frac{v_t^2}{rg} \tag{41}$$

After calculating our fit in Matlab, we see that our new SSR and standard deviation values are:

$$SSR = 0.00049 \tag{42}$$
$$s = 0.0084 \tag{43}$$

which would suggest that this model is actually better than our original model, at least for this data set. Moreover, if we look at the graph of this models predictions versus the actual experimental values shown below:



It is clear, comparing this graph to the original model's graph, that this model is a better fit for the data.

As a continuation of this project, we may want to consider what factors lead to better dimensionless products, or find new dimensionless products by introspection in order to further improve our model.

5 Appendix

Project1.m

```
% calculating rref of matrix
A = [1 0 1 1 0 0; 0 1 -3 -1 1 1; 0 0 0 -1 -2 -1]
rref(A)

% graviational acceleration
g = 9.81;
% densities with corn syrup first and shampoo second
p = [1427.1 1427.1 1427.1 1427.1 1427.1 1018.0 1018.0 1018.0 1018.0 1018.0];
% viscosities with shampoo first and corn syrup second
mu = [5 5 5 5 5 20 20 20 20 20];
% masses of marbles 1-5 twice
m = [0.003 0.005 0.013 0.022 0.054 0.003 0.005 0.013 0.022 0.054];
% radii of marbles 1-5 twice
r = [0.007 0.008 0.011 0.013 0.0175 0.007 0.008 0.011 0.013 0.0175];
%velocities with corn syrup first and shampoo second
v = [0.026 0.030 0.046 0.070 0.076 0.009 0.010 0.016 0.019 0.021];

% pi1, pi2, pi3 values
pi1 = r.^3 .* p ./ (m)
pi2 = (m.^2 .* g) ./ (r.^3 .* mu.^2)
pi3 = (v .^ 2) ./ (r .* g)
% Fit
F=[ones(size(pi3))' pi1' pi2']
phi_vals=F\pi3'
x = 1:1:10;
pi3 = phi_vals(1) + phi_vals(2) .* pi1 + phi_vals(3) .* pi2;
v_t = sqrt(r .* g .* pi3);
% plotting exact values of v
plot(x, v, 'o')
hold on
% plotting predicted values of v
plot(x, v_t, 'x')
hold off
title("Terminal velocities of different marbles in corn syrup and shampoo");
ylabel("Terminal velocity (m/s)");
xlabel("Trial number");
legend('Actual terminal velocities', 'Predicted terminal velocities');
% SSR of fit
E = sum((v - v_t).^2)
s = sqrt(E ./ (10 - 3))
```

Project1Prediction.m

```matlab
% This file allows a user to create a prediction for the terminal velocity
% of a marble in a viscous liquid given
phi_vals;

% graviational acceleration
g = 9.81;
% density
prompt = "Please enter the density of the liquid in kg/m^3\n";
p = input(prompt);
% viscosity
prompt = "Please enter the viscosity of the liquid in kg/m/s\n";
mu = input(prompt);
% mass
prompt = "Please enter the mass of the marble in kg\n";
m = input(prompt);
% radius
prompt = "Please enter the radius of the marble in m\n";
r = input(prompt);
% We will now use our model to predict terminal velocity of this marble

% pi1, pi2, pi3 values
pi1 = (r.^3 .* p) ./ m;
pi2 = (m.^2 .* g) ./ (r.^3 .* mu.^2);
pi3 = v.^2 ./ (r .* g);

phi = phi_vals(1) + phi_vals(2) .* pi1 + phi_vals(3) .* pi2;
v_t_prediction = sqrt(r .* g .* phi)
```