MATH 442 Poker Project

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1 Introduction

Poker is a game for which the basics are easy to learn, but an effective strategy is hard to derive. Elements adding to the complexity of the game include position, betting strategy, betting limits, bankroll, etc. In this project, we attempt to use game theory, combinatorics, and probability theory to find optimal situational solutions to the game of Texas Hold'em.

We give an in-depth analysis of unsuited connectors, flop play, runner-runner possibilities, turn play, bluffing and check-raising. The analysis is aided by examples of Rose and Colin, two poker players who are faced with common limit Hold'em situations. Through detailed mathematical explanations, we help them make decisions on their hands. All decisions are purely based on game theory and we do not account for reading body language or opponents' hands.

2 Unsuited connectors

We first consider unsuited connector starting hands. Considering that there are $\binom{52}{2} = 1326$ possible starting hands, we must calculate how many of these starting hands are unsuited connectors with a top card in the range 5-J. First, there are 7 ways to choose ranks. Then, there are 4 ways to choose the suit of the top card. Then, given that we have already chosen 1 suit for the top card and the bottom card must be of a different suit, there are 3 ways to choose a suit for the bottom card. This means that there are $7 \cdot 4 \cdot 3 = 84$ different unsuited connector starting pairs with a top card in the range 5-J. This means that the odds of getting an unsuited connector starting hand with a top card in the range 5-J is $\frac{84}{1326} = 0.063348$.

Now, we compute probabilities of possible hands after the flop given that we have an unsuited pair of connectors in the range 5-J. First, we compute the total number of possible 5 card hands after the flop. After we have selected our two unsuited connectors, notice that we have 50 cards left in the deck and we need to choose any 3 of these. Therefore, the total number of possible 5 card hands is: $\binom{50}{3} = 19600$.

3 Unsuited connectors at the flop

1. Straight:
   For each pair of unsuited connectors, in order to get a straight, we need to either have the flop cards be the three cards below, two cards below and one above, one card below and two cards above, or three cards above. So there are 4 rank options. After we have chosen the ranks of the cards in the straight, notice that each of the three flop cards has 4 suit options. Therefore, the total number of straights under the given conditions is: $4 * 4 * 4 * 4 = 256$. The probability of getting one of these hands is therefore: $\frac{4*4*4*4}{19600} = \frac{4^4}{19600} = \frac{256}{19600}$.
2. 8-out straight draw with a pocket card pairing:
   Since the probabilities for each of the unsuited connector starting hands are the same, it is helpful to consider only one pair, say a 7-8 pair. Given that this is our starting hand, notice that there are 6 options that give us a pocket pair: 3 suits of rank 7 and 3 suits of rank 8. For the remaining two cards in the flop, the following ranks generate 8-out straight draws:

   $$5-6$$
   $$6-9$$
   $$9-10$$

   So there are 3 rank possibilities. For each of these ranks, there are 4 suit options for the lower card, and 4 suit options for the upper card. This means that there are $6 * 3 * 4 * 4 = 288$ total hands with an 8-out straight draw and a pocket card pairing. The probability of getting one of these hands is therefore: $\frac{288}{19600}$.

3. <u>8-out straight draw without a pocket card pairing:</u>
This probability is similar to the one above in that we still have the same 3 rank possibilities and 16 suit possibilities for the flop cards generating the 8-out straight draw. However, now we must consider possibilities for our final flop card: It cannot match the top or bottom card in our hand since this would be a pocket card pairing. It also cannot be either of the two ranks directly surrounding the generated 8-out straight draw since this would lead to a straight. This leaves us with 7 rank possibilities for which any suit can be chosen, and 2 rank possibilities for which only 3 suits can be chosen (the ranks selected to generate the 8-out straight draw). In other words, the total number of possibilities for this last flop card is: $7 * 4 + 3 * 2 = 28 + 6 = 34$. This means that there are $34 * 3 * 4 * 4 = 1632$ total hands with an 8-out straight draw and no pocket card pairing. The probability of getting one of these hands is therefore: $\frac{1632}{19600}$.

4. <u>4-out straight draw with a pocket card pairing:</u>
Since the probabilities for each of the unsuited connector starting hands are the same, it is helpful to consider only one pair, say a 7-8 pair. Given that this is our starting hand, notice that there are 6 options that give us a pocket pair: 3 suits of rank 7 and 3 suits of rank 8. For the remaining two cards in the flop, the following ranks generate 4-out straight draws:

$$4 - 5$$
$$4 - 6$$
$$5 - 9$$
$$6 - 10$$
$$9 - J$$
$$10 - J$$

So there are 6 rank possibilities. For each of these ranks, there are 4 suit options for the lower card, and 4 suit options for the upper card. This means that there are $6 * 6 * 4 * 4 = 576$ total hands with an 8-out straight draw and a pocket card pairing. The probability of getting one of these hands is therefore: $\frac{576}{19600}$.

5. <u>4-out straight draw without a pocket card pairing:</u>
This probability is similar to the one above in that we still have the same 6 rank possibilities and 16 suit possibilities for the two flop cards generating the 4-out straight draw. However, now we must consider possibilities for our final flop card: It cannot match the top or bottom card in our hand since this would be a pocket card pairing. It also cannot be of the singular rank that would create a straight. This leaves us with 8 rank possibilities for which any suit can be chosen, and 2 rank possibilities for which only 3 suits can be chosen (the ranks selected to generate the 4-out straight draw). In other words, the total number of possibilities for this last flop card is: $8 * 4 + 3 * 2 = 32 + 6 = 38$. This means that there are $38 * 6 * 4 * 4 = 3648$ total hands with an 8-out straight draw and no pocket card pairing. The probability of getting one of these hands is therefore: $\frac{3648}{19600}$.

6. <u>Trip or quad the top card (includes possible full house):</u>
Notice that there is clearly only one way to get a quad with the top card since all three of the flop cards must be of the same rank as the top card. To get a trip with the top card, there are $\binom{3}{2} = 3$ ways to do so since there are three possible suits with a rank equal to the top card and we need to choose 2 of them. For the final card of the flop, we cannot choose the rank of this card to be that of the top card, but, since full house is possible, this card could be the rank of the bottom card or any other rank. This leaves us with $52 - 4 - 1 = 47$ rank options for this final card. So the total number of ways we can get a trip or a quad with the top card is $1 + 3 * 47 = 142$. So the probability of getting such a hand is $\frac{142}{19600}$.

7. <u>Trip or quad the bottom card (includes possible full house):</u>
The number of possible hands for this condition is the same as the previous condition. So the total number of ways we can get a trip or a quad with the top card is $1 + 3 * 47 = 142$. So the probability of getting such a hand is $\frac{142}{19600}$.

8. <u>Pair both cards (no trips):</u>
There are 3 suits we can use to pair the top card and 3 suits we can use to pair the bottom card. Since the third card can be of any rank besides the two ranks in our unsuited pair and any suit, there are $52 - 8 = 44$ options for this last card. Therefore, the total number of hands for this condition is $3^2 * 44 = 396$, and the probability of getting such a hand is $\frac{396}{19600}$.

9. <u>Top pair with top card (no straight draw, board pair possible):</u>
First, there are clearly 3 ways to pair the top card. Notice that if this pair is a top pair, the other two

cards in the flop must be of a lower rank than this card. Also, since we want these conditions to be disjoint, notice that we cannot have another card of the same rank as the top card (since this would be a top card trip), and we also cannot have another card of the same rank as the bottom (since this would be two pairs). The last two cards in the flop must be below the two cards in our unsuited pair. Since an ace can either be high or low, if one of the last two flop cards is an ace, then we no longer have a top pair, since if another player has an ace in their hand, their pair beats ours. Therefore, consider ace to be ranked above king for this problem. With this in mind, for any top card rank, the number of ranks possible for the last two flop cards is top card rank $-$ 3. Each of these ranks has 4 suits. Since we need to choose 2 of these, the number of ways to do this is: $\binom{(\text{top}-3)*4}{2}$. However, some of these pairs of final two flop cards will overlap with the 4-outs with a pocket pair and 8-outs with a pocket pair computed earlier and must be removed to preserve disjointedness. For top cards of rank 6 - J, final two flop card pairings to be removed are of ranks:

$$\begin{gathered}(\text{top - 3}) - (\text{top - 4})\\(\text{top - 2}) - (\text{top - 3})\\(\text{top - 2}) - (\text{top - 4})\end{gathered}$$

and since (top - 4) is undefined if the top card is of rank 5 (ace is high), if the top card is of rank 5, we only remove

$$(\text{top - 2}) - (\text{top - 3})$$

Each of these pair rankings has $4^2 = 16$ suit combinations, so, for top cards of rank 6-J, the total number of final two flop pairs we need to remove is: $3 * 16 = 48$ and for a top card of rank 5, we only need to remove 16. This implies that the total number of 5 card hands with top paired top card and no straight draw is:

$$\begin{cases}3 * [\binom{(\text{top}-3)*4}{2} - 48] & \text{if top card rank is 6-J}\\ 3 * [\binom{(\text{top}-3)*4}{2} - 16] & \text{if top card rank is 5}\end{cases}$$

and the probability of getting such a hand is:

$$\begin{cases}\frac{3*[\binom{(\text{top}-3)*4}{2}-48]}{19600} & \text{if top card rank is 6-J}\\ \frac{3*[\binom{(\text{top}-3)*4}{2}-16]}{19600} & \text{if top card rank is 5}\end{cases}$$

10. Low pair with top card (no straight draw, board pair possible):

First, there are clearly 3 ways to pair the top card. Since we want these conditions to be disjoint, notice that we cannot have another card of the same rank as the top card (since this would be a top card trip), and we also cannot have another card of the same rank as the bottom card (since this would be two pairs). One of the last two flop cards must be of a rank above the top card, and the other can be or any rank either above the top card or below the bottom card. Again, consider ace to be ranked above king for this problem. Recall that the total number of ways to choose our final two flop cards is $\binom{44}{2}$. In the previous calculation, we concluded that the number of ways to place the final two flop cards below the bottom card was $\binom{(\text{top}-3)*4}{2}$. Therefore, the total number of ways to place either one card below the bottom card and one card above the top card or both cards above the top card is: $\binom{44}{2} - \binom{(\text{top}-3)*4}{2}$. However, notice that there are still some combinations included in this count that lead to either 4-outs or 8-outs. In order to remove these, we must first find the ranks which would lead to such a situation. For a top card of rank 6-J these are:

$$\begin{gathered}(\text{top} + 1) - (\text{top - 3})\\(\text{top} + 1) - (\text{top - 2})\\(\text{top} + 1) - (\text{top} + 2)\\(\text{top} + 1) - (\text{top} + 3)\\(\text{top} + 2) - (\text{top - 2})\\(\text{top} + 2) - (\text{top} + 3)\end{gathered}$$

These final flop pair rank combinations also lead to 4-outs or 8-outs if the top card is a 5, but notice that if the top card is indeed a 5, final flop pair rank combinations of A-2 and A-3 both lead to 4-outs as well.

Each of these pair rankings has $4^2 = 16$ suit combinations, so the total number of final two flop pairs we need to remove if the top card is 6-J is: $6 * 16 = 96$, and the total number of final two flop pairs we need to remove if the top card is a 5 is $8 * 16 = 128$. This implies that the total number of 5 card hands with low paired top card and no straight draw is:

$$\begin{cases} 3 * [\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 96] & \text{if top card rank is 6-J} \\ 3 * [\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 128] & \text{if top card rank is 5} \end{cases}$$

and the probability of getting such a hand is:

$$\begin{cases} \frac{3*[\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 96]}{19600} & \text{if top card rank is 6-J} \\ \frac{3*[\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 128]}{19600} & \text{if top card rank is 5} \end{cases}$$

11. Top pair with bottom card (no straight draw, board pair possible):
Notice that the number of hands with a top pair with bottom card is no different than the number of hands with a top pair with top card. Recall that this number was:

$$\begin{cases} 3 * [\binom{(\text{top}-3)*4}{2} - 48] & \text{if top card rank is 6-J} \\ 3 * [\binom{(\text{top}-3)*4}{2} - 16] & \text{if top card rank is 5} \end{cases}$$

and the probability of getting such a hand was:

$$\begin{cases} \frac{3*[\binom{(\text{top}-3)*4}{2} - 48]}{19600} & \text{if top card rank is 6-J} \\ \frac{3*[\binom{(\text{top}-3)*4}{2} - 16]}{19600} & \text{if top card rank is 5} \end{cases}$$

12. Low-pair with bottom card (no straight draw, board pair possible).:
Notice that the number of hands with a low pair with bottom card is no different than the number of hands with a low pair with top card. Recall that this number was:

$$\begin{cases} 3 * [\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 96] & \text{if top card rank is 6-J} \\ 3 * [\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 128] & \text{if top card rank is 5} \end{cases}$$

and the probability of getting such a hand was:

$$\begin{cases} \frac{3*[\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 96]}{19600} & \text{if top card rank is 6-J} \\ \frac{3*[\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 128]}{19600} & \text{if top card rank is 5} \end{cases}$$

4 Checking section 3 calculations

First, we compute the number of ways that the top card can be paired, no trip, no quad, and no pair on the bottom card (board pair possible). There are 3 possible suits for the top card. Then, since we want no pairs, trips, or quads, the remaining two flop cards cannot be of the same rank as either the top or bottom card in our hand, leaving 44 possible cards to choose from. So the total number of ways that the top card can be paired, no trip, no quad, and no pair on the bottom card (board pair possible) is:

$$3 * \binom{44}{2} = 2838 \tag{1}$$

For the following calculations, lets first exclude the possibility that the top card in our hand is a 5, since complications arise here. So for a top card in the range 6-J, we check that $N_9 + N_{10}$ remains constant:

$$\begin{aligned} N_9 + N_{10} &= 3 * [\binom{(\text{top}-3)*4}{2} - 48] + 3 * [\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 96] \\ &= \cancel{3 * \binom{(\text{top}-3)*4}{2}} - 144 + 3 * \binom{44}{2} \cancel{- 3 * \binom{(\text{top}-3)*4}{2}} - 288 \\ &= 3 * \binom{44}{2} - 432 \\ &= 3 * 946 - 432 \\ &= 2406 \end{aligned}$$

Now, assuming that the top card is of rank 5, we again get that:

$$\begin{aligned}
N_9 + N_{10} &= 3 * [\binom{(\text{top}-3)*4}{2} - 16] + 3 * [\binom{44}{2} - \binom{(\text{top}-3)*4}{2} - 128] \\
&= \cancel{3 * \binom{(\text{top}-3)*4}{2}} - 48 + 3 * \binom{44}{2} - \cancel{3 * \binom{(\text{top}-3)*4}{2}} - 384 \\
&= 3 * \binom{44}{2} - 432 \\
&= 3 * 946 - 432 \\
&= 2406
\end{aligned}$$

so we see that $N_9 + N_{10}$ will remain constant at 2406 possibilities regardless of the top card rank. Indeed, this implies that:

$$\frac{N_2 + N_4}{2} + N_9 + N_{10} = \frac{288 + 576}{2} + 2406 = 432 + 2406 = 2838 \tag{2}$$

for any top card of a rank in the range 5-J, so it is likely that we have done our calculations in section 3 correctly.

5 Starting-hand play with unsuited connectors

First, we need to decide what hands would be considered strong, middle-tier, and weak. We will consider the 12 hands in section 3 enumerated 1-12 which we will denote $H_1 - H_{12}$, respectively. Let hand types $H_1 - H_9$ & H_{11} be the weak set of hands we are willing to play if we don't believe other players have strong hands. Let hand types $H_1 - H_7$ be the middle-tier set of hands we will be willing to play if we believe some other players may have strong hands. Let hand types $H_1, H_2, H_4, H_6,$ & H_7 be the strong set of hands we are willing to play no matter what we believe to be the strength of the other players' hands. Then the following table gives the probabilities that we will have a hand in the previously mentioned categories given we start the hand with an unsuited pair of connectors.

Unsuited Connector Probabilities

Unsuited connector	$H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{11}$	$H_1, H_2, H_3, H_4, H_5, H_6, H_7$	H_1, H_2, H_4, H_6, H_7
10-J	0.498367	0.341020	0.071633
9-10	0.462245	0.341020	0.071633
8-9	0.0.431020	0.341020	0.071633
7-8	0.404694	0.341020	0.071633
6-7	0.383265	0.341020	0.071633
5-6	0.366734	0.341020	0.071633
4-5	0.364897	0.341020	0.071633

Starting hand 1:
Nothing in the betting so far suggests that anyone has a particularly strong hand. Therefore, we can work with the probabilities in the second column of the above table (column with $H_1 - H_9$ & H_{11}). This gives us that $P(R) = 0.405306$. Then, we compute use the expected value equation for a call:

$$E[W_c] = 18 \cdot 0.498367 - 4(1 - 0.498367) = 6.964074 \tag{3}$$

so Rose should certainly at least call. Then,

$$\frac{P(R^c)}{P(R)} = \frac{1 - 0.498367}{0.498367} = 1.006553 \tag{4}$$

so Rose requires at least two more people to call in order to raise. Since everyone seems content just calling, Rose has no reason to believe that she will get two callers, and should likely just call.

Starting hand 2:
Nothing in the betting so far suggests that anyone has a particularly strong hand. Therefore, we can work with the probabilities in the second column of the above table (column with $H_1 - H_9$ & H_{11}). This gives us that $P(R) = 0.405306$. Then, we compute use the expected value equation for a call:

$$E[W_c] = 26 \cdot 0.071633 - 8(1 - 0.071633) = -5.564478 \tag{5}$$

so Rose should certainly fold.

6 Flush draws

Suppose Rose holds any two suited cards in her pocket. The probability that she flops a flush is:

$$\frac{\binom{11}{3}}{19600} \tag{6}$$

and the probability that she flops a flush draw is:

$$\frac{\binom{11}{2} * 39}{19600} = \frac{2145}{19600} \tag{7}$$

7 Play at the flop

Note that for the following hands, probabilities are calculated without taking the river card and river betting round into account.

Hand 1:
Rose has a 4-out straight draw and a low pair possibility. Given that there is a raise, we can reasonably expect at least one player to have a good hand. If this is the case, then we need a strong hand in order to call the current bet of 8. Our decision equation is:

$$q > \frac{B}{P+B} = \frac{8}{36+8} = \frac{8}{44} = 0.181818.. \tag{8}$$

so Rose should call if she has reason to believe that she has at least an 18% chance of having the best hand. However, a straight is her only chance of having a strong hand since someone likely has a pair and possibly top pair. The chance she completes her straight draw on the turn is $\frac{4}{47} = 0.085106$. So Rose's decision equation suggests that she should fold.

Hand 2:
Through almost two full rounds of betting, no one has shown any strength, but no one has shown any weakness either. With the given flop cards, we can guarantee that we currently have one of the strongest hands with top pair. Our decision equation is:

$$q > \frac{B}{P+B} = \frac{4}{24+4} = \frac{4}{28} = 0.142857 \tag{9}$$

so Rose should call if she has reason to believe that she has at least a 14% chance of having a strong hand. There are many river cards that would keep her hand the strongest. These include: both remaining tens, three remaining queens, any 2, any 3, any 5, any 6, any 7, any 9. This is a total of 29 cards out of a possible 47. So Rose estimates her chance of winning to be $\frac{29}{47} = 0.617021$. So she should clearly call. Furthermore, $\frac{P(R^c)}{P(R)} = \frac{1-0.617021}{0.617021} = 0.620690$. So Rose only needs 1 player to call in order to raise. Since every player is still in the hand, there is a decent chance that this will happen, so we conclude that Rose should raise.

8 Raising at the flop to protect a hand

Suppose that the bet level is 4 and there is P currently in the pot. Suppose that Rose decides to call. Then Colin's decision equation is:

$$q > \frac{4}{P+8} \tag{10}$$

Colin's probability of having enough outs to call in this situation is:

$$q = 1 - \frac{47-x}{47} \cdot \frac{46-x}{46} \tag{11}$$

and plugging this value of q into equation (10), we get that:

$$1 - \frac{47-x}{47} \cdot \frac{46-x}{46} > \frac{4}{P+8} \implies -\frac{1}{2162}x^2 + \frac{93}{2162}x > \frac{4}{P+8} \tag{12}$$

Now, suppose that Rose decides to raise. Then Colin's decision equation is:

$$q > \frac{8}{P+16} \tag{13}$$

Colin's probability of having enough outs to call in this situation is:

$$q = 1 - \frac{47 - x}{47} \cdot \frac{46 - x}{46} \tag{14}$$

and plugging this value of q into equation (10), we get that:

$$1 - \frac{47 - x}{47} \cdot \frac{46 - x}{46} > \frac{8}{P + 16} \implies -\frac{1}{2162}x^2 + \frac{93}{2162}x > \frac{8}{P + 16} \tag{15}$$

Let the value of the pot $P = 44$. Using our inequality from equations (12), we get that the number of outs Colin would need in order to call if Rose calls is:

$$-\frac{1}{2162}x^2 + \frac{93}{2162}x > \frac{4}{44 + 8} \implies x = 1.82402 \quad \text{and} \quad x = 91.17597 \tag{16}$$

Since this quadratic equation is negative, it is clear that Colin needs any number of outs greater than 2 to call Rose's call. Using our inequality from equations (12),, we get that the number of outs Colin would need in order to call if Rose raises is:

$$-\frac{1}{2162}x^2 + \frac{93}{2162}x > \frac{8}{44 + 16} \implies x = 3.21047 \quad \text{and} \quad x = 89.78952 \tag{17}$$

Since this quadratic equation is negative, it is clear that Colin needs any number of outs greater than 4 to call Rose's raise. So, if Rose suspects that Colin has less than 2 outs, she can either call or raise and expect Colin to fold. If Rose suspects that Colin has either 2 or 3 outs, she should raise to get Colin to fold and protect her hand. If Rose suspects that Colin has greater than or equal to 4 outs, she should likely just call since Colin will call a raise and has a strong hand.

9 Runner-runner hands

7a There are six runner-runner card combinations that would give a straight by the river. These are (first card on the river and second card on the turn):

$$\begin{array}{c} 4 - 5 \\ 5 - 4 \\ 5 - 9 \\ 9 - 5 \\ 9 - 10 \\ 10 - 9 \end{array}$$

The probability of any one of these events occurring is:

$$P(\text{flush by river}) = 6 \cdot \frac{4}{47} \cdot \frac{4}{46} = 0.044403 \tag{18}$$

7b The probability that we get a club as the river card is $\frac{10}{47}$. and the probability that we also get a club as the turn card is $\frac{9}{47}$. So the probability that she hits a runner-runner flush draw by the river is:

$$\frac{10}{47} \cdot \frac{9}{46} = 0.041628 \tag{19}$$

The runner-runner possibility would increase Rose's chance of winning flop hand 1 from Item 5 to 12.673412%. However, she needed at least an 18.181818% chance of winning in order to call, so we conclude that the runner-runner possibility does not change her decision to fold the hand.

10 Play at the turn

Hand 1:
Since Rose has a gut-shot straight draw with a 5 high card, only a 3 on the river will give her a good hand. The probability of Rose hitting her straight on the river is $\frac{4}{46} = 0.086957$. Our decision equation is:

$$E[W_c] = (80 + 8)0.086957 - 8(1 - 0.086957) = 0.347826 \tag{20}$$

so Rose can certainly call. Now we can check if Rose should raise:

$$\frac{P(R^c)}{P(R)} = \frac{1 - 0.086957}{0.086957} = 10.499937 \tag{21}$$

Since Rose is going head to head, she will not get 11 callers and should not raise.

Hand 2:
So far in this hand, Rose has top pair with the highest kicker since she paired her Queen and holds an unpaired Ace. In order to defend her hand, Rose has to account for a King (someone could pair their King), or a card that would help complete a straight or flush. Hence, Rose should be worried about 4 Kings, 11 spades, 3 Jacks, 3 Sixes, 3 Eights and 3 Nines (3 of each since spades are accounted for in the 11 spades). Hence, this leaves Rose with 46 - 27 = 19 cards that are good for her. The probability of one of those 19 cards showing up on the river is $\frac{19}{46} = 0.413043$. This leaves us with a decision equation that looks like:

$$E[W_c] = 106 \cdot 0.413043 - 8(1 - 0.413043) = 39.086902 \tag{22}$$

so Rose should definitely call due to such a high expected value for calling. Should Rose raise?:

$$\frac{P(R^c)}{P(R)} = \frac{1 - 0.413043}{0.413043} = 1.421055 \tag{23}$$

There are 5 players remaining at the table and Rose can expect 2 out of the 5 to call her raise and therefore, she should definitely raise.

11 A mathematician's view of bluffing

Consider the following game with two players Rose and Colin: Rose and Colin each place a \$1.00 ante into a pot, and then each is dealt a single card from a limitless deck consisting only of queens, kings, and aces. After looking at his card, Colin must decide either to bet \$2.00 or to fold. If Colin bets, Rose must decide either to fold or call. If Rose calls there is a showdown: aces beat kings and queens, kings beat queens, and no money changes hands in the event of a tie. We find optimal strategies for Rose and Colin in the following way:

First, we consider the reasonable strategies for both Rose and Colin. In this game, the reasonable strategies for both players include:

A. - only bet if dealt an ace

B. - bet if dealt and ace or a king

C. - bet with any card (ace, king, or queen)

Next, let W_{XY} by a random variable denoting Rose's winnings under strategy X, Y where X = Rose's strategy and Y = Colin's strategy. Notice that there are 9 such random variables:

$$W_{AA}, W_{AB}, W_{AC}, W_{BA}, W_{BB}, W_{BC}, W_{CA}, W_{CB}, W_{CC}$$

We will construct a payoff matrix for this game by computing the expected values of each of these random variables. For the following calculations, the possible deals to Rose and Colin are (A, A), (A, K), (A, Q), (K, A), (K, K), (K, Q), (Q, A), (Q, K), and (Q, Q) where the first letter in the parentheses denotes the card Rose receives and the second letter in the parentheses denotes the card that Colin receives. Since each of these deals occurs with probability $\frac{1}{9}$, all of the following expected values are calculated in the form:

$$\begin{aligned} E[W_{XY}] = \frac{1}{9}[&\text{Rose's payoff given Rose plays strategy X and Colin plays strategy Y and the deal is (A, A)}+ \\ &\text{Rose's payoff given Rose plays strategy X and Colin plays strategy Y and the deal is (A, K)} + ...+ \\ &\text{Rose's payoff given Rose plays strategy X and Colin plays strategy Y and the deal is (Q, Q)}] \end{aligned}$$

The expected values are:

$$
\begin{aligned}
E[W_{AA}] &= \frac{1}{9}[0+1+1-1+1+1-1+1+1] = \frac{1}{9}[4] = \frac{4}{9}\\
E[W_{BA}] &= \frac{1}{9}[0+1+1-3+1+1-1+1+1] = \frac{1}{9}[2] = \frac{2}{9}\\
E[W_{CA}] &= \frac{1}{9}[0+1+1-3+1+1-3+1+1] = \frac{1}{9}[0] = 0\\
E[W_{AB}] &= \frac{1}{9}[0+3+1-1-1+1-1-1+1] = \frac{1}{9}[2] = \frac{2}{9}\\
E[W_{BB}] &= \frac{1}{9}[0+3+1-3+0+1-1-1+1] = \frac{1}{9}[1] = \frac{1}{9}\\
E[W_{CB}] &= \frac{1}{9}[0+3+1-3+0+1-3-3+1] = \frac{1}{9}[-3] = \frac{-3}{9}\\
E[W_{AC}] &= \frac{1}{9}[0+3+3-1-1-1-1-1-1] = \frac{1}{9}[0] = 0\\
E[W_{BC}] &= \frac{1}{9}[0+3+3-3+0+3-1-1-1] = \frac{1}{9}[3] = \frac{3}{9}\\
E[W_{CC}] &= \frac{1}{9}[0+3+3-3+0+3-3-3+0] = \frac{1}{9}[0] = 0
\end{aligned}
$$

which gives us the following payoff matrix:

		Colin A	Colin B	Colin C
Rose	A	$\frac{4}{9}$	$\frac{2}{9}$	0
	B	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{3}{9}$
	C	0	$\frac{-3}{9}$	0

However, notice that Rose A dominates Rose C so that Rose has no reason to ever play Rose C. Similarly, Colin B dominates Colin A so that Colin has no reason to ever play Colin A. If we eliminate these two strategies from the table above, we are left with the following 2x2 game:

		Colin B	Colin C
Rose	A	$\frac{2}{9}$	0
	B	$\frac{1}{9}$	$\frac{3}{9}$

We can then use the method of oddments to compute how often each player should play each of their strategies. Rose's oddments are:

For Rose A:

$$\frac{|\frac{1}{9}-\frac{3}{9}|}{|\frac{1}{9}-\frac{3}{9}|+|\frac{2}{9}-0|} = \frac{\frac{2}{9}}{\frac{4}{9}} = \frac{1}{2} \tag{24}$$

For Rose B:

$$\frac{|\frac{2}{9}-0|}{|\frac{1}{9}-\frac{3}{9}|+|\frac{2}{9}-0|} = \frac{\frac{2}{9}}{\frac{4}{9}} = \frac{1}{2} \tag{25}$$

and Colin's oddments are:

For Colin B:

$$\frac{|0-\frac{3}{9}|}{|\frac{2}{9}-\frac{1}{9}|+|0-\frac{3}{9}|} = \frac{\frac{3}{9}}{\frac{4}{9}} = \frac{3}{4} \tag{26}$$

For Colin C:

$$\frac{|\frac{2}{9}-\frac{1}{9}|}{|\frac{2}{9}-\frac{1}{9}|+|0-\frac{3}{9}|} = \frac{\frac{1}{9}}{\frac{4}{9}} = \frac{1}{4} \tag{27}$$

So Rose should play strategy Rose A 50% of the time and should play strategy Rose B 50% of the time as well. Colin, on the other hand, should play Colin B 75% of the time and should play Colin C 25% of the time. In order to follow these strategies, both Rose and Colin should pick their strategies randomly with the given probabilities prior to looking at their card, and should place their bet or fold according to the strategy chosen. Note that we can compute the value of this game as follows:

$$E[C_B] = \frac{1}{2} \cdot -\frac{2}{9} + \frac{1}{2} \cdot -\frac{1}{9} = -\frac{1}{9} - \frac{1}{18} = -\frac{2}{18} - \frac{1}{18} = -\frac{3}{18} = -\frac{1}{6} \tag{28}$$

and since this is Colin's expected value, the value of the game is the additive inverse of Colin's expected value: $\frac{1}{6}$. So the game is in Rose's favor and we expect her to win about 33¢ every time they play the game.

12 A gambler's view of bluffing

Notice that since it is assumed that Rose holds a King, Rose's only options are to fold or call. So we will denote Rose's strategies:

$$A - \text{call}$$
$$B - \text{fold}$$

and Colin's strategies remain the same (although we will see that we could remove his strategy A since this strategy is no better than betting with king and ace). Colin's strategies are:

$$A - \text{bet only with an ace}$$
$$B - \text{bet with king or ace}$$
$$C - \text{bet with queen, king or ace}$$

Next, let W_{XY} by a random variable denoting Rose's winnings under strategy X, Y where X = Rose's strategy and Y = Colin's strategy. Notice that there are 6 such random variables:

$$W_{AA}, W_{AB}, W_{AC}, W_{BA}, W_{BB}, W_{BC} \tag{29}$$

We will construct a payoff matrix for this game by computing the expected values of each of these random variables. For the following calculations, the possible deals to Roes and Colin are $(K, A), (K, K), (K, Q)$ where the first letter in the parentheses denotes the card Rose receives and the second letter in the parentheses denotes the card that Colin receives. Since each of these deals occurs with probability $\frac{1}{3}$, all of the following expected values are calculated in the form:

$$\begin{aligned} E[W_{XY}] = \frac{1}{3}[&\text{Rose's payoff given Rose plays strategy X and Colin plays strategy Y and the deal is (K, A)}+ \\ &\text{Rose's payoff given Rose plays strategy X and Colin plays strategy Y and the deal is (K, K)}+ \\ &\text{Rose's payoff given Rose plays strategy X and Colin plays strategy Y and the deal is (K, Q)}] \end{aligned}$$

The expected values are:

$$\begin{aligned} E[W_{AA}] &= \frac{1}{3}[-3 + 1 + 1] = -\frac{1}{3} \\ E[W_{AB}] &= \frac{1}{3}[-3 + 0 + 1] = -\frac{2}{3} \\ E[W_{AC}] &= \frac{1}{3}[-3 + 0 + 3] = 0 \\ E[W_{BA}] &= \frac{1}{3}[-1 + 1 + 1] = \frac{1}{3} \\ E[W_{BB}] &= \frac{1}{3}[-1 - 1 + 1] = -\frac{1}{3} \\ E[W_{BC}] &= \frac{1}{3}[-1 - 1 - 1] = -1 \end{aligned}$$

which gives us the following payoff matrix:

		Colin		
		A	B	C
Rose	A	$-\frac{1}{3}$	$-\frac{2}{3}$	0
	B	$\frac{1}{3}$	$-\frac{1}{3}$	-1

However, notice that Colin B dominates Colin A, so we can reduce our payoff matrix to the following 2x2 matrix:

		Colin B	Colin C
Rose	A	$-\frac{2}{3}$	0
	B	$-\frac{1}{3}$	-1

We can then use the method of oddments to compute how often each player should play each of their strategies. Roses oddments are:

For Rose A:

$$\frac{|-\frac{1}{3}-(-1)|}{|-\frac{1}{3}-(-1)|+|-\frac{2}{3}-0|} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2} \tag{30}$$

For Rose B:

$$\frac{|-\frac{2}{3}-0|}{|-\frac{1}{3}-(-1)|+|-\frac{2}{3}-0|} = \frac{\frac{2}{3}}{\frac{4}{3}} = \frac{1}{2} \tag{31}$$

and Colin's oddments are:

For Colin B:

$$\frac{|0-(-1)|}{|-\frac{2}{3}-(-\frac{1}{3})|+|0-(-1)|} = \frac{1}{\frac{4}{3}} = \frac{3}{4} \tag{32}$$

For Colin C:

$$\frac{|-\frac{2}{3}-(-\frac{1}{3})|}{|-\frac{2}{3}-(-\frac{1}{3})|+|0-(-1)|} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{1}{4} \tag{33}$$

So Colin and Rose's strategies have not changed. Rose should still play Rose A 50% of the time and Rose B 50% of the time. Colin should still play Colin B 75% of the time and Colin C 25% of the time. In essence, given no information on what card Rose was dealt, Colin should assume that Rose holds a king. The game theoretic value can be calculated using:

$$E[C_B] = \frac{1}{2}\cdot\frac{2}{3}+\frac{1}{2}\cdot\frac{1}{3} = \frac{1}{3}+\frac{1}{6} = \frac{2}{6}+\frac{1}{6} = \frac{1}{2} \tag{34}$$

So the game theoretic value of the game is $-\frac{1}{2}$.

13 Bluffing on the river

We are given that Rose's options are:

$$A - \text{call}$$
$$B - \text{fold}$$

Colin's options are:

$$A - \text{bet w/ strong hand only}$$
$$B - \text{bet w/ weak or strong hand}$$

Next, let W_{XY} by a random variable denoting Rose's winnings under strategy X, Y where X = Rose's strategy and Y = Colin's strategy. Notice that there are 4 such random variables:

$$W_{AA}, W_{AB}, W_{BA}, W_{BB} \tag{35}$$

We will construct a payoff matrix for this game by computing the expected values of each of these random variables. For the following calculations, the possible deals to Rose and Colin are (medium hand, strong hand), (medium-hand, weak hand). The expected values of each of these variables are:

$$E[W_{AA}] = q(-B-\frac{P}{2})+(1-q)(\frac{P}{2}) = -qB-q\frac{P}{2}+\frac{P}{2}-q\frac{P}{2} = -qB-qP+\frac{P}{2}$$
$$E[W_{AB}] = q(-B-\frac{P}{2})+(1-q)(B+\frac{P}{2}) = -qB-q\frac{P}{2}+B+\frac{P}{2}-qB-q\frac{P}{2} = -2qB-qP+B+\frac{P}{2}$$
$$E[W_{BA}] = q(-\frac{P}{2})+(1-q)(\frac{P}{2}) = -qP+\frac{P}{2}$$
$$E[W_{BB}] = q(-\frac{P}{2})+(1-q)(-\frac{P}{2}) = -\frac{P}{2}$$

which gives us the following payoff matrix:

		Colin A	Colin B
Rose	A	$-qB - qP + \frac{P}{2}$	$-2qB - qP + B + \frac{P}{2}$
	B	$-qP + \frac{P}{2}$	$-\frac{P}{2}$

We can then use the method of oddments to compute how often each player should play each of their strategies. Roses oddments are:

For Rose A:

$$\frac{|P(1-q)|}{|P(1-q)| + |B(1-q)|} = \frac{P(1-q)}{(P+B)(1-q)} = \frac{P}{P+B} \tag{36}$$

For Rose B:

$$\frac{|B(1-q)|}{|P(1-q)| + |B(1-q)|} = \frac{B(1-q)}{(P+B)(1-q)} = \frac{B}{P+B} \tag{37}$$

and Colin's oddments are:

For Colin A:

$$\frac{|(1-2q)B + (1-q)P|}{|(1-2q)B + (1-q)P| + |-qB|} = \frac{(1-2q)B + (1-q)P}{(1-q)B + (1-q)P} \tag{38}$$

For Colin B:

$$\frac{|-qB|}{|-2qB - qP + B + P| + |-qB|} = \frac{qB}{(1-q)B + (1-q)P} \tag{39}$$

River hand 1:

First, we compute the probability that Colin wins the hand. This probability is (assuming Rose does not have the ace of diamonds): $q = \frac{9}{44} = 0.204545$. Note that the bet level $B = 8$ and the pot level is $P = 80$. So we can compute the frequency with which Colin should bluff using the formula for Colin B above:

$$\frac{\frac{9}{44} \cdot 8}{(1 - \frac{9}{44}) \cdot 8 + (1 - \frac{9}{44}) \cdot 80} = \frac{9}{385} = 0.023377 \tag{40}$$

so Colin should bluff roughly 2.3% of the time.

14 More bluffing on the river

$$-q(\frac{P}{2} + B) + r(\frac{P}{2} + B) + (1 - r - q)\frac{P}{2} \tag{41}$$

Notice that Rose and Colin's options still remain the same. The only difference in this problem from the last is a change in the calculations for absolute values. These are now:

$$\begin{aligned}
E[W_{AA}] &= q(-B - \frac{P}{2}) + (1-q)(\frac{P}{2}) = -qB - q\frac{P}{2} + \frac{P}{2} - q\frac{P}{2} = -qB - qP + \frac{P}{2} \\
E[W_{AB}] &= q(-\frac{P}{2} - B) + r(\frac{P}{2} + B) + (1 - r - q)\frac{P}{2} = -qP + -qB + rB + \frac{P}{2} = P(\frac{1}{2} - q) + B(r - q) \\
E[W_{BA}] &= q(-\frac{P}{2}) + (1-q)(\frac{P}{2}) = -qP + \frac{P}{2} \\
E[W_{BB}] &= q(-\frac{P}{2}) + r(-\frac{P}{2}) + (1 - r - q)\frac{P}{2} = -qP - rP + \frac{P}{2}
\end{aligned}$$

which gives us the following payoff matrix:

		Colin A	Colin B
Rose	A	$-qB - qP + \frac{P}{2}$	$P(\frac{1}{2} - q) + B(r - q)$
	B	$-qP + \frac{P}{2}$	$-qP - rP + \frac{P}{2}$

We can then use the method of oddments to compute how often each player should play each of their strategies. Roses oddments are:

For Rose A:

$$\frac{rP}{rP + rB} \tag{42}$$

For Rose B:

$$\frac{rB}{rP + rB} \tag{43}$$

and Colin's oddments are:

For Colin A:

$$\frac{|B(r-q) + rP|}{|B(r-q) + rP| + |-qB|} = \frac{B(r-q) + rP}{B(r-q) + rP + qB} \tag{44}$$

For Colin B:

$$\frac{|-qB|}{|B(r-q) + rP| + |-qB|} = \frac{qB}{B(r-q) + rP + qB} \tag{45}$$

<u>River-hand 2</u>:
We need to compute the probability that Colin beats Rose straight out. To do this, he would need an ace as his river card. Since we assume Rose has a pair of queens, there are 44 cards left in the deck for the river card. This means that $q = \frac{2}{44}$. Next, we need to decide which cards Colin can bluff with. It would be reasonable for Colin to bluff trip kings, or a straight. Therefore, we could bluff with any 4, 5, 6, 9, 10, J, or any of the three remaining kings. We could also bluff a flush, but the only two cards not already counted and not a queen that we could use to do this are the 2 of hearts and the three of hearts. This is a total of 29 cards. Therefore, we compute $r = \frac{29}{44}$. Plugging our computed q, r, and the given B, P into the formula for Colin B, we get that the frequency with which Colin should bluff in this situation is:

$$\frac{\frac{2}{44} \cdot 8}{8(\frac{29}{44} - \frac{2}{44}) + \frac{29}{44} \cdot 50 + \frac{2}{44} \cdot 8} = 0.009512 \tag{46}$$

so Colin should make this bluff approximately 1% of the time.

15 Check-raising

Suppose that a bet has a value of B and that any raise is for another B. The possible events are:

$$\begin{aligned} A &= \text{bet and and opponent calls bet} \\ B &= \text{check and opponent bets but does not call raise} \\ C &= \text{check and opponent bets and calls raise} \end{aligned}$$

Let X be a random variable denoting the event that occurs. Let P be a random variable denoting our profit (pot money can be assumed to be already in our bank). Next, lets consider our profit for each of the aforementioned events:

$$P = \begin{cases} B & \text{if} \quad X = A \\ B & \text{if} \quad X = B \\ 2B & \text{if} \quad X = C \end{cases}$$

Finally, lets consider the expected values of a check-raise and a bet in terms of the probabilities of events A, B, and C occurring. These are:

$$\begin{aligned} E[winnings\ w/\ check-raise] &= B \cdot P(B) + 2B \cdot P(C) \\ E[winnings\ w/\ bet] &= B \cdot P(A) \end{aligned}$$

Check-raising is profitable if the expected value of a check-raise is higher than the expected value of a bet. This is the case when:

$$\begin{aligned} B \cdot P(B) + 2B \cdot P(C) &> B \cdot P(A) \\ P(B) + 2 \cdot P(C) &> P(A) \end{aligned}$$

so we have confirmed Sklansky's math.

16 Conclusion

In this thorough analysis, we looked at real-life poker situations and made decisions based on game theory, conditional expected value and combinatorics. We thoroughly analyzed each stage of limit Hold'em play and utilized probabilities of having a good/bad hand, pot size and bet size to calculate expected value of a call or raise. We noticed that each stage (pre-flop, flop, turn and river) of Hold'em is different due to three main factors: pot size, bet size and remaining cards. As you go through each stage, the pot and bet size likely increase while the remaining cards start to affect your hand more and more, changing the expected value of continuing play.

On a surface level, one can see the importance of ins and outs in poker. But digging deeper, we can notice that game theory can be used in multiple different scenarios like sports betting or even making daily life decisions. Overall, studying poker in a detailed manner gives a good practical understanding on probability theory.