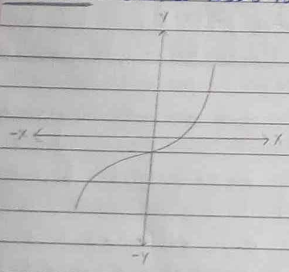
Date January 20, 2021

WEEK 7

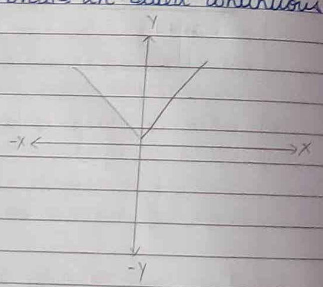
Graph of Polynomials

Characterisation of graphs of Polynomial Functions

- Polynomial functions of second degree or higher have graphs that do not have sharp corners. That is, the graphs are smooth curves.
- Polynomial fn also display graphs that have no breaks. Curves with no breaks are called continuous.



This graph is a graph of polynomial f^n.



This is not a graph of polynomial f^n.

Zeroes Of Poly. F^n

Zeros of Polynomial Functions

- If f is a polynomial function, the values of x for which $f(x) = 0$ are called zeroes of f.
- If the eqn of the polynomial fn can be factored, we can set each factor equal to zero and solve for the zeroes.
- Also any value $x = a$ that is a zero of a poly. fn yields a factor of the polynomial, of the form $(x-a)$.
- Given, the eqn of a polynomial fn, we can use this method to find x-intercepts because at the x-intercepts we find the input values whose output value is zero.

For general polynomials, this can be a challenging prospect. However, quadratic fn can be solved using the quadratic formula.

- The corresponding formulas for cubic & fourth degree polynomials are not simple enough to remember. And formulas do not exist for general higher degree polynomials.

Factoring

- The polynomial can be factored using known methods:

a) greatest common factor

b) factor by grouping

(c) trinomial factoring

· The polynomial is given in factored form.

· Technology is used to determine the intercepts.

x-intercept of Polynomial Function by factoring

1. set $f(x) = 0$

2. If the polynomial f^n is not given in factored form:
   a) Factor out any common monomial factors
   b) factor any factorable binomials or trinomials.

3. Set each factor equal to zero and solve to find the x-intercepts.

Example: Find x intercept of x^6 $f(x) = x^6 - 8x^4 + 16x^2$

Solution: Set $f(x) = 0$

$\therefore\ x^6 - 8x^4 + 16x^2 = 0$
$x^2(x^4 - 8x^2 + 16) = 0$
$x^2(x^2 - 4)^2 = 0$

Either $x = 0$ or $x^2 - 4 = 0$
$x^2 = 4$
$x = \pm 2$

$\therefore\ x = 0, 2, -2$ are the x intercepts of f.

Example: Find x intercepts of $f(x) = x^3 - 4x^2 - 3x + 12$

Solution: Set $f(x) = 0$

$\therefore\ x^3 - 4x^2 - 3x + 12 = 0$
$x^2(x-4) - 3(x-4) = 0$
$(x^2-3)(x-4) = 0$

Either $x^2 - 3 = 0$ or $x - 4 = 0$
$x^2 = 3$ or $x = 4$
$x = \pm\sqrt{3}$

Thus, $x = 4, \sqrt{3}, -\sqrt{3}$ are x intercepts of f.

Example: Find the y & x intercepts of $g(x) = (x-1)^2(x+3)$

Sol^n: set $g(x) = 0$

$\therefore\ (x-1)^2(x+3) = 0$

Either $(x-1)^2 = 0$ or $x + 3 = 0$
$x - 1 = 0$ $x = -3$
$x = 1$

$\therefore\ x = 1, -3$ are x-intercepts of f.

Now, put $x = 0$ in $g(x)$

$\Rightarrow g(0) = (0-1)^2(0+3) = 1(3) = 3$

Thus, y intercept is 3.

x-intercept of Polynomial Function using Graph.

Find x intercept of $f(x) = x^3 + 4x^2 + x - 6$

In this case, the polynomial is not in a factored form, has no common factors, and does not appear to be factorable using techniques previously discussed.

The only option is to generated the pair of values as done in quadratic case.

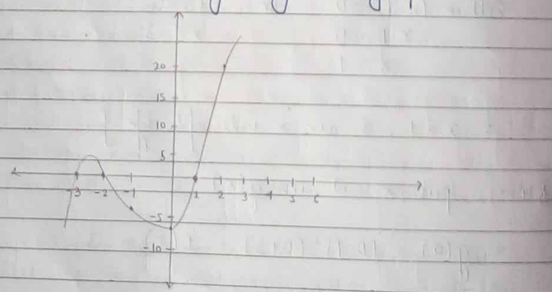
x	y
-2	0
-1	-4
0	-6
1	0
2	20

From table,
$x = -2, 1$ are the x intercepts of f.

The third zero can be found by dividing $f(x)$ by $(x+2)(x-1)$

The third zero of $f(x)$ is $x = -3$

$\therefore$ Join the points smoothly to get the graph



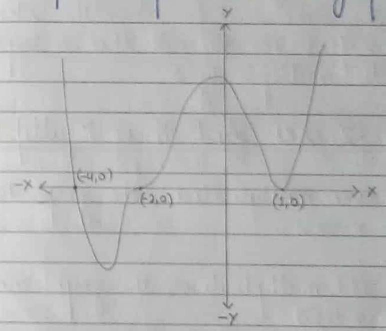
Identification of Zeros & Their Multiplicates

Graphs behave differently at various x-intercepts.

Sometimes, the graph will cross over the horizontal axis at an intercept.

Other times, the graph will touch the horizontal axis and 'bounce off'.

Suppose, for example, we ~~have~~ graph the $f(x) = (x-1)^2(x+2)^3(x+4)$



Identifying zeroes and their multiplicities

$\Rightarrow$ The x-intercept -4 is the solution of the eqn $(x+4) = 0$. The graph passes directly through the x-intercept at $x = -4$. The factor is linear (degree 1), so the behaviour near the intercept is like that of a line – it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of f^n.

:) The x-intercept 1 is the repeated solution of the equation $(x-1)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behaviour near the intercept is like that of a quadratic – it bounces of the horizontal axis at the intercept.

:) The x intercept −2 is the repeated solⁿ of the eqⁿ $(x+2)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behaviour near the intercept is like that of a cubic – with the same S shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

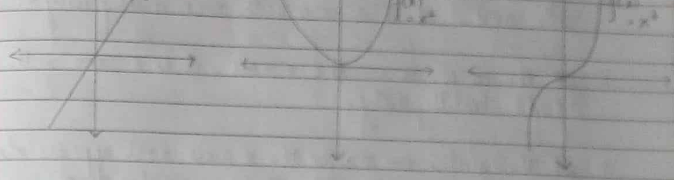
Identifying Zeroes And Their Multiplicities

→ For zeros with even multiplicities, the graph touch or are tangent to the x-axis.

→ For zeros with odd multiplicities, the graphs cross or intersect, the x-axis.

→ For higher even powers, such as 4, 6 & 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches & leaves the x-axis.

→ For higher odd powers, such as 5, 7, 9 ... the graph will still cross through the horizontal axis but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.



Graphical Behaviour of Polynomials at X-Intercepts

If a polynomial contains a factor of the form $(x-a)^m$, the behaviour near the x-intercept 'a' is determined by the exponent m. We say that,

$x = a$ is zero of multiplicity m

- The graph of a polynomial fⁿ will touch but not cross the x-axis at zeros with even multiplicities.

- The graph will cross the x-axis at zeros with odd multiplicities.

- The sum of the multiplicities is no greater than the degree of a polynomial fⁿ.

Given, the graph of a polynomial of degree n, how can one identify zeros & their multiplicities?

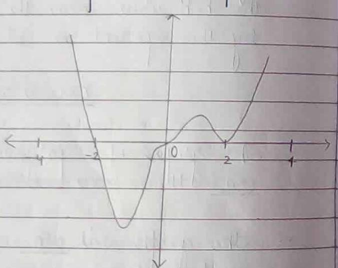
1. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.

2. If the graph crosses the x-axis, it is a zero with odd multiplicity.

3. If the graph ~~cro~~ crosses the x-axis and appears almost linear at the intercept, it is a single zero.

4. The sum of all the multiplicities is no greater than n.

Example : Use the graph of the function of degree 6 to identify the zeros of the fn & their possible multiplicities.

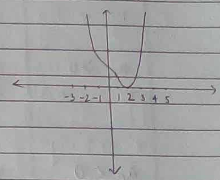


From graph,
$x = -2, 0, 2$

~~$(x-2)$ is linear, deg 1~~
$x = -2$, linear, 1
$x = 0$, odd deg, 3 or 5
$x = 2$, even deg, 2 or 4

Sum = 6
∴ $x = 0$ is with multiplicity 3
$x = 2$ is with multiplicity 2
$x = -2$ with multiplicity 1.

Example : Use the graph of the fn of degree 4 to identify the zeroes of the fn and their possible multiplicities.



$x = 2$

$x = 2$, even degree, 2 or 4

Hence the f^n must have a factor $(x-2)^2$

End Behaviour Of Poly.

As, we have already observed, the behaviour of a polynomial function,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

is either increasing or decreasing as the value of x increases which is mainly due to the fact that the leading terms dominate the behaviour of polynomial. This behaviour is known as End Behaviour of f^n.

As observed in quad. eqns, if the leading term of a polynomial function, $a_n x^n$, is an even power function and $a_n > 0$, then as x increases or decreases, $f(x)$ increases and is unbounded.
When the leading term is an odd power f^n, as x decreases, $f(x)$ also decreases and is unbounded;

as x increases, f(x) also increases and is unbounded.

	EVEN DEGREE	ODD DEGREE
$a_n > 0$	$x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to \infty$	$x \to \infty, f(x) \to \infty$ $x \to -\infty, f(x) \to -\infty$
$a_n < 0$	$x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to -\infty$	$x \to \infty, f(x) \to -\infty$ $x \to -\infty, f(x) \to \infty$

Relationship Between the Degree & Turning Points

As seen in quadratic case, a polynomial of degree 2 has one turning point.

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

We can hypothesize that a polynomial of degree n can have at most (n-1) turning points.

Find the maximum possible no. of turning points of each polynomial function.

1. $f(x) = 1 + x^2 + 4x^5$
   $n = 5$
   $\therefore$ Turning pts $= n-1 = 5-1 = 4$ (max. possible)

2. $f(x) = (x-1)^3(x+2)$
   $n = 4$
   $\therefore$ Turning pts $= n-1 = 4-1 = 3$ (max. possible)

Graphing a Poly. f^n

1. Find the x- and y- intercepts if possible.
2. Check for symmetry. If the f^n is an even f^n, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a f^n is an odd function, its graph is symmetrical about the origin, that is $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behaviour of the polynomial at the x-intercepts.
4. Determine the end behaviour by examining the leading ~~term~~ term.
5. Use the end behaviour and the behaviour at the intercepts to sketch a graph.

6. Ensure that the no. of turning points does not exceed one less than the degree of the polynomial.

7. Optionally, use graphing tools to check the graph.

Example: Sketch a graph of $f(x) = -(x+2)^2(x-5)$

x-intercepts are $x = -2, 5$

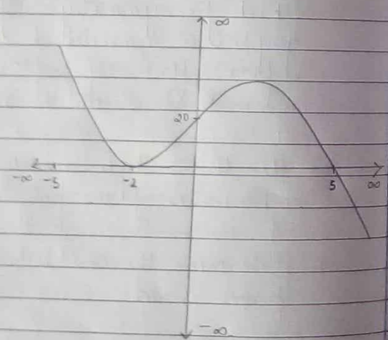
$x = -2$ has multiplicity 2, quadratic
$x = 5$ has multiplicity 1, linear

Y intercept $f(0) = 20$.

The leading term is $-x^3$. Therefore the odd degree polynomial with negative leading coefficient has the following end behaviour.

$x \to \infty$, $f(x) \to -\infty$
$x \to -\infty$, $f(x) \to \infty$

f can have at most $3-1 = 2$ turning pts.



INTERMEDIATE VALUE THEOREM

Let f be a polynomial function. The Intermediate Value Theorem states that if f(a) and f(b) have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

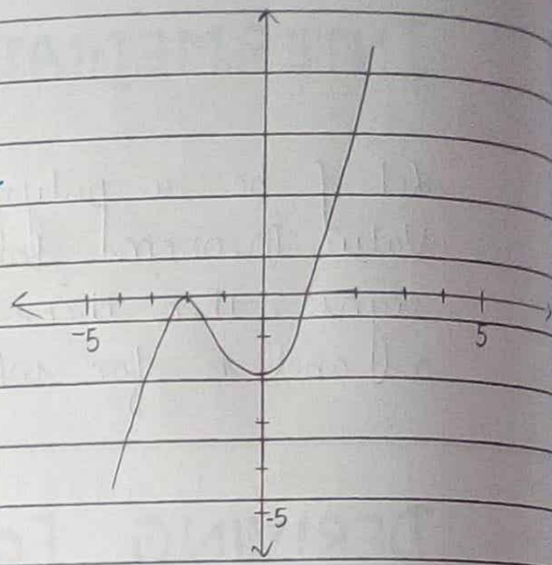
DERIVING FORMULA FOR POLYNOMIAL F

Given, the graph, how to find the formula for polynomial functions?

1. Find the x-intercepts of the graph to find the factor of the polynomial.
2. Understand the behaviour of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example: Write the formula for polynomial given in the graph.

$x = -2, 1$ are the x-intercepts and the fn has 2 turning points. The end behaviour is similar to odd degree polynomial with +ve leading term, i.e. it may be a polynomial of degree 3.



The behaviour at $x=1$ is linear and $x=-2$ is of even degree and hence quadratic.
The resultant polynomial is of degree 3 withe zeroes -2 and $+1$ with multiplicities 2 and 1 resp.

The polynomial has form, $f(x) = a(x+2)^2(x-1)$

To determine a, use y-intercept.
From graph, $f(0) = -2$
From the form, $f(0) = -4a$
$\Rightarrow a = \frac{1}{2}$

Hence, the fn must be, $f(x) = \frac{1}{2}(x+2)^2(x-1)$