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UNIVARIATE

WEEK - 2 CALCULUS

CONTINUITY

* A function $f(x)$ is continuous at a point x_0 if for any sequence of x_i converging to x_0, $f(x_i)$ converges to $f(x_0)$

A function $f(x)$ is continuous at $x = a$ if

$$\boxed{f(a) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)}$$

* A function is continuous in $\mathbb{R}$ if it is continuous at all points in $\mathbb{R}$

DIFFERENTIABILITY

→ means that the derivative of the function exists.

A function $f(x)$ is differentiable at $x = a$ if:

$$\boxed{f'(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}}$$

What is derivative? → Rate of change of function

NOTE: If a function is differentiable, it is continuous BUT if a function is continuous, it may or may not be differentiable.

LINEAR APPROXIMATION

The linear approximation $L(x)$ of a function $f(x)$ at a point a is given by:

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$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

The plot of linear approximation of a function f at a point v is TANGENT to f at the point $(v, f(v))$

This is the equation of a tangent line :

$$y - y_1 = m(x - x_1)$$
$$y = y_1 + m(x - x_1)$$

If $x_1 = a$, $y_1 = f(a)$ and $m = f'(a)$, we get,

$$y = f(a) + f'(a)(x-a)$$

HIGHER ORDER APPROXIMATIONS

QUADRATIC APPROXIMATION :

$$\boxed{L(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}$$

HIGHER ORDER APPROXIMATIONS :

$$\boxed{L(x) = f(a) + f^{(1)}(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3 \cdot 2}(x-a)^3 + \dots}$$

- Higher order approximation are BETTER and more ACCURATE than linear approximations.

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CRITICAL POINT

A critical point of a continuous function f is a point at which the derivative is 0 or undefined.
Critical points are the points on the graph where the function's rate of change is altered — either a change from increasing to decreasing or vice versa.

Critical points are useful for determining EXTREMA (minima or maxima) and solving optimization problems.

To find the value of critical point of a differentiable f^n, f, put $f'(x) = 0$.

SECOND DERIVATIVE TEST

- If $f''(x) = 0$, point is saddle
- If $f''(x) < 0$, point is maximum
- If $f''(x) > 0$, point is minimum.

DERIVATIVE

For a function $f(x)$ its derivative function $f'(x)$ or $\frac{d}{dx} f(x)$ is,

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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PRODUCT RULE : $(fg)'(a) = f'(a) \cdot g(a) + g'(a) \cdot f(a)$

CHAIN RULE : $\left(f(g)\right)'(x) = f'(g(x)) \cdot g'(x)$

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* Two important limits : (i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$

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Linear Algebra

- What is a vector space?

A vector space V over $\mathbb{R}$ is a set along with two functions:

→ Vector Addition : $V + V \longrightarrow V$
→ Scalar Multiplication : $\mathbb{R} \cdot V \longrightarrow V$

- What are the conditions of a set of vectors W to become a subspace of vector space V, i.e, $W \subseteq V$?

(1) $0 \in W$
(2) If $v_1, v_2 \in W$, then $v_1 + v_2 \in W$
(3) If $v \in W$ and $c \in \mathbb{R}$, then $cv \in W$

- ~~How~~ Representation of a line through a point $u \in \mathbb{R}^d$ along the vector $v \in \mathbb{R}^d$ is given –

$$\{x \in \mathbb{R}^d : x = u + \alpha v \quad \forall \alpha \in \mathbb{R}\}$$

- Representation of a line through two points u & $u' \in \mathbb{R}^d$:

$$\{x \in \mathbb{R}^d : x = u + \alpha(u' - u) \quad \forall \alpha \in \mathbb{R}\}$$

- Representation of a hyperplane normal to the vector $w \in \mathbb{R}^d$ with value $b \in \mathbb{R}$ is :

$$\{ x \in \mathbb{R}^d : w^T x = b \}$$

- When are two vectors x & y perpendicular to each other?

When their dot product is zero, i.e, $x \cdot y = 0$ or $x^T y = 0$

Multivariate Calculus

★ Multivariate functions are $f : \mathbb{R}^d \rightarrow \mathbb{R}$

Partial Derivatives

Suppose, we have a function $f(x,y)$ which depends on two variables x and y, where x and y are independent of each other. Then we say that the function f partially depends on x and y.

Now, if we calculate the derivative of f, then that derivative is known as the partial derivative of f.

• HOW TO COMPUTE A PARTIAL DERIVATIVE :

If we differentiate the function f wrt x, then take y as constant & if we differentiate f wrt y, then take x as constant.

Consider this function : $f(x,y) = x^2 y^3$

$$\frac{\partial f}{\partial x} = 2xy^3 \quad (\text{wrt } x) \rightarrow \text{taking } y \text{ constant}$$

$$\frac{\partial f}{\partial y} = 3y^2x^2 \quad (\text{wrt } y) \rightarrow \text{taking } x \text{ constant}$$

Gradient

The gradient of a multivariable function $f(x, y, \ldots)$, denoted ∇f, is the collection of all its partial derivatives into a vector.

$$\boxed{\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \ldots\ldots \end{bmatrix}^T}$$

→ If you imagine standing at a point $(x_0, y_0, \ldots)$ in the input space of f, the vector $\nabla f(x_0, y_0, \ldots)$ tells you which direction you should travel to increase the value of f most rapidly, which means

The gradient of a function points in the direction of STEEPEST ASCENT.

→ Gradient of a function at a point is PERPENDICULAR to the contour through the point.

Linear Approximation of Functions involving Multivariables

The linear approximation of a function f of two variables x and y in the neighbourhood of (a, b) is :

$$\boxed{L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)\,(x - a) + \frac{\partial f}{\partial y}(a, b)\,(y - b)}$$

OR

$$\boxed{f(y_1, y_2) \approx f(v_1, v_2) + \nabla f(v)^T (y - v)}$$

Directional Derivatives

NOTATION : $D_{\vec{u}}[f](v)$ → directional derivative of f at the point v, along u.

- $f_x(x,y) = \frac{\partial f}{\partial x}(x,y)$ → Rate of change of f as we vary x (keeping y fixed)
- $f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$ → Rate of change of f as we vary y (keeping x fixed)
- $D_{\vec{u}}[f](x,y)$ → Rate of change of f if we allow both x and y to change simultaneously (in some direction u)

$$D_{\vec{u}} f(x,y) = \nabla f \cdot u = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \cdot [u_1, u_2]$$

$$\Rightarrow D_{\vec{u}} f(x,y) = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y}$$

gradient

$$\boxed{D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}}$$

unit vector

Directional derivative can be considered to be a weighted sum of partial derivatives.

HOW TO COMPUTE : (EXAMPLE)

- What is the directional derivative of $f(x,y) = x^2 - xy$ at the point $(2,-3)$ along the vector $[3,4]$?

$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$

Let's calculate the gradient : $\nabla f = \begin{bmatrix} 2x - y \\ \cancel{x} - x \end{bmatrix}$ at $(2,-3)$

$\therefore$ ∇f at $(2,-3) = \begin{bmatrix} 2(2) - (-3) \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$

Now, the unit vector of $[3,4]$:

$$\vec{u} = \left[\frac{3}{\sqrt{3^2+4^2}}, \frac{4}{\sqrt{3^2+4^2}}\right] = \left[\frac{3}{5}, \frac{4}{5}\right]$$

Thus, $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = \begin{bmatrix} 7 \\ -2 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix}$

$= 21/5 - 8/5$

$= \frac{13}{5}$ Ans. or 2.6 Ans.

Direction of Steepest Ascent

The direction of steepest ascent is used to find a direction u, that maximises the rate of change of f as you move from v along u.

$$\boxed{u = \frac{\nabla f}{\|\nabla f\|}}$$

Direction of Steepest Descent

To find a direction u, that minimizes the rate of change of f.

$$\boxed{u = -\frac{\nabla f}{\|\nabla f\|}}$$

Cauchy - Schwarz Inequality

$$\boxed{-\|a\| \cdot \|b\| \leq a^T b \leq \|a\| \cdot \|b\|}$$

where $a, b \in \mathbb{R}^d$.

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Important Formulae from Week 2

- Linear Approximation for Univariable

$$L(x) = f(a) + f'(a)(x-a)$$

- Linear Approximation for Multivariables

$$L(x,y) = f(v) + \nabla f(v)^T (y-v)$$

- Gradient : collections of partial derivatives as vector

$$\nabla f = [\partial f/\partial x \quad \partial f/\partial y \quad \ldots\ldots]^T$$

- Directional Derivative

$$D = \nabla f \cdot \vec{u}$$

- Steepest Ascent

$$u = \frac{\nabla f}{\|\nabla f\|}$$

- Cauchy - Schwarz Inequality

$$-\|a\| \cdot \|b\| \le a^T b \le \|a\| \cdot \|b\|$$