Date
October 3, 2020

WEEK 1

Natural Numbers & Their Operation

\# Numbers keep a count of objects, i.e, 1, 2, 3, 4...

\# 0 to represent no objects at all.

\# Natural numbers : $\mathbb{N} = \{0, 1, 2, 3 \dots\}$

· sometimes $\mathbb{N}_0$ is used to emphasize 0 is included.

$\boxed{\text{Operations}}$ —

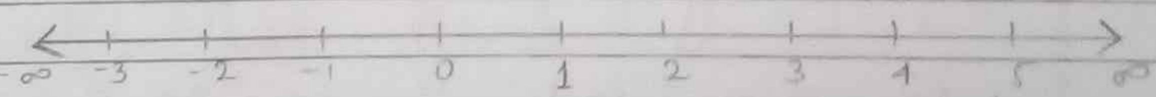
\# Addition, Subtraction, Multiplication, Division

· Which of these always produce a natural no. as the answer?

Subtraction :

1) → 5 - 6 is not a natural no. (Subtraction fails)
→ Extend the natural no. with -ve no.s
→ -1, -2, -3 ....
→ **INTEGERS** : $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3 \dots\}$

Number Line :-



⟶ increase

decrease ⟵

★ MULTIPLICATION & EXPONENTIATION

(2) → 7x4 – make 4 groups of 7

→ $m \times n = \underbrace{m + m + \dots + m}_{n \text{ times}}$

→ NOTATION : $m \times n$, $m \cdot n$, mn

→ Multiplication is repeated addition.

→ Sign rule for multiplying negative no.s

- → $-m \times n = -(m \cdot n)$
- → $-m \times -n = m \cdot n$

→ $m \times m = m^2$ – m squared

→ $m \times m \times m = m^3$ – m cubed

→ $m^k = \underbrace{m \times m \times m \times \dots m}_{k \text{ times}}$ – m to the power k

\# Multiplication is repeated addition.
Exponentiation is repeated multiplication.

★ DIVISION

(3) → You have 20 mangoes to distribute to 5 friends. How many do you give to each of them?

- Give them 1 each. You have 20 - 5 = 15 left
- Another round. You have 15 - 5 = 10 left
- Third round. You have 10 - 5 = 5 left
- Fourth round. You have 5 - 5 = 0 left
- 20 ÷ 5 = 4

→ Division is repeated subtraction.

→ What if you had only 19 mangoes to start with?

- After distributing 3 to each, you have 4 left.
- Can't distribute another round.
- The quotient of 19 ÷ 5 is 3
- The remainder of 19 ÷ 5 is 4
  - → 19 mod 5 = 4

★ FACTORS

(4) → a divides b if b mod a is 0

- $a|b$
- b is a multiple of a   b is divisible by a

→ $4|20$, $7|63$, $32|1024$, ... $a|b \rightarrow 4|16$
b mod a

→ $4 \nmid 19$, $9 \nmid 100$

→ a is a factor of b if $a|b$

→ Factors occur in pairs : factors of 12 are {1, 12}, {2, 6}, {3, 4}

→ ... unless the no. is a perfect square : factors of 36 are {1, 36}, {2, 18}, {3, 12}, {4, 9}, {6}.

★ PRIME NUMBERS

(5) → p is prime if it has only 2 factors $\{1, p\}$
· 1 is not a prime no. – only one factor.

→ Prime numbers are 2, 3, 5, 7, 11, 13 ......
· Sieve of Eratosthenes – remove multiples of p.

→ Every No. can be decomposed into prime factors.
→ $12 = 2 \times 2 \times 3 = 2^2 \times 3$
→ $126 = 2 \times 3 \times 3 \times 7 = 2 \times 3^2 \times 7$

→ This decomposition is unique – Prime Factorisation.

Summary

· **N** : natural numbers = $\{0, 1, 2, 3, \ldots\}$

· **Z** : integers = $\{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$

· Arithmetic Operations : $+, -, \times, \div, m^n$

· Quotient, remainder, a mod b

· Divisibility, $a|b$

· Factors

· Prime Numbers & Prime Factorisation

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RATIONAL NUMBERS

Division

→ Cannot represent $19 \div 5$ as integer

→ Fractions : $3\frac{4}{5}$

→ Rational Numbers : p/q, p and q are integers
· Numerator 'p', denominator 'q', $q \neq 0$
· Use $\mathbb{Q}$ to denote rational no.'s

→ The same no. can be written in many ways
· $3/5 = 6/10 = 30/50 \ldots$

→ Useful to add, subtract, compare rational no.'s

· $3/5 + 3/4 = 12/20 + 15/20 = 27/20$

· $3/5 < 3/4$ because $12/20 < 15/20$

→ Representation is not unique

· $3/5 = 6/10 = 30/50 \ldots$

→ Reduced form : $\frac{p}{q}$, where p, q have no common factors.

· Reduced form of $\frac{18}{60}$ is $\frac{3}{10}$

→ Greatest common divisor : gcd(18,60) = 6

- Recall prime factorisation
- 18 = 2·3·3 , 60 = 2·2·3·5
- Common prime factors are 2·3
- Can find gcd(m, n) more efficiently

Density

→ For each integer, we have a next integer and a previous integer

→ For m, next is m+1, previous is m−1

→ Next : No integer between m and m+1
Previous : No integer between m−1 and m.

→ Not possible for rationals
- Between any 2 rationals we can find another one.
- Suppose, $\frac{m}{n} < \frac{p}{q}$

Their average $\left(\frac{m}{n} + \frac{p}{q}\right)/2$ lies b/w them.

→ Rationals are dense, integers are discrete.

Summary

→ $\mathbb{Q}$: rational no.s
→ p/q, where p, q are integers
→ Representation is not unique
p/q = n·p/n·q
→ Reduced form, gcd(p,q) = 1
→ Rationals are dense - can't talk of next or prev.

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REAL AND COMPLEX NO.s

Beyond Rationals

→ Rational numbers are dense
- B/w any 2 rationals we can find another one

→ Is every point on the number line a rational no?

→ For an integer m, its square is $m^2 = m \cdot m$

→ Square root of m, $\sqrt{m}$, is r such that $r \cdot r = m$

→ Perfect squares : 1, 4, 16, 9, 25, 36, 49, ... 256

→ Square roots : 1, 2, 4, 3, 5, 6, 7 16

→ What about integers that are not perfect squares?

→ $\sqrt{2}$ cannot be written as $\frac{p}{q}$

→ Yet, we can draw a line of length $\sqrt{2}$
- Diagonal of a square whose sides have length 1.

→ $\sqrt{2}$ is irrational

→ **Real Numbers** : $\mathbb{R}$ - all rational & irrational no.s

→ Like rationals, real no.s are dense
- If $r < r'$, then $\left(\frac{r+r'}{2}\right)$ lies b/w r & r'.

Beyond Reals

→ Some well known irrationals
- $\pi = 3.1415927\ldots$
- $e = 2.7182818\ldots$

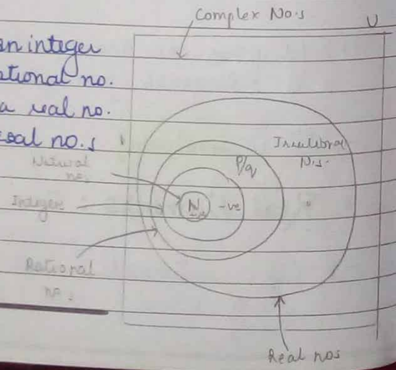
→ Can we stop with real no.s ?
- What about $\sqrt{-1}$
- For any real number x, x^2 must be +ve — (law of signs for multiplication)

→ $\sqrt{-1}$ is a complex number

Summary

→ Real no.s extend rational no.s
→ Typical irrational no.s – sq. root of integers that are not perfect sq.
→ Real numbers are dense like rationals.

- Every natural no. is an integer
- Every integer is a rational no.
- Every rational no. is a real no.
- Complex no. extend real no.s



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SET THEORY

Sets

→ A set is a collection of items
- Days of the week : {Sun, Mon, Tues, Wed, Thurs, Fri, Sat}
- Factors of 24 : {1, 2, 3, 4, 6, 8, 12, 24}
- Prime no.s below 15 : {2, 3, 5, 7, 11, 13}

→ Sets may be infinite
- Different types of numbers : $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

→ No requirement that members of a set have uniform type.
- Sets of objects in a painting
- Sets of objects in a room

Order, Duplicates, Cardinality

→ Sets are unordered
- {Kohli, Dhoni, Pujara}
- {Pujara, Kohli, Dhoni}

→ Duplicates don't matter
- {Kohli, Dhoni, Pujara, Dhoni}

→ Cardinality : number of items in a set
- For finite sets, count the items
- {1, 2, 3, 4, 6, 8, 12, 24} has cardinality 8.

→ What about infinite sets?
- Is $\mathbb{Q}$ bigger than $\mathbb{Z}$?
- Is $\mathbb{R}$ bigger than $\mathbb{Q}$?

Describing Sets, Membership

→ Finite sets can be listed out explicitly
- {Kohli, Dhoni, Pujara}
- {1, 2, 3, 4, 6, 8, 12, 24}

→ Infinite sets cannot be listed out
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ is not formal notation

→ Not every collection of items is a set
- Collection of all sets is not a set
- Russell's Paradox

→ Items in a set are called elements.
- Membership : $x \in X$, x is an element of X
- $5 \in \mathbb{Z}$, $\sqrt{2} \notin \mathbb{Q}$

Subsets

→ X is a subset of Y
if Every element of X is also an element of Y

→ NOTATION : $X \subseteq Y$

→ Examples
- {Kohli, Pujara} $\subseteq$ {Kohli, Pujara, Dhoni}
- Primes $\subseteq \mathbb{N}$, $\mathbb{N} \subseteq \mathbb{Z}$, $\mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \subseteq \mathbb{R}$

→ Every set is a subset of itself : $X \subseteq X$
- $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$

→ Proper Subset : $X \subseteq Y$ but $Y \neq X$
- Notation : $X \subset Y$ or $X \subsetneq Y$
- $\mathbb{N} \subsetneq \mathbb{Z}$, $\mathbb{Z} \subsetneq \mathbb{Q}$, $\mathbb{Q} \subsetneq \mathbb{R}$

The Empty Set & The Power set

→ The empty set has no elements - ϕ

→ $\phi \subseteq X$ for every set X
- Every element of ϕ is also in X

→ A set can contain other sets

→ POWERSET - set of subsets of a set
- $X = \{a, b\}$
- Powerset is $\{\phi, \{a\}, \{b\}, \{a, b\}\}$

→ Set with n elements has 2^n subsets
- $X = \{x_1, x_2, x_3, \ldots x_n\}$
- In a subset, either include or exclude each x_i
- 2 choices per element, $\underbrace{2 \cdot 2 \cdot 2 \ldots 2}_{n \text{ times}} = 2^n$ subsets

Subsets & Binary Numbers

→ $X = \{x_1, x_2, \ldots\ldots, x_n\}$

→ n bit binary numbers
  ⇾ 3 bits : 000, 001, 010, 011, 100, 110, 111

→ Digit i represents whether x_i is included in a subset
  ⇾ $X = \{a, b, c, d\}$
  ⇾ 0101 is $\{b, d\}$
  ⇾ 0000 is ϕ and 1111 is X

→ 2^n n bit binary numbers.

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Construction Of Subsets And Set Operations

Constructing Subsets

★ Set Comprehension (Set Builder Form)

→ The subset of even integers
$\{x \mid x \in \mathbb{Z}, x \bmod 2 = 0\}$

  ⇾ Begin with an existing set, $\mathbb{Z}$
  ⇾ Apply a condition to each element in that set
   . $x \in \mathbb{Z}$ such that $x \bmod 2 = 0$
  ⇾ Collect all the elements that match the condition

→ EXAMPLES
  ⇾ The set of perfect squares
  $\{m \mid m \in \mathbb{N}, \sqrt{m} \in \mathbb{N}\}$

  ⇾ The set of rationals in reduced form
  $\{p/q \mid p, q \in \mathbb{Z}, \gcd(p, q) = 1\}$

★ Intervals

→ Integers from −6 to +6 : $\{z \mid z \in \mathbb{Z}, -6 \leq z \leq 6\}$

→ Real numbers between 0 and 1

→ Closed Interval $[0,1]$ − include endpoints

→ $\{x \mid x \in \mathbb{R},\ 0 \le x \le 1\}$

→ Open Interval $(0,1)$ – exclude endpoints

→ $\{x \mid x \in \mathbb{R},\ 0 < x < 1\}$

→ Left Open $(0,1]$

→ $\{x \mid x \in \mathbb{R},\ 0 < x \le 1\}$

→ Right Open $[0,1)$

→ $\{x \mid x \in \mathbb{R},\ 0 \le x < 1\}$

Union, Intersection, Complement

→ Union – combine X and Y

$X \cup Y$

$\{a,b,c\} \cup \{c,d,e\} = \{a,b,c,d,e\}$

→ Intersection – elements common to X and Y

$X \cap Y$

$\{a,b,c,d\} \cap \{d,a,e,f\} = \{a,d\}$

→ Set Difference – elements in X that are not in Y,

$X - Y$ or $X \setminus Y$

$\{a,b,c,d\} - \{a,d,e,f\} = \{b,c\}$

→ Complement – elements not in X, $\bar{X}$ or X^c

→ Define complement relative to larger set, universe

→ Compliment of prime no.s in $\mathbb{N}$ are composite nos

Summary

→ Sets are a standard way to represent collections of mathematical objects.

→ Sets may be finite or infinite

→ Can carve out interesting subsets of sets

→ Set operations : union, intersection, difference, compliment.

October 6, 2020

SETS : Examples

Set Comprehension

→ square of the even integers $\{0, 4, 16, 36, 64, \dots\}$
$\{x^2 \mid x \in \mathbb{Z},\ x \bmod 2 = 0\}$

- Generate Elements drawn from existing set
  -2, -1, 0, 1, 2, 3 ....
- Filter select elements that satisfy a constraint
  -2 0 2 4 6 ....
- Transform Modify selected elements
  4 0 4 16 36 ....

→ More Filters

- Rationals in reduced form
  $\{\frac{p}{q} \mid p/q \in \mathbb{Q},\ \gcd(p,q) = 1\}$
- Reals in interval $[-1, 2)$
  $\{x \mid x \in \mathbb{R},\ -1 \le x < 2\}$

→ cubes of first 5 natural numbers

$Y = \{n^3 \mid n \in \{0, 1, 2, 3, 4\}\}$

→ Cube of first 500 natural numbers ?

$Y = \{n^3 \mid n \in \{0, 1, 2, 3 \dots 498, 499\}\}$

→ Use set comprehension to define first 500 nat. nos

$X = \{n \mid n \in \mathbb{N},\ n < 500\}$

→ Now a more readable version.

$X = \{n \mid n \in \mathbb{N},\ n < 500\}$
$Y = \{n^3 \mid n \in X\}$

Perfect Squares

→ Integers whose square root is also an integer
$\{z \mid z \in \mathbb{Z},\ \sqrt{z} \in \mathbb{Z}\}$

→ All squares are positive, so this is the same as
$\{n \mid n \in \mathbb{N},\ \sqrt{n} \in \mathbb{N}\}$

→ Alternatively, generate all the perfect squares
$\{n^2 \mid n \in \mathbb{N}\}$

→ Extend the definition to rationals

- $9/16 = (3/4)^2$ is a square. $1/2 \neq (p/q)^2$ for any p, q
- $\{q \mid q \in \mathbb{Q},\ \sqrt{q} \in \mathbb{Q}\}$ or $\{q^2 \mid q \in \mathbb{Q}\}$

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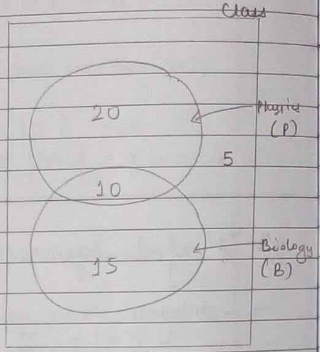
COUNTING PROBLEMS

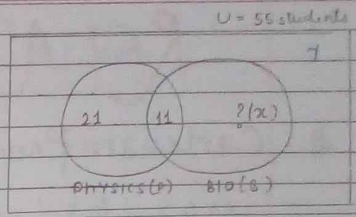
\# In a class, 30 students took Physics, 25 took Biology and 10 took both, 5 took neither. How many students are there in class?



- Draw sets for Physics (P) & Biology (B)
- 10 students are in $P \cap B$
- This leaves 20 students in $P \setminus B$. (Took Physics but didn't take Biology)
- Likewise 15 students in $B \setminus P$ (Took Biology but not Physics)
- 5 students in $\overline{P \cup B}$ (In the class but neither took Physics nor Biology.

→ Class Strength : $20 + 10 + 15 + 5 = 50$

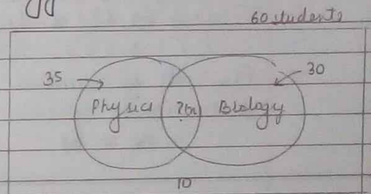
\# In a class of 55 students, 32 students took Physics, 11 students took both Physics and Biology, and 7 took neither.
How many students took Biology but not Physics?



Solution :
$55 = x + 11 + 21 + 7$
$55 = x + 39$
$x = 55 - 39$
$x = 16$

∴ 16 students took Biology but not Physics.

\# In a class of 60 students, 35 students took Physics, 30 took Biology, and 10 took neither. How many took both Physics and Biology?



- $P \cup B = n(P) + n(B) - n(P \cap B)$
- $P \cup B = 60 - 10 = 50$
- $n(P) = 35$
  $n(B) = 30$
  ∴ $n(P) + n(B) = 35 + 30 = 65$

Thus, $P \cap B = 65 - 50 = 15$

Summary :
- Set notation is useful way to concisely describe collection of objects
- Set comprehension combines generators, filters & transformations to produce new sets from old
- Venn diagrams can be useful to workout problems involving sets.

October 7, 2020

RELATIONS

Cartesian Product

→ $A \times B = \{ (a,b) \mid a \in A, b \in B \}$

- Pair up elements from A and B
- $A = \{0,1\}$, $B = \{2,3\}$
- $A \times B = \{ (0,2), (0,3), (1,2), (1,3) \}$

→ In a pair, the order is important
- $(0,1) \neq (1,0)$

→ For sets of numbers, visualize product as two dimensional space
- $\mathbb{N} \times \mathbb{N}$

Binary Relations

→ Select some pairs from the Cartesian Product

→ Combine cartesian product with set comprehension

→ $\{ (m,n) \mid (m,n) \in \mathbb{N} \times \mathbb{N}, n = m+1 \}$
- $\{ (0,1), (1,2), (2,3) \ldots (17,18) \ldots \}$

→ Pairs (d,n) where d is a factor of n
- $\{ (d,n) \mid (d,n) \in \mathbb{N} \times \mathbb{N}, d \mid n \}$
- $\{ (1,1), \ldots (2,82), \ldots (14,56), \ldots \}$

→ BINARY RELATION $R \subseteq A \times B$

→ Notation : $(a,b) \in R$, aRb

More Relations

→ Points at a distance 5 from $(0,0)$
- Distance from $(0,0)$ to (a,b) is $\sqrt{a^2+b^2}$
- $\{ (a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R}, \sqrt{a^2+b^2} = 5 \}$
- $\{ (0,5), (5,0), (3,4), (-3,-4), \ldots \}$
- A circle with centre at $(0,0)$

→ Rationals in reduced form
- A subset of $\mathbb{Q}$
  - $\{ \frac{p}{q} \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}, \gcd(p,q) = 1 \}$
- .. but also a relation on $\mathbb{Z} \times \mathbb{Z}$
  - $\{ (p,q) \mid (p,q) \in \mathbb{Z} \times \mathbb{Z}, \gcd(p,q) = 1 \}$

Types of Binary Relations

→ Identity Relation $I \subseteq A \times A$
- $I = \{ (a,b) \mid (a,b) \in A \times A, a = b \}$

- $I = \{(a,a) \mid (a,a) \in A \times A\}$
- $I = \{(a,a) \mid a \in A\}$

→ Reflexive Relations

- $R \subseteq A \times A$, $I \subseteq R$
- $\{(a,b) \mid (a,b) \in N \times N,\ a,b > 0,\ a|b\}$
  - $a|a$ for all $a > 0$

→ Symmetric Relations

- $(a,b) \in R$ if and only if $(b,a) \in R$
- $\{(a,b) \mid (a,b) \in N \times N,\ gcd(a,b) = 1\}$
- $\{(a,b) \mid (a,b) \in N \times N,\ |a-b| = 2\}$

→ Transitive Relations

- If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
- $\{(a,b) \mid (a,b) \in N \times N,\ a|b\}$
  - If $a|b$ and $b|c$ then $a|c$
- $\{(a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R},\ a < b\}$
  - If $a < b$ & $b < c$ then $a < c$

→ Antisymmetric Relations

- If $(a,b) \in R$ and $a \neq b$, then $(b,a) \notin R$
- $\{(a,b) \mid (a,b) \in \mathbb{R} \times \mathbb{R},\ a < b\}$
  - If $a < b$ then $b \nless a$
- $M \subseteq P \times P$ relates mothers to children
  - If $(p,c) \in M$ then $(c,p) \notin M$

Equivalence Relations

→ Reflexive, symmetric & transitive

→ Same remainder modulo 5

- 7 mod 5 = 2 , 22 mod 5 = 2
- If a mod 5 = b mod 5 then $(b-a)$ is a multiple of 5
- $\mathbb{Z}$ mod 5 = $\{(a,b) \mid a,b \in \mathbb{Z},\ (b-a) \bmod 5 = 0\}$
- Divides integers into 5 groups based on remainder when divided by 5

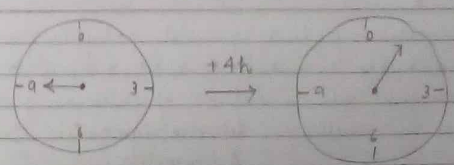
→ An equivalence relation partitions a set

→ Group of equivalent elements are called Equivalence Classes.

★ Measuring Time

clock displays hours modulo 12

2:00 am is equivalent to 2:00 pm



Beyond Binary Relations

→ Cartesian products of more than two sets

→ Pythagorean triplets

- Square on the hypotenuse is the sum of the squares on the opp. sides.
- $\{(a,b,c) \mid (a,b,c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N},\ a,b,c > 0,\ a^2+b^2=c^2\}$

→ Corners of squares

- A corner is a point $(x,y) \in \mathbb{R} \times \mathbb{R}$
- $((x_1,y_1),(x_2,y_2),(x_3,y_3),(x_4,y_4))$ are related if they are four corners of a square.
- For instance :
  - $((0,0),(0,2),(2,2),(2,0))$
  - $((0.5,0),(0,0.5),(0.5,1),(1,0.5))$
- $Sq \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$

SUMMARY : → Cartesian products generate n-tupples from n-sets
- $(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times X_3 \dots \times X_n$

→ A relation picks out a subset of cartesian product

→ Properties of relations : Reflexive, Symmetric, Transitive, Asymmetric.

→ Equivalence relation partitions a set.

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FUNCTIONS

→ A rule to map inputs to outputs

→ convert x to x^2

- The rule $x \longrightarrow x^2$
- Give it a name : $sq(x) = x^2$
  - Input is a parameter

→ Need to specify the input and output sets

- Domain : Input set
  - $\text{domain}(sq) = \mathbb{R}$
- codomain : Output set of possible values
  - $\text{codomain}(sq) = \mathbb{R}$
- Range : Actual values that the output can take
  - $\text{range}(sq) = \mathbb{R}_{\geq 0} = \{r \mid r \in \mathbb{R},\ r \geq 0\}$

→ $f : X \to Y$, domain of f is X, codomain is Y

Functions & Relations

→ Associate a relation R_f with each function f

→ $R_{sq} = \{(x,y) \mid x,y \in \mathbb{R},\ y = x^2\}$

- Additional notation : $y \stackrel{?}{=} x^2$

→ $R_f \subset \text{domain}(f) \times \text{range}(f)$

→ Properties of R_f
- defined on the entire domain
  - → for each $x \in$ domain(f), there is a pair $(x,y) \in R_f$
- Single Valued
  - → for each $x \in$ domain(f), there is exactly one $y \in$ codomain(f) such that $(x,y) \in R_f$

→ Drawing f as a graph is plotting R_f.

Lines

→ $f(x) = 3.5x + 5.7$
- → 3.5 is the slope
- → 5.7 is intercept where line crosses y-axis when $x=0$

→ Changing the slope and intercept produce different lines
- → $f(x) = 3.5x - 1.2$
- → $f(x) = 2x + 5.7$
- → $f(x) = -4.5x + 2.5$

→ In all these cases
- Domain = $\mathbb{R}$
- codomain = Range = $\mathbb{R}$

More Functions

→ $x \rightarrow \sqrt{x}$

→ Is this a function?
- → $5^2 = (-5)^2 = 25$
- → $\sqrt{25}$ = gives two options
- → By convention take the square root

→ What is the domain?
- → Depends on codomain
- → Negative no.s do not have real sq. roots
- → If codomain is R, domain is $\mathbb{R} \geq 0$
- → If codomain is the set $\mathbb{C}$ of complex no.s, domain is $\mathbb{R}$.

Types Of Functions (on the basis of mapping)

→ Injective : Different inputs produces diff outputs (one-one)
- → If $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$
- → $f(x) = 3x + 5$ is injective
- → $f(x) = 7x^2$ is not ; for any a, $f(a) = f(-a)$

→ Surjective : Range is the codomain - (onto)
- → For every $y \in$ codomain(f), there is an $x \in$ domain(f) such that $f(x) = y$
- → $f(x) = -7x + 10$ is surjective
- → $f(x) = 5x^2 + 3$ is not surjective for codomain $\mathbb{R}$
- → $f(x) = 7\sqrt{x}$ is not surjective for codomain $\mathbb{R}$.

→ Bijective : 1-1 correspondance between domain and codomain.

∴ Every $x \in$ domain(f) maps to a distinct $y \in$ codomain
∴ Every $y \in$ codomain(f) has a unique pre-image $x \in$ domain(f) such that $y = f(x)$

THEOREM
A function is bijective if and only if it is injective and surjective.

→ From the definition, if a function is bijective it is injective and surjective.

→ Suppose a function f is injective and surjective

∴ Injectivity gurantees that f satisfies the first condition of bijection.
∴ Surjectivity says every $y \in$ codomain(f) has a pre image. Injectivity gurantees this pre-image is unique.

Bijections & Cardinality

→ For finite sets, we can count the items

→ What if we have two large sacks filled with marbles?

· Do we need to count the marbles in each sack?

· Pull out marbles in ~~each~~ ~~sack~~ pairs, one from each sack
· Do both sets become empty simultaneously?
· Bijection between the marbles in the sacks

→ For infinite sets

· No. of lines is the same as $\mathbb{R} \times \mathbb{R}$
· Every line $y = mx + c$ is determined uniquely by (m, c) and vice versa.

→ For every pair of points (x_1, y_1) and (x_2, y_2), there is a unique line passing through both points.

→ No. of lines is same as cardinality of $\mathbb{R} \times \mathbb{R}$

→ Does this show that $(\mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) = \mathbb{R}^2 \times \mathbb{R}^2$ has the same cardinality as $\mathbb{R} \times \mathbb{R}$?

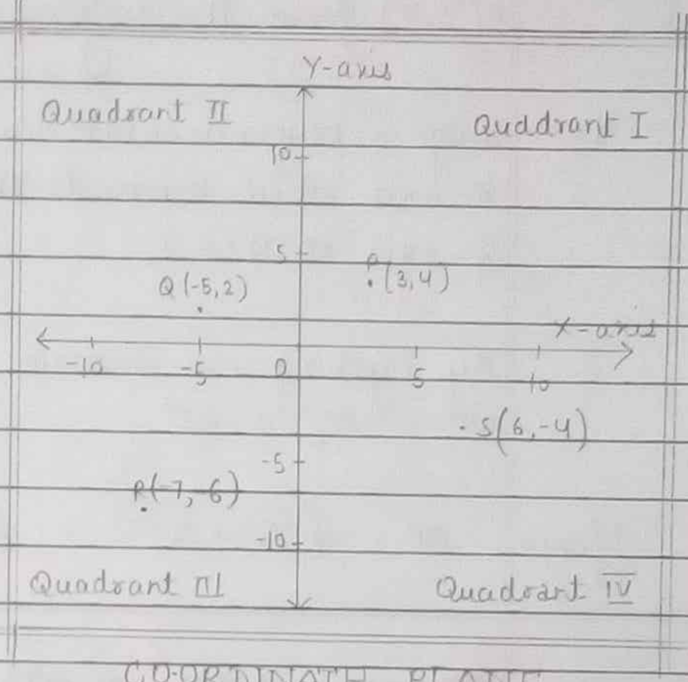
→ The correspondance is not a bijection - many pairs of points describe the same line

Summary ∴ A fn is given by a rule mapping inputs to outputs.
∴ Define the domain, codomain & range.
∴ Associate a relation R_f with each fn f.
∴ Properties of fn : Injective (one-one), Surjective (onto)
∴ Bijections : injective & surjective
∴ A bijection establishes that domain & codomain have same cardinality.

WEEK 2

Rectangular Co-ordinate System

- The horizontal line is called X-axis.
- The vertical line is called Y-axis
- The point of intersection of these two lines is called Origin.
- Any point on the co-ordinate plane can be represented by an ordered pair (x, y).



COORDINATE PLANE

- For example, P(3,4), Q(-5,2).
- The coordinate axes split the coordinate plane into four quadrants and two axes.

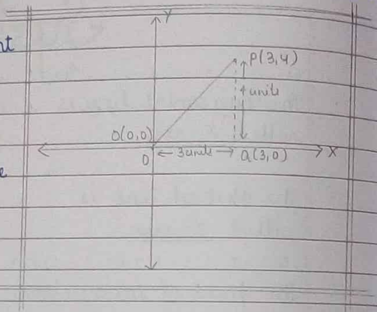
→ Quadrant I : (+, +)
→ Quadrant II : (−, +)
→ Quadrant III : (−, −)
→ Quadrant IV : (+, −)

→ X-axis : (±, 0)
→ Y-axis : (0, ±)
→ Origin : (0, 0)

October 19, 2020

Distance Formula

DISTANCE OF A POINT FROM ORIGIN



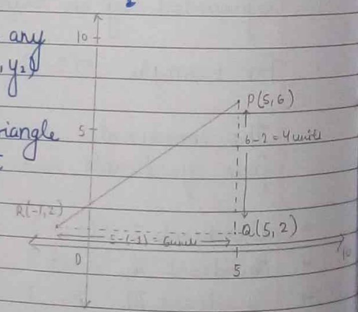
Goal : To find the distance of point P(3,4) from the origin.

1. Drop a perpendicular on X-axis which intersects the X-axis at Q(3,0).

2. By Pythagorean Theorem,
$OP^2 = OQ^2 + QP^2$

Hence, $OP = \sqrt{OQ^2 + QP^2} = \sqrt{3^2 + 4^2} = 5$

DISTANCE BETWEEN ANY 2 POINTS



Goal : To find the distance between any two points $P(x_1, y_1)$ and $R(x_2, y_2)$

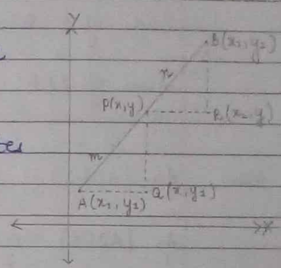
1. Construct a right-angled triangle with right angle at point $Q(x_1, y_2)$

2. By Pythagoras Theorem,
$PR^2 = QR^2 + PQ^2$
$\Rightarrow PR = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ ⟶ Distance Formula

October 19, 2020

Section Formula

Given that, the point P cuts the line segment AB in the m : n ratio. Our goal is to find the coordinates of P.



Let the co-ordinates of A and B are (x_1, y_1) and (x_2, y_2), respectively.

Assume that P has ~~no~~ the coordinates (x, y).

Observe the $\Delta AQP \sim \Delta PRB$,
Hence,

$$\frac{m}{n} = \frac{AP}{PB} = \frac{AQ}{PR} = \frac{PQ}{BR}$$

$$\Rightarrow \frac{m}{n} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y}$$

$$\therefore x = \frac{mx_2 + nx_1}{m+n}, \quad y = \frac{my_2 + ny_1}{m+n}$$

October 19, 2020

Area of a Triangle

Goal : To find the area of ΔABC with known coordinates.

Let the coordinates of the vertices be $A(x_1, y_1)$, $B(x_2, y_2)$ & $C(x_3, y_3)$.

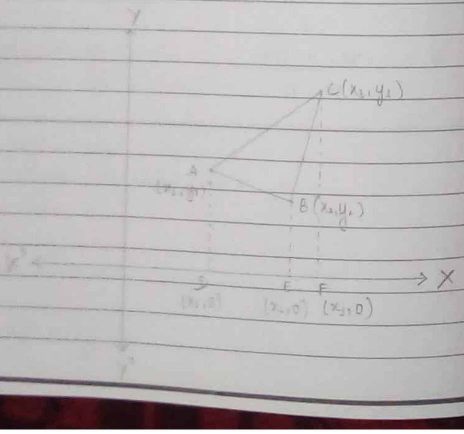
Area of ΔABC = Area of trap. ADFC − Ar. of trap ADEB − Ar. of trap. BEFC

Now, $Ar.(ADFC) = \frac{1}{2}(AD + CF) \times DF = \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$

$Ar.(ADEB) = \frac{1}{2}(AD + EB) \times DE = \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$

$Ar.(BEFC) = \frac{1}{2}(BE + CF) \times EF = \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$

Thus, $Ar.(\Delta ABC) = \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$

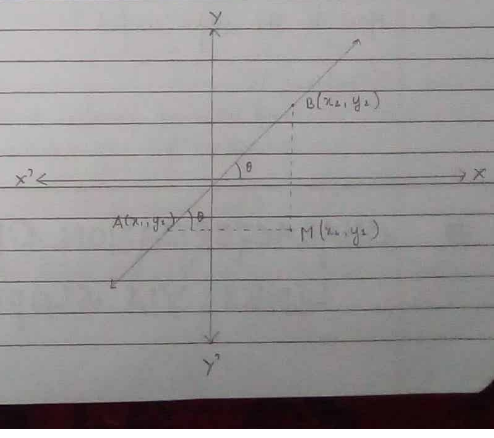


October 19, 2020

Slope of A Line

Goal : To find the slope of a line, given on a coordinate plane

- Identify two points on the line, say, $A(x_1, y_1)$ and $B(x_2, y_2)$.
- Construct a right angled triangle with a right angle at the point $M(x_2, y_1)$.
- Define
$$m = \frac{MB}{AM} = \frac{y_1 - y_2}{x_1 - x_2} = \tan\theta$$
- The 'm' is called slope of a line.
- θ is called the inclination of line with positive X-axis, measured in anti-clockwise direction.
- $0^\circ \le \theta \le 180^\circ$



- Observe that the lines parallel to X-axis have inclination of $0°$.
Hence the slope $m = \tan 0 = 0$

- The inclination of a vertical line is $90°$.
Hence, the slope m is undefined.

Definition: If θ is the inclination of a line l, then $\tan\theta$ is called the slope or gradient of line l.

If $\theta \neq 90°$, then $m = \tan\theta$

For obtuse,

$$m = \tan(180-\theta) = -\tan\theta = \frac{y_1 - y_2}{x_1 - x_2}$$

\# Can a slope of a line uniquely determine a line?
→ No, it can not uniquely determine the line.

\# How is the slope useful?

→ To explore:
- conditions of parallel lines
- conditions for perpendicular lines.

CHARACTERIZATION OF PARALLEL LINES VIA SLOPE

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclination α and β resp.

- If l_1 is parallel to l_2, then $\alpha = \beta$
- It is clear that $\tan\alpha = \tan\beta$
- Hence, $m_1 = m_2$

- Assume, $m_1 = m_2$. Then $\tan\alpha = \tan\beta$
- Since $0° \leq \theta \leq 180°$, $\alpha = \beta$.
- Therefore, $l_1 \parallel l_2$.

Two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

CHARACTERIZATION OF PERPENDICULAR LINES VIA SLOPE

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β resp.

- If l_1 is perpendicular to l_2, then $90° + \alpha = \beta$
- Now, $\tan\beta = \tan(90° + \alpha) = -\cot\alpha = -1/\tan\alpha$
- Hence, $m_2 = -1/m_1 \Rightarrow m_1 m_2 = -1$

- Assume $m_1 m_2 = -1$. Then $\tan\alpha \tan\beta = -1$
- $\tan\alpha = -\cot\beta = \tan(90+\beta)$ or $\tan(90-\beta)$
- Hence, α and β differ by $90°$ which proves l_1 is perpendicular to l_2.

Two non-vertical lines l_1 and l_2 are perpendicular if and only if $m_1 m_2 = -1$.

ANGLES B/W TWO LINES

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α_1 and α_2 respectively.

Suppose l_1 and l_2 intersect and let ϕ and θ be the adjacent angles formed by l_1 and l_2.

Now, $\theta = \alpha_2 - \alpha_1$, for $\alpha_1, \alpha_2 \neq 90°$

Then, $\tan\theta = \tan(\alpha_2 - \alpha_1)$

$$\Rightarrow \tan\theta = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 \tan\alpha_1} = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Now, $\tan\phi = \tan(180 - \theta) = -\tan\theta = -\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$

COLLINEARITY OF POINTS

Consider 3 points A, B and C.

If slope of AB = slope of BC (common point B)

Thus, the three points A, B & C are collinear.

October 20, 2020

Representation of a Line

- How to represent a line uniquely?
- Given a point, how to decide whether the point lies on a line?

In other words, for a given line l, we should have a definite expression that describes the line in terms of coordinate plane.

If the coordinates of a given point P, satisfy the expression for the line l, then the point P lies on the line l.

HORIZONTAL & VERTICAL LINES

Horizontal Lines : A line is a horizontal line only if it is parallel to x-axis.

- To locate such a line, we need to specify the value it takes on Y-axis.
- That is, the expression for such a line is of the form $y = a$
- Then all points that lie on this line are of the form (x, a).

Vertical Lines : A line is a vertical line only if it is parallel to Y-axis.

- To locate such a line we need to specify the value it takes on X-axis.
- That is, the expression for such a line is of the form $x = b$.
- Then, all points that lie on this line are of the form (b, y).

EQUATION OF LINE

(1) Point Slope Form

For a non-vertical line l, with slope m and a fixed point $P(x_0, y_0)$ on the line, can we find the equation (algebraic representation) of the line?

- Let $Q(x, y)$ be an arbitrary point on line l. Then, the slope of the line is given by

$$m = \frac{y - y_0}{x - x_0}$$

$$\Rightarrow \boxed{y - y_0 = m(x - x_0)}$$

Any point $P(x, y)$ is on line l, if and only if the coordinates of P satisfy the above equation.

(2) Two point Form

Let the line l pass through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Assume that $R(x, y)$ is an arbitrary point on the line l.

Then, the points P, Q & R are collinear.

Hence, Slope of PR = Slope of PQ

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

Any point $R(x, y)$ is on line l, if and only if, the coordinates of R satisfy the above equation.

(3) Slope - Intercept Form

→ Let a line l with slope m cut Y-axis at c. Then c is called y-intercept of the line l. That is, the point $(0, c)$ lies on the line l.

$\therefore$ By Point slope Form, $y - c = mx \Rightarrow \boxed{y = mx + c}$

→ Let a line l with slope m cut X-axis at d. Then d is called x-intercept of the line l. That is, the point $(d, 0)$ lies on the line l.

By point slope form, $\boxed{y = m(x-d)}$

(4) Intercept Form

Suppose a line makes x-intercept at a and y-intercept at b. Then the two points on the line are (a,0) & ~~b,0~~ (0,b).

Using two point form,
$$y-0 = \frac{b-0}{0-a}(x-a)$$

$$\Rightarrow \boxed{\frac{x}{a} + \frac{y}{b} = 1}$$

Date: October 31, 2020

WEEK 3

General Equation of Line

	Different Forms Of Equation of Line	Representation
1.	Slope - Point Form	$m = \frac{y - y_0}{x - x_0}$
2.	Slope - Intercept Form	(a) $y = mx + c$ (b) $y = m(x-d)$
3.	Two Point Form	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$
4.	Intercept Form	$\frac{x}{a} + \frac{y}{b} = 1$
5.	Normal Form	$x \cos w + y \sin w = p$

General Form Of Line Equation

$$\boxed{Ax + By + C = 0}$$

(1) Slope - point form

$$m = \frac{y - y_0}{x - x_0} \Rightarrow mx - mx_0 = y - y_0$$

$\Rightarrow \quad mx - y - mx_0 + y_0 = 0$

comparing with $Ax + By + C = 0$

$$\boxed{\begin{aligned} &\rightarrow m = \frac{-A}{B} \\ &\rightarrow y_0 - mx_0 = \frac{-C}{B} \end{aligned}}$$

(2) Slope Intercept Form

$y = mx + c$ $\qquad\qquad$ $y = m(x-d) \Rightarrow y = mx - md$

$\Rightarrow mx - y + c = 0$ $\qquad\qquad$ $\Rightarrow mx - y - md = 0$

comparing with $Ax + By + C = 0$,

$$\boxed{m = \frac{-A}{B} \;,\; c = \frac{-C}{B} \;\text{ or }\; d = \frac{-C}{A}}$$

(3) Two Point Form

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow (y-y_1)(x_2-x_1) = (x-x_1)(y_2-y_1)$$

$$\Rightarrow x_2 y - x_1 y - x_2 y_1 + \cancel{x_1 y_1} = x(y_2 - y_1) - x_1 y_2 + \cancel{x_1 y_1}$$

$$\Rightarrow y(x_2 - x_1) - x(y_2 - y_1) - x_2 y_1 + x_1 y_2 = 0$$

Comparing with $Ax + By + C = 0$

$$\boxed{\frac{y_2 - y_1}{x_2 - x_1} = \frac{-A}{B} \;,\; y_1 + \frac{A}{B} x_1 = \frac{-C}{B}}$$

4. Intercept Form

$$\frac{x}{a} + \frac{y}{b} = 1$$

$\Rightarrow bx + ay = ab$

$\Rightarrow bx + ay - ab = 0$

Comparing with $Ax + By + C = 0$

$$\boxed{a = \frac{-C}{A} \;,\; b = \frac{-C}{B}}$$

Conditions for A Parallel & Perpendicular Line

If the lines given are $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$

then, for the lines to be PARALLEL,

$$\boxed{a_1 b_2 = a_2 b_1}$$

and, for the lines to be PERPENDICULAR,

$$\boxed{a_1 a_2 + b_1 b_2 = 0}$$

Date
November 2, 2020

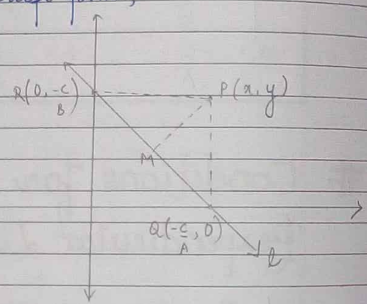
DISTANCE OF A POINT FROM A LINE

Goal: To find the distance of a point $P(x_1, y_1)$ from the line l having equation $Ax + By + C = 0$

$\therefore$ For $A, B \neq 0$, using intercept form,

x- intercept $= \frac{-C}{A}$

y- intercept $= -\frac{C}{B}$



$Ar.(\Delta PQR) = \frac{1}{2}(QR \times PM)$

$\Rightarrow PM = 2 \frac{Area(\Delta PQR)}{QR}$

$$A(\Delta PQR) = \frac{1}{2}\left| x_1\left(\frac{-C}{B}\right) - \frac{C}{A}\left(y_1 + \frac{C}{B}\right)\right| = \frac{1}{2}\frac{|C|}{|AB|}|Ax_1 + By_1 + C|$$

$$QR = \sqrt{\frac{C^2}{A^2} + \frac{C^2}{B^2}} = \frac{|C|}{|AB|}\sqrt{A^2+B^2}$$

Thus $PM = \frac{2\,A(\Delta PQR)}{QR} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$

DISTANCE B/W TWO PARALLEL LINES

Let l_1 and l_2 be two parallel lines with slopes m.

l_1 : $y = mx + c_1$. Comparing with general form, we get x-intercept at $\left(\frac{-c_1}{m}\right)$.

l_2 : $y = mx + c_2$. Comparing with general form, we get $A = -m$, $B = 1$ and $C = -c_2$

By using distance of a point from a line formula, where point is $\left(\frac{-c_1}{m}, 0\right)$, we get

$$\frac{|A(-c_1/m) + B(0) + C|}{\sqrt{A^2 + B^2}} = \frac{|c_1 - c_2|}{\sqrt{1 + m^2}}$$

For general form, $m = \frac{-A}{B}$, $c_1 = -\frac{C_1}{B}$ and $c_2 = \frac{-C_2}{B}$

then

$$d = \frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$$

Distance of a Set of Points from a line

Apart from perpendicular distance, we can also talk about the distance which is parallel to Y-axis.

Consider the set of points $\{(x_i, y_i) \mid i = 1, 2, \ldots, n\}$ and a line with equation $y = mx + c$.

Then the squared sum of the distance of set of points from the line is defined as

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Least Squares Motivation

- In general, this raises the following question
- Given a set of points, how to find the line that fits the given set of points?
- In other words, what is the equation of the best fit line for given set of points?

In other words, if I need to find the equation of line $y = mx + c$, then the question can be reframed into two questions.

- What is the value of m & c that best fits the given set of points.
- What is the meaning of best fit?

BEST FIT : Given a set of n points, $\{(x_i, y_i) \mid i = 1, 2, \ldots n\}$ define

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Find the value of m & c that minimizes SSE.

Week 4

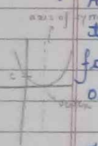
Quadratic Function

- A quadratic function is described by an equation of the form

$$f(x) = \underbrace{ax^2}_{\text{Quadratic Term}} + \underbrace{bx}_{\text{Linear Term}} + \underbrace{c}_{\text{constant Term}}, \quad \text{where } a \neq 0$$

- The graph of any quadratic function is called PARABOLA.

IMPORTANT OBSERVATIONS



- All parabolas have an axis of symmetry. That is, if the graph paper containing the graph of parabola is folded along the axis of symmetry the portion of parabola on either sides will exactly match each other.
- The point at which the axis of symmetry intersects the parabola is called the vertex.
- The y-intercept of a quadratic function is c.

Let $f(x) = ax^2 + bx + c$, where $a \neq 0$

→ The y-intercept : $y = a(0)^2 + b(0) + c = c$
→ The eqn of axis of symmetry : $x = -b/2a$
→ The x-coordinate of vertex : $\dfrac{-b}{2a}$

Maximum & Minimum Values

The y-coordinate of the vertex of a given quadratic function is the minimum or maximum value attained by the function.

The graph of a quadratic function $f(x) = ax^2 + bx + c$, where $a \neq 0$ is:

- Opens up and has a minimum value, if $a > 0$.
- Opens down and has maximum value, if $a < 0$
- The range of a quadratic function is

$$R \cap \{f(x) \mid f(x) \geq f_{min}\} \quad \text{or}$$

$$R \cap \{f(x) \mid f(x) \leq f_{max}\}$$

Slope of a Quadratic Function

Slope of f = $2ax + b$

Slope denotes the rate of change of y with respect to x.

Hence, slope = 0 means the fn has either maximum or minimum which happens when

$$2ax + b = 0 \Rightarrow \boxed{x = \frac{-b}{2a}}$$

Roots of Equations & Zeroes of Functions

The solution to a quadratic equation are called roots of the equation.

One method for finding the roots of a quadratic eqn is to find zeroes of the related quadratic fn.

Observe that the zeroes of a function are x-intercepts of its graph and these are the solutions of related eqn. as $f(x) = 0$ at these points.

Date: January 6, 2021

WEEK 5

Quadratic Equations

Solutions of A Quadratic Equation Using Factorisation

★ Quadratic Function : Intercept Form

Let $y = f(x) = a(x-p)(x-q)$, where p and q represents x-intercepts for the function.
Then the form $y = a(x-p)(x-q)$ is called the INTERCEPT FORM.

★ Intercept form to the Standard form

Changing intercept form to standard form requires us to use FOIL method which can be described as follows:

The product of 2 binomials is the sum of the products of the first (F) terms, the outer (O), the inner (I) and the last terms.

$$(ax+b)(cx+d) = \underbrace{ax \cdot cx}_{F} + \underbrace{ax \cdot d}_{O} + \underbrace{cx \cdot b}_{I} + \underbrace{b \cdot d}_{L}$$

Quick Observations :
→ The product of coefficient of x^2 and the constant is abcd.
→ The product of two terms in the coefficient of x is also abcd.

Example : Write a quadratic eq^n with roots, $\frac{2}{3}$ and -4, in the standard form.

Recall: by standard form we mean $ax^2+bx+c=0$; $a,b,c \in \mathbb{Z}$

We have, by intercept form, $\left(x-\frac{2}{3}\right)(x+4) = 0$

$\therefore$ By Foil,
The standard form will be,

$$x^2 + 4x - \frac{2}{3}x - \frac{8}{3} = 0$$

$$\Rightarrow x^2 + x\left(4-\frac{2}{3}\right) - \frac{8}{3} = 0$$

$$\Rightarrow x^2 + \frac{10}{3}x - \frac{8}{3} = 0$$

$$\Rightarrow \boxed{3x^2 + 10x - 8 = 0}$$

★ Standard to Intercept Form

Example : Convert the fn $f(x) = 5x^2 - 13x + 6$ to intercept form.

Solution : $f(x) = 5x^2 - 13x + 6$ ← standard form

$= 5x^2 - 10x - 3x + 6$

$= 5x(x-2) - 3(x-2)$

$= (5x-3)(x-2)$

$= 5\left(x-\frac{3}{5}\right)(x-2)$ — intercept form

SOLUTIONS OF A QUADRATIC EQ^N USING SQUARE

★ Solving a quadratic Equation by completing The Square method.

Example : $x^2 + 10x = 24$

Observe that $(x+a)^2 = x^2 + 2ax + a^2$.
Using this, write $10 = 2\times5$ and add 25 on both the sides of the equation to get,

$$x^2 + 25 + 10x = 24 + 25$$
$$(x+5)^2 = 49$$
$$\Rightarrow x+5 = \pm\sqrt{49}$$

Either $x+5 = -7$ or $x+5 = 7$

$\Rightarrow \boxed{x = -12}$ or $\boxed{x = 2}$

Quadratic Formula

$$ax^2 + bx + c = 0$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Rightarrow x^2 + \left(\frac{b}{a}\right)x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

discriminant

$$\boxed{D = \sqrt{b^2 - 4ac}}$$

$$\Rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

finding roots by Quadratic Formula

If $D > 0$, 2 roots
$D < 0$, no real root
$D = 0$, 1 root (2 equal roots)

Value of Discriminant	Type & no. of roots
$b^2 - 4ac > 0$ (perfect square)	→ 2 real, rational roots
$b^2 - 4ac > 0$ (not perfect square)	→ 2 real, irrational roots
$b^2 - 4ac = 0$	→ 1 real root
$b^2 - 4ac < 0$	→ No real roots

graph will not touch x-axis

Summary of Concepts

METHOD	CAN BE USED	WHEN PREFERRED
Graphing	Occasionally	Best used to verify the answer found algebraically
Factoring	Occasionally	If constant term is 0 or factors are easy to find.
Completing the square	Always	Use when b is Even
Quadratic Formula	Always	Use when other methods fail.

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WEEK 6

Polynomials

What is a Polynomial?

A Layman's Perspective : a polynomial is one kind of mathematical expression which is a sum of several mathematical terms.

→ Each term in this expression is called 'monomial' and the term can be a number, a variable or a product of several variables.

A Mathematician's Perspective : a polynomial is an algebraic expression in which the only arithmetic is addition, subtraction, multiplication and "natural" exponents of variables.

Examples : $4x^3 + 9x + 3 = 0$; $2x + 1 = 8$ ... etc.

Why do we call them Polynomials?

The word 'polynomial' is derived from two words

Poly + Nomen

many — name / term

- Each term is called monomial
- A polynomial having 2 terms is called binomial.
- A polynomial with three terms is called trinomial.

E.g: A polynomial in 1 variable can be treated as,

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = \sum_{m=0}^{n} a_m x^m$$

exponent, variable, coeff. of term

Identification of Polynomials

→ Identify whether the following are polynomials or not.

1. $x^2 + 4x + 2$ → Yes
2. $x + x^{1/2}$ → No
3. $x + y + xy + x^3$ → Yes

Types of Polynomials

1. Polynomials in one variable
   E.g. $x^4 + 1$
2. Polynomials in two variables
   E.g. $x^4 + y^5 + xy$
3. Polynomials in more than 2 variables
   E.g. $xyz + x^2 z^5$

3. $p(x) = x^3 + 2x^2 + x$, $q(x) = x^2 + 2x + 2$

$$
\begin{array}{rcl}
p(x) & = & x^3 + 2x^2 + x + 0 \\
q(x) & = & 0x^3 + x^2 + 2x + 2 \\
\hline
p(x) + q(x) & = & x^3 + 3x^2 + 3x + 2
\end{array}
$$

Thus, Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{j=0}^{m} b_j x^j$

Then $\boxed{p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k) x^k}$

Subtraction Of Polynomials

Ques. Subtract the foll. polynomials :

1. $p(x) = x^2 + 4x + 4$, $q(x) = 10$

$$
\begin{array}{rcl}
p(x) & = & x^2 + 4x + 4 \\
-q(x) & = & 0x^2 - 0x - 10 \\
\hline
p(x) - q(x) & = & x^2 + 4x - 6
\end{array}
$$

2. $p(x) = x^4 + 4x$, $q(x) = x^3 + 1$

$$
\begin{array}{rcl}
p(x) & = & x^4 + 0x^3 + 4x + 0 \\
-q(x) & = & 0x^4 - 1x^3 - 0x - 1 \\
\hline
p(x) - q(x) & = & x^4 - x^3 + 4x - 1
\end{array}
$$

3. $p(x) = x^3 + 2x^2 + x$, $q(x) = x^2 + 2x + 2$

$$
\begin{array}{rcl}
p(x) & = & x^3 + 2x^2 + x + 0 \\
-q(x) & = & -0x^3 - x^2 - 2x - 2 \\
\hline
p(x) - q(x) & = & x^3 + x^2 - x - 2
\end{array}
$$

$\therefore$ Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{j=0}^{m} b_j x^j$

Then, $\boxed{p(x) - q(x) = \sum_{k=0}^{m \vee n} (a_k - b_k) x^k}$

Multiplication of Polynomials

Ques. Multiply the foll. polynomials.

1. $p(x) = x^2 + x + 1$ and $q(x) = 2x^3$

Solⁿ. $p(x) \cdot q(x) = (x^2 + x + 1)(2x^3)$
$= 2x^5 + 2x^4 + 2x^3$

2. $p(x) = x^2 + x + 1$ and $q(x) = 2x + 1$

$p(x) \cdot q(x) = (x^2 + x + 1)(2x + 1)$
$= 2x^3 + x^2 + 2x^2 + x + 2x + 1$
$= 2x^3 + 3x^2 + 3x + 1$

3. $p(x) = a_2 x^2 + a_1 x + a_0$ & $q(x) = b_1 x + b_0$

Sol^n: $p(x)\cdot q(x) = (a_2x^2 + a_1x + a_0)(b_1x + b_0)$

$= a_2 b_1 x^3 + a_2 b_0 x^2 + a_1 b_1 x^2 + a_1 b_0 x + a_0 b_1 x + a_0 b_0$

$= a_2 b_1 x^3 + (a_1 b_1 + a_2 b_0)x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$

Let $p(x) = \sum_{k=0}^{m} a_k x^k$ and $q(x) = \sum_{j=0}^{n} b_j x^j$, Then,

$$\boxed{p(x)\cdot q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j b_{k-j}) x^k}$$

Ques. Multiply $p(x) = x^2+x+1$ & $q(x) = x^2+2x+1$.

Let $p(x) = \sum_{k=0}^{n} a_k x^k$ & $q(x) = \sum_{j=0}^{m} b_j x^j$

$p(x)\cdot q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j b_{k-j}) x^k$

k	a_k	b_j
0	1	1
1	1	2
2	1	1

k	Coefficient	Calculations
0	a_0b_0	1
1	$a_1b_0 + a_0b_1$	$1+2=3$
2	$a_0b_2 + a_1b_1 + a_2b_0$	$1+2+1=4$
3	$a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$	$0+1+2+0=3$
4	$a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0$	$0+0+1+0+0=1$

The resultant polynomial is,

$p(x)\cdot q(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$

Division of Polynomials

- Division of a polynomial by a monomial

$$\frac{3x^2+4x+3}{x} = 3x + 4 + \frac{3}{x}$$

- Division of a polynomial by another polynomial

$$\frac{3x^2+4x+3}{2x+1} = ??$$

Let us divide, $\frac{3x^2+4x+1}{x+1}$

$$\Rightarrow \frac{3x^2+3x+x+1}{x+1} = \frac{3x(x+1)+1(x+1)}{(x+1)} = \frac{(3x+1)(x+1)}{x+1}$$

$\Rightarrow 3x+1$ Ans.

Ques. Divide $p(x) = x^4 + 2x^2 + 3x + 2$ by $q(x) = x^2 + x + 1$

$$\frac{p(x)}{q(x)} = \frac{x^4+2x^2+3x+2}{x^2+x+1} = \frac{x^4+x^3+x^2-x^3+x^2+3x+2}{x^2+x+1}$$

$$= \frac{x^4+x^3+x^2}{x^2+x+1} - \frac{x^3+x^2+3x+2}{x^2+x+1} = x^2 + \frac{(-x^3+x^2+3x+2)}{x^2+x+1}$$

$$= (x^2) + \left(\frac{-x^3-x^2-x}{x^2+x+1}\right) + \left(\frac{x^2+x+x^2+3x+2}{x^2+x+1}\right) = x^2 - x + \frac{2x^2+4x+2}{x^2+x+1}$$

$$= x^2 - x + \frac{2(x^2 + x + x + 1)}{x^2 + x + 1}$$

$$= x^2 - x + \frac{2(x^2 + x + 1)}{(x^2 + x + 1)} + \frac{2x}{x^2 + x + 1}$$

$$= x^2 - x + 2 + \frac{2x}{x^2 + x + 1} \quad \text{Ans.}$$

Algorithm for Division of Polynomials

$$\underset{\text{Divisor}}{\underset{\text{Dividend}}{\frac{p(x)}{q(x)}}} = \underbrace{x^2 - x + 2}_{\text{Quotient}} + \frac{\overbrace{2x}^{\text{Remainder}}}{q(x)}$$

Step 1: Arrange the terms in descending order of the degree and add the missing exponents with 0 as coefficient.

Step 2: Divide the first term of the dividend by the first term of the divisor ~~to~~ and get the monomial.

Step 3: Multiply the monomial with divisor and subtract the result from the dividend.

Step 4: Check if the resultant polynomial has degree less than divisor. If true, write the remainder else Go to step 2.

Question: Divide : $\frac{2x^3 + 3x^2 + 1}{2x + 1}$ by long division method.

$$\begin{array}{r|l}
 & x^2 + x - 1/2 \quad \leftarrow \text{Quotient} \\
\hline
\text{Divisor} \rightarrow 2x+1 & 2x^3 + 3x^2 + 1 \quad \leftarrow \text{Dividend} \\
 & -\ 2x^3 + x^2 \\
\hline
 & 2x^2 + 1 \\
 & -\ x + 2x^2 + 0 \\
\hline
 & -x + 1 \\
 & -\ -x - 1/2 \\
\hline
 & 3/2 \quad \leftarrow \text{Remainder} \\
\hline
\end{array}$$

$\frac{2x^3}{2x} = x^2(2x+1)$

$\frac{2x^2}{2x} = x(2x+1)$

$\frac{-x}{2x} = \frac{-1}{2}(2x+1)$

$$\therefore \frac{2x^3 + 3x^2 + 1}{2x + 1} = x^2 + x - \frac{1}{2} + \frac{3}{2(2x+1)}$$