NOTES BY GAGNEET KAUR

Date: February 7, 2023

WEEK 3

FOUR FUNDAMENTAL VECTOR SUBSPACES

1. COLUMN SPACE → subspace of $\mathbb{R}^m$

The column space of a matrix A is the vector space formed by the columns of A, essentially meaning all linear combinations of the columns of A.

$$C(A) = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \}$$

Suppose $A = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$

$$C(A) = \text{span}(u_1, u_2, \ldots, u_n)$$

SOLVING $Ax = b$:

For what b does $Ax = b$ have a solution?

→ $b \in C(A)$

RANK of a matrix = dimensions of its column space

For example, consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 0 & 6 & 2 \\ 3 & 4 & 9 & 7 \end{bmatrix}$$

Then the column vectors are : $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$

which form a vector space.

After RREF of A, we get,

$$RREF(A) = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The the corresponding columns in matrix A to the pivotal columns of RREF(A) form the basis of C(A).

$$\Rightarrow C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \right)$$

$\Rightarrow$ Rank of A = 2 = Dimension of C(A)

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2. NULL SPACE

subspace of $\mathbb{R}^n$

The nullspace of a matrix A is set of all vectors x for which

$$Ax = 0$$

($A \rightarrow m \times n$ matrix)

$$N(A) = \{ \vec{x} \mid Ax = 0 \text{ and } \vec{x} \in \mathbb{R}^n \}$$

Consider the foll. matrix,

$$A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 0 & 6 & 2 \\ 3 & 4 & 9 & 7 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

To find the null space of the matrix A, we need to find solutions to $A\vec{x} = 0$.

Firstly, $RREF(A) = \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

According to this, we have,

$$x_1 + 3x_3 + x_4 = 0 \Rightarrow x_1 = -3x_3 - x_4$$
$$x_2 + x_4 = 0 \Rightarrow x_2 = -x_4$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Here, we have 2 free variables $\rightarrow$ x_3 & x_4

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Now, Basis of Null space of A = $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Thus, we have

$$N(A) = \text{Span}\left(\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\text{Dim}(N(A)) = 2$$

If A is invertible, then N(A) has zero vector only.
$\Rightarrow$ $Ax = b$ has a unique solⁿ.
$\Rightarrow$ C(A) is whole space

Dimension of Null space of A = No. of free variables in RREF(A)

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3. ROW SPACE → subspace of $\mathbb{R}^n$

- The row space of mat(A) is the column space of A^T.
- $R(A) = C(A^T)$
- Rank of matrix $= \text{Dim}(R(A)) = \text{Dim}(C(A))$

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4. LEFT NULL SPACE → subspace of $\mathbb{R}^m$

- The left null space of mat(A) is the null space of A^T.

Page No. ____

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Date: ________

- $N(A^T)$ → left null space
- Dimensions of $N(A^T)$ = no. of rows - rank of matrix

Important Facts :

1. DIMENSIONS

Let A be matrix, A is $m \times n$ matrix.

→ Dim. of $C(A)$ = Dim. of $R(A)$ = r ← rank of matrix

→ Dim. of $N(A)$ = $n - r$

→ Dim. of $N(A^T)$ = $m - r$

Rank → no. of pivot columns
Nullity → no. of free variables

2. ORTHOGONALITY

→ $N(A) \perp R(A)$

→ $N(A^T) \perp C(A)$

3. Rank-Nullity Theorem

→ $\underbrace{\text{Rank}(A)}_{\text{dim. of } C(A)} + \underbrace{\text{Nullity}(A)}_{\text{dim. of } N(A)}$ = No. of columns of A

→ $\text{Rank}(A) + \text{Nullity}(A^T)$ = No. of rows of A

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ORTHOGONAL VECTORS AND SUBSPACES

Length of a vector

$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x} = x^T x$ ← Dot product

Orthogonal Vectors

Two vectors $\vec{a}$ & $\vec{b}$ are said to be orthogonal ($\theta = 90°$) when,

$$\boxed{\vec{a} \cdot \vec{b} = 0}$$

Facts : 1. $\vec{0}$ is orthogonal to every vector

2. If $\{v_1, v_2, \dots, v_k\}$ are mutually orthogonal 'non-trivial' set of vectors, then $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set.

Orthonormal Vectors

Two vectors $\vec{x}$ & $\vec{y}$ are orthonormal when,

$\vec{x} \cdot \vec{y} = 0$ and $\|\vec{x}\| = \|\vec{y}\| = 1$

Orthogonal Subspaces

Two subspaces S_1 and S_2 are orthogonal if,

$$x^T y = 0 \quad , \forall x \in S_1 \; , \forall y \in S_2$$

Orthogonality wrt Four Fundamental Spaces

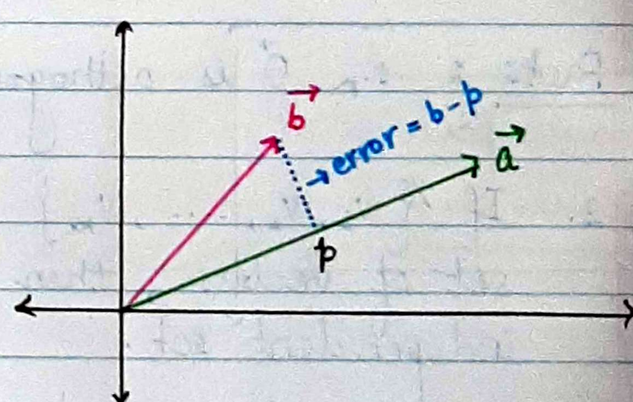
$R(A) \perp N(A)$

$C(A) \perp N(A^T)$

PROJECTIONS

(1) Projection of $\vec{b}$ on $\vec{a}$,

$$\text{Proj} = \left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} \right) \vec{a}$$



(2) Projection Matrix (Matrix of vector on which we project)

$$\mathbb{P} = \frac{\vec{a}\,\vec{a}^T}{a^T a}$$

Projection of $\vec{b}$ on $\vec{a}$ is $\mathbb{P}\vec{b}$.

proj. matrix

Properties of Projection Matrix of vector $\vec{a}$

1. $\mathbb{P}$ is symmetric

2. $\mathbb{P}^2 = \mathbb{P}$

3. Column space of $\mathbb{P}$, $C(\mathbb{P})$ = line through $\vec{a}$

4. Null space of $\mathbb{P}$, $N(\mathbb{P})$ = plane orthogonal to $\vec{a}$

5. Rank, $r(\mathbb{P}) = 1$

- Consider $Ax = b$. The projection matrix of A is an identity matrix if,

$$b \in C(A)$$

- $P^T = \mathbb{P}$

February 9, 2023

PROJECTIONS ONTO A SUBSPACE

Let V be a subspace.

⇒ The orthogonal complement of V, ie, $V^{\perp}$ is also a subspace

→ Projection of $\vec{x}$ on V = unique vector $\vec{v} \in V$ such that $\vec{x} = \vec{v} + \vec{w}$ where $\vec{w} \in V^{\perp}$

($\because \vec{x} - \vec{v} = \vec{w} \leftarrow$ error)

Similar to:

→ $Proj_V(\vec{x})$ = some unique vector in V such that $\vec{x} - Proj_V \vec{x}$ is orthogonal to every member of V

$$\boxed{Proj_V(\vec{x}) = A(A^TA)^{-1}A^T\vec{x}}$$

where A is the basis of Subspace 'V'

LEAST SQUARES APPROXIMATION

- It often happens that $A\vec{x} = \vec{b}$ has NO solution.

→ the usual reason is: too many equations

→ The matrix A has more rows than columns

→ There are more equations than unknowns.

Then columns span a small part of m-dim space.

- We can't always get the error $e = b - Ax$ down to zero.

When e is zero, $\vec{x}$ is an exact sol^n to $A\vec{x} = \vec{b}$.

$\therefore$ When the length of e is as small as possible, $\hat{x}$ is a least squares solution.

LEAST SQUARES METHOD :

$$A^T A \hat{x} = A^T b$$

Solving the above eqn, we get $\hat{x} = \begin{bmatrix} \text{slope} \\ \text{intercept} \end{bmatrix}$

The slope & the intercept for the best fit line !!

Important Facts About Least Squares Approximation

- If $A\theta = b$ has a solution then, $\| A\theta - b \|^2 = 0$
- The least squares solution to a system of linear equations $A\theta = b$ aims to minimize $\|A\theta - b\|$
- The least squares solution to a system of linear equations $A\theta = b$ projects b onto $C(A)$.