Week 5

Complex Vectors

$x \in \mathbb{C}^n$ $\qquad$ $x = \begin{bmatrix} 3-2i \\ -2+i \\ -4-3i \end{bmatrix} \in \mathbb{C}^3$

→ $c = a + ib$

→ $\bar{c} = a - ib$

→ $c \cdot \bar{c} = a^2 - (ib)^2 = a^2 + b^2$

→ $\|c\|^2 = a^2 + b^2$

→ $\mathbb{R} \in \mathbb{C}$

→ Inner product : $x \cdot y = \bar{x}^T y = \overline{x^T} y \in \mathbb{C}$

→ $x \cdot x$ = Real No. $\qquad$ where $x \in \mathbb{C}$

Properties :

1. $x \cdot y = \overline{y \cdot x}$
2. $(x+y) \cdot z = x \cdot z + y \cdot z$
3. $x \cdot cy = c(x \cdot y)$
4. $cx \cdot y = \bar{c}(x \cdot y)$
5. $x \cdot x = \|x\|^2 \in \mathbb{R}$
6. $cx \cdot cy = |c|^2 (x \cdot y)$

COMPLEX MATRICES

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$$

A matrix is Hermitian if : $A^* = A$

where $A^* = \bar{A}^T$ or $\overline{A^T}$

$\Rightarrow a_{ij} = \overline{a_{ji}}$

→ The above matrix A is Hermitian

→ diagonal elements in Hermitian $\in \mathbb{R}$

Properties of Hermitian Matrices

- All eigenvalues λ_i are real
- Eigenvectors are orthogonal if $\lambda_i \neq \lambda_j$ for $i \neq j$

UNITARY MATRICES (similar to orthogonal matrix)

Real Case

$Q^T Q = I$

Complex Case

$U^* U = I$ (where $U^* = \bar{U}^T = \overline{U^T}$)

Properties of Unitary Matrices

- it preserves length and angle of vectors
- Therefore, eigenvalues are $|\lambda_i| = 1$
- Eigenvectors are orthogonal if $\lambda_i \neq \lambda_j$ for $i \neq j$
- Unitary matrices need not necessarily be Hermitean.
- There exists unitary matrix that diagonalizes a Hermitian matrix.

Example of unitary matrix

$$U = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$U^T = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

$$U * U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

DIAGONALIZATION OF HERMITIAN MATRIX

Schur's Theorem

Any $n \times n$ matrix is similar to upper triangular matrix T, ie, $A = UTU^*$

Spectral Theorem

Any Hermitian matrix is similar to diagonal matrix D, i.e, $A = UDU^*$

Week 6

~~[illegible]~~ Quadratic Function

- Function : $f(x,y) = ax^2 + 2bxy + cy^2$

In quadratic form,

$$f(x,y) = [x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Another notation,

$$f(x,y) = v^T A v \quad , \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} , \quad v = \begin{bmatrix} x \\ y \end{bmatrix}$$

Partial Derivatives of A Function

- First order partial derivatives at point (p,q) :

$$\nabla f = \begin{bmatrix} \partial f/\partial x \ (p,q) \\ \partial f/\partial y \ (p,q) \end{bmatrix}$$

→ put it to zero to find the value of critical point OR if $\nabla f = 0$ at (p, q), then (p, q) is a stationary point.

- **Second Order partial derivatives at (p, q)**

$$f_{xx} = \frac{\partial f^2}{\partial x^2}(p,q) \qquad f_{xy} = \frac{\partial f^2}{\partial xy}(p,q)$$

$$f_{yy} = \frac{\partial f^2}{\partial y^2}(p,q)$$

Matrix : $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

Stationary point of a Function

→ At a stationary point (p, q) : first order derivatives vanishes

$$f'(x) = 0 \quad , \quad f'(y) = 0$$

→ Determinant :

$$D(p,q) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

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→ Second partial Derivative Test :

Stationary Points	Conditions
Minima	$f_{xx} > 0$, $D > 0$
Maxima	$f_{xx} < 0$, $D > 0$
Saddle	$D < 0$
Inconclusive	$D = 0$

Definiteness of a $n \times n$ real symmetric matrix

- Trace (A) = sum of all eigen values of A
  = sum of diagonal elements of A
- Det (A) = product of all eigenvalues of A
- Definiteness of the matrix A for all $x \neq 0$ in R^n.

Definiteness	Function Form	Eigen Values
+ve definite	$f = x^T A x > 0$	all positive
+ve semi definite	$f = x^T A x \geq 0$	non negative
-ve definite	$f = x^T A x < 0$	all negative
-ve semi definite	$f = x^T A x \leq 0$	non positive
indefinite	both $f > 0$, $f < 0$	both -ve & +ve

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

Definiteness	Function	Condition
+ve definite	$f(x,y) > 0$	$a > 0$, $D > 0$
+ve semi-def	$f \geq 0$	$a > 0$, $D = 0$
-ve definite	$f < 0$	$a < 0$, $D > 0$
-ve semi-def	$f \leq 0$	$a < 0$, $D = 0$
indefinite	$f > 0$, $f < 0$	$D < 0$

- How to find singular values of matrix A?

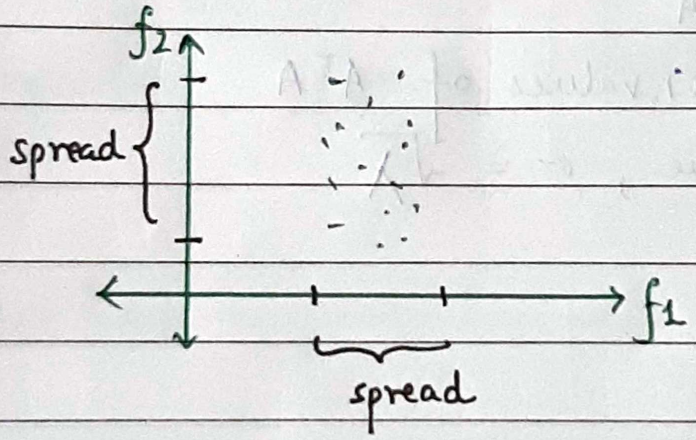
→ Calculate $A^T A$
→ Find the eigenvalues of $A^T A$
→ singular value, $\sigma = \sqrt{\lambda}$

Week 7

Principle Component Analysis

- WHY?
  - → for vizualization
  - → $d' < d$ for model training

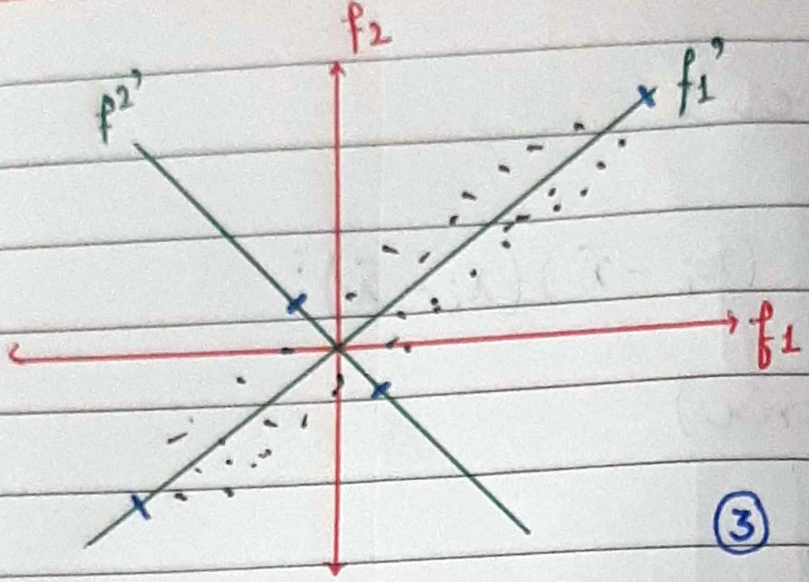
- Geometric Intuition



→ To reduce dimensionality from 2D to 1D, we can skip f1.

→ By reducing f1 we are loosing less information

→ Preserving direction with maximum spread



① ⇒ Find $f_1' \perp f_2'$
spread on $f_2' << f_1'$

② Drop f_2'

③ Project x_i' onto f_1'
2D → 1D

So what we want :

→ To find a direction f_1' such that the variance of x_i's projected on f_1' is maximum.

TASK :

- Minimize Reconstruction Error

$$d_i^2 = \|x_i\|^2 - (\mu_i^T x_i)^2$$

Problem
→ minimize $\Sigma \left(\|x_i\|^2 - (u_i^T x_i)^2 \right)$
contrained $\mu_i^T u_i = 1$

solution
→ u_1 → corresponding to maximum λ_1

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Things To Remember :

1. $$C = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^T$$

(Covariance Matrix)

2. $u_1, u_2, u_3 \rightarrow$ corresponding to $\lambda_1 > \lambda_2 > \lambda_3 \ldots$

3. Projected Data's $x_i' = \sum_i (x_i^T u_i) u_i$

4. Reconstruction Error

$$J^* = \frac{1}{n} \sum_{i=1}^{n} \| x_i - \tilde{x}_i \|^2$$