June 18, 20

Date
Page

WEEK 6

Permutations & Combination

LEARNING OBJECTIVES

(1) Understanding differences between counting with order (permutation) and counting without regard to order (combination)

(2) Use permutations and combinations to answer real life applications

---

PERMUTATION

A permutation is an ordered arrangement of all or some of n objects.

We will learn about :
Permutations when objects are distinct
(i) repetition not allowed
(ii) repetition allowed

Permutation Formula

The no. of possible permutation of 'r' objects from a collection of 'n' distinct objects is given by the formula,

$$n \times (n-1) \times \ldots \times (n-r+1)$$

and is denoted by ${}^nP_r$.

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$$

SPECIAL CASES :

(1) $${}^nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

There is only one ordered arrangement of 0 objects.

(2) $${}^nP_1 = \frac{n!}{(n-1)!} = n$$

There are 'n' ways of choosing one object from 'n' objects.

(3) $${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

We can arrange n distinct objects in n! ways - multiplication principle of counting.

Application

Question: From a committee of 8 persons, in how many ways can we choose a chairman & a vice chairman assuming one person can not hold more than one position?

Solution: $n = 8$
$r = 2$

$\therefore {}^nP_r = \frac{n!}{(n-r)!} = \frac{8!}{(8-2)!} = \frac{8!}{6!} = 8 \times 7 = 56$ ways

---

Question: Find the no. of 4-digit no.s that can be formed using the digits 1, 2, 3, 4, 5 if no digit is repeated.

Solution: $n = 5$
$r = 4$

$\therefore {}^nP_r = \frac{n!}{(n-r)!} = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 5! = 120$

Question: Six people go to the cinema. They sit in a row with 10 seats. Find how many ways can this be done if

(i) they can sit anywhere
(ii) all the empty seats are next to each other

Solution: (i) $n = 10$
$r = 6$

$\therefore {}^nP_r = \frac{n!}{(n-r)!} = \frac{10!}{4!} = 10 \times 9 \times 8 \times 7 \times 6 \times 5$ ways

(ii) Box method $\square\square\square\square\square\square\square$

$\cancel{7}$

$\cancel{6!}$

$\therefore$ No. of ways $= \cancel{6! \times 5!}\ \cancel{4! \times 5!} \cancel{= 120 \times 24}$

$= {}^7P_6 = 5040$

Permutation Formula (Repetition is allowed)

The no. of possible permutations of r objects from a collection of n distinct objects when repetition is allowed is given by the formula

$n \times n \times n \times \ldots \times n$

and is denoted by n^r.

For example : Take A,B, C, . What are the possible arrangements taken two at a time.

$n = 3 \qquad r = 2 \qquad \therefore n^r = 3^2 = 9$

i.e, AB, AC, BA, BC, CA, CB, AA, BB, CC.

Date
July 6, 2021

PERMUTATIONS WHEN OBJECTS ARE NOT DISTINCT

EXAMPLE: Rearranging letters

- Suppose we want to rearrange the letters in the word "DATA". How many ways can it be done?
- There are three distinct letters : D, A, T
- As seen in the example, we can treat the two A's in DATA as distinct. Say A_1 & A_2.
- If they are treated as distinct objects, then based on the earlier formula,
  Total no. of arrangements $= 4! = 24$
- Also A_1 & A_2 can be arranged among themselves in $2!$ ways
- A_1 & A_2 are essentially the same. Hence the total no. of ways the letters in "DATA" can be arranged as $\frac{4!}{2!} = 12$

PERMUTATION FORMULA

The no. of permutations of n objects when p of them are of one kind and rest distinct is equal to $\rightarrow$ $\frac{n!}{p!}$

Example : Suppose we want to rearrange the letters in the word ~~of~~ "STATISTICS". How many ways can it be done?

→ Total of 10 letters of which there are 5 distinct letters : S, T, A, I, C.

→ 'S' appears 3 times, 'T' appears 3 times, 'A' once, 'I' twice, and 'C' once

Permutation Formula

The no. of permutations of n objects where p_1 is of one kind, p_2 is of second kind, and so on p_k of k^{th} kind is given by

$$\frac{n!}{p_1!\, p_2! \ldots p_k!}$$

→ Applying the above formula to the word "STATISTICS"

$n = 10$, $p_1 = 3$, $p_2 = 3$, $p_3 = 1$, $p_4 = 2$, $p_5 = 1$

Hence, the total no. of ways $= \dfrac{10!}{3! \times 3! \times 1! \times 2! \times 1!}$

$= 50,400$

CIRCULAR PERMUTATIONS

EXAMPLE:

- How many ways can 4 people sit in a round table?

- We consider 2 cases: each selection is called a combination of 3 different objects taken 2 at a time

  # Clockwise & anticlockwise are different
  # clockwise & anticlockwise are same.

#1. CLOCKWISE & ANTICLOCKWISE ARE DIFFERENT

The number of ways n distinct objects can be arranged in a circle (clockwise & anticlockwise are diffrent) is equal to $(n-1)!$

#2 CLOCKWISE & ANTICLOCKWISE ARE SAME

The number of ways n distinct objects can be arranged in a circle (clockwise & anticlockwise are same) is equal to $\frac{(n-1)!}{2}$

SOLVING FOR n & r USING PERMU-TATION FORMULA

1. Find the value of n if ${}^nP_4 = 20\ {}^nP_2$

Sol. $\dfrac{n!}{(n-4)!} = 20 \times \dfrac{n!}{(n-2)!}$

$\Rightarrow \dfrac{(n-2)!}{(n-4)!} = 20 \Rightarrow \dfrac{(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}} = 20$

$\Rightarrow n^2 - 3n - 2n + 6 = 20$

$\Rightarrow n^2 - 5n - 14 = 0$

$\Rightarrow n^2 - (7-2)n - 14 = 0$

$\Rightarrow n^2 - 7n + 2n - 14 = 0$

$\Rightarrow (n-7)(n+2) = 0$

$\therefore \boxed{n = 7}$ or $n = -2$ (rejected)

Ans.

---

2. Find the value of n if $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3}$

Sol. ${}^nP_4 \times 3 = 5 \times {}^{n-1}P_4$

$\Rightarrow \dfrac{n!}{(n-4)!}(3) = (5)\dfrac{(n-1)!}{(n-1-4)!}$

$\Rightarrow 3 \times \dfrac{n\cancel{(n-1)!}}{(n-4)\cancel{(n-5)!}} = 5 \times \dfrac{\cancel{(n-1)!}}{\cancel{(n-5)!}}$

$\Rightarrow 3n = 5n - 20 \Rightarrow 2n = 20 \Rightarrow \boxed{n = 10}$

3. Find r if $^5P_r = 2 \cdot {}^6P_{r-1}$

Sol. $\dfrac{5!}{(5-r)!} = 2 \times \dfrac{6!}{(6-(r-1))!}$

$\Rightarrow \cancel{5-r}\ \dfrac{(7-r)!}{(5-r)!} = 2 \times \dfrac{6!}{5!}$

$\Rightarrow \dfrac{(7-r)!}{(5-r)!} = 12$

$\Rightarrow \dfrac{(7-r)(6-r)\cancel{(5-r)!}}{\cancel{(5-r)!}} = 12$

$\Rightarrow 42 - 7r - 6r + r^2 = 12$

$\Rightarrow r^2 - 13r + 30 = 0$

$\Rightarrow r^2 - (10+3)r + 30 = 0$

$\Rightarrow r^2 - 10r + 3r + 30 = 0$

$\Rightarrow (r-10)(r-3) = 0$

$r = 10$ or $r = 3$

We know, $r \leq n$ $\therefore r = 10$ (rejected)

Thus, $\boxed{r = 3}$ Ans.

July 6, 2021

Date ____
Page ____

COMBINATIONS

\# Combinations → selection of objects

\# Unlike permutations, here ORDER is not important.

Example : Consider A, B, C - possible combinations taking two at a time

AB , AC , BC

→ Note each combination gives rise to 2! arrangements
→ All combinations give $3 \times 2 = 6$ arrangements.
→ No. of combinations $\times$ 2! = No. of permutations

\# NOTATION AND FORMULA

→ In general, each combination of r objects from n objects can give rise to $r!$ arrangements.

→ The no. of possible combinations of r objects from a collection of n distinct objects is denoted by ${}^nC_r$ and is given by

$$ {}^nC_r = \frac{n!}{r!\,(n-r)!} $$

→ Another common notation is $\binom{n}{r}$ which is also referred to as the binomial coefficient.

Some Useful Results

1. $^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,r!} = {}^nC_{(n-r)}$

In other words, selecting r objects from n objects is the same as rejecting n-r objects.

2. $^nC_n = 1$

3. $^nC_0 = 1$

4. $^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \;;\; \leq r \leq n$

Example : Choosing questions in an exam :

In an examination, a question paper consists of 12 questions divided into two parts i.e, Part I and Part II, containing 7 and 5 questions, resp. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions?

Solution: $^7C_3\,{}^5C_5 + {}^7C_4\,{}^5C_4 + {}^7C_5\,{}^5C_3$

$= 35 + 175 + 210$

$= 420$

---

classmate

Date
Page

Example : GAME OF CARDS

Lets consider the case of choosing four cards from a deck of 52 cards.

Sol. → Total number of ways of choosing four cards from 52 cards $= {}^{52}C_4 = \frac{52!}{4!\,48!} = 2,70,725$

→ All four cards are of the same suit

$${}^{4}C_1 \times {}^{13}C_4 = 4 \times \frac{13!}{4!\;9!} = 2860$$

→ Cards are of same colour $\;{}^{2}C_1 \times {}^{26}C_4 = 2 \times \frac{26!}{4!\,22!}$

$= 2,99,00$

Example : Choosing a cricket team

Select a cricket team of 11 from 17 players in which only 5 players can bowl. The requirement is the cricket team of 11 must include exactly 4 bowlers. How many ways can the selection be done?

Solution. → Total no. of players available for selection = 17
No. of bowlers = 5

→ Need four bowlers = This selection can be done in ${}^{5}C_4$ ways

→ Remaining seven players can be selected from remaining 12 players in $^{12}C_7$ ways

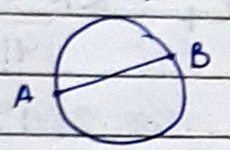
→ Total no. of ways the selection can be done is

$$^5C_4 \times {}^{12}C_7 = 5 \times 792 = 3960 \text{ ways}$$

Example : DRAWING LINES IN A CIRCLE

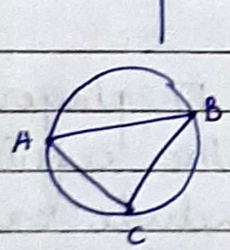
Given n points on a circle, how many lines can be drawn connecting these points?

Sol. → n = 2 points, one line can be drawn connecting the points.



line segment : AB

→ n = 3 points, three lines can be drawn connecting the points



line segments : AB, AC, BC

→ In general, given n points,

No. of line segments that can be drawn connecting the points $= {}^nC_2$

July 6, 2021

Applications Of PnC

Important to distinguish between situations involving combinations and situation involving permutations

PERMUTATIONS : order matters
COMBINATIONS : order does not matter

EXAMPLE 1 → Finishing A Race

consider the situation of 8 athletes participating in a 100 m race in a competition with several rounds.

(1) How many different ways can you award the Gold, Silver & Bronze medals?

(2) How many different ways can you choose the top 3 athletes to proceed to the next round in the competition?

Solution: (1) Order is important
Hence we need permutation

$$\therefore {}^nP_r = {}^8P_3 = \frac{8!}{(8-3)!} = 8 \times 7 \times 6 = 336 \text{ ways}$$

(2) Order is not important
Hence we need combination

$$ {}^{n}C_{r} = {}^{8}C_{3} = \frac{8!}{3!\,5!} = \frac{8 \times 7 \times \cancel{6}^{2}}{\cancel{3} \times \cancel{2}} = 56 \text{ ways} $$

EXAMPLE 2 : Selecting a Team

consider the situation of a class with 40 students.

(1) How many different ways can we choose 2 leaders ?

(2) How many different ways can we choose a captain & a vice captain ?

Solution: (1) Order not important
Hence we need combination.

$$ \therefore {}^{n}C_{r} = {}^{40}C_{2} = \frac{40!}{2!\,38!} = \frac{\cancel{40}^{20} \times 39}{\cancel{2}} = 780 \text{ ways} $$

(2) Order is important
Hence, permutation.

$$ \therefore {}^{n}P_{r} = \frac{n!}{(n-r)!} = \frac{40!}{38!} = 40 \times 39 = 1560 \text{ ways} $$

EXAMPLE 3 : Drawing lines in a circle

Given n points on a circle, how many lines can be drawn connecting these points?

Solution : 1. If the segment has a direction line segment AB is different from BA.
Order is important.
Hence, total no. of ways $= {}^nP_2$

2. If line segment has no direction. Line segment AB is same as BA.
Order is not important.
Hence, total no. of ways is $= {}^nC_2$

Tutorials

1. If all the permutations of the word 'GRAPH' are arranged in dictionary order, then what will be the 73^{rd} word?

2. There a 5 special dishes in a collection of 10 dishes. In how many ways can we have 7 dishes in a sequence such that at least three special dishes are served and special dishes are served consecutively?

3. Choose the correct options:

(a) If a coin is to be tossed 7 times, the no. of outcomes in which utmost 3 heads appears are 64.

(b) If a fair die is rolled thrice, the no. of outcomes of in which the sum of the results is odd is 36.

(c) The no. of ways of selecting at least one Indian and at least 1 American for a debate from a group comprising 3 Indians and 4 Americans is 105.

(d) Two adults and 3 children can sit around a circular table in 12 ways such that the adults always sit together.

classmate

Date

Page

4. There are 4 black dogs and 16 brown dogs are running on ground such that no three dogs are in a straight line. Suddenly 2 of the brown dogs started following one black dog making a st. line, and the remaining 3 black dogs started following one of the remaining brown dogs. How many st lines can be formed passing through the dogs?

5. Find the no. of ways to arrange r people from n people along a circular table considering that seating is same if each person has same neighbours no matter which side he/she is seated?

6. In how many ways we can arrange the 26 letters of the alphabet such that the first letter is a vowel and there are exactly 8 letters between A & B?

Solutions