Turbulence modelling, a computational analysis of the Burgers' equations

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Abstract — This report is developed with the purpose of giving the student a better understanding of what is turbulence modelling and its analysis. A brief introduction about the Burger's equation and the theories behind the models used later on the report. The study consist in the analysis of the Burger's equation in the Fourier space analysing the behaviour of the total energy as a function of the Reynolds Number [Re], the number of modes [N] and finally the effect of these two on the number of iterations of the model. Finally a comparison between LES [Large Eddy Simulation] and DNS [Direct Numerical Simulation] is made in order to analysis when the increase in computational cost is necessary.

1 Introduction

This report consists in an analysis of the energy spectrum or energy cascade E k for each wave number k using different simulations methods. Energy cascade refers to the transfer of energy from large scales of motion to the small scales.

Whereas in the upstream flow, large eddies are enough to describe the behaviour of the flow; in the downstream flow or in the zones in which the flow is interacting with an object small eddies at the Kolmogorov scale are needed [1].

2 Burger's equation [2]

The Brugers equation is a simplification of the Navier-Stokes equations for studies on the field of turbulent and nonlinear dynamics and the Burgers' momentum equation in one dimension can be written as:

$$\delta_t u + u\delta_x u = \frac{1}{Re}\delta_{xx} u + f \tag{1}$$

where δ_t is the time partial derivative, δ_x is the first order spatial partial derivative, δ_{xx} is the second order spatial partial derivative, u is the horizontal velocity, Re is the Reynolds number, and f is the source term.

The analysis continues with the transformation of Burgers equation into a Fourier mode. The transformation of terms from the Burgers equation into Fourier space are the following:

$$\text{Transient: } \sum_{-\infty}^{+\infty} \frac{\partial \hat{u_k}}{\partial t} e^{ikx} \tag{2}$$

$$\text{Diffusive: } \sum_{-\infty}^{+\infty} -k^2 \hat{u_k} e^{ikx} \tag{3}$$

$$\text{External forces: } \sum_{-\infty}^{+\infty} F_k e^{ikx} \tag{4}$$

$$\text{Convective: } \sum_{q=-\infty;p=-\infty}^{q=+\infty;p=+\infty} \hat{u_q} i \hat{u_p} e^{i(p+q)x} \tag{5}$$

Finally the full Fourier equation would look like:

$$\frac{\partial \hat{u_k}}{\partial t} + \sum_{p+q=k} \hat{u_q} i \hat{u_p} e^{i(p+q)x} = -k^2 \hat{u_k} + F_k \tag{6}$$

The spatial discretization is done taking attention into convective term. A given a mode k, the pair of modes which contribute to the convective term are those which satisfy:

$$k = p+q; |p| < N; |q| < N \ ; |p+q| < N \tag{7}$$

Because of the symmetry of the problem, only the positive modes, k ¿ 0, are solved. The integration is performed using a fully explicit Euler scheme with the following time step condition. Moreover a comparison between the Euler and the Adam-Basforth is made later on the report.The coefficient C_1 is between 0 and 1. In this case, it has been tested 0.01. The low the coefficient the littler is the time step used in every iteration.

$$\Delta t = C_1 \frac{Re}{N^2} \tag{8}$$

3 Problem definition

Solve the Burgers' equation for Re = 40. As initial condition it has been $\hat{u}$ k = k 1 . Since mode k = 0 has no interactions with other modes, we assume $\hat{u_0} = 0$ (no mean flow).

Transfer of energy from large scales of motion (low-frequency modes) to the small scales (high-frequency modes). The larger eddies contain most of the kinetic energy, whereas the smallest eddies are responsible for the viscous dissipation of turbulence kinetic energy. Energy spectrum at inertial range can be approximated by Kolmogorov's theory as:

$$E(k) = C_k e^{2/3} k^{-5/3} \tag{9}$$

While, the energy of each mode is computed as follows:

$$E_k = u_k \bar{u_k} \tag{10}$$

3.1 DNS, Direct Numerical Simulation

This approach to the problem solves numerically wihtout any modelling the full solution of the Burgers' equation. This approach is computationally costly but produces better result. According to Kolmogrov the DNS memory requirements grows as $Re^{9/4}$ and the computational cost as $Re^{11/4}$.

3.2 LES, Large Eddy Simulation

In Large Eddy Simulation (LES) the large scales are resolved, but the effects of the smaller ones (sub-grid-scales, for example) must be modelled. The modified Burgers' equation can be written as:

$$\delta_t u + u \delta_x u = (\nu + \nu_t) \delta_{xx} u + f \tag{11}$$

Where ν_t is defined as:

$$\nu_t(\frac{k}{k_n}) = \nu_t^{+\infty}(\frac{E_{kn}}{k_N})^{1/2} \nu_t * * \tag{12}$$

$$\nu_t^{+\infty} = 0.31 \frac{5-m}{m+1} \sqrt{3-m} C_k^{-3/2} \tag{13}$$

$$\nu_t^*(\frac{k}{k_n}) = 1 + 34.5 e^{-3.03 k_n/k} \tag{14}$$

where E_k N is the energy at the cut-off frequency, k_N is the cut-off frequency, ν_t^* is a non-dimensional eddy-viscosity, m is the slope of the energy spectrum, and C_k is

the Kolmogorov constant for 1D Burgers equations which is equal to $C_k \approx 0.4523$. Note that for this project also a value of $C_k = 0.05$ has been used in order to see the effect of this constant on the final solution.

4 Results

This section will show the obtained results for the following analysis:

- Difference between LES and DNS solutions.
- Computational impact of the number of modes and the Re.
- Impact of the Re on the final solution.

4.1 LES vs DNS

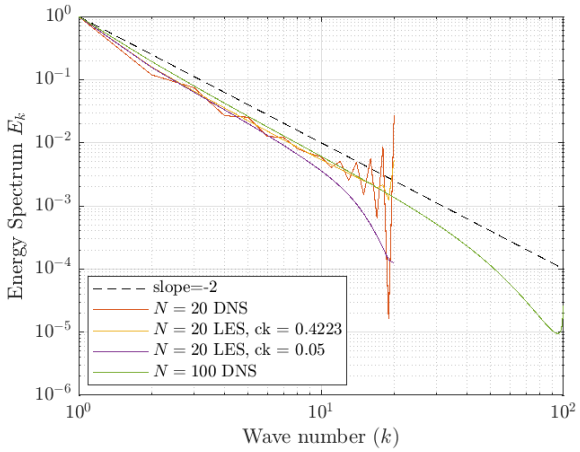

Figure 1: DNS and LES solution for $Re = 40$ and $C_t = 0.01$

The difference between LES and DNS has already been explained, therefore, what it is expected from the solution between the two computational methodologies is that the DNS one should be more accurate than the LES solution.

As it can be seen in (Fig. 1) the DNS solutions for 100 modes shows the expected result where the energy decreases and being bent for smaller scales. It can be noted how the number of modes modifies the final behaviour of the curve. Finally between the LES and DNS approach for N = 20 it can be seen how the DNS modeled solution shows a simmilar approach using a $C_k = 0.4223$, but for a different Kolmogrov constant the accuracy decays for small scales. This is natural as the LES solutions models the small scales, thus if C_k is not accurate enough the solution will lose its accuracy for this last part of the solution.

4.2 Influence of the Reynolds number

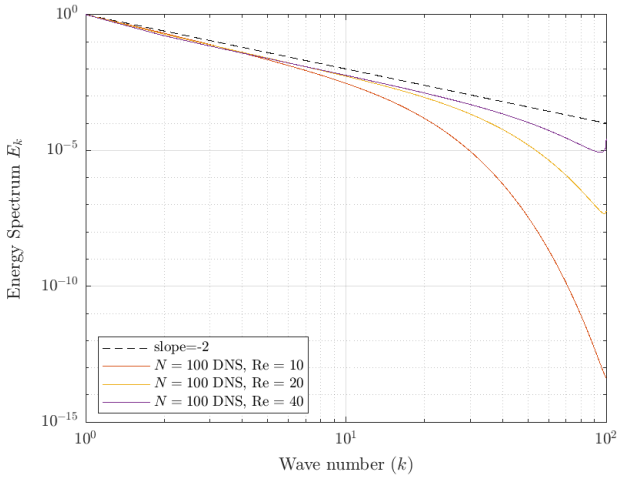

Figure 2: Impact of the Re in the final solution using a DNS solver

The Re has big impact on the solution as it is directly inverse to the viscosity of the fluid. As it can be seen in the (Eq. 11) the viscosity is key within the diffusive impact of the solution. Thus, as the Re numbers grows the diffusive part of the equation will have a lower impact on the final solution.

As it can be seen on the (Fig. 2), the previously described behaviour is shown. It can be seen how for a Re number of 40 the dissipation of energy is lower than for a Re of 10. Notice the dissipation is produces on the low scales area, this makes sense as the

diffusive part of the equation plays a key role in low scales meanwhile for large scales it can almost be neglected.

4.3 Computational impact of the number of modes and the Re

Previously, it has been seen the impact of the number of modes [N] and the Re for the Energy Spectrum. It is also interesting to see how these variables affect the computational time and the number of iterations made by the solver.

4.3.1 Impact of Re and N for DNS

Firstly (Fig. 5) image shows the impact of both the Re and the N using DNS. The simulation was done using a uniform discretization of both the number of modes and the Re using 100 of steps between extremes, giving a mesh resolution of 100^2.

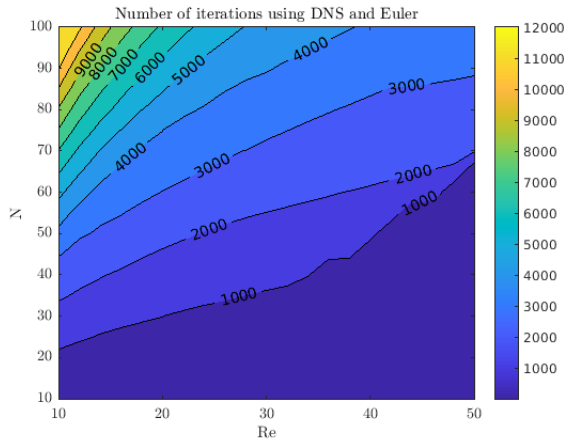

Figure 3: Impact of Re and N on the number of iterations

it is interesting to see how the lowe the Re the faster the increase on the number of iterations. It can be seen how for a Re = 50 and 100 modes the number of iterations in order to convergence is two times lower than for the case of N =100 and Re = 10.

Similar to the number of iterations the time to produce these iterations shows a similar behaviour. This, can be seen in the following picture:

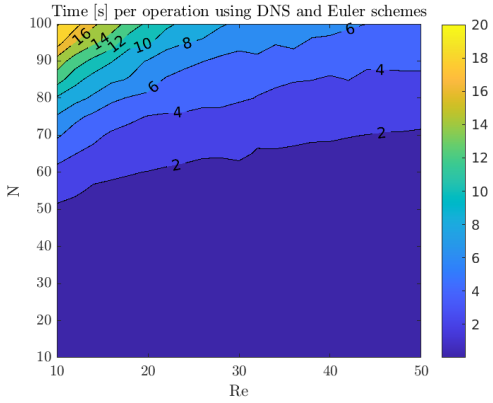

Figure 4: Impact of Re and N on the convergence time

4.3.2 Impact of Re and N for LES

Using the same approach as for the previous case the simulation was done using a LES modelization of $C_k = 0.4223$, a mesh resolution of 100^2 and a $C_t = 0.01$. Similarly to the previous results, the ones obtained using a LES model show the same behaviour.

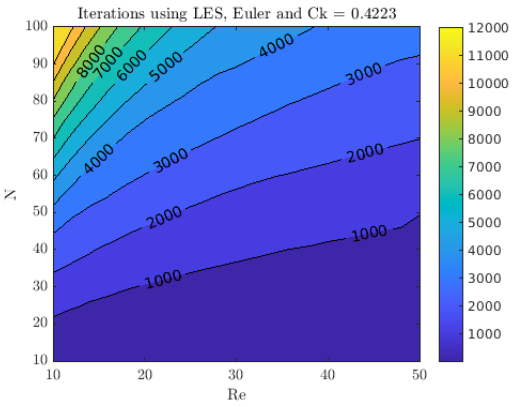

Figure 5: Impact of Re and N on the convergence time

Interestingly, the number of iteration increases slightly on the zones for a higher Re number and slight decreases for lower Re.

This behaviour could possible be due to the addition of ν_t to the diffusive term.

As for the time performance it can be seen a slightly improve for the complete domain. This improvement can be seen on the bellow figure (Fig.6):

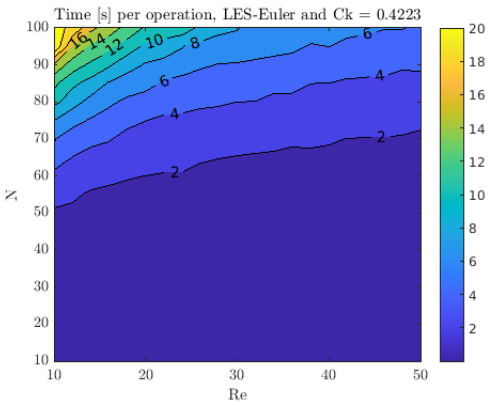

Figure 6: Impact of Re and N on the convergence time

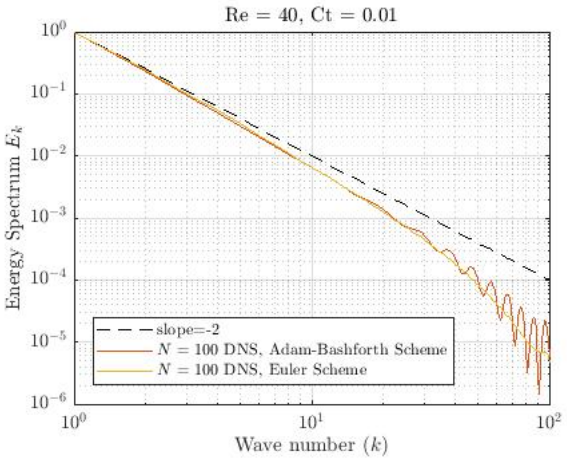

Figure 7: Adam-Bashforth vs Euler time integration schemes

The Adam-Bashroth script has not been validated and there could be an error wihtin the solution. Therefore further development on this part should be made in order to write down stronger conclusions.

4.4 Adam-bashforth vs Euler schemes

Finally a short comparison between the Adam-Bashorth and the Euler integration's schemes has been made. Unexpectedly the resolution of the AB[1] showed a worse behaviour for smaller scales. Even the AB is a second-order time integration scheme, showed a worse resolution in front for the Euler one, which a one-order time integration scheme. Several C_t where tested and only for bigger time steps constants the solution for both schemes showed the same behaviour.

The solution found for both schemes can be found In the following figure:

[1] Adam-Bashorth

5 Code's algorithm

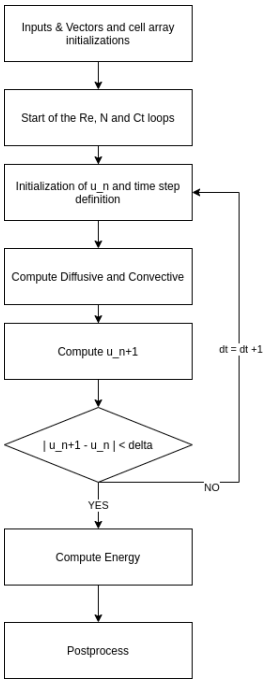

Figure 8: Algorithm of the source code

The algorithm of the developed code can be found on the previous image. It is quite simply as there is no iteration needed within each time step and it is a 1D problem. The source code can be also found in the following Github repository: Burgers1D Solver

References

[1] Frisch, U. *The Legacy of A.N.Kolmogorov.* Cambridge University Press, year =.

[2] TECNIO. Burgers equation in fourier space. https://atenea.upc.edu/pluginfile.php/3551423/mod_resource/content/1/Burgers.pdf, 2008. Accessed: 2020-28-10.