Kepler's equation analysis using Euler and Newton-Raphson iteration methods

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Abstract — This report is developed with the purpose of giving the lecturer a better understanding of the Keplers' equation resolution using different iterative methodologies. The body of the project briefly explains both Euler and Newton rapson methodologies and their equations. Finally an analysis of three different initial conditions is made for both methodologies and their computational cost.

1 Keplers'Equation

For ellipse orbits typically we have the **Mean Anomaly M** instead of the True Anomaly θ. In order to solve Kepler equation we must deal with this particular equation: [1]

$$M = E - e * sin(E) \tag{1}$$

Also called transcendental equation since since is not trivial to get **E** as a function of **e** and **M**. Hence in order to solve this equation we must apply a iterative methods.
The first iterative method that we are asked to use is Newton-Raphson method. And the second one we will use has been chosen by us and it is Euler iterative method. [3]

- Therefore for Newton-Raphson method we make use of the fist derivative: [2]

$$f'(x_i) = \frac{0 - f(x_i)}{x_{i+1} - x_i} \tag{2}$$

$$f(E) = E - e * sin(E) - M \tag{3}$$

$$f'(E) = 1 - e * cos(E) \tag{4}$$

Substituting we get new value of **E** when $E_o = M$ is updated by:

$$E_{i+1} = E_i - \frac{E_i - e * sin(E_i) - M}{1 - e * cos(E_i)} \tag{5}$$

- Meanwhile for the Euler method the update of value **E** will be:

$$E_{i+1} = M + e * sin(E_i) \tag{6}$$

To solve this equation by these methods we must give an initial guess for **Eccentric Anomaly** and a convergence criteria, the initial values of our problem are:

1. $E_o = M$
2. $E_o = \pi$
3. $E_o = M + e * cos(M)$

Depending on what is the initial guess the method will converge or diverge affecting to its performance.

Consequently we will study how many iterations each method needs to converge for a given **M** depending on **e** for each initial guess.

2 Results

In this section we will analyse how many iterations the program need to converge for any **mean anomaly M** from 0 to 360 degrees depending on the **eccentricity** from 0 to 1.

Algorithm parameters		
\textbf{delta}	10e-6	% Maximum admisible error
\textbf{N}	1000	% Discretization parameter
\textbf{maxIter}	20	% Maximum iterations

Table 1: Input parameters used for simulator.

2.1 Case 1, initial guess: $E_0 = M$

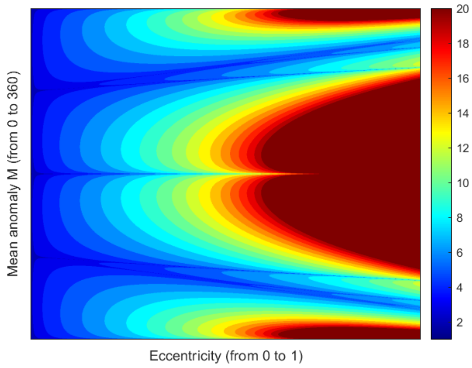

Figure 1: Euler method case1.

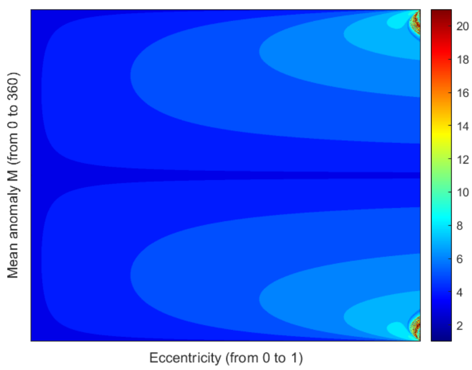

Figure 2: Newton-Raphson method case1.

For both solvers we can see that regions are symmetric with respect to **mean anomaly M** and the number of iterations increase with the **eccentricity**.Watching that with Euler method the equations need as an average three times more iterations to converge. Convertging at the 20th iteration in the Newton-Raphson method when **M** value is 330º and 30º with a eccentricity bigger than 0.98.

2.2 Case 2, initial guess: $E_0 = \pi$

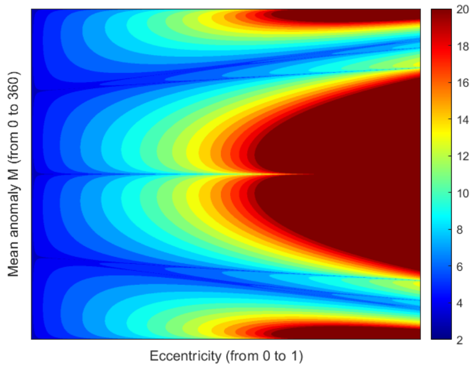

Figure 3: Euler method case2.

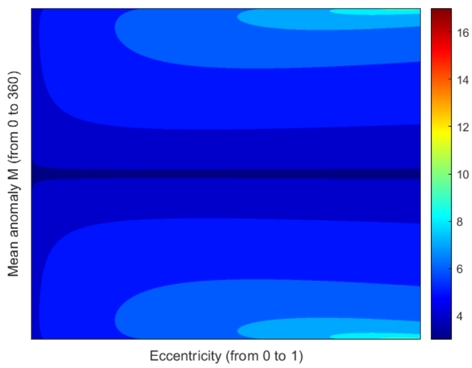

Figure 4: Newton-Raphson method case2.

For the initial guess of $E_0 = \pi$ the results are quite similar to the case when $E_0 = M$ with the difference that in this case Newton-Raphson method converge faster.

2.3 Case 3, initial guess: $E_0 = E + e * cos(E)$

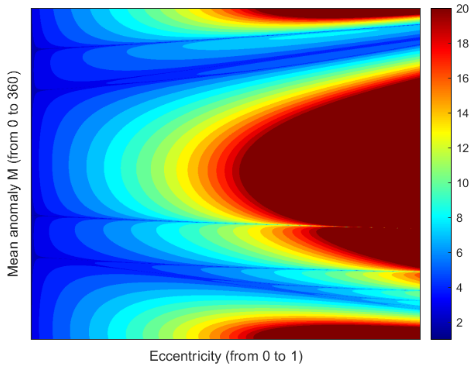

Figure 5: Euler method case3.

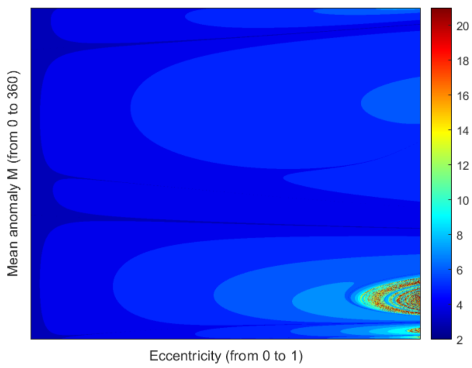

Figure 6: Newton-Raphson method case3.

In this case results are not symmetric with respect **M**. Needing Newton-Raphson method almost 20 iteration to converge when **M** is 60º and eccentricity bigger than 0.85.

2.4 Algorithm execution time analysis

Profile Summary	Calls	Execution time
Main	1	422.17 s
Euler Solver	3000000	60.94 s
Newton-Raphson solver	3000000	29.04 s

Table 2: Execution time summary.

So Euler solver takes 60.94 s as total execution time by being called 3000000 what is 0.2ms meanwhile Newton-Raphson need as an average 0.0096ms per call. After the analysis of the algorithm performance we can conclude that in terms of execution performance the most optimum solver is Newton-Raphson since is twice faster than Euler solver.The speed of execution of each method is determined by the number of iterations they need to do to converge the equations.

2.5 Conclusion

To conclude the report we must define Newton-Raphson as a faster iteration method since it needs less iteration to converge. And depending on the initial guess our algorithm will give a better or worse performance by converging faster or even diverging.

It is also interesting to notice how the NW[1] algorithm gives a better solution regardless the initial conditions. As it has been seen the NW converges faster and for a wider range of eccentricities.

As the Euler method only takes into account the equation itself it wasn't expected to see high changes depending on the initial conditions. Interestingly as the NW takes into account the derivative of the solution in order to iterate the process is much faster and can be clearly seen the differences between the three different initial conditions.

3 Source code

The source code of the project can be found in the following Github repository: Kepler's Solver

References

[1] Curtis, H. D. Chapter 3 - orbital position as a function of time.

[1] Newton-Rapson

In *Orbital Mechanics for Engineering Students (Third Edition)*, H. D. Curtis, Ed., third edition ed. Butterworth-Heinemann, Boston, 2014, pp. 145 – 186.

[2] Danchick, R. Gauss meets newton again: How to make gauss orbit determination from two position vectors more efficient and robust with newton–raphson iterations., 2008.

[3] Wanner, G. Kepler, newton and numerical analysis. *Acta Numerica 19* (05 2010), 561–598.

A Theoretical approach - Basics of the problem

Kepler's Equation allows to know the position of an object in an orbit for a given time by relating this time with an angular position in a two-body Keplerian orbit. [1]
Starting with orbit equation from first law of Kepler:

$$r = \frac{h^2}{\mu}\frac{1}{1+e*cos(\theta)} \tag{7}$$

This equation describes the path of the body m_2 around m_1 relative to m_1. Being 'e': Eccentricity, 'h': Angular momentum,'μ': Gravitational parameter,'θ': True anomaly.
Therefore time since periapsis:

$$h = r * v_{\perp} = r * (r * \dot{\theta}) \tag{8}$$

So we can relate time and angular position:

$$h = \frac{h^4}{\mu^2}\frac{1}{(1+e*cos(\theta))^2}\frac{d\theta}{dt} \tag{9}$$

$$\frac{\mu^2}{h^3}\int_{tp=0}^{t} dt = \int_0^{\theta}\frac{dx}{(1+e*cos(x))^2} \tag{10}$$

⇒Thus for circular orbits (e=0):

$$\frac{\mu^2}{h^3}t = \theta \tag{11}$$

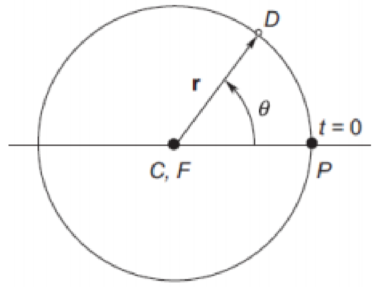

Figure 7: Circular orbit

⇒ For elliptical orbit (0 < e < 1):
By second Kepler law we see that the period of the orbit only depend on the semi-major axis, describing third Kepler law:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{12}$$

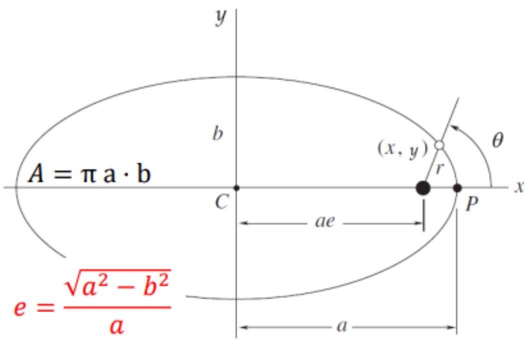

Figure 8: Elliptical orbit

Developing all these equations we get:

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{\frac{3}{2}}}M \tag{13}$$

Where M is the **Mean anomaly M** that is related with the orbital period T as:

$$M = \frac{2\pi}{T}t = nt \tag{14}$$

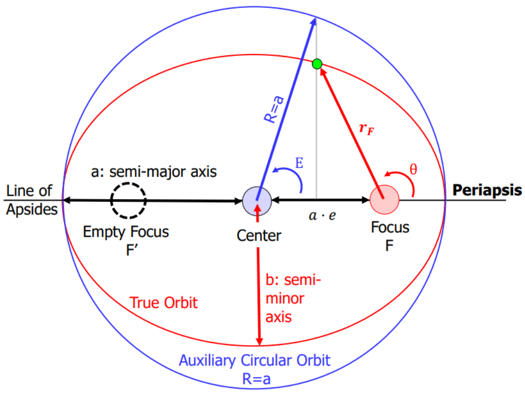

Figure 9: Orbital Anomalies

From this schema we see the orbital parameters where E is the **Eccentric anomaly E** and is related with the **Mean anomaly M** by using Kepler equation:

$$M = E - sin(E) \tag{15}$$

And θ is the **True anomaly** θ that can be related with **eccentric anomaly E** by:

$$tan(\frac{E}{2}) = \sqrt{\frac{1-e}{1+e}} tan(\frac{\theta}{2}) \tag{16}$$