What Practitioners Need to Know . . .

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. . . About Estimating Volatility
Part 1

Volatility is important to financial analysts for several reasons. Perhaps most obvious, estimates of volatility, together with information about central tendency, allow us to assess the likelihood of experiencing a particular outcome. For example, we may be interested in the likelihood of achieving a certain level of wealth by a particular date, depending on our choice of alternative investment strategies. In order to assess the likelihood of achieving such an objective, we must estimate the volatility of returns for each of the alternative investment strategies.

Financial analysts are often faced with the task of combining various risky assets to form efficient portfolios—portfolios that offer the highest expected return at a particular level of risk.[1] Again, it is necessary to estimate the volatility of the component assets. Also, the valuation of an option requires us to estimate the volatility of the underlying asset. These are but a few examples of how volatility estimates are used in financial analysis.

Historical Volatility

The most commonly used measure of volatility in financial analysis is standard deviation. Standard deviation is computed by measuring the difference between the value of each observation in a sample and the sample's mean, squaring each difference, taking the average of the squares and then determining the square root of this average.

Suppose, for example, that during a particular month we observe the daily returns shown in column 1 in Table I. The average of the returns in column 1 equals 0.28 per cent. Column 2 shows the difference between each observed return and this average return. Column 3 shows the squared values of these differences. The average of the squared differences—0.0167 per cent—equals the variance of the returns. (In computing the variance, we divide by the number of observations less one, because we used up one degree of freedom to compute the average of the returns.) The square root of the variance—1.2904 per cent—equals the standard deviation of the daily returns.

1. Footnotes appear at end of article.

Table I Standard Deviations of Return

Day	*1* *Return (%)*	*2* *Return − Average (%)*	*3* *Squared Difference (%)*
1	1.00	0.72	0.0052
2	1.50	1.22	0.0149
3	2.10	1.82	0.0332
4	−0.40	−0.68	0.0046
5	1.00	0.72	0.0052
6	−1.40	−1.68	0.0281
7	0.45	0.17	0.0003
8	−0.75	−1.03	0.0106
9	1.00	0.72	0.0052
10	1.40	1.12	0.0126
11	−2.00	−2.28	0.0519
12	1.00	0.72	0.0052
13	−1.50	−1.78	0.0316
14	0.35	0.07	0.0001
15	−0.30	−0.58	0.0033
16	1.00	0.72	0.0052
17	0.00	−0.28	0.0008
18	−0.60	−0.88	0.0077
19	−1.20	−1.48	0.0218
20	2.90	2.62	0.0688
Average	0.28		0.0167
Square Root			1.2904

In this example, the standard deviation measures the volatility of daily returns. It is typical in financial analysis to annualize the standard deviation. Unlike rates of return, which increase proportionately with time, standard deviations increase with the square root of time. We can take two approaches to converting a daily standard deviation into an annual standard deviation.

We can reconvert the standard deviation back into a variance by squaring it. Then we can multiply the variance by 260 (the number of trading days in a year) and take the square root of this value to get the annualized standard deviation:

$$0.012904^2 = 0.000167$$

$$0.000167 \cdot 260 = 0.0433$$

$$\sqrt{0.0433} = 0.2081.$$

Table II Standard Deviations of Logarithmic Returns

Day	1 *Log(1 + Return) (%)*	2 *Log − Average (%)*	3 *Squared Difference (%)*
1	1.00	0.73	0.0053
2	1.49	1.22	0.0149
3	2.08	1.81	0.0327
4	−0.40	−0.67	0.0045
5	1.00	0.73	0.0053
6	−1.41	−1.68	0.0282
7	0.45	0.18	0.0003
8	−0.75	−1.02	0.0104
9	1.00	0.73	0.0053
10	1.39	1.12	0.0126
11	−2.02	−2.29	0.0524
12	1.00	0.73	0.0053
13	−1.51	−1.78	0.0317
14	0.35	0.08	0.0001
15	−0.30	−0.57	0.0032
16	1.00	0.73	0.0053
17	0.00	−0.27	0.0007
18	−0.60	−0.87	0.0076
19	−1.21	−1.48	0.0218
20	2.86	2.59	0.0671
Average (%)	0.27		0.0166
Square Root (%)			1.2867
Annualized Standard Deviation (%)			20.75

Alternatively, we can multiply the daily standard deviation by the square root of 260 to determine the annualized value. This is algebraically equivalent to the first approach:

$$\sqrt{260} = 16.1245$$

$$0.012904 \cdot 16.1245 = 0.2081.$$

The approach we have just described for estimating the standard deviation from historical returns yields an estimate that is approximately correct, but not exact. The inexactitude arises because the dispersion of investment returns conforms to a lognormal distribution, rather than a normal distribution, owing to the effect of compounding. We can attain a more precise estimate of standard deviation by calculating the logarithms of one plus the returns, squaring the differences of these values from their average, and taking the square root of the average of the squared differences. Table II shows these calculations.

Comparing the standard deviation from Table I with the standard deviation from Table II, we see that it does not make much difference which approach we use given this short measurement interval. As the measurement interval lengthens, however, the distinction becomes more important.

Implied Volatility

As we have seen, estimating volatility from historical data is fairly straightforward. Unfortunately, the result may not be the best estimate if volatility is unstable through time. In the fall of 1979, for example, the Federal Reserve changed its operating policy with respect to its management of the money supply and interest rates. Over the 10 years ending in 1978, the annualized standard deviation for long-term corporate bonds was a little less than 8 per cent. In the subsequent 10-year period, from 1979 through 1988, the annualized standard deviation for long-term corporate bonds rose to more than 13 per cent. Clearly, historical precedent made a poor guide for estimating bond market volatility in the 1980s.

As an alternative to historical data, we can infer the investment community's consensus outlook for the volatilities of many assets by examining the prices at which options on these assets trade. These implied volatilities presumably reflect all current information that impinges on an asset's volatility.

The value of an option depends on five factors—the current price of the underlying asset; the strike price, or price at which the option can be exercised; the time remaining until expiration; the riskless rate of interest; and the volatility of the underlying asset. We know the strike price and the time remaining until expiration from the terms of the option contract. The price of the underlying asset and the riskless rate of interest can be determined from a variety of market quote services. The only factor that we do not know with certainty is the volatility of the underlying asset. In order to determine volatility, we can substitute various values into the option pricing formula until the solution to this formula equals the price at which the option is trading.

Consider a call option with 90 days to expiration and a strike price of \$295.00, written on an underlying asset currently priced at \$300.00. Suppose the annualized riskless rate of interest is 8 per cent and the option trades for \$15.00. In order to determine the standard deviation, or implied volatility, consistent with the price of this option, we start by assuming some value for volatility and use this value in the following Black-Scholes option-pricing formula:

$$C = S \cdot N(D) - Xe^{-rT} \cdot N(D - \sigma\sqrt{T}) \tag{1}$$

where

- S = price of underlying asset,
- X = strike price,
- T = time remaining until expiration,
- r = instantaneous riskless rate of interest, $\ln(1 + r)$,[2]
- D = $(\ln(S/X) + (r + \sigma^2/2) \cdot T)/(\sigma \cdot \sqrt{T})$
- $\ln(\ )$ = natural log,
- σ = standard deviation (volatility) of underlying asset and
- $N(\ )$ = cumulative normal distribution function.[3]

Suppose we use the historical volatility over the previous 90 days—say, 20 per cent. By substituting this value and the values we assigned earlier into Equation (1), we find that D equals 0.401. Thus C, the option value, is calculated as:

$$C = 300 \cdot N(0.401) - 295\, e^{-0.077(90/365)} \cdot N(0.401 - 0.2 \cdot 90/365)$$

$$C = 17.69.$$

This estimated value is greater than the price at which the option currently trades. We must therefore lower our estimate for volatility. Suppose we next try a value of 12 per cent. Based on this volatility assumption, the option value equals $13.50; this is less than the actual price of $15.00, implying that we should raise our estimate.

If we continue substituting various volatility values into the Black-Scholes formula, we will eventually discover that a volatility estimate of 14.96 per cent is consistent with an option value of $15.00—the price at which the option is currently trading. This 14.96 per cent is the implied volatility, given the current values for the underlying asset, the option and the riskless rate and given the terms of the option contract.

Newton-Raphson Method

Of course, as we are solving for the implied volatility, the prices of the underlying asset and the option may be changing. We need a reasonably quick way to arrive at the implied volatility. The Newton-Raphson Method is one way.

According to the Newton-Raphson Method, we start with some reasonable estimate for volatility and evaluate the option using this estimate. Unless we are unusually lucky, however, we will not arrive at the correct value for implied volatility on our first try. We therefore revise our initial volatility estimate by subtracting an amount equal to the estimated option value minus the option's actual price, divided by the derivative of the option formula with respect to volatility evaluated at our estimate for volatility. This derivative is shown below:

$$\partial C/\partial \sigma = S \cdot \sqrt{T} \cdot (1/\sqrt{2\pi})\, e^{-D^2/2} \tag{2}$$

where

$\pi = 3.1416$ and
$e = 2.7183$.

C, S, T and D are as defined earlier.

Our earlier example used an assumed volatility of 20 per cent. Using this assumption, the derivative for the Newton-Raphson Method is:

$$\partial C/\partial \sigma = 300 \cdot \sqrt{90/365} \cdot (1/\sqrt{2\pi}) e^{-0.401^2/2},$$

$$\partial C/\partial \sigma = 54.64.$$

The volatility of 20 per cent resulted in an option value of $17.69. We compute the Newton-Raphson volatility estimate as follows:

$$\text{N-R Estimate} = 0.20 - (17.69 - 15)/54.64$$

$$= 0.1507.$$

A volatility estimate of 15.07 per cent results in an option value of $15.06. One more iteration yields a volatility estimate of 14.96 per cent, for an option price of $15.00.

Method of Bisection

The efficiency of the Newton-Raphson Method depends to a certain extent on the choice of the initial volatility estimate. An alternative search procedure, which tends to be less sensitive to the initial volatility estimate, is called the Method of Bisection.[4] This approach is more intuitive. We start by choosing a low estimate for volatility corresponding to a low option value and a high estimate for volatility corresponding to a high option value.

For example, suppose we start with a low estimate of 10 per cent and a high estimate of 30 per cent. Given the assumptions from our previous example, the corresponding option values are $12.56 and $23.27. Our next estimate for volatility is found by interpolation, as shown below:

$$\text{New Estimate} = \sigma_L + (C - C_L) \cdot (\sigma_H - \sigma_L)/(C_H - C_L), \tag{3}$$

$$0.1456 = 0.1 + (15 - 12.56) \cdot (0.3 - 0.1)/(23.27 - 12.56),$$

$$0.1494 = 0.1456 + (15 - 14.79) \cdot (0.3 - 0.1456)/(23.27 - 14.79),$$

$$0.1496 = 0.1494 + (15 - 14.99) \cdot (0.3 - 0.1494)/(23.27 - 14.99).$$

If the option value corresponding to our interpolated estimate for volatility is below the actual option price, we replace our low volatility estimate with the interpolated estimate and repeat the calculation. If the estimated option value is above the actual option price, we replace the high volatility estimate with the interpolated estimate and proceed accordingly. When the option value corresponding to our volatility estimate equals the actual price of the option, we have arrived at the implied volatility for that option.

Historical vs. Implied Volatility

Is it better to estimate volatility from historical observations or to infer it from the prices at which options trade? The answer, of course, depends on the quality of the inputs. If volatility is stationary through time, and we have reliable prices from which to estimate returns, then historical volatility is a reasonably good indicator of subsequent volatility. Unfortunately, and especially over short measurement intervals, nonrecurring events or conditions often cause volatility to shift up or down temporarily, so that historical volatility will over or underestimate subsequent volatility. To the extent that the investment community recognizes the transitory nature of these nonrecurring events, implied volatility may provide a superior estimate of subsequent volatility. In estimating implied volatility, however, we must use contemporaneous observations for the inputs to the Black-Scholes formula.

The following procedure provides an intuitively appealing test of whether historical or implied volatility is a superior forecaster of actual volatility. First, estimate historical volatility in periods one through n. Then estimate implied volatility as of the end of periods one through n. Next estimate actual volatility in periods two through n plus one. Two simple linear regressions can then be performed—one in which the independent variable is the historical volatility and the dependent variable is the actual volatility in the subsequent period, and the other in which the independent variable is the implied volatility and the dependent variable is the actual volatility in the subsequent period.

It may be tempting to conclude that the regression equation with the higher R-squared reveals the better predictor, but this conclusion may very well be wrong. The regression equation with implied volatility as the independent variable may have a higher R-squared, but the slope of the regression line may be significantly greater or less than one, or the intercept may be significantly greater or less than zero. In these cases, the forecasts would be biased, although the R-squared alone would not reveal this. The regression equation with historical volatility as the independent variable might have the weaker R-squared but a slope and intercept closer to one and zero, respectively. In this contrived example, historical volatility may be the better predictor of subsequent volatility, even though it has a lower R-squared than implied volatility.[5]

An alternative method of comparison would be to examine the tracking errors associated with historical and implied volatilities. Tracking error is computed by squaring the differences between actual values and historical or implied values, taking the average of these differences, and then calculating the square root of the average.

A relatively recent innovation for estimating volatility uses a technique known as ARCH, an acronym for Autoregressive Conditional Heteroscedasticity.[6] Essentially, ARCH and related models incorporate the time-series dynamics of past volatility to forecast subsequent volatility. We will introduce ARCH models in Part 2 of "What Practitioners Need to Know About Estimating Volatility."

Footnotes

1. See, for example, M. Kritzman, "What Practitioners Need to Know: The Nobel Prize," *Financial Analysts Journal*, January/February 1991.
2. To be precise, we should use the continuously compounded riskless rate of interest. Thus, if the rate is quoted as simple interest, we should use the natural log of one plus the interest rate.
3. For a discussion of the cumulative normal distribution function, see M. Kritzman, "What Practitioners Need to Know About Uncertainty," *Financial Analysts Journal*, March/April 1991.
4. For an excellent review of this approach, see S. Brown, "Estimating Volatility," in Figlewski, Silber and Subrahmanyam, eds., *Financial Options: From Theory to Practice* (Homewood, IL: Business One Irwin, 1990).
5. For further elucidation, see L. Canina and S. Figlewski, "The Information Content of Implied Volatility" (Working paper number S-91-20, Salomon Center, New York University).
6. See R. Engle, "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation," Econometrica 17, pp. 5–26.

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from about 10 to 18—is unlikely to be the key factor determining the attractiveness of the stock market. In this area, contrarian logic has demonstrated little value, and factors other than P/Es are likely to provide the clue to the outlook for stock prices. In contrast, P/Es at the extremes—over 20 and under 8—are much more difficult to ignore. Very low or very high P/Es—although they do not guarantee the outcomes shown in Figure C—suggest that the contrarian argument be accorded considerable weight.