. . . About Monte Carlo Simulation

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As financial analysts, we are often required to anticipate the future. Monte Carlo simulation is a numerical technique that allows us to experience the future with the aid of a computer. This column reviews some of the important concepts that are relevant to Monte Carlo simulation and describes the procedures one would undertake to perform such a simulation.

Deterministic vs. Stochastic Models

In order to forecast the future, we build models that define the relationship between a set of inputs and a description of the future. Models that assume a fixed relationship between the inputs and the output are called *deterministic;* the inputs lead unambiguously to the answer. Models that depend on inputs that are influenced by chance or estimated with uncertainty are called *stochastic*. Stochastic models do not yield unambiguous solutions; instead, they provide a *distribution* of probable answers.

A model that predicts an eclipse, for example, is deterministic, because it relies on known fixed laws governing the motions of the earth, the moon and the sun. You are unlikely to hear an astronomer say there is a 30% chance of an eclipse next Wednesday. A model that predicts tomorrow's weather, however, is stochastic, because many uncertain elements influence the weather.

A variable that is influenced by chance is called a random variable. In a deterministic model, we assign a single value to a variable; in a stochastic model, we assign a distribution of probable values to a random variable. We often represent the random variable's distribution by its mean and standard deviation.

We solve deterministic models analytically, which simply means that we represent the model with a mathematical formula and then solve the formula. Many stochastic models can also be solved analytically. It is sometimes inefficient or not possible to solve a stochastic model analytically, however. In such a case, we must resort to a numerical solution.

To solve a model numerically, we try out various values for the model's parameters and variables. When the values we use come from a succession of random numbers, the numerical solution is called a Monte Carlo simulation.

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The term Monte Carlo simulation was introduced by John von Neumann and Stanislaw Ulam when they both worked on the Manhattan Project at the Los Alamos National Laboratory. Ulam invented the procedure of substituting a sequence of random numbers into equations to solve problems relating to the physics of nuclear explosions. The two of them used the term as a code name for the secret work they were conducting.[1] The choice of the name was obviously inspired by the gambling casinos in Monte Carlo.

Random Numbers

In order to perform a Monte Carlo simulation, we need access to a sequence of numbers that are distributed uniformly and are independent of each other. Ideally, the numbers should be random. Unfortunately, truly random numbers are almost impossible to obtain except by some mechanical process that is typically costly or time-consuming. We could generate random numbers by rolling dice and recording the results, for example, but we would probably tire long before we had a sufficiently long sequence of numbers. Moreover, the dice might have a slight, undetectable bias.

As an alternative to mechanically generated random numbers, we can use mathematical techniques to generate pseudorandom numbers. We might start by squaring the last four digits of our phone number. Then we could extract the middle four digits and proceed by squaring the middle four digits we just extracted and extracting the middle four digits from the new squared value. We could continue in this manner for as long as necessary or until the middle four digits contain a sequence of zeros that terminates the process.

This approach, invented by John von Neumann, is called the mid-square method.[2] The sequence of numbers it generates are called pseudorandom numbers because they come from a purely deterministic process. As long as they are uniform and independent, however, they will suffice for most Monte Carlo simulations. By uniform, we mean that all the numbers have an equal chance of occurrence. Independence implies that the numbers are unrelated to one another.

There are many methods for generating pseudorandom numbers that are independent and uniformly distributed. Most statistics packages and spreadsheet software include random-number generators.

In many financial analysis applications, the random variables of interest are not distributed uniformly; rather, they conform to some other distribution. In order to perform a Monte Carlo simulation of a model in which the random variables are normally distributed, we must transform our sequence of uniformly distributed random numbers into a sequence of normally distributed random numbers.

Monte Carlo Simulation of the Stock Market

Suppose we allocate $100,000 to an S&P 500 index fund in which the dividends are reinvested, and we wish to forecast the value of our investment 10 years from now. We can start by generating a series of 10 random numbers that are uniformly distributed within the range of 0 to 1. We assume, however, that the S&P's returns are normally distributed.[3] Hence we need to transform our uniform sequence into a random sequence that is normally distributed.[4]

There is a convenient way to accomplish this transformation. According to the Central Limit Theorem, if we sum or average a group of independent random variables, which themselves are not normally distributed, the sum or average will be normally distributed if the group is sufficiently large. Most advanced statistics books contain a proof of this important theorem. For those who do not require a formal proof, I will provide a simple intuitive demonstration.

Suppose the value of a random variable, X, is determined by the toss of a fair die. There is a one-in-six chance that X will take on the value 1, a one-in-six chance that it will take on the value 2, and the same chance that it will take on the value 3, 4, 5 or 6. Clearly, X is uniformly distributed because there is an equal probability of experiencing each outcome, and its values are independent because they are determined purely by chance. Now consider a second random variable, Y, whose value is determined by the toss of another die. Y has the same uniform distribution as X, although their outcomes are independent of one another.

Now consider a third random variable—the average of the values of X and Y. The distribution of the third random variable, called (X + Y)/2, is not uniform. There is a greater likelihood it will take on a value close to 3 or 4 than one close to 1 or 6. The obvious reason is that there are more ways to experience a 3 or a 4 than there are a 1 or a 6. Table 1 shows the relative frequencies of the values for each of the three random variables.

The relative frequencies for the random variable (X + Y)/2 are computed by summing the probabilities for all the combinations of X and Y that yield the relevant average value. For example, there is a 5/36 chance that the random variable (X + Y)/2 will take on the value 3, because there are five combinations of X and Y whose average equals 3, each of which has a 1/36 chance of occurrence: X = 1 and Y = 5; X = 5 and Y = 1; X = 2 and Y = 4; X = 4 and Y = 2; and X = 3 and Y = 3.

It is easy to see from Table 1 that although neither X nor Y individually is normally distributed, their sum or average begins to approach a normal distribution. This result suggests that we can generate a sequence of random numbers that is normally distributed simply by averaging together many sequences of random numbers that are uniformly distributed.

Table 1. Demonstration of the Central Limit Theorem

	Relative Frequencies		
Value	X	Y	(X + Y)/2
1.0	1/6	1/6	1/36
1.5	0	0	1/18
2.0	1/6	1/6	1/12
2.5	0	0	1/9
3.0	1/6	1/6	5/36
3.5	0	0	1/6
4.0	1/6	1/6	5/36
4.5	0	0	1/9
5.0	1/6	1/6	1/12
5.5	0	0	1/18
6.0	1/6	1/6	1/36

Table 2 shows two sequences of random numbers. The first sequence is uniformly distributed between the values 0 and 1. The second sequence is the average of 30 uniformly distributed sequences.[5] Figure A plots the relative frequencies of both sequences. It is easy to see that the average of the 30 uniformly distributed sequences approaches a normal distribution.

Table 2. Random Numbers

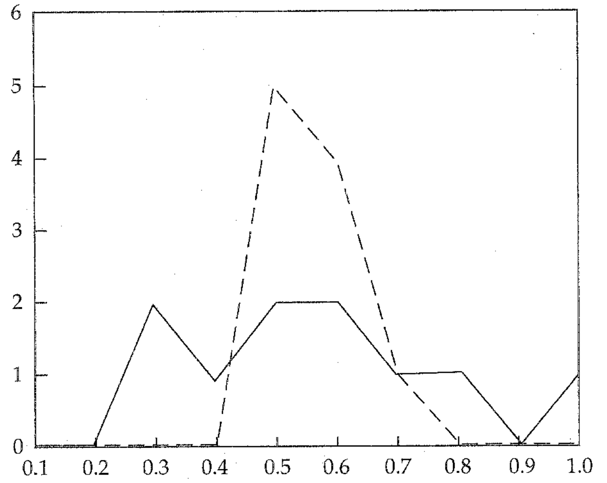
Uniform Sequence	Average of 30 Uniform Sequences
.6471	.4965
.4162	.4336
.5691	.5747
.2006	.4477
.4685	.5014
.7442	.5126
.9439	.4930
.5556	.6040
.2480	.5267
.3644	.4721

Although we now have a sequence of random numbers that is normally distributed, we are not yet ready to simulate the performance of our investment in the S&P 500 index fund. The average value of our normally distributed sequence equals 0.5062 and its standard deviation equals 0.0499. Our assumptions about the S&P's average return and standard deviation are likely to be quite different.

In order to proceed, we should scale our normally distributed sequence so that it has a mean of 0 and a standard deviation of 1. We can carry out this transformation by dividing each observation by 0.05, the theoretical standard deviation, and then subtracting 10, the theoretical mean of this new sequence, from each of its observations. Table 3 shows this transformation.

Next we need to rescale our sequence to reflect our views about the mean return and standard deviation of

Figure A. Relative Frequencies



the S&P 500. Suppose we believe that the average return of the S&P 500 is 12% and that its standard deviation is 20%. We rescale our standardized normal sequence by multiplying each observation by our assumption for the S&P's standard deviation and adding to this value our assumption for its average return. Finally, we have a sequence of returns that we can use to simulate our investment's performance. Table 4 gives this sequence.

In order to carry out the simulation, we simply link the sequence of random returns and multiply this result by 100,000 to derive an estimate of our investment's value 10 years forward. The simulated returns in Table 4, for example, yield a terminal value of $333,810.

We repeat the entire process beginning with the step in which we generate and average the 30 random sequences. We proceed in this fashion until we generate a sufficiently large number of estimates. The distribution of these estimates is the solution to our problem. Figure B illustrates a frequency distribution of the terminal value of $100,000 10 years hence resulting from 100 simulations performed as I have just described.

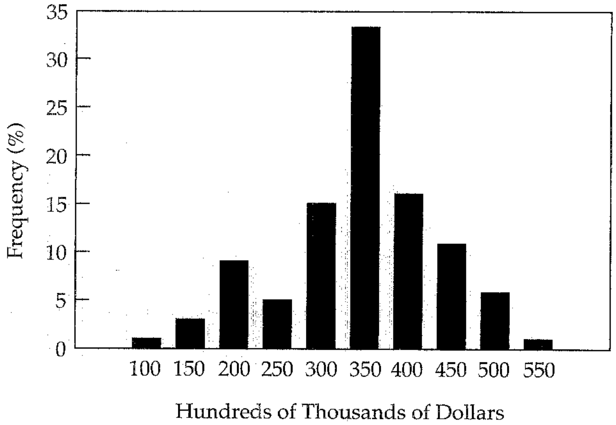
Table 3. Standardizing a Normally Distributed Sequence

	A Normally Distributed Sequence	B A/0.05	C B − 10
	.4965	9.9300	−0.0700
	.4336	8.6720	−1.3280
	.5747	11.4940	1.4940
	.4477	8.9540	−1.0460
	.5014	10.0280	−0.0280
	.5126	10.2520	0.2520
	.4930	9.8600	−0.1400
	.6040	12.0800	2.0800
	.5267	10.5340	0.5340
	.4721	9.4420	−0.5580
Average	.5062	10.1246	0.1246
Standard Deviation	.0499	0.9937	0.9973

Table 4. Normally Distributed Random Sequence of S&P 500 Returns

.1060	.1704
−.1456	.0920
.4188	.5360
−.0892	.2268
.1256	−.0084
Average	14.49%
Standard Deviation	19.95%

Figure B. Frequency Distribution for $100,000 10 Years Forward Based on 100 Simulations



From this information, we can estimate the likelihood that our fund's value will equal or exceed various wealth goals. We simply count the relative frequency of outcomes above or below the value we are interested in.

Simulation vs. Analytical Solution

The example I have just described is intended to illustrate the steps one would take to perform a Monte Carlo simulation. There is nothing about this particular problem that prevents us from solving it analytically. Given the assumptions that the S&P's returns are random and normally distributed, we can simply calculate the normal deviate associated with alternative values for terminal wealth and find the corresponding probability of achieving that value in a normal distribution table.[6]

There are many problems, however, that are more amenable to a Monte Carlo simulation than to an analytical solution. Suppose, for example, that we plan to contribute one-half of our investment's return each year to our favorite charity. It is considerably easier to simulate our investment's growth after accounting for these conditional disbursements than it would be to devise an analytical solution. We simply reduce our fund's value each year in each simulation by one-half of the fund's randomly generated return. Try to figure out an analytical solution to this problem.

Monte Carlo simulation is a valuable tool for forecasting events, especially for problems that are too complex to be described by equations. It is important, however, to repeat the simulation enough times to obtain a reliable result.[7]

Footnotes

1. M. Browne, "Coin-Tossing Computers Found to Show Subtle Bias," *New York Times*, January 12, 1993.
2. R. Rubinstein, *Simulation and the Monte Carlo Method* (New York: John Wiley & Sons, Inc., 1981), pp. 20–21. This book also describes more advanced methods for generating pseudorandom numbers. Moreover, it is an excellent resource for those who wish to pursue this topic.
3. Actually, it is more acceptable to assume that the logarithms of the wealth relatives of the S&P's geometric returns are normally distributed. The returns themselves are usually assumed to be lognormally distributed. The simulation approach I describe, however, should produce a distribution of terminal wealth that is lognormally distributed.
4. The 10 numbers selected from the uniform distribution are not themselves perfectly uniformly distributed because they comprise too small a sample. Nor, for the same reason, will the 10 transformed numbers be perfectly normally distributed. Thus, when I describe a sequence of 10 numbers as uniformally or normally distributed, I mean that they are drawn from respective distributions with uniform or normal distributions.
5. The uniformly distributed random sequences were generated by using the @rand function in Lotus 123.
6. For a review of this analytical approach, see M. Kritzman, "What Practitioners Need to Know About Uncertainty," *Financial Analysts Journal*, March/April 1991.
7. I thank George Chow and Richard Tanenbaum for their helpful comments.