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In Defense of Optimization: The Fallacy of 1/*N*

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Previous research has shown that equally weighted portfolios outperform optimized portfolios, which suggests that optimization adds no value in the absence of informed inputs. This article argues the opposite. With naive inputs, optimized portfolios usually outperform equally weighted portfolios. The ostensible superiority of the 1/N approach arises not from limitations in optimization but, rather, from reliance on rolling short-term samples for estimating expected returns. This approach often yields implausible expectations. By relying on longer-term samples for estimating expected returns or even naively contrived yet plausible assumptions, optimized portfolios outperform equally weighted portfolios out of sample.

Benartzi and Thaler (2001) showed that more than a third of direct contribution plan participants allocate their assets equally among investment options. Although this approach to portfolio construction may seem naive, research has shown that 1/*N* portfolios outperform optimized portfolios out of sample.[1] The study by DeMiguel, Garlappi, and Uppal (DGU 2009) is ostensively convincing because the authors evaluated 14 models on seven empirical datasets. Their models included advances in Bayesian estimation and moment restrictions designed to reduce estimation error. Nevertheless, none of the optimized portfolios consistently outperformed 1/*N*. DGU's results show that 1/*N* produces Sharpe ratios that are 50 percent higher, on average, than those of the mean–variance-optimized portfolios. DGU's results also show that 1/*N* outperforms the market portfolio.

On the one hand, certain features of the 1/*N* approach may help explain its superior performance:

1. It avoids concentrated positions.
2. On rebalancing dates, it sells high and buys low and thus capitalizes on any mean-reversion effect.
3. It never underperforms the worst performing asset.
4. It always invests in the best performing asset.
5. It captures the size alpha because it overweights small-cap stocks and underweights large-cap stocks.

On the other hand, equally weighted diversification assumes that we have no investment knowledge of the component assets. If we have at least some information on the expected returns, riskiness, and diversification properties of the assets, limited though it may be, why should we not expect optimization to improve on a naively diversified portfolio? Optimization cynics claim that optimizers maximize input errors. Indeed, a great deal of effort has been devoted to ameliorating this problem:

- Jorion (1986, 1991) proposed the Bayes–Stein approach, which compresses expected returns toward the minimum-variance portfolio.
- Michaud (1998) used resampling as a method to reduce estimation error. (But Scherer [2002] demonstrated that resampling converges to the mean–variance solution when short sales are allowed.)
- Black and Litterman (1992) combined views with equilibrium expected returns.
- Chow (1995) augmented the mean–variance objective function with a benchmark-tracking-error term.
- DeMiguel, Garlappi, Nogales, and Uppal (2009) constrained the sum of absolute portfolio weights to be smaller than or equal to a given threshold.
- Chevrier and McCulloch (2008) used advances in Bayesian estimation to ensure that optimization inputs did not violate basic economic theory.

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For our study, we discounted the notion that error maximization explains the finding that $1/N$ produces better portfolios than does optimization. In our view, those who reject optimization on the ground that it maximizes errors misunderstand the issue (Kritzman 2006). Consider, for example, optimization among assets that have similar expected returns and risk. Errors in the estimates of these values may substantially misstate optimal allocations, but the return distributions of both the correct and the incorrect portfolios will nevertheless likely be very similar. Now consider optimization among assets that have dissimilar expected returns and risk. Errors in these estimates will have little impact on optimal allocations; again, therefore, the return distributions of the correct and incorrect portfolios will not differ much.

In our study, we posited that poor optimization results often arise from reliance on trailing 60- and 120-month historical returns in modeling expected returns. For example, DGU (2009) reported results for various models that relied on trailing 60-month returns. No thoughtful investor would blindly extrapolate historical means estimated over such short samples, especially if those returns were outright implausible.

Consider the following facts. As of the end of 2008, the 200 largest U.S. defined benefit pension plans, on average, allocated 60 percent of their funds to stocks and 40 percent to bonds. As of February 2009, the 60-month trailing return of the S&P 500 Index was –5.66 percent, as compared with 3.80 percent for the Barclays Capital U.S. Aggregate Bond Index. This average allocation, together with the belief that these historical returns characterize future returns, implies that the average risk aversion of these funds was –3.45 (Appendix A shows how we calculated implied risk aversion). A negative value for risk aversion is surprising. Would the largest U.S. funds prefer *more* risk to less risk for the same expected return? Of course not, but the trailing-return methodology implies that investors often seek risk.

Markowitz (1952) was clear that investors' views of the future should not be dictated by past performance:

> The process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. (p. 77)

Markowitz stated that optimization is about the choice of a portfolio given a set of beliefs and is not about how to form those beliefs. Moreover, he explicitly acknowledged that beliefs are derived from observation and experience, neither of which limits the formation of beliefs to extrapolation of historical values.

Data and Methodology

The purpose of our study was to demonstrate that optimization is superior to $1/N$. We did not require clever models of expected returns. We used simple models of expected returns that assumed no forecasting skill, and we used no constraints other than the long-only constraint. With 13 datasets comprising 1,028 data series, we constructed more than 50,000 optimized portfolios and evaluated their out-of-sample performance. We grouped the portfolios into three categories: asset class, beta, and alpha. This classification corresponds to the investment process that most institutional investors follow—first, asset/liability management; then, beta allocation; and finally, the search for alpha. **Table 1** shows our datasets, and Appendix B provides additional information on constituents and data sources. We used monthly data except for the 500 stocks, for which we used daily data to accommodate the shorter period and the larger covariance matrix.

For each dataset, we compared the out-of-sample performance of the market portfolio, the $1/N$ portfolio, and the optimized portfolios. To measure the performance of the optimized portfolios, we first forecasted return and risk only on the basis of information available at the time of portfolio construction. Thus, every forecast was out of sample. We then invested in the optimal allocation that maximized the trade-off between expected return and risk. Specifically, we solved for the weights (w_i) that maximized

Table 1. Datasets

Dataset	Start Date	End Date
Asset/liability management		
7 asset classes + liability proxy	Feb 1973	Dec 2008
Betas		
10 industries	Jul 1926	Dec 2008
30 industries	Jul 1926	Dec 2008
10 size deciles	Jul 1926	Dec 2008
10 book-to-market deciles	Jul 1926	Dec 2008
10 dividend yield deciles	Jul 1927	Dec 2008
10 momentum deciles	Jan 1927	Dec 2008
10 long-term-reversal deciles	Jan 1931	Dec 2008
10 short-term-reversal deciles	Feb 1926	Dec 2008
Alphas		
500 stocks	Dec 1998	Dec 2008
21 commodities	Jan 1971	Dec 2008
14 hedge fund styles	Jan 1996	Dec 2008
15 asset managers	May 1987	Dec 2008

$$\sum_{i=1}^{n} w_i E(r_i) - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij},$$

subject to

$$\sum_{i=1}^{n} w_i = 1 \text{ and } w_i \geq 0, \forall i,$$

where $E(r_i)$ is the expected return for asset i and σ_{ij} is the covariance between asset i and asset j.[2] In the following period, we

1. revised our forecasts on the basis of new information,
2. reoptimized the portfolio, and
3. computed realized total return and risk for each strategy.

For the asset/liability simulations, we assumed five-year holding periods with annual rebalancing and measured average performance for all five-year holding periods. For all other simulations, we reoptimized monthly. We used the Sharpe ratio as our measure of performance. As in DGU (2009), we included cash in the optimizations to ensure that the optimal portfolio maximized the expected Sharpe ratio.[3] We did not impose any constraints (other than a long-only constraint) for two reasons: (1) We wanted our results to be directly comparable to those of previous studies, and (2) we wanted to evaluate the performance of optimization in its simplest form.

Our tests required us to estimate expected returns, volatilities, and correlations. A common approach for estimating expected returns is to use time-series econometrics. Econometric models, however, often suffer from two limitations: (1) the possibility of data mining and (2) the lack of applicability. The data-mining problem arises when the researcher tries a large number of models and specifications until one is found that produces a satisfying outcome (this problem is also called overfitting the data). Too often, econometric models perform poorly out of sample. The lack-of-applicability problem refers to the gap between statistical significance and economic significance. These models, even though statistically significant, often fail to produce satisfying profits in a portfolio management context. These limitations explain why many money managers continue to use simple models to discriminate among investment opportunities.

We used three approaches to estimate expected returns. Although extremely simple, these expected returns have an important difference from most of the expected returns used in previous studies: They do not rely on short rolling samples of realized returns, which often imply implausible expectations.

- First, we generated the minimum-variance portfolio.[4] We designed this experiment to determine whether we could improve risk-adjusted performance by simply optimizing to reduce risk on the basis of pure extrapolation of the covariance matrix. In this experiment, expected returns were constant for all assets.
- Second, for each asset, we estimated a risk premium over a long data sample and assumed that it remained constant throughout the backtest. To do so, we used data available before the backtest start date. **Table 2** shows our assumptions for the asset/liability optimizations. For the betas, we used simply the first 50 years of each dataset.
- Third, in the spirit of classical statistics, we used a growing sample that included all available out-of-sample data.

To estimate expected volatilities and correlations, we used the monthly rolling 5-, 10-, and 20-year covariance matrices, as well as the all-data approach. All covariance matrices were equally weighted.

Table 3 provides a summary of our experiments. Owing to a lack of available data, we focused on the minimum-variance approach and the five-year covariance matrix for the alpha portfolios; given the large number of securities involved, we used a daily covariance matrix for the security selection experiment.

Table 2. Risk Premium Expected Returns for Asset/Liability Optimization

Asset Class	Expected Return	Assumption
Domestic equity	10.36%	Annualized U.S. stock return (1926–Jan 1978)
Foreign equity	10.36	Annualized U.S. stock return (1926–Jan 1978)
Government bonds	3.37	U.S. government bond return (Jan 1926–Jan 1978)
Corporate bonds	4.09	U.S. government bonds + credit spread (Jan 1950–Jan 1978)
REITs	10.36	Annualized U.S. stock return (1926–Jan 1978)
Commodities	3.89	Annualized change in U.S. CPI (Jan 1957–Jan 1978)
Cash	2.41	Annualized return of U.S. cash (Jan 1926–Jan 1978)

Note: CPI = Consumer Price Index.

Table 3. Summary of Experiments

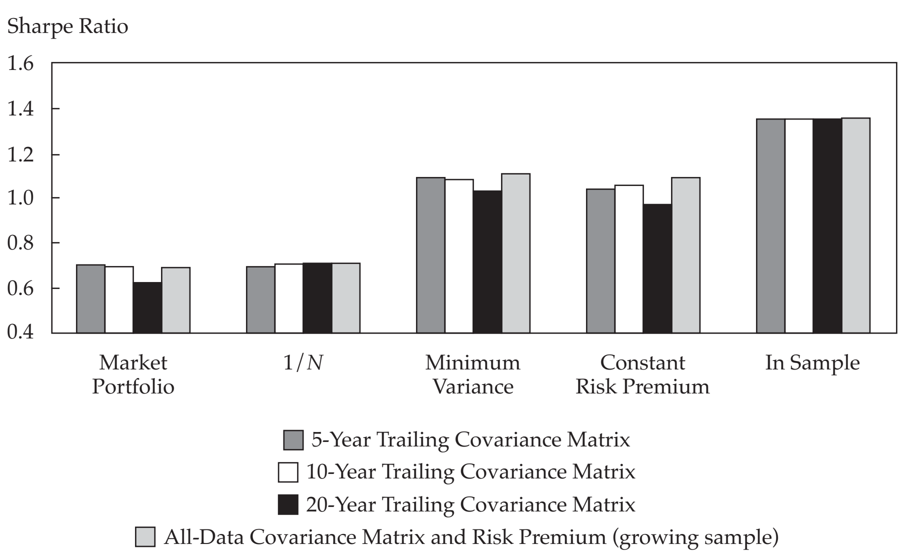
	Expected Returns			Covariance Matrices (years)			
	Min. Var.	RP	All Data	5	10	20	All Data
Asset/liability management	●	●	●	●	●	●	●
Betas							
10 industries	●	●	●	●	●	●	●
30 industries	●	●	●	●	●	●	●
10 size deciles	●	●	●	●	●	●	●
10 book-to-market deciles	●	●	●	●	●	●	●
10 dividend yield deciles	●	●	●	●	●	●	●
10 momentum deciles	●	●	●	●	●	●	●
10 long-term-reversal deciles	●	●	●	●	●	●	●
10 short-term-reversal deciles	●	●	●	●	●	●	●
Alphas							
500 stocks	●			Daily			
21 commodities	●			●			
14 hedge fund styles	●			●			
15 asset managers	●			●			

Note: Min. Var. = minimum-variance portfolio; RP = long-term risk premium.

Results

Figure 1 reports the results for our asset/liability management optimizations. We included in-sample optimization results, which reveal how we would perform if we knew the true parameters of the distribution. Our results show that optimized portfolios significantly outperform the $1/N$ portfolio. (Transaction costs have a minimal effect because all strategies have low turnover.) We found that even without any ability to forecast returns, optimization of the covariance matrix by itself adds value.

Figure 1. Asset/Liability Management: Simulation Results



Notes: Assuming 40 bps in round-trip transaction costs, the estimated impact of transaction costs on Sharpe ratios is –0.02 for $1/N$, –0.04 for minimum variance, and –0.05 for constant risk premium. The market portfolio is 25 percent U.S. equities, 25 percent foreign equities, 15 percent government bonds, 15 percent corporate bonds, and 20 percent commodities.

In practice, data are not always available to measure long-term risk premiums. Optimization, however, outperforms $1/N$ with any reasonable set of expected returns that do not rely on short samples of realized returns. For example, suppose that instead of the expectations presented in Figure 1 (the all-data approach), we make a judgment call and arbitrarily decide on the following expected returns, simply because they appear reasonable:

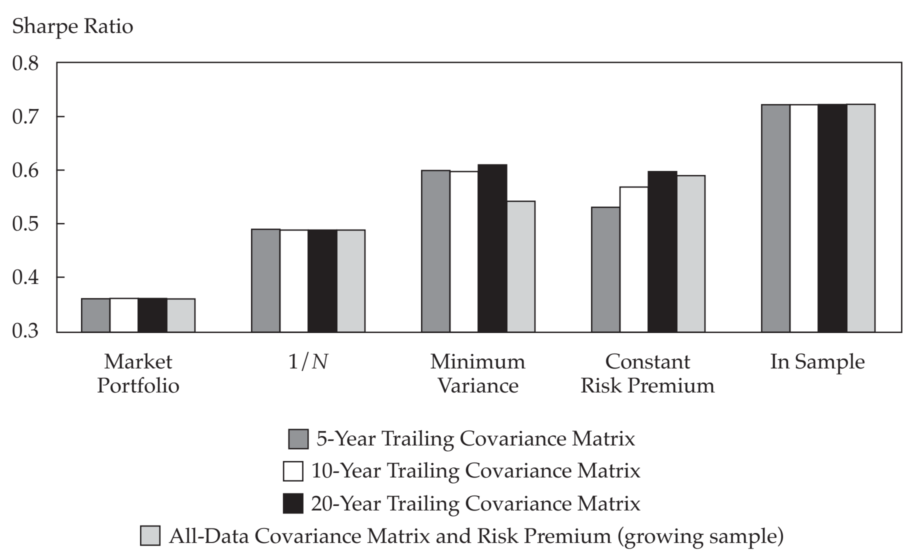
- U.S. equity: 8 percent
- Foreign equity: 8 percent
- U.S. government bonds: 5 percent
- U.S. corporate bonds: 6 percent
- REITs: 8 percent
- Commodities: 4 percent
- Cash: 3 percent

Using these expected returns throughout the backtest generates an average Sharpe ratio that is higher than those reported for the minimum-variance and constant risk premium models. We mention this *ad hoc* example to illustrate a simple point: Rolling five-year realized returns can generate implausible expectations—they represent only one realization of a grand distribution. For example, we might forecast that cash will outperform stocks because it did so over the past five years. Why would this particular realization be a good forecast of the next one? We could torture this rolling sample of realized returns with all the mathematical tools in the world, including both Bayesian and shrinkage models, but none of them would produce a reasonable return forecast. We should not use these past data; all we need is a reasonable forecast tied to economic intuition.

Figure 2 shows the results for our beta universes. For these backtests, we averaged Sharpe ratios across beta universes. The results in DGU (2009) were based on this dataset, which is notorious for the exceptional performance of the $1/N$ portfolio, as evidenced by its high Sharpe ratio as compared with that of the market portfolio. Nonetheless, the optimized portfolios outperformed $1/N$. Because we report averages only, we note that in some cases (e.g., when allocating among size deciles), $1/N$ outperformed optimization. But out-of-sample results are often noisy; hence, we are mostly interested in the average performance across backtests.

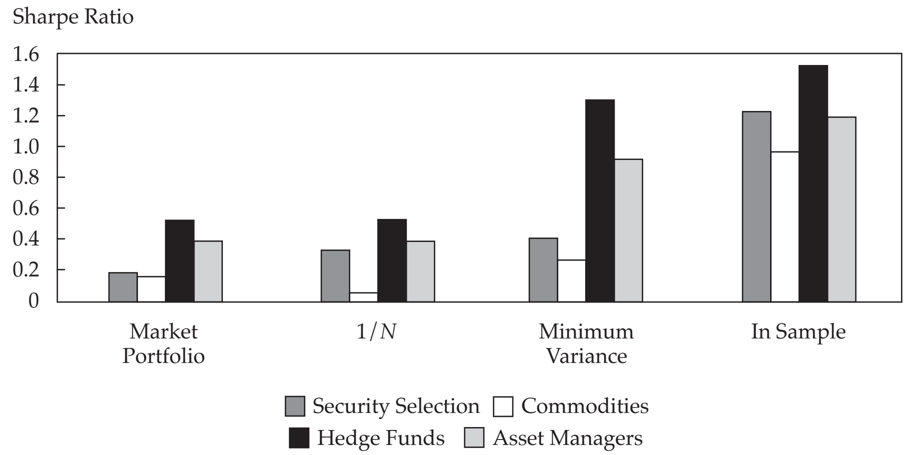
Figure 3 shows the results for our alpha universes. We assumed that the market portfolios for hedge funds and asset managers were equally weighted. We used the S&P 500 as the market portfolio for security selection and the S&P GSCI as the market portfolio for commodities. We found that optimization of the covariance matrix, without estimates of expected return, once again significantly improved out-of-sample performance as compared with the $1/N$ portfolio.[5] Clarke, De Silva, and Thorley (2006) reported similar results for security selection. They found that minimum-variance portfolios, constructed on 1,000 U.S. stocks, experienced returns that were comparable to or better than those of the market and reduced risk by 25 percent.

Figure 2. Betas: Simulation Results



Notes: Assuming 40 bps in round-trip transaction costs, the estimated impact of transaction costs on Sharpe ratios is –0.002 for $1/N$, –0.008 for minimum variance, and –0.009 for constant risk premium. The market portfolio is the S&P 500.

Figure 3. Alphas: Simulation Results



Notes: Assuming 40 bps in round-trip transaction costs, for minimum variance, the estimated impact of transaction costs on Sharpe ratios is –0.04 for security selection, –0.02 for commodities, –0.17 for hedge funds, and –0.06 for asset managers; for $1/N$, the estimated impact is –0.01 in all cases.

The minimum-variance portfolio performed well in our asset class, beta, and alpha simulations. In some cases, it outperformed optimization with expected return. This result is plausible for several reasons. First, our expected return models did not assume any forecasting skill. In an efficient market, past information should not be predictive of future returns, except for a basic risk–return relationship. Nothing in the literature, however, suggests that past risk cannot be predictive of future risk. Optimization with expected return should outperform the minimum-variance portfolio to the extent that return forecasts contain additional information beyond the fundamental risk–return relationship.

Second, introducing expected returns does not necessarily increase the Sharpe ratio of high-return–high-risk portfolios when leverage is not allowed. Consider a simple example with two assets: Assume that stocks have an expected return of 8 percent and a risk of 20 percent and that bonds have an expected return of 6 percent and a risk of 8 percent. Assume also that the two assets are 15 percent correlated. As we move from the risk-minimizing portfolio (90 percent bonds and 10 percent stocks) to the highest-risk portfolio (100 percent stocks), the Sharpe ratio first rises until we reach 81 percent bonds and 19 percent stocks, and then it decreases.

Third, Blitz and Van Vliet (2007) suggested a few explanations for why a minimum-variance portfolio would outperform in an inefficient market. They conjectured that investors overpay for risky assets because (1) retail investors like to gamble and (2) investment managers like to outperform at the same time their asset class outperforms (to attract money flowing into the asset class), which means that they prefer high-beta stocks. Blitz and Van Vliet also conjectured that leverage restrictions prevent investors from arbitraging away the minimum-risk portfolio's attractive Sharpe ratio.

Portfolio Weights

Are optimal portfolio weights reasonable? Corner solutions were rare in our simulations. For example, with cash included as an asset, the percentage of corner solutions—defined as the portfolio hitting a maximum weight constraint—was only 9 percent. Moreover, when assets are highly correlated (as in the beta simulations), concentrated weights are not necessarily a problem. *Risk* concentration matters more. We defined risk concentration as the portfolio's volatility divided by the weighted average of the individual asset volatilities such that a portfolio invested 100 percent in one asset has a risk concentration of 100 percent. On average, the optimized portfolios had a risk concentration of 56 percent, compared with 63 percent for the $1/N$ portfolios.[6] As Chevrier and McCulloch (2008) argued, using reasonable expectations yields reasonable portfolio weights.

Other Considerations

In our simulations, we used the equally weighted covariance matrix. In many periods, however, no significant events occur that should cause prices to change; hence, the returns we observed merely reflect the fact that prices are noisy. In other periods (e.g., August 1998 and September 2008), prices legitimately shifted in response to significant events.

Despite this difference in return generation, the formulas often used to calculate standard deviation and correlation assign as much importance to periods with no events as they do to periods with significant events. Consequently, risk parameters estimated from full samples provide unreliable measures of the hedging and diversification properties of assets during turbulent markets. Consider this example: When both U.S. and world ex-U.S. stock markets are one standard deviation above their means, the correlation between them is –17 percent. When both markets are one standard deviation below their means, the correlation between them is 76 percent (Chua, Kritzman, and Page 2009).

To improve the performance of optimization, we can partition historical returns into two subsamples—one associated with quiet periods and one characterizing turbulent times. We can then compute standard deviations and correlations that are specific to these regimes and overweight the turbulent regime's covariance matrix. Doing so improves the portfolio's resilience to significant market events (Chow, Jacquier, Kritzman, and Lowry 1999; Kritzman, Lowry, and Van Royen 2001).

Although we focused on mean–variance optimization, new technology in portfolio construction allows for increased flexibility. For example, full-scale optimization identifies the optimal portfolio for any set of return distributions, including highly non-normal distributions. Moreover, it allows for any description of investor preferences. It thus yields the truly optimal portfolio in sample, whereas mean–variance optimization provides an approximation of the in-sample truth (Cremers, Kritzman, and Page 2005; Sharpe 2007). Adler and Kritzman (2007) showed that full-scale optimization outperforms mean–variance optimization out of sample as well.

Conclusion

Previous research has created the misimpression that equally weighted portfolios outperform optimized portfolios, a result that some attribute to the sensitivity of optimization to estimation error. Our research showed the opposite to be true. Using naive but plausible estimates of expected returns, volatilities, and correlations, we demonstrated in a variety of applications that optimized portfolios generate superior out-of-sample performance compared with equally weighted portfolios. We showed that reliance on small historical samples for estimating expected returns often leads to views that would be rejected by any thoughtful investor. In our opinion, the misimpression that equal weighting is superior to optimization arises from this reliance on obviously implausible assumptions.

Although investors should certainly avoid implausible assumptions when using optimization, they need not rely solely on naive extrapolations of long historical samples. Instead, investors may benefit by conditioning optimization inputs on subsamples of turbulent and quiet regimes in accordance with their expectations of future market conditions. Alternatively, investors can use a technique called full-scale optimization, which accommodates a wider range of investor preferences and features of return distributions than does mean–variance optimization. In our view, $1/N$ is not a viable alternative to thoughtful optimization but is, rather, a capitulation to cynicism.

This article qualifies for 1 CE credit.

Appendix A. Calculation of Implied Risk Aversion

To calculate implied risk aversion, we began with the assumption that investors derive the optimal allocation to stocks and bonds by maximizing mean–variance expected utility. For a portfolio of stocks and bonds, expected utility can be expressed as

$$E(U) = \mu_S W_S + \mu_B W_B - \lambda(\sigma_S^2 W_S^2 + \sigma_B^2 W_B^2 + 2\rho\sigma_S W_S \sigma_B W_B),$$

where

μ_S = expected return of stocks
W_S = weighting of stocks
μ_B = expected return of bonds
W_B = weighting of bonds
σ_S = standard deviation of stocks
σ_B = standard deviation of bonds
ρ = correlation of stocks and bonds
λ = risk aversion

The partial derivative of expected utility with respect to the stock weight or bond weight represents the change in expected utility that would result from a one-unit increase in the stock or bond allocation, respectively. We computed these partial derivatives as follows:

$$\partial E(U)/\partial W_S = \mu_S - \lambda(2\sigma_S^2 W_S + 2\rho\sigma_S\sigma_B W_B),$$

and

$$\partial E(U)/\partial W_B = \mu_B - \lambda(2\sigma_B^2 W_B + 2\rho\sigma_S W_S \sigma_B).$$

For a given level of risk aversion, expected utility is maximized when these two derivatives are equal. In this example, we used the 60-month historical estimates for expected returns, together with the

120-month historical estimates for standard deviations and correlation. By setting these two derivatives equal to each other and solving for risk aversion, we obtained a formula for implied risk aversion as a function of the expected returns, standard deviations, and correlation of stocks and bonds, together with the investor's chosen allocation:

$$\lambda = (\mu_S - \mu_B)/(2\sigma_S{}^2 W_S + 2\rho\sigma_S\sigma_B W_B - 2\sigma_B{}^2 W_B - 2\rho\sigma_S W_S \sigma_B).$$

Appendix B. Constituents and Data Sources

This appendix presents additional information on constituents and data sources.

Exhibit B1. Asset Classes (asset/liability management)

Asset Class	Source
Domestic equity	S&P 500
Foreign equity	MSCI EAFE Index
Government bonds	Barclays U.S. Gov.
Corporate bonds	Barclays U.S. Corp.
REITs	FTSE NAREIT
Commodities	S&P GSCI
Cash	Ken French (1-month T-bill)
Liabilities (remeaned)	Barclays U.S. Gov. Long

Note: We remeaned the long bond liability proxy to reflect a more realistic growth rate of 3 percent.

Exhibit B2. Betas

Universe	Source
10 industries	CRSP/Ken French
30 industries	CRSP/Ken French
10 size deciles	CRSP/Ken French
10 book-to-market deciles	CRSP/Ken French
10 dividend yield deciles	CRSP/Ken French
10 momentum deciles	CRSP/Ken French
10 long-term-reversal deciles	CRSP/Ken French
10 short-term-reversal deciles	CRSP/Ken French

Notes: We downloaded the data directly from Ken French's website (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html). This dataset provided us with returns for deciles formed on variables that represent risk premiums. For example, for the book-to-market deciles, portfolios were constructed by ranking all NYSE, Amex, and NASDAQ stocks for which book equity (BE) and market equity (ME) data were available and then constructing 10 deciles from high to low BE/ME. The ranking occurred once a year, and performance for each decile was calculated out of sample.

Exhibit B3. Alphas

Asset Class	Source
500 stocks[a]	S&P 500 constituents
21 commodities	S&P GSCI
14 hedge fund styles	HFRI
15 asset managers	Yahoo!

[a]For the 500 stocks, we used historical constituents; thus, the sample is free of survivorship bias.

Exhibit B4. Constituents for Commodities, Hedge Fund Styles, and Asset Managers

Commodities

Wheat	Cotton	Platinum	Live cattle
Live hogs	Crude oil	Corn	Copper
Cocoa	Soybeans	Gold	Aluminum
Grains	Coffee	Zinc	Silver
Petroleum	Nickel	Sugar	Heating oil
Natural gas			

Hedge fund styles

Equity market neutral	Private issues–Reg D	Corporates
Quant directional	Merger arbitrage	Asset backed
Short bias	Systematic diversified	Yield alternatives
Tech–health care	Multistrategy	
Energy	Convertible arbitrage	

Asset managers[a]

American Funds Fundamental Investors	Fidelity Growth Company	Fidelity Equity-Income
Davis NY Venture A	American Funds Inv. Co. of America	Columbia Acorn Z
Vanguard	American Funds Wash. Mutual A	T. Rowe Price Mid-Cap Growth
American Funds Growth	Dodge & Cox Stock	Fidelity Value
Fidelity Contrafund	Vanguard Windsor II	Perkins Mid Cap Value Investor

[a]To create the asset manager universe, we used a Yahoo! mutual fund screener and extracted U.S. stock funds with manager tenures greater than 10 years and net assets above $5 billion. That screen identified 36 funds. We removed redundant funds and ended up with 15 funds.

Notes

1. See, for example, Jobson and Korkie (1981); DeMiguel, Garlappi, and Uppal (2009); and Duchin and Levy (2009).
2. Our asset/liability management simulations differed from our beta and alpha simulations in that the former included an embedded short position in the liability proxy with a weight of −100 percent, which corresponds to a funding ratio of 1. Investors who manage portfolio assets from a surplus perspective often use asset/liability optimization. This framework is simply a variation of the theoretical foundations used in asset-only optimization.
3. Technically speaking, portfolios optimized with very low risk aversion did not maximize the Sharpe ratio because we did not allow borrowing. Our results, however, are robust to the inclusion or exclusion of cash (results without cash are not reported).
4. DGU (2009) also generated the minimum-variance portfolio.
5. The Sharpe ratio might not be an appropriate measure of success for hedge fund portfolios because they tend to have highly non-normal distributions. Also, because mean–variance optimization assumes normality, it might not be the best approach for hedge fund optimization. As we discuss later in this article, full-scale optimization is a more appropriate approach for hedge funds.
6. Results for the percentage of corner solutions and risk concentration are reported for the 10-year covariance matrix for the asset/liability optimization and beta simulations and the 5-year covariance matrix for the alpha simulations. Similar results for the other covariance matrices are available from the authors upon request.

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