. . . About Future Value

Mark Kritzman

Suppose we want to estimate the future value of an investment based on its return history. This problem, at first glance, might seem pedestrian. Yet it involves subtleties that confound many financial analysts.

Some analysts argue that the best guide for estimating future value is the arithmetic average of past returns. Others claim that the geometric average provides a better estimate of future value. The correct answer depends on what it is about future value that we want to estimate.

Let us proceed with a quick review of the arithmetic and geometric averages.

Averages

The arithmetic average is simply the sum of the holding-period returns divided by the number of returns in the sample.

The geometric average is calculated by adding 1 to the holding-period returns, multiplying these values together, raising the product to the power of 1 divided by the number of returns, and then subtracting 1. The geometric average is sometimes called the constant rate of return, or the annualized return.

We can also compute the geometric average by converting holding-period returns into continuous returns. A continuous return, when compounded continuously, yields the same wealth we would achieve by investing at the holding-period return without compounding. It equals the natural logarithm of the quantity 1 plus the holding-period return. If the holding-period return equals 10%, for example, the continuous return equals 9.53%. If we were to invest $1.00 at an annual rate of 9.53% compounded continuously throughout the year, it would grow to $1.10 by the end of the year.

We calculate the geometric average from continuous returns by raising e (2.7182), the base of the natural logarithm, to the power of the arithmetic average of the continuous returns and subtracting 1.

Mark Kritzman is a Partner of Windham Capital Management.

Table 1 shows how to compute the arithmetic and geometric averages. We compute the arithmetic average by summing the values in the first column and dividing by 4. It equals 8.00%.

Table 1. Arithmetic and Geometric Averages

	Holding-Period Return (HPR)		1 + HPR		ln(1 + HPR)
	12.00%		1.1200		11.33%
	−6.00		0.9400		−6.19
	28.00		1.2800		24.69
	−2.00		0.9800		−2.02
Sum:	32.00%	Product:	1.3206	Sum:	27.81%
Average:	8.00	4th Root − 1:	7.20%	Average:	6.95
				Exponential − 1:	7.20

We can compute the geometric average in two ways. We can multiply the values in the second column, which yields 1.3206, then take the fourth root of 1.3206 and subtract 1 to arrive at the geometric average, 7.20%. Alternatively, we can compute the arithmetic average of the third column, raise e to this value, and subtract 1 to arrive again at 7.20%. It follows, therefore, that the arithmetic average of the logarithms of the quantities 1 plus the holding-period returns equals the logarithm of the quantity 1 plus the geometric average.

Here is how to interpret the geometric average. If we invest $1.00 in this sequence of returns, our dollar will grow to $1.3206. We would achieve the same terminal value by investing $1.00 at a constant rate of 7.20% for the four periods.

Expected Value

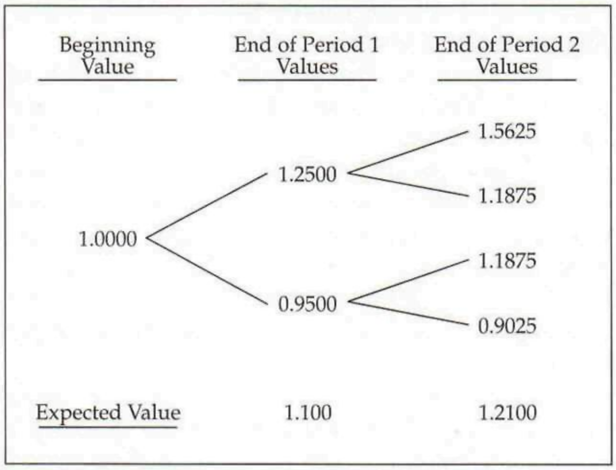
Now consider our earlier question. Which average should we use to estimate future value, assuming we wish to base our estimate on past returns? The question as I have posed it is too vague. We must be more precise about what we wish to know about future value.

If our goal is to estimate an investment's expected value either one period forward or many periods forward, we should use the arithmetic average of holding-period returns. We estimate expected value from past returns by adding 1 to this average and compounding this quantity forward.

In order to see why the arithmetic average is used to estimate expected value, consider an investment that has a 50% chance of increasing by 25% and a 50% chance of decreasing by 5%. After one period, there is an even chance that a dollar will grow to $1.25 or decline to $0.95. The expected value after one period thus equals $1.10, which in turn equals 1 plus the arithmetic average of the two possible returns.

After two periods, there are four equally likely outcomes. The investment can increase to $1.25 after the first period and then increase to $1.5625 after the second period or decrease to $1.1875. It can first decrease to $0.95 after the first period and then increase to $1.1875 or decrease further to $0.9025. Figure A diagrams these four possible paths.

Figure A. Possible Paths of $1.00 Investment



The expected value after two periods, which equals the probability-weighted outcome, equals 1.2100. It corresponds precisely to the quantity 1 plus the arithmetic average of 10% raised to the second power. The geometric average of a 25% increase followed by a 5% decrease or a 5% decrease followed by a 25% increase equals 8.9725%. If we add 1 to the geometric average and compound it forward for two periods, we arrive at a terminal value of 1.1875, which does not equal the expected value.

The expected value is higher than the value we would have achieved had we invested in the returns on which the arithmetic average is based.

This result might seem paradoxical. The intuition is as follows. The expected value assumes that there is an equal chance of experiencing any of the possible paths. A path of high returns raises the expected value over multiple periods more than a path of equal-magnitude low returns lowers it. This disproportionate effect is the result of compounding. Suppose the high return is 10% while the low return is −10%. Two consecutive high returns produce a 21% increase in value, while two consecutive low returns produce a decrease in value of only 19%.[1]

Here's how to interpret expected value. Suppose we observe 10 years of monthly returns and wish to estimate how much wealth we should expect to achieve if we were to draw randomly from these 120 monthly returns, *replacing each of the returns that is drawn*. If we were to invest in the 120 returns that we selected from the sample (without yet observing them), we should expect our investment to grow at a rate equal to the quantity 1 plus the arithmetic average of the sample of monthly returns, raised to the 120th power minus 1 or, equivalently, 1 plus the arithmetic average of the yearly returns from the sample, raised to the 10th power minus 1.

If we were to repeat this experiment many times, the average of the cumulative wealths generated from the sequences of randomly selected returns would indeed converge to the wealth predicted by the compounded arithmetic average. You can verify this result with a random-number generator.

Distribution of Future Value

Suppose we ask the following questions about future value. What is the likelihood or probability that an investment will grow or fall to a particular value? Or what value should we be 50% confident of achieving or failing to achieve? The answers to these questions depend on the geometric average.

Let's start with the assumption that the logarithms of the quantities 1 plus the holding-period returns are normally distributed.[2] This assumption implies that the returns themselves are lognormally distributed. It follows that the normal deviate used to estimate the probability of achieving a particular future value is calculated from the mean (arithmetic average) and standard deviation of these logarithms.[3]

A normal deviate measures distance from the mean in standard deviation units. It is the number that we look up in a normal distribution table to estimate the probability of achieving or falling short of a particular value.

Suppose we wish to estimate the likelihood that \$1 million will grow to equal \$1.5 million over five years, based on the past annual returns of a particular investment. If we believe that the logarithms of the quantities 1 plus these returns are normally distributed, we can proceed by computing the mean and standard deviation of these logarithms, as Table 2 shows.

Table 2. Past Investment Returns

	Annual Return	ln(1 + Return)
	−3.0200%	−3.0665%
	6.9500	6.7191
	27.1500	24.0197
	12.2700	11.5736
	−9.1400	−9.5850
	17.3000	15.9565
	2.1900	2.1664
	5.6500	5.4962
	−7.9100	−8.2404
	21.7200	19.6553
Arithmetic Average	7.3160	6.4695
Standard Deviation	11.6779	10.8744

We compute the normal deviate as follows:

$$Z = \frac{ln(1500000/1000000) - 0.064695 \times 5}{0.108744 \times \sqrt{5}}$$

$$Z = 0.3372.$$

The logarithm of the quantity 1.5 million divided by 1 million (40.5465%) is the continuous five-year return required in order for \$1 million to grow to \$1.5 million. It corresponds to an annualized continuous growth rate of 8.1093%. The quantity 5 times 6.4695% is the expected five-year continuous return. The normal deviate measures how far away the required five-year continuous return is from the expected five-year continuous return. It is 0.3372 standard-deviation units away. If we look up this value in a normal distribution table, we see that there is a 38.6% chance that \$1 million will grow to \$1.5 million, based on the past returns of this investment.

Suppose we wish to know the likelihood that our investment will generate a loss over five years. We compute the normal deviate as:

$$Z = \frac{ln(1000000/1000000) - 0.064695 \times 5}{0.108744 \times \sqrt{5}}$$

$$Z = -1.3303.$$

There is a 9.17% chance that this investment will lose money, on average, over five years.

What value should we expect to equal or exceed with 50% confidence? This value is called the median. Half the values are expected to exceed the median and half are expected to fall short of the median. A 50% probability of occurrence corresponds to a normal deviate of 0.00. The normal deviate equals 0.00 only when the required continuous return equals the expected continuous return. Thus we should expect with 50% confidence to equal or exceed the value that corresponds to the expected five-year continuous return. We find this value by raising e, the base of the natural logarithm, to the power 5 times the expected annualized continuous return. Thus the median wealth equals \$1,381,924.26. There is a 50% chance that the value in five years will exceed this value and a 50% chance that it will fall short of this value.

The geometric average is relevant for estimating probabilities, because the average of the logarithms equals the logarithm of the quantity 1 plus the geometric average.

The center of the probability distribution of terminal values is found by compounding the initial value at the geometric average. When we compound at the geometric average, we determine the future value for which there is an equal chance of exceeding or failing to exceed. Here is the logic behind this result.

1. The logarithms of the quantities 1 plus the holding-period returns are assumed to be normally distributed.

2. The mean (arithmetic average) of these logarithms equals the logarithm of the quantity 1 plus the geometric average of the holding-period returns.

3. The expected multiperiod continuous return therefore equals the number of periods times the logarithm of the quantity 1 plus the geometric average.

4. We convert the expected multiperiod continuous return into median wealth by raising e to the power of the multiperiod continuous return.

5. The quantity e raised to the power of the multiperiod continuous return is exactly equal to

the initial wealth compounded forward at the geometric average.

The bottom line is that we should compound at the *arithmetic* average if we wish to estimate an investment's expected value. We should compound at the *geometric* average, however, if we wish to estimate the likelihood that an investment will exceed or fall below a target value.

Some Formulas for Estimating Future Value

Here are some formulas that you should carry on your person at all times in the event you are called upon to discuss future value.

Expected future value is calculated as:

$$EV = B(1 + Ra)^n \tag{1}$$

or

$$EV = Be^{(Rc + S^2/2)n}. \tag{2}$$

Median future value is calculated as:

$$MV = B(1 + Rg)^n \tag{3}$$

or

$$MV = Be^{Rc(n)}. \tag{4}$$

The normal deviate for estimating the probability of achieving a target future value is:

$$Z = \frac{ln(T/B) - ln(1 + Rg)n}{S\sqrt{n}}. \tag{5}$$

In these formulas:

B = beginning value,
T = target value,
Ra = arithmetic average of holding-period returns,
Rg = geometric average of holding-period returns,
Rc = arithmetic average of the logarithms of the quantities 1 plus the holding-period returns,
S = standard deviation of the logarithms of the quantities 1 plus the holding-period returns,
e = base of the natural logarithm, 2.7128,
ln = natural logarithm and
n = number of periods.

When deciding whether to compound at the arithmetic average to estimate expected value or the geometric average to estimate the distribution of future value, you should be aware of the following distinction. More often than not, an investment will fail to achieve its expected value (compounded arithmetic average), which in some sense implies that the expected value is not to be expected.[4] Because of the effect of compounding, however, the outcomes that exceed the expected value have a greater impact, *per outcome*, than the more frequent outcomes that fall below the expected value.

Half the outcomes should exceed the median value (compound geometric average) and half should fall below this value. Those values that are above the median, however, will *on average* exceed the median by a magnitude greater than the magnitude by which the below-median values fall short of the median. Hence the expected value will exceed the median value.[5]

Footnotes

1. I thank Alan Marcus for this intuition.
2. For a review of this assumption, see M. Kritzman, "What Practitioners Need to Know About Lognormality," *Financial Analysts Journal*, July/August 1992.
3. The normal deviate, in effect, rescales the distribution to have a mean of 0 and a standard deviation of 1. For a review of this topic, see M. Kritzman, "What Practitioners Need to Know About Uncertainty," *Financial Analysts Journal*, March/April 1991.
4. Robert Ferguson deserves credit for the comment, "the expected value is not to be expected."
5. I thank George Chow and Richard Tanenbaum for their helpful comments.