What Practitioners Need to Know . . .

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. . . About Uncertainty

The primary challenge to the financial analyst is to determine how to proceed in the face of uncertainty. Uncertainty arises from imperfect knowledge and from incomplete data. Methods for interpreting limited information may thus help analysts measure and control uncertainty.

Long ago, natural scientists noticed the widespread presence of random variation in nature. This led to the development of laws of probability, which help predict outcomes. As it turns out, many of the laws that seem to explain the behavior of random variables in nature apply as well to the behavior of financial variables such as corporate earnings, interest rates, and asset prices and returns.

Table I S&P 500 Annual Returns

1951	24.0%	1961	26.9%	1971	14.3%	1981	−4.9%
1952	18.4%	1962	−8.7%	1972	19.0%	1982	21.4%
1953	−1.0%	1963	22.8%	1973	−14.7%	1983	22.5%
1954	52.6%	1964	16.5%	1974	−26.5%	1984	6.3%
1955	31.6%	1965	12.5%	1975	37.2%	1985	32.2%
1956	6.6%	1966	−10.1%	1976	23.8%	1986	18.8%
1957	−10.8%	1967	24.0%	1977	−7.2%	1987	5.3%
1958	43.4%	1968	11.1%	1978	6.6%	1988	16.6%
1959	12.0%	1969	−8.5%	1979	18.4%	1989	31.8%
1960	0.5%	1970	4.0%	1980	32.4%	1990	−3.1%

Source: Data through 1981 from R. Ibbotson and R. Sinquefield, *Stocks, Bonds, Bills and Inflation: The Past and the Future* (Charlottesville, VA: The Financial Analysts Research Foundation, 1982).

Relative Frequency

A random variable can be thought of as an event whose outcome in a given situation depends on chance factors. For example, the toss of a coin is an event whose outcome is governed by chance, as is next year's closing price for the S&P 500. Because an outcome is influenced by chance does not mean that we are completely ignorant about its possible values. We may, for example, be able to garner some insights from prior experiences.

Suppose we are interested in predicting the return of the S&P 500 over the next 12 months. Should we be more confident in predicting that it will be between 0 and 10 per cent than between 10 and 20 per cent? The past history of returns on the index can tell us how often returns within specified ranges have occurred. Table I shows the annual returns over the last 40 years.

We can simply count the number of returns between 0 and 10 per cent and the number of returns between 10 and 20 per cent. Dividing each figure by 40 gives us the *relative frequency* of returns within each range. Six returns fall within the range of 0 to 10 per cent, while 10 returns fall within the range of 10 to 20 per cent. The relative frequencies of these observations are 15 and 25 per cent, respectively, as Table II shows.

Figure A depicts this information graphically in what is called a *discrete probability distribution.* (It is discrete because it covers a finite number of observations.) The values along the vertical axis represent the probability (equal here to the relative frequency) of observing a return within the ranges indicated along the horizontal axis.

The information we have is limited. For one thing, the return ranges (which we set) are fairly wide. For another, the sample is confined to annual returns, and covers only the past 40 years, a period which excludes two world wars and the Great Depression.

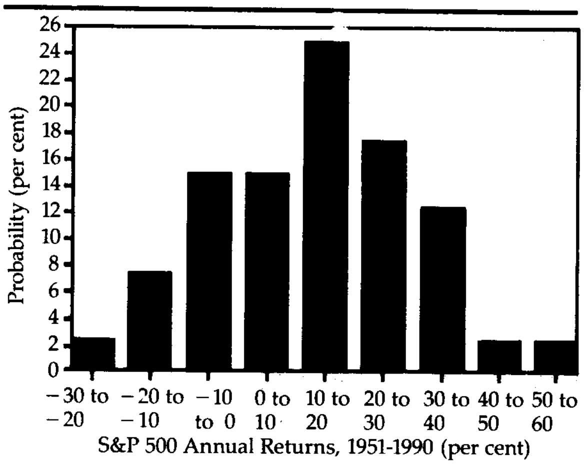
We can nonetheless draw several inferences from this limited information. For example, we may assume that we are about two-thirds more likely to observe a return within the range of 10 to 20 per cent than a return within the range of 0 to 10 per cent. Furthermore, by summing the relative frequencies for the three ranges below 0 per cent, we can also assume that there is a 25 per cent chance of experiencing a negative return.

If we wanted to make more precise inferences, we would have to augment our sample by extending the

Table II Frequency Distribution

Range of Return	*Frequency*	*Relative Frequency*
−30% to −20%	1	2.5%
−20% to −10%	3	7.5%
−10% to 0%	6	15.0%
0% to 10%	6	15.0%
10% to 20%	10	25.0%
20% to 30%	7	17.5%
30% to 40%	5	12.5%
40% to 50%	1	2.5%
50% to 60%	1	2.5%

Figure A Discrete Probability Distribution



measurement period or by partitioning the data into narrower ranges. If we proceed along these lines, the distribution of returns should eventually conform to the familiar pattern known as the bell-shaped curve, or *normal distribution.*

Normal Distribution

The normal distribution is a *continuous probability distribution;* it assumes there are an infinite number of observations covering all possible values along a continuous scale. Time, for example, can be thought of as being distributed along a continuous scale. Stocks, however, trade in units that are multiples of one-eighth, so technically stock returns cannot be distributed continuously. Nonetheless, for purposes of financial analysis, the normal distribution is usually a reasonable approximation of the distribution of stock ranges, as well as the returns of other financial assets.

The formula that gives rise to the normal distribution was first published by Abraham de Moivre in 1733. Its properties were investigated by Carl Gauss in the 18th and 19th centuries. In recognition of Gauss' contributions, the normal distribution is often referred to as the Gaussian distribution.

The normal distribution has special appeal to natural scientists for two reasons. First, it is an excellent approximation of the random variation of many natural phenomena.[1] Second, it can be described fully by only two values—(1) the mean of the observations, which measures location or central tendency, and (2) the variance of the observations, which measures dispersion.

For our sample of S&P 500 annual returns, the *mean return* (which is also the expected return) equals the sum of the observed returns times their probabilities of occurrence:

$$\bar{R} = R_1 \cdot P_1 + R_2 \cdot P_2 + \ldots + R_n \cdot P_n.$$

$\bar{R}$ equals the mean return. $R_1, R_2, \ldots R_n$ equal the observed returns in years one through n. $P_1, P_2, \ldots P_n$ equal the probabilities of occurrence (or relative frequencies) of the returns in years one through n. This computation yields the arithmetic mean. (The arithmetic mean ignores the effects of compounding; we will discuss later how to modify this calculation to account for compounding.)

The *variance* of returns is computed as the average squared difference from the mean. To compute the variance, we subtract each annual return from the mean return, square this value, sum these squared values, and then divide by the number of observations (or n, which in our example equals 40).[2] The formula for variance, V, is:

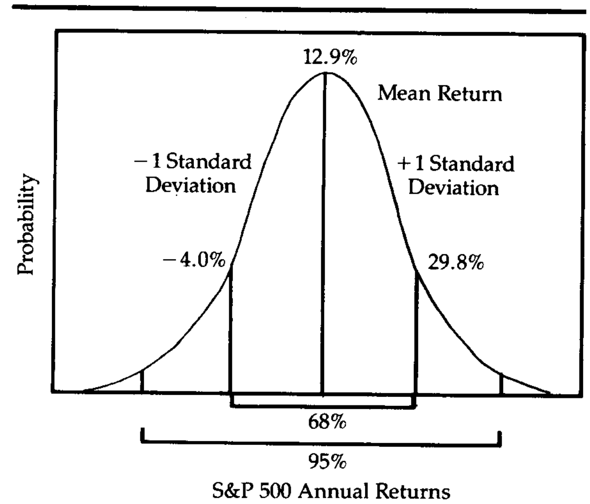
$$V = \frac{(R_1 - \bar{R})^2 + (R_2 - \bar{R})^2 + \ldots + (R_n - \bar{R})^2}{n}.$$

The square root of the variance, which is called the *standard deviation,* is commonly used as a measure of dispersion.

If we apply these formulas to the annual returns in Table I, we find that the mean return for the sample equals 12.9 per cent, the variance of returns equals 2.9 per cent, and the standard deviation of returns equals 16.9 per cent. These values, together with the assumption that the returns of the S&P 500 are normally distributed, enable us to infer a normal probability distribution of S&P 500 returns. This is shown in Figure B.

The normal distribution has several important characteristics. First, it is symmetric around its mean;

Figure B Normal Probability Distribution



1. Footnotes appear at end of article.

50 per cent of the returns are below the mean return and 50 per cent of the returns are above the mean return. Also, because of this symmetry, the mode of the sample—the most common observation—and the median—the middle value of the observations—are equal to each other and to the mean.

Note that the area enclosed within one standard deviation on either side of the mean encompasses 68 per cent of the total area under the curve. The area enclosed within two standard deviations on either side of the mean encompasses 95 per cent of the total area under the curve, and 99.7 per cent of the area under the curve falls within plus and minus three standard deviations of the mean.

From this information we are able to draw several conclusions. For example, we know that 68, 95 and 99.7 per cent of returns, respectively, will fall within one, two and three standard deviations (plus and minus) of the mean return. It is thus straightforward to measure the probability of experiencing returns that are one, two or three standard deviations away from the mean.

There is, for example, about a 32 per cent (100 per cent minus 68 per cent) chance of experiencing returns one standard deviation above or below the mean return. Thus there is only a 16 per cent chance of experiencing a return below −4.0 per cent (mean of 12.9 per cent minus standard deviation of 16.9 per cent) and about an equal chance of experiencing a return greater than 29.8 per cent (mean of 12.9 per cent plus standard deviation of 16.9 per cent).

Standardized Variables

We may, however, be interested in the likelihood of experiencing a return of less than 0 per cent or a return of greater than 15 per cent. In order to determine the probabilities of these returns (or the probability of achieving *any* return, for that matter), we can *standardize* the target return. We do so by subtracting the mean return from the target return and dividing by the standard deviation. (By standardizing returns we, in effect, rescale the distribution to have a mean of 0 and a standard deviation of 1.) Thus, to find the area under the curve to the left of 0 per cent (which is tantamount to the probability of experiencing a return of less than 0 per cent), we subtract 12.9 per cent (the mean) from 0 per cent (the target) and divide this quantity by 16.9 per cent (the standard deviation):

$$\frac{0\% - 12.9\%}{16.9\%} = -0.7633.$$

This value tells us that 0 per cent is 0.7633 standard deviation below the mean. This is much less than a full standard deviation, so we know that the chance of experiencing a return of less than 0 per cent must be greater than 16 per cent.

In order to calculate a precise probability directly, we need to evaluate the integral of the standardized normal density function. Fortunately, most statistics books include tables that show the area under a standardized normal distribution curve that corresponds to a particular standardized variable. Table III is one example.

To find the area under the curve to the left of the standardized variable we read down the left column to the value −0.7 and across this row to the column under the value −0.06. The value at this location—0.2236—equals the probability of experiencing a return of less than 0 per cent. This, of course, implies that the chance of experiencing a return greater than 0 per cent equals 0.7764 (= 1 − 0.2236). (The probability of experiencing a negative return as estimated from the discrete probability distribution in Table II equals 25 per cent.)

Suppose we are interested in the likelihood of experiencing an annualized return of less than 0 per cent *on average* over a five-year horizon? First, we'll assume that the year-by-year returns are mutually independent (that is, this year's return has no effect on next year's return). We can then convert the standard deviation back to the variance (by squaring it), divide the variance by five (the years in the horizon) and use the square root of this value to standardize the difference between 0 per cent and the mean return. Alternatively, we can simply divide the standard deviation by the square root of five and use this value to standardize the difference:

$$\frac{0\% - 12.9\%}{16.9\%/\sqrt{5}} = -1.71.$$

Again, by referring to Table III, we find that the likelihood of experiencing an annualized return of less than 0 per cent on average over five years equals only 0.0436, or 4.36 per cent. This is much less than the probability of experiencing a negative return in any one year. Intuitively, we are less likely to lose money on average over five years than in any particular year because we are diversifying across time; a loss in any particular year might be offset by a gain in one or more of the other years.

Now suppose we are interested in the likelihood that we might lose money in *one or more* of the five years. This probability is equivalent to one minus the probability of experiencing a positive return in every one of the five years. Again, if we assume independence in the year-to-year returns, the likelihood of experiencing five consecutive yearly returns each greater than 0 per cent equals 0.7764 raised to the fifth power, which is 0.2821. Thus the probability of experiencing a negative return in at least one of the five years equals 0.718 (= 1 − 0.2821).

Over extended holding periods, the normal distribution may not be a good approximation of the distribution of returns because short-holding-period returns are compounded, rather than cumulated, to derive long-holding-period returns. Because we can

Table III Normal Distribution Table (probability that standardized variable is less than z)

z	−.00	−.01	−.02	−.03	−.04	−.05	−.06	−.07	−.08	−.09
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0017	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0275	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0300	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0560
−1.4	.0808	.0793	.0778	.0764	.0750	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1921	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3400	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3694	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

represent the compound value of an index as a simple accumulation when expressed in terms of logarithms, it is the logarithms of one plus the holding-period returns that are normally distributed. The actual returns thus conform to a lognormal distribution. A lognormal distribution assigns higher probabilities to extremely high values than it does to extremely low values; the result is a skewed distribution, rather than a symmetric one. This distinction is usually not significant for holding periods of one year or less. For longer holding periods, the distinction can be important. For this reason, we should assume a lognormal distribution when estimating the probabilities associated with outcomes over long investment horizons.[3]

Caveats

In applying the normal probability distribution to measure uncertainty in financial analysis, we should proceed with caution. We must recognize, for example, that our probability estimates are subject to sampling error. Our example assumed implicitly that the experience from 1951 through 1990 characterized the mean and variance of returns for the S&P 500. This period, in fact, represents but a small sample of the entire universe of historical returns and may not necessarily be indicative of the central tendency and dispersion of returns going forward.

As an alternative to extrapolating historical data, we can choose to estimate S&P 500 expected returns based on judgmental factors. We can infer the investment community's consensus prediction of the standard deviation from the prices of options on the S&P 500.[4]

Finally, we must remember that the normal distribution and the lognormal distribution are not perfect models of the distribution of asset returns and other financial variables. They are, in many circumstances, reasonable approximations. But in reality stock prices do not change continuously, as assumed by the normal distribution, or even, necessarily, by small increments. October 19, 1987 provided sobering evidence of this fact. Moreover, many investment strategies, especially those that involve options or dynamic trading rules, often generate return distributions that are significantly skewed, rather than symmetric. In these instances, the assumption of a normal distribution might result in significant errors in probability estimates.[5]

The normal distribution can be applied in a wide variety of circumstances to help financial analysts measure and control uncertainty associated with financial variables. Nonetheless, the normal distribution is not a law of nature. It is, rather, an inexact model of reality.

Footnotes

1. Although no set of measurements conforms exactly to the specifications of the normal distribution, such diverse phenomenon as noise in electromagnetic systems, the dynamics of star clustering and the evolution of ecological systems behave in accordance with the predictions of a normal distribution.
2. To be precise, we should divide by the number of observations less one, because we lose one degree of freedom by using the same data to calculate the mean. This correction yields a so-called unbiased estimate of the variance, which typically is of little practical consequence.
3. For a more detailed discussion of the lognormal distribution, see S. Brown and M. Kritzman, *Quantitative Methods for Financial Analysis*, Second Edition (Homewood, IL: Dow Jones-Irwin, 1990), pp. 235–238.
4. The value of an option depends on the price of the underlying asset, the exercise price, the time to expiration, the risk-free return and the standard deviation of the underlying asset. All these values except the standard deviation are known, and the standard deviation can be inferred from the price at which the option trades. The implied value for the standard deviation is solved for iteratively. For a review of this technique, see M. Kritzman, *Asset Allocation for Institutional Portfolios* (Homewood, IL: Business One Irwin, 1990), pp. 184–185.
5. For an excellent discussion of this issue, see R. Bookstaber and R. Clarke, "Problems in Evaluating the Performance of Portfolios with Options," *Financial Analysts Journal*, January/February 1985.