Abstract

Notes I wrote summer 2025 as I was studying for my preliminary oral exam. Often informal/ schematic and may contain mistakes. A few almost verbatim proofs from Bieri [1] and Brown [2].

Contents

1 Fundamentals of algebraic topology

1.1 CW complexes and adjunction spaces

Definition 1.1 (CW topology)**.** CW complexes are given the weak topology: a subset $U \subseteq X = \bigcup_k X^{(k)}$ is open if and only if $U \cap X^{(k)}$ is open in $X^{(k)}$ for each k.

Lemma 1.2 (Some properties / defining axioms of CW complexes)**.**

1. *Each closed cell $\overline{e} \in X$ intersects only finitely many open cells.*
2. *A map $f : X \to Y$ between CW complexes is continuous if and only if its restrictions to each closed cell of X are continuous.*
3. *A map $f : X \to Y$ between CW complexes is continuous if and only if its restriction to each n-skeleton of X is continuous.*

4. *A homotopy of maps $F: I \times X \to Y$ is continuous if and only if its restrictions $F|_{I\times\overline{e^n_\alpha}}$ to each closed cell $\overline{e^n_\alpha}$ are continuous.*

Example 1.3 (Non-example of a CW complex)**.** A 2-disk consisting of one open 2-cell and uncountable many 0-cells in the boundary is *not* a CW complex: the closure of the 2-cell meets infinitely many cells in X, contradicting the closure-finiteness axiom.

Lemma 1.4 (Homotopies from identification spaces)**.** *A homotopy $X/\sim \times I \to Y$ is continuous if and only if it arises from a continuous map $X \times I \to Y$ which is, for each fixed $t \in I$, constant on equivalence classes in X.*

Proof. We use the following theorem: Let $p : X \to X/\sim$ be a quotient map (surjective, continuous, image of saturated opens/ closed are open/closed), and let C be a locally compact Hausdorff space. Then $p \times \mathrm{id}_C : X \times C \to X/\sim \times C$ is a quotient map.

Suppose we have a continuous map $H : X \times I \to Y$ such that for each fixed $t \in I$, the map is constant on equivalence classes of X. So the map $H_t : X \to Y$ factors through a continuous map

$$\begin{tikzcd} X \arrow[r, "H_t"] \arrow[dr, "\rho"'] & Y \\ & X/\sim \arrow[u, "\tilde{H}_t"'] \end{tikzcd}$$

for each $t \in I$. So we have a continuous map $\tilde{H} : X/\sim \times I \to Y$ induced by H.

On the other hand, suppose we have a continuous homotopy $\tilde{H} : X/\sim \times I \to Y$. Then the composition $X \times I \to X/\sim \times I \to Y$ is continuous, since I is a locally compact Hausdorff space. If we fix $t \in I$, then the map is $H_t : X \to X/\sim \to Y$, which is clearly constant on the equivalence classes of X (since we factor through $X/\sim$). ◻

Theorem 1.5 (HE spaces and homotopic attaching maps yield HE spaces)**.** *Let X be a CW complex and $A \subseteq X$ a subcomplex. Suppose $h : Y \to Y'$ is a homotopy equivalence of topological spaces. Let $f : A \to Y$, $f' : A \to Y'$ be continuous maps. If f' is homotopic to $h \circ f$, then h extends to a homotopy equivalence $H : X \sqcup_f Y \to X \sqcup_{f'} Y'$. Any homotopy inverse g of h extends to a homotopy inverse of H.*

Corollary 1.6. *If X is a CW complex and $D \subseteq X$ a contractible subcomplex, then the quotient map $X \to X/D$ is a homotopy equivalence.*

Proof. Take $h : D \to \{d\}$ a homotopy equivalence. Then $\mathrm{id}_D : D \to D$ and $f' : D \to \{d\}$ are continuous maps, and $f' = h \circ \mathrm{id}_D$, so by the theorem 1.5 we know that h extends to a homotopy equivalence $H : X \sqcup_{\mathrm{id}_D} D = X \to X \sqcup_{f'} \{d\} = X/D$. ◻

Corollary 1.7. *Let X be a CW complex and let $\iota : A \overset{X}{\hookrightarrow}$ be the inclusion of a subcomplex. Then $X/A \simeq X \sqcup_\iota CA$*

Proof. Recall that $X/A = X \sqcup_f \{a\}$ where $f : A \to \{a\}$ is a constant map. Since the cone on A is contractible, we have a homotopy equivalence $h : CA \to \{a\}$. Let $f' : A \to CA$ be the inclusion of A as the bottom of the cone CA (recall that $CA = (A \times I)/((a,1) \sim (a',1))$ for all $a, a' \in A$). Then $f = h \circ f'$, so by 1.5 $X \sqcup_f \{a\} = X/A \simeq X \sqcup_{f'} CA = X \sqcup_\iota CA$. ◻

Corollary 1.8. *Let $\iota : A \overset{X}{\hookrightarrow}$ be a subcomplex. Suppose ι is nullhomotopic. Then $X/A \simeq X \vee SA$.*

Proof. We know from the previous corollary that $X/A \simeq CA \sqcup_\iota X$, where here $\iota : A \overset{X}{\hookrightarrow}$ views A first as a subcomplex of CA and includes it as a subcomplex of X. Since $\iota : A \to X$ is nullhomotopic, there exists a constant map $c : A \to \{x\} \in X$ such that ι and c are homotopic. Hence $X/A \simeq CA \sqcup_\iota X \simeq CA \sqcup_c X = (X \sqcup CA)/((a, 0) \sim x)$ for all $a \in A$. This is precisely $X \vee SA$. $\square$

Corollary 1.9. *Let Y be a space and let $D^n \sqcup_\phi Y$ be obtained by attaching an n-disk D^n along its boundary via the map $\phi : \partial D^n \to Y$. If ϕ is nullhomotopic, then $D^n \sqcup_\phi Y$ is homotopy equivalent to the wedge $Y \vee S^n$.*

Proof. Recall that $Y \vee S^n = D^n \sqcup_f Y$ where $f : \partial D^n \to Y$ sends the entire boundary ∂D^n to a single point in Y. Since ϕ is nullhomotopic, we know $\phi \simeq f$, and thus by Theorem 1.5 we have a homotopy equivalence $D^n \sqcup_\phi Y \simeq D^n \sqcup_f Y = Y \vee S^n$. $\square$

Corollary 1.10. *Let Y be a CW complex. Let A be a CW complex such that $\iota_Y : A \hookrightarrow Y$ and $\iota_D : A \hookrightarrow D^n$ are inclusions of A as subcomplexes of Y and the n-disk D^n. If A is contractible, then $D^n \sqcup_{\iota_Y} Y \simeq Y$.*

Proof. Notice that $D^n \sqcup_{\iota_Y} Y \simeq Y \sqcup_{\iota_D} D^n$, since in both cases we take the disjoint union of D^n and Y and identify the subcomplex A in both. Since A is contractible, we know that the identity map on A is homotopic to the constant map. In particular, the inclusion of A into D^n (the map ι_D) restricts to the identity map on A, so it is homotopic to a constant map c. Since D^n is also contractible, we have a homotopy equivalence $h : D^n \to *$, and the maps h, c, ι_D satisfy $c = h \circ \iota_D$. Hence 1.5 yields

$$D^n \sqcup_{\iota_Y} Y \simeq Y \sqcup_{\iota_D} D^n \simeq Y \sqcup_c * = Y/A \simeq Y$$

where the last homotopy equivalence follows from the fact that A is contractible. $\square$

Definition 1.11 (Covering space action)**.** Say G acts as a covering space action on a space Y if for all $y \in Y$, there exists a neighborhood U of y such that $gU \cap U \neq \varnothing$ implies $g = e$ (so the nonidentity translates of U are all disjoint).

Prop 1.12. *If G acts by a covering space action 1.11 on Y, then*

- *the quotient map $Y \to Y/G$, $p(y) = Gy$ is a normal covering space.*
- *G is the group of deck transformations of this covering space if Y is path-connected.*
- *If Y is path-connected and locally path-connected, then $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$.*

1.2 Homotopy Theory

1.2.1 Relative homotopy groups, excision, and Hurewicz

Definition 1.13. $\pi_n(X, A, x_0)$. Several ways to define (using triples $(I^n, \partial I^n, J^{n-1})$ or (D^n, S^{n-1}, s_0)). I prefer the latter one. Note that $J^{n-1} = \overline{\partial I^n \backslash I^{n-1}}$, and that collapsing J^{n-1} to a point turns the triple $(I^n, \partial I^n, J^{n-1})$ into the triple (D^n, S^{n-1}, s_0). We define

$$\pi_n(X, A, x_0) = [(D^n, S^{n-1}, s_0), (X, A, x_0)].$$

Have an addition given by

$$([f] + [g])(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) : & 0 \leqslant t_1 \leqslant \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) : & \frac{1}{2} \leqslant t_1 \leqslant 1 \end{cases}$$

if we view f, g as maps from I^n. Alternatively, if we view f and g as maps from D^n, we can define a sum by first collapsing $D_{n-1} \subset D^n$ to a point (yielding $D^n \to D^n \vee D^n$), and then applying f and g to each hemisphere.

Prop 1.14 (Compression Criterion). *A map $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ if and only if it is homotopic rel S^{n-1} to a map with image contained in A.*

Proof. Suppose we have such a homotopy to a map g. Then $g : (D^n, S^{n-1}, s_0) \to (A, A, x_0)$, and composing g with a deformation retraction $\phi : D^n \times I \to s_0$ of D^n onto s_0 gives a homotopy of g to the constant map at s_0 through maps of the form $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$, so $[f] = [g] = 0$. On the other hand, if $[f] = 0$ via $F : D^n \times I \to X$, finish this!!! □

Lemma 1.15. *A CW pair (X, A) is n-connected if the cells in $X \backslash A$ have dimension $> n$. In particular $(X, X^{(n)})$ is n-connected. (Consequence: the inclusion $X^{(n)} \hookrightarrow X$ induces isomorphisms on π_i for $i < n$ and a surjection on π_n.)*

Proof. By the cellular approximation theorem, every map $(D^i, \partial D^i) \to (X, A)$ is homotopic to a cellular map (by maps of the same form). Since $X \backslash A$ only has cells in dimension greater than n, then for $i \leqslant n$ our cellular map $(D^i, \partial D^i) \to (X, A)$ lands in A. By the compression criterion (1.14), $\pi_i(X, A) = 0$ for $i \leqslant n$. The consequence follows from the long exact homotopy sequence of a pair. □

Prop 1.16 (Hatcher prop 4.15, converse to 1.15). *If a pair (X, A) is n-connected, then X is homotopy equivalent rel A to a CW complex Z having only cells in dimension greater than n in $Z \backslash A$.*

Prop 1.17 (LES for homotopy groups of a pair). *If (X, A) is a CW pair, the homotopy groups fit into a long exact sequence*

$$\cdots \to \pi_n(A, x_0) \xrightarrow{\iota_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0).$$

where i_* is induced by the inclusion $A \hookrightarrow X$, j_* is induced by the inclusion $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$, and ∂ sends a map $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to the restriction $(S^{n-1}, s_0) \to (A, x_0)$.

Definition 1.18 (HLP for pairs). The map $p : E \to B$ is said to have the homotopy lifting property for a pair (X, A) if each homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{f}_t : X \to E$ starting with a given lift $\tilde{f}_0$ and extending a given lift $\tilde{f}_t : A \to E$.

Theorem 1.19 (Thm 4.23 in [5]). *Let $X = A \cup B$ be a CW complex with A, B subcomplexes having nonempty <u>connected</u> intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected, $m, n \geqslant 0$, then the map $\pi_i(A, C) \to \pi_i(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$ (i.e., is $m + n$-connected).*

Definition 1.20 (Hurewicz homomorphism). Let X be a topological space and $x_0 \in X$ a chosen basepoint. For each n, there is a natural map $h : \pi_n(X, x_0) \to H_n(X)$ given as follows: for u a chosen generator of $H_n(S^n) \cong \mathbb{Z}$ and $f : (S^n, *) \to (X, x_0)$ a representative of $[f] \in \pi_n(X, x_0)$, we map $h([f]) = f_*(u)$.

Theorem 1.21 (Hurewicz). *Let X be a space, $A \subseteq X$, and $x_0 \in X$.*

(i) If X is path-connected, the Hurewicz map $\pi_1(X, x_0) \to H_1(X)$ is surjective and identifies $H_1(X)$ with the abelianization of $\pi_1(X, x_0)$.

(ii) Let $n \geqslant 2$. If X is $(n-1)$-connected, then $\tilde{H}_i(X) = 0$ for $i < n$, and the Hurewicz map $h : \pi_n(X, x_0) \to H_n(X)$ is an isomorphism.

(iii) Let $n \geqslant 2$. Suppose A is simply-connected, and (X, A) is $(n-1)$-connected (so $\iota : A \hookrightarrow X$ induces isomorphisms on π_i for $i < n-1$, and a surjection on π_{n-1}). Then $H_i(X, A) = 0$ for $i < n$.

(iv) Let $n \geqslant 2$. If A is simply-connected and (X, A) is $(n-1)$-connected, then $\tilde{H}_i(X/A) = 0$ for $i < n$ and there is an isomorphism $\pi_n(X/A) \cong H_n(X/A)$.

1.2.2 Whitehead's Theorem & corollaries

Theorem 1.22 (Whitehead's Theorem, Part I)**.** *Let X, Y be connected CW complexes. If $\iota : X \to Y$ is the inclusion of a subcomplex with the property that $\iota_* : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is an isomorphism for all $n \geqslant 0$, then Y deformation retracts onto the subcomplex X.*

Proof. We wish to exhibit a deformation retraction $F : Y \times I \to Y$ of Y onto X. We'll proceed by induction on the skeleta of Y. We wish to construct homotopies $F^{(n)} : Y \times I \to Y$ satisfying the following:

1. When $n = 0$, $F^{(0)}|_{t=0}$ is the identity map $\mathrm{id}_Y : Y \to Y$.

2. When $n \geqslant 1$, $F^{(n)}|_{t=0} = F^{(n-1)}|_{t=1}$ (so that we may glue the homotopies).

3. The image of $X \cup Y^{(n)}$ under $F^{(n)}|_{t=1}$ is contained in X (so at the end of the n^{th} homotopy, the n skeleton of Y lands in X.

4. $F^{(n)}$ is stationary on $X \cup Y^{(n-1)}$ (so all the homotopies are stationary on X, which is good because we want to show a deformation retraction. Also we only push n-cells into X, without messing up lower-dimensional cells (which we've already moved?!).

We first construct $F^{(0)} : Y \times I \to Y$ with $F^{(0)}|_0 = \mathrm{id}_Y$ and $F^{(0)}|_{t=1}(X \cup Y^{(0)}) \subseteq X$ (and $F^{(0)}|_X(t) = \mathrm{id}_X$ for all t). Since $\iota_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism, every path-component of Y comes from the inclusion of a path-component of X. In particular this means that for every $y \in Y^{(0)}$, there exists an $x \in X^{(0)}$ and a path between x and y. Since paths are contractible subcomplexes, we can contract Y along these paths (until the point in $Y^{(0)}$ lands in X, and doesn't move within X. This is possible since X is closed (?)) and get a homotopy equivalent space. At the end of this contraction, $Y^{(0)} \cup X \subset X$ (and the points in X never moved). Since $Y^{(0)} \cup X$ is a subcomplex of Y, then $(Y, Y^{(0)} \cup X)$ is a good pair (satisfies homotopy extension property), and we can extend the contraction $G : Y^{(0)} \cup X \times I \to Y$ to all of $Y \times I \to Y$. Call this extended map $F^{(0)}$. Note that $F^{(0)}|_0 = \mathrm{id}_Y$, $F^{(0)}|_{t=1}(X \cup Y^{(0)}) \subseteq X$, and $F^{(0)}$ is stationary on $X \cup Y^{(-1)} = X$.

Now suppose $\iota_* : \pi_d(X) \to \pi_d(Y)$ is surjective, and $\iota_* : \pi_{d-1}(X) \to \pi_{d-1}(Y)$ is injective. We claim that any map $(D^d, \partial D^d) \to (Y, X)$ can be homotoped rel ∂D^d to a map with image contained in X. Consider the long exact sequence of homotopy groups

$$\cdots \to \pi_d(X) \xrightarrow{\iota_*} \pi_d(Y) \to \pi_d(Y, X) \to \pi_{d-1}(X) \xrightarrow{\iota_*} \pi_{d-1}(Y) \to \cdots$$

This yields a short exact sequence

$$0 \to \mathrm{Im}(\pi_d(Y) \to \pi_d(Y, X)) \to \pi_d(Y, X) \to \ker(\pi_{d-1}(X) \to \pi_{d-1}(Y)) \to 0.$$

Since $\iota_* : \pi_d(X) \to \pi_d(Y)$ is surjective, then its cokernel (which is isomorphic to $\mathrm{Im}(\pi_d(Y) \to \pi_d(Y, X))$ is 0, and since $\iota_* : \pi_{d-1}(X) \to \pi_{d-1}(Y)$ is injective, its kernel is 0. Hence $\pi_d(Y, X) = 0$. By 1.14, every map $f : (D^d, S^{d-1}) \to (Y, X)$ is homotopic rel S^{d-1} to a map whose image lies in X. In particular, since we have by assumption that $\iota_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all n, then every map $\phi : (D^n, S^{n-1}) \to (Y, X)$ is homotopic rel S^{n-1} to one whose image lies in X.

We use this fact to inductively construct our $F^{(n)}$. Suppose $F^{(n)} : Y \times I \to Y$ has been constructed satisfying properties 1-4 above. Let e^{n+1} be an $(n+1)$-cell in Y and $\phi^{n+1} : (D^{n+1}, S^n) \to (Y, Y^{(n)} \cup X)$ the corresponding characteristic map. Composing this map with $F_1^{(n)}$ (which sends $Y^{(n)} \cup X$ into X), we obtain a map $F_1^{(n)} \circ \phi : (D^{n+1}, S^n) \to (Y, X)$, which is homotopic rel S^n to a map $D^{n+1} \to X$. If we define $H_t : (D^{n+1}, S^n) \to (Y, X)$ to be this homotopy, then we claim that we have a well-defined homotopy

$F_t^{(n+1)} = H_t \cup F_1^{(n)}$ on the adjunction $X \cup Y^{(n)} \cup_\phi e^{n+1}$, since the homotopy H_t is rel S^n, whose image lies in $Y^{(n)} \cup X$ (thus $F_t^{(n+1)}$ is continuous). Really we are homotoping $F_1^{(n)}$ to be such that $X \cup Y^{(n)} \cup_\phi e^{n+1}$ lands in X, without moving $F_1^{(n)}(X \cup Y^{(n)})$. Doing this for all cells e_α^{n+1} simultaneously, we obtain a homotopy of $F_1^{(n)}$ on $X \cup Y^{(n+1)}$ which ends by sending $X \cup Y^{(n+1)}$ into X, and which fixes $F_1^{(n)}(X \cup Y^{(n)})$ throughout. Let this homotopy be called $F^{(n+1)}$. By the homotopy extension property, we can extend this homotopy to all of Y. Abusing notation, we call this extended homotopy $F^{(n+1)}$.

We compose these homotopies into one by performing the homotopy $F^{(n)}$ in the time interval $[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}]$. This is continuous since $F_1^{(n)} = F_0^{(n+1)}$. Also, since $F_0 = F_0^{(0)}(Y) = \mathrm{id}_Y$ and $F_1(Y) \subset X$ with $F_t(X) = F_t^{(n)}(X)$ stationary for all t and n, we conclude that F is a deformation retraction of Y onto X. □

Theorem 1.23 (Whitehead's Theorem, Part II)**.** *If $f : X \to Y$ is any map inducing isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n \geqslant 0$, then f is a homotopy equivalence.*

Proof. Let M_f be the mapping cylinder of $f : X \to Y$. Since M_f deformation retracts onto Y, then $\pi_n(M_f) \to \pi_n(Y)$ is an isomorphism for all $n \geqslant 0$, and since $f : X \to Y$ can be given as the composition of the inclusion $\iota : X \to M_f$ and the retraction $r : M_f \to Y$, we get that $\pi_n(X) \to \pi_n(M_f) \to \pi_n(Y)$ is an isomorphism. In particular $\iota_* : \pi_n(X) \to \pi_n(M_f)$ is an isomorphism for all $n \geqslant 0$, so by Whitehead's theorem part 1 1.22, X and M_f are homotopy equivalent via ι. But M_f and Y are also homotopy equivalent via r, so $X \simeq Y$ via f. □

Corollary 1.24. *A weakly contractible CW complex is contractible.*

Proof. Suppose X is a weakly contractible CW complex, that is, $\pi_n(X) = 0$ for all $n \geqslant 0$. Then letting $f : X \to \{*\}$ be a constant map, we see that the induced map $f_* : \pi_n(X) \to \pi_n(\{*\})$ is necessarily an isomorphism for all n, which means that f is a homotopy equivalence by 1.23. □

Definition 1.25 ($K(G, 1)$)**.** A connected CW complex X with $\pi_1(X) = G$ and $\pi_n(X) = 0$ for $n \geqslant 2$ is called a $\underline{K(G, 1)}$.

Prop 1.26 ($K(G, 1)$'s have contractible universal covers.)**.** *A connected CW complex X is a $K(\pi, 1)$ if and only if its universal cover $\tilde{X}$ is contractible.*

Proof. Suppose X is a connected CW complex with $\pi_n(X) = 0$ for $n \geqslant 2$. Since p_* induces an injection on homotopy groups, we see that $\pi_n(\tilde{X}) = 0$ for all $n \geqslant 2$. Since we also know that $\pi_1(\tilde{X}) = 0$, then Whitehead's theorem implies that $\tilde{X}$ is contractible.

On the other hand, if $\tilde{X}$ is contractible, then $\pi_n(\tilde{X}) = 0$ for all $n \geqslant 1$. By the lifting criterion (since S^n is simply-connected for $n \geqslant 2$), every map $\phi : S^n \to X$ with $n \geqslant 2$ lifts to a map $\tilde{\phi} : S^n \to \tilde{X}$ such that $p \circ \tilde{\phi} = \phi$:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, dashed, "\tilde{\phi}"] \arrow[r, "\phi"'] & X \end{tikzcd}$$

As $\tilde{\phi}$ must be nullhomotopic ($\pi_n(\tilde{X}) = 0$), composing this nullhomotopy with p gives a nullhomotopy of ϕ. We conclude that $\pi_n(X) = 0$ for $n \geqslant 2$. (Alternatively, can use the fact that if $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a universal cover, then $p_* : \pi_n(\tilde{X}, \tilde{x_0}) \to \pi_n(X, x_0)$ is an iso for all $n \geqslant 2$.) □

Remark 1.27 (Technicality in Whitehead). We remark that the maps specifically induced by $f : X \to Y$ must be isomorphisms on homotopy groups in order to conclude that f is a homotopy equivalence. That is, two spaces can have abstractly isomorphic homotopy groups but not be homotopy equivalent, as there is not a specific map between them inducing the isomorphisms on homotopy groups. Consider the following example:

Example 1.28 ($\mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^\infty$ have isomorphic homotopy groups but are not homotopy equivalent)**.** Both $\mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^\infty$ are connected CW complexes, and have $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$, $\pi_1(S^2 \times \mathbb{R}P^\infty) \cong \pi_1(S^2) \times \pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$. Now S^2 is the universal cover of $\mathbb{R}P^2$, so $\pi_n(S^2) \cong \pi_n(\mathbb{R}P^2)$ for all $n \geqslant 2$ ($p_* : \pi_n(S^2) \to \pi_n(\mathbb{R}P^2)$ is always injective, and surjectivity follows from the lifting criterion, since S^n is simply connected for $n \geqslant 2$). On the other hand, $S^2 \times S^\infty$ is the universal cover of $S^2 \times \mathbb{R}P^\infty$, so we have $\pi_n(S^2 \times \mathbb{R}P^\infty) \cong \pi_n(S^2 \times S^\infty) \cong \pi_n(S^2)$ as S^∞ is contractible.

But $\mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^\infty$ are not homotopy equivalent, for their homology groups differ: $H_2(\mathbb{R}P^2) = 0$ since $\mathbb{R}P^2$ is non-orientable. On the other hand, by Kunneth 1.73, we know that

$$\begin{aligned} H_2(S^2 \times \mathbb{R}P^\infty) &\cong \bigoplus_{i+j=2} H_i(S^2) \otimes H_j(\mathbb{R}P^\infty) \oplus \bigoplus_{i+j=2} \operatorname{Tor}(H_{i-1}(S^2), H_j(\mathbb{R}P^\infty)) \\ &\cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \oplus \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \\ &\cong \mathbb{Z} \end{aligned}$$

since $\mathbb{Z}$ is a flat $\mathbb{Z}$-module (so $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) = 0$).

Theorem 1.29 (Whitehead for homology)**.** *Let X and Y be simply connected CW complexes. If a map $f : X \to Y$ induces isomorphisms on homology $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ for all n, then f is a homotopy equivalence.*

Proof. Let M_f be the mapping cylinder of f, and consider the inclusion $X \hookrightarrow M_f$. Recall that M_f deformation retracts onto Y, and that the map f factors as

$$X \overset{\iota}{\hookrightarrow} M_f \xrightarrow{r} Y.$$

Since $f_* : H_n(X) \xrightarrow{\cong} H_n(Y)$ and $r_* : H_n(M_f) \xrightarrow{\cong} H_n(Y)$ (the retraction r is a homotopy equivalence, hence induces isomorphisms on homology), then $\iota_* : H_n(X) \to H_n(M_f)$ is an isomorphism for all $n \geqslant 0$. If we can show that this implies $\iota_* : \pi_n(X) \to \pi_n(M_f)$ is an isomorphism for all n, then Whitehead's theorem 1.22 will imply that ι is a homotopy equivalence between X and M_f (and thus $f = r \circ \iota$ is a homotopy equivalence between X and Y).

Since X is simply-connected, the Hurewicz theorem 1.21 tells us that (assuming X is $(n-1)$-connected, with $n \geqslant 2$) $h : \pi_i(X, x_0) \to H_i(X)$ is an isomorphism for $i = n$ and is surjective for $i = n+1$. In particular, we know that $h : \pi_2(X) \to H_2(X)$ is an isomorphism, and that $h : \pi_2(M_f) \to H_2(M_f)$ is an isomorphism. Since the Hurewicz homomorphism is natural, the map induced by ι on homotopy groups $\pi_2(X) \to \pi_2(M_f)$ is the composition

$$\begin{tikzcd} H_2(X) \arrow[r, "\iota_*"] & H_2(M_f) \arrow[d, "h^{-1}"] \\ \pi_2(X) \arrow[u, "h"] \arrow[r, dashed] & \pi_2(M_f) \end{tikzcd}$$

and thus is an isomorphism. Now suppose that $\iota_* : \pi_k(X) \to \pi_k(M_f)$ is an isomorphism for $k < n$. By definition, this means the pair (M_f, X) is $(n-1)$-connected. Since X and M_f are simply-connected (the latter of which holds since $M_f \simeq Y$, which is assumed to be simply-connected), the relative Hurewicz theorem 1.21 gives that $h : \pi_n(M_f, X) \to H_n(M_f, X)$ is an isomorphism. Since $H_i(M_f, X) = 0$ for all i (LES sequence of a pair), then $\pi_n(M_f, X) = 0$, which implies that $\iota_* : \pi_n(X) \to \pi_n(M_f)$ is an isomorphism. □

Corollary 1.30. *Let X be a simply connected CW complex. If the reduced homology of X vanishes in all degrees, then X is contractible.*

Proof. Since $\tilde{H}_n(X) = 0$ for all $n \geqslant 0$, then $f_* : H_n(X) \to H_n(\{*\})$ is an isomorphism for all $n \geqslant 0$. Since X and $\{*\}$ are simply-connected, 1.29 implies that $X \simeq \{*\}$. □

Corollary 1.31. *Let X be a simply-connected CW complex and $n \geqslant 2$. Suppose the reduced homology of X is nonzero in exactly one degree (degree n), and is free abelian,*

$$\tilde{H}_k(X) \cong \begin{cases} 0 & k \neq n \\ \bigoplus_I \mathbb{Z} & k = n \end{cases}$$

Then X is homotopy equivalent to a wedge of n-spheres, $X \simeq \bigvee_I S^n$.

Proof. Since $\tilde{H}_k(X) = 0$ for $k < n$, then by Hurewicz 1.21, it must be the case that $h : \pi_n(X) \to H_n(X)$ is an isomorphism. Thus $\pi_n(X) \cong \bigoplus_I \mathbb{Z}$, so for each generator α of $\bigoplus_I \mathbb{Z}$ we let $f_\alpha : (S^n, s_0) \to (X, x_0)$ be a map representation α. Notice that all maps f_α agree on the basepoint s_0, so they induce a well-defined map $\bigvee_I f_\alpha : \bigvee_I S^n \to X$. This in turn induces an isomorphism on reduced homology in all degrees, so by 1.29 we know it is a homotopy equivalence (as X and $\bigvee S^n$ are both simply-connected for $n \geqslant 2$). □

Corollary 1.32. *Let $n \geqslant 1$. Any $(n-1)$-connected CW complex of dimension n is either contractible or homotopy equivalent to a wedge of n-spheres.*

Proof. If $n = 1$, then the statement is equivalent to one that every connected graph is either contractible (if a tree) or homotopy equivalent to a wedge of circles, which we know is true by contracting contractible subcomplexes. If $n \geqslant 2$, then X is simply-connected. By the Hurewicz theorem 1.21, X is also $(n-1)$-homologically connected, so the only nonzero reduced homology group can be in dimension n (since X is n-dimensional, X has no cells in dimension $> n$, which implies that $H_i(X) = 0$ for $i \geqslant n+1$). We also note that $\tilde{H}_n(X)$ must be either 0 (if $\ker(C_n(X) \to C_{n-1}(X)) = 0$) or free abelian, being a subgroup of a free abelian group $C_n(X)$. By corollaries 1.30 and 1.31, we conclude that X is either contractible or homotopy equivalent to a wedge of spheres. □

Corollary 1.33. *(a) Suppose that a CW complex X is $(d-1)$-connected. For $0 \leqslant k \leqslant d$, its k-skeleton is either contractible or homotopy equivalent to a wedge of k-spheres.*

(b) Compute the homotopy type of the k-skeleton of a standard n-simplex.

Proof. (a) Since the k-skeleton of X is k-dimensional, and $\pi_i(X) = \pi_i(X^{(i+1)})$, then $\pi_i(X^{(k)}) = 0$ for $i \leqslant k-1$. By corollary 1.32, we see that X is either contractible or a wedge of k-spheres.

(b) See example 3.81.

□

Hatcher page 73. An exercise at the end of the section is to show that for actions on Hausdorff spaces, freeness plus proper discontinuity implies condition (*) (covering space action!!)

1.2.3 Fibrations

Definition 1.34. Let E (total space) and B (base space) be topological spaces. Say a map $p : E \to B$ has the *homotopy lifting property* (HLP) with respect to a space X if, for every map $\tilde{f} : X \to E$ and homotopy

$F : X \times I \to B$ such that $F_0 = p \circ \tilde{f}$ (so $\tilde{f}$ is a lift of F_0), there exists a (not necessarily unique) lift $\tilde{F} : X \times I \to E$ such that $\tilde{F}_0 = \tilde{f}$ and $F = p \circ \tilde{F}$.

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "F_t"] \arrow[ur, dashed, "\tilde{F}"] & B
\end{tikzcd}$$

Definition 1.35. Say $p : E \to B$ is a *(Hurewicz) fibration* if it has the HLP with respect to every space X. Call it a *Serre fibration* if it has the HLP with respect to disks $D^n, n \geqslant 0$. (This implies HLP with respect to CW pairs).

For a fibration, all fibers $p^{-1}(b_0)$ ($b_0 \in B$) are homotopy equivalent. So we can pick a basepoint $* \in B$ and call $F = p^{-1}(*)$ "the" fiber of p. Express the situation as a sequence

$$F \to E \xrightarrow{p} B.$$

Example 1.36. A covering map $p : \tilde{X} \to X$ (for X locally path-connected) is a fibration with discrete fibers.

Proof. Recall that covering maps have the HLP. □

Prop 1.37. *Given a fibration $F \overset{\iota}{\hookrightarrow} E \xrightarrow{\pi} B$ we have a long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F) \xrightarrow{\iota_*} \pi_n(E) \xrightarrow{\pi_*} \pi_n(B) \xrightarrow{\delta} \pi_{n-1}(F) \to \cdots \to \pi_1(B) \xrightarrow{\delta} \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 0.$$

Proof. We need to construct the connecting homomorphism δ. Let $\gamma : I \to B$ be a based loop (at $b \in B$, where $F = \pi^{-1}(b)$) representative in $\pi_1(B)$. Since π has the homotopy lifting property (and hence the path lifting property), the loop $\gamma : I \to B$ lifts to a path $\tilde{\gamma} : I \to E$ with $\pi(\tilde{\gamma})(0) = \gamma(0) = b$. Pick the lift $\tilde{\gamma}$ for which $\tilde{\gamma}(0)$ is the basepoint of F. We can then define $\delta(\gamma) = \tilde{\gamma}(1) \in \pi_0(F)$. (Is this well-defined? Need to check: if I pick another representative $\alpha : I \to B$ of $[\gamma]$, does δ map the lift of α based at f to the same thing as it did the lift of γ based at f? We need to show $\tilde{\alpha}(1)$ and $\tilde{\gamma}(1)$ lie in the same path component of F, so it suffices to give a path between them. Consider $\overline{\tilde{\alpha}} \cdot \tilde{\gamma} : I \to E$ which starts at $\tilde{\alpha}(1)$ and ends at $\tilde{\gamma}(1)$.)

More generally, given a representative $\gamma : I^n \to B \in \pi_n(B)$ (so $\partial I^n \mapsto b$), we know that $I^{n-1} \times \{0\}$ maps to $b \in B$, and hence we can define $\tilde{\gamma}$ on $I^{n-1} \times \{0\}$ to be the constant map to the basepoint f of F in E.

$$\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, "\mathrm{cnst}_f"] \arrow[d, hook] & E \arrow[d, "\pi"] \\
I^{n-1} \times I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & B
\end{tikzcd}$$

The homotopy lifting property extends this lift to a map $\tilde{\gamma} : I^{n-1} \times I \to E$, and we can define $\delta([\gamma]) = [\tilde{\gamma}(1)] \in \pi_{n-1}(F)$. □

See 1.5.4 for the construction of an associated spectral sequence for fibrations.

1.3 Approximation theorems

Theorem 1.38 (Cellular approximation theorem)**.** *Every continuous map $f : X \to Y$ of CW complexes is homotopic to a cellular map (one for which $f(X^{(n)}) \subseteq Y^{(n)}$). If f is already cellular on a subcomplex $A \subseteq X$, the homotopy may be taken to be stationary on A.*

Theorem 1.39 (Simplicial approximation theorem)**.** *If $f : K \to L$ is a continuous map between simplicial complexes, there exists some subdivision* 3.47 *K' of K and a simplicial map $g : K' \to L$ such that $f \simeq g$.*

Theorem 1.40 (Zeeman's relative simplicial approximation theorem)**.** *Let $f : K \to M$ be a continuous map between finite simplicial complexes, and let $L \subseteq K$ be a subcomplex such that $f|_L$ is a simplicial map. Then there exists some subdivision K' of K (keeping L fixed, so a subdivision of $K \backslash L$) and a simplicial map $g : K' \to M$ such that $g \simeq f$ keeping L fixed, and $g|_L = f|_L$.*

1.4 Semisimplicial objects & trisps

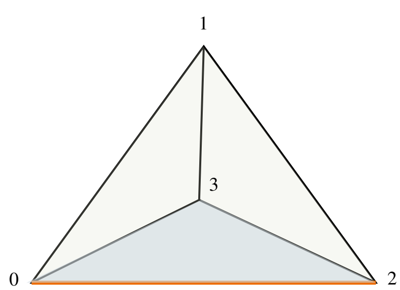
References: [4] and [6].

Definition 1.41 (Semisimplicial category)**.** Let Δ_+ be the semisimplicial category with

- objects = $[n]_0 = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{Z}_{\geqslant 0}$.
- morphisms = strictly increasing maps. For $i = 0, \ldots, n+1$, can write δ_i for the strictly increasing map $[n]_0 \to [n+1]_0$ which misses the element i, that is,

$$\delta_i(j) = j \ \forall j < i$$
$$\delta_i(j) = j + 1 \ \forall j \geqslant i$$

These morphisms satisfy $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$ for $i < j$. Pictorally: consider the orange 1-simplex.



It gets included via δ_1 into the beige simplex, and then via δ_3 into the entire tetrahedron. On the other hand, it gets included via δ_2 into the blue simplex, and then via δ_1 into the entire tetrahedron. We see then that $\delta_3 \circ \delta_1 = \delta_1 \circ \delta_2$, as claimed. Any strictly increasing map can be written as a composition of such maps δ_i, uniquely up to this identity.

Definition 1.42 (Semisimplicial object)**.** Let C be a category. A semisimplicial object in C is a *covariant* functor

$$X_\bullet : \Delta_+^{\mathrm{op}} \to C$$

where $\mathrm{Hom}_{\Delta_+^{\mathrm{op}}}([p]_0, [p-1]_0) = \mathrm{Hom}_{\Delta_+}([p-1]_0, [p]_0)$. The objects $X_p := X([p]_0)$ are called the p-simplices of X. The face maps $d_i : X_p \to X_{p-1}$ are the functorial image of the maps $\delta_i : [p-1]_0 \to [p]_0$ which "miss" i. By contravariance, we obtain the relation $d_i \circ d_j = d_{j-1} \circ d_i$.

Definition 1.43 (Map of semisimplicial objects)**.** A map of semisimplicial objects $F : X_\bullet \to Y_\bullet$ in a category C is a natural transformation F of semisimplicial objects. Hence for every object $[p]_0 \in \Delta_+^{\text{op}}$ and morphism $\delta_i : [p+1]_0 \to [p]_0$ in Δ_+^{op}, have a commutative diagram

$$\begin{tikzcd} X_{p+1} \arrow[r, "F_{p+1}"] \arrow[d, "X(\delta_i)=d_i"'] & Y_{p+1} \arrow[d, "d_i=Y(\delta_i)"] \\ X_p \arrow[r, "F_p"'] & Y_p \end{tikzcd}$$

Definition 1.44 (Semisimplicial set)**.** A semisimplicial object with $C = \underline{\text{Set}}$. Then elements of $X_p = X([p]_0)$ should be thought of as topological p-simplices, each with their faces ordered 0 through p. Then the face maps $d_i = X(\delta_i) : X_p \to X_{p-1}$ tell us gluing instructions (how to identify the i^{th} face of each p-simplex with a $(p-1)$-simplex). The geometric realization $||X_\bullet||$ is a topological space with quotient topology

$$||X_\bullet|| = (\bigsqcup_{p=0}^{\infty} X_p \times \Delta^p)/\sim$$

where $(\sigma, d^i(t)) \sim (d_i(\sigma), t)$ for all $\sigma \in X_p$ and $t \in \Delta^{p-1}$, and $d^i : \Delta^{p-1} \hookrightarrow \Delta^p$ is the inclusion of the i^{th} face (opposite vertex i). (Here we can imagine X_p as a *labeling* for the p-simplices.)

The n-skeleton $||X_\bullet||^{(n)}$ of $||X_\bullet||$ is the image of $\bigsqcup_{p=0}^n X_p \times \Delta^p$ under the quotient map.

Remark 1.45. The geometric realization of a semisimplicial set need not be a simplicial complex, as simplices need not be specified by their vertices.

To compute the homology of $||X_\bullet||$, we take the chain complex of p-chains $\mathbb{Z}X_p$ and differential given by the alternating sum of face maps $d = \sum_{i=0}^p (-1)^i d_i : \mathbb{Z}X_p \to \mathbb{Z}X_{p-1}$.

Definition 1.46 (Semisimplicial space)**.** Let $X_\bullet$ denote a semisimplicial space (a functor $X_\bullet : \Delta_+^{\text{op}} \to \underline{\text{Top}}$). For each $p \geqslant 0$ we obtain a *topological space* $X_p = X([p]_0)$ parameterizing p-simplices of $X_\bullet$.

Remark 1.47. While the geometric realization of a semisimplicial set *is* a CW complex (the set X_n can be seen as a topological space with the discrete topology, so we are really just taking a disjoint union of copies of Δ^n for each n and gluing), the geometric realization of a semisimplicial *space* need not be a CW complex. Here the p-simplices X_p are topological spaces and carry their own topology, complicating matters. Consider, for instead, if $X_1 = S^1$, in which case $X_1 \times \Delta^1$ is an annulus, and the gluing is more complicated. (We can't glue in simplices one by one in this case!)

Definition 1.48 (Levelwise properties)**.** Say that a map of semisimplicial spaces $f_\bullet : X_\bullet \to Y_\bullet$ has a property levelwise if each $f_p : X_p \to Y_p$ has that property.

Lemma 1.49. *For $m \geqslant n$ the inclusion $||X_\bullet||^{(n)} \to ||X_\bullet||^{(m)}$ is n-connected, and the inclusion $||X_\bullet||^{(n)} \to ||X_\bullet||$ is n-connected.*

Lemma 1.50. *If $f_\bullet : X_\bullet \to Y_\bullet$ is a map of semisimplicial spaces which is a levelwise homotopy equivalence, then $||f_\bullet|| : ||X_\bullet|| \to ||Y_\bullet||$ is a homotopy equivalence.*

Prop 1.51. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map of semi-simplicial spaces. If $f_p : X_p \to Y_p$ is $(k-p)$-connected for all p, then $||f_\bullet||$ is k-connected.*

Proof. By 1.49, it suffices to show that $||f_\bullet||^{(n)} : ||X_\bullet||^{(n)} \to ||Y_\bullet||^{(n)}$ is k-connected for each n, as then the maps on π_i induced by the vertical maps

$$\begin{tikzcd} ||X_\bullet||^{(n)} \arrow[r, "||f_\bullet||^{(n)}"] \arrow[d] & ||Y_\bullet||^{(n)} \arrow[d] \\ ||X_\bullet|| \arrow[r, dashed] & ||Y_\bullet|| \end{tikzcd}$$

are isomorphisms for $i < n$ and the maps on π_i induced by $||f_\bullet||^{(n)}$ are isomorphisms for $i \leqslant k$, so that the maps on π_i induced by the bottom horizontal map is an isomorphism for $i \leqslant k$ (don't we only need the top map to be an isomorphism for some $n \geqslant k$??. □

Definition 1.52 (Homotopy quotient). Let X be a space with a G-action. The homotopy quotient $X /\!\!/ G$ is the balanced product $X \times_G \mathrm{EG}$ where EG is a contractible space with free and properly discontinuous G-action.

Remark 1.53. In the definition above (1.52), taking $X = *$ gives $* \times_G \mathrm{EG} = \mathrm{EG}/G = \mathrm{BG}$.

Some properties of the homotopy quotient:

Theorem 1.54 (Properties of Homotopy Quotient). *(a) If $X \to Y$ is an equivariant map of G-spaces, there is an induced map $X /\!\!/ G \to Y /\!\!/ G$.*

(b) If $X \to Y$ is an equivariant map of G-spaces which is homologically d-connected then the induced map $X /\!\!/ G \to Y /\!\!/ G$ is homologically d-connected.

(c) If $X_\bullet$ is a semisimplicial G-space, then $||X_\bullet /\!\!/ G|| \cong ||X_\bullet|| /\!\!/ G$.

(d) If S is a transitive G-set, then $S /\!\!/ G \cong B\,\mathrm{stab}_G(s)$ for any $s \in S$.

1.5 Spectral sequences

There are many ways in which spectral sequences arise in nature.

1.5.1 Double complexes

Cohomologically graded: Start with a collection of, say, vector spaces $E^{p,q}$ with horizontal (rightward) and vertical (upward) differentials

$$\begin{tikzcd} E^{p,q+1} \arrow[r, "d_>"] & E^{p+1,q+1} \\ E^{p,q} \arrow[u, "d_\wedge"] \arrow[r, "d_>"'] & E^{p+1,q} \arrow[u, "d_\wedge"'] \end{tikzcd}$$

which anticommute, that is, $d_> \circ d_\wedge + d_\wedge \circ d_> = 0$, and $d_\wedge^2 = 0$, $d_>^2 = 0$. A first quadrant cohomologically graded double complex has $E^{p,q} = 0$ for $p, q < 0$.

From the double complex we form a single complex called the total complex $\mathrm{Tot}(E^\bullet)$. This has chain groups $E^n = \bigoplus_{p+q=n} E^{p,q}$ with differential

$$E^n \xrightarrow{d_> + d_\wedge} E^{n+1}.$$

Check that this is truly a differential ($(d_> + d_\wedge)^2 = 0$). The two spectral sequences arising from the double complex both help us compute the cohomology of the total complex. We don't always care about this cohomology, but instead the information that arises from the fact that both ways converge to the same thing.

The two spectral sequences that arise: one with "rightwards" orientation and one with "upwards" orientation, depending on whether we take homology with respect to the rightwards or upwards morphisms first. Say we take "rightwards" orientation first. This means that ${}_{>}E_1$ is produced by taking horizontal homology, and looks like

$$
\begin{tikzcd}
q \\
& {}_{>}E_1^{0,2} & {}_{>}E_1^{1,2} & {}_{>}E_1^{2,2} \\
& {}_{>}E_1^{0,1} \arrow[u] & {}_{>}E_1^{1,1} \arrow[u] & {}_{>}E_1^{2,1} \arrow[u] \\
& {}_{>}E_1^{0,0} \arrow[u] & {}_{>}E_1^{1,0} \arrow[u] & {}_{>}E_1^{2,0} \arrow[u] \\
& & & & p
\end{tikzcd}
$$

where the vertical differentials are induced by the original vertical maps on the E_0 page (the original double complex). It is not too hard to check that the original vertical differentials descend to the homology groups present on the ${}_{>}E_1$ page. When we flip the page, we get

$$
\begin{tikzcd}
q \\
& {}_{>}E_2^{0,2} & {}_{>}E_2^{1,2} & {}_{>}E_2^{2,2} \\
& {}_{>}E_2^{0,1} & {}_{>}E_2^{1,1} & {}_{>}E_2^{2,1} \\
& {}_{>}E_2^{0,0} & {}_{>}E_2^{1,0} \arrow[uul] & {}_{>}E_2^{2,0} \arrow[uul] \\
& & & & p
\end{tikzcd}
$$

where now the diagonal map $d_2 : {}_{>}E_2^{p,q} \to {}_{>}E_2^{p-1,q+2}$ is induced by a diagram chase on the original double complex as follows:

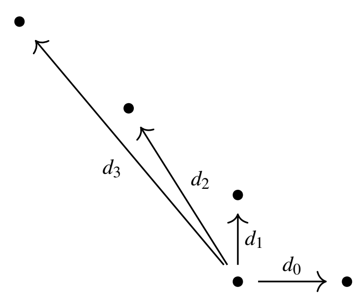
$$\begin{tikzcd}
d_\wedge(z) & E_0^{p-1,q+2} & & & \\
& & \exists z \arrow[rr, mapsto] & & d_\wedge^{p,q}(x) \\
z \arrow[uu, mapsto] & E_0^{p-1,q+1} \arrow[uu, "d_\wedge"] \arrow[r, "d_>"'] & E_0^{p,q+1} & & \\
& & E_0^{p,q} \arrow[u, "d_\wedge"] & & x \arrow[uu, mapsto]
\end{tikzcd}$$

where $x \in E_0^{p,q}$ is a representative of $\overline{x} \in \ker d_1^{p,q}$. Since $d_1^{p,q}(\overline{x}) = \overline{0} \in_> E_1^{p,q+1} = \ker d_>^{p,q+1} / \operatorname{Im} d_>^{p-1,q+1}$, then $d_\wedge(x) \in \operatorname{Im} d_>^{p-1,q+1}$, giving the z in the diagram.

Continuing this way, if our original double complex is bounded, then eventually, for every fixed p, q, there exists an $r \gg 0$ such that every arrow d_r is the 0 map (will come from a 0 vector space or go to one). If there exists r for which this holds for all p, q simultaneously, then taking homology yields nothing new, and our spectral sequence collapses (or stabilizes to E_∞). We say this spectral sequence abuts or converges to $H^\bullet(E^\bullet)$, which means there exists a filtration of $H^\bullet(E^\bullet)$

$$0 = F^{-1}H^n(E^\bullet) \subset F^0 H^n(E^\bullet) \subset \cdots F^n H^n(E^\bullet)$$

with successive quotients $F^p H^n / F^{p-1} H^n \cong_> E_\infty^{p,n-p}$. Key for the arrows:



Suppose instead we took the upwards orientation, so the ${}_\wedge E_1$ page is formed by taking homology with respect to the vertical differentials. We get that ${}_\wedge E_1$ looks like

$$\begin{tikzcd}
q & & & \\
& {}_\wedge E_1^{0,2} \arrow[r] & {}_\wedge E_1^{1,2} \arrow[r] & {}_\wedge E_1^{2,2} \\
& {}_\wedge E_1^{0,1} \arrow[r] & {}_\wedge E_1^{1,1} \arrow[r] & {}_\wedge E_1^{2,1} \\
& {}_\wedge E_1^{0,0} \arrow[r] & {}_\wedge E_1^{1,0} \arrow[r] & {}_\wedge E_1^{2,0} \\
& & & & p
\end{tikzcd}$$

where the horizontal differentials are induced by the original rightwards maps. Turn the page and obtain

$$\begin{tikzcd}
{}_{\wedge}E_2^{0,2} \arrow[rrd] & {}_{\wedge}E_2^{1,2} & {}_{\wedge}E_2^{2,2} \\
{}_{\wedge}E_2^{0,1} \arrow[rrd] & {}_{\wedge}E_2^{1,1} & {}_{\wedge}E_2^{2,1} \\
{}_{\wedge}E_2^{0,0} & {}_{\wedge}E_2^{1,0} & {}_{\wedge}E_2^{2,0}
\end{tikzcd}$$

(axes: q vertical, p horizontal)

and so on. Again this d_2 map is induced by a diagram chase on the original double complex:

$$\begin{tikzcd}
x \arrow[rr, mapsto] & & d_>^{p,q}(x) & \\
E_0^{p,q} \arrow[r, "d_>"'] & E_0^{p+1,q} & & \\
& E_0^{p+1,q-1} \arrow[u, "d_\wedge"] \arrow[rr, "d_>"'] & \exists z \arrow[uu, mapsto] & E_0^{p+2,q-1} \\
& z \arrow[rr, mapsto] & & d_>(z)
\end{tikzcd}$$

where $x \in E_0^{p,q}$ is a representative of $\overline{x} \in \ker d_1^{p,q}$. Since $d_1(\overline{x}) = \overline{d_>(x)} \in E_1^{p+1,q} = \ker d_\wedge^{p+1,q} / \operatorname{Im} d_\wedge^{p+1,q-1}$, we know that $d_>(x) \in \operatorname{Im} d_\wedge^{p+1,q-1}$. Of course one should check that everything is well-defined, since many choices are being made. But this is tedious.

Key for the arrows:

$$\begin{tikzcd}
\bullet & & & \\
\bullet \arrow[u, "d_0"] \arrow[r, "d_1"] \arrow[rrd, "d_2"] \arrow[rrrdd, "d_3"'] & \bullet & & \\
& & \bullet & \\
& & & \bullet
\end{tikzcd}$$

We now get a filtration of the cohomology of the total complex (opposite to that for the rightwards orientation):

$$0 = F^{-1}H^n \subset F^0 H^n \subset \cdots \subset F^n H^n = H^n(E^\bullet)$$

where now $F^pH^n/F^{p-1}H^n \cong_{\wedge} E_\infty^{n-p,p}$. (Notice: in this case we actually get what looks like a cohomologically graded spectral sequence, so the filtration here makes sense (is opposite to our usual homological filtration!!) Note that we should not expect ${}_{\wedge}E_\infty^{p,q} \cong_{>} E_\infty^{p,q}$ in general.

Applications of the two filtrations. Let's see some ways that these two filtrations help us solve problems.

Example 1.55 (Snake Lemma)**.** Let

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows. There is an exact sequence

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0.$$

Proof. View the given commutative diagram as a double complex. Taking the *rightwards* orientation first, since rows are exact, we see that the ${}_{>}E_1$ page is (almost) entirely zero:

$$\begin{tikzcd} 0 & 0 & 0 & \operatorname{coker}(E \to F) & 0 \\ 0 \arrow[u] & \ker(A \to B) \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

(axes q vertical, p horizontal)

All arrows are now zero, so our spectral sequence stabilizes on this page. Hence ${}_{>}E_1^{p,q} =_{>} E_\infty^{p,q}$, and the cohomology of the total complex is 0 in all but possibly two degrees. In particular, we see that $H^1(E^\bullet) = \ker(A \to B)$ and $H^4(E^\bullet) = \operatorname{coker}(E \to F)$.

Taking the upwards orientation now, we flip to the ${}_{\wedge}E_1$ page and get

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma \arrow[r] & 0 \\ 0 \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[r] & 0 \end{tikzcd}$$

(axes q vertical, p horizontal)

When we flip to the ${}_{\wedge}E_2$ page we get

$$\begin{tikzcd}[column sep=small]
0 \arrow[drr] & \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta) \arrow[drr, "g"] & \ker(\operatorname{coker}\beta \to \operatorname{coker}\gamma)/\operatorname{Im}(\operatorname{coker}\alpha \to \operatorname{coker}\beta) \arrow[drr] & \operatorname{coker}(\operatorname{coker}\beta \to \operatorname{coker}\gamma) & \\
0 & \ker(\ker\alpha \to \ker\beta) & \ker(\ker\beta \to \ker\gamma)/\operatorname{Im}(\ker\alpha \to \ker\beta) & \operatorname{coker}(\ker\beta \to \ker\gamma) & 0
\end{tikzcd}$$

so that all arrows but one are now zero. When we make one final flip we get the stable page ${}_{\wedge}E_3 = {}_{\wedge}E_\infty$:

$$\begin{tikzcd}[column sep=small]
0 & \ker g \arrow[ddrrr] & \ker(\operatorname{coker}\beta \to \operatorname{coker}\gamma)/\operatorname{Im}(\operatorname{coker}\alpha \to \operatorname{coker}\beta) & \operatorname{coker}(\operatorname{coker}\beta \to \operatorname{coker}\gamma) & \\
0 & \ker(\ker\alpha \to \ker\beta) & \ker(\ker\beta \to \ker\gamma)/\operatorname{Im}(\ker\alpha \to \ker\beta) & \operatorname{coker} g & 0 \\
 & & & & 0
\end{tikzcd}$$

Since we know from the first orientation that $H^\bullet(E^\bullet) = 0$ in all but two degrees, this means that

(i) $\ker(\ker\beta \to \ker\gamma)/\operatorname{Im}(\ker\alpha \to \ker\beta) = 0$. This gives exactness at $\ker\alpha \to \ker\beta \to \ker\gamma$.

(ii) Both $\ker g = 0$ and $\operatorname{coker} g = 0$. Hence g is an isomorphism, which means that we have an exact sequence

$$\begin{tikzcd}
\ker\beta \arrow[r] & \ker\gamma \arrow[rr, dashed] \arrow[dr] & & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \\
 & & \operatorname{coker}(\ker\beta \to \ker\gamma) \cong \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta) \arrow[ur] \arrow[d, bend right] & & \\
 & & 0 \arrow[u, bend right] & &
\end{tikzcd}$$

If in addition $\ker(A \to B) = 0$, then $\ker(\ker\alpha \to \ker\beta) = 0$. This gives an exact sequence $0 \to \ker\alpha \to \ker\beta$. If in addition $\operatorname{coker}(E \to F) = 0$, then $\operatorname{coker}(\operatorname{coker}\beta \to \operatorname{coker}\gamma) = 0$, so $\operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$ is exact, and we are done. $\square$

Example 1.56 (Five Lemma)**.** Suppose

$$\begin{tikzcd}
F \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & J \\
A \arrow[r] \arrow[u, "\alpha"] & B \arrow[r] \arrow[u, "\beta"] & C \arrow[r] \arrow[u, "\gamma"] & D \arrow[r] \arrow[u, "\delta"] & E \arrow[u, "\varepsilon"]
\end{tikzcd}$$

is a commutative diagram with exact rows. If β, δ are injective, and α is surjective, then γ is injective. If β, δ are surjective, and ε is injective, then γ is surjective. In particular: if α is surjective, ε is injective, and β, δ are isomorphisms, then γ is an isomorphism.

Proof. We prove the first statement, since the second follows analogously (and the third is an immediate corollary of the first two). We start with the rightwards orientation. Since rows are exact, the ${}_{>}E_1$ page looks like

$$\begin{tikzcd}
q & & & & & \\
& ? & 0 & 0 & 0 & ? \\
& ? \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & ? \arrow[u] \\
& & & & & p
\end{tikzcd}$$

When we turn the page all maps are 0, so ${}_{>}E_2 =_{>} E_\infty$:

$$\begin{tikzcd}
q & & & & & \\
& ?? & 0 & 0 & 0 & ?? \\
& ?? & 0 \arrow[uul] & 0 \arrow[uul] & 0 & ?? \arrow[uul] \\
& & \arrow[uul] & \arrow[uul] & & p
\end{tikzcd}$$

Note in particular that $H^2(E^\bullet) = 0$.

On the other hand, taking the upwards orientation yields an ${}_{\wedge}E_1$ page of

$$\begin{tikzcd}
q & & & & & \\
& 0 \arrow[r] & ? \arrow[r] & ? \arrow[r] & ? \arrow[r] & ? \\
& ? \arrow[r] & 0 \arrow[r] & \ker\gamma \arrow[r] & 0 \arrow[r] & ? \\
& & & & & p
\end{tikzcd}$$

When we turn the page some of the single question marks stay the same, while the others change. The ${}_{\wedge}E_2$ page looks like:

$$\begin{tikzcd}
q & & & & & \\
& 0 \arrow[drr] & ?? \arrow[drr] & ?? \arrow[drr] & ?? & ?? \\
& ? \arrow[drr] & 0 & \ker\gamma \arrow[drr] & 0 & ? \\
& & & & & p
\end{tikzcd}$$

All but one of these maps are zero. We turn the page again and get that ${}_\wedge E_3 = {}_\wedge E_\infty$. In particular, the diagonal corresponding to $H^2(E^\bullet)$ does not change when we flip the page to ${}_\wedge E_3$, and this diagonal has $\ker\gamma$ as its only (possibly) nonzero entry. We conclude that $0 = H^2(E^\bullet) = \ker\gamma$. □

Example 1.57 (SES induces LES)**.** Use spectral sequences to show that a short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology.

Proof. We really just use the snake lemma here. Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of cochain complexes. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \ker d_A^{i+1} \arrow[r] & \ker d_B^{i+1} \arrow[r] & \ker d_C^{i+1} \\
& A^i/\operatorname{Im} d_A^{i-1} \arrow[r] \arrow[u, "\overline{d_A^i}"] & B^i/\operatorname{Im} d_B^{i-1} \arrow[r] \arrow[u, "\overline{d_B^i}"] & C^i/\operatorname{Im} d_C^{i-1} \arrow[r] \arrow[u, "\overline{d_C^i}"'] & 0
\end{tikzcd}$$

By the snake lemma, we have an exact sequence

$$\ker\overline{d_A^i} \to \ker\overline{d_B^i} \to \ker\overline{d_C^i} \to \operatorname{coker}\overline{d_A^i} \to \operatorname{coker}\overline{d_B^i} \to \operatorname{coker}\overline{d_C^i}.$$

But $\ker\overline{d_A^i} = \ker d_A^i/\operatorname{Im} d_A^{i-1} = H^i(A^\bullet)$, and $\operatorname{coker}\overline{d_A^i} = \ker d_A^{i+1}/\operatorname{Im} d_A^i = H^{i+1}(A^\bullet)$, so our exact sequence becomes

$$H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to H^{i+1}(C^\bullet).$$

□

Example 1.58 (Mapping cone)**.** Suppose $\mu : A^\bullet \to B^\bullet$ is a morphism of cochain complexes. Let $C^\bullet$ be the single complex associated to the double complex $A^\bullet \to B^\bullet$ ($C^\bullet$ is the *mapping cone* of μ). There exists a long exact sequence of complexes

$$\cdots \to H^{i-1}(C^\bullet) \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

Proof. Writing the cochain complex $A^\bullet$ in the 0^{th} column and $B^\bullet$ in the 1^{st} column, we get that $C^n = A^n \oplus B^{n-1}$ with differentials

$$A^n \oplus B^{n-1} \xrightarrow{d_\wedge + d_>} A^{n+1} \oplus B^n$$

stemming from the horizontal differentials $\mu^n : A^n \to B^n$ and the vertical differentials of the cochain complexes. For each n we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B^n \arrow[r, "\iota"] & A^{n+1} \oplus B^n \arrow[r, "\pi"] & A^{n+1} \arrow[r] & 0 \\
0 \arrow[r] & B^{n-1} \arrow[r, "\iota"] \arrow[u, "d_\wedge"'] & A^n \oplus B^{n-1} \arrow[r, "\pi"] \arrow[u, "d_\wedge + d_>"'] & A^n \arrow[r] \arrow[u, "d_\wedge"] & 0
\end{tikzcd}$$

and hence a short exact sequence of cochain complexes $0 \to B^\bullet \to C^\bullet \to A^{\bullet+1} \to 0$. Applying the previous example gives the desired conclusion. □

1.5.2 Filtered complexes

Reference: Brown Cohomology of Groups [2].

Suppose we have a chain complex C and a subcomplex C'. Have a long exact sequence giving information about $H_*(C)$ from $H_*(C')$ and $H_*(C/C')$ (as a short exact sequence of complexes

$$0 \to C'_\bullet \to C_\bullet \to (C/C')_\bullet \to 0$$

induces a long exact sequence in homology).

What if instead we're given a *sequence of subcomplexes* $\{F_pC\}_{p\in\mathbb{Z}}$ with $F_{p-1}C \subseteq F_pC \subseteq C$? We want info on $H_*(C)$ in terms of $H_\bullet(F_pC/F_{p-1}C)$. We obtain a series of successive approximations E^r ($r \geqslant 0$) to $H_\bullet(C)$, so E^r consists of $H_\bullet(F_pC/F_{p-1}C)$.

When we say $F_{p-1}C \subseteq F_pC$, we mean that the differential respects the filtering on the complex; that is, $\partial(F_pC_n) \subset F_pC_{n-1}$. A natural situation in which this happens: let X be a CW complex, and consider a filtration by skeleta:

$$X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots$$

Remark 1.59. Knowledge of $E_r^{*,*}$ and d_r determines $E_{r+1}^{*,*}$ but does *not* determine d_{r+1}. If some differential is unknown, some other principle is required to proceed (typically using the geometry of the space).

Definition 1.60. Given a ring R and an R-module M, an *increasing filtration* on M is a family of submodules F_pM such that $F_pM \subseteq F_{p+1}M$. We see this filtration is *finite* if $F_pM = 0$ for p sufficiently small and $F_pM = M$ for p sufficiently large. Have an *associated graded module* $\mathrm{Gr}M$ whose p^{th} graded piece is $\mathrm{Gr}_pM = F_pM/F_{p-1}M$.

If furthermore M is graded (and each F_pM is a graded submodule), then for all $n \in \mathbb{Z}$ there exists a filtration $\{F_pM_n\}$ on M_n, so we can associate a *bigraded* module $\mathrm{Gr}M$ to M with

$$\mathrm{Gr}_{pq}M = F_pM_{p+q}/F_{p-1}M_{p+q}.$$

Call p the *filtration degree*. Call q the *complementary degree*. Call $p+q$ the *total degree*.

Say we have a filtered chain complex $C = (C_n)_{n\in\mathbb{Z}}$ where each F_pC is a subcomplex, and $\{F_pC_n\}_{p\in\mathbb{Z}}$ is dimension-wise finite for each n. Obtain an induced filtration on the homology $H(C)$ via

$$F_pH(C) = \mathrm{Im}(H(F_pC) \to H(C)).$$

More explicitly,

$$F_pH_i(C) = \{\alpha \in H_i(C) : \exists x \in F_pC_i | \alpha = [x]\}$$

This has associated graded pieces $G_pH_i(C) = F_pH_i(C)/F_{p-1}H_i(C)$.

Q: How to actually use this to compute $H_i(C)$?
A: Let (F_pC, ∂) be a filtered chain complex. Denote the associated graded by

$$E^0_{p,q} = G_pC_{p+q} = F_pC_{p+q}/F_{p-1}C_{p+q}.$$

The differential on C induces the differential on the associated graded, since $\partial(F_pC_{p+q}) \subseteq F_pC_{p+q-1}$:

$$\begin{tikzcd}
E^0_{p,q} \arrow[d] \\
E^0_{p,q-1}
\end{tikzcd}$$

Then $E^1_{p,q} = H_{p+q}(G_pC)$. How to define the next differential?

$$\begin{tikzcd}
E^1_{p-1,q} & E^1_{p,q} \arrow[l]
\end{tikzcd}$$

Let $\alpha \in E^1_{p,q}$ be a homology class. This is represented by a cycle in $E^0_{p,q}$, that is, an $\overline{x} \in G_pC_{p+q}$ such that $\partial\overline{x} = 0$. This in turn is represented by an $x \in F_pC_{p+q}$ such that $\partial x \in F_{p-1}C_{p+q-1}$. So we can define $d^1(\alpha) = [\overline{\partial x}]$. Check: d^1 is well-defined and $d^1 \circ d^1 = 0$.

The E^∞ page gives the associated graded of the homology $H(C)$, that is, $E^\infty_{p,q} = F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C)$. *If C_i has a bounded filtration for each i, then our spectral sequence converges to the associated graded as given above.* For example:

Example 1.61 (LES in homology from SES of chain complexes)**.** Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of chain complexes. Let $\{F_pC\}$ be the filtration such that $F_0C = 0$, $F_1C = C'$, and $F_2C = C$ (increasing filtration). The E^0 page looks like

$$\begin{tikzcd}
 & q \\
2 & C'_3 \arrow[d] & C''_4 \arrow[d] & 0 \\
1 & C'_2 \arrow[d] & C''_3 \arrow[d] & 0 \\
0 & C'_1 & C''_2 & 0 & & p \\
 & 1 & 2 & 3
\end{tikzcd}$$

The vertical maps are just the differentials on the chain complexes C' and C''. Hence when we turn the page we get that E^1 looks like

$$
\begin{tikzcd}
 & q & & & \\
0 & H_3(C') \arrow[l] & H_4(C'') \arrow[l, "\delta"'] & 0 \arrow[l] & \\
0 & H_2(C') \arrow[l] & H_3(C'') \arrow[l, "\delta"'] & 0 \arrow[l] & \\
0 & H_1(C') \arrow[l] & H_2(C'') \arrow[l, "\delta"'] & 0 \arrow[l] & p \\
0 & H_0(C') \arrow[l] & H_1(C'') \arrow[l, "\delta"'] & 0 \arrow[l] & \\
 & 0 & H_0(C'') \arrow[l] & 0 \arrow[l] &
\end{tikzcd}
$$

Taking homology then yields

$$
\begin{tikzcd}
 & q & & & \\
0 & & & & \\
0 & & & & \\
0 & \operatorname{coker}\delta & \ker\delta \arrow[ull] & 0 & p \\
0 & \operatorname{coker}\delta & \ker\delta \arrow[ull] & 0 & \\
 & 0 & H_0(C'') & 0 \arrow[ull] &
\end{tikzcd}
$$

and we see that the spectral sequence collapses on the E^2 page. If $p = 2$ and $q = -2$, the associated graded of the homology of C tells us that

$$H_0(C'') = E^{\infty}_{2,-2} = F_2 H_0(C)/F_1 H_0(C) = H_0(C)/\operatorname{Im}(H_0(C') \to H_0(C))$$

and hence we have the start of our long exact sequence

$$H_0(C') \to H_0(C) \to H_0(C'') \to 0.$$

Next, if $p = 1$ and $q = -1$, we get

$$H_0(C')/\operatorname{Im}(H_1(C'') \to H_0(C')) = \operatorname{coker}\delta = E^{\infty}_{1,-1} = F_1 H_0(C)/F_0 H_0(C) = \operatorname{Im}(H_0(C') \to H_0(C))$$

giving exactness

$$H_1(C'') \to H_0(C') \to H_0(C) \to H_0(C'') \to 0$$

and so on.

Homologically graded spectral sequence. Let $E = \{E^r\}$ consist of a sequence of differential $\mathbb{Z}$-bigraded R-modules $E^r = \{E^r_{p,q}\}$ together with differentials

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

(so arrows point left r units and up $r-1$ units) such that $E^{r+1} \cong H_*(E^r)$ and $d^r \circ d^r = 0$. Letting $Z^r = \ker d^r$ be the cycles and $B^r = \operatorname{Im} d^r$ the boundaries, we have that d^{r+1} is a map

$$Z^r/B^r \xrightarrow{d^{r+1}} Z^r/B^r.$$

This map has kernel Z^{r+1}/B^r and image B^{r+1}/B^r. Hence we have a sequence of submodules

$$0 = B^0 \subset B^1 \subset \cdots \subset Z^2 \subset Z^1 \subset Z^0 = E^1.$$

Let $Z^\infty = \bigcap_{r\geqslant 1} Z^r$, $B^\infty = \bigcup_{r\geqslant 1} B^r$, and $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$. If we consider an element in Z^r as "surviving to the r^{th} stage", then Z^∞ consists of the submodule of E^2 of elements that survive forever (are cycles at every stage). While B^∞ consists of elements of E^2 which "eventually bound." The E^∞ page of the spectral sequence is generally the goal of the computation.

Best-case scenario: the differentials are zero from some point on (say $r \geqslant N$), in which case we have a short exact sequence

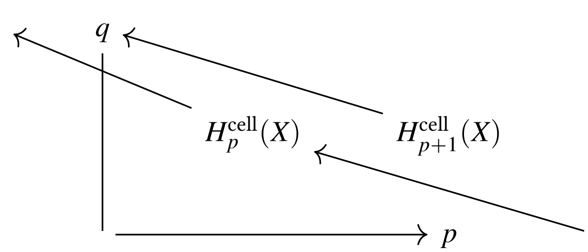
$$0 \to Z^r/B^{r-1} \to Z^{r-1}/B^{r-1} \xrightarrow{d^r} B^r/B^{r-1} \to 0.$$

The fact that $d^r = 0$ forces $B^r = B^{r-1}$ and $Z^r = Z^{r-1}$. Hence $E^\infty = E^N$.

Example 1.62. Singular homology of a CW complex agrees with its cellular homology. Let $C_\bullet X$ denote the singular chain complex of X. Define an increasing filtration $F_p C_\bullet X = C_\bullet(X^{(p)})$. Have an associated graded $E^0_{p,q} = C_{p+q}(X^{(p)})/C_{p+q}(X^{(p-1)})$. By definition, the homology of this is the relative homology

$$E^1_{p,q} = H_{p+q}(X^{(p)}, X^{(p-1)}) = \begin{cases} C^{\text{cell}}_p(X) & q = 0 \\ 0 & q \neq 0. \end{cases}$$

The cellular differential $C^{\text{cell}}_p(X) \to C^{\text{cell}}_{p-1}(X)$ is the map induced by $H_p(X^{(p)}, X^{(p-1)}) \to H_{p-1}(X^{(p-1)}, X^{(p-2)})$ in the long exact sequence of the triple $(X^{(p)}, X^{(p-1)}, X^{(p-2)})$. This agrees with the d^1 differential. Thus the E^2 page is concentrated along the $q = 0$ row, and is given by the cellular homology:



so the spectral sequence collapses on the E^2 page. Then

$$E^2_{p,q} = E^\infty_{p,q} = \operatorname{Gr}_p H_{p+q}(C_\bullet(X))$$

so $H_p^{\text{cell}}(X) = E^2_{p,0} = \operatorname{Gr}_p H_p(X) = F_pH_p(X)/F_{p-1}H_p(X)$. Then

$$F_pH_p(X)/F_{p-1}H_p(X) = \operatorname{Im}(H_p(C_\bullet(X^{(p)})) \to H_p(X))/\operatorname{Im}(H_p(C_\bullet(X^{(p-1)})) \to H_p(X))$$

But $H_p(X^{(p)}) \twoheadrightarrow H_p(X)$ and $H_p(X^{p-1}) = 0$, so we obtain

$$H_p^{\text{cell}}(X) = H_p(X).$$

Cohomologically graded spectral sequence. Suppose now we have a cochain complex $C^\bullet$ with a *decreasing* filtration, so $F^pC^\bullet \subset F^{p-1}C^\bullet$ and such that the differentials in the cochain complex respect the filtration, in the sense that $\delta(F^pC^n) \subset F^pC^{n+1}$. Have a spectral sequence with

$$E_0^{p,q} = \operatorname{Gr}_p C^{p+q} = F^pC^{p+q}/F^{p+1}C^{p+q}$$
$$E_1^{p,q} = H^{p+q}(\operatorname{Gr}_p C^\bullet) = H^{p+q}(F^pC^{p+q}/F^{p+1}C^{p+q})$$

converging to an associated graded on cohomology, where $F^pH^n(C^\bullet)$ consists of those cohomology classes $[\varphi]$ with a cocycle representative coming from F^pC^n. Since we want homology classes, this means that

$$\operatorname{Gr}_p H^n(C^\bullet) = F^pH^n(C^\bullet)/F^{p+1}H^n(C^\bullet) = \frac{\ker(\delta : F^pC^n \to C^{n+1})}{\ker(\delta : F^{p+1}C^n \to C^{n+1}) + \delta(F^pC^{n-1})}.$$

1.5.3 Geometric realization spectral sequence

Let $X_\bullet$ be a semisimplicial space as in 1.46. We can filter the geometric realization $||X_\bullet||$ by its skeleta $||X_\bullet||^{(n)}$, where $||X_\bullet||^{(0)} = X_0$, and

$$||X_\bullet||^{(n)} = ||X_\bullet||^{(n-1)} \cup_{X_n \times \partial\Delta^n} X_n \times \Delta^n.$$

Then each map $K \to ||X_\bullet||$ from a compact Hausdorff space K factors through some finite stage [4]. Let R be a ring. By the spectral sequence associated to skeletal filtrations 1.62, we have a spectral sequence with E^1 page

$$E^1_{p,q} = H_{p+q}(||X_\bullet||^{(p)}, ||X_\bullet||^{(p-1)}; R) \implies H_{p+q}(||X_\bullet||; R).$$

Let $B = X_p \times \Delta^p$ and let $A = ||X_\bullet||^{(p-1)}$. The description of the skeletal filtration above yields $A \cap B = X_p \times \partial\Delta^p$, so by excision we know the natural map

$$H_{p+q}(X_p \times \Delta^p, X_p \times \partial\Delta^p; R) \to H_{p+q}(||X_\bullet||^{(p)}, ||X_\bullet||^{(p-1)}; R)$$

is an isomorphism. By the relative Kunneth theorem 1.75, we have a Kunneth map

$$H_q(X_p; R) \otimes_R H_p(\Delta^p, \partial\Delta^p; R) \to H_{p+q}(X_p \times \Delta^p, X_p \times \partial\Delta^p; R)$$

which is an isomorphism since $H_i(\Delta^p, \partial\Delta^p; R)$ is a free R-module for all i. Since $H_p(\Delta^p, \partial\Delta^p; R) \cong R$, this means that $H_q(X_p; R) \cong H_{p+q}(||X_\bullet||^{(p)}, ||X_\bullet||^{(p-1)}; R)$. We thus have a convergent spectral sequence

$$E^1_{p,q} = H_q(X_p; R) \implies H_{p+q}(||X_\bullet||; R).$$

The d^1 differential $d^1_{p,q} : H_q(X_p; R) \to H_q(X_{p-1}; R)$ is the map $\sum_{i=0}^p (-1)^i (d_i)_*$ induced by the $d_i : X_p \to X_{p-1}$.

1.5.4 Leray-Serre Spectral Sequences

Let G be an abelian group (coefficients) and $F \hookrightarrow E \xrightarrow{\pi} B$ a fibration, with path-connected base space B and connected fiber F. Then there exists a first quadrant spectral sequence $\{E^r_{*,*}, d^r\}$ converging to $H_*(E;G)$ with

$$E^2_{p,q} \cong H_p(B; H_q(F;G)),$$

where $H_p(B; H_q(F;G))$ is the homology of B with local coefficients in homology of the fiber of π. We also have a first quadrant cohomological spectral sequence $\{E_r^{*,*}, d_r\}$ converging to $H^*(E;G)$ with

$$E_2^{p,q} = H^p(B; H^q(F;G)).$$

Construction of the spectral sequence. The fibration $F \to E \xrightarrow{\pi} B$ enjoys the homotopy lifting property. Suppose B has the homotopy type of a CW complex (WLOG, is a CW complex). Then B has a filtration given by the skeleta. Can lift this to a filtration on E by letting $E^s = \pi^{-1}(B^{(s)})$, the "subspace of E lying over the s-skeleton of B." Note: $B^{(s)} \subseteq B^{(s+1)}$ implies that $\pi^{-1}(B^{(s)}) \subseteq \pi^{-1}(B^{(s+1)})$, so $E^s \subseteq E^{s+1}$ is an increasing filtration.

$(B, B^{(p)})$ is p-connected (that is, $\pi_k(B, B^{(p)}) = 0$ for all $1 \leqslant k \leqslant p$). This follows from lemma 1.15. Claim: this implies (E, E^p) is p-connected. This follows from the fact that $E \xrightarrow{\pi} B$ has the HLP with respect to disks means it has the HLP with respect to pairs (D^n, S^{n-1}) (meaning 1.18). Let $\tilde{F}_0 : D^n \to E$ be any map with $\tilde{F}_0(S^{n-1}) \subseteq E^p$. Since $(B, B^{(p)})$ is p-connected, every map $(D^n, S^{n-1}) \to (B, B^{(p)})$ is homotopic rel S^{n-1} to a map whose image is in $B^{(p)}$. We have the map $F_0 = \pi \circ \tilde{F}_0 : (D^n, S^{n-1}) \to (B, B^{(p)})$. Let $F_t : D^n \to B$ be such a homotopy. Letting $\tilde{F}_t : S^{n-1} \to E^p$ be $\tilde{F}_0|_{S^{n-1}}$, the HLP with respect to pairs gives a lift $\tilde{F}_t : D^n \to E$ extending $\tilde{F}_t : S^{n-1} \to E^p$. Hence every map $\tilde{F}_0 : (D^n, S^{n-1}) \to (E, E^p)$ is homotopic rel S^{n-1} to a map with image in E^p (as $\pi \circ \tilde{F}_1(D^n) = F_1(D^n) \subseteq B^{(p)}$).

Hence $\pi_i(E, E^p) = 0$ for all $i \leqslant n$. LES of a pair yields $\pi_i(E) \cong \pi_i(E^p)$ for $i < p$ and we have a surjection $\pi_p(E^p) \twoheadrightarrow \pi_p(E)$. to-do: finish this section!!!

Example 1.63 (Integral cohomology ring of $K(\mathbb{Z}, 3)$)**.** The integral cohomology of a $K(\mathbb{Z}, n)$ is a bit involved. Let's see the situation for $K(\mathbb{Z}, 3)$. We have a path space fibration

$$\Omega K(\mathbb{Z}, 3) \to PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$$

where $PK(\mathbb{Z}, 3)$ is the (based) path space of a $K(\mathbb{Z}, 3)$ and $\Omega K(\mathbb{Z}, 3)$ is the based loop space, given as the fiber of the fibration $PK(\mathbb{Z}, 3) \to K(\mathbb{Z}, 3)$, $\gamma \mapsto \gamma(1)$. (Recall: path space PX is the space of continuous paths $\{\gamma : I \to X\}$ topologized using compact-open topology: a subbase for the compact open topology consists of $V(U, K)$, which is the set of paths $\gamma : I \to X$ such that K is compact in I, U is open in X, and $\gamma(K) \subseteq U$.) Since $PK(\mathbb{Z}, 3)$ is contractible (we can homotope paths which start at the basepoint through paths starting at the basepoint until we get the constant path), then the long exact sequence of homotopy groups associated to a fibration (Prop 1.37) yields

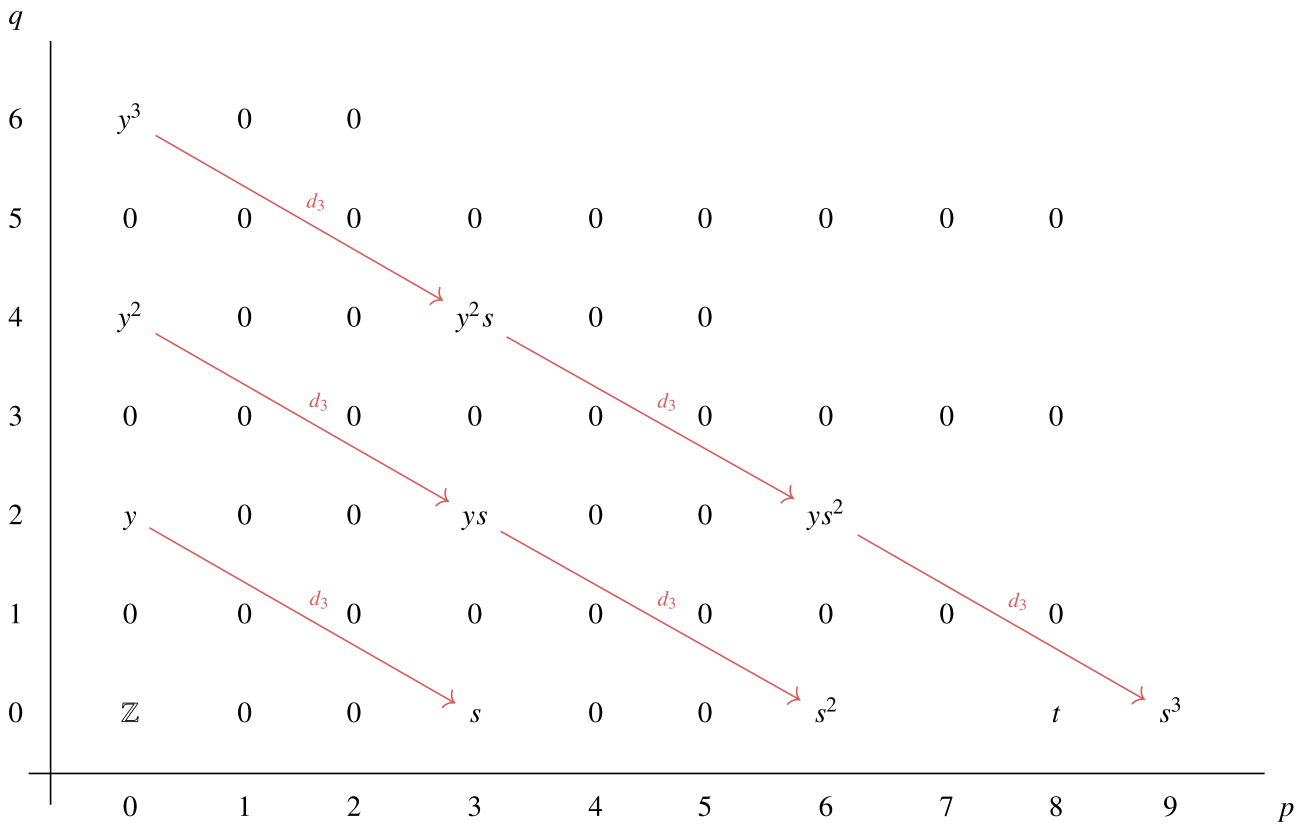
$$\cdots \to 0 \to \pi_3(\Omega K(\mathbb{Z}, 3)) \to 0 \to \pi_3(K(\mathbb{Z}, 3)) \to \pi_2(\Omega K(\mathbb{Z}, 3)) \to 0 \to \cdots$$

which shows that $\Omega K(\mathbb{Z}, 3)$ is weakly equivalent to a $K(\mathbb{Z}, 2)$ (which implies homotopy equivalent to a $K(\mathbb{Z}, 2)$, since it's a CW complex? By a result of Milnor, the space of maps from a finite CW complex to any

CW complex is homotopy equivalent to a CW complex). We know that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$, and has integral cohomology $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[y]$ with $|y| = 2$. The Leray-Serre cohomological spectral sequence has

$$E_2^{p,q} = H^p(K(\mathbb{Z},3); H^q(\mathbb{C}P^\infty)) \Rightarrow H^{p+q}(PK(\mathbb{Z},3)) = \begin{cases} \mathbb{Z} & p,q=0 \\ 0 & \text{else.} \end{cases}$$

Now the E_3 page is isomorphic to the E_2 page (no nontrivial differentials in E_2 page):



Since $PK(\mathbb{Z},3)$ is contractible, we need nonzero elements on the bottom row of the E_3 page which will help kill off the nonzero elements on the left most column of the E_3 page. For instance, we need an element $s \neq 0 \in E_3^{3,0} = E_2^{3,0} = H^3(K(\mathbb{Z},3);\mathbb{Z})$ for which $d_3(y) = s$ (else y would never get killed off). Then

$$d_3(y^2) = d_3(y)\cdot y + (-1)^0 y \cdot d_3(y) = sy + ys = 2ys$$

since $sy = (-1)^0 ys$. Also $d_3(ys) = d_3(y)\cdot s + (-1)^{|y|} y \cdot d_3(s) = s^2$. If $s^2 = 0$, then $ys \in \ker d_3$, and flipping the page yields a $\mathbb{Z}/2$ in $E_4^{3,2}$ which cannot get killed off by any later differential, which is a problem. Hence $s^2 \neq 0$. Instead for the $E_3^{3,2}$ element ys to die we need $\ker d_3 = \operatorname{Im} d_3 = 2ys$, so we have $d_3(2ys) = 2s^2 = 0$. So far this means that $H^3(K(\mathbb{Z},3)) = \langle s\rangle \cong \mathbb{Z}$ and $H^6(K(\mathbb{Z},3)) = \langle s^2\rangle/(2s^2) \cong \mathbb{Z}/2$. One could continue on to calculate other cohomology groups.

The situation is nicer if we change to rational coefficients:

Example 1.64 (Rational cohomology ring $H^*(K(\mathbb{Z},n);\mathbb{Q})$).

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[z]/(z^2) & |z| = n,\ n \text{ odd} \\ \mathbb{Q}[z] & |z| = n,\ n \text{ even} \end{cases}$$

The cases of $n = 1$ ($K(\mathbb{Z}, 1) \simeq S^1$) and $n = 2$ ($K(\mathbb{Z}, 2) \simeq \mathbb{C}P^\infty$) are clear. We can do the rest by induction using the path space fibration

$$K(\mathbb{Z}, n-1) \to PK(\mathbb{Z}, n) \to K(\mathbb{Z}, n).$$

When $n = 3$, the E_2 (and E_3) pages look like the one for the example above, except with a $\mathbb{Q}$ in the bottom left corner. This time the multiplication by 2 map $d_3(y^2) = 2ys$ is an isomorphism (since 2 is invertible in $\mathbb{Q}$), so both y^2 and ys vanish when we flip the page. We also have that $s^2 = 0$, since $s \cdot s = (-1)^3 s \cdot s$ implies that $2s^2 = 0$ in $\mathbb{Q}$. So $H^6(K(\mathbb{Z}, 3); \mathbb{Q}) = \langle s^2 \rangle / \langle 2s^2 \rangle = 0$. We deduce that $H^*(K(\mathbb{Z}, 3); \mathbb{Q}) \cong \mathbb{Q}[s]/s^2$ with $|s| = 3$.

Hence the cohomology of $K(\mathbb{Z}, 3)$ with rational coefficients vanishes everywhere except in degrees 0 and 3. This allows us to set up the spectral sequence to compute $H^*(K(\mathbb{Z}, 4); \mathbb{Q})$ as

$$\begin{tikzcd}
q \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & y \arrow[ddddrrrr, "d_4"] & 0 & 0 & 0 & ys \arrow[ddddrrrr, "d_4"] & 0 & 0 & 0 & ys^2 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Q} & 0 & 0 & 0 & s & 0 & 0 & 0 & s^2 \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & p
\end{tikzcd}$$

The d_4 differentials must be isomorphisms, and in particular $s^2 \neq 0$. The rest of the picture would show that $s^k \neq 0$ for any $k \geqslant 1$, so $H^*(K(\mathbb{Z}, 4); \mathbb{Q}) \cong \mathbb{Q}[s]$ with $|s| = 4$. An analogous analysis would be used to prove the claim for any odd or even n.

1.5.5 Poincaré Series & Euler Characteristics

Maybe not that important/ relevant. But:

Prop 1.65. *$\chi(E_r^{*,*}) = \chi(H^*)$ for all $r \geqslant 2$. So the Euler characteristic of H^* is determined by the E_2 page.* *to-do: write this up. See July 1 2024 notes.*

Corollary 1.66. *If we want the Euler characteristic of a manifold M, and we have a spectral sequence converging to $H^*(M; k)$, we can compute $\chi(M)$ via $\chi(E_2^{*,*})$, even if $H^*(M; k)$ is not known.* *to-do: write this up. See July 1 2024 notes.*

Example 1.67 (Wang sequence)**.** Suppose $\{E_r^{*,*}, d_r\}$ is a first quadrant spectral sequence of *cohomological* type converging to H^*. Suppose further that $E_2^{p,q} = 0$ unless $p = 0$ or $p = n$ for some $n \geqslant 2$. There is a (Wang) sequence

$$\cdots \to H^k \to E_2^{0,k} \xrightarrow{d_n} E_2^{n,k-n+1} \to H^{k+1} \to E_2^{0,k+1} \to \cdots$$

For instance, if $W^* = H^*(S^n; k)$ and V^* is any graded vector space with $E_2^{p,q} = W^p \otimes_k V^q$. Then the only nontrivial differential is $d_n : E_2^{0,k} \to E_2^{n,k-n+1}$. So the pages before are all isomorphic: $E_2 \cong E_3 \cong \cdots \cong E_n$, and $E_\infty = E_{n+1} = H(E_n, d_n)$ (because the remaining differentials become trivial too– only one nonzero differential). to-do: work out this example. (UGSS Ch.1 Informal Intro Exercise 3)

1.6 (Co)limits

Schematic: we're dealing with cones (limits) and "co-cones" (colimits). When we cone something off we usually add a point at the top (tip of the cone, so imagine a cone in the categorical sense the same way, and always want arrows going down!). A limit of a diagram (diagrams are functors!! from small categories, which are those whose Hom sets are actual sets, and not "proper classes") is the "shallowest cone":

Definition 1.68 (Cone)**.** Let $F : \mathcal{J} \to C$ be a diagram from a small category to a category. Say that (N, ϕ) is a cone to F if N is an object of C such that for every morphism $f : X \to Y$ between objects in $\mathcal{J}$, we have morphisms $\phi_X : N \to F(X)$ and $\phi_Y : N \to F(Y)$ making the diagram commute.

$$\begin{tikzcd} & N \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\ F(X) \arrow[rr, "F(f)"] & & F(Y) \end{tikzcd}$$

The fact that the limit is the *shallowest* cone is reflected in the following definition.

Definition 1.69 (Limit)**.** Suppose N is a cone to $F : \mathcal{J} \to C$. The limit $\lim F \in C$ is the shallowest cone to F, in the sense that it is the *terminal* object in the category of cones to F.

$$\begin{tikzcd} & N \arrow[d, dashed, "!u"] \arrow[ddl, "\phi_X"'] \arrow[ddr, "\phi_Y"] & \\ & \lim F \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\ F(X) \arrow[rr, "F(f)"] & & F(Y) \end{tikzcd}$$

Every cone to F has a unique map to the limit $\lim F$. If the limit exists, it is unique up to unique isomorphism!

Definition 1.70 (Cocone)**.** Let $F : \mathcal{J} \to C$ be a diagram from a small category to a category. Say that (N, ϕ) is a cocone to F if $N \in C$ and for every morphism $f : X \to Y$ between objects in $\mathcal{J}$, we have morphisms $\phi_X : F(X) \to N$ and $\phi_Y : F(Y) \to N$ making the diagram commute.

$$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"'] & & F(Y) \arrow[dl, "\phi_Y"] \\ & N & \end{tikzcd}$$

The colimit of a diagram F is the *shallowest* cocone to F:

Definition 1.71 (Colimit)**.** The <u>colimit</u> of a diagram F is the initial object in the category of cocones to F:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\varphi_X"] \arrow[ddr, "\phi_X"'] & & F(Y) \arrow[dl, "\varphi_Y"'] \arrow[ddl, "\phi_Y"] \\
& \operatorname{colim} F \arrow[d, dashed, "!\mu"] & \\
& N &
\end{tikzcd}$$

1.7 Fundamental theorems in (co)homology

Mostly just a collection of useful results without proofs.

1.7.1 Kunneth Theorems

Theorem 1.72. *The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y;R)$ is a finitely generated free R-module for all k.*

Theorem 1.73 (Kunneth)**.** *Let R be a PID and let X, Y be topological spaces. The homology of this product is determined by the following (noncanonically) split short exact sequence of R-modules*

$$0 \to \bigoplus_{i+j=n} H_i(X;R) \otimes H_j(Y;R) \to H_n(X \times Y;R) \to \bigoplus_{i+j=n-1} \operatorname{Tor}_1^R(H_i(X;R), H_j(Y;R)) \to 0$$

Theorem 1.74 (Relative Kunneth for cohomology, 3.18 in [5])**.** *For CW pairs (X,A) and (Y,B) the cross product homomorphism*

$$H^*(X,A;R) \otimes_R H^*(Y,B;R) \to H^*(X \times Y, A \times Y \cup X \times B;R)$$

is an isomorphism of rings if $H^k(Y,B;R)$ is a finitely generated free R-module for each k.

Theorem 1.75 (Relative Kunneth for homology)**.** *Let R be a PID, and let (X,A), (Y,B) be CW pairs. There exists a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(X,A;R) \otimes_R H_j(Y,B;R) \to H_n(X \times Y, A \times Y \cup X \times B;R) \to \bigoplus_{i+j=n-1} \operatorname{Tor}^R(H_i(X,A;R), H_j(Y,B;R)) \to 0.$$

1.7.2 Universal Coefficient Theorems

Given knowledge of $H_*(X;\mathbb{Z})$ for a space X, we can compute $H^n(X;A)$ and $H_n(X;A)$ for other coefficients (abelian groups) A.

Theorem 1.76 (UCT for homology)**.** *Let X be a topological space and A an abelian group. We have a (non-natural) split short exact sequence*

$$0 \to H_n(X) \otimes_{\mathbb{Z}} A \to H_n(X;A) \to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), A) \to 0.$$

So $H_n(X;A) \cong H_n(X) \otimes A \oplus \operatorname{Tor}_1(H_{n-1}(X;\mathbb{Z}), A)$.

Theorem 1.77 (UCT for cohomology)**.** *Let X be a topological space and A an abelian group. We have a (non-natural) split short exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), A) \to H^n(X;A) \to \operatorname{Hom}_{\mathbb{Z}}(H_n(X;\mathbb{Z}), A) \to 0.$$

1.7.3 Poincaré-Lefschetz duality

Definition 1.78 (Cap product)**.** For $k \geqslant \ell$, we define

$$\begin{aligned}\frown\colon C_k(X;R) \times C^\ell(X;R) &\to C_{k-\ell}(X;R)\\ \sigma \frown \varphi &= \varphi(\sigma|_{[v_0,\dots,v_\ell]}) \cdot \sigma|_{[v_\ell,\dots,v_k]}.\end{aligned}$$

One can check that the cap product satisfies $\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$. In particular this means that the cap product of a cycle σ and a cocycle φ yields a cycle. If σ is a cycle (so $\partial\sigma = 0$), then $\partial(\sigma \frown \varphi) = \pm(\sigma \frown \delta\varphi)$, so the cap product of a cycle and a coboundary is a boundary. Finally, if φ is a cocycle (so $\delta\varphi = 0$), then $\partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi)$, so the cap product of a cocycle and a boundary is a boundary. This means that we have an induced map

$$\frown\colon H_k(X;R) \times H^\ell(X;R) \to H_{k-\ell}(X;R).$$

There's also a relative version:

$$\frown\colon H_k(X,A;R) \times H^\ell(X,A;R) \to H_{k-\ell}(X;R)$$

using the fact that the cap product $\frown\colon C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ restricts to 0 on the submodule $C_k(A;R) \times C^\ell(X,A;R)$.

Theorem 1.79 (Poincaré-Lefschetz duality)**.** *Let M be an orientable, compact manifold of dimension n with boundary ∂M, and let $z \in H_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$ be a fundamental class (generator of $H_n(M,\partial M;\mathbb{Z})$). The cap product*

$$\frown\colon H_n(M,\partial M;\mathbb{Z}) \times H^k(M,\partial M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$$

induces an isomorphism of $H^k(M,\partial M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$ for all k, and the dual cap product

$$\frown\colon H^n(M,\partial M;\mathbb{Z}) \times H_k(M,\partial M;\mathbb{Z}) \to H^{n-k}(M;\mathbb{Z})$$

induces an isomorphism of $H_k(M,\partial M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$ for all k.

Remark 1.80 (Dispensing of compactness)*.* We do not need to assume that M is compact if we instead use cohomology with compact support. Suppose M is a (non-compact) orientable n-dimensional manifold.

2 Group (co)homology

2.1 Classifying spaces

2.1.1 Nerves and the bar construction

Reference: Hatcher 2.3 [5].

Let C be a category. Associate a to C a Δ-complex $\mathcal{B}C$ whose n-simplices are strings of composable morphisms in C

$$X_0 \to X_1 \to \cdots \to X_n.$$

The faces of this simplex are obtained by deleting an X_i and composing the adjacent morphisms when $i \neq 0, n$. A functor $F : C \to \mathcal{D}$ induces a map $\mathcal{B}C \to \mathcal{B}\mathcal{D}$.

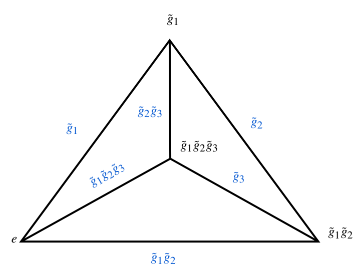
If C has a single object (say, C is a group G), and the morphisms of C form a group G (say, by taking the *morphisms* of C to be the *elements* of G), then we claim $\mathcal{B}C$ is the same as the Δ-complex BG which we construct as a $K(G, 1)$ by taking EG $/G$ (with G acting on EG, the space of tuples of G, by left multiplication). This is mostly easily seen when we consider the *bar construction*. Recall that we take EG to be the contractible Δ-complex whose n-simplices are $(n + 1)$ tuples of elements $(g_0, \dots, g_n)$ and the boundary maps forget vertices (attaching in the usual, order-preserving way). G acts freely (and properly discontinuously, basically by construction) on EG by left multiplication. Then BG $=$ EG $/G$ is the orbit space of this action. Pictorally: suppose we take a simplex $[g_0, g_1, g_2, g_3]$ in the universal cover EG (ordered tuple, comes equipped with an orientation!). We can write this as

$$[g_0, g_1, g_2, g_3] = g_0[e, g_0^{-1}g_1, g_0^{-1}g_2, g_0^{-1}g_3].$$

Let $\tilde{g}_i := g_{i-1}^{-1}g_i$. Then the simplex becomes

$$[g_0, g_1, g_2, g_3] = g_0[e, \tilde{g}_1, \tilde{g}_1\tilde{g}_2, \tilde{g}_1\tilde{g}_2\tilde{g}_3].$$

In BG, this simplex thus gets identified as



Its boundary is computed in the standard way as

$$\partial([e, \tilde{g}_1, \tilde{g}_1\tilde{g}_2, \tilde{g}_1\tilde{g}_2\tilde{g}_3]) = [\tilde{g}_1, \tilde{g}_1\tilde{g}_2, \tilde{g}_1\tilde{g}_2\tilde{g}_3] - [e, \tilde{g}_1\tilde{g}_2, \tilde{g}_1\tilde{g}_2\tilde{g}_3] + [e, \tilde{g}_1, \tilde{g}_1\tilde{g}_2\tilde{g}_3] - [e, \tilde{g}_1, \tilde{g}_1\tilde{g}_2]$$

Denote $[e, \tilde{g}_1, \tilde{g}_1\tilde{g}_2, \dots, \tilde{g}_1 \cdots \tilde{g}_n]$ using bars $[\tilde{g}_1| \cdots |\tilde{g}_n]$, and take such simplices to be the *basis* of the free $\mathbb{Z}G$-module F_n. Note that the boundary map for our simplex above then corresponds to

$$\partial([\tilde{g}_1| \cdots |\tilde{g}_3]) = \tilde{g}_1[\tilde{g}_2|\tilde{g}_3] - [\tilde{g}_1\tilde{g}_2|\tilde{g}_3] + [\tilde{g}_1|\tilde{g}_2\tilde{g}_3] - [\tilde{g}_1|\tilde{g}_2].$$

In general, the face map

$$d_i : [g_0, \dots, g_n] \mapsto [g_0, \dots, \hat{g}_i, \dots, g_n]$$

corresponds to a composition in the bar notation

$$d_i : [g_1 | \cdots | g_n] \mapsto [g_1 | \cdots | g_i g_{i+1} | \cdots g_n].$$

for $1 \leqslant i \leqslant n-1$, while $d_0([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n]$ and $d_n([g_0|\cdots|g_n]) = [g_1|\cdots|g_{n-1}]$. In the bar notation, our differentials act as face maps of simplices analogous to those described for $\mathcal{B}C$. We conclude that the notions are the same.

Remark 2.1. The bar construction yields a classifying space that is large, and often not computationally tractable.

Definition 2.2 (The (unnormalized) bar resolution)**.**

$$\to \cdots B_n \to \cdots \to B_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where each B_i is a free $\mathbb{Z}G$-module. Let $[]$ denote $1 \in \mathbb{Z}G$. Then $\varepsilon([]) = 1$. Each B_n is the free $\mathbb{Z}G$-module on symbols $[g_1 \otimes \cdots \otimes g_n]$. Define differentials

$$d_i([g_1 \otimes \cdots \otimes g_n]) = \begin{cases} g_1[g_2 \otimes \cdots \otimes g_n] & i = 0 \\ [g_1 \otimes \cdots \otimes g_i g_{i+1} \otimes \cdots \otimes g_n] & 1 \leqslant i \leqslant n-1 \\ [g_1 \otimes \cdots \otimes g_{n-1}] & i = n \end{cases}$$

Then $d = \sum_{i=0}^{n} (-1)^i d_i$.

2.1.2 Principal G-bundles and models for BG

Definition 2.3 (Principal G-bundles)**.** Suppose B is a topological space with trivial G-action, P is a right G-space equipped with a surjective G-map $\pi : P \to B$. Say P is a *principal G-bundle* over B if B has a cover of open sets U such that there exist G-equivariant homeomorphisms $\phi_U : \pi^{-1}(U) \to U \times G$ satisfying commutativity of

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

where $U \times G$ has right G-action $(u, g) \cdot \tilde{g} = (u, g\tilde{g})$. We also have G acts freely and transitively on fibers of π. Note that the orbit space of P is homeomorphic to B. Why? Let $x \in B$. The fiber $P_x = \pi^{-1}(\{x\})$ has a G-action since G takes $y \in P_x$ to $y \cdot g \in P_x$ by the fact that $\pi(y \cdot g) = \pi(y)$. This action is free and transitive by assumption, so we can identify the fiber P_x with G. Since π is surjective, and the map $\pi : P \to B$ factors through G, then $P/G \cong B$.

The quotient $\mathrm{EG} \xrightarrow{\pi} \mathrm{BG}$ is a principal G-bundle, and is universal in the sense that for any other principal G-bundle $Y \to X$, there exists a map $f : X \to \mathrm{BG}$ such that $Y \simeq$ the pullback of π along f:

$$\begin{tikzcd} Y \simeq \{(x,e) : f(x) = \pi(e)\} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathrm{EG} \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & \mathrm{BG} \end{tikzcd}$$

This yields a bijection

$$\{\text{principal } G\text{-bundles } Y \to X \text{ up to isomorphism}\} \leftrightarrow \{\text{maps } X \to \mathrm{BG} \text{ up to homotopy }\}.$$

Remark 2.4. In fact, we call B a classifying space for G whenever we have a principal G-bundle $P \to B$ with P weakly contractible (homotopy groups vanish). In this case we call P a universal G-bundle.

Definition 2.5 (Model for BG)**.** A particular choice of B (and principal G-bundle $P \to B$ with P weakly contractible) is called a model for BG.

Definition 2.6 (Admissible subgroups)**.** If G is a topological group, call H an admissible subgroup if $G \to G/H$ is a principal H-bundle.

Lemma 2.7. *If G is a discrete group, every subgroup H is admissible.*

Proof. We need to show that $\pi : G \to G/H$ is a principal H-bundle. First we note that G/H has a trivial H-action. Note also that H is also a discrete group, so we can take an open cover of G/H by singletons. Clearly G is a right H-space, and for every $\overline{x} \in G/H$ the fiber $\pi^{-1}(\overline{x})$ is freely and transitively permuted by H on the right. We have obvious homeomorphisms $\phi_{\overline{x}} : \pi^{-1}(\overline{x}) \to \overline{x} \times H$ making the diagram in the definition commute. □

Question: Given a group homomorphism $G \to H$, how to produce a map $\mathrm{BG} \to \mathrm{BH}$? And if $1 \to K \to G \to G/K \to 1$ is a short exact sequence of groups, how do we obtain a fibration $\mathrm{BG} \to B(G/K)$ with fiber homotopy equivalent to BK?

Lemma 2.8. *If $\pi : P \to B$ is a principal G bundle and X is a left G-space, then $P \times X \to P \times_G X$ is a principal G-bundle.*

Proof. $P \times X$ has the structure of a right G-space via $(p, x) \cdot g = (p \cdot g, g^{-1} \cdot x)$. Notice that G acts freely on $P \times X$ since it acts freely on P. If we identify the elements of the orbit of this action then we see that $(p \cdot g, g^{-1} \cdot x) \sim (p, x)$, which is precisely the identification occurring in $P \times_G X$. Hence $(P \times X)/G \cong P \times_G X$. G permutes the fibers of this map transitively. Finally, the local triviality condition follows from the fact that our map $P \times X \to P \times_G X$ is a quotient map, and the preimage of an open neighborhood in $P \times_G X$ is precisely a G-orbit of open neighborhoods in $P \times X$? Which is clearly homeomorphic to the original neighborhood times G... □

Remark 2.9. Not really sure where B comes into play here. Really just used the fact that P is a free G-space...

Corollary 2.10. *Taking $P = \mathrm{EG}$ and X a weakly contractible G-space, we have that $\mathrm{EG} \times_G X$ is a model for* BG *(since $\mathrm{EG} \times X \to \mathrm{EG} \times_G X$ is a principal G-bundle, and $\mathrm{EG} \times X$ is weakly contractible).*

Attached to a group homomorphism $\theta : H \to G$ and a principal H-bundle $P \to B$ is a principal G-bundle $P \times_{H,\theta} G \to B$, where G is a left H-space via $h \cdot g = \theta(h)g$, and $P \times_H G$ is a right G-space via $(p, g) \cdot \tilde{g} = (p, g\tilde{g})$. Hence we have a natural transformation

$$P_\theta : \mathcal{P}_H B \to \mathcal{P}_G B.$$

Since $\mathcal{P}_H B \cong [B, \mathrm{BH}]$, Yoneda's lemma implies that there exists a unique homotopy class $B\theta : \mathrm{BH} \to \mathrm{BG}$ inducing P_θ.

Definition 2.11 (Classifying map)**.** Given a principal G-bundle $f : X \to B$, a weakly contractible G-space P, and the pullback bundle $Q = f^*(P)$, say that f is a classifying map for $Q \to P$ since it is covered by a principal G-bundle map to P:

$$\begin{tikzcd} Q = f^*P \arrow[r] \arrow[d] & P \arrow[d] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

Apparently $B\theta$ is a classifying map for $\mathrm{EH} \times_{H,\theta} G \to \mathrm{BH}$.

Lemma 2.12. *We have a functor from the category of topological groups to the homotopy category of CW complexes via $G \mapsto \mathrm{BG}$.*

Proof. Omitted for now. $\square$

Prop 2.13 (Model for Bi)**.** *If $i : H \to G$ is the inclusion of an admissible subgroup, then $\pi : \mathrm{EG} \times_G G/H \to \mathrm{BG}$ is a model for Bi.*

Proof. Because H is admissible, $G \to G/H$ is a principal H-bundle. Also $\mathrm{EG} \to \mathrm{EG}/H$ is a universal principal H-bundle (EG is weakly contractible) so EG/H is a classifying space for H. We have a map of principal G-bundles

$$\begin{tikzcd}
\mathrm{EG} \times_H G \arrow[r, "\tilde{\pi}"] \arrow[d] & \mathrm{EG} \arrow[d] \\
\mathrm{BH} = \mathrm{EG}/H \arrow[r, "\pi"'] & \mathrm{BG} = \mathrm{EG}/G
\end{tikzcd}$$

where $\tilde{\pi}[e, g] = eg$. Recall that $\mathrm{EG}/H = \mathrm{EG} \times_G (G/H)$ (because $\mathrm{EG}/H = \mathrm{EG} \times_H * = \mathrm{EG} \times_G G \times_H * = \mathrm{EG} \times_G (G/H)$). Because the map $\tilde{\pi}$ also covers the map $\mathrm{BH} = \mathrm{EG} \times_G (G/H) \to \mathrm{BG} = \mathrm{EG}/G$, then this second map is a *model* for Bi. $\square$

Prop 2.14. *If $H \subseteq G$ is an admissible normal subgroup, then attached to a group extension $H \xrightarrow{i} G \xrightarrow{q} G/H$ is a fiber sequence $\mathrm{BH} \xrightarrow{\mathrm{B}\,i} \mathrm{BG} \xrightarrow{\mathrm{B}\,q} \mathrm{B}(G/H)$.*

Proof. Let $X = E(G/H)$. Then X is weakly contractible, and by lemma 2.8 we have that $\mathrm{EG} \times_G E(G/H)$ is a model for BG. We have a natural map

$$\mathrm{EG} \times_G E(G/H) \to (E(G/H))/G.$$

Modding out by the G-orbits is the same as modding out by G/H-orbits since H acts trivially on G/H, and hence $(E(G/H))/G = (E(G/H))/(G/H) = \mathrm{B}(G/H)$. This map is covered by a (G/H)-bundle map

$$\begin{tikzcd}
\mathrm{EG} \times_{G,q} (G/H) \arrow[r, "\tilde{\pi}"] \arrow[d] & E(G/H) \arrow[d] \\
\mathrm{EG} \times_G E(G/H) \arrow[r, "\pi"'] & E(G/H)/(G/H)
\end{tikzcd}$$

???? So is a model for $\mathrm{B}\,q$? $\square$

Prop 2.15. *If a finite-dimensional CW complex X is a $K(G, 1)$, then the group $G = \pi_1(X)$ must be torsion-free.*

Proof. If G had torsion, let $H \leqslant G$ be a torsion subgroup. Then $H_n(H; \mathbb{Z})$ would be nonzero for infinitely many n, which would imply that H, and hence G, has infinite cohomological dimension. But if X is a finite-dimensional $K(G, 1)$, then $H_*(G; \mathbb{Z}) = H_*(X; \mathbb{Z})$ would vanish above the dimension of X, a contradiction. $\square$

Theorem 2.16 (Whitehead ([2] II.7.3))**.** *Suppose*

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_1"] \arrow[d, hook, "\iota_2"'] & G_1 \arrow[d] \\
G_2 \arrow[r] & G
\end{tikzcd}$$

is an amalgamation diagram of groups, with ι_1, ι_2 injective. This can be realized by an amalgamation diagram

$$\begin{tikzcd} Y \arrow[r, hook, "\iota_1"] \arrow[d, hook, "\iota_2"'] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

of $K(\pi, 1)$'s under the functor $\pi_1 : \underline{\text{Top}} \to \underline{\text{Grp}}$, where $Y = X_1 \cap X_2$.

Proof. Associated to an amalgamation of groups is a pushout of $K(\pi, 1)$'s. Taking mapping cylinders if necessary (replace X with the homotopy equivalent space M_f), the maps out of Y are inclusions.

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, hook] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

By Van Kampen's theorem, we know $\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \cong G_1 *_A G_2$. It remains to show that X has a contractible (acyclic) universal cover. We need the following lemma:

Lemma 2.17 ([2]II.7)**.** *Let $X' \subseteq X$ be an inclusion of connected CW-complexes such that the induced map $\pi' \to \pi$ of fundamental groups is injective. Let $p : \tilde{X} \to X$ be the universal cover of X. Then each connected component of $p^{-1}(X')$ is simply-connected (hence is a copy of the universal cover of X').*

Proof. For any basepoint in $p^{-1}(X')$, we have a diagram

$$\begin{tikzcd} \pi_1(p^{-1}(X')) \arrow[r, dashed] \arrow[d, hook, "p_*"'] & \pi_1(\tilde{X}) \arrow[d, hook, "p_*"] \\ \pi_1(X') = \pi' \arrow[r, hook] & \pi \end{tikzcd}$$

where the dotted horizontal map is induced by the inclusion of $p^{-1}(X')$ into $\tilde{X}$. Since $\pi_1(\tilde{X}) = 0$, then $\pi_1(p^{-1}(X')) = 0$ for every choice of basepoint in $p^{-1}(X')$, and hence every path-component is simply-connected. □

Using this lemma: Let $p : \tilde{X} \to X$ be the universal cover, and let $\tilde{X}_1, \tilde{X}_2$, and $\tilde{Y}$ be the preimages of X_1, X_2, Y under p. Since X_1, X_2, Y are $K(\pi, 1)$'s, their universal covers are acyclic, and lemma 2.17 gives that $\tilde{X}_1, \tilde{X}_2$, and $\tilde{Y}$ are acyclic (because they are copies of the universal covers of X_1, X_2, Y, by the lemma!). The Mayer-Vietoris sequence associated to

$$\begin{tikzcd} \tilde{Y} \arrow[r, hook] \arrow[d, hook] & \tilde{X}_1 \arrow[d, hook] \\ \tilde{X}_2 \arrow[r, hook] & \tilde{X} \end{tikzcd}$$

yields that $\tilde{X}$ is acyclic. □

there's an adjunction: $[*, K(G, n)]$ and $H^n(*, G)$

2.2 Homological algebra basics

Lemma 2.18 (tensor-hom adjunction mnemonic)**.** *If X is an (R, S)-bimodule, Y is a right R-module and Z is a right S-module, then*

$$\mathrm{Hom}_S(Y \otimes_R X, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z)).$$

Remark 2.19. Mnemonic to remember: in the category of finite sets, if A and N are finite sets then $\mathrm{Hom}(A, N)$ has the isomorphism type of A^N ($|A|^{|N|}$ many elements), and $A \otimes N$ has the isomorphism type of AN ($|A||N|$ many elements). The tensor-hom adjunction encodes the simple exponent law: $Z^{YX} \cong (Z^X)^Y$.

Computing Tor and Ext in a few cases.

- Always have $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$. If M or N is a flat R-module, then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.
- Always have $\mathrm{Ext}_R^0(M, N) \cong \mathrm{Hom}_R(M, N)$. If M is a projective R-module or N is an injective R-module, then $\mathrm{Ext}_R^i(M, N) = 0$ for all $i > 0$.
- $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(m, n)$: Let

  $$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

  be a free resolution of $\mathbb{Z}/n$ over $\mathbb{Z}$. Then

  $$0 \to \mathbb{Z} \otimes \mathbb{Z}/m \xrightarrow{\cdot n \otimes \mathrm{id}} \mathbb{Z} \otimes \mathbb{Z}/m \to 0$$

  is a chain complex whose first homology gives $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m)$. But this is precisely $\ker(\cdot n \otimes \mathrm{id}) = \ker(\mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m)$ which consists of those $\overline{x} \in \mathbb{Z}/m\mathbb{Z}$ which are multiples of $\frac{m}{\gcd(m,n)}$. There are precisely $\gcd(m, n)$ such multiples.
- $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, \mathbb{Z}/m) \cong \mathbb{Z}/\gcd(n, m)$. Again let

  $$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

  be a free resolution of $\mathbb{Z}/n$ over $\mathbb{Z}$. Then

  $$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \xrightarrow{\cdot n} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/m) \to 0$$

  is a cochain complex whose first cohomology is $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, \mathbb{Z}/m) = (\mathbb{Z}/m)/(\mathrm{Im}\,\mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m)$. Since $\mathrm{Im}(\mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m)$ is isomorphic to $\gcd(n, m)\mathbb{Z}/m\mathbb{Z}$, we have $(\mathbb{Z}/m)/(\gcd(n, m)\mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(n, m)$.
- $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Same reasoning as the last bullet, just now have $\mathbb{Z}/\mathrm{Im}(\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}) = \mathbb{Z}/n$. More generally: $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n, A) \cong A/nA$ for any abelian group A.
- $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, A) = \{a \in A : na = 0\}$ for A an abelian group. Since $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, A) = \ker(A \xrightarrow{\cdot n} A)$.

Prop 2.20 (Fundamental Lemma of Homological Algebra)**.** *Let (C, ∂) and (C', ∂') be chain complexes and let r be an integer. Let $(f_i : C_i \to C_i')_{i \leqslant r}$ be a family of maps such that $\partial_i' \circ f_i = f_{i-1} \circ \partial_i$ for $i \leqslant r$. If C_i is projective for $i > r$ and $H_*(C') = 0$ for $i \geqslant r$ (so C' is acyclic), then $(f_i)_{i \leqslant r}$ extends to a chain map $f : C \to C'$, and is unique up to homotopy.*

Example 2.21. One can also compute $\mathrm{Tor}(M, N)$ by taking a projective resolution $P_\bullet \to M$ and a projective resolution $Q_\bullet \to N$ and taking the homology of $P_\bullet \otimes Q_\bullet$ total complex. Recall that by the double complex spectral sequence 1.5.1, we can fix p, and then take homology with respect to $Q_\bullet$ to get

$$E^1_{p,q} = H_q(P_p \otimes_R Q\bullet) \cong P_p \otimes_R H_q(Q_\bullet)$$

since P_p is a flat R-module, and so commutes with homology. Now $H_q(Q_\bullet) = 0$ for $q \neq 0$ and $H_0(Q_\bullet) = N$, so the spectral sequence collapses on the E^2 page and we have $E^\infty_{p,0} = H_*(P_p \otimes_R N) = \mathrm{Tor}^R_p(M, N)$. Since the double complex spectral sequence converges to the homology of the total complex (and the E^2 page has *one* nonzero group in each diagonal), we see that $H_p(P \otimes_R Q) \cong \mathrm{Tor}^R_p(M, N)$.

Lemma 2.22 (Eilenberg trick)**.** *If P is a projective module over an arbitrary ring R, there exists a free R-module F such that $P \oplus F \cong F$.*

Proof. (Note: usually F will not be finitely generated). Since P is projective, there exists a free R-module F and a projective R-module Q such that $F \cong P \oplus Q$. Taking a direct sum of countably many copies of F, we obtain a free module F' with

$$F' = \bigoplus (P \oplus Q)$$

which means that F' is a direct sum of countably many copies of P and countably many copies of Q. Adding a copy of P does not change this, so $F' \oplus P \cong F'$. □

Lemma 2.23 (Schanuel's lemma)**.** *Suppose $0 \to K \to P \xrightarrow{\pi} M \to 0$ and $0 \to K' \to P' \xrightarrow{\pi'} M \to 0$ are two exact sequences of R-modules with P, P' projective. Then $K \oplus P' \cong K' \oplus P$.*

Proof. Define an R-module Q to be the pullback of $P \xrightarrow{\pi} M$ along $P' \xrightarrow{\pi'} M$, that is, $Q = \{(p, p') \in P \oplus P' : \pi(p) = \pi'(p)\}$. We have a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K' \arrow[r, equal] \arrow[d] & K' \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & Q \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

since the projection from Q to its first factor has kernel those elements of P' for which $\pi'(p') = e$, which is precisely the module K'. Since P is projective, the left vertical short exact sequence splits, so that $Q \cong K' \oplus P$. Since P' is projective, the top horizontal short exact sequence splits, so that $Q \cong K \oplus P'$. □

Corollary 2.24 (Generalized Schanuel's)**.** *Let $0 \to P_n \to \cdots \to P_0 \to M \to 0$ and $0 \to P'_n \to \cdots \to P'_0 \to M \to 0$ be exact sequences with P_i, P'_i projective for $i \leqslant n-1$. Then $P_0 \oplus P'_1 \oplus \cdots \cong P'_0 \oplus P_1 \oplus \cdots$. Consequently, if P_i and P'_i are finitely generated for $i \leqslant n-1$, then P_n is finitely generated if and only if P'_n is finitely generated.*

Proof. Assume the claim holds for sequences of length $n-1$. Let $K = \ker(P_0 \to M)$ and let $K' = \ker(P'_0 \to M)$. Then $0 \to K \to P_0 \to M \to 0$ and $0 \to K' \to P'_0 \to M \to 0$ are exact. By Schanuel's lemma 2.23, we know $K \oplus P'_0 \cong K' \oplus P_0$. Now

$$0 \to P_n \to \cdots \to P_2 \to P_1 \oplus P'_0 \to K \oplus P'_0 \to 0$$

and

$$0 \to P'_n \to \cdots \to P'_2 \to P'_1 \oplus P_0 \to K' \oplus P_0 \to 0$$

are both exact and of length $n-1$ (with $K \oplus P'_0 \cong K' \oplus P_0$), so by the inductive hypothesis, the claim holds. □

Prop 2.25 (Extension of scalars is left adjoint to restriction of scalars)**.** *Let $R \to S$ be a ring map. We have two functors $F : \mathrm{Mod}_R \to \mathrm{Mod}_S, M \mapsto S \otimes_R M$, and $G : \mathrm{Mod}_S \to \mathrm{Mod}_R, N \mapsto N$ (the forgetful functor) satisfying the universal mapping property: given any S-module N and R-module map $f : M \to N$, there exists a unique S-module map $g : S \otimes_R M \to N$ such that $g \circ i = f$:*

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "f"'] & S \otimes_R M \arrow[dl, dashed, "g"] \\ N & \end{tikzcd}$$

Hence $\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, N)$.

Prop 2.26 (Extension of scalars preserves projectivity)**.** *Extension of scalars takes projective R-modules to projective S-modules.*

Proof. Let M be a projective R-module. We wish to show that $\mathrm{Hom}_S(S \otimes_R M, -)$ is an exact functor on S-modules. By the extension of scalars- restriction of scalars adjunction, we know $\mathrm{Hom}_S(S \otimes_R M, -) \cong \mathrm{Hom}_R(M, -)$, and since M is a projective R-module, the right hand functor is exact. □

Prop 2.27 (Flat restriction preserves injectivity)**.** *If S is flat as a right R-module, restriction of scalars takes injective S-modules to injective R-modules.*

Proof. Since S is flat as a right R-module, the functor $S \otimes_R -$ is exact. Let N be an injective S-module. Then the functor $\mathrm{Hom}_S(S \otimes_R -, N) \cong \mathrm{Hom}_R(-, N)$ is exact, so N is an injective R-module. □

Definition 2.28 (Coextension of scalars)**.** Note that $\mathrm{Hom}_R(S, M)$ is a left S-module via $(s \cdot f)(s') = f(s's)$ (right translation). Call this coextension of scalars. We have a natural map $\pi : \mathrm{Hom}_R(S, M) \to M, f \mapsto f(1)$. Given $\alpha : R \to S$ a ring map, have $\pi(\alpha(r) \cdot f) = (\alpha(r) \cdot f)(1) = f(\alpha(r)) = r \cdot f(1) = r \cdot \pi(f)$, so π is an R-module homomorphism. Given an S-module N and an R-module map $f : N \to M$, there exists a unique S-module map $g : N \to \mathrm{Hom}_R(S, M)$ such that $\pi \circ g = f$:

$$\begin{tikzcd} N \arrow[r, dashed, "g"] \arrow[d, "f"'] & \mathrm{Hom}_R(S, M) \arrow[dl, "\pi"] \\ M & \end{tikzcd}$$

$(f(n) = \pi(g(n))$, so each n yields a hom $g(n) : S \to M$ with $g(n)(1) = f(n)$. This defines $g(n)$ on $\alpha(R) \subset S$, since $g(n)(\alpha(r)) = r \cdot g(n)(1) = r \cdot f(n) = f(rn)$.) We thus have an isomorphism

$$\mathrm{Hom}_R(N, M) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)).$$

This means coextension of scalars is right adjoint to restriction of scalars.

Prop 2.29 (Projective restriction preserves projectivity)**.** *If S is projective as a left R-module, restriction of scalars takes projective S-modules to projective R-modules.*

Proof. We know $\mathrm{Hom}_R(S, -)$ is exact. Let N be a projective S-module. Then $\mathrm{Hom}_R(S, -)$ is exact and $\mathrm{Hom}_S(N, -)$ is exact implies that $\mathrm{Hom}_R(N, -) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, -))$ is exact. □

Corollary 2.30 (Change of rings for Ext)**.** *Let R be a ring. Then $\mathrm{Ext}^*_{R[G]}(R, A) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A)$ for all $R[G]$-modules A. In particular $H^*(G; A) \cong \mathrm{Ext}^*_{R[G]}(R, A)$.*

Proof. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}G$-modules. We have a ring map $\varphi : \mathbb{Z}G \to R[G]$ given by sending $\alpha(e) \in \mathbb{Z}G$ to $\alpha(e) \otimes 1 \in R[G]$. Since extension of scalars preserves projectivity, we have that $P_\bullet \otimes_{\mathbb{Z}G} R[G]$ is a projective $R[G]$ module. Since the resolution splits as a resolution of abelian groups (projective $\mathbb{Z}G$-modules are in particular free abelian groups), tensoring with R over $\mathbb{Z}$ (which is isomorphic to tensoring with $R[G]$ over $\mathbb{Z}G$, since $R[G] \cong R \otimes_{\mathbb{Z}} \mathbb{Z}G$) preserves exactness. Thus

$$P_\bullet \otimes_{\mathbb{Z}G} R[G] \to \mathbb{Z} \otimes_{\mathbb{Z}G} R[G] = \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}} R \cong R$$

is a projective resolution of R over $R[G]$. By the tensor-hom adjunction, we have

$$\begin{aligned} \mathrm{Ext}^*_{R[G]}(R, A) = H_*(\mathrm{Hom}_{R[G]}(P_\bullet \otimes_{\mathbb{Z}G} R[G], A)) &\cong H_*(\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, \mathrm{Hom}_{R[G]}(R[G], A))) \\ &\cong H_*(\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A)) \\ &= \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A). \end{aligned}$$

□

Corollary 2.31. *Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module (localizations are flat), we can compute $H^*(G; V) = \mathrm{Ext}^*_{\mathbb{Q}G}(\mathbb{Q}, V)$ for any $\mathbb{Q}G$-module (representation) V.*

Corollary 2.32 (Change of rings for Tor)**.** *Let R be a ring. Then $\mathrm{Tor}^{R[G]}_*(R, A) \cong \mathrm{Tor}^{\mathbb{Z}G}_*(\mathbb{Z}, A)$ for all $R[G]$-modules A. In particular $H_*(G; A) \cong \mathrm{Tor}^{R[G]}_*(R, A)$.*

Proof. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Tensoring with R over $\mathbb{Z}$ preserves exactness, as the original sequence splits over $\mathbb{Z}$. Hence $P_\bullet \otimes_{\mathbb{Z}} R \to R$ is a projective resolution of R over $R[G]$ (extension of scalars preserve projectivity, by 2.26, and $P_\bullet \otimes_{\mathbb{Z}} R \cong P_\bullet \otimes_{\mathbb{Z}G} R[G]$). Tensoring the truncated complex with A and taking homology yields $\mathrm{Tor}^{R[G]}_*(R, A) = H_*(P_\bullet \otimes_{\mathbb{Z}G} R[G] \otimes_{R[G]} A) = H_*(P_\bullet \otimes_{\mathbb{Z}G} A) = \mathrm{Tor}^{\mathbb{Z}G}_*(\mathbb{Z}, A)$. □

Lemma 2.33 (Hom commutes with direct limits)**.** *Let P be a finitely generated projective R-module. Then*

$$\mathrm{Hom}_R(P, \varinjlim A_j) \cong \varinjlim \mathrm{Hom}_R(P, A_j)$$

Prop 2.34 (kG-modules for G finite. Brown III.5.5)**.** *If G is a finite group and k is a field, a kG-module is injective if and only if it is projective.*

Proof. First we show that kG is itself an injective kG-module. Since G is finite, then $[G : e] < \infty$, and hence $\operatorname{Ind}_e^G M \cong \operatorname{Coind}_e^G M$ for any kG-module M by 2.56. This means that

$$\begin{aligned} kG \otimes_k M &\cong \mathbb{Z}G \otimes_{\mathbb{Z}} k \otimes_k M \cong \mathbb{Z}G \otimes_{\mathbb{Z}} M \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, M) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}G, \operatorname{Hom}_k(k, M)) \cong \operatorname{Hom}_k(\mathbb{Z}G \otimes_{\mathbb{Z}} k, M) \cong \operatorname{Hom}_k(kG, M) \end{aligned}$$

by the tensor-hom adjunction. Taking $M = k$ shows that $kG \cong \operatorname{Hom}_k(kG, k)$. Recall that we wish to show $\operatorname{Hom}_{kG}(-, kG)$ is exact. We have

$$\operatorname{Hom}_{kG}(-, kG) \cong \operatorname{Hom}_{kG}(-, \operatorname{Hom}_k(kG, k)) \cong \operatorname{Hom}_k(- \otimes_{kG} kG, k) \cong \operatorname{Hom}_k(-, k)$$

again by the tensor-hom adjunction. But k is an injective k-module (as fields are PIDs, and fields are divisible, which over a PID implies injectivity), so $\operatorname{Hom}_k(-, k)$ is exact, and so is $\operatorname{Hom}_{kG}(-, kG)$.

Since G is finite, the ring kG is finitely generated as an abelian group (and I think kG must be Noetherian?). If kG is Noetherian then direct sums of injective modules are injective, so that any free kG-module is injective. Suppose now that P is a projective kG-module. We claim that direct summands of injective modules are injective: suppose $P \oplus Q \cong F$ for F an injective kG-module, and suppose $i : M \to M'$ is an injection. Since F is injective, the induced map $\operatorname{Hom}_{kG}(M', F) \to \operatorname{Hom}_{kG}(M, F)$ is surjective. Now suppose we have a map $\varphi \in \operatorname{Hom}_{kG}(M, P)$. We can extend this naturally to a kG-map $\psi : M \to F$ via including P into the first component of F. Hence there exists a kG-map $\tilde{\psi} : M' \to F$ such that $\tilde{\psi} \circ i = \psi$.

$$\begin{tikzcd}
M \arrow[d, "\varphi"'] \arrow[dr, "i"] & \\
P \arrow[d, "\iota"'] & M' \arrow[dl, "\tilde{\psi}"] \\
F \arrow[d, "\pi_1"'] & \\
P &
\end{tikzcd}$$

Composing $\tilde{\psi}$ with $\pi_1 : F \to P$ yields a map such that $\pi_1 \circ \tilde{\psi} \circ i = \pi_1 \circ \iota \circ \varphi = \varphi$, proving the desired surjectivity of $\operatorname{Hom}_{kG}(M', P) \xrightarrow{i_*} \operatorname{Hom}_{kG}(M, P)$. This gives that P is injective as a kG-module.

For any kG-module M, we have a canonical injection

$$\begin{aligned} M &\xrightarrow{i} kG \otimes_k M \\ m &\mapsto 1 \otimes m. \end{aligned}$$

Since M is a k-vector space, it has a k-basis $\{m_i\}$. Since kG is a k-vector space, it has a k-basis $\{g\}_{g \in G}$. Then $g \otimes m_i$ forms a k-basis for $kG \otimes_k M$, and in fact it forms a kG-basis, since $\tilde{g} \cdot g \otimes m = \tilde{g}g \otimes m$ is another basis element. We conclude that $kG \otimes_k M$ is a free $\mathbb{Z}G$-module. If M is an injective kG-module, the short exact sequence

$$0 \to M \xrightarrow{i} kG \otimes_k M \to \operatorname{coker} i \to 0$$

splits, so that M is a direct summand of a free kG-module, and hence is projective. $\square$

2.3 Group cohomology basics

Definition 2.35 (Algebraic def of group (co)homology)**.** Let G be a group and M a $\mathbb{Z}G$-module. Define

$$H_*(G;M) := \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, M)$$
$$H^*(G;M) := \mathrm{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, M).$$

Definition 2.36 (Topological def of group (co)homology)**.** Let G be a group and let X be a $K(G,1)$ (connected CW complex with $\pi_1 = G$ and vanishing higher homotopy groups (aspherical); equivalently, the universal cover $\tilde{X}$ is contractible). Then $H^*(X;\mathbb{Z}) = H^*(G;\mathbb{Z})$ and $H_*(X;\mathbb{Z}) = H_*(G;\mathbb{Z})$.

Remark 2.37. Why does this second definition align with the algebraic one? Recall that if X is a $K(G,1)$, then the universal cover is a contractible, free G-CW complex (so G permutes the cells of $\tilde{X}$ freely). Let S be a set of representatives of the G-orbits of n-cells of $\tilde{X}$, and let β be a ($\mathbb{Z}$) basis of n-cells of $\tilde{X}$. Then

$$C_n^{CW}(\tilde{X}) = \mathbb{Z}\{\beta\} = \mathbb{Z}[\sqcup_{\sigma\in S} G\cdot\sigma] \cong \bigoplus_{\sigma\in S}\mathbb{Z}[G\cdot\sigma] \cong \bigoplus_{\sigma\in S}\mathbb{Z}[G/G_\sigma] = \bigoplus_{\sigma\in(X^{(n)}\setminus X^{(n-1)})/G}\mathbb{Z}[G]$$

since G acts freely on $\tilde{X}$ and hence $G_\sigma = \{e\}$ for every n-cell σ. (For the isomorphism between the group ring of the disjoint union and the direct sum of group rings, note that the group ring $R[G]$ is characterized as the set of maps $f : G \to R$ with finite support, and maps from a disjoint union factor as maps from each component of the disjoint union.) Since $\tilde{X}$ is contractible, it is acyclic, and thus its augmented cellular chain complex

$$\cdots \to C_n^{CW}(\tilde{X}) \to C_{n-1}^{CW}(\tilde{X}) \to C_{n-2}^{CW}(\tilde{X}) \to \cdots \to C_0^{CW}(\tilde{X}) \to \mathbb{Z} \to 0$$

is exact. Since we saw above that $C_n^{CW}(\tilde{X})$ is a free $\mathbb{Z}G$-module, we get a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ in this way. Truncating and tensoring with $\mathbb{Z}$ over $\mathbb{Z}G$ yields

$$\cdots \to \bigoplus_{\sigma\in(X^{(n)}\setminus X^{(n-1)})/G}\mathbb{Z} \to \bigoplus_{\sigma\in(X^{(n-1)}\setminus X^{(n-2)})/G}\mathbb{Z} \to \cdots \to \bigoplus_{\sigma\in X^{(0)}/G}\mathbb{Z} \to 0$$

and the homology of this would yield the group homology $H_*(G;\mathbb{Z})$. On the other hand, since $X = \tilde{X}/G$, the n-cells of X are given as precisely the G-orbits of n-cells in $\tilde{X}$, which means that the complex above is isomorphic to the cellular chain complex of X. Hence taking homology of the above complex gives $H_*(X;\mathbb{Z}) \cong H_*(G;\mathbb{Z})$. Similar story with cohomology! In summary: $C_n(X)_G \cong C_n(X/G)$ as abelian groups (or as trivial $\mathbb{Z}G$-modules). And $C^n(\tilde{X})^G \cong C^n(\tilde{X}/G) = C^n(X)$ via the isomorphism $\mathrm{Hom}_{\mathbb{Z}G}(C_n(\tilde{X}),\mathbb{Z}) \cong (\mathrm{Hom}_{\mathbb{Z}}(C_n(\tilde{X}),\mathbb{Z}))^G$ of 2.41.

Lemma 2.38 (Invariants and coinvariants)**.** *As functors,* $(-)_G \cong \mathbb{Z}\otimes_{\mathbb{Z}G} -$, *and* $(-)^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$.

Proof. Let M be a $\mathbb{Z}G$-module. Recall that

$$M_G = M/\langle m - g\cdot m : m \in M, g \in G\rangle.$$

Define

$$M \xrightarrow{\varphi} \mathbb{Z}\otimes_{\mathbb{Z}G} M$$
$$m \mapsto 1 \otimes m.$$

Since $\mathbb{Z}$ is a trivial $\mathbb{Z}G$-module, then for all $m \in M$ and $g \in G$, we have

$$\varphi(g\cdot m) = 1\otimes gm = 1\cdot g\otimes m = 1\otimes m = \varphi(m).$$

Hence φ factors through M_G, is clearly surjective (since φ is), and is injective: if $\overline{\varphi}(\overline{m}) = \overline{\varphi}(\overline{n})$, then $1 \otimes m = 1 \otimes n$, which means $n = gm$ for some $g \in G$ (and thus $\overline{m} = \overline{n}$). Similarly, recall that

$$M^G = \{m \in M : gm = m \ \forall g \in G\}.$$

Define a map

$$\begin{aligned} \psi : \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) &\to M^G \\ \phi &\mapsto \phi(1). \end{aligned}$$

The fact that $\phi(1) \in M^G$ follows from the fact that $g \cdot \phi(1) = \phi(g \cdot 1) = \phi(1)$. Since such a map ϕ is uniquely determined by where it sends 1, ψ is clearly surjective and injective. □

Lemma 2.39 (H_0 and H^0). *$H_0(G; M) = M_G$ (coinvariants) and $H^0(G; M) = M^G$ (invariants).*

Proof. Let

$$\cdots \to F_1 \to F_0 \to M \to 0$$

be a free resolution of M over $\mathbb{Z}G$. Truncate and tensor with $\mathbb{Z}$ over $\mathbb{Z}G$ (same as taking coinvariants, by 2.38), take homology to obtain

$$H_0(G; M) = (F_0)_G / \mathrm{Im}(F_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \to F_0 \otimes_{\mathbb{Z}G} \mathbb{Z}).$$

On the other hand, by exactness of $F_1 \to F_0 \to M \to 0$ and the fact that tensoring is right-exact, we have an exact sequence

$$F_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \to F_0 \otimes_{\mathbb{Z}G} \mathbb{Z} \to M \otimes_{\mathbb{Z}G} \mathbb{Z} \to 0.$$

This shows that $M_G \cong (F_0)_G / \mathrm{Im}(F_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \to F_0 \otimes_{\mathbb{Z}G} \mathbb{Z}) = H_0(G; M)$.

Similarly, starting with a resolution $\cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$ over $\mathbb{Z}G$, we can truncate and apply $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ to get the cochain complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_1, M) \to \cdots$$

Then $H^0(G; M) = \ker(\mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_1, M))$. But since $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ is contravariant left exact, we know that

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_0, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_1, M)$$

is exact, so $H^0(G; M) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \cong M^G$ (again by 2.38). □

Prop 2.40 (Computing $\mathrm{Tor}^G_*(M, N)$). *Let M, N be G-modules. If M is $\mathbb{Z}$-torsion-free (=free if finitely generated), then $\mathrm{Tor}^G_*(M, N) = H_*(G; M \otimes N)$, where G acts diagonally on $M \otimes N$.*

Proof. Let $\varepsilon : F \to \mathbb{Z}$ be a projective $\mathbb{Z}G$-module resolution, and consider the resolution $\varepsilon \otimes M : F \otimes M \to M$. If M is $\mathbb{Z}$-torsion-free, then M is a flat $\mathbb{Z}$-module. We wish to show that $(F \otimes M) \otimes_{\mathbb{Z}G} -$ is an exact functor, so that $F \otimes M$ is a flat $\mathbb{Z}G$-module resolution of M. First we note that $(F \otimes M) \otimes_G - \cong (F \otimes M \otimes -)_G$, since G acts diagonally on $M \otimes N$, and hence for any G-module N, the map

$$\begin{aligned} f : M \otimes N &\to M \otimes_G N \\ m \otimes n &\mapsto m \otimes n \end{aligned}$$

factors through $(M \otimes N)_G$, since $f(g \cdot m \otimes n) = f(gm \otimes gn) = gm \otimes gn = m \otimes n = f(m \otimes n)$. Letting

$$\overline{f} : (M \otimes N)_G \to M \otimes_G N$$

be the map from the quotient. We can also define a map $h : M \otimes N \to (M \otimes N)_G$, $m \otimes n \mapsto m \otimes n$, and notice that h is G-balanced: $h(gm \otimes n)) = h(mg^{-1} \otimes n) = mg^{-1} \otimes n = m \otimes gn = h(m \otimes gn)$, so h factors through $M \otimes_G N$, and it is clear that f and h are inverses. Since M is a flat $\mathbb{Z}$-module, then $M \otimes -$ is exact, and since F is a flat $\mathbb{Z}G$-module, then $F \otimes_G (M \otimes -)$ is exact. From the isomorphisms

$$(F \otimes M)_G - \cong (F \otimes M \otimes -)_G \cong F \otimes_G (M \otimes -)$$

we conclude that $(F \otimes M)_G-$ is exact, and that $F \otimes M$ is a flat $\mathbb{Z}G$-module resolution of M. Tensoring with N over $\mathbb{Z}G$, we obtain

$$\operatorname{Tor}_*^G(M, N) = H_*((F \otimes M) \otimes_G N) = H_*(F \otimes_G (M \otimes N)) = H_*(G; M \otimes N).$$

$\square$

Prop 2.41 (Computing $\operatorname{Ext}_G^*(M, N)$)**.** *Let M, N be left G-modules. If M is $\mathbb{Z}$-free, then* $\operatorname{Ext}_G^*(M, N) \cong H^*(G; \operatorname{Hom}(M, N))$, *where G acts diagonally on* $\operatorname{Hom}(M, N)$ *(meaning $(g \cdot f)(m) = g \cdot f(g^{-1} \cdot m)$).*

Proof. Let $\varepsilon : F \to \mathbb{Z}$ be a projective $\mathbb{Z}G$-module resolution. Recall that because of the diagonal action on $\operatorname{Hom}(M, N)$, we have a map

$$h : \operatorname{Hom}_{\mathbb{Z}G}(F, N) \to (\operatorname{Hom}(F, N))^G$$
$$\varphi \mapsto \varphi$$

(since if $\varphi \in \operatorname{Hom}_{\mathbb{Z}G}(F, N)$, then $\varphi(g^{-1} \cdot f) = g^{-1} \cdot \varphi(f)$, which means that the diagonal action $(g \cdot \varphi)(f) = g \cdot \varphi(g^{-1}(f)) = g \cdot g^{-1} \cdot \varphi(f) = \varphi(f)$ fixes φ). This map is an isomorphism.

We claim that $F \otimes M \to M$ is a projective $\mathbb{Z}G$-module resolution of M when M is $\mathbb{Z}$-free. Firstly, since M is a free (and hence flat) $\mathbb{Z}$-module, tensoring $F \to \mathbb{Z}$ with M is exact. The fact that $F \otimes M$ is projective as a $\mathbb{Z}G$-module follows from the fact that F is a projective $\mathbb{Z}G$-module and M is $\mathbb{Z}$-free, so $M \cong \bigoplus_I \mathbb{Z}$, and $F \otimes M \cong F \otimes \bigoplus_I \mathbb{Z} \cong \bigoplus_I (F \otimes \mathbb{Z}) \cong \bigoplus_I F$, which is a projective $\mathbb{Z}G$-module. To compute $\operatorname{Ext}_G^*(M, N)$, we apply $\operatorname{Hom}_{\mathbb{Z}G}(-, N)$ to our projective $\mathbb{Z}G$-module resolution of M and then take homology:

$$\begin{aligned}
\operatorname{Ext}_G^*(M, N) = H_*(\operatorname{Hom}_{\mathbb{Z}G}(F \otimes M, N)) &\cong H_*((\operatorname{Hom}(F \otimes M, N)^G)) \\
&\cong H_*(\operatorname{Hom}(F, \operatorname{Hom}(M, N))^G) \\
&\cong H_*(\operatorname{Hom}_{\mathbb{Z}G}(F, \operatorname{Hom}(M, N))) \\
&= H_*(G; \operatorname{Hom}(M, N))
\end{aligned}$$

where one of the isomorphisms comes from the tensor-hom adjunction $\operatorname{Hom}_S(F \otimes_R M, N) \cong \operatorname{Hom}_R(F, \operatorname{Hom}_S(M, N))$.

$\square$

2.4 Tools for computations

2.4.1 (Co)homological Mayer-Vietoris

Lemma 2.42. *Given an amalgamation of groups*

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_1"] \arrow[d, hook, "\iota_2"'] & G_1 \arrow[d] \\
G_2 \arrow[r] & G
\end{tikzcd}$$

there exists a short exact sequence of permutation modules $0 \to \mathbb{Z}[G/A] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z} \to 0$.

Proof. Let $G = G_1 *_A G_2$. We'll construct a tree X on which G acts, with the property that there exists an edge e on vertices v, w which forms a fundamental domain for the action (every edge is a translate of e under G), and such that the isotropy group of e is A, of v is G_1, and of w is G_2. The action will also be such that no translate of v under G can be a translate of w, so that the quotient X/G is a single edge (and not a loop).

Let the vertices of X be given by cosets in $G/G_1 \sqcup G/G_2$. Attach edges in G/A to the vertices using the canonical maps $\alpha : G/A \to G/G_1$ and $\beta : G/A \to G/G_2$ induced by the inclusions $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$. That is, given an $e \in G/A$, we attach e to the vertices $\alpha(e) \in G/G_1$ and $\beta(e) \in G/G_2$. We say that a path in X is *reduced* if $e_{i+1} \neq \overline{e_i}$ (the edge e_i traversed backwards). We note that X cannot have any nontrivial reduced loops: for if we had, say, a nontrivial reduced loop

$$\begin{tikzcd} v \arrow[r, dash, "e"] \arrow[rrr, dash, bend right, "ghke"'] & w \arrow[r, dash, "ge"] & gv \arrow[r, dash, "ghe"] & ghw \end{tikzcd}$$

then because $ghkv \neq ghw$ (as elements of G can only permute vertices in G/G_1 and vertices in G/G_2 within themselves), it must be the case that $ghkv = v$ and $ghkw = ghw$. This implies that our bottom edge is a translation of the right most edge by k, so we also know that $gkv = gv$, which means $kv = v$. Since $ghv = gv$, then $hv = v$, which, combined with the fact that $ghkv = v$ and $kv = v$, tells us that $gv = v$, contradicting the fact that our loop was assumed to be *reduced.*

In particular we see that X must be a tree. Why is X connected? Let f be another edge in X. We know that $ge = f$ for some $g \in G$. Writing $g = g_1 \cdots g_n$ where $g_1 \in G_1$ and $g_2 \in G_2$ (and so on) without loss of generality. Since $G_1 = G_v$ and $G_2 = G_w$, we have a path of edges (say, if $n = 3$)

$$\begin{tikzcd} v \arrow[r, dash, "e"] \arrow[d, dash, "g_1e"'] & w \\ g_1w \arrow[r, dash, "g_1g_2e"'] & g_2v \arrow[d, dash, "ge=f"] \\ & g_1g_2g_3w \end{tikzcd}$$

This gives that X is acyclic, and so the augmented chain complex

$$0 \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0$$

is exact. As $C_1(X) = \mathbb{Z}[G/G_A]$ and $C_0(X) = \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]$, we are done. $\square$

Prop 2.43 (Mayer-Vietoris for amalgamations). *Given an amalgamation of groups $G = G_1 *_A G_2$ (with $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$ inclusions) and a $\mathbb{Z}G$-module M, there exists a long exact sequence in cohomology*

$$0 \to H^0(A; M) \to H^0(G_1; M) \oplus H^0(G_2; M) \to H^0(G; M) \xrightarrow{\delta} H^1(A; M) \to \cdots$$

Proof. Using the short exact sequence of permutation modules in 2.42, we note that this sequence splits as a short exact sequence of abelian groups (since $\mathbb{Z}$ is a free abelian group). Thus for any $\mathbb{Z}G$-module M, applying $\mathrm{Hom}_{\mathbb{Z}}(-, M)$ to the short exact sequence of permutation modules yields a short exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2], M) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/A], M) \to 0$$

of $\mathbb{Z}G$-modules, where each respective Hom set is a $\mathbb{Z}G$-module under the diagonal action $(g \cdot \varphi)(x) = g \cdot \varphi(g^{-1} \cdot x)$. Let $F_\bullet \to \mathbb{Z}$ be a free resolution over $\mathbb{Z}G$. Then $\mathrm{Hom}_{\mathbb{Z}G}(F_n, -)$ is an exact functor for each n, and we have a short exact sequence of chain complex complexes

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, M) \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2], M)) \to \mathrm{Hom}_{\mathbb{Z}G}(F_\bullet, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G/A], M)) \to 0$$

By the usual methods (snake lemma), this induces a long exact sequence in cohomology

$$0 \to H^0(G; \mathrm{Hom}(\mathbb{Z}[G/A], M)) \to H^0(G; \mathrm{Hom}(\mathbb{Z}[G/G_1], M)) \oplus H^0(G; \mathrm{Hom}(\mathbb{Z}[G/G_2], M)) \to H^0(G; M) \to H^1(G; \mathrm{Hom}(\mathbb{Z}[G/A], M)) \to \cdots$$

Since $\mathbb{Z}[G/H] = \mathrm{Ind}_H^G \mathbb{Z} = \mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}$, we have

$$\mathrm{Hom}(\mathbb{Z}[G/H], M) = \mathrm{Hom}(\mathbb{Z}G \otimes_{\mathbb{Z}H} \mathbb{Z}, M) \cong \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)) \cong \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) = \mathrm{Coind}_H^G M.$$

By Shapiro's lemma 2.57, $H^n(G; \mathrm{Coind}_H^G M) \cong H^n(H; M)$, so the long exact sequence above becomes

$$0 \to H^0(A; M) \to H^0(G_1; M) \oplus H^0(G_2; M) \to H^0(G; M) \to H^1(A; M) \to \cdots$$

as desired. □

2.4.2 (Co)induction & Shapiro's lemma

We can apply the constructions in section 2.2 to ring homomorphisms $\mathbb{Z}H \hookrightarrow \mathbb{Z}G$ for $H \subseteq G$. Now "extension of scalars" becomes "induction" while "coextension of scalars" becomes coinduction. That is:

Definition 2.44 (Induction)**.** If $H \subseteq G$ and M is a left $\mathbb{Z}H$-module, we obtain a left $\mathbb{Z}G$-module via induction: $\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$.

Notice that $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module, with a basis the set of coset representatives gH. So as an abelian group,

$$\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M \cong \bigoplus_{g \in G/H} g \otimes M \cong \bigoplus_{g \in G/H} M.$$

So M is an H-submodule of $\mathrm{Ind}_H^G M$, and $\mathrm{Ind}_H^G M$ is the direct sum of the transforms gM, where g ranges over any set of coset representatives for the left cosets of H in G. (Note that the subgroup gM of $\mathrm{Ind}_H^G M$ depends only on the class of g in G/H, because M is an H-module, and hence any element of H maps M onto itself bijectively.) We claim that this description completely characterizes G-modules of the form $\mathrm{Ind}_H^G M$.

Theorem 2.45. *Suppose N is a G-module whose underlying abelian group is a direct sum $\bigoplus_{i \in I} M_i$. Assume that the G-action transitively permutes the summands, in the sense that there exists a transitive action of G on I such that $g \cdot M_i = M_{gi}$ for $g \in G$ and $i \in I$. Let M be one of the summands M_i, and let $H \subseteq G$ be the isotropy subgroup of i. Then M is an H-module and $N \cong \mathrm{Ind}_H^G M$.*

Proof. It's clear that $M = M_i$ is an H-submodule of N, since $h \cdot M_i = M_{h \cdot i} = M_i$ for all $h \in H$, by construction of H as the isotropy subgroup of i. The inclusion $M \hookrightarrow N$ extends to a G-map $\mathrm{Ind}_H^G M \xrightarrow{\varphi} N$ via the adjunction of extension and restriction of scalars

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, hook, "f"'] & \mathbb{Z}G \otimes_{\mathbb{Z}H} M \arrow[dl, dashed, "\varphi"] \\ N & \end{tikzcd}$$

and φ is an iso as it maps the summand $g \otimes M$ of $\mathrm{Ind}_H^G M$ to the corresponding summand $M_{g \cdot i}$ of N. □

Corollary 2.46. *Let N be a G-module whose underlying abelian group is of the form $\bigoplus_{i\in I} M_i$. Assume that the G-action permutes the summands according to some action of G on I. Let G_i be the isotropy group of i and let E be a set of representatives for I mod G. Then M_i is a G_i-module, and there is a G-isomorphism*

$$N \cong \bigoplus_{i\in E} \operatorname{Ind}_{G_i}^G M_i.$$

Proof. Writing $I = \bigsqcup_{i\in E} G\cdot i$, we obtain $N = \bigoplus_{i\in E}\bigoplus_{j\in G\cdot i} M_j$. Now G transitively permutes $G\cdot i$, and the isotropy subgroup of M_i is G_i, so by the theorem above we have that $\bigoplus_{j\in G\cdot i} M_j \cong \operatorname{Ind}_{G_i}^G M_i$. Hence $N \cong \bigoplus_{i\in E}\operatorname{Ind}_{G_i}^G M_i$. $\square$

Example 2.47 (Permutation modules)**.** The permutation module $\mathbb{Z}[G/H]$ is isomorphic to $\operatorname{Ind}_H^G \mathbb{Z}$, with H acting trivially on $\mathbb{Z}$. This is because $\mathbb{Z}[G/H] \cong \bigoplus_{g\in G/H}\mathbb{Z}\alpha(g)$. This is a G-module, and G transitively permutes the summands. Let M be the summand $\mathbb{Z}\alpha(e)\cong\mathbb{Z}$. The isotropy subgroup of $\overline{e}$ in G/H is H, so by the theorem we have $\mathbb{Z}[G/H]\cong\operatorname{Ind}_H^G\mathbb{Z}$. (Alternatively: $\operatorname{Ind}_H^G\mathbb{Z} = \mathbb{Z}G\otimes_{\mathbb{Z}H}\mathbb{Z}\cong\mathbb{Z}[G/H]$.

Example 2.48. Let X be a G-CW complex and consider the G-module $C_n(X)$. We express $C_n(X) = \bigoplus_{\sigma\in X^{(n)}\setminus X^{(n-1)}}\mathbb{Z}$. These summands are permuted by the G-action. Letting Σ_n be a set of representatives for the G-orbits of n-cells. The corollary above implies

$$C_n(X)\cong\bigoplus_{\sigma\in\Sigma_n}\operatorname{Ind}_{G_\sigma}^G\mathbb{Z}_\sigma$$

where $\mathbb{Z}_\sigma$ is the permutation module of σ (G acts by ±1 depending on whether it preserves or reverses the orientation of σ) and G_σ is the setwise stabilizer of σ.

Prop 2.49 (Exercise 2(a) of III.5 in [2]- Induction commutes with tensoring on the right)**.** *For any H-module M and G-module N, we have that $N\otimes\operatorname{Ind}_H^G M\cong\operatorname{Ind}_H^G((\operatorname{Res}_H^G N)\otimes M)$, where the left hand tensor product has a diagonal G-action and the right hand tensor product has a diagonal H-action.*

Proof. Write $\operatorname{Ind}_H^G M\cong\bigoplus_{i\in G/H} i\cdot M$. Then $N\otimes\operatorname{Ind}_H^G M\cong\bigoplus_{i\in G/H} N\otimes iM$. Since $g\cdot N = N$, we have that $g\cdot(N\otimes iM) = gN\otimes giM = N\otimes giM$, and hence G transitively permutes the summands of $\bigoplus_{i\in G/H} N\otimes iM$. The isotropy group of $e\in G/H$ under this action is isomorphic to H, so by the theorem, $N\otimes\operatorname{Ind}_H^G M\cong\operatorname{Ind}_H^G((\operatorname{Res}_H^G N)\otimes M)$. $\square$

Prop 2.50 (Exercise 2(b) of III.5 in [2])**.** *Prove similar results for* $\operatorname{Hom}(\operatorname{Ind}_H^G M, N)$ *and* $\operatorname{Hom}(N,\operatorname{Coind}_H^G M)$.

Proof. Since $\operatorname{Ind}_H^G M\cong\bigoplus_{\sigma\in G/H}\sigma M$ as abelian groups, we have

$$\operatorname{Hom}(\operatorname{Ind}_H^G M, N)\cong\operatorname{Hom}(\bigoplus_{\sigma\in G/H}\sigma M, N)\cong\bigoplus_{\sigma\in G/H}\operatorname{Hom}(\sigma M, N)$$

as abelian groups. Now G acts on $\operatorname{Hom}(\sigma\otimes M, N)$ diagonally and transitively, since if $\varphi\in\operatorname{Hom}(\sigma\otimes M, N)$ then $g\cdot\varphi\in\operatorname{Hom}(g\sigma\otimes M, N)$ (as $(g\cdot\varphi)(g\sigma\otimes m) = g\cdot\varphi(g^{-1}\cdot(g\sigma\otimes m)) = g\cdot\varphi(\sigma\otimes m)\in N$). Thus by 2.45, we have $\operatorname{Hom}(\operatorname{Ind}_H^G M, N)\cong\operatorname{Ind}_H^G(\operatorname{Hom}(M,N),\operatorname{Res}_H^G N)$.

Similarly, we have

$$\operatorname{Hom}(N,\operatorname{Coind}_H^G M)\cong\operatorname{Hom}(N,\prod_{\sigma\in G/H}\sigma M)\cong\prod_{\sigma\in G/H}\operatorname{Hom}(N,\sigma M)\cong\operatorname{Coind}_H^G(\operatorname{Hom}(\operatorname{Res}_H^G N, M))$$

by 2.55. $\square$

Corollary 2.51 (Prop III.5(6) In [2]). *Let N be a G-module. Then $\operatorname{Ind}_H^G \operatorname{Res}_H^G N \cong N \otimes \mathbb{Z}[G/H]$, where G acts diagonally on the tensor product.*

Proof. Immediate application of prop 2.49, with $M = \mathbb{Z}$. □

Corollary 2.52. *Let M be a G-module and M_0 its underlying abelian group. Then $\mathbb{Z}G \otimes M$ with diagonal G-action is canonically isomorphic to the induced module $\mathbb{Z}G \otimes_{\mathbb{Z}e} M_0 = \mathbb{Z}G \otimes M_0$. In particular, if M_0 is a free $\mathbb{Z}$-module, then $\mathbb{Z}G \otimes M$ is a free $\mathbb{Z}G$-module.*

Proof. Letting $H = \{e\}$, we have $\operatorname{Ind}_{\{e\}}^G \operatorname{Res}_{\{e\}}^G M \cong M \otimes \mathbb{Z}G \cong \mathbb{Z}G \otimes M$ (with diagonal G-action). □

Definition 2.53 (Coinduction). If $H \subseteq G$ and M is a left $\mathbb{Z}H$-module, we have a left $\mathbb{Z}G$-module via coinduction: $\operatorname{Coind}_H^G M = \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$.

Prop 2.54. *As an abelian group, $\operatorname{Coind}_H^G M \cong \prod_{\sigma H \in G/H} M$.*

Proof. Notice that each $\varphi \in \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ is determined by where it maps a set of representatives in G/H: say $\{\gamma_i\}_{i\in I}$ are a set of representatives S for the left cosets of G/H. If $g \in \gamma_i H$, then $\gamma_i^{-1} g \in H$, so $\varphi(\alpha(g)) = \varphi(\alpha(\gamma_i \gamma_i^{-1} g)) = \gamma_i^{-1} g \cdot \varphi(\alpha(\gamma_i))$. We get a map

$$\operatorname{Coind}_H^G M \xrightarrow{\psi} \prod_{\sigma H \in G/H} M$$

if we specify group homomorphisms $\operatorname{Coind}_H^G M \to M$ for each $\sigma \in S$ (then $\psi(\varphi) = (\varphi(\alpha(\gamma_i))_{i\in I})$. Note that $\varphi \in \ker \psi$ if and only if $\varphi(\alpha(\gamma_i)) = 0$ for all i, which holds if and only if $\varphi \equiv 0$. Hence ψ is injective. To see that ψ is also surjective, let $(m_i)_{i\in I} \in \prod_{\sigma H \in G/H} M$. We wish to construct a $\mathbb{Z}H$-module homomorphism $\varphi : \mathbb{Z}G \to M$ such that $\varphi(\alpha(\gamma_i)) = m_i$ for all i. Using the H-module action on M, we can extend φ to an element of $\operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ by defining, for $g \in \gamma_i H$,

$$\varphi(\alpha(g)) := \gamma_i^{-1} g \cdot \varphi(\alpha(\gamma_i)).$$

□

Prop 2.55. *Suppose N is a G-module, which as an abelian group admits a direct product decomposition $(\pi_i : N \twoheadrightarrow M_i)_{i\in I}$. Suppose that there is a transitive right G-action on I such that $\pi_i \cdot g \sim \pi_{ig}$ (meaning $\ker \pi_i = \ker \pi_{ig}$). Let $\pi : N \to M$ be one of the π_i and $H \subseteq G$ the isotropy group of i. Then M admits an H-module structure from N, and $N \cong \operatorname{Coind}_H^G M$.*

Proof. The proof is analogous to that of 2.45. Recall that if $\pi : N \twoheadrightarrow M$ is a surjection and N has a G-module structure, we define $(\pi \cdot g)(x) = \pi(gx)$. Let $M = M_i$ and let H be the isotropy group of π_i, that is $\pi_{ih} \sim \pi_i$ for all $h \in H$. Easy to see that $M = M_i$ admits an H-module structure, since $N \twoheadrightarrow M$ is surjective, and thus if $m \in M$ with $\pi(n) = m$, we can define $m \cdot h = \pi(h \cdot n) = (\pi \cdot h)(n)$ (note that if n, n' are both such that $\pi(n) = \pi(n') = m$, then $n - n' \in \ker \pi = \ker(\pi \cdot h)$, so $(\pi \cdot h)(n) = (\pi \cdot h)(n')$, and our action is well-defined). Since we have a natural H-module map $\operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \to M$, the H-module map $N \to M$ descends to a map of G-modules

$$\begin{tikzcd} N \arrow[r, dashed] \arrow[d, "\pi"'] & \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \arrow[dl] \\ M & \end{tikzcd}$$

by the universal property of coextension of scalars 2.28. This gives us the desired isomorphism. □

Prop 2.56 (Equivalence of (co)induction for finite index subgroups)**.** *If $[G : H] < \infty$, then there exists an isomorphism of $\mathbb{Z}G$-modules $\operatorname{Ind}_H^G M \cong \operatorname{Coind}_H^G M$ for any $\mathbb{Z}H$-module M.*

Proof. Consider the map

$$\begin{aligned} \phi : M &\to \operatorname{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) = \operatorname{Coind}_H^G M \\ m &\mapsto \left(\begin{cases} g \mapsto g \cdot m & g \in H \\ g \mapsto 0 & g \notin H \end{cases} \right). \end{aligned}$$

This is an H-module map, since $\phi(h \cdot m)(g) = g \cdot h \cdot m$ if $g \in H$, and $\phi(h \cdot m)(g) = 0$ if $g \notin H$. But $(h \cdot \phi(m))(g) = \phi(m)(gh) = ghm$ if $gh \in H$ (which holds if and only if $g \in H$), and $\phi(m)(gh) = 0$ if $gh \notin H$ (which holds if and only if $g \notin H$). Since extension of scalars is left adjoint to restriction of scalars, there exists a unique G-module map $\varphi : \operatorname{Ind}_H^G M \to \operatorname{Coind}_H^G M$ such that

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[d, "\phi"'] & \operatorname{Ind}_H^G M \arrow[dl, dashed, "\varphi"] \\ \operatorname{Coind}_H^G M & \end{tikzcd}$$

commutes (with $i : M \to \operatorname{Ind}_H^G M, m \mapsto 1 \otimes m$). Explicitly, $\varphi(\alpha(g) \otimes m) = g \cdot \phi(m)$. We claim that φ is an isomorphism. Since $[G : H] < \infty$ by assumption, the finite direct sum $\bigoplus_{\sigma H \in G/H} M$ and the finite direct product $\prod_{\sigma H \in G/H} M$ are isomorphic as abelian groups. The G-module homomorphism φ sends the summand corresponding to the representative $\gamma_i H$ ($\alpha(\gamma_i) \otimes M$) to the summand corresponding to γ_i, since $\varphi(\alpha(\gamma_i) \otimes M) = \gamma_i \cdot \phi(M)$, and $\gamma_i \cdot \phi(M)$ consists of precisely those $\mathbb{Z}H$-module homomorphisms $\mathbb{Z}G \to M$ with support in $\gamma_i H$. □

Prop 2.57 (Shapiro's Lemma)**.** *Let $H \leqslant G$, M an H-module. Then*

$$H_*(H; M) \cong H_*(G; \operatorname{Ind}_H^G M), \qquad H^*(H; M) \cong H^*(G; \operatorname{Coind}_H^G M).$$

as $\mathbb{Z}H$-modules.

Proof. This can be proven purely using adjunctions. Let $P_\bullet$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Since $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module (and a projective restriction of scalars preserves projectivity), then $P_\bullet$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ as well. Since

$$H_*(P_\bullet \otimes_{\mathbb{Z}G} \operatorname{Ind}_H^G M) = H_*(P_\bullet \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong H_*(P_\bullet \otimes_{\mathbb{Z}H} M)$$

which shows that $H_*(G; \operatorname{Ind}_H^G M) \cong H_*(H; M)$. On the other hand, we have

$$H_*(\operatorname{Hom}_{\mathbb{Z}G}(P_\bullet, \operatorname{Coind}_H^G M)) = H_*(\operatorname{Hom}_{\mathbb{Z}H}(\operatorname{Res}_H^G P_\bullet, M))$$

so that $H^*(G; \operatorname{Coind}_H^G M) \cong H^*(M; H)$.

It's not clear that these isomorphisms hold as $\mathbb{Z}H$-modules. Let $\alpha : H \hookrightarrow G$ be an inclusion, let M be an H-module, and let $i : M \to \operatorname{Ind}_H^G M$ and $\pi : \operatorname{Coind}_H^G M \to M$ be the canonical H-maps. The isomorphisms of Shapiro's lemma are given by $(\alpha, i)_* : H_*(H, M) \to H_*(G, \operatorname{Ind}_H^G M)$ and $(\alpha, \pi)^* : H^*(G; \operatorname{Coind}_H^G M) \to H^*(H; M)$.

Proof. Let $P_\bullet$ be a projective $\mathbb{Z}G$-module resolution of $\mathbb{Z}$. Viewing it as a projective $\mathbb{Z}H$-module resolution via restriction of scalars, we have an H-chain map $\tau = \mathrm{id}$

$$\begin{tikzcd}
\cdots \arrow[r] & P_n \arrow[r] \arrow[d, dashed, "\tau_n"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\tau_0"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

since $\tau(h \cdot x) = \alpha(h)\tau(x)$. Since i is an H-module map, we obtain a well-defined H-chain map $\tau \otimes_H i : P_\bullet \otimes_{\mathbb{Z}H} M \to P_\bullet \otimes_{\mathbb{Z}G} \mathrm{Ind}_H^G M$ via $(p \otimes_H m) \mapsto p \otimes_G (1 \otimes_H m)$. This chain map induces the map $(\alpha, i)_* : H_*(H; M) \to H_*(G; \mathrm{Ind}_H^G M)$ on homology. Notice that our chain map has an H-chain homotopy inverse! We can construct a map $P_\bullet \times (\mathrm{Ind}_H^G M) \xrightarrow{\gamma} P_\bullet \otimes_{\mathbb{Z}H} M$ via $(p, g \otimes_H m) \mapsto pg \otimes_H m$. This map is G-balanced since

$$\gamma(pg, \tilde{g} \otimes_H m) = pg\tilde{g} \otimes_H m = \gamma(p, g\tilde{g} \otimes_H m)$$

and hence we get a well-defined map $\gamma : P_\bullet \otimes_{\mathbb{Z}G} \mathrm{Ind}_H^G M \to P_\bullet \otimes_{\mathbb{Z}H} M$. This is an H-chain map since

$$\begin{aligned}
\gamma(h \cdot (p \otimes_G (g \otimes_H m))) = \gamma(ph^{-1} \otimes_G (g \otimes_H m)) &= \gamma(p \otimes_G h^{-1}(g \otimes_H m)) \\
&= \gamma(p \otimes_G (gh \otimes_H m)) \\
&= pgh \otimes_H m \\
&= h \cdot \gamma(p \otimes_G (g \otimes_H m)) ?h^{-1}
\end{aligned}$$

Thus (α, i_*) is actually an isomorphism of H-modules. Similarly, letting τ be the identity, we have a well-defined H-chain map

$$\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, \mathrm{Coind}_H^G M) \to \mathrm{Hom}_{\mathbb{Z}H}(P_\bullet, M)$$

FINISH THIS!!! □

□

2.4.3 Transfer maps

Lemma 2.58. *If $p : X \to B$ is a k-fold cover, there is a map $\tau : \tilde{H}_*(B; \mathbb{Z}) \to \tilde{H}_*(X; \mathbb{Z})$ so that $p_* \circ \tau = k \cdot \mathrm{id}$.*

Proof. Let $\sigma : \Delta^n \to B$ be a singular simplex. Then σ can be thought of as an oriented cell of B. We have k oriented cells $\tilde{\sigma}_i$ lying over σ, since $p : X \to B$ is a k-fold cover, and $\sigma : \Delta^n \to B$ satisfies the lifting criterion by virtue of Δ^n being simply-connected:

$$\begin{tikzcd}
 & X \arrow[d, "p"] \\
\Delta^n \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & B.
\end{tikzcd}$$

We obtain a map $\sigma \mapsto \sum_{i=1}^k \tilde{\sigma}_i$ on $C_n(B) \xrightarrow{f_\#} C_n(X)$, and thus a map $\tilde{H}_n(B) \xrightarrow{\tau} \tilde{H}_n(X)$, $[\sigma] \mapsto \sum_{i=1}^k [\tilde{\sigma}_i]$. Then $p_* \circ \tau([\sigma]) = p_*(\sum_{i=1}^k [\tilde{\sigma}_i]) = k \cdot [\sigma]$. □

Lemma 2.59. *If G is a discrete group and $H \leqslant G$ is a subgroup of finite index, there exists a model of* BH *which is a $[G : H]$-fold cover of* BG.

Proof. Let EG be a contractible CW complex with a free and properly discontinuous G-action (by properly discontinuous, we mean that for every compact subset $K \subseteq \mathrm{EG}$, the set of $g \in G$ for which $g \cdot K \cap K \neq \varnothing$ is finite). Such a space always exists for discrete groups G. Recall from covering space theory that for every subgroup $H \leqslant G$ of finite index, we can construct a covering space $\tilde{X}_H \xrightarrow{p} \mathrm{BG} = \mathrm{EG}/G$ from the universal cover $\tilde{X} \to \mathrm{BG}$ such that $p_*(\pi_1(\tilde{X}_H)) = H \leqslant G = \pi_1(\mathrm{BG})$. We also know that this cover is a $[G:H]$-sheeted cover. Hence $\tilde{X}_H$ is a model of BH which is a $[G:H]$-fold cover of BG. □

Prop 2.60. *For a finite group G, $\tilde{H}_*(\mathrm{BG}; \mathbb{Z}[\frac{1}{|G|}]) = 0$. For $H \leqslant G$, the map $\iota_* : \tilde{H}_*(\mathrm{BH}; \mathbb{Z}[\frac{1}{[G:H]}]) \to \tilde{H}_*(\mathrm{BG}; \mathbb{Z}[\frac{1}{[G:H]}])$ admits a right inverse τ (that is, $\iota_* \circ \tau = \mathrm{id}$).*

Proof. By the universal coefficient theorem, we have an isomorphism

$$H_*(\mathrm{BH}; \mathbb{Z}[1/[G:H]]) \cong H_*(\mathrm{BH}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/[G:H]] \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{*-1}(\mathrm{BH}), \mathbb{Z}[1/[G:H]]).$$

Since $\mathbb{Z}[1/[G:H]]$ is a localization of $\mathbb{Z}$, and localizations are flat, we have that $\mathbb{Z}[1/[G:H]]$ is a flat $\mathbb{Z}$-module and the Tor term vanishes. Letting $p : \mathrm{BH} \to \mathrm{BG}$ be the $[G:H]$-fold cover provided by lemma 2.59, we have that ι_* can be described by

$$\begin{aligned} H_*(\mathrm{BH}) \otimes \mathbb{Z}[1/[G:H]] &\xrightarrow{\iota_*} H_*(\mathrm{BG}) \otimes \mathbb{Z}[1/[G:H]] \\ [\sigma] \otimes 1 &\mapsto p_*([\sigma]) \otimes 1. \end{aligned}$$

Let $\tau : \tilde{H}_*(\mathrm{BG}) \to \tilde{H}_*(\mathrm{BH})$ be the map described in lemma 2.58. The composite

$$\iota_* \circ (\tau \otimes \frac{1}{[G:H]}) : \tilde{H}_*(\mathrm{BG}) \otimes \mathbb{Z}[1/[G:H]] \to \tilde{H}_*(\mathrm{BH}) \otimes \mathbb{Z}[1/[G:H]] \to \tilde{H}_*(\mathrm{BG}) \otimes \mathbb{Z}[1/[G:H]]$$

$$[\sigma] \otimes 1 \mapsto \sum_{i=1}^{[G:H]} [\tilde{\sigma}_i] \otimes \frac{1}{[G:H]} \mapsto \sum_{i=1}^{[G:H]} p_*([\tilde{\sigma}_i]) \otimes \frac{1}{[G:H]} = [G:H] \cdot [\sigma] \otimes \frac{1}{[G:H]} = [\sigma] \otimes 1.$$

is the identity, as desired. In particular, applying this to the subgroup $H = \{e\}$, we have that $B\{e\}$ is given by a single point, so the composite map

$$\tilde{H}_*(\mathrm{BG}; \mathbb{Z}[1/|G|]) \to \tilde{H}_*(*; \mathbb{Z}[1/|G|]) \to \tilde{H}_*(\mathrm{BG}; \mathbb{Z}[1/|G|])$$

is the identity. Since the middle group is trivial, it must be the case that $\tilde{H}_*(\mathrm{BG}; \mathbb{Z}[1/|G|]) = 0$. □

Corollary 2.61. *If G is a finite group, then $\tilde{H}_*(G; \mathbb{Z})$ has $|G|$-torsion (is annihilated by multiplication by $|G|$).*

Proof. We have a $|G|$-fold cover $p : \mathrm{EG} \to \mathrm{BG}$ (noting that EG being a contractible CW complex (?) means that EG is a classifying space for $\{e\}$). We know there exists a map $\tau : \tilde{H}_*(\mathrm{BG}; \mathbb{Z}) \to \tilde{H}_*(\mathrm{EG}; \mathbb{Z})$ with $p_* \circ \tau = |G| \cdot \mathrm{id}$. But τ is the 0 map since $\tilde{H}_*(\mathrm{EG}; \mathbb{Z}) = 0$, so multiplication by $|G|$ on $\tilde{H}_*(\mathrm{BG}; \mathbb{Z}) = \tilde{H}_*(G; \mathbb{Z})$ is the 0 map. □

Note that the proof would have worked for any $\mathbb{Z}G$-module A.

Corollary 2.62. *If G is a finite group and A is a $\mathbb{Z}G$-module in which $|G|$ is invertible (say $A = \mathbb{Q}$ or $\mathbb{Z}[\frac{1}{|G|}]$), then $H_*(G; A) = H^*(G; A) = 0$ for $* \neq 0$.*

Proof. By 2.61, we know that $\tilde{H}_*(G; A)$ is annihilated by multiplication by $|G|$. But if we multiply by $|G|$

$$\begin{aligned} \tilde{H}_*(G; A) &\to \tilde{H}_*(G; A) \\ [\sigma] &\mapsto |G|[\sigma] \end{aligned}$$

and $|G|$ is invertible in A, then the product $|G|[\sigma]$ being 0 tells us that $[\sigma] = 0$, and thus that $\tilde{H}_*(G; A) = 0$. A similar proof works for cohomology. □

Definition 2.63 (Transfer maps)**.** If $H \leqslant G$ is a subgroup of finite-index, then $\mathrm{Ind}_H^G M \cong \mathrm{Coind}_H^G M$ for any H-module M 2.56, and hence we have natural maps $M \to \mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M$ and $\mathrm{Coind}_H^G M \cong \mathrm{Ind}_H^G M \to M$. This gives natural maps $H_*(G; M) \to H_*(G; \mathrm{Ind}_H^G M) \cong H_*(H; M)$ by Shapiro's lemma, and $H^*(H; M) \cong H^*(G; \mathrm{Coind}_H^G M) \to H^*(G; M)$.

2.4.4 (Co)restriction, (co)inflation

Viewing $H^*(-;-)$ and $H_*(-;-)$ as covariant functors in the first variable. Parts of this follow from Weibel [10] section 6.7, parts follow from Brown [2] III.8.

If $\rho : H \to G$ is a group homomorphism, let $\rho^\# : \mathrm{Mod}_G \to \mathrm{Mod}_H$ be the *forgetful functor*, which is exact.

This functor remembers G-modules as H-modules (restriction of scalars). If A is a G-module, we obtain an H-module $\rho^\#(A)$ via $h \cdot \rho^\#(A) = \rho^\#(\rho(h) \cdot A)$. We have a natural surjection

$$\begin{aligned} (\rho^\# A)_H &\to A_G \\ \overline{\rho^\#(x)} &\mapsto \overline{x}. \qquad \text{(coinvariants)} \end{aligned}$$

We need to verify that this is a well-defined surjection. Suppose $\overline{\rho^\#(x)} = \overline{\rho^\#(y)}$. Then there exists some $h \in H$ such that $h \cdot (\rho^\#(x) - \rho^\#(y)) = \rho^\#(x) - \rho^\#(y)$, which means that $\rho^\#(\rho(h) \cdot (x-y)) = \rho^\#(x-y)$. Since $\rho^\#$ is injective, we conclude that $\rho(h)(x-y) = x-y$, so $\overline{x} = \overline{y} \in A_G$. The fact that this map is a surjection is clear, as $\rho^\#$ is defined for every $x \in A$.

For similar reasons we have a natural injection

$$\begin{aligned} A^G &\to (\rho^\# A)^H \\ a &\mapsto \rho^\#(a) \qquad \text{(invariants)} \end{aligned}$$

Our coinvariants surjection extends to a morphism $\rho_* = \mathrm{cor}_H^G$ (corestriction).

Definition 2.64 (Corestriction)**.** We *start* with "restricted scalars" and go up to full scalars, hence "corestriction." Let $\rho : H \to G$ be a group homomorphism, and let A be a $G - \mathrm{Mod}$.

$$\mathrm{cor}_H^G : H_*(H; \rho^\# A) \to H_*(G; A).$$

To construct it, recall that $H_*(G; A) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, A)$. Start with a projective resolution $Q_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}H$ and a flat resolution $P_\bullet$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Viewing $P_\bullet$ as $\mathbb{Z}H$-modules via ρ, we potentially lose flatness but still keep acyclicity. By the fundamental lemma of homological algebra 2.20, there exists an H-chain map $\tau : Q_\bullet \to P_\bullet$ satisfying

$$\tau(h \cdot x) = \rho(h) \cdot \tau(x)$$

which is unique up to homotopy. Note that we have an H-map $\rho^\# A \xrightarrow{f} A$ via $\rho^\#(a) \mapsto a$, since $h \cdot \rho^\#(a) = \rho^\#(\rho(h) \cdot a) \mapsto \rho(h) \cdot a$.

$$\begin{tikzcd} \cdots \arrow[r] & Q_2 \arrow[r] \arrow[d, "\tau_2"'] & Q_1 \arrow[r] \arrow[d, "\tau_1"'] & Q_0 \arrow[r] \arrow[d, "\tau_0"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

We have an induced chain map $\tau \otimes f : Q_\bullet \otimes_H \rho^\# A \to P_\bullet \otimes_G A$, inducing a well-defined homomorphism $(\tau \otimes f)_* : H_*(H; \rho^\# A) \to H_*(G; A)$. We should check that the map $\tau \otimes f$ is well-defined (H-balanced). Recall that we see $Q_\bullet$ and $P_\bullet$ as right $\mathbb{Z}H$-modules via "$q \cdot h$" $= h^{-1} \cdot q$. Then

$$\begin{aligned}(\tau \otimes f)(h \cdot q \otimes \rho^\#(a)) = \tau(h \cdot q) \otimes a &= (\rho(h) \cdot \tau(x)) \otimes a \\ &= \tau(x) \otimes \rho(h)^{-1} \cdot a \\ &= \tau(x) \otimes \rho(h^{-1}) \cdot a \\ &= (\tau \otimes f)(q \otimes h^{-1} \cdot \rho^\#(a)).\end{aligned}$$

Definition 2.65 (Restriction)**.** Similar story, but this time we have an induced chain map $\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, A) \to \mathrm{Hom}_{\mathbb{Z}H}(Q_\bullet, \rho^\# A)$ given by

$$(\alpha : P_n \to A) \mapsto (\beta : Q_n \xrightarrow{\tau_n} P_n \xrightarrow{\alpha} A \to \rho^\#(A)).$$

Induces a map $\mathrm{res}^G_H : H^*(G; A) \to H^*(H; \rho^\# A)$.

Example 2.66 (Application of Functoriality (Conjugation))**.** Let $H \leqslant G$ be a subgroup, M a G-module, and $g \in G$. We have an isomorphism $c(g) : (H, M) \to (gHg^{-1}, M)$ via $h \mapsto ghg^{-1}$ and $m \mapsto g \cdot m$. How do we compute $c(g)_* : H_*(H; M) \to H_*(gHg^{-1}; M)$ on the chain level? Let $F_\bullet$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Use the fact that $F_\bullet$ is naturally a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$ and $\mathbb{Z}gHg^{-1}$ as well. Let $\tau : F \to F$ be the map given by multiplication by g ($\tau(x) = g \cdot x$). This induces a map

$$\begin{aligned}\tau \otimes f : F \otimes_H M &\to F \otimes_{gHg^{-1}} M \\ x \otimes m &\mapsto \tau(x) \otimes f(m) = gx \otimes gm = xg^{-1} \otimes gm\end{aligned}$$

We check that this map is H-balanced, and hence well-defined:

$$\begin{aligned}(\tau \otimes f)(x \cdot h^{-1} \otimes m) = x \cdot h^{-1} g^{-1} \otimes gm &= x \cdot g^{-1} g h^{-1} g^{-1} \otimes gm \\ &= x \cdot g^{-1} \otimes g h^{-1} g^{-1} g m \\ &= g \cdot x \otimes g \cdot h^{-1} m \\ &= (\tau \otimes f)(x \otimes h^{-1} \cdot m)\end{aligned}$$

where we are freely moving between the identification of F as a left and right $\mathbb{Z}$-module via $g \cdot x = x \cdot g^{-1}$.

The map induced by g is now given by $c(g)_* : H_*(H; M) \to H_*(gHg^{-1}; M), z \mapsto c(g)_*(z)$. Notice: if $h \in H$, then $h \cdot z = z$ for all $z \in H_*(H; M)$. This is because if $h \in H$, then $hHh^{-1} = H$, and the map $\tau \otimes f : F \otimes_H M \to F \otimes_H M$ is just the identity (as $xh^{-1} \otimes hm = x \otimes m$). We see then that conjugation induces a well-defined action of G/H on $H_*(H; M)$.

Note also in cohomology, a map $\alpha : G \to G'$ and a map $f : M' \to M$ such that $f(\alpha(g)m') = gf(m')$ induces a map $H^*(G'; M') \to H^*(G; M)$. In the case of conjugation ($H \leqslant G$), we see that

$$\begin{aligned}(H, M) \xrightarrow{c_g, \cdot g} (gHg^{-1}, M) &\xrightarrow{c_{g^{-1}}, \cdot g^{-1}} (H, M) \\ (h, m) &\mapsto (h, m)\end{aligned}$$

is the identity, so $(c_g)^* : H^*(gHg^{-1}; M) \to H^*(H; M)$ is invertible, and we can define $g \cdot z = ((c_g)^*)^{-1}(z)$ for $z \in H^*(H; M)$. Alternatively, if we look at $(c_{g^{-1}})^*$, we see that the induced map on the cochain level is given by (recalling that $\tau(x) = g^{-1}x$, so that $\tau(ghg^{-1}x) = hg^{-1}x = h\tau(x)$)

Definition 2.67 (Inflation)**.** Given $H \xrightarrow{\iota} G \xrightarrow{\pi} G/H$, we can compose maps to obtain (for a G-module A),

$$\inf = \inf_{G/H}^{G} : H^*(G/H; A^H) \xrightarrow{\operatorname{res}_G^{G/H}} H^*(G; \pi^\#(A^H)) \to H^*(G; A).$$

Definition 2.68 (Coinflation)**.** Given $H \xrightarrow{\iota} G \xrightarrow{\pi} G/H$, we can compose maps to obtain (for a G-module A),

$$\operatorname{coinf} = \operatorname{coinf}_{G/H}^{G} : H_*(G; A) \to H_*(G; A_H) \xrightarrow{\operatorname{cor}_G^{G/H}} H_*(G/H; A_H)$$

2.4.5 Lyndon-Hochschild-Serre Spectral Sequence

Theorem 2.69 (LHS SS)**.** *For every normal subgroup H of a group G, there are two convergent first quadrant spectral sequences corresponding to the extension $H \to G \to G/H$ and the coefficient $\mathbb{Z}G$-module A (which can also be canonically viewed as a $\mathbb{Z}H$-module):*

Homological: $E^2_{p,q} = H_p(G/H; H_q(H; A)) \Longrightarrow H_{p+q}(G; A)$

Cohomological: $E_2^{p,q} = H^p(G/H; H^q(H; A)) \Longrightarrow H^{p+q}(G; A)$.

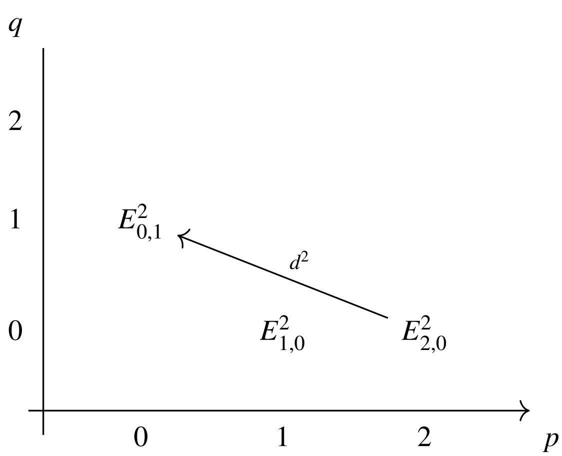
Corollary 2.70. *There exists a five term exact sequence*

$$H_2(G; M) \xrightarrow{\text{coinf}} H_2(G/H; M_H) \xrightarrow{\text{transgression}} (H_1(H; M))_{G/H} \xrightarrow{\text{cor}} H_1(G; M) \xrightarrow{\text{coinf}} H_1(G/H; M_H) \to 0.$$

Proof. Since $E^\infty = \operatorname{Gr} H(G; M)$, and we have a homology filtration $0 = F_{-1}H_1(G; M) \to F_0H_1(G; M) \to F_1H_1(G; M) = H_1(G; M)$ with $F_pH_{p+q}(G; M)/F_{p-1}H_{p+q}(G; M) \cong E^\infty_{p,q}$. Hence $F_0H_1(G; M) \cong E^\infty_{0,1}$, and $H_1(G; M)/E^\infty_{0,1} \cong E^\infty_{1,0}$. This means we have a short exact sequence

$$0 \to E^\infty_{0,1} \to H_1(G; M) \to E^\infty_{1,0} \to 0.$$

Recall that $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$, so for $r \geqslant 2$, there are no nontrivial differentials to or from $E^r_{1,0}$.



Hence $E^\infty_{1,0} = E^2_{1,0} = H_1(G/H; H_0(H; M)) = H_1(G/H; M_H)$. The only nontrivial differentials involving $E^r_{0,1}$ or $E^r_{2,0}$ for $r \geqslant 2$ is $d^2 : E^2_{2,0} \to E^2_{0,1}$ (and $E^\infty_{2,0} = \ker d_2$, $E^\infty_{0,1} = \operatorname{coker} d_2$), so we have an exact sequence

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to E^\infty_{0,1} \to 0.$$

Using the short exact sequence from earlier in the proof, we can extend this exact sequence to

$$\begin{tikzcd} 0 \arrow[r] & E^\infty_{2,0} \arrow[r] & E^2_{2,0} \arrow[r, "d^2"] & E^2_{0,1} \arrow[r, dashed] \arrow[dr, two heads] & H_1(G;M) \arrow[r, two heads] & E^\infty_{1,0} \arrow[r] & 0 \\ & & & & E^\infty_{0,1} \arrow[u, hook] & & \end{tikzcd}$$

where the dashed map is the composition of the two below it. Since $E^\infty_{2,0}$ is a quotient of $H_2(G;M)$ (and hence surjects onto $H_2(G;M)$), we have a map $H_2(G;M) \to E^2_{2,0} = H_2(G/H;H_0(H;M)) = H_2(G/H;M_H)$. Since $\ker d^2 = E^\infty_{2,0}$, and the image of $H_2(G;M) \to E^2_{2,0}$ is the kernel of d_2, since for each $u \in H_2(G;M)$ we pick a lift $x \in E^\infty_{2,0}$, and the map $H^2(G;M) \to E^2_{2,0}$ sends $u \mapsto x$ (where x is now included in $E^2_{2,0}$). Notice that the inclusion $E^\infty_{2,0} \hookrightarrow E^2_{2,0}$ is precisely as the kernel of $d^2 : E^2_{2,0} \to E^2_{0,1}$. Finally, since $E^2_{0,1} = H_0(G/H;H_1(H;M)) \cong (H_1(H;M))_{G/H}$, we conclude that we have an exact sequence

$$H_2(G;M) \xrightarrow{\text{coinf}} H_2(G/H;M_H) \xrightarrow{\text{transgression}} (H_1(H;M))_{G/H} \xrightarrow{\text{cor}} H_1(G;M) \xrightarrow{\text{coinf}} H_1(G/H;M_H) \to 0.$$

□

Example 2.71 (Homology of D_n for $n \geqslant 3$ odd.)**.** Let $D_n = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle$ be the dihedral group of order $2n$, with $n \geqslant 3$ odd. We can express D_n as a semidirect product $C_n \rtimes C_2$ with $C_n = \langle r \rangle$ and $C_2 = \langle s \rangle$. We thus have a short exact sequence of groups

$$1 \to C_n \to D_n \to C_2 \to 1.$$

The homological LHS tells us that we have a spectral sequence with $E^2_{p,q} = H_p(C_2;H_q(C_n;\mathbb{Z})) \implies H_{p+q}(D_n;\mathbb{Z})$. What groups can we figure out with relative ease? Along the p-axis ($q = 0$), we have

$$E^2_{p,0} = H_p(C_2;H_0(C_n;\mathbb{Z})) = H_p(C_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}/2 & p \text{ odd} \\ 0 & p > 0 \text{ even.} \end{cases}$$

When $q > 0$ is even, we know that $H_q(C_n;\mathbb{Z}) = 0$, so $E^2_{p,q} = 0$. Along the q-axis ($p = 0$), we have

$$E^2_{0,q} = H_0(C_2;H_q(C_n;\mathbb{Z})) = (H_q(C_n;\mathbb{Z}))_{C_2}$$

the C_2-invariants of the $\mathbb{Z}C_2$-module $H_q(C_n;\mathbb{Z})$ (recalling that the generator of C_2 acts on the generator of C_n by conjugation). When $q > 0$ is even this is trivial, as $H_q(C_n;\mathbb{Z}) = 0$ in this case. When $q = 0$ we obtain $H_0(C_n;\mathbb{Z}) = \mathbb{Z}$ is the trivial C_2-module, so $E^2_{0,0} = \mathbb{Z}$. When q is odd some more analysis is needed. We know that

$$(H_q(C_n;\mathbb{Z}))_{C_2} = H_q(C_n;\mathbb{Z})/\langle [x] - g \cdot [x] | g \in C_2, [x] \in H_q(C_n;\mathbb{Z}) \rangle.$$

Simplifying this depends heavily on how C_2 acts on $H_q(C_n;\mathbb{Z})$. We make the following claim:

Claim 2.72. *For $i \geqslant 1$, C_2 acts on $H_{2i-1}(C_n;\mathbb{Z})$ by multiplication by $(-1)^i$.*

Proof. Let $C_2 = \langle s \rangle$ and let $\rho : C_n \to C_n$ be given by $r \mapsto srs^{-1} = r^{-1}$. We have a resolution P of $\mathbb{Z}$ by $\mathbb{Z}C_n$ -modules

$$\cdots \to \mathbb{Z}C_n \xrightarrow{\cdot N} \mathbb{Z}C_n \xrightarrow{\cdot(1-r)} \mathbb{Z}C_n \xrightarrow{\cdot N} \mathbb{Z}C_n \xrightarrow{(1-r)} \mathbb{Z}C_n \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where the augmentation map sends the generator $r \mapsto 1$ (here we are identifying the generator of $\mathbb{Z}C_n$ with the generator of C_n). By the fundamental lemma of homological algebra 2.20, there is an augmentation-preserving C_n-chain map $\tau : P \to P$ (so we have commuting squares) satisfying

$$\tau(g \cdot x) = \rho(g) \cdot \tau(x),$$

unique up to chain-homotopy. Thus any such map works for computing the induced map on homology. We need to find such a τ which works for

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"] \arrow[d] & \mathbb{Z}C_n \arrow[r, "\cdot N"] \arrow[d] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"] \arrow[d] & \mathbb{Z}C_n \arrow[r, "\varepsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"'] & \mathbb{Z}C_n \arrow[r, "\cdot N"'] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"'] & \mathbb{Z}C_n \arrow[r, "\varepsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

It turns out that the following τ works:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"] \arrow[d, "r^{-2}\cdot\mathrm{inv}"'] & \mathbb{Z}C_n \arrow[r, "\cdot N"] \arrow[d, "-r^{-1}\cdot\mathrm{inv}"'] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"] \arrow[d, "-r^{-1}\cdot\mathrm{inv}"'] & \mathbb{Z}C_n \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{inv}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"'] & \mathbb{Z}C_n \arrow[r, "\cdot N"'] & \mathbb{Z}C_n \arrow[r, "\cdot(1-r)"'] & \mathbb{Z}C_n \arrow[r, "\varepsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{inv} : C_n \to C_n$ sends $r^k \mapsto r^{-k}$. Let's check that this works: For the first square on the right, commutativity follows from the fact that C_n acts trivially on $\mathbb{Z}$ (and the augmentation maps only see the $\mathbb{Z}$-coefficients of elements in $\mathbb{Z}C_n$). We also have

$$\tau(r \cdot x) = \mathrm{inv}(rx) = (rx)^{-1} = x^{-1}r^{-1} = \rho(r) \cdot \tau(x).$$

Let's check commutativity of the next square, for good measure. We have $\mathrm{inv}((1-r)\cdot r^k) = \mathrm{inv}(r^k - r^{k+1}) = r^{-k} - r^{-k-1} = (1-r)\cdot(-r^{-1}\cdot \mathrm{inv}(r^k)) = (1-r)\cdot(-r^{-k-1}) = r^{-k} - r^{-k-1}$, as desired (for $x = r^k$). We also have

$$\tau(r^k \cdot x) = -r^{-1} \cdot \mathrm{inv}(r^k x) = -r^{-1} \cdot x^{-1} r^{-k} = \rho(r^k) \cdot -r^{-1} \cdot \mathrm{inv}(x) = \rho(r^k) \cdot \tau(x).$$

Taking coinvariants, we see that τ has the effect of sending an element to plus or minus itself, depending on the degree. The induced map on homology sends $[x \otimes a] \mapsto [(-1)^i \cdot r^i \cdot x \otimes a] = [(-1)^i x \otimes a] = (-1)^i [x \otimes a]$. We conclude that C_2 acts on $H_{2i-1}(C_n;\mathbb{Z})$ via multiplication by $(-1)^i$. □

We are now set to finish computing the E^2 page. By the claim, we now have that

$$\begin{aligned}
H_{2i-1}(C_n;\mathbb{Z})_{C_2} &= H_{2i-1}(C_n;\mathbb{Z})/\langle [x] - (-1)^i [x] \rangle \\
&= (\mathbb{Z}/n)/\langle [x] - (-1)^i [x] \rangle = \begin{cases} \mathbb{Z}/n & i \text{ even} \\ 0 & i \text{ odd} \end{cases}
\end{aligned}$$

which means that $E^2_{0,q} = \mathbb{Z}/n$ when q is 3 mod 4, $E^2_{0,0} = \mathbb{Z}$, and $E^2_{0,q} = 0$ otherwise. The E^2 page as we've computed it so far looks like

$$
\begin{array}{c|ccccc}
q & & & & & \\
4 & 0 & 0 & 0 & 0 & 0 \\
3 & \mathbb{Z}/n & ? & ? & ? & ? \\
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & ? & ? & ? & ? \\
0 & \mathbb{Z} & \mathbb{Z}/2 & 0 & \mathbb{Z}/2 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 \quad p
\end{array}
$$

It turns out that the question marks are all zeros as well: To compute $H_p(C_2; H_q(C_n;\mathbb{Z})) = \mathrm{Tor}_p^{\mathbb{Z}C_2}(\mathbb{Z}, H_q(C_n;\mathbb{Z}))$ when q is odd, we may first resolve $\mathbb{Z}$ by $\mathbb{Z}C_2$ modules in the usual way

$$\cdots \longrightarrow \mathbb{Z}C_2 \xrightarrow{(1-s)} \mathbb{Z}C_2 \xrightarrow{(1+s)} \mathbb{Z}C_2 \xrightarrow{(1-s)} \mathbb{Z}C_2 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

and then tensor with $H_q(C_n;\mathbb{Z})$ over $\mathbb{Z}C_2$, truncating at the end to obtain

$$\cdots \longrightarrow \mathbb{Z}C_2 \otimes_{\mathbb{Z}C_2} H_q(C_n) \xrightarrow{(1-s)\otimes \mathrm{id}} \mathbb{Z}C_2 \otimes_{\mathbb{Z}C_2} H_q(C_n) \xrightarrow{(1+s)\otimes \mathrm{id}} \mathbb{Z}C_2 \otimes_{\mathbb{Z}C_2} H_q(C_n) \xrightarrow{(1-s)\otimes \mathrm{id}} \mathbb{Z}C_2 \otimes_{\mathbb{Z}C_2} H_q(C_n) \longrightarrow 0$$

Letting s be the generator of C_2 (and identifying it with the generator of $\mathbb{Z}C_2$), we know that s acts by $(-1)^i$ when $q = 2i-1$, so the maps above become

$$\cdots \longrightarrow H_{2i-1}(C_n) \xrightarrow{1-(-1)^i} H_{2i-1}(C_n) \xrightarrow{1+(-1)^i} H_{2i-1}(C_n) \xrightarrow{1-(-1)^i} H_{2i-1}(C_n) \longrightarrow 0.$$

This yields the chain complex

$$\cdots \to \mathbb{Z}/n \xrightarrow{\cdot 2} \mathbb{Z}/n \xrightarrow{0} \mathbb{Z}/n \xrightarrow{\cdot 2} \mathbb{Z}/n \to 0.$$

In both even and odd degrees we get trivial homology: if p is even, the homology of this complex is $(\mathbb{Z}/n)/2(\mathbb{Z}/n) \cong 0$, and since the kernel of the multiplication by 2 map is trivial, we get trivial homology when p is odd as well. The E^2 page thus has nontrivial homology only on the axes:

$$
\begin{array}{c|ccccc}
q & & & & & \\
4 & & & & & \\
3 & \mathbb{Z}/n & & & & \\
2 & & & & & \\
1 & & & & & \\
0 & \mathbb{Z} & \mathbb{Z}/2 & & \mathbb{Z}/2 & \\
\hline
 & 0 & 1 & 2 & 3 & 4 \quad p
\end{array}
$$

and it is clear that the spectral sequence collapses here. We can immediately conclude that $H_i(D_n;\mathbb{Z}) = \mathbb{Z}$ when $i = 0$, $H_i(D_n;\mathbb{Z}) = 0$ when $i > 0$ is even, and $H_i(D_n;\mathbb{Z}) = \mathbb{Z}/2$ when i is $1 \mod 4$. When i is $3 \mod 4$, the exact sequence

$$1 \to \mathbb{Z}/n \to H_i(D_n;\mathbb{Z}) \to \mathbb{Z}/2 \to 1$$

gives that $H_i(D_n;\mathbb{Z}) \cong \mathbb{Z}/n \times \mathbb{Z}/2 \cong \mathbb{Z}/2n$.

Example 2.73. We can now quickly compute the cohomology of D^n as well, using the universal coefficient theorem: Since

$$H^i(D_n;\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_i(D_n;\mathbb{Z}),\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(D_n;\mathbb{Z}),\mathbb{Z})$$

we have

$$H^i(D_n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = \text{odd} \\ \mathbb{Z}/2 & i = 2 \mod 4 \\ \mathbb{Z}/2n & i = 0 \mod 4. \end{cases}$$

Remark 2.74. Weibel page 196: The edge maps $H^*(G/H;A^H) \to H^*(G;A)$ and $H^*(G;A) \to (H^*(H;A))^{G/H}$ in the second spectral sequence are induced from the inflation and restriction map.

2.4.6 Cup products in cohomological spectral sequence

Classes in cohomological spectral sequences can be multiplied to obtain more classes via cup product structures on the underlying spaces. For instance, consider the cohomological Serre spectral sequence 1.5.4, which has E_2 page $E_2^{p,q} = H^p(B;H^q(F;R))$. How can I multiply an element $x \in E_2^{p,q}$ with an element $y \in E_2^{s,t}$ (to get an element $xy \in E_2^{p+s,q+t}$)? Suppose $\phi \in \mathrm{Hom}(C_p(B),H^q(F;R))$ is a cocycle and $\psi \in \mathrm{Hom}(C_s(B),H^t(F;R))$ is a cocycle. Given $\sigma : \Delta^{p+s} \to B$ a $(p+s)$-singular simplex, we can define

$$(\phi \cup \psi)(\sigma) := \phi(\sigma|_{[v_0,\ldots,v_p]}) \cup \phi(\sigma|_{[v_p,\ldots,v_{p+s}]})$$

where the latter cup product goes from $H^q(F;R) \times H^t(F;R) \to H^{q+t}(F;R)$.

2.5 Application: $\mathbb{Q}$-hom of orbit spaces with finite stabilizers

We saw previously: if X is a contractible simplicial complex with a *free* simplicial G-action, then $H_*(G) \cong H_*(X/G)$ (taking the simplicial chain complex of X provides a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, since $X \simeq *$ implies that homology vanishes. Taking G-coinvariants gives a chain complex isomorphic to $C_*(X/G)$, so $H_*(C(X)_G) = H_*(X/G)$). In fact, have $H_*(G;A) \cong H_*(X/G;A)$ for any abelian group A. Turns out that if we want to compute $H_*(G;\mathbb{Q})$, it's enough to assume G acts simplicially on a contractible complex X with *finite stabilizers.*

Lemma 2.75. *Let G be a group, X a simplicial complex on which G acts simplicially, and let $Y \subseteq X$ be a subcomplex preserved by the G-action (i.e., $g \cdot Y \subseteq Y$ for all $g \in G$). For some $n \geqslant 0$, assume that $G_\sigma = \{g \in G : g\sigma = \sigma\}$ is finite for every n-simplex of $X \backslash Y$. Then the $\mathbb{Q}G$-module $C_n(X, Y; \mathbb{Q})$ of relative simplicial n-chains is flat.*

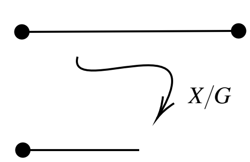
Proof. For a simplex σ, define M_σ to be the $\mathbb{Q}G$-module generated by σ. Let I index the G-orbits of n-simplices σ_i of X not contained in Y (note that Y being stable under G action implies that $X^{(n)} \backslash Y^{(n)}$ is also stable under the G-action). As $\mathbb{Q}G$-modules, we have $C_n(X, Y; \mathbb{Q}) \cong \bigoplus_{i \in I} M_{\sigma_i}$. Note that M_σ decomposes (as an abelian group) as a direct sum $\bigoplus_{g \in G} \mathbb{Q}\langle g \cdot \sigma \rangle$. Since G acts transitively on the summands, and the isotropy group of $\langle \sigma \rangle$ is G_σ, we have by Corollary 2.46 that $M_\sigma \cong \mathrm{Ind}_{G_\sigma}^G \mathbb{Q}\langle \sigma \rangle$ as $\mathbb{Q}G$-modules. Notice that G_σ acts on $\mathbb{Q}\langle \sigma \rangle$ by either ± 1, depending on whether elements in G_σ preserve or reverse the orientation of σ. We let $\mathbb{Q}_\sigma$ denote this one-dimensional vector space with associated G_σ action, so that $M_\sigma \cong \mathrm{Ind}_{G_\sigma}^G \mathbb{Q}_\sigma$.

Now $\mathbb{Q}_\sigma$ is a 1-dimensional G_σ-module, hence an irreducible representation of G_σ (more generally, though, we just need the fact that it *is* a representation of G_σ, so it is a direct summand of finitely many copies of the regular representation $\mathbb{Q}[G_\sigma]$. Implicitly we're using the fact that G_σ is finite!). Thus $\mathrm{Ind}_{G_\sigma}^G \mathbb{Q}_\sigma$ is a direct summand of $\mathrm{Ind}_{G_\sigma}^G \mathbb{Q}[G_\sigma] = \mathbb{Z}G \otimes_{\mathbb{Z}G_\sigma} \mathbb{Q}G_\sigma \cong \mathbb{Q}G$ (since $M \otimes_R S^{-1}R \cong S^{-1}M$), and so $M_\sigma = \mathrm{Ind}_{G_\sigma}^G \mathbb{Q}G_\sigma$ is a flat (projective, actually) $\mathbb{Q}G$-module. As a direct sum of flat modules is flat, we conclude that $C_n(X, Y; \mathbb{Q})$ is a flat $\mathbb{Q}G$-module. (Note: since $\mathbb{Q}G$ is flat over $\mathbb{Z}G$ (localizations are flat), then $C_n(X, Y; \mathbb{Q})$ is also flat as a $\mathbb{Z}G$-module.) ◻

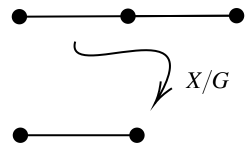
Lemma 2.76 (Pointwise stabilizers)**.** *Let G be a group and X a simplicial complex on which G acts simplicially. After possibly barycentrically subdividing, we may assume that the setwise stabilizer G_σ fixes σ pointwise.*

Proof. Let $g \in G_\sigma$ be a setwise stabilizer of σ. By lemma 3.62, since X is a simplicial G-complex, then $\mathrm{sd}(X)$ satisfies condition (S): g fixes $\sigma \cap g \cdot \sigma$ pointwise. In particular, if $g \in G_\sigma$, then g fixes $\sigma \cap \sigma = \sigma$ pointwise, as desired. ◻

Remark 2.77. Note that this assumption prevents the orbit space from having weird properties (and not being a CW complex). For instance, if σ is a 1-simplex and $g \in G$ flips σ (swapping the two vertices), then the midpoint of σ is fixed by the action, and the quotient looks like



But the edge is not really an edge– its boundary is not attached to 0-cells (only one part of its boundary is attached). Notice however that if we begin by subdividing the original simplex, then the original $g \in G_\sigma$ is no longer a setwise stabilizer, but instead identifies these two simplices to yield



which is now a valid CW complex (and is in this case simplicial. Normally we need to subdivide again to ensure we get a simplicial action).

Lemma 2.78. *If every setwise stabilizer of a simplex is a pointwise stabilizer, the orbit space X/G inherits a CW structure.*

Proof. We claim that n-simplices in X/G correspond to G-orbits of n-simplices in X. Given a characteristic map $\phi_\alpha : \Delta^n_\alpha \to X$, we can compose it with the quotient $q : X \to X/G$ to get a characteristic map $\overline{\phi_\alpha} : \Delta^n_\alpha \to X/G$. Note that if an element g fixes a point in an open simplex, then it fixes the entire closed simplex. Let Ge represent the orbit $\{ge : g \in G\}$ of an open cell e, and note that because X/G is Hausdorff (??), the closure of Ge is the orbit of the closed cell $\overline{e}$ idk!!! ugh!!!($\overline{Ge} = G\overline{e}$).
If $g\sigma = \tau$ for two n-simplices in X, we claim that their characteristic maps in X/G are the same:

How to check that the result is a CW complex? □

Theorem 2.79 ($\mathbb{Q}$ cohomology of G-complexes with finite stabilizers)**.** *Let G be a group and let X be a contractible simplicial complex on which G acts simplicially. Assume that G_σ is finite for every simplex σ of X. Then $H_*(G;\mathbb{Q}) \cong H_*(X/G;\mathbb{Q})$.*

Proof. Let $Y = \varnothing$. By lemma 2.75, for each $n \geqslant 0$, $C_n(X;\mathbb{Q})$ is a flat $\mathbb{Q}G$-module. Hence the augmented chain complex

$$\cdots \to C_n(X;\mathbb{Q}) \to C_{n-1}(X;\mathbb{Q}) \to \cdots \to C_0(X;\mathbb{Q}) \xrightarrow{\varepsilon} \mathbb{Q}$$

is a flat $\mathbb{Q}G$-module resolution of $\mathbb{Q}$. Now $H_*(G;\mathbb{Q}) = \operatorname{Tor}_*^{\mathbb{Q}G}(\mathbb{Q},\mathbb{Q})$, so tensoring the above resolution with $\mathbb{Q}$ over $\mathbb{Q}G$ yields a chain complex

$$\cdots \to C_n(X;\mathbb{Q}) \otimes_{\mathbb{Q}G} \mathbb{Q} \to C_{n-1}(X;\mathbb{Q}) \otimes_{\mathbb{Q}G} \mathbb{Q} \to \cdots \to$$

On the other hand, the n-simplices of X/G correspond to G-orbits of n-simplices of X, so $C_n(X/G;\mathbb{Q}) = \bigoplus_{\sigma \in X^{(n)}/G} \mathbb{Q}$ as abelian groups ($C_n(X/G;\mathbb{Q})$ has a $\mathbb{Q}$-basis of simplices in $X^{(n)}/G$). Since $\mathbb{Q}$ has a trivial $\mathbb{Q}G$-action, the tensor product $C_n(X;\mathbb{Q}) \otimes_{\mathbb{Q}G} \mathbb{Q}$ identifies the G-orbit of an n-cell. Hence

$$\begin{aligned} H_*(C_n(X;\mathbb{Q}) \otimes_{\mathbb{Q}G} \mathbb{Q}) &= H_*(C_n(X/G;\mathbb{Q})) \\ &\implies H_*(G;\mathbb{Q}) \cong H_*(X/G;\mathbb{Q}). \end{aligned}$$

□

Remark 2.80. The theorem implies that, with rational coefficients (or coefficients in a $\mathbb{Q}G$-module), it suffices to find contractible simplicial complexes on which G acts with finite stabilizers! We don't need G to act freely.

2.6 Finiteness conditions

2.6.1 Cohomological dimension & Serre's theorem

Definition 2.81 (Projective dimension)**.** Let R be a ring and M and R-module. The projective dimension of M, denoted $\operatorname{pd}_R(M)$, is the minimal n such that there exists a projective resolution of length n

$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$

Theorem 2.82 (Characterizations of $\operatorname{pd}_R M$)**.** *Let M be an R-module. The following are equivalent:*

(i) $\operatorname{pd}_R M \leqslant n$.

(ii) $\operatorname{Ext}^i_R(M, -) = 0$ *for* $i > n$.

(iii) $\operatorname{Ext}^{n+1}_R(M, -) = 0$.

(iv) If $0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ *is exact with each* P_i *projective, then* K *is projective.*

Proof. $(i) \implies (ii)$: Suppose there exists a projective resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0.$$

Applying $\operatorname{Hom}_R(P_\bullet, N)$ for any R-module N and truncating yields a chain complex

$$0 \to \operatorname{Hom}_R(P_0, N) \to \operatorname{Hom}_R(P_1, N) \to \cdots \to \operatorname{Hom}_R(P_n, N) \to 0.$$

Then $\operatorname{Ext}^i_R(M, N)$ is the homology of this chain complex. For $i > n$, it is clear that $\operatorname{Ext}^i_R(M, N) = 0$.

$(ii) \implies (iii)$: Obvious.
$(iii) \implies (iv)$: Let

$$P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be a partial projective resolution of M over R. Let $K = \ker(P_{n-1} \to P_{n-2})$. Complete this to a projective resolution

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+2} \arrow[r, "d_{n+2}"] & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[dr, two heads] & P_n \arrow[r, "d_n"] \arrow[dr, two heads] & P_{n-1} \arrow[r, "d_{n-1}"] & P_{n-2} \arrow[r] & \cdots \\ & & & L \arrow[u, hook] & K \arrow[u, hook] & & \end{tikzcd}$$

where $\operatorname{coker} d_{n+2} = L$, and $P_n/L \cong K$. Since we have a short exact sequence of R-modules

$$0 \to L \to P_n \to K \to 0$$

with P_n projective, to show that K is projective it suffices to show that this sequence splits. For any R-module N, we apply $\operatorname{Hom}_R(-, N)$ to the resolution to obtain the cochain complex

$$\cdots \longrightarrow \operatorname{Hom}_R(P_{n-1}, N) \xrightarrow{d_n^*} \operatorname{Hom}_R(P_n, N) \xrightarrow{d_{n+1}^*} \operatorname{Hom}_R(P_{n+1}, N) \xrightarrow{d_{n+2}^*} \operatorname{Hom}_R(P_{n+2}, N) \longrightarrow \cdots$$

Let φ be an $(n+1)$-cocycle, so $\varphi \in \ker d^*_{n+2}$. Then $\varphi \circ d_{n+2} = 0$, so φ vanishes on the image of d_{n+2}. So $\varphi : P_{n+1} \to N$ factors through $P_{n+1}/\operatorname{Im}(d_{n+2}) = \operatorname{coker} d_{n+2} = L$, and hence we can view it as a map $L \to N$. Since $\operatorname{Ext}^{n+1}_R(M, -) = 0$, then φ must be a coboundary, which means $\varphi \in \operatorname{Im} d^*_{n+1}$. We conclude that every R-module homomorphism $L \to N$ extends to an R-module homomorphism $P_n \to N$, for every R-module N. In particular this holds for the identity map $L \to L$, which shows that $0 \to L \to P_n \to K \to 0$ splits, and hence K is projective.
$(iv) \to (i)$: Clear. $\square$

Definition 2.83 (Cohomological dimension)**.** Let G be a group. The cohomological dimension of G, denoted $\operatorname{cd}(G)$, is defined to be $\operatorname{pd}_{\mathbb{Z}G} \mathbb{Z}$.

Prop 2.84 (Characterization of $\mathrm{cd}(G)$).

$$\begin{aligned}\mathrm{cd}(G) = \mathrm{pd}_{\mathbb{Z}G}\,\mathbb{Z} &= \inf\{n : \mathbb{Z} \textit{ admits a projective resolution of length } n \textit{ over } \mathbb{Z}G\}\\ &= \inf\{n : H^i(G;-) = 0, i > n\}\\ &= \sup\{n : H^n(G;M) \neq 0 \textit{ for some } \mathbb{Z}G\textit{-module } M\}\end{aligned}$$

Proof. $(i) = (ii)$: Suppose $\mathbb{Z}$ admits a projective resolution of length n over $\mathbb{Z}G$:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

Then $H^i(G;-) := \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},-)$, and we know by the characterization of projective dimension that $\mathrm{pd}_{\mathbb{Z}G}\,\mathbb{Z} \leqslant n$ implies that $\mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},-) = 0$ for $i > n$. We also know that if $\mathrm{Ext}^{n+1}_{\mathbb{Z}G}(\mathbb{Z},-) = 0$, then $\mathrm{pd}_{\mathbb{Z}G}\,\mathbb{Z} \leqslant n$.

$(ii) = (iii)$: Let k be the infimum of n for which $H^i(G;-) = 0$ for $i > n$. That is, we have $H^k(G;-) \neq 0$, but $H^i(G;-) = 0$ for $i > k$. Let M be a $\mathbb{Z}G$-module for which $H^k(G;M) \neq 0$. It is clear that $k = \sup\{n : H^n(G;M) \neq 0$ for some $\mathbb{Z}G$-module $M\}$. ◻

Example 2.85 (Trivial group)**.** $\mathrm{cd}(G) = 0 \iff G$ is trivial.

Proof. Suppose G is trivial. The $X = *$ is a classifying space for G, so $0 \to C_0(X) \to \mathbb{Z} \to 0$ is a length 0 projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, showing that $\mathrm{cd}(G) = 0$. On the other hand, if $0 \to P_0 \to \mathbb{Z} \to 0$ is a length 0 projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then $P_0 \cong \mathbb{Z}$, which means $\mathbb{Z}$ is a projective $\mathbb{Z}G$-module, and so the exact sequence of $\mathbb{Z}G$-modules

$$0 \to \ker\varepsilon \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

splits. Explicitly there exists some $\mathbb{Z}G$-module morphism $j : \mathbb{Z} \to \mathbb{Z}G$ such that $\varepsilon \circ j = \mathrm{id}_{\mathbb{Z}}$. Suppose $j(1) = \alpha(e)$. Since $\mathbb{Z}$ is a trivial $\mathbb{Z}G$-module, this means $\alpha(e) = j(1) = j(g\cdot 1) = g\cdot j(1) = g\cdot\alpha(e) = \alpha(g)$ for all $g \in G$. We conclude that $G = \{e\}$. ◻

Example 2.86 (Free groups)**.** If G is a nontrivial free group, then $\mathrm{cd}(G) = 1$.

Proof. Write $G = F(S)$ for some generating set S. Recall that $Y = \bigvee_{s\in S} S^1_s$ is a 1-dimensional classifying space for G (its universal cover is a tree, hence contractible). Letting $\tilde{Y}$ be the universal cover of Y, we have that $0 \to C_1(\tilde{Y}) \to C_0(\tilde{Y}) \to \mathbb{Z}$ gives a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. This shows that $\mathrm{cd}(G) \leqslant 1$. Since G is nontrivial, we know $\mathrm{cd}(G) > 0$, and hence $\mathrm{cd}(G) = 1$. ◻

Remark 2.87. It is a theorem of Stallings and Swan that every group of cohomological dimension 1 is free.

Example 2.88 (Finite cyclic groups)**.** Finite cyclic groups have infinite cohomological dimension. Recall that $\mathrm{cd}(G) = \inf\{n : H^i(G;-) = 0 \text{ for } i > n\}$. If $G \cong \mathbb{Z}/n$, then $H^{2k}(\mathbb{Z}/n;\mathbb{Z}) = \mathbb{Z}/n$ for all $k \geqslant 1$, which shows that $\mathrm{cd}(G) = \infty$.

Corollary 2.89. *If G has a torsion element,* $\mathrm{cd}(G) = \infty$.

Proof. If G has a torsion element then G contains a finite cyclic subgroup H. This means $\mathrm{cd}(H) = \infty$, and since $\mathrm{cd}(G) \geqslant \mathrm{cd}(H)$, we obtain that $\mathrm{cd}(G) = \infty$. ◻

Prop 2.90 (Suffices to check vanishing with free module coefficients)**.** *Suppose G has finite cohomological dimension. Then* $\mathrm{cd}(G) = \sup\{n : H^n(G;F) \neq 0$ *for some free* $\mathbb{Z}G$*-module* $F\}$.

Proof. Let $n = \mathrm{cd}(G)$. Then $H^n(G; M) \neq 0$ for some $\mathbb{Z}G$-module M (while $H^i(G; 0) = 0$ for $i > n$). Since every module is a quotient of a free module, let F be a free $\mathbb{Z}G$-module of which M is a quotient. We have a short exact sequence of $\mathbb{Z}G$-modules

$$0 \to K \to F \to M \to 0.$$

It suffices to show that $H^n(G; -)$ is right exact, for then $H^n(G; F)$ surjects onto $H^n(G; M) = 0$, and we are done. Recall that we have a (covariant) long exact sequence in cohomology

$$\cdots \to H^n(G; K) \to H^n(G; F) \to H^n(G; M) \xrightarrow{\delta} H^{n+1}(G; K) \to \cdots$$

and since $\mathrm{cd}(G) = n$, we know $H^{n+1}(G; K) = 0$. This proves our claim. □

Prop 2.91 (Finite cohomological dimension and finite-index subgroups). *Let $H \leqslant G$ be a finite-index subgroup, and suppose* $\mathrm{cd}(G) < \infty$. *Then* $\mathrm{cd}(H) = \mathrm{cd}(G)$.

Proof. We know $\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$, so it suffices to show that $\mathrm{cd}(G) \leqslant \mathrm{cd}(H)$. Let $\mathrm{cd}(G) = n$. By prop 2.90, there exists some free $\mathbb{Z}G$-module F such that $H^n(G; F) \neq 0$. As $F \cong \bigoplus \mathbb{Z}G$ and $\mathbb{Z}G = \mathrm{Ind}_H^G \mathbb{Z}H$, then by Shapiro's lemma 2.57 (and the fact that $H^n(G; F) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, F) = \prod \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ind}_H^G \mathbb{Z}H) = \prod H^n(G; \mathrm{Ind}_H^G \mathbb{Z}H)$), we have

$$H^n(H; \bigoplus \mathbb{Z}H) = \prod H^n(H; \mathbb{Z}H) \cong \prod H^n(G; \mathrm{Coind}_H^G \mathbb{Z}H) \cong \prod H^n(G; \mathrm{Ind}_H^G \mathbb{Z}H) \neq 0$$

where the isomorphism $\mathrm{Coind}_H^G \mathbb{Z}H \cong \mathrm{Ind}_H^G \mathbb{Z}H$ follows from the fact that $[G : H] < \infty$ (see prop 2.56). □

Prop 2.92 (Cohomological dimensions of SES). *If* $1 \to K \to G \to Q \to 1$ *is a short exact sequence of groups, then* $\mathrm{cd}(G) \leqslant \mathrm{cd}(K) + \mathrm{cd}(Q)$.

Proof. We use the cohomological Lyndon-Hochschild-Serre spectral sequence, which says that for a group extension

$$1 \to K \to G \to Q \to 1$$

there is an associated first quadrant spectral sequence with $E_2^{p,q} = H^p(Q; H^q(K; A)) \implies H^{p+q}(G; A)$ for A a G-module. If $\mathrm{cd}(K) = \infty$ or $\mathrm{cd}(Q) = \infty$, the claim is obvious. Thus suppose that $\mathrm{cd}(K) = m$ and $\mathrm{cd}(Q) = n$. The E_2 page has trivial groups for $q \geqslant m + 1$ and $p \geqslant n + 1$. Recall that we wish to show that $\mathrm{cd}(G) \leqslant m + n$. For this it suffices to show that $H^i(G; -) = 0$ for all $i \geqslant m + n + 1$. Note that if $p + q \geqslant m + n + 1$, then either $p \geqslant m + 1$ or $q \geqslant n + 1$ (else $p \leqslant m$ and $q \leqslant n$ implies $p + q \leqslant m + n$, a contradiction). Thus every group along the $p + q \geqslant m + n + 1$ diagonal is 0 on the E_2 page, and hence is 0 on the E_∞ page. Since the E_∞ page determines $H^{p+q}(G; A)$ (up to extension), we conclude that $H^i(G; A) = 0$ for all G-modules A and $i \geqslant m + n + 1$, as desired. □

Prop 2.93 (Cohomological dimension of amalgamated product). *Suppose $G = G_1 *_A G_2$, where $A \hookrightarrow G_1$ and $A \hookrightarrow G_2$. Then* $\mathrm{cd}(G) \leqslant \max\{\mathrm{cd}(G_1), \mathrm{cd}(G_2), \mathrm{cd}(A) + 1\}$.

Proof. The claim is clear if any of $\mathrm{cd}(G_1)$, $\mathrm{cd}(G_2)$, or $\mathrm{cd}(A) = \infty$. Hence suppose $n = \max\{\mathrm{cd}(G_1), \mathrm{cd}(G_2), \mathrm{cd}(A) + 1\}$ is finite. Then $H^i(G_1; -) = H^i(G_2; -) = H^i(A; -)$ for $i > n$. From the cohomological Mayer-Vietoris sequence 2.43

$$\cdots \to H^i(A; -) \to H^i(G_1; -) \oplus H^i(G_2; -) \to H^i(G; -) \to H^{i+1}(A; -) \to \cdots$$

we see that if $i > n$, the sequence reduces to

$$\cdots \to 0 \to 0 \to H^i(G; -) \to 0 \to \cdots$$

so that $H^i(G; -) = 0$. This proves that $\mathrm{cd}(G) \leqslant n$. □

Prop 2.94. *[Results for $K(G,1)$-manifolds] Suppose Y is a d-dimensional $K(G,1)$-manifold (possibly with boundary).*

(a) $\mathrm{cd}(\Gamma) \leqslant d$*, with equality if and only if Y is closed (compact without boundary).*

(b) If Y is compact then Γ *is of type* FL.

Sketch. (a) Suffices to show that Y has the homotopy type of a CW complex of dimension $\leqslant d$ (as then the chain complex of the universal cover of Y, which is also $\leqslant d$-dimensional, provides a finite-length free resolution of $\mathbb{Z}$ over $\mathbb{Z}$).

(b) If Y is compact, then one can show that Y has the homotopy type of a finite CW complex, so the cellular chain complex of its universal cover provides a finite length, finite type free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (as a $\mathbb{Z}G$-module, each $C_n(\tilde{Y})$ has a basis corresponding to the n-cells of Y, of which there are finitely many by assumption).

□

Corollary 2.95 (Ex VIII.8 in [2]). *Let* Γ *be a group such that there exists a closed* $K(\Gamma,1)$*-manifold. If* $\Gamma' \subset \Gamma$ *is a subgroup of infinite index, deduce that* $\mathrm{cd}\,\Gamma' < \mathrm{cd}\,\Gamma$.

Proof. Since $\Gamma' < \Gamma$ is of infinite index, we have an infinite-sheeted covering space $\mathrm{B}\,\Gamma' \to \mathrm{B}\,\Gamma$, with $\mathrm{B}\,\Gamma'$ being a $K(\Gamma',1)$-manifold. In particular, $\mathrm{B}\,\Gamma'$ is not compact, so not closed, and we have by 2.94 that $\mathrm{cd}(\Gamma') < \dim(\mathrm{B}\,\Gamma') = \dim(\mathrm{B}\,\Gamma) = \mathrm{cd}(\gamma)$. □

Theorem 2.96 (Finite cohomological dimension implies finite-dimensional classifying space). *Let G be an arbitrary group. Let $n = \max\{\mathrm{cd}(G), 3\}$. There exists an n-dimensional $K(G,1)$ complex Y. In particular, if $\mathrm{cd}(G) < \infty$, there exists a finite dimensional $K(G,1)$.*

Proof. We construct the $K(G,1)$ complex Y inductively by construct its skeleta. Let Y^2 be the presentation 2-complex associated to some presentation of G (start with a single vertex, attach 1-cells for each generator of G, and attach 2-cells to the generators according to each relation). Then $\pi_1(Y^2) = G$. Note that the universal cover X^2 of Y^2 has $H_i = 0$ for $0 < i < 2$: the universal cover of a presentation complex X_G is the Cayley complex $\tilde{X}_G$, which has vertices the elements of G and an edge between g and gg_α for each of the chosen generators g_α. This is the *Cayley graph*, and it's connected (there's a path in the graph connecting every vertex to the identity e). We now attach a 2-cell for every relation r_β among the generators, which is given by various loops in the graph. The group G acts on $\tilde{X}_G$ by left multiplication, sending a vertex $\tilde{g}$ to $g\tilde{g}$ and an edge $g' - g'g_\alpha$ to the edge $gg' - gg'g_\alpha$. This is a *free* covering space action, and the orbit space is just the presentation complex X_G, since the vertices form a single orbit, edges collapse to loops labeled by the generators, and the various loops in the graph representing the same relation r_β get collapsed to a single loop (and hence a single 2-cell for each relation). In other words, $\tilde{X}_G/G \cong X_G$. Note that $\tilde{X}_G$ is simply connected, since every loop in the Cayley complex (based at e) corresponds to a relation in G, and we have attached 2-cells whose boundaries are precisely the relations in G. Thus $H_1(X^2) = \mathrm{Ab}(\pi_1(X^2, e)) = \mathrm{Ab}(\pi_1(\tilde{X}_G, e)) = 0$.

Note that if X^2 were *contractible*, we would be done at this point (as then Y^2 would be a $K(G,1)$.) But X^2 is not necessarily contractible, so we must continue to attach higher-dimensional cells (which don't affect the fundamental group, since the fundamental group is determined by the 2-skeleton). Suppose Y^{k-1} has been constructed inductively to be a complex whose universal cover X^{k-1} is a G-complex with $H_i = 0$ for $0 < i < k-1$. We construct Y^k according to the following recipe:

1. Since X^{k-1} is a G-complex, $H_{k-1}(X^{k-1})$ is naturally a G-module. Let $\{z_\alpha\}$ be a set of generators for $H_{k-1}(X^{k-1})$. By the Hurewicz theorem 1.21, we know that $h : \pi_{k-1}(X^{k-1}) \to H_{k-1}(X^{k-1})$ is an isomorphism, so for each z_α there exists a map $f_\alpha : S^{k-1} \to X^{k-1}$ such that $(f_\alpha)_* : H_{k-1}(S^{k-1}) \to H_{k-1}(X^{k-1})$ sends $1 \mapsto z_\alpha$.

2. Set $Y^k = Y^{k-1} \cup \bigcup_\alpha e^k_\alpha$, where each k-cell is attached to Y^{k-1} via the composite map $S^{k-1} \xrightarrow{f_\alpha} X^{k-1} \xrightarrow{p} Y^{k-1}$ where $p : X^{k-1} \to Y^{k-1}$ is the universal cover of Y^{k-1}.

3. Let X^k be the universal cover of Y^k. We wish to show that $H_i(X^k) = 0$ for $0 < i < k$. First we claim that X^{k-1} is the $(k-1)$-skeleton of X^k, so that $H_i(X^k) = H_i((X^k)^{(k-1)}) = H_i(X^{k-1}) = 0$ for $0 < i < k-1$ (at which point it will suffice to show that $H_{k-1}(X^k) = 0$). Since Y^k is obtained from Y^{k-1} by attaching cells according to the attaching maps f_α, then X^k is obtained from X^{k-1} by attaching k-cells via the maps f_α and their transforms under the action of G (i.e., lift $|G|$ many copies of each k-cell living downstairs).

4. We need to show that $H_{k-1}(X^k) = 0$. From the long exact sequence of the pair (X^k, X^{k-1}), we have an exact sequence

$$H_k(X^k, X^{k-1}) \xrightarrow{\partial} H_{k-1}(X^{k-1}) \to H_{k-1}(X^k) \to H_{k-1}(X^k, X^{k-1}) = 0$$

where $H_{k-1}(X^k, X^{k-1}) = 0$ since (X^k, X^{k-1}) is a good pair, and $X^k/X^{k-1} \simeq \bigvee S^k$. If we can show that ∂ is surjective, then $H_{k-1}(X^k) \cong H_{k-1}(X^{k-1})/\operatorname{Im}\partial = 0$. Recall that if $p : X^k \to Y^k$ is a universal cover, then for each cell $\sigma \in Y^k$ we have that $p^{-1}(\sigma)$ is an orbit of cells permuted freely and transitively by $G = \operatorname{Deck}(X^k) = \pi_1(Y^k)$. As a $\mathbb{Z}G$-module, we have that $H_k(X^k, X^{k-1}) = C^{CW}_k(X^k)$ has a basis in bijection with the k-cells of Y^k (as a $\mathbb{Z}$-module, we have $|G|$ times as many cells, and hence as many basis elements). The basis (v_α) is obtained as follows: for a characteristic map $\chi_\alpha : (D^k, S^{k-1}) \to (X^k, X^{k-1})$ for a cell attached via f_α, let $v_\alpha \in H_k(X^k, X^{k-1}) \cong \prod_\alpha \mathbb{Z}$ be the image under $H_k(\chi_\alpha)$ of a generator of $H_k(D^k, S^{k-1})$. We have a commutative diagram

$$\begin{tikzcd}
H_k(D^k, S^{k-1}) \arrow[r, "\partial"] \arrow[d, "H_k(\chi_\alpha)"'] & H_{k-1}(S^{k-1}) \arrow[d, "H_{k-1}(f_\alpha)"] \\
H_k(X^k, X^{k-1}) \arrow[r, "\partial"'] & H_{k-1}(X^{k-1})
\end{tikzcd}$$

by naturality of the long exact sequence of a pair. Choosing the generator of $H_k(D^k, S^{k-1})$ and $H_{k-1}(S^{k-1})$ compatibly (so that ∂ of a preferred generator of $H_k(D^k, S^{k-1})$ gives the preferred generator of $H_{k-1}(S^{k-1})$), we have that $\partial(H_k(\chi_\alpha)(1)) = H_{k-1}(f_\alpha)(1)$, so $\partial(v_\alpha) = z_\alpha$. We conclude that ∂ is surjective.

5. If $\operatorname{cd}(G) = \infty$, continue this process indefinitely, and take $Y = \bigcup_k Y^k$ to be the desired $K(G,1)$. If $\operatorname{cd}(G) = n < \infty$, consider X^{n-1} (since $n \geqslant 3$, then $n-1 \geqslant 2$). We have a cellular chain complex of free $\mathbb{Z}G$-modules

$$C_{n-1}(X^{n-1}) \to C_{n-2}(X^{n-1}) \to \cdots \to C_0(X^{n-1}) \to \mathbb{Z} \to 0$$

which is a partial resolution of length $n-1$. Since $\operatorname{cd}(G) = n$, we know that $\ker(C_{n-1}(X^{n-1}) \to C_{n-2}(X^{n-1})) = H_{n-1}(X^{n-1})$ is a projective $\mathbb{Z}G$-module (note that $C_n(X^{n-1}) = 0$). If we can force $H_{n-1}(X^{n-1})$ to be a *free* $\mathbb{Z}G$-module, then the boundary map $\partial : H_{n-1}(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ is an isomorphism. Since we have a long exact sequence of the pair (X^n, X^{n-1})

$$H_n(X^n) \to H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}),$$

we could conclude that $H_n(X^n) = 0$ (and since $H_i(X^n) = 0$ for $i > n$, we know that Y^n is a $K(G,1)$).

6. Thus we replace $H_{n-1}(X^{n-1})$ with a free $\mathbb{Z}G$-module via the Eilenberg trick (2.22). Let F be a free $\mathbb{Z}G$-module such that $H_{n-1}(X^{n-1}) \oplus F \cong F$. Replace Y^{n-1} with $\overline{Y^{n-1}} = Y^{n-1} \vee \bigvee S^{n-1}$, where we wedge on a copy of S^{n-1} for every basis element of F. The universal cover $\overline{X^{n-1}}$ now has $H_{n-1}(\overline{X^{n-1}}) \cong H_{n-1}(X^{n-1}) \oplus F$, which is free.

7. Attach n-cells e^n_α to $\overline{Y^{n-1}}$ corresponding to basis elements z_α of $H_{n-1}(\overline{X^{n-1}})$. By the remark in step 5, we know that $\overline{X^n}$ (the universal cover of $\overline{Y^n}$) has vanishing homology above degree 0. Since it is also simply connected, Whitehead's theorem for homology implies that $\overline{X^n}$ is contractible. We conclude that $\overline{Y^n}$ is an n-dimensional $K(G,1)$.

□

Prop 2.97. *If G is finitely presented and of type* FP *(resp.* FL*), then the $K(G,1)$ in 2.96 can be taken to be finitely dominated (resp. finite).*

Proof. Let's trace where these assumptions come into the proof. If G is finitely presented, our presentation 2-complex Y^2 (and hence its universal cover X^2, the Cayley complex) is a finite complex.

1. Suppose Y^{k-1} is a finite complex. We need to show that Y^k can be made finite, which holds if $H_{k-1}(X^{k-1})$ is finitely generated as a $\mathbb{Z}G$-module. Since Y^{k-1} is finite, the cellular chain complex of X^{k-1} (its universal cover) is a partial free resolution of finite type

$$C_{k-1}(X^{k-1}) \to \cdots \to C_0(X^{k-1}) \to \mathbb{Z} \to 0$$

Then $H_{k-1}(X^{k-1}) = \ker(C_{k-1}(X^{k-1}) \to C_{k-2}(X^{k-1}))$ is finitely generated, since G is of type FP 2.115.

2. We also know that $H_{k-1}(X^{k-1})$ is a projective $\mathbb{Z}G$-module, so by the Eilenberg trick 2.22. By a later proposition (2.128), $H_{k-1}(X^{k-1})$ is stably free. So there exists a free module F of finite rank such that $H_{k-1}(X^{k-1}) \oplus F$ is free (of finite rank), and we can proceed as in theorem 2.96.

□

Remark 2.98. We can also assume the $K(G,1)$ in the theorems above is simplicial, by assuming inductively that Y^{k-1} is simplicial, then using the simplicial approximation theorem to get that the attaching map $f_\alpha : S^{k-1} \to Y^{k-1}$ is simplicial relative to some triangulation of S^{n-1}.

Definition 2.99 (Geometric dimension)**.** The minimal dimension of a $K(G,1)$ is the geometric dimension of G.

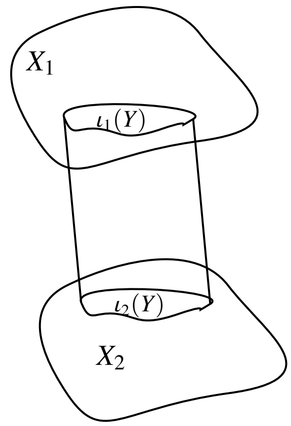
Remark 2.100. We know that $\mathrm{cd}(G) \leqslant \mathrm{gd}(G)$ in general, but the above theorem shows that when $\mathrm{cd}(G) \geqslant 3$ we actually have equality. We also have equality when $\mathrm{cd}(G) = 0$ (as then G is trivial, and $X = *$ is a 0-dimensional $K(G,1)$) or when $\mathrm{cd}(G) = 1$ (as then G is free, and a $K(G,1)$ is given by a graph). The only unknown case of equality is when $\mathrm{cd}(G) = 2$.

Prop 2.101 (Facts about geometric dimension ([2] VIII.2.4))**.** *(a)* $\mathrm{gd}(G_1 *_A G_2) \leqslant \max\{\mathrm{gd}\, G_1, \mathrm{gd}\, G_2, 1 + \mathrm{gd}\, A\}$ *(cf 2.93). (b) If* $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ *is exact, then* $\mathrm{gd}(\Gamma) \leqslant \mathrm{gd}(\Gamma') + \mathrm{gd}(\Gamma'')$.

Proof. (a) By 2.16, we know we can realize a $K(G,1)$ of $G = G_1 *_A G_2$ as a pushout of $K(\pi,1)$'s:

$$\begin{tikzcd} Y \arrow[r, "\iota_1"] \arrow[d, "\iota_2"'] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

where X is the adjunction space $(X_1 \sqcup X_2)/(\iota_1(y) \sim \iota_2(y))$ for all $y \in Y$. The maps ι_1 and ι_2 might not be inclusions. Hence we take the mapping cylinders of ι_1 and ι_2, which now become inclusions, and form the adjunction $X = (X_1 \sqcup X_2 \sqcup Y \times I)/((y,0) \sim \iota_1(y), (y,1) \sim \iota_2(y))$:



It is clear that $\dim(X) \leqslant \max\{\dim(X_1), \dim(X_2), \dim(Y) + 1\} = \max\{\mathrm{gd}(G_1), \mathrm{gd}(G_2), \mathrm{gd}(A) + 1\}$.

(b) My guess: take a balanced product? Let X be an n-dimensional $K(G', 1)$ and let Y be a $K(G'', 1)$. Should we take $Z = X \times_G Y$ or something? The higher homotopy groups will vanish, but is $\pi_1(X \times_G Y) \cong G$? What do we know about short exact sequences of groups? Splitting?

□

Theorem 2.102 (Serre's Theorem)**.** *If G is a torsion-free group and H is a finite-index subgroup,* $\mathrm{cd}(H) = \mathrm{cd}(G)$.

Proof. By prop 2.91, we know that if $\mathrm{cd}(G) < \infty$ and H is a finite-index subgroup of G, then $\mathrm{cd}(H) = \mathrm{cd}(G)$. It remains to show this when $\mathrm{cd}(G) = \infty$. Equivalently, we may show that if $\mathrm{cd}(H) < \infty$, then $\mathrm{cd}(G) < \infty$.

Proof 1: Topological (Serre 1971). Since $\mathrm{cd}(H) < \infty$ by assumption, Theorem 2.96 guarantees a finite-dimensional $K(H, 1)$ which we'll call X. The universal cover $\tilde{X}$ of X is contractible and a free H-complex. We construct from $\tilde{X}$ a finite-dimensional, contractible, free G-complex via "coinduction": Define

$$X' = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, \tilde{X}),$$

a left G-complex via $(g \cdot \varphi)(\tilde{g}) = \varphi(\tilde{g}g)$. Since $[G : H] = n < \infty$, we choose coset representatives $s_1, \ldots, s_n$. This gives rise to a bijection

$$\begin{aligned} X' &\xrightarrow{\psi} \prod_{i=1}^{n} \tilde{X} \\ \varphi &\mapsto (\varphi(s_1), \ldots, \varphi(s_n)). \end{aligned}$$

(Note: this is precisely the bijection between a coinduced module and the product of modules.) The product on the right has a CW structure given by products of cells in $\tilde{X}$ (and topologized with the CW topology). The bijection ψ gives X' a CW structure. The CW structure is independent on the choice of coset reps: if $s'_1, \ldots, s'_n$ is a new set of representatives, with $s'_i = s_i h_i$ for $h_i \in H$, then the new bijection ψ is obtained from the old one via composition with the CW isomorphism

$$\begin{aligned} \prod_{i=1}^{n} h_i : \prod_{i=1}^{n} \tilde{X} &\to \prod_{i=1}^{n} \tilde{X} \\ (\varphi(s_1), \ldots, \varphi(s_n)) &\mapsto (h_1 \cdot \varphi(s_1), \ldots, h_n \cdot \varphi(s_n)) = (\varphi(s'_1), \ldots, \varphi(s'_n)). \end{aligned}$$

Now the G-action on X' preserves the CW structure, since for any $g \in G$, we have a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\cdot g"] \arrow[dr, "\psi"'] & X' \arrow[d, "\psi'"] \\ & \prod_{i=1}^{n} \tilde{X} \end{tikzcd}$$

where ψ is defined by the coset reps $s_1, \ldots, s_n$, and ψ' is defined using the coset reps $s_1 \cdot g, \ldots, s_n \cdot g$ (as then $\psi'(g \cdot \varphi) = \psi(\varphi)$ for all $\varphi \in X'$ and $g \in G$). This shows that X' is a well-defined G-complex. Since X' is a product of finite-dimensional complexes, it is also finite-dimensional (of dimension $\dim(\tilde{X}) \cdot [G : H] = \mathrm{cd}(H) \cdot [G : H]$, though this is not necessarily the minimal dimensional such complex), and since it is the product of contractible complexes, it is also contractible.

It remains to show that G acts *freely* on X' (for then X' is a finite dimensional $K(G, 1)$, and we conclude that $\mathrm{cd}(G) < \infty$). We have a canonical map

$$X' \xrightarrow{\pi} \tilde{X}$$
$$\varphi \mapsto \varphi(e)$$

which is H-equivariant, since $\pi(h \cdot \varphi) = (h \cdot \varphi)(e) = \varphi(h) = h \cdot \varphi(e)$, as $\varphi \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, \tilde{X})$. Since H acts freely on $\tilde{X}$, then H acts freely on X'. (If $h \cdot \varphi = \varphi$ for some $h \neq e$, then $\varphi(e) = \pi(\varphi) = \pi(h \cdot \varphi) = h \cdot \varphi(e)$ implies that $h \cdot \varphi(e) = \varphi(e)$, a contradiction.) This map also takes cells to cells, as it projects a product of cells in $\tilde{X}$ to a single cell (the one corresponding to the e representative summand). Given a cell $\sigma \in X'$, we know that $G_\sigma \cap H = \{e\}$. Recall that if H, K are subgroups of G, then

$$[K : K \cap H] \leqslant [G : H].$$

Letting $K = G_\sigma$, we see that $|G_\sigma| \leqslant [G : H] < \infty$. If $G_\sigma \neq \{e\}$, then G contains a nontrivial finite subgroup and hence has torsion, a contradiction. We conclude that $G_\sigma = \{e\}$, so G acts freely on X'.

□

Prop 2.103 (Free resolutions of length $\mathrm{cd}(G)$)**.** *For any group G, there is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules of length equal to* $\mathrm{cd}(G)$.

Proof. If $\mathrm{cd}(G) = \infty$, the statement is equivalent to the existence of a free resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules, which we know exists. Suppose $\mathrm{cd}(G) = n < \infty$. Let

$$F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

be a partial free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. As $\mathrm{cd}(G) = n$, prop 2.82 implies that this partial resolution may be completed to

$$0 \to K \xrightarrow{\partial} F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

with $K = \ker(F_{n-1} \to F_{n-2})$ projective. By lemma 2.22, there exists a free $\mathbb{Z}G$-module F with $K \oplus F \cong F$. Replacing K and F_i with $K \oplus F$ and $F_i \oplus F$, and defining $\partial|_F = 0$, we have a free $\mathbb{Z}G$-module resolution

$$0 \to K \oplus F \xrightarrow{\partial} F_{n-1} \oplus F \to \cdots \to F_0 \oplus F \to \mathbb{Z} \to 0$$

since $K \oplus F \cong F$.

□

2.6.2 Virtual notions

Definition 2.104 (Virtually torsion-free)**.** Say a group G is virtually torsion-free if it contains a torsion-free finite-index subgroup.

Prop 2.105 (Virtual cohomological dimension)**.** *If G is virtually torsion-free, all finite-index torsion-free subgroups of G have the same cohomological dimension (called the virtual cohomological dimension, or vcd).*

Proof. Let H, K be two finite-index torsion-free subgroups of G. Then $H \cap K$ is clearly torsion-free, and it has finite index in both H and K since $[H : H \cap K], [K : H \cap K] \leqslant [G : H \cap K] \leqslant [G : H] \cdot [G : K] < \infty$. By Serre's theorem, $\mathrm{cd}(K) = \mathrm{cd}(H \cap K) = \mathrm{cd}(H)$ (as H and K are torsion-free, and $H \cap K$ is a finite-index subgroup of both). □

Prop 2.106 (Torsion-free $K(H, 1)$)**.** *Suppose G is virtually torsion-free and acts simplicially on a contractible simplicial complex X. Furthermore suppose that the stabilizer G_σ of any simplex σ is finite. Let H be a torsion-free subgroup of G. Then X/H is a $K(H, 1)$.*

Proof. We claim that H acts freely and properly discontinuously on X, so that $X \to X/H$ is a covering space with $\pi_1(X/H) = H$ (and since X is contractible, this means X/H is a $K(H, 1)$). Note that G_σ is finite for all $\sigma \in X$, and hence $G_\sigma \cap H$ is a finite subgroup of H. But H is torsion-free, so $G_\sigma \cap H = \{e\}$, and we conclude that H acts freely on X. □

Definition 2.107 (Rational cohomological dimension)**.** Define $\mathrm{cd}_{\mathbb{Q}}(G) := \mathrm{pd}_{\mathbb{Q}G}(\mathbb{Q})$.

Prop 2.108 (Rational cohomological dimension)**.** *For any group G, $\mathrm{cd}_{\mathbb{Q}}(G) = \max\{k : H^k(G; V) \neq 0$ for some $\mathbb{Q}G$-module $V\}$.*

Proof. Recall that

$$\begin{aligned}\mathrm{pd}_{\mathbb{Q}G}(\mathbb{Q}) &= \inf\{n : \mathrm{Ext}^i_{\mathbb{Q}G}(\mathbb{Q}, -) = 0 \ \ \forall i > n\} \\ &= \sup\{n : \mathrm{Ext}^n_{\mathbb{Q}G}(\mathbb{Q}, V) \neq 0 \text{ for some } \mathbb{Q}G\text{-module } \ V\}.\end{aligned}$$

□

Prop 2.109. *For any group G, $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant \mathrm{cd}(G)$.*

Proof. By prop 2.31, we know $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, A) \cong \mathrm{Ext}^*_{\mathbb{Q}G}(\mathbb{Q}, A)$ for any $\mathbb{Q}G$-module (representation) A (which is also naturally a $\mathbb{Z}G$-module). If $\mathrm{Ext}^k_{\mathbb{Q}G}(\mathbb{Q}, A) \neq 0$ for some $\mathbb{Q}G$-module A, then $\mathrm{Ext}^k_{\mathbb{Z}G}(\mathbb{Z}, A) \neq 0$, which shows that $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant \mathrm{cd}(G)$. (Note: every $\mathbb{Q}G$-module A is a $\mathbb{Z}G$-module by restriction of scalars, but not every $\mathbb{Z}G$-module is a $\mathbb{Q}G$-module, which is why we don't necessarily have equality.) □

Prop 2.110 (Rational cohomological dimension agrees with finite-index subgroups)**.** *Let G be a group (not necessarily torsion-free). Then $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(H)$ for any finite-index subgroup H.*

Proof. If $\mathrm{cd}_{\mathbb{Q}}(H) = \infty$, then $\mathrm{cd}_{\mathbb{Q}}(H) \leqslant \mathrm{cd}_{\mathbb{Q}}(G)$ implies that $\mathrm{cd}_{\mathbb{Q}}(G) = \infty$ as well. It remains to show that if $\mathrm{cd}_{\mathbb{Q}}(H) < \infty$, then $\mathrm{cd}_{\mathbb{Q}}(H) = \mathrm{cd}_{\mathbb{Q}}(G)$. Proof strategy courtesy of [1]: First notice that every subgroup of finite index in G contains a normal subgroup of finite index (if $H \leqslant G$ has $[G : H] < \infty$, take $N = \bigcap_{gH \in G/H} gHg^{-1}$. Then $N \subseteq H$ is normal in G, and is an intersection of finitely many finite-index subgroups, hence is finite-index). Thus it suffices to prove the claim for $H \trianglelefteq G$.

Let $F_\bullet \to \mathbb{Q} \to 0$ be a free $\mathbb{Q}H$-module resolution of $\mathbb{Q}$. If $[G : H] = n$, let $E_\bullet = \bigotimes_{i=1}^n F_\bullet$ be the n-fold tensor product of $F_\bullet$ over $\mathbb{Q}$. Then $E_\bullet \to \mathbb{Q} \to 0$ is a free $\mathbb{Q}[\prod_{i=1}^n H]$-module resolution of $\mathbb{Q}$. Define a G-action on $E_\bullet$ which is compatible with the differential: First choose coset representatives $x_1, \ldots, x_n$ so that $G = \bigsqcup x_i H$. If $g \in G$, then $g^{-1}x_i \in x_{v_i}H$ for some coset representative x_{v_i}, and hence there exists some $h_{v_i} \in H$ such that $g^{-1}x_i = x_{v_i}h_{v_i}^{-1}$. Define a G-action

$$g \cdot (p_1 \otimes \cdots \otimes p_n) = (-1)^{\alpha} h_{v_1} p_{v_1} \otimes \cdots \otimes h_{v_n} p_{v_n}$$

for $p_k \in P_{i_k}$ ($k = 1, \ldots, n$), and $\alpha = \sum i_r i_s$ over all pairs $r < s$ with $v_r > v_s$. Is this compatible with the differentials? Recall: differentials in a tensor product of complexes finish this later. an annoying detail... Suppose we've shown that this action commutes with the differential. We wish to show that $E_\bullet$ is an exact complex of projective $\mathbb{Q}G$-modules. Let $\{b_\sigma\}_{\sigma \in J}$ be a $\mathbb{Q}H$-basis for F. Then $\{hb_\sigma\}_{h \in H, \sigma \in J}$ is a $\mathbb{Q}$-basis for F, so E has a $\mathbb{Q}$-basis consisting of all elements

$$w = h_1 b_{\sigma_1} \otimes \cdots \otimes h_n b_{\sigma_n}.$$

Since G permutes the $\mathbb{Q}w$, we have that E is isomorphic, as a $\mathbb{Q}G$-module, to a direct sum $\bigoplus_i \mathbb{Q}G \cdot w_i$ for some basis elements w_i. It thus suffices to show that $\mathbb{Q}G \cdot w_i$ is a projective $\mathbb{Q}G$-module. Let G_{w_i} be the stabilizer of w_i (so $g \cdot w_i = \pm w_i$ for all $g \in G_{w_i}$). We claim that $G_{w_i} \cap H = \{e\}$. If $h \in H$, then $h \cdot x_i = x_i(x_i^{-1}h^{-1}x_i)^{-1}$, where $x_i^{-1}h^{-1}x_i \in H$ since $H \trianglelefteq G$. So for $h \in H$, we have that $x_{v_i} = x_i$ and $h_{v_i} = x_i^{-1}h^{-1}x_i$, showing that

$$\begin{aligned} h \cdot w_i = \pm w_i &\implies h \cdot (h_1 b_{\sigma_1} \otimes \cdots \otimes h_k b_{\sigma_k}) = \pm h_1 b_{\sigma_1} \otimes \cdots \otimes h_k b_{\sigma_k} \\ &\implies (-1)^0 x_1^{-1}h^{-1}x_1 h_1 b_{\sigma_1} \otimes \cdots \otimes x_k^{-1}h^{-1}x_k h_k b_{\sigma_k} = \pm h_1 b_{\sigma_1} \otimes \cdots \otimes h_k b_{\sigma_k} \\ &\implies h = e. \end{aligned}$$

In particular this implies that G_{w_i} is a finite subgroup of G (as $G_{w_i} \leqslant G/H$). Let $m = |G_{w_i}|$, which is a unit in $\mathbb{Q}$. We can define a $\mathbb{Q}G$-module homomorphism

$$\begin{aligned} p : \mathbb{Q}G \cdot w_i &\to \mathbb{Q}G \\ p(\lambda \cdot w_i) &= \frac{1}{m}\lambda \cdot (\varepsilon_1 g_1 + \cdots + \varepsilon_m g_m) \end{aligned}$$

where $g_j \cdot w_i = \varepsilon_j w_i$. We claim that p splits the projection $\mathbb{Q}G \to \mathbb{Q}G \cdot w_i$. Indeed, if $\lambda \cdot w_i \in \mathbb{Q}G \cdot w_i$, then

$$p(\lambda \cdot w_i) \cdot w_i = \frac{1}{m}\lambda \cdot (\varepsilon_1 g_1 + \cdots + \varepsilon_m g_m) \cdot w_i = \frac{1}{m}\lambda \cdot m w_i = \lambda \cdot w_i.$$

□

Corollary 2.111 ($\mathrm{cd}_{\mathbb{Q}}(G)$ bounded above by vcd)**.** *Let G be virtually torsion-free. Then $\mathrm{cd}_{\mathbb{Q}}(G) \leqslant \mathrm{vcd}(G)$. Thus $\mathrm{vcd}(G)$ gives an upper bound on the degree k such that $H^k(G;V)$ can be nonzero for a $\mathbb{Q}G$-module V.*

Proof. Let H be a torsion-free finite-index subgroup of G. We have that $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(H) \leqslant \mathrm{cd}(H) = \mathrm{vcd}(G)$. In particular, if V is a $\mathbb{Q}G$-module, then

$$H^i(G;V) = 0 \quad \forall i > \mathrm{vcd}(G)$$

(since this holds for all $i > \mathrm{cd}_{\mathbb{Q}}(G)$). If $H^{\mathrm{vcd}(G)}(G;V) \neq 0$ for some $\mathbb{Q}G$-module G, then $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{vcd}(G)$ and we conclude that

$$\mathrm{vcd}(G) = \max\{k : \exists \mathbb{Q}G\text{-module } V | H^k(G;V) \neq 0\}.$$

□

2.6.3 Groups of type FP_n, FP_∞, FP, FL

Definition 2.112 (finite type)**.** A resolution or partial resolution of R-modules is of finite type if each term is finitely generated.

Definition 2.113 (FP_n)**.** An R-module M is of type FP_n if it admits a partial *projective* resolution of finite type

$$P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Definition 2.114. A group G is of type FP_n if $\mathbb{Z}$ is a $\mathbb{Z}G$-module of FP_n.

Prop 2.115 (Equivalent conditions for FP_n)**.** *Fix an R-module M and $n \geqslant 0$. The following are equivalent:*

(i) M admits a partial resolution

$$F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

with each F_i free of finite rank.

(ii) M is of type FP_n.

(iii) M is finitely generated, and for every partial projective resolution

$$P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$$

of finite type with $k < n$, the kernel $\ker\{P_k \to P_{k-1}\}$ is finitely generated.

Proof. $(i) \implies (ii)$: obvious. Suppose (ii) holds, and let $P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$ be a partial projective resolution of finite type with $k < n$. Since P_0 surjects onto M and P_0 is finitely generated, then M is finitely generated. Since M is of type FP_n, we know there exists some other partial projective resolution

$$P'_{k+1} \to P'_k \to \cdots \to P'_0 \to M \to 0$$

of finite type. Let $K = \ker(P_k \to P_{k-1})$ and let $K' = \ker(P'_k \to P'_{k-1}) = \mathrm{Im}(P'_{k+1} \to P'_k)$. Since the homomorphic image of a finitely generated module is finitely generated, we have that K' is finitely generated. By the generalized Schanuel's lemma 2.24, we know $K \oplus P'_k \oplus P_{k-1} \oplus \cdots \cong K' \oplus P_k \oplus P'_{k-1} \oplus \cdots$. Hence K is finitely generated, giving (iii). To see that $(iii) \implies (i)$, we use the fact that M is finitely generated, and hence there exists a partial free resolution $F_0 \to M \to 0$, where F_0 has finite rank. If $0 < n$, the kernel $K = \ker(F_0 \to M)$ is finitely generated, so there exists a surjection $F_1 \twoheadrightarrow K$ from a free module of finite rank. One continues building the partial resolution $F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ in this way, with each F_i free of finite rank. ◻

Prop 2.116 (Finite-index subgroups of type FP_n)**.** *Let G be a group and H a finite-index subgroup. Show G has type FP_n for some $0 \leqslant n \leqslant \infty$ if and only if H does.*

Proof. Suppose G has type FP_n. Then we have a partial resolution $F_n \to \cdots \to F_0 \to \mathbb{Z} \to 0$ of finite rank free $\mathbb{Z}G$-modules. As free $\mathbb{Z}G$-modules are free $\mathbb{Z}H$-modules, and $[G : H] < \infty$, then $\mathbb{Z}G$ is a finite rank free $\mathbb{Z}H$-module, and we conclude that $F_n \to \cdots \to F_0 \to \mathbb{Z} \to 0$ is a free $\mathbb{Z}H$-module resolution of finite rank. Hence H has type FP_n.

Suppose H has type FP_n, and let $P_k \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ be a partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of finite type for some $k \geqslant 0$. Then each P_i is a projective $\mathbb{Z}H$-module of finite type, so if $k < n$ we know that $\ker(P_k \to P_{k-1})$ is finitely generated as a $\mathbb{Z}H$-module. Hence it is finitely generated as a $\mathbb{Z}G$-module, so G has type FP_n. ◻

Prop 2.117 (Equiv conditions for type FP_∞). *An R-module M is of type FP_∞ if the following equivalent conditions holds:*

(i) M admits a free resolution of finite type.

(ii) M admits a projective resolution of finite type.

(iii) M is of type FP_n for all $n \geqslant 0$.

Proof. $(i) \implies (ii)$ is trivial. $(ii) \implies (iii)$ is also trivial. Suppose (iii) holds. Let $F_n \to \cdots \to F_0 \to M \to 0$ be a partial resolution of finite rank free modules. Then $K = \ker(F_n \to F_{n-1})$ is finitely generated, so there exists a free module F_{n+1} of finite rank which surjects onto K, and the composition $F_{n+1} \to \cdots \to F_0 \to M \to 0$ gives a larger partial resolution of finite rank free modules. Continue indefinitely. □

Definition 2.118 (FP). A group G is of type FP if it admits a *finite* length projective resolution of finite type.

Prop 2.119 ($\mathrm{FP} = \mathrm{FP}_\infty +$ finite $\mathrm{cd}(G)$). *A group G is of type* FP *if and only if it has finite cohomological dimension and is of type* FP_∞.

Proof. Suppose G is of type FP. Then $\mathbb{Z}$ has a finite length projective resolution over $\mathbb{Z}G$, which implies that $\mathrm{cd}(G) < \infty$. Also each module in the resolution has finite type, so G is of type FP_∞.

Suppose G is of type FP_∞ and has finite cohomological dimension, with $\mathrm{cd}(G) = n$. Let $P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ be a finite type partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Since $\mathrm{cd}(G) = n$, we know that $K = \ker(P_{n-1} \to P_{n-2})$ is projective by 2.82, and since G is of type FP_n, we know that K is also finitely generated (2.115). Hence $0 \to K \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ is a finite type projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, showing that G is of type FP. □

Prop 2.120 (Suffices to check vanishing with $\mathbb{Z}G$ coefficients). *If G is a group of type* FP, *then* $\mathrm{cd}(G) = \max\{n : H^n(G; \mathbb{Z}G) \neq 0\}$.

Proof. By prop 2.90, we know that if $\mathrm{cd}(G) < \infty$, then $\mathrm{cd}(G) = \max\{n : H^n(G; F) \neq 0$ for F a free $\mathbb{Z}G$-module$\}$. We now wish to show that if furthermore G is of type FP, then we can take F to simply be $\mathbb{Z}G$. Notice that if F is free of finite rank (so $F \cong \bigoplus_{i=1}^k \mathbb{Z}G$), then

$$H^n(G; F) = H^n(G; \bigoplus_{i=1}^k \mathbb{Z}G) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \bigoplus_{i=1}^k \mathbb{Z}G) \cong \bigoplus_{i=1}^k \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G) = \bigoplus_{i=1}^k H^n(G; \mathbb{Z}G)$$

so that $H^n(G; F) \neq 0$ implies that $H^n(G; \mathbb{Z}G) \neq 0$. On the other hand, if F is a free $\mathbb{Z}G$-module of infinite rank, we can express it as generated by a basis $(e_i)_{i \in I}$. Then for all finite subsets $J \subset I$, we can define $F_J = \langle e_i \rangle_{i \in J}$, so $F = \lim_{\to} F_J$ (this follows from the fact that finite subsets of I form a directed system under inclusion, as "being a subset of" is reflexive, transitive, and any two sets have an upper bound (their union)). Hence it remains to show that if G is of type FP, then $H^n(G; -) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, -)$ commutes with direct limits, for then

$$H^n(G; F) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \lim_{\to} F_J) \cong \lim_{\to} \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, F_J) \neq 0 \implies \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, F_J) \neq 0$$

for some $J \subset I$ finite, which implies that $H^n(G; \mathbb{Z}G) \neq 0$, as we saw above. Since G is of type FP, we have a projective resolution of finite type over $\mathbb{Z}G$:

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

Since each P_i is projective and *finitely generated* (the key), $\mathrm{Hom}_{\mathbb{Z}G}(P_i, -)$ commutes with direct limits (2.33), and we have a cochain complex

$$0 \to \varinjlim \mathrm{Hom}_{\mathbb{Z}G}(P_0, F_J) \to \varinjlim \mathrm{Hom}_{\mathbb{Z}G}(P_1, F_J) \to \cdots$$

Finally, since homology commutes with direct limits, we have that $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, \lim_\to F_J) \cong \lim_\to \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, F_J)$. □

Definition 2.121 (FL)**.** A group G is of type FL if there exists a finite length and finite type *free* resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Remark 2.122. Note that if G is of type FP, then for a finite type free resolution $F_n \to \cdots \to F_0 \to \mathbb{Z} \to 0$ over $\mathbb{Z}G$, we are only guaranteed that $K = \ker(F_n \to F_{n-1})$ is *projective* of finite type, and not free. Hence there may be groups of type FP which are not of type FL.

Lemma 2.123. *If there exists a $K(G,1)$ which is a finite complex, then G is of type* FL.

Proof. The universal cover $\tilde{X}$ will also be a finite complex, and since it its cellular chain complex is of free $\mathbb{Z}G$-modules, we're done. □

Definition 2.124 (Finitely dominated)**.** A space Y is finitely dominated if there exists a finite complex K such that Y is a retract of K in the homotopy category, that is, we require maps $i : Y \to K$ and $r : K \to Y$ with $r \circ i \simeq \mathrm{id}_Y$.

Lemma 2.125. *If there exists a finitely dominated $K(G,1)$, then G is of type* FP.

Remark 2.126. The existence of groups of type FP which are not of type FL is equivalent to the existence of finitely dominated $K(G,1)$'s which are not finite. Examples of finitely dominated complexes which are not homotopy equivalent to finite complex exist, but none of the known ones are $K(G,1)$'s!

Definition 2.127 (Stably free)**.** A finitely generated projective module P is stably free if there exists some finite rank free-module F for which $P \oplus F$ is free (then necessarily of finite rank).

One can ask whether there exist finitely generated projectives which are not stably free. The answer is yes over arbitrary rings, and even over group rings (known examples over $\mathbb{Z}[\mathbb{Z}/23]$). But no known examples over group rings $\mathbb{Z}G$ for G torsion-free. And G has to be torsion-free for G to have type FP!

Prop 2.128. *If G is of type* FL *and $\mathrm{cd}(G) = n$, $\mathbb{Z}$ admits a finite free resolution over $\mathbb{Z}G$ of length n.*

Proof. If G is of type FL, there exists a finite length and finite type free resolution $0 \to F_\ell \to F_{\ell-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}G$. Let $\tilde{F_{n-1}} \to \cdots \to \tilde{F}_0 \to \mathbb{Z} \to 0$ be a partial free resolution of $\mathbb{Z}$ of finite type over $\mathbb{Z}G$. Then $P_n = \ker(\tilde{F}_{n-1} \to \tilde{F}_{n-2})$ must be finitely generated and projective by 2.82 and 2.115. Adding zeros to one of the resolutions to make them the same length, we have by generalized Schanuel's lemma 2.24 that $\tilde{F}_0 \oplus F_1 \oplus \cdots \cong F_0 \oplus \tilde{F}_1 \oplus \cdots$, and hence P_n is stably free. Adding the free module F for which $P_n \oplus F$ is free of finite rank to the second resolution gives our desired finite free resolution of length n. □

Prop 2.129 (Ex VIII.6.4 in [2])**.** *Let Γ be a group of type* FP*, let $n = \mathrm{cd}\,\Gamma$, and suppose $H^n(\Gamma; \mathbb{Z}\Gamma)$ is finitely generated as an abelian group. If $\Gamma' < \Gamma$ is a subgroup of infinite index, then $\mathrm{cd}\,\Gamma' < n$.*

Proof. By Shapiro's lemma 2.57, we know that for any $\mathbb{Z}\Gamma'$-module M, we have $H^n(\Gamma'; M) \cong H^n(\Gamma; \mathrm{Coind}^\Gamma_{\Gamma'} M)$. We wish to show that this latter module must be 0. If G is a group of type FP and $n = \mathrm{cd}(G)$, then $H^n(G; M) \cong H_0(G; H^n(G; \mathbb{Z}G) \otimes M)$ by 2.134. Hence $H^n(\Gamma; \mathrm{Coind}^\Gamma_{\Gamma'} M) \cong H_0(\Gamma; H^n(\Gamma; \mathbb{Z}\Gamma) \otimes \mathrm{Coind}^\Gamma_{\Gamma'} M) \cong (H^n(\Gamma; \mathbb{Z}\Gamma) \otimes \mathrm{Coind}^\Gamma_{\Gamma'} M)_G$. Proof follows from fact (not proven, but see exercise 4(b) of [2] III.5) that if G is finitely generated and $[G : H] = \infty$, then $(\mathrm{Coind}^G_H M)_G = 0$. Since Γ is of type FP, it is in particular finitely generated, so $(\mathrm{Coind}^\Gamma_{\Gamma'} M)_\Gamma = 0$. Since $H^n(\Gamma; \mathbb{Z}\Gamma)$ is finitely generated as an abelian group, IDK FINISH LATER!!!! □

2.6.4 The module $H^*(G;\mathbb{Z}G)$

Prop 2.130 (The module $H^n(G;\mathbb{Z}G)$)**.** *The groups $H^n(G;\mathbb{Z}G)$ have the structure of a right $\mathbb{Z}G$-module.*

Proof. The coefficient module $\mathbb{Z}G$ has the structure of a right G-module via right translation: $\alpha(g)\cdot g' = \alpha(g)\alpha(g') = \alpha(gg')$. This action induces a right action on $H^*(G;\mathbb{Z}G)$ as follows: Given a free resolution of $\mathbb{Z}$ by right $\mathbb{Z}G$-modules

$$\cdots \to F_1 \xrightarrow{\partial_1} F_0 \to \mathbb{Z} \to 0,$$

we truncate and apply $\mathrm{Hom}_{\mathbb{Z}G}(-,\mathbb{Z}G)$ to get the cochain complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(F_0,\mathbb{Z}G) \xrightarrow{\partial_1^*} \mathrm{Hom}_{\mathbb{Z}G}(F_1,\mathbb{Z}G) \to \cdots .$$

The modules in this cochain complex have a right G-action via $(f\cdot g)(c_i) = f(c_i)\cdot g$ for $f \in \mathrm{Hom}_{\mathbb{Z}G}(F_i,\mathbb{Z}G)$ and $c_i \in F_i$. Check that this is indeed a right G-action:

(i) $(f\cdot e)(c_i) = f(c_i)\cdot e = f(c_i)$.

(ii) $((f\cdot g_1)\cdot g_2)(c_i) = (f\cdot g_1)(c_i)\cdot g_2 = f(c_i)\cdot g_1\cdot g_2 = (f\cdot g_1g_2)(c_i)$.

We can also see that this G-action commutes with our differentials, as

$$\begin{aligned}(\partial_{i+1}^*(f\cdot g))(c_{i+1}) = (f\cdot g)(\partial_{i+1}(c_{i+1})) &= f(\partial_{i+1}(c_{i+1}))\cdot g\\ &= (\partial_{i+1}^*(f))(c_{i+1})\cdot g\\ &= (\partial_{i+1}^*(f)\cdot g)(c_{i+1}).\end{aligned}$$

This action descends to an action on the homology, so that $H^*(G;\mathbb{Z}G) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}G)$ has the structure of a right $\mathbb{Z}G$-module. ◻

Remark 2.131. Note that if we instead consider $H_*(G;\mathbb{Z}G)$, then because $H_*(G;\mathbb{Z}G) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}G)$, and $\mathbb{Z}G$ is a free $\mathbb{Z}G$-module, then $H_*(G;\mathbb{Z}G) = 0$ for $* > 0$ (and $H_0(G;\mathbb{Z}G) = (\mathbb{Z}G)_G \cong \mathbb{Z}\otimes_{\mathbb{Z}G}\mathbb{Z}G \cong \mathbb{Z}$ by lemma 2.39). So this is not a very interesting G-module (trivial).

Prop 2.132. *For any left $\mathbb{Z}G$-module M, there exists a map $\phi : H^*(G;\mathbb{Z}G)\otimes_{\mathbb{Z}G} M \to H^*(G;M)$ defined on the level of cochains by mapping $u\otimes m$ ($u \in \mathrm{Hom}_{\mathbb{Z}G}(P_i,\mathbb{Z}G)$) to the cochain ($x \mapsto u(x)\cdot m \in \mathrm{Hom}_{\mathbb{Z}G}(P_i,M)$).*

Proof. Let

$$\cdots \to P_{n+1} \xrightarrow{\partial_{n+1}} P_n \to \cdots \xrightarrow{\partial_1} P_0 \to \mathbb{Z} \to 0$$

be a projective $\mathbb{Z}G$-module resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Truncate and apply $\mathrm{Hom}_{\mathbb{Z}G}(-,\mathbb{Z}G)$ to get the cochain complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(P_0,\mathbb{Z}G) \xrightarrow{\partial_1^*} \mathrm{Hom}_{\mathbb{Z}G}(P_1,\mathbb{Z}G) \xrightarrow{\partial_2^*} \cdots$$

Recall that $H^n(G;\mathbb{Z}G)$ has the structure of a right G-module induced by the right G-module structure of $\mathrm{Hom}_{\mathbb{Z}G}(P_n,\mathbb{Z}G)$. Define a map

$$\begin{aligned}\phi : H^n(G;\mathbb{Z}G)\times M &\to H^n(G;M)\\ [u]\times m &\mapsto ([x\mapsto u(x)\cdot m])\end{aligned}$$

where $u \in \ker \partial_{n+1}^*$ is a cocycle. First we check that this is well-defined: if $[u] = [v]$, then $u - v \in \operatorname{Im} \partial_n^*$. Let $f \in \operatorname{Hom}_{\mathbb{Z}G}(P_{n-1}, \mathbb{Z}G)$ be such that $\partial_n^*(f) = u - v$. Consider the map $g(x) = f(x) \cdot m \in \operatorname{Hom}_{\mathbb{Z}G}(P_{n-1}, M)$. Viewing ∂_n^* as a differential in the cochain complex defining $H^n(G; M)$, we have that $\partial_n^*(g)(x) = g(\partial_n(x)) = f(\partial_n(x)) \cdot m = (\partial_n^*(f))(x) \cdot m = (u - v)(x) \cdot m$. This shows that $[u] \times m$ and $[v] \times m$ map to the same element under ϕ.

We now show that ϕ is G-balanced, so that it extends uniquely to a map $H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \to H^n(G; M)$:

$$\phi([u] \cdot g, m) = ([x \mapsto u(x) \cdot g \cdot m]) = ([x \mapsto u(x) \cdot (gm)]) = \phi([u], g \cdot m).$$

□

Prop 2.133. *If G is of type* FP *and $n = \operatorname{cd}(G)$, the map $\phi : H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \to H^n(G; M)$ from 2.132 is an isomorphism.*

Proof. Recall that when $n = \operatorname{cd}(G)$, the functor $H^n(G; -)$ is right exact (see the proof in 2.90), and since tensor products are right exact, we have that $H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} -$ is right exact. Thus if we have an exact sequence $F' \to F \to M \to 0$ with F', F free $\mathbb{Z}G$-modules, we have a commutative diagram

$$\begin{tikzcd}
H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} F' \arrow[r] \arrow[d, "\phi"] & H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} F \arrow[r] \arrow[d, "\phi"] & H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \arrow[r] \arrow[d, "\phi"] & 0 \\
H^n(G; F') \arrow[r] & H^n(G; F) \arrow[r] & H^n(G; M) \arrow[r] & 0
\end{tikzcd}$$

If we can show that the left two vertical maps are isomorphisms, then the right vertical map must be an isomorphism as well. It therefore suffices to prove the claim when $M = F$ is a free $\mathbb{Z}G$-module. Since tensor products commute with direct limits, and so does $H^n(G; -)$ when $n = \operatorname{cd}(G)$ and G has type FP (see 2.120), it suffices to prove the claim when $F = \mathbb{Z}G$. That is, we show

$$\phi : H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}G \to H^n(G; \mathbb{Z}G)$$

is an isomorphism. Recall that $\phi([u] \otimes m) = [x \mapsto u(x) \cdot m]$. We define an inverse map $\psi : H^n(G; \mathbb{Z}G) \to H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}G$, $[v] \mapsto [v] \otimes \alpha(e)$. Check:

$$\begin{aligned}
\psi(\phi([u] \otimes \alpha(g))) = \psi([x \mapsto u(x) \cdot \alpha(g)]) &= [x \mapsto u(x) \cdot \alpha(g)] \otimes \alpha(e) \\
&= [(x \mapsto u(x)) \cdot g] \otimes \alpha(e) \\
&= [u] \otimes \alpha(g)
\end{aligned}$$

and

$$\phi(\psi([u])) = \phi([u] \otimes \alpha(e)) = [x \mapsto u(x) \cdot \alpha(e)] = [u].$$

□

Corollary 2.134 (Codimension 0 duality iso)**.** *Let G be a group of type* FP*, $n = \operatorname{cd}(G)$, and $D = H^n(G; \mathbb{Z}G)$. Then $H^n(G; M) \cong H_0(G; D \otimes_{\mathbb{Z}} M)$.*

Proof. By prop 2.133, we know $H^n(G; M) \cong H^n(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} M = D \otimes_{\mathbb{Z}G} M$ for all $\mathbb{Z}G$-modules M. By lemma 2.39, we know $H_0(G; D \otimes M) = (D \otimes M)_G$, so it remains to show that $(D \otimes M)_G \cong D \otimes_{\mathbb{Z}G} M$ (where $D \otimes \mathbb{Z}$ has the diagonal action of G). But this was shown in the proof of prop 2.40. □

A slightly weaker result than the one given in 2.133 (which required G to be of type FP, and which gave a sort of "universal coefficient theorem" isomorphism for *any* $\mathbb{Z}G$-module M) holds for groups of type FP_∞, if we restrict coefficient modules to be *flat* $\mathbb{Z}G$-modules. The nice thing about the following isomorphism is that it holds for all degrees, whereas the isomorphism in prop 2.133 only holds for $n = \mathrm{cd}(G)$. We first need the following lemma:

Lemma 2.135. *If P is a finitely generated projective right $\mathbb{Z}G$-module and M is any $\mathbb{Z}G$-module, then*

$$\mathrm{Hom}_{\mathbb{Z}G}(P, M) \cong P^* \otimes_{\mathbb{Z}G} M$$

where $P^* = \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G)$.

Proof. Recall that if P is a right $\mathbb{Z}G$-module, we have a right $\mathbb{Z}G$-module structure on $\mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G)$ via $(\varphi \cdot g)(p) = \varphi(p) \cdot g$. Define a map

$$\begin{aligned} \psi : \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G) \times M &\to \mathrm{Hom}_{\mathbb{Z}G}(P, M) \\ (\varphi, m) &\mapsto (p \mapsto \varphi(p) \cdot m). \end{aligned}$$

This map is $\mathbb{Z}G$-balanced since $\psi(\varphi \cdot g, m) = (p \mapsto (\varphi \cdot g)(p) \cdot m) = (p \mapsto \varphi(p) \cdot g \cdot m) = \psi(\varphi, g \cdot m)$. Hence it descends to a map

$$\psi : \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \to \mathrm{Hom}_{\mathbb{Z}G}(P, M).$$

Since P is a finitely generated projective $\mathbb{Z}G$-module, it is a direct summand of $(\mathbb{Z}G)^k$ for some k. If we can prove that ψ is an isomorphism when $P = \mathbb{Z}G$, then additivity will yield the desired isomorphism for $P = (\mathbb{Z}G)^k$, and then we could verify that the map restricts to direct summands. The fact that ψ is an isomorphism for $P = \mathbb{Z}G$ follows from the fact that ψ factors through the canonical isomorphisms $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \cong M$ and $M \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M)$, where the first isomorphism sends $\varphi \otimes m$ to $\varphi(\alpha(e)) \cdot m$, and the second isomorphism sends m to $(\alpha(e) \mapsto m)$. □

Using the lemma, we now have the following result:

Prop 2.136. *If G is of type FP_∞, then $H^*(G; F) \cong H^*(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} F$ for any flat $\mathbb{Z}G$-module F.*

Proof. Since G is of type $\mathbb{F}P_\infty$, we have a projective resolution of finite type

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

over $\mathbb{Z}G$. Apply $\mathrm{Hom}_{\mathbb{Z}G}(-, F)$ to get a cochain complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(P_0, F) \to \mathrm{Hom}_{\mathbb{Z}G}(P_1, F) \to \cdots$$

Taking homology of this cochain complex gives $H^*(G; F)$. On the other hand, using the isomorphism in 2.135, this cochain complex is isomorphic to

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(P_0, \mathbb{Z}G) \otimes_{\mathbb{Z}G} F \to \mathrm{Hom}_{\mathbb{Z}G}(P_1, \mathbb{Z}G) \otimes_{\mathbb{Z}G} F \to \cdots$$

Since F is a flat $\mathbb{Z}G$-module, it commutes with homology. Hence the homology of the above complex is $H^*(G; \mathbb{Z}G) \otimes_{\mathbb{Z}G} F$, giving the desired isomorphism. □

Prop 2.137 (Brown VIII.5.5)**.** *Let Γ be a free product $\Gamma_1 * \Gamma_2$, where Γ_1 and Γ_2 are infinite and of type FP_∞. For any integer i there is a map of (right) Γ-modules*

$$H^i(\Gamma; \mathbb{Z}\Gamma) \to \mathrm{Ind}_{\Gamma_1}^{\Gamma} H^i(\Gamma_1; \mathbb{Z}\Gamma_1) \oplus \mathrm{Ind}_{\Gamma_2}^{\Gamma} H^i(\Gamma_2; \mathbb{Z}\Gamma_2)$$

which is an isomorphism if $i > 1$ and an epimorphism if $i = 1$ with kernel a free $\mathbb{Z}\Gamma$-module of rank 1.

Proof. By the cohomological Mayer-Vietoris sequence 2.43, we have a long exact sequence

$$\cdots \to H^{i-1}(\{e\};\mathbb{Z}\Gamma) \to H^i(\Gamma;\mathbb{Z}\Gamma) \to H^i(\Gamma_1;\mathbb{Z}\Gamma) \oplus H^i(\Gamma_2;\mathbb{Z}\Gamma) \to H^i(\{e\};\mathbb{Z}\Gamma) \to \cdots$$

Since Γ_1 and Γ_2 are both of type FP_∞, and $\mathbb{Z}\Gamma$ is a flat $\mathbb{Z}\Gamma_1$- and $\mathbb{Z}\Gamma_2$-module (being a free module over these rings), prop 2.136 yields $H^i(\Gamma_j;\mathbb{Z}\Gamma) \cong H^i(\Gamma_j;\mathbb{Z}\Gamma_j) \otimes_{\mathbb{Z}\Gamma_j} \mathbb{Z}\Gamma = \mathrm{Ind}_{\Gamma_j}^{\Gamma} H^i(\Gamma_j;\mathbb{Z}\Gamma_j)$ for $j = 1, 2$. The natural map from the Mayer-Vietoris sequence $H^i(\Gamma;\mathbb{Z}\Gamma) \to H^i(\Gamma_1;\mathbb{Z}\Gamma) \oplus H^i(\Gamma_2;\mathbb{Z}\Gamma)$ is then the desired map. If $i > 1$, $H^{i-1}(\{e\};\mathbb{Z}\Gamma) = H^i(\{e\};\mathbb{Z}\Gamma) = 0$, so exactness means that our aforementioned map is an isomorphism. If $i = 1$, we still have $H^1(\{e\};\mathbb{Z}\Gamma) = 0$, so the map is an epimorphism. The kernel is $H^0(\{e\};\mathbb{Z}\Gamma) = (\mathbb{Z}\Gamma)^{\{e\}} = \mathbb{Z}\Gamma$. □

Topological interpretation of right $\mathbb{Z}G$-modules $H^*(G;\mathbb{Z}G)$.

Assume G is finitely presented and of type FL.

Lemma 2.138 (Brown 7.4 [2])**.** *Let G be a group and let M be a left $\mathbb{Z}G$-module. Let $\mathrm{Hom}_C(M,\mathbb{Z})$ consist of abelian group homomorphisms $f : M \to \mathbb{Z}$ such that for all $m \in M$, $f(g \cdot m) = 0$ for all but finitely many $g \in G$. Then there exists a natural isomorphisms of right $\mathbb{Z}G$-modules*

$$\mathrm{Hom}_G(M,\mathbb{Z}G) \cong \mathrm{Hom}_C(M,\mathbb{Z})$$

where G acts on right $\mathrm{Hom}_G(M,\mathbb{Z}G)$ via its right action on $\mathbb{Z}G$ ($(\varphi \cdot g)(m) = \varphi(m) \cdot g$), and G acts on $\mathrm{Hom}_C(M,\mathbb{Z})$ via its left action on M ($(\varphi \cdot g)(m) = \varphi(g \cdot m)$).

Proof. A $\mathbb{Z}$-module map $F : M \to \mathbb{Z}G$ has the form $F(m) = \sum_{g\in G} f_g(m) \cdot g$ where $f_g : M \to \mathbb{Z}$, and for each $m \in M$, $f_g(m) = 0$ for almost all $g \in G$. We need another lemma first:

Lemma 2.139. *F is a $\mathbb{Z}G$-module homomorphism if and only if $f_g(m) = f_e(g^{-1}m)$ for all $g \in G$.*

For suppose F is a $\mathbb{Z}G$-module homomorphism. Then

$$\sum_{g\in G} f_g(m) \cdot hg = h \cdot \sum_{g\in G} f_g(m) \cdot g = h \cdot F(m) = F(h \cdot m) = \sum_{g\in G} f_g(h \cdot m) \cdot g.$$

Writing $g = h^{-1}hg$ and comparing the two expressions above, we see that $f_g(h \cdot m) = f_{h^{-1}g}(m)$. Therefore $f_g(g \cdot m) = f_e(m)$ for all $g \in G$, which means $f_g(m) = f_e(g^{-1}m)$. On the other hand, suppose $f_g(m) = f_e(g^{-1}m)$ for all $g \in G$. Then

$$F(h \cdot m) = \sum_{g\in G} f_g(h \cdot m) \cdot g = \sum_{g\in G} f_e(g^{-1}hm) \cdot g = \sum_{g\in G} f_{h^{-1}g}(m) \cdot g = \sum_{h\cdot x\in G} f_x(m) \cdot hx$$

where $x = h^{-1}g$ (so $g = hx$). Pulling out the h from the last sum yields

$$F(h \cdot m) = h \cdot \sum_{h\cdot x\in G} f_x(m) \cdot x = h \cdot \sum_{x\in G} f_x(m) \cdot x = h \cdot F(m).$$

Using this lemma, we construct a map

$$\Psi : \mathrm{Hom}_G(M,\mathbb{Z}G) \to \mathrm{Hom}_C(M,\mathbb{Z})$$
$$F \mapsto f_e$$

which has inverse

$$\begin{aligned}\Phi : \mathrm{Hom}_C(M, \mathbb{Z}) &\to \mathrm{Hom}_G(M, \mathbb{Z}G)\\ f &\mapsto (m \mapsto \sum_{g \in G} f(g^{-1}m) \cdot g).\end{aligned}$$

These maps are inverses since

$$(m \mapsto \sum_{g \in G} f_e(g^{-1}m) \cdot g) = (m \mapsto \sum_{g \in G} f_g(m) \cdot g) = F$$

and since

$$f_e(m) = f(e^{-1}m) = f(m).$$

These maps are also compatible with the right G-action. Note that if $F(m) = \sum_{g \in G} f_g(m) \cdot g$, then $(F \cdot \tilde{g})(m) = F(m) \cdot \tilde{g} = \sum_{g \in G} f_g(m) \cdot g\tilde{g} = \sum_{x \in G} f_{x\tilde{g}^{-1}}(m) \cdot x$, so $\Psi(F \cdot \tilde{g}) = f_{\tilde{g}^{-1}} = \Psi(F) \cdot \tilde{g}$ since $(f_e \cdot \tilde{g})(m) = f_e(\tilde{g}m) = f_{\tilde{g}^{-1}}(m)$. Similarly, $\Phi(f \cdot \tilde{g})(m) = \sum_{g \in G}(f \cdot \tilde{g})(g^{-1}m) \cdot g = \sum_{g \in G} f(\tilde{g}g^{-1}m)) \cdot g$, while $(\Phi(f) \cdot \tilde{g})(m) = \Phi(f)(m) \cdot \tilde{g} = \sum_{g \in G} f(g^{-1}m) \cdot g\tilde{g} = \sum_{x \in G} f(\tilde{g}x^{-1}m) \cdot x$, so the actions commute. ◻

Definition 2.140 (Cohomology with compact support)**.** Let X be a contractible, free G-complex with finite orbit space X/G. The cohomology of X is called the cohomology of X with compact support, denoted $H^*_C(X; \mathbb{Z})$.

Recall: If G is finitely presented and of type FL, then we've seen there exists a contractible, free G-complex X with X/G finite (X/G being our finite $K(G, 1)$). Then $C_\bullet(X)$ is a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, and $H^\bullet(G; \mathbb{Z}G)$ is the cohomology of $\mathrm{Hom}_{\mathbb{Z}G}(C(X), \mathbb{Z}G)$. By lemma 2.138, we know

$$\mathrm{Hom}_{\mathbb{Z}G}(C(X), \mathbb{Z}G) \cong \mathrm{Hom}_C(C(X), \mathbb{Z}) \subseteq \mathrm{Hom}(C(X), \mathbb{Z})$$

with the isomorphism compatible with the right G-action and the coboundary operator. Since $C(X)$ has a $\mathbb{Z}$-basis with one element per cell σ of X, and these basis elements are permuted freely by G (by assumption) and fall into finitely many orbits, then $\mathrm{Hom}_C(C(X), \mathbb{Z})$ consists of those cochains $f \in \mathrm{Hom}(C(X), \mathbb{Z})$ such that $f(\sigma) = 0$ for all but finitely many σ. (For orbit representatives $\sigma_1, \ldots, \sigma_n$, we know $f(g \cdot \sigma_i) = 0$ for all but finitely many $g \in G$.) We have established:

Prop 2.141. *If X is a contractible, free G-complex with compact quotient X/G, then there exists an isomorphisms*

$$H^*(G; \mathbb{Z}G) \cong H^*_C(X; \mathbb{Z})$$

of right G-modules, where the right action of G on $H^*_C(X; \mathbb{Z})$ is induced by the left action of G on X.

Remark 2.142. With $\mathbb{Q}$-coefficients, we do not need G to act freely on a contractible space X to make the same conclusion as in prop 2.141, we just need G to act on X properly-discontinuously (implies with finite stabilizers) and cocompactly. We then have

$$H^*(G; \mathbb{Q}G) \cong H^*_C(X; \mathbb{Q}).$$

2.7 Duality groups

2.7.1 Bieri-Eckmann

Generalization of Poincaré duality in the context of group cohomology.

Definition 2.143 (Bieri-Eckmann duality groups)**.** A group G of type FP is called a (Bieri-Eckmann) duality group if there exists an integer n and a $\mathbb{Z}G$-module D such that

$$H^i(G;M) \cong H_{n-i}(G;D\otimes_{\mathbb{Z}} M)$$

for all $\mathbb{Z}G$-modules M and integers i. Here G acts diagonally on $D\otimes_{\mathbb{Z}} M$.

In fact, if G is a duality group, then $n = \mathrm{cd}(G)$ and $D \cong H^n(G;\mathbb{Z}G)$. (Why? $H^n(G;\mathbb{Z}G) \cong H_0(G;D\otimes_{\mathbb{Z}}\mathbb{Z}G) = (D\otimes_{\mathbb{Z}}\mathbb{Z}G)_G \cong D\otimes_{\mathbb{Z}G}\mathbb{Z}G \cong D$.)

Theorem 2.144 (Characterization of Bieri-Eckmann duality)**.** *Let G be a group of type* FP*. The following are equivalent:*

(i) G is a duality group, that is, there exists $n\in\mathbb{Z}$ and a $\mathbb{Z}G$-module D such that $H^i(G;M)\cong H_{n-i}(G;D\otimes M)$ for all $\mathbb{Z}G$-modules M and degrees i.

(ii) There exists n such that $H^i(G;\mathbb{Z}G\otimes A) = 0$ for all $i\neq n$ and all abelian groups A.

(iii) There exists n such that $H^i(G;\mathbb{Z}G) = 0$ for all $i\neq n$ and $H^n(G;\mathbb{Z}G)$ is a torsion-free abelian group.

(iv) There exist natural isomorphisms $H^i(G;-)\cong H_{n-i}(G;D\otimes -)$ where $n=\mathrm{cd}(G)$ and $D = H^n(G;\mathbb{Z}G)$, which are compatible with the connecting homomorphisms in homology and cohomology associated to a short exact sequence of modules.

Proof. $(i) \implies (ii)$: Suppose (i) holds. Let A be an abelian group and let $M = \mathbb{Z}G\otimes A$. By (i), we know that $H^i(G;\mathbb{Z}G\otimes A)\cong H_{n-i}(G;D\otimes\mathbb{Z}G\otimes A)$ for all i. It suffices to show that $H_*(G;D\otimes\mathbb{Z}G\otimes A)$ for $*>0$. Since $D\otimes\mathbb{Z}G\otimes A\cong\mathbb{Z}G\otimes(D\otimes A)$, consider $(D\otimes A)_0$ as an abelian group (forgetting G-module structure), and recall that by corollary 2.52, the induced module $\mathbb{Z}G\otimes(D\otimes A)_0$ is canonically isomorphic to $\mathbb{Z}G\otimes(D\otimes A)$. Since induced modules are H_* acyclic, the claim holds.

$(ii) \implies (iii)$: Suppose (ii) holds. Let $A=\mathbb{Z}$. Then there exists n such that $H^i(G;\mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}) = H^i(G;\mathbb{Z}G) = 0$ for all $i\neq n$. It remains to show that $H^n(G;\mathbb{Z}G)$ is torsion-free abelian. For this we show that no elements of $H^n(G;\mathbb{Z}G)$ have order m for any $m\geqslant 2$. We have a short exact sequence

$$0\to\mathbb{Z}\xrightarrow{\cdot m}\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}\to 0.$$

Since $\mathbb{Z}G$ is a free abelian group, tensoring with $\mathbb{Z}G$ over $\mathbb{Z}$ is exact, and so we have a short exact sequence

$$0\to\mathbb{Z}G\xrightarrow{\cdot m}\mathbb{Z}G\to\mathbb{Z}G\otimes_{\mathbb{Z}}\mathbb{Z}/m\to 0.$$

This short exact sequence of coefficient modules induces a long exact sequence

$$\cdots\to H^{n-1}(G;\mathbb{Z}G\otimes\mathbb{Z}/m)\to H^n(G;\mathbb{Z}G)\xrightarrow{\cdot m} H^n(G;\mathbb{Z}G)\to H^n(G;\mathbb{Z}G\otimes\mathbb{Z}/m)\to\cdots$$

and since $H^{n-1}(G;\mathbb{Z}G\otimes\mathbb{Z}/m)=0$ by assumption ($A=\mathbb{Z}/m$), the multiplication by m map on cohomology is injective. Since this holds for all $m\geqslant 2$, we conclude that $H^n(G;\mathbb{Z}G)$ is torsion-free.

$(iii) \implies (iv)$: Note that the integer n in (iii) must be $\mathrm{cd}(G)$ since we have previously shown that $H^{\mathrm{cd}(G)}(G;\mathbb{Z}G) \neq 0$ for a group of type FP. We know there exists a finite projective resolution $P = (P_i)_{0\leqslant i\leqslant n}$ of finite type of $\mathbb{Z}$ over $\mathbb{Z}G$. Consider the dual complex $P^* = \mathrm{Hom}_{\mathbb{Z}G}(P,\mathbb{Z}G)$. Since $H^i(G;\mathbb{Z}G) = 0$ for $i \neq n$, then P^* is exact except at P_n^*. So we actually have a projective resolution

$$0 \to P_0^* \to \cdots \to P_n^* \to H^n(G;\mathbb{Z}G) \to 0.$$

(Each P_i^* is a projective $\mathbb{Z}G$-module: Since P_i is projective and finitely generated, there exists a free $\mathbb{Z}G$-module F of finite rank and a projective $\mathbb{Z}G$-module Q_i with $P_i \oplus Q_i = F$. Then $\mathrm{Hom}_{\mathbb{Z}G}(F,\mathbb{Z}G) \cong \mathrm{Hom}_{\mathbb{Z}G}(P_i,\mathbb{Z}G) \oplus \mathrm{Hom}_{\mathbb{Z}G}(Q_i,\mathbb{Z}G)$, and $\mathrm{Hom}_{\mathbb{Z}G}(F,\mathbb{Z}G)$ is free since $\mathrm{Hom}_{\mathbb{Z}G}(-,-)$ is additive.) Tensoring the truncated cochain complex with a $\mathbb{Z}G$-module A over $\mathbb{Z}G$ yields a cochain complex

$$0 \to P_0^* \otimes_{\mathbb{Z}G} A \to P_1^* \otimes_{\mathbb{Z}G} A \to \cdots \to P_n^* \otimes_{\mathbb{Z}G} A \to 0.$$

By 2.135, $P_i^* \otimes_{\mathbb{Z}G} A = \mathrm{Hom}_{\mathbb{Z}G}(P_i,\mathbb{Z}G) \otimes_{\mathbb{Z}G} A \cong \mathrm{Hom}_{\mathbb{Z}G}(P_i,A)$. Hence the cochain complex above becomes

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(P_0,A) \to \cdots \to \mathrm{Hom}_{\mathbb{Z}G}(P_n,A) \to 0.$$

Note the opposite indexing. Taking homology yields

$$H^i(G;A) \cong \mathrm{Tor}_{n-i}^{\mathbb{Z}G}(H^n(G;\mathbb{Z}G),A) \cong H_{n-i}(G;H^n(G;\mathbb{Z}G) \otimes A)$$

where the last isomorphism follows from 2.40.

$(iv) \implies (i)$: Clear. □

Definition 2.145 (Duality groups over R)**.** A group G of type FP is called a duality group over R if there exists an $n \in \mathbb{Z}$ and a right $R[G]$-module D such that $H^i(G;M) \cong H_{n-i}(G;D \otimes_R M)$ for all $R[G]$-modules M and all integers i. Here G acts diagonally on $D \otimes_R M$.

Prop 2.146 (Duality groups over $\mathbb{Z}$ are duality groups over R)**.** *If G is a duality group over $\mathbb{Z}$ with dualizing module D, then it is a duality group over R with dualizing module $D \otimes R$.*

Proof. Suppose G is of type FP, $n = \mathrm{cd}(G)$, and D is its dualizing module. Then

$$H^i(G;M) \cong H_{n-i}(G;D \otimes M) \cong H_{n-i}(G;D \otimes R \otimes_R M)$$

since $D \otimes M$ and $D \otimes R \otimes_R M$ are isomorphic as $\mathbb{Z}G$-modules. □

Lemma 2.147. *If G is of type* FP *and F is a flat $R[G]$-module, then $H^*(G;F) \cong H^*(G;R[G]) \otimes_{R[G]} F$.*

Proof. By the same reason as in prop 2.136. □

Prop 2.148. *If G is a duality group over a ring R with* $\mathrm{cd}(G) = n$*, its dualizing module is $D = H^n(G;R[G])$ (a right $R[G]$-module).*

Proof. By definition, we have that there exists some n and a right $R[G]$-module D such that

$$H^n(G;M) \cong H_0(G;D \otimes_R M)$$

for all $R[G]$-modules M. In particular, taking $M = R[G]$ gives

$$\begin{aligned} H^n(G;R[G]) \cong H_0(G;D \otimes_R R[G]) \cong (D \otimes_R R[G])_G &\cong D \otimes_R R \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \\ &\cong D \otimes_{\mathbb{Z}} \mathbb{Z} \\ &\cong D. \end{aligned}$$

Prop 2.149 (Dualizing module is flat ([1] 9.2)). *Let G be a duality group over R, with $n = \mathrm{cd}_R(G)$ and $D = H^n(G; R[G])$. Then D is a flat R-module.*

Proof. Let $0 \to L \to L' \to L'' \to 0$ be a short exact sequence of $R[G]$-modules. Since $R[G]$ is R-free, we have that $0 \to L \otimes_R R[G] \to L' \otimes_R R[G] \to L'' \otimes_R R[G] \to 0$ is exact. We have a long exact sequence in cohomology

$$\cdots \to H^{n-1}(G; L'' \otimes_R R[G]) \to H^n(G; L \otimes_R R[G]) \to H^n(G; L' \otimes_R R[G]) \to H^n(G; L'' \otimes_R R[G]) \to \cdots$$

Since $\mathrm{cd}_R(G) = n$, then $H^i(G; M) = 0$ for $i > n$ and all $R[G]$-modules M. Since G is a duality group over R, then $H^i(G; A) \cong H_{n-i}(G; D \otimes_R A)$ for all i and $R[G]$-modules A. In particular, when $i \geqslant 1$ and $A = L \otimes_R R[G]$ is an induced module, we have by Shapiro's lemma 2.57 that $H_{n-i}(G; D \otimes_R L \otimes_R R[G]) \cong H_{n-i}(\{e\}; D \otimes_R L) = 0$. Hence the above long exact sequence becomes

$$0 \to H^n(G; L \otimes_R R[G]) \to H^n(G; L' \otimes_R R[G]) \to H^n(G; L'' \otimes_R R[G]) \to 0.$$

By 2.133, we have natural isomorphisms (the vertical maps)

$$\begin{tikzcd}
H^n(G; L \otimes_R R[G]) \arrow[r] & H^n(G; L' \otimes_R R[G]) \arrow[r] & H^n(G; L'' \otimes_R R[G]) \\
H^n(G; R[G]) \otimes_{R[G]} (R[G] \otimes_R L) \arrow[r] \arrow[u, "\cong"] & H^n(G; R[G]) \otimes_{R[G]} (R[G] \otimes_R L') \arrow[r] \arrow[u, "\cong"] & H^n(G; R[G]) \otimes_{R[G]} (R[G] \otimes_R L'') \arrow[u, "\cong"] \\
H^n(G; R[G]) \otimes_R L \arrow[u, "\cong"] & H^n(G; R[G]) \otimes_R L' \arrow[u, "\cong"] & H^n(G; R[G]) \otimes_R L'' \arrow[u, "\cong"]
\end{tikzcd}$$

which proves that $H^n(G; R[G])$ is a flat R-module. □

□

Prop 2.150 (Dualizing modules are of type FP over $\mathbb{Z}G$). *If G is an n-dimensional duality group with dualizing module $D = H^n(G; \mathbb{Z}G)$, then D is a $\mathbb{Z}G$-module of type* FP *(admits finite projective resolution) and of projective dimension n.*

Proof. Follows from the proof of $(iii) \implies (iv)$ in 2.144. We have a projective resolution

$$0 \to P_0^* \to \cdots \to P_n^* \to H^n(G; \mathbb{Z}G) \to 0$$

of finitely generated $\mathbb{Z}G$-modules. □

2.7.2 Virtual duality groups

Prop 2.151. *Let G be a torsion-free group and H a finite-index subgroup.*

1. *G is a duality group if and only if H is.*
2. *If H and G are duality groups, they have the same dualizing module (under restriction of scalars from $\mathbb{Z}G$ to $\mathbb{Z}H$).*

Proof. Suppose G is a duality group. Then G has type FP implies H has type FP (prop 2.91, 2.116, and 2.119). In fact, by 2.91, we know that $\mathrm{cd}(G) = \mathrm{cd}(H) = n$. Since G is a duality group, its dualizing module is $D = H^n(G; \mathbb{Z}G)$. By our equivalent conditions for Bieri-Eckmann duality 2.144, it suffices to show that $H^i(H; \mathbb{Z}H) = 0$ for $i \neq n$ and that $H^n(H; \mathbb{Z}H)$ is torsion-free abelian. By Shapiro's lemma 2.57, we have

$H^i(H;\mathbb{Z}H) \cong H^i(G;\mathrm{Coind}_H^G \mathbb{Z}H) \cong H^i(G;\mathbb{Z}G) = 0$ for $i \neq n$, and is torsion-free abelian for $i = n$.

If H has type FP, then by Serre's theorem (using the fact that G is a torsion-free group) 2.102, we obtain $\mathrm{cd}(G) = \mathrm{cd}(H) < \infty$. We also know that H is of type FP_n if and only if G is, so G must also be of type FP. The same argument as above yields that G is a duality group, and furthermore, since $H^n(G;\mathbb{Z}G) \cong H^n(H;\mathbb{Z}H)$ as $\mathbb{Z}H$-modules (this follows from the proof of Shapiro's lemma: $H^n(G;\mathbb{Z}G) = H^n(G;\mathrm{Ind}_H^G \mathbb{Z}H) \cong H^n(G;\mathrm{Coind}_H^G \mathbb{Z}H) \cong H^n(H;\mathbb{Z}H)$ (since $[G : H] < \infty$), they must have the same dualizing module. ◻

Definition 2.152 (Virtual duality group)**.** A group G is a virtual duality group if some finite-index subgroup is a duality group.

Prop 2.153. *A group G is a virtual duality group if and only if the following are both satisfied:*

(a) G has a finite-index subgroup of type FP.

(b) There is an integer n such that $H^i(G;\mathbb{Z}G) = 0$ for all $i \neq n$ and $H^n(G;\mathbb{Z}G)$ is torsion-free abelian.

In this case, every torsion-free subgroup of finite-index is a duality group with dualizing module $H^n(G;\mathbb{Z}G)$.

Proof. Suppose G is a virtual duality group. Then G has some finite-index subgroup H which is a duality group, so H has type FP and (a) is satisfied. By Shapiro's lemma, (b) is satisfied (since it's satisfied for $H^i(H;\mathbb{Z}H)$, by virtue of H being a duality group).

On the other hand, suppose (a) and (b) are satisfied. Again Shapiro's lemma implies $H^i(H;\mathbb{Z}H) = 0$ for $i \neq n$ and $H^n(H;\mathbb{Z}H)$ is torsion-free abelian. Combined with the fact that H is of type FP, we see that H is a duality group, and hence G is a virtual duality group.

Suppose K is a torsion-free subgroup of finite-index, and G is a virtual duality group with $H \leqslant G$ a finite-index subgroup of type FP. We wish to show that K is a duality group (has type FP and there exists an integer n such that $H^n(K;\mathbb{Z}K)$ is torsion-free abelian, $H^i(K;\mathbb{Z}K) = 0$ for $i \neq n$). Firstly, we have that $\mathrm{cd}(K) = \mathrm{cd}(H) = n = \mathrm{vcd}(G)$: Since $H \cap K$ is a finite-index subgroup of K ($H \cap K$ is a finite-index subgroup of G, given that the intersection of finite-index subgroups is finite-index), and since K is torsion-free (and so is H, necessarily), then by Serre's theorem we know that $\mathrm{cd}(K) = \mathrm{cd}(H \cap K) = \mathrm{cd}(H)$. Now H has type FP implies that $H \cap K$ has type FP, which implies that K has type FP since $H \cap K$ is a finite-index subgroup of K 2.116. By Shapiro's lemma, we know that K is a duality group with dualizing module $H^{\mathrm{vcd}(G)}(G;\mathbb{Z}G)$. ◻

Prop 2.154. *Let G be a virtual duality group. Then G is a duality group over $\mathbb{Q}$. In particular, for every i and every $\mathbb{Q}G$-module V,*

$$H^{\mathrm{vcd}(G)-i}(G;V) \cong H_i(G;D \otimes_{\mathbb{Q}} V)$$

where $D \cong H^{\mathrm{vcd}(G)}(G;\mathbb{Q}G) = H^{\mathrm{vcd}(G)}(G;\mathbb{Z}G) \otimes \mathbb{Q}$ is the common rational dualizing module of G and its finite index subgroups.

Proof. Since G is a virtual duality group, there exists a finite-index subgroup H such that H is a duality group. Let $n = \mathrm{cd}(H) = \mathrm{vcd}(G)$. Since H is a duality group, we know that $H^{\mathrm{vcd}(G)}(H;\mathbb{Z}H) \cong H^{\mathrm{vcd}(G)}(G;\mathbb{Z}G)$ is torsion-free abelian and $H^i(H;\mathbb{Z}H) \cong H^i(G;\mathbb{Z}G) = 0$ for $i \neq \mathrm{vcd}(G)$. We also know that H is a duality group over $\mathbb{Q}$ with dualizing module $H^{\mathrm{cd}_{\mathbb{Q}}(H)}(H;\mathbb{Q}H)$ by prop 2.148 and dualizing module $H^{\mathrm{vcd}(G)}(H;\mathbb{Z}H) \otimes \mathbb{Q}$ by 2.146, so

$$H^{\mathrm{cd}_{\mathbb{Q}}(H)}(H;\mathbb{Q}H) \cong H^{\mathrm{vcd}(G)}(H;\mathbb{Z}H) \otimes \mathbb{Q}.$$

In particular this means that $\mathrm{cd}_{\mathbb{Q}}(H) = \mathrm{vcd}(G)$. To see that G is a duality group over $\mathbb{Q}$, it suffices to show that G is of type FP over $\mathbb{Q}$, and that $H^{\mathrm{cd}_{\mathbb{Q}}(G)}(G; \mathbb{Q}G)$ is flat as a $\mathbb{Q}$-module (which it must be, since it is free), and $H^i(G; \mathbb{Q}G) = 0$ for $i \neq \mathrm{cd}_{\mathbb{Q}}(G)$. Note that since $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{cd}_{\mathbb{Q}}(H)$ by 2.110, then $\mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{vcd}(G)$, and Shapiro's lemma gives

$$H^{\mathrm{vcd}(G)}(G; \mathbb{Q}G) \cong H^{\mathrm{vcd}(G)}(H; \mathbb{Q}H)$$

is flat. Since we also know that H is of type FP_n over $\mathbb{Q}$ if and only if G is (slight modification of 2.116), then G has type FP over $\mathbb{Q}$. Finally, Shapiro's lemma again shows that $H^i(G; \mathbb{Q}G) = 0$ for $i \neq \mathrm{cd}_{\mathbb{Q}}(G) = \mathrm{vcd}(G)$, so we are done. □

2.7.3 Poincaré duality groups

If G is a group which has a $K(G, 1)$ space equal to a closed orientable manifold, we can see that Bieri-Eckmann duality achieves the same result as Poincaré duality. For example:

Example 2.155 (Bieri-Eckmann duality for $G = \mathbb{Z}$)**.** When $G = \mathbb{Z}$, we can take a $K(G, 1)$ to be S^1 (a closed (compact + no boundary) orientable 1-manifold). Recall that we have a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$ via

$$0 \to \mathbb{Z}[t, t^{-1}] \xrightarrow{\cdot(t-1)} \mathbb{Z}[t, t^{-1}] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

In particular this shows that $G = \mathbb{Z}$ is of type FP (and $\mathrm{cd}(G) \leqslant 1$. Since $\mathbb{Z}$ is nontrivial we know that $\mathrm{cd}(G) = 1$). Truncating and applying $\mathrm{Hom}_{\mathbb{Z}[t,t^{-1}]}(-, \mathbb{Z}[t, t^{-1}])$ yields a cochain complex

$$0 \to \mathbb{Z}[t, t^{-1}] \xrightarrow{\cdot(t-1)} \mathbb{Z}[t, t^{-1}] \to 0$$

We need to compute $D = H^1(\mathbb{Z}; \mathbb{Z}[t, t^{-1}]) = \mathbb{Z}[t, t^{-1}]/(t-1)\mathbb{Z}[t, t^{-1}]$. Quotienting out by $(t-1)$ has the effect of killing off t, so $H^1(\mathbb{Z}; \mathbb{Z}[t, t^{-1}]) \cong \mathbb{Z}$, which is torsion-free abelian. By (iii) of the characterization of Bieri-Eckmann duality groups in theorem 2.144, it remains to show that $H^*(\mathbb{Z}; \mathbb{Z}[t, t^{-1}]) = 0$ for $* \neq 1$. Certainly this is the case for $* > 1$. Now $H^0(\mathbb{Z}; \mathbb{Z}[t, t^{-1}]) = \ker(\cdot(t-1)) = \{p(t) \in \mathbb{Z}[t, t^{-1}] : t \cdot p(t) = p(t)\} = 0$, as desired.

By theorem 2.144, we now have that $H^i(\mathbb{Z}; M) \cong H_{1-i}(\mathbb{Z}; D \otimes M) = H_{1-i}(\mathbb{Z}; M)$ for all i and all $\mathbb{Z}[t, t^{-1}]$-modules M. Let's reconcile this with classical Poincaré duality: Since S^1 is a $K(\mathbb{Z}, 1)$, we know

$$H^*(\mathbb{Z}; M) = H^*(S^1; M).$$

Recall Poincaré duality: If M is a closed R-orientable manifold of dimension n with fundamental class $[M] \in H_n(M; R)$ (an element whose image in $H_n(M, M \backslash \{x\}; R)$ is a generator for all $x \in M$), then the cap product

$$\begin{aligned} D_M : H_c^k(M; R) &\to H_{n-k}(M; R) \\ \alpha &\mapsto [M] \cap \alpha \end{aligned}$$

is an isomorphism for all k. (Recall how this is defined: View $[M]$ as a singular simplex $\Delta^n \to M$. View α as an element of $\mathrm{Hom}(C_k(M), R)$. Then $[M] \cap \alpha : \Delta^{n-k} \to M$ is the map $\alpha([M]|_{[v_0, \ldots, v_k]}) \cdot [M]_{[v_k, \ldots, v_n]}$, since $\alpha([M]|_{[v_0, \ldots, v_k]}) \in R$.) In our case, we have that $H^k(S^1; R) \cong H_{1-k}(S^1; R)$ for all R, as desired.

Definition 2.156 (Poincaré duality group)**.** Say G is a Poincaré duality group of dimension n if the dualizing module $D = H^n(G; \mathbb{Z}G)$ is infinite cyclic. If G acts trivially on D we call G orientable, and nonorientable otherwise.

Remark 2.157. If G is a Poincaré duality group, then $D \cong \mathbb{Z}$, so Bieri-Eckmann duality gives $H^i(G; M) \cong H_{n-i}(G; M)$ for all i and $\mathbb{Z}G$-modules M (like classical Poincaré duality).

Remark 2.158. If G has a $K(G, 1)$ which is a closed orientable n-manifold, then G is a Poincaré duality group. However, it is not known whether every finitely presented Poincaré duality group has a $K(G, 1)$ which is a closed orientable manifold. Note that Poincaré duality groups need not be finitely presented, but the fundamental groups of closed manifolds must be. WHY?

Example 2.159 (n-torus). $G = \mathbb{Z}^n$ is a Poincaré duality group, since the n-fold torus $Y = S^1 \times \cdots \times S^1$ has fundamental group $\mathbb{Z}^n$ and is a closed orientable n-manifold. We can also compute directly that $H^*(\mathbb{Z}^n; \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]) \cong \mathbb{Z}$ when $* = n$ and is 0 otherwise.

Example 2.160 (f.g. free groups are duality groups). If G is a free group on k generators ($1 \leqslant k < \infty$), then G is a 1-dimensional duality group. We know that a 1-dimensional $K(G, 1)$ exists by taking a wedge of k circles, so $\operatorname{cd}(G) = 1$. Certainly $H^i(G; \mathbb{Z}G \otimes A) = 0$ for all $i \geqslant 2$ and all abelian groups A, so it remains to show that $H^0(G; \mathbb{Z}G \otimes A) = 0$. Recall that $H^0(G; \mathbb{Z}G \otimes A) \cong (\mathbb{Z}G \otimes A)^G = (\operatorname{Ind}_{\{e\}}^G(A))^G$. We claim that this must be 0 by the following lemma:

Lemma 2.161 (Induced invariants vanish). *If $[G : H]$ is infinite, then $(\operatorname{Ind}_H^G M)^G = 0$ for all H-modules M.*

Proof. Write $\operatorname{Ind}_H^G M \cong \bigoplus_{\sigma \in G/H} \sigma M$, and let $x = \sum_{i=1}^k \sigma_i \otimes m_i \in \operatorname{Ind}_H^G M$. Since $[G : H] = \infty$, there exists $g \in G$ such that $g \notin \sigma_1 H, \ldots, \sigma_k H$. But then for $\tilde{g} = g \cdot \sigma_1^{-1} \in G$, we have $\tilde{g} \cdot x = \sum_{i=1}^k g \cdot \sigma_1^{-1} \sigma_i \otimes m_i = g \otimes m_1 + \sum_{i=1}^2 g \cdot \sigma_1^{-1} \sigma_i \otimes m_i$. This cannot equal x, since the summand $g \otimes m_1$ was not previously a summand of x (given that $g \notin \sigma_1 H, \ldots, \sigma_k H$). Since for every $x \in \operatorname{Ind}_H^G M$ we can find such a g, we conclude that $(\operatorname{Ind}_H^G M)^G = 0$. □

In particular, when $G = F(k)$, we have that $(\operatorname{Ind}_e^G(A))^G = 0$ for every abelian group A (as $|G| = \infty$). Hence $H^0(G; \mathbb{Z}G \otimes A) = 0$, and we conclude that free groups on k generators are duality groups.

Unless $k = 1$, the group $G = F(k)$ is *not* a Poincaré duality group. If $D = H^1(G; \mathbb{Z}G)$ were infinite cyclic, then Bieri-Eckmann duality would yield

$$H^1(F(k); \mathbb{Z}) \cong H_0(F(k); D) = H_0(F(k); \mathbb{Z}) \cong \mathbb{Z}$$

since $\bigvee_{i=1}^k S^1$ is a $K(F(k), 1)$. But $H^1(\bigvee_{i=1}^k S^1; \mathbb{Z}) \cong \prod_{i=1}^k H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}^k$, which clearly does not equal $\mathbb{Z}$ for $k > 1$.

Alternatively, when $G = F(2)$ we have (by prop 2.137) a surjection

$$H^1(F(2); \mathbb{Z}[F(2)]) \twoheadrightarrow \operatorname{Ind}_{F(1)}^{F(2)} H^1(F(1); \mathbb{Z}[F(1)]) \oplus \operatorname{Ind}_{F(1)}^{F(2)} H^1(F(1); \mathbb{Z}[F(1)]) \cong \bigoplus_{\sigma \in \mathbb{Z} * \mathbb{Z}/\mathbb{Z}} \mathbb{Z} \oplus \bigoplus_{\sigma \in \mathbb{Z} * \mathbb{Z}/\mathbb{Z}} \mathbb{Z}$$

which proves that $H^1(F(2); \mathbb{Z}[F(2)])$ is not infinite cyclic (in fact, it ends up being a free abelian group of countable rank). By a repeated repeated application of prop 2.137, we obtain that $H^1(F(k); \mathbb{Z}[F(k)])$ is free abelian of countable rank for $k \geqslant 2$. Hence $F(k)$ is *not* a Poincaré duality group.

Prop 2.162 (Ex VIII.10.6 in [2]). *If Γ is a Poincaré duality group and $\Gamma' \subset \Gamma$ is a subgroup of infinite index, then* $\operatorname{cd} \Gamma' < \operatorname{cd} \Gamma$.

(Note: if it turns out that all Poincaré duality groups appear as the fundamental groups of $K(G, 1)$-manifolds, then this exercise is the same as 2.95. But this is an open question!)

Proof. Let $n = \operatorname{cd}(\Gamma)$. Since Γ is a Poincaré duality group, then $H^n(\Gamma; \mathbb{Z}\Gamma) \cong \mathbb{Z}$ is finitely generated as an abelian group. The result follows by 2.129 □

2.8 Homological stability

Main reference: [9].

Theorem 2.163 (Quillen's approach to homological stability). *Let $\{G_n\}$ be a family of groups, $n \geqslant 1$, with inclusions $\varphi_n : G_n \to G_{n+1}$. For each n let $(Y_n)_\bullet$ be a semisimplicial set with a G_n-action, satisfying*

(i) The spaces $||(Y_n)_\bullet||$ are $\frac{n-2}{k}$-connected for some $k \geqslant 2$.

(ii) For each p, G_n acts transitively on the set of p-simplices $(Y_n)_p$.

(iii) $\operatorname{stab}(\sigma_p)$ of a p-simplex σ_p is isomorphic to G_{n-p-1}, and is conjugate to the distinguished copy $G_{n-p-1} \subseteq G_n$ determined by the inclusion maps φ_i.

(iv) $\operatorname{stab}(\sigma_p)$ fixes σ_p pointwise for each simplex σ_p, not just setwise.

(v) The inclusion of a face $\tau_q \hookrightarrow \sigma_p$ induces an inclusion on groups $G_{n-p-1} \hookrightarrow G_{n-q-1}$ (via their stabilizers) conjugate to the distinguished inclusion.

Then, given $i \in \mathbb{N}$, the inclusion $G_n \xhookrightarrow{\varphi_n} G_{n+1}$ induces isomorphisms on homology $H_i(G_n) \cong H_i(G_{n+1})$ for $n \geqslant ki+1$, and induces an epimorphism for $n \geqslant ki$.

Proof. Step 1: Computing the homology of a semisimplicial set. Given a semisimplicial set

$$X_\bullet : \Delta_+^{\mathrm{op}} \to \underline{\mathrm{Set}}$$

the homology groups of its geometric realization $||X_\bullet||$ are calculated as the homology of the chain complex

$$\cdots \to \mathbb{Z}X_{p+1} \xrightarrow{\partial} \mathbb{Z}X_p \xrightarrow{\partial} \mathbb{Z}X_{p-1} \to \cdots$$

where $X_p = X([p]_0)$ and the differential is given as the alternating sum of face maps

$$\partial = \sum_{i=0}^{p} (-1)^i d_i : \mathbb{Z}X_p \to \mathbb{Z}X_{p-1}$$

Step 2: Consider the double complex

$$E_\bullet G_{n+1} \otimes_{G_{n+1}} \tilde{C}_*(Y_{n+1})$$

where $\tilde{C}_*(Y_n)$ is the augmented cellular chain complex of Y_n, and $E_\bullet G_n$ is a free $\mathbb{Z}G_n$-module resolution of $\mathbb{Z}$. Since $E_\bullet G_{n+1} \otimes_{G_{n+1}} \tilde{C}_*(Y_{n+1})$ is the total complex of abelian groups $(E_p G_{n+1} \otimes_{G_{n+1}} \tilde{C}_q(Y_{n+1}))$, we have two spectral sequences associated to this double complex 1.5.1 which both converge to $H(E_\bullet G_{n+1} \otimes_{G_{n+1}} \tilde{C}_*(Y_{n+1}))$.

On the one hand, keeping p fixed and taking the homology with respect to $\tilde{C}_*(Y_n)$ first yields $E^1_{p,q} = H_q(E_p G_n \otimes_{G_n} \tilde{C}_*(Y_n))$. Since $E_p G_n$ is a free $\mathbb{Z}G_n$-module, tensoring with $E_p G_n$ over G_n commutes with homology to yield

$$E^1_{p,q} = E_p G_{n+1} \otimes_{G_{n+1}} H_q(\tilde{C}_*(Y_{n+1})) \cong E_p G_{n+1} \otimes_{G_{n+1}} H_q(Y_{n+1}).$$

But $||(Y_{n+1})_\bullet||$ is $\frac{n-1}{k}$-connected (and thus $\frac{n-1}{k}$-homologically connected), so $H_q(Y_{n+1}) = H_q(||(Y_{n+1})_\bullet||) = 0$ for $q \leqslant \frac{n-1}{k}$, and the homology of the total complex of the double complex must be 0 in degrees $* \leqslant \frac{n-1}{k} - 1$

(as $p \geqslant -1$, and we'll get $H_{p+q}(\mathrm{Tot}) = 0$ when $q \leqslant \frac{n-1}{k}$, and thus when $p + q \leqslant \frac{n-1}{k} - 1$). On the other hand, keeping p fixed and computing the homology with respect to $E_\bullet G_{n+1}$ gives

$$E^1_{p,q} = H_q(E_\bullet G_{n+1} \otimes_{G_{n+1}} \tilde{C}_p(Y_{n+1})) = H_q(G_{n+1}; \tilde{C}_p(Y_{n+1})).$$

Recall: we assumed that for each p, G_{n+1} acts transitively on the set of p-simplices $(Y_{n+1})_p$, and that $\mathrm{stab}_{G_{n+1}}(\sigma_p)$ of a *distinguished* p-simplex σ_p is isomorphic to G_{n-p}. As an abelian group, we have $\tilde{C}_p(Y_{n+1}) \cong \bigoplus_{\sigma_p \in (Y_{n+1})_p} \mathbb{Z}$, while as a G_{n+1}-module, we have (by theorem 2.45, and letting $\mathrm{stab}_{G_{n+1}}(\sigma_p)$ be the stabilizer of a p-simplex σ_p),

$$\tilde{C}_p(Y_{n+1}) \cong \mathrm{Ind}^{G_{n+1}}_{\mathrm{stab}_{G_{n+1}}(\sigma_p)} \mathbb{Z}.$$

By assumption, $\mathrm{stab}_{G_{n+1}}(\sigma_p) \cong G_{n-p}$, so $\tilde{C}_p(Y_{n+1}) \cong \mathrm{Ind}^{G_{n+1}}_{G_{n-p}} \mathbb{Z}$. Shapiro's lemma 2.57 then yields

$$E^1_{p,q} = H_q(G_{n+1}; \mathrm{Ind}^{G_{n+1}}_{G_{n-p}} \mathbb{Z}) \cong H_q(G_{n-p}; \mathbb{Z}) \qquad -1 \leqslant p \leqslant n.$$

Recall, again, that we know this spectral sequence converges to 0 in degrees $p + q \leqslant \frac{n-1}{k} - 1$.

Step 3: We need to understand the d^1 differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$. Since we obtained $E^1_{p,q} = H_q(G_{n-p}; \mathbb{Z})$ by taking the homology of our double complex with respect to $E_\bullet G_{n+1}$, then the d^1 differential is the one induced by the differential in $\tilde{C}_*(Y_{n+1})$. By step 1, we know that this differential is given by the alternating sum of the $d_i : (Y_{n+1})_p \to (Y_{n+1})_{p-1}$. Since Shapiro's lemma 2.57 gives natural isomorphisms, we have a diagram

$$\begin{tikzcd}
H_q(G_{n+1}; \tilde{C}_p(Y_{n+1})) \arrow[r, "\sum_{i=0}^p (-1)^i (d_i)_*"] \arrow[d, "\cong"'] & H_q(G_{n+1}; \tilde{C}_{p-1}(Y_{n+1})) \arrow[d, "\cong"] \\
H_q(\mathrm{stab}_{G_{n+1}}(\sigma_p)) \arrow[r, "?"] & H_q(\mathrm{stab}_{G_{n+1}}(\sigma_{p-1}))
\end{tikzcd}$$

How do we actually compute the bottom differential? Recall that we chose a distinguished p-simplex σ_p for each p so that the inclusion of the stabilizer

$$\mathrm{stab}_{G_{n+1}}(\sigma_p) \cong G_{n-p} \hookrightarrow G_{n+1}$$

is the distinguished inclusion given by the composition of stabilization maps

$$G_{n-p} \hookrightarrow G_{n-p+1} \hookrightarrow \cdots \hookrightarrow G_{n+1}.$$

Although $d_i\sigma_p$ is a $(p-1)$-simplex for each i, it is *not* true that $d_i\sigma_p$ will be the distinguished $(p-1)$-simplex σ_{p-1}! However, because G_{n+1} acts transitively on the p-simplices of Y_{n+1}, there does exist (many) $h_i \in G_{n+1}$ such that $h_i d_i \sigma_p = \sigma_{p-1}$. Furthermore, one can choose such an h_i so that h_i centralizes (commutes with the elements of) $\mathrm{stab}_{G_{n+1}}(\sigma_p)$. Details of this are in [9]. However, in the application to the symmetric groups Σ_n, it will be clear that this is possible. Notice: conjugation by h_i yields a map

$$\begin{aligned} c_{h_i} : \mathrm{stab}_{G_{n+1}}(d_i\sigma_p) &\to \mathrm{stab}_{G_{n+1}}(\sigma_{p-1}) \\ g &\mapsto h_i g h_i^{-1} \end{aligned}$$

for if $g d_i \sigma_p = d_i \sigma_p$, then $h_i g h_i^{-1} \sigma_{p-1} = h_i g d_i \sigma_p = h_i d_i \sigma_p = \sigma_{p-1}$. We thus have a composite map

$$\mathrm{stab}_{G_{n+1}}(\sigma_p) \xrightarrow{\iota} \mathrm{stab}_{G_{n+1}}(d_i\sigma_p) \xrightarrow{c_{h_i}} \mathrm{stab}_{G_{n+1}}(\sigma_{p-1})$$

where the first map is the inclusion (if g stabilizes σ_p pointwise, it must stabilize the faces of σ_p pointwise). The composite of these maps then sends $g \mapsto h_i g h_i^{-1}$. But since $g \in \mathrm{stab}_{G_{n+1}}(\sigma_p)$, and h_i was chosen to commute with $\mathrm{stab}_{G_{n+1}}(\sigma_p)$, then $h_i g h_i^{-1} = g$, and the composite map above is just the identity. It follows that the induced map on homology

$$\begin{tikzcd}
H_q(\mathrm{stab}_{G_{n+1}}(\sigma_p)) \arrow[r, "(c_{h_i})_* \circ \iota_*"'] \arrow[d, "\cong"] & H_q(\mathrm{stab}_{G_{n+1}}(\sigma_{p-1})) \arrow[d] \\
H_q(G_{n-p}) \arrow[r] & H_q(G_{n-p+1})
\end{tikzcd}$$

is that induced by the distinguished inclusion (stabilization map) $G_{n-p} \xrightarrow{\iota_{n-p}} G_{n-p+1}$! Since this holds for each i, and we are taking an alternating sum $\sum_{i=0}^{p}(-1)^i (d_i)_*$, then we obtain

$$d^1 : E^1_{p,q} \to E^1_{p-1,q} = \begin{cases} \iota_{n-p} & p \text{ is even} \\ 0 & p \text{ is odd.} \end{cases}$$

In particular, when $p = 0$, we obtain

$$d^1 : E^1_{0,q} \to E^1_{-1,q} = \iota_n : H_q(G_n;\mathbb{Z}) \to H_q(G_{n+1};\mathbb{Z}).$$

Step 4: Let $r_n = \frac{n-1}{k}$, for $k \geqslant 2$. (Note: this approach to homological stability only gives isomorphisms in this slope 2 range, that is, this proof shows that we get isomorphisms $H_i(G_n) \to H_i(G_{n+1})$ induced by the stabilization maps when $n \geqslant ki + 1$, for $k \geqslant 2$. To potentially get improved slopes, other methods are needed...) We wish to show that the differential $d^1 : E^1_{0,i} = H_i(G_n;\mathbb{Z}) \to H_i(G_{n+1};\mathbb{Z}) = E^1_{-1,i}$ is an epimorphism for $n \geqslant ki$ and an isomorphism for $n \geqslant ki+1$. We prove this by *strong* induction on n. Consider the statements:

(E_I): d^1 is an epimorphism for all $i \leqslant I$ and all n such that $n \geqslant ki$.

(I_I): d^1 is a monomorphism for all $i \leqslant I$ and all n such that $n \geqslant ki + 1$.

We show the base case of (E_0) and (I_0), then show $(E_{I-1}) + (I_{I-1}) \implies (E_I)$, and finally $(E_{I-1}) + (I_{I-1}) \implies (I_I)$.

Base case: when $I = 0$, we need to show that $d^1 : E^1_{0,0} \to E^1_{-1,0}$ (which is the map $H_0(G_n) \to H_0(G_{n+1})$) is an epimorphism for all $n \geqslant 0$ and a monomorphism for all $n \geqslant 1$. Since $H_0(G_n) \cong (\mathbb{Z})_{G_n} = \mathbb{Z}$ as $\mathbb{Z}$ is a trivial G_n-module, the map $H_0(G_n) \to H_0(G_{n+1})$ is just $(\mathbb{Z})_{G_n} \to (\mathbb{Z})_{G_{n+1}}$, which ends up being an identity.

Step $4.\overline{3}$: Suppose (E_{I-1}) and (I_{I-1}). We claim that $d^1 : E^1_{0,i} \to E^1_{-1,i}$ is an epimorphism for all $n \geqslant ki$, for all $i \leqslant I$. We need two subclaims. For all $i \leqslant I$:

Subclaim 1: $E^\infty_{-1,i} = 0$.

Subclaim 2: $E^2_{p,q} = 0$ for $p + q = i$ with $q < i$.

Let's suppose we've proven these two subclaims. Note that our spectral sequence has $E^1_{p,q} = 0$ for $p < -1$, for the augmented chain complex of Y_{n+1} has $\tilde{C}_p(Y_{n+1}) = 0$ for all $p \leqslant -2$. Thus all the differentials leaving $E^r_{-1,i}$ are 0 for all $r \geqslant 1$, and the only hope of killing off $E^1_{-1,i}$ (which must happen eventually, since we have by assumption that $E^\infty_{-1,i} = 0$) is if some differential with target $E^r_{-1,i}$ is surjective. Subclaim 2 implies that $E^r_{p,q} = 0$ for $p + q = i$ and $q < i$, for all $r \geqslant 2$, and thus no differential hitting $E^r_{-1,i}$ can be surjective for $r \geqslant 2$. It must be the case then that $d^1 : E^1_{0,i} \to E^1_{-1,i}$ is surjective, as desired.

Let's prove the subclaims. By assumption, we have $E^{\infty}_{p,q} = 0$ for $p + q \leqslant \frac{n-1}{k}$ (by connectivity of Y_{n+1}). If $n \geqslant ki$, then $i \leqslant \frac{n}{k}$, so $i - 1 \leqslant \frac{n-k}{k} \leqslant \frac{n-1}{k}$ when $k \geqslant 1$. Thus $E^{\infty}_{-1,i} = 0$. For subclaim 2, we claim that the map induced by inclusion of stabilizers $\mathrm{stab}_{G_{n+1}}(\sigma_{p'}) \hookrightarrow G_{n+1}$

$$E^1_{p',q} = H_q(\mathrm{stab}_{G_{n+1}}(\sigma_{p'}); \mathbb{Z}) \to H_q(G_{n+1}; \mathbb{Z})$$

is an isomorphism for $p' \leqslant p$ and an epimorphism if $p' = p + 1$. First, since $\sigma_{p'}$ is the distinguished p'-simplex, this map can be identified with a composition of stabilization maps (the distinguished inclusions)

$$H_q(G_{n-p'}; \mathbb{Z}) \to H_q(G_{n-p'+1}; \mathbb{Z}) \to \cdots \to H_q(G_{n+1}; \mathbb{Z}) \to \cdots \to H_q(G_{n+1}; \mathbb{Z}).$$

By (I_{I-1}), we know each of these maps is an epimorphism if $n - p' \geqslant kq$ and an isomorphism if $n - p' \geqslant kq + 1$ (recall that $q < i \leqslant I$ by assumption, so $q \leqslant I - 1$). We check that these conditions are satisfied in the case of $p' \leqslant p$ or $p' = p + 1$.

- If $p' + q \leqslant p + q = i \leqslant \frac{n}{k}$, then $n \geqslant kp' + kq \geqslant p' + kq + 1$ for $k \geqslant 2$ and $p' \geqslant 1$. This means $n - p' \geqslant kq + 1$, and we get the desired isomorphisms.
- If $p' = 0$, then $q \leqslant i - 1 \leqslant \frac{n}{k} - 1$ gives $n \geqslant kq + k \geqslant kq + 1$ for $k \geqslant 1$.
- If $p' + q = i + 1$ (so $p' \geqslant 2$, as $q \leqslant i - 1$) and $k \geqslant 2$, then $k(p' + q) = k(i + 1) \leqslant n + k$. Hence $n + k(1 - p') \geqslant kq$. But $-p' \geqslant k - kp' = k(1 - p')$, so $kq \leqslant n - p'$, which means our maps above are epimorphisms.

Now we have a commutative diagram induced by stabilization maps

$$\begin{tikzcd}
H_q(\mathrm{stab}_{G_{n+1}}(\sigma_{p-1}); \mathbb{Z}) \arrow[d] & H_q(\mathrm{stab}_{G_{n+1}}(\sigma_p); \mathbb{Z}) \arrow[l, "\varphi_1"'] \arrow[d] & H_q(\mathrm{stab}_{G_{n+1}}(\sigma_{p+1}); \mathbb{Z}) \arrow[l, "\varphi_2"'] \arrow[d] \\
H_q(G_{n+1}; \mathbb{Z}) & H_q(G_{n+1}; \mathbb{Z}) \arrow[l] & H_q(G_{n+1}; \mathbb{Z}) \arrow[l]
\end{tikzcd}$$

where the top two maps are alternately the 0 map and the stabilization map, depending on the parity of p. Suppose $p + q = i$ and $q < i$. Then the left two vertical maps are isomorphisms and the right vertical map is an epimorphism, as we just proved. If φ_1 is the stabilization map, then the map below it must be the identity (which has trivial kernel). We then have that φ_2 is the zero map, and $\ker \varphi_1 = 0 = \mathrm{Im}\, \varphi_2$, yielding $E^2_{p,q} = 0$. On the other hand, if $\varphi_1 = 0$ and φ_2 is the stabilization map, then the map below it must be the identity, and we see that φ_2 must be surjective. But then $\ker \varphi_1 = \mathrm{Im}\, \varphi_2$, so $E^2_{p,q} = 0$ here too.

Step $4.\overline{6}$: Suppose (E_{I-1}) and (I_{I-1}). We wish to show that $d^1 : E^1_{0,i} \to E^1_{-1,i}$ is a monomorphism for $i \leqslant I$ and all $n \geqslant ki + 1$. We have three subclaims:

Subclaim 1: $E^{\infty}_{0,i} = 0$.

Subclaim 2: $E^2_{p,q} = 0$ with $p + q = i + 1$ and $q < i$.

Subclaim 3: $d^1 : E^1_{1,i} \to E^1_{0,i}$ is the zero map.

Assume these subclaims are shown. Then because $E^1_{0,i}$ lies in the vanishing range of $E^{\infty}_{0,i} = 0$ (subclaim 1), and subclaim 2 implies that every differential $d^r : E^r_{r,-r+i+1} \to E^r_{0,i}$ has source 0 for $r \geqslant 2$, then the only way for $E^{\infty}_{0,i}$ to die is if $d^1 : E^1_{1,i} \to E^1_{0,i}$ has image equal to the kernel of $d^1 : E^1_{0,i} \to E^1_{-1,i}$. The former of these maps is 0, since $p = 1$ is odd, and thus the latter of these maps must be injective, as desired.

It remains to show the subclaims. The first two follow from the same reasons as in step $4.\overline{3}$. The last one follows from the fact that $p = 1$. □

Example 2.164 (Homological stability for the symmetric groups)**.** Let Σ_n be the symmetric group on n letters. We have canonical inclusions yielding stabilization maps

$$\Sigma_n \xrightarrow{\iota_n} \Sigma_{n+1}$$

with image the permutations of $[n+1]$ which fix $\{n+1\}$. We claim that the symmetric groups satisfy homological stability with slope $k = 2$, that is, the induced map

$$H_i(\Sigma_n;\mathbb{Z}) \to H_i(\Sigma_{n+1};\mathbb{Z})$$

is an isomorphism for $n \geqslant 2i+1$ and an epimorphism for $n \geqslant 2i$ (and $n \geqslant 3$). Using Quillen's argument 2.163, it suffices to check that for each n, there exists a sufficiently highly-connected semisimplicial set $(Y_n)_\bullet$ such that: Σ_n acts on the p-simplices of $(Y_n)_\bullet$ transitively; for each p, there is a distinguished p-simplex $\sigma_p \in (Y_n)_p$ such that $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p) \cong \Sigma_{n-p}$ includes in Σ_{n+1} as the composition of stabilization maps; for every face $d_i\sigma_p \in (Y_n)_{p-1}$, there exists an $h_i \in \Sigma_{n+1}$ which centralizes (commutes with every element of) $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p)$ and such that $h_i d_i \sigma_p = \sigma_{p-1}$, so that the inclusion $d_i\sigma_p \hookrightarrow \sigma_p$ induces an inclusion $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p) \to \mathrm{stab}_{\Sigma_{n+1}}(d_i\sigma_p)$ which is conjugate to the distinguished inclusion $\Sigma_{n-p} \to \Sigma_{n-p+1}$; every element in Σ_{n+1} which stabilizes a simplex σ_p setwise stabilizes it pointwise.

Consider the complex of injective words $(I_n)_\bullet$ given in section 3.8.1. It is clear that:

- Σ_{n+1} acts transitively on $(I_{n+1})_p$ for each p, and the distinguished p-simplex $\sigma_p = (n-p+1,\dots,n+1)$ is given by the last $(p+1)$-elements of $[n+1]$, in order.
- Elements in $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p)$ must fix σ_p pointwise, since we have *distinct* p-simplices for every permutation of the letters in $(n-p+1,\dots,n+1)$. Furthermore $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p)$ consists of those permutations of $\{1,\dots,n-p\}$, and is therefore isomorphic to Σ_{n-p} and includes into Σ_{n+1} by the distinguished inclusion.
- For each $i = 0,\dots,p$, there exists $h_i \in \Sigma_{n+1}$ such that $h_i d_i \sigma_p = \sigma_{p-1}$. We need h_i to replace $n-p+1$ with $n-p+i+1$ (so take the transposition $(n-p+1, n-p+i+1)$), and then permute the resulting coordinates to get σ_{p-1}, which only involves a permutation of the last p letters of $[n+1]$. Hence h_i can be chosen to not include any letters of $[n-p]$, which shows that h_i can be chosen to commute with $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p) \cong \Sigma_{n-p}$, as desired.
- The inclusion $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p) \to \mathrm{stab}_{\Sigma_{n+1}}(d_i\sigma_p)$ is conjugate to the distinguished inclusion $\mathrm{stab}_{\Sigma_{n+1}}(\sigma_p) \to \mathrm{stab}_{\Sigma_{n+1}}(\sigma_{p-1})$, since if $g \in \mathrm{stab}_{\Sigma_{n+1}}(d_i\sigma_p)$, then $h_i g h_i^{-1} \in \mathrm{stab}_{\Sigma_{n+1}}(\sigma_{p-1})$.
- $||(I_n)_\bullet||$ is $\frac{n-2}{2}$-connected, for $n \geqslant 3$: We show in 3.94 that $||(I_n)_\bullet||$ is $(n-2)$-connected for $n \geqslant 3$, so in particular it is $\frac{n-2}{2}$-connected.

3 Combinatorial topology

3.1 (Abstract) simplicial complexes

Definition 3.1 (ASC)**.** An abstract simplicial complex X consists of

(i) A vertex set $V(X)$

(ii) A set $S(X)$ consisting of finite (nonempty) subsets of $V(X)$, called simplices

subject to: every $v \in V(X)$ is a simplex in $S(X)$, and if $\sigma \in S(X)$ with $\tau \subseteq \sigma$, then $\tau \in S(X)$.

Lemma 3.2. *An abstract simplicial complex determines the n-cells and attaching map data of a simplicial complex.*

Proof. Clear!

We form the geometric realization of a (finite) abstract simplicial complex as follows: Let $S_n(X)$ label the n-simplices of X ($n+1$ element subsets of $V(X)$). Face maps $d_i : S_n(X) \to S_{n-1}(X)$ give an identification with a label in $S_n(X)$ and a label in $S_{n-1}(X)$. Form

$$||X|| := (\bigsqcup_{j\geqslant 0} S_j(X) \times \Delta^j)/((t, d^i(\sigma)) \sim (d_i(t), \sigma)) \qquad \text{for } \sigma \in \Delta^{j-1}$$

for all $1 \leqslant i \leqslant n$, where $d^i : \Delta^{j-1} \to \Delta^j$ is the inclusion of the i^{th} face. □

Remark 3.3 (Topology of geometric realization). Weak topology: If K is an ASC and $|K|$ is its geometric realization, a subset S of $|K|$ is closed if and only if $S \cap |\sigma|$ is closed for every simplex $|\sigma|$.

Definition 3.4 (Simplicial map)**.** A simplicial map $f : X \to Y$ of abstract simplicial complexes is a function of their vertex sets $f : V(X) \to V(Y)$ with the property that, whenever $\sigma \subset V(X)$ is a simplex of X, then $f(\sigma)$ is a simplex of Y. (Note that f need not be injective, so it can take higher-dimensional simplices to lower-dimensional ones.)

Prop 3.5. *A simplicial map $f : X \to Y$ of abstract simplicial complexes induces a continuous simplicial map $|X| \to |Y|$ on their geometric realizations. Conversely, every simplicial map $|f| : |X| \to |Y|$ arises in this way.*

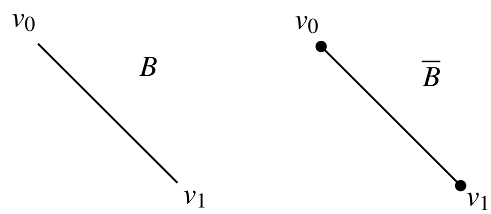
Proof. Let $\{v_0, \dots, v_p\}$ be a p-simplex in X. By definition, $\{f(v_0), \dots, f(v_p)\}$ is a simplex in Y. Since we form the geometric realization by taking our simplices to be the convex hull of their defining vertices, we can extend f to a linear map $f(\sum_{i=0}^p \lambda_i v_i) = \sum_{i=0}^p \lambda_i f(v_i)$ for $\lambda_i \geqslant 0$ and $\sum_{i=0}^p \lambda_i = 1$. By the weak topology, it suffices to check that the restriction of this map to closed simplices is continuous, and indeed this holds since f is linear on simplices (and restricts to linear maps between faces).

On the other hand, if $|f| : |X| \to |Y|$ is a simplicial map, then $|f|$ takes a p-simplex of $|X|$ to a simplex of $|Y|$. Then f takes vertices of $|X|$ to vertices of $|Y|$ appropriately. □

Definition 3.6 (Subcomplex of ASC)**.** A subcomplex A of an ASC X is a subset of $V(X)$ and a subcollection of $S(X)$ closed under the conditions of a simplicial complex.

Definition 3.7 (Closure)**.** Given an ASC X and a collection of simplices $B \subset S(X)$, the closure of B is the smallest subcomplex containing B, that is, the collection of all nonempty subsets of elements of B.

Remark 3.8. One can view the elements of $S(X)$ as open simplices, and then the closure of B is the *topological closure.* For instance, if B contains a simplex $\{v_0, v_1\}$, then the closure $\overline{B}$ also contains $\{v_0\}, \{v_1\}$:



Definition 3.9 (Full subcomplex)**.** A subcomplex A of an ASC X is full if, whenever vertices $v_0, \dots, v_p \in A$ form a simplex $\{v_0, \dots, v_p\}$ in X, then they form a simplex in A.

3.1.1 Stars, links, & other important subcomplexes

Definition 3.10 ((Closed) star)**.** Let X be an ASC with simplex σ. The (closed) star of σ is the subcomplex consisting of the closures of all simplices containing σ as a face.

$$\mathrm{Star}_X(\sigma) = \overline{\{\tau \in S(X) | \tau \cup \sigma \in S(X)\}}$$

Definition 3.11 (Deletion)**.** The deletion of σ in X is the subcomplex of X not containing σ:

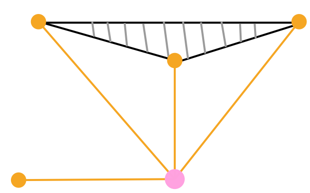
$$\mathrm{dl}_X(\sigma) = \{\tau \in S(X) | \sigma \nsubseteq \tau\}.$$

Definition 3.12 (Link)**.** The link of σ in X is the subcomplex

$$\mathrm{Lk}_X(\sigma) = \{\tau \in S(X) | \tau \cap \sigma = \varnothing, \tau \cup \sigma \in S(X)\}.$$

Lemma 3.13. *Let σ be a simplex in X. Give conditions for the star, deletion, and link of σ a full subcomplex of X.*

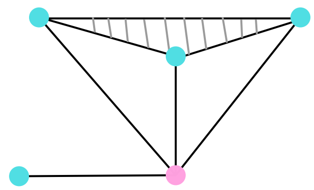
Proof. If σ is a facet (maximal dimensional simplex in X), then $\mathrm{Star}_X(\sigma)$ consists of the faces of σ. This is a full subcomplex. In general, however, $\mathrm{Star}_X(\sigma)$ need not be a full subcomplex. Consider σ the pink vertex, whose closed star is given in orange:



Then every vertex of the 2-simplex belongs to $\mathrm{Star}_X(\sigma)$, and the 2-simplex belongs to $S(X)$, but it does not belong to $\mathrm{Star}_X(\sigma)$.

The deletion is not necessarily a full subcomplex. For suppose $\sigma \subseteq \{v_0, \dots, v_p\} \in S(X)$ but σ does not equal any of the vertices. Then $\sigma \nsubseteq \{v_i\}$ for any v_i, which shows that $v_0, \dots, v_p \in \mathrm{dl}_X(\sigma)$. But $\{v_0, \dots, v_p\} \notin \mathrm{dl}_X(\sigma)$. But if $\sigma = \{v\}$ for some vertex $v \in V(X)$, then all vertices not equal to v belong to $\mathrm{dl}_X(\sigma)$, so if $\{v_0, \dots, v_p\} \in S(X)$, then $\{v\} \nsubseteq \{v_0, \dots, v_p\}$. Hence $\{v_0, \dots, v_p\} \in \mathrm{dl}_X(\{v\})$, and we conclude that $\mathrm{dl}_X(\{v\})$ is a full subcomplex of X.

Suppose $v_0, \dots, v_p \in \mathrm{Lk}_X(\sigma)$ such that $\{v_0, \dots, v_p\} \in S(X)$. Then $\tau \cap \{v_0, \dots, v_p\} = \varnothing$, but it is not necessarily the case that $\tau \cup \{v_0, \dots, v_p\} \in S(X)$. Again we consider the case above, where now the link of the pink vertex is in blue:

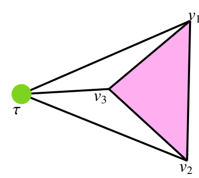


Again we see that the top three blue vertices all belong to $\mathrm{Lk}_X(\sigma)$ and they span a simplex in X, but do not span a simplex in $\mathrm{Lk}_X(\sigma)$. ◻

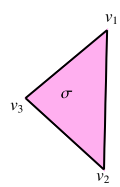
Lemma 3.14 (Links of unions)**.** $\mathrm{Lk}_X(\sigma \cup \tau) = \mathrm{Lk}_{\mathrm{Lk}_X(\sigma)}(\tau)$.

Proof. Suppose $\gamma \in \mathrm{Lk}_X(\sigma \cup \tau)$. Then $\gamma \cap (\sigma \cup \tau) = \varnothing$, and $\gamma \cup \sigma \cup \tau \in S(X)$. In particular we see that $\gamma \cap \tau = \varnothing$, and that $\gamma \cup \tau$ is a simplex in $\mathrm{Lk}_X(\sigma)$. This shows $\gamma \in \mathrm{Lk}_{\mathrm{Lk}_X(\sigma)}(\tau)$. On the other hand, if $\gamma \in \mathrm{Lk}_{\mathrm{Lk}_X(\sigma)}(\tau)$, then $\gamma \cap \tau = \varnothing$ and $\gamma \cup \tau \in \mathrm{Lk}_X(\sigma)$, which means that $(\gamma \cup \tau) \cup \sigma \in S(X)$ and $(\gamma \cup \tau) \cap \sigma = \varnothing$. In particular this means $\gamma \cap \sigma = \varnothing$, and since we already know that $\gamma \cap \tau = \varnothing$, then $\gamma \cap (\sigma \cup \tau) = \varnothing$. Hence $\gamma \in \mathrm{Lk}_X(\sigma \cup \tau)$. ◻

Example 3.15 (Link of intersection is not intersection of links)**.** $\mathrm{Lk}_X(\sigma) \neq \bigcap_{v \in \sigma} \mathrm{Lk}_X(v)$. Let σ be in pink, and let τ be in green. Then $\tau \in \mathrm{Lk}_X(v_1) \cap \mathrm{Lk}_X(v_2) \cap \mathrm{Lk}_X(v_3)$, but $\tau \notin \mathrm{Lk}_X(\sigma)$ since $\tau \cup \sigma \notin S(X)$.



Example 3.16 (The link is not just the star intersect the deletion)**.** $\mathrm{Lk}_X(\sigma) \neq \mathrm{Star}_X(\sigma) \cap \mathrm{dl}_X(\sigma)$. Let σ be



Then $v_1 \in \mathrm{Star}_X(\sigma) \cap \mathrm{dl}_X(\sigma)$ but $v_1 \notin \mathrm{Lk}_X(\sigma)$.

3.1.2 Posets and their order complexes

Definition 3.17 (Order complex of a poset)**.** Let $P = (P, \leqslant)$ be a poset. The <u>order complex</u> $\Delta(P)$ (or $|P|$) is a Δ-complex whose vertices are elements of P and whose simplices are chains of elements of P under $\leqslant$.

Lemma 3.18 (Quillen)**.** *Let $f : A \to B$ be a map of posets. Assume*

- *f is strictly increasing; $a < b$ implies $f(a) < f(b)$.*
- *$|B|$ is CM of dimension d (see 3.32).*
- *For all $b \in B$, the geometric realization $|f_{\leqslant b}|$ of the downwards fiber $f_{\leqslant b} := \{a \in A : f(a) \leqslant b\} \subset A$ is CM of dimension $\mathrm{ht}(b)$.*

Then $|A|$ is CM, and the map $f_* : \tilde{H}_d(|A|) \to \tilde{H}_d(|B|)$ surjects.

Remark 3.19. Actually, the conclusion is that f_* is homologically d-connected (induces isos on homology for $i < d$, and a surjection for $i = d$). Also have a generalization in Church-Putman [3] when the fiber is $\mathrm{ht}(b) + m$-connected (giving $d + m$-connectedness of f_*).

Lemma 3.20 (Quillen fiber lemma, II)**.** *If $f : A \to B$ is an increasing map of posets, and the fibers $|f_{\leqslant b}|$ are contractible for every $b \in B$, then f induces a homotopy equivalence $|A| \to |B|$.*

Prop 3.21 (Nerve lemma)**.** *Let Δ be a simplicial complex. Suppose we can write Δ as the union of subcomplexes $\Delta = \bigcup \Delta_i$ such that every simplex of Δ is in finitely many Δ_i. Let $\mathcal{N}$ be the nerve of the covering, a simplicial complex whose vertices correspond to Δ_i and simplices are given by $\{\Delta_0, \dots, \Delta_p\}$ when $\cap_i \Delta_i \neq \emptyset$. If all the finite intersections of the Δ_i are contractible, then $\Delta \simeq \mathcal{N}$.*

Proof. Define a map $f : \mathcal{P}(\Delta) \to \mathcal{P}(\mathcal{N})$ sending a vertex σ (a simplex in Δ) which belongs to $\Delta_{i_1}, \dots, \Delta_{i_m}$ to the simplex $\{\Delta_{i_1}, \dots, \Delta_{i_m}\}$ (since $\sigma \in \Delta_{i_1}, \dots, \Delta_{i_m}$ implies $\bigcap_{j=1}^m \Delta_{i_j} \neq \emptyset$). This map extends over simplices and is order-reversing (if $s_0 \subset s_1$, then the Δ_i which contain s_1 certainly contain s_0). Thus instead of checking that the downwards fibers are contractible, we'll check that the upwards fibers are contractible: let $V = \{\Delta_1, \dots, \Delta_m\}$ be a simplex in $\mathcal{N}$. Then $|f_{\geqslant V}| = \bigcap_{i=1}^m \Delta_i$, since if $f(\sigma) \geqslant V$, then σ is contained in $\Delta_1, \dots, \Delta_m$, and if $\sigma \in \bigcap_{i=1}^m \Delta_i$, then $f(\sigma) \geqslant V$. But $\bigcap_{i=1}^m \Delta_i$ is contractible by assumption, so by Quillen's fiber lemma 3.20, f induces a homotopy equivalence between $|\mathcal{P}(\Delta)|$ and $|\mathcal{P}(\mathcal{N})|$ (which are homeomorphic to $\mathrm{sd}(\Delta)$ and $\mathrm{sd}(\mathcal{N})$). Since a complex is homeomorphic to its barycentric subdivision, we conclude that $\Delta \simeq \mathcal{N}$. □

Definition 3.22 (n-connected poset map)**.** A poset map $f : X \to Y$ is n-connected if the induced map on the geometric realization of the nerves is n-connected.

Lemma 3.23 (van der Kallen)**.** *A poset map $f : X \to Y$ is n-connected if there is a function $t : Y \to \mathbb{Z}$ such that for every $y \in Y$, we have*

1. *$Y_{>y} := \{y' \in Y : y' > y\}$ is $(n - t(y) - 2)$-connected.*
2. *$f_{\leqslant y} := \{x \in X : f(x) \leqslant y\}$ is $(t(y) - 1)$-connected.*

Definition 3.24 (Flag complex)**.** A flag complex is an (abstract) simplicial complex X with the following property: Let $S \neq \emptyset$ be a subset of $V(X)$. If every pair of vertices of S span an edge, then S is a simplex of X.

Lemma 3.25. *The order complex of a poset is a flag complex.*

Proof. Suppose $S \subseteq V(X)$ with every pair of vertices spanning an edge in $S(X)$. Since simplices in an order complex correspond to chains in a poset, then $\{v_i, v_j\} \in S(X)$ means that $v_i \leqslant v_j$ or $v_j \leqslant v_i$. That is, every pair of vertices in S is comparable, and hence the vertices $v_0, \dots, v_p$ form a chain (totally ordered poset). But this means $\{v_0, \dots, v_p\} \in S(X)$. □

Lemma 3.26. *If X is a flag complex, then $\mathrm{Lk}_X(\sigma)$ is a full subcomplex for $\sigma \in S(X)$.*

Proof. Suppose $v_0, \dots, v_p \in \mathrm{Lk}_X(\sigma)$ such that $\{v_0, \dots, v_p\} \in S(X)$. Since $\sigma \cup \{v_i\} \in S(X)$ for all i, then every pair of vertices in $V(\sigma) \cup \{v_0, \dots, v_p\}$ forms an edge. Since X is a flag complex, this means $\sigma \cup \{v_0, \dots, v_p\} \in S(X)$. Since we also know that $\sigma \cap \{v_0, \dots, v_p\} = \emptyset$, then $\{v_0, \dots, v_p\} \in \mathrm{Lk}_X(\sigma)$. □

Remark 3.27. The same proof shows that $\mathrm{Star}_X(\sigma)$ is a full subcomplex of X, when X is a flag complex.

3.1.3 Generalized simplicial complexes

Definition 3.28 (Generalized simplicial complex)**.** A CW complex X is a generalized simplicial complex if its n-cells are parameterized as n-simplices Δ^n_α, and the attaching maps $\phi^n_\alpha : \partial\Delta^n_\alpha \to X^{(n-1)}$ restrict to linear isomorphisms from the $(n-1)$-faces of Δ^n_α to characteristic maps of $(n-1)$-simplices $\Delta^{n-1}_\beta \to X^{(n-1)}$.

Remark 3.29. Note that the isomorphisms from the $(n-1)$-faces to the characteristic maps of $(n-1)$-simplices do not need to respect orientation (which must hold for the more restrictive Δ-*complex* or trisp).

Definition 3.30 (Regular generalized simplicial complex)**.** A generalized simplicial complex is regular if $\forall n \geqslant 1$, the attaching maps $\phi_\alpha^n : \partial\Delta_\alpha^n \to X$ of any n-simplex is an embedding of the boundary of $\overline{\Delta_\alpha^n}$ into X (so ϕ_α^n is a homeomorphism when restricted to $\partial\Delta_\alpha^n$).

In particular, regular generalized simplicial complexes have $n+1$ distinct vertices for any n-simplex. This in fact characterizes them:

Lemma 3.31 (Regular generalized simplicial complexes)**.** *A generalized simplicial complex is regular if and only if every n-simplex of X has $(n+1)$ distinct vertices, for all n. In particular, simplicial complexes are regular generalized simplicial complexes.*

Proof. Suppose X is a regular generalized simplicial complex, and let σ be an n-simplex. Let Δ_σ^n be a standard n-simplex. Then $\partial\Delta_\sigma^n$ has $n+1$ distinct vertices, and since X is regular, the attaching map $\phi_\sigma : \partial\Delta_\sigma^n \to X$ of σ is injective. Thus σ has $n+1$ distinct vertices.

On the other hand, suppose every n-simplex has $n+1$ distinct vertices, for all n. Let $\phi_\alpha : \partial\Delta_\alpha^1 \to X$ be an attaching map of a 1-simplex. Since ϕ_α is injective on vertices by assumption, and $\partial\Delta_\alpha^1$ consists of two vertices, then ϕ_α is injective on $\partial\Delta_\alpha^1$. Now suppose that the attaching map of every n-simplex is an embedding of the boundary of Δ^n into X. Let $\phi_\beta : \partial\Delta_\beta^{n+1} \to X$ be the attaching map of an $(n+1)$-simplex. We know that on each face of Δ_β^{n+1}, ϕ_β restricts to a linear isomorphism with a Δ^n composed with a characteristic map $\phi_\gamma : \Delta_\gamma^n \to X$. Since each characteristic map maps to a unique n-simplex (and two faces of Δ^{n+1} can't glue down using the same characteristic map ϕ_γ, since they don't share at least one vertex and our original attaching map is injective on vertices). ◻

Definition 3.32. [Cohen-Macaulay] A d-dimensional simplicial complex X is Cohen-Macaulay if X is $(d-1)$-connected and $\mathrm{Lk}_X(\sigma)$ is $(d-2-\dim(\sigma))$-connected for all $\sigma \in S(X)$.

3.2 Morse theory lemma

Theorem 3.33. *Let Y be a simplicial complex, and $X \subseteq Y$ a subcomplex. Assume X and Y satisfy the following.*

(i) X is a full subcomplex of Y.

(ii) X is d-connected.

(iii) For all vertices y_1, y_2 in $Y \backslash X$, there is no edge $\{y_1, y_2\}$.

(iv) For all vertices $y \in Y\backslash X$, the link $\mathrm{Lk}_Y(y)$ is $(d-1)$-connected.

Then Y is d-connected.

Proof. Let $\varphi : (S^k, s_0) \to (Y, x_0)$ be a based map for $1 \leqslant k \leqslant d$. By the simplicial approximation theorem, φ is homotopic to a simplicial map with respect to some simplicial structure on S^k, so without loss of generality we may take φ to be simplicial. If φ has image contained in X then it is nullhomotopic by virtue of X being d-connected, so we suppose there exists some $y \in Y\backslash X$ such that $y \in \varphi(S^k)$. Since S^k is compact, the compact image $\varphi(S^k)$ meets Y in finitely many simplices, so in particular meets finitely many vertices of $Y\backslash X$.

Note that X being a full subcomplex of Y and condition (iii) together imply that for all $y \in Y\backslash X$, $\mathrm{Lk}_Y(y) \subseteq X$. For, if $y' \neq y \in Y\backslash X$ lies in the link of y, then $\{y, y'\}$ is a simplex in Y, contradicting (iii)! So all vertices in

$\mathrm{Lk}_Y(y)$ must lie in X. And if such vertices form a simplex in Y, then X being full implies that this simplex must lie in X.

Consider the full subcomplex of Y consisting of X and the (finitely many) $\mathrm{Star}_Y(y)$ of $y \in Y \backslash X$ lying in the image of φ. This complex is obtained inductively from X by coning off the subcomplex $\mathrm{Lk}_Y(y)$ for $y \in Y \backslash X$. Thus it suffices to show that $X \cup_\iota \mathrm{Star}_Y(y)$ is d-connected (Note: easy to check to that this is a full subcomplex of Y, so we can run the same argument inductively). By Thm 1.19 (excision for homotopy groups), if we can show that $(\mathrm{Star}_Y(y), \mathrm{Lk}_Y(y))$ is d-connected and $(X, \mathrm{Lk}_Y(y))$ is 0-connected (note that $\mathrm{Lk}_Y(y) \subseteq X$ since if $y' \in Y \backslash X$, we do not have an edge $\{y, y'\} \in S(Y)$, by assumption), then the map induced by inclusion $\pi_i(\mathrm{Star}_Y(y), \mathrm{Lk}_Y(y)) \to \pi_i(X \cup \mathrm{Star}_Y(y), X)$ is an isomorphism for $i < d$ and surjective for $i = d$. Since $\mathrm{Star}_Y(y)$ is contractible (being the cone of $\mathrm{Lk}_Y(y)$) and $\mathrm{Lk}_Y(y)$ is $(d-1)$-connected by assumption, the long exact sequence of the pair $(\mathrm{Star}_Y(y), \mathrm{Lk}_Y(y))$ gives that it is d-connected. Since a pair (X, A) being 0-connected means that A intersects every connected component of X, it is trivial that $(X, \mathrm{Lk}_Y(y))$ is 0-connected. We conclude that $(X \cup \mathrm{Star}_Y(y), X)$ is d-connected. From the long exact sequence

$$\cdots \to \pi_i(X) \to \pi_i(X \cup \mathrm{Star}_Y(y)) \to \pi_i(X \cup \mathrm{Star}_Y(y), X) \to \pi_{i-1}(X) \to \cdots$$

we obtain that $\pi_i(X \cup \mathrm{Star}_Y(y)) = 0$ for $i \leqslant d$. Continuing inductively by attaching finitely many stars, we see that $X \cup_i \mathrm{Star}_Y(y)$ is d-connected. □

3.3 Joins

Definition 3.34 (Join)**.** Let X and Y be topological spaces. The join of X and Y is

$$X * Y = X \sqcup X \times Y \times I \sqcup Y / (x, y, 0) \sim x, \quad (x, y, 1) \sim y.$$

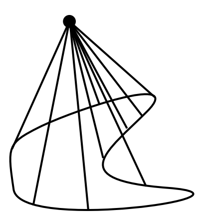
Hence if X and Y are disjoint subsets of Euclidean space $\mathbb{R}^n$, then $X * Y \cong \{tx + (1-t)y | x \in X, y \in Y, t \in [0, 1]\}$.

Lemma 3.35. *Let X be a topological space. Let S^n denote the n-sphere and let Δ^n denote an n-simplex. Then*

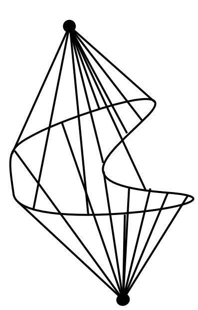
(a) $X * \Delta^0 \cong CX$.

(b) $X * S^0 \cong SX$.

Proof. (a) $X * \Delta^0$ is just the space of lines from the point Δ^0 to X, which is precisely the cone CX.



(b) $X * S^0$ is the space of lines from the two points S^0 to X, which is precisely the unreduced suspension SX.



□

Corollary 3.36. *$(S^0)^{*n+1} \cong S^n$ and $(\Delta^0)^{*n+1} \cong D^n$.*

Lemma 3.37. *If X and Y are CW complexes, the join $X * Y$ has a CW structure with X and Y as subcomplexes. If X and Y are CW complexes such that the product and weak topologies on $X \times Y$ agree, then the weak topology on the join $X * Y$ agrees with the defining quotient topology.*

Proof. Since I is Hausdorff, both $\{0\}$ and $\{1\}$ are closed points in I, and hence $X \times Y \times \{0\}$ (which is identified with X in the quotient) and $X \times Y \times \{1\}$ (which is identified with Y in the quotient) come back to this!!! □

Prop 3.38 (Homotopy type of a join)**.** *Suppose there exist homotopy equivalences $X \simeq X'$, $Y \simeq Y'$. Then there exists a homotopy equivalence $X * Y \simeq X' * Y'$.*

Proof. Let $f_1 : X \to X'$ and $f_2 : Y \to Y'$ be homotopy equivalences, and let $g_1 : X' \to X$, $g_2 : Y' \to Y$ be their respective homotopy inverses (so $f_1 \circ g_1 \simeq \mathrm{id}_{X'}$, and so on). We define maps

$$\begin{aligned} F : X * Y &\to X' * Y' \\ [(x, y, t)] &\mapsto [(f_1(x), f_2(y), t)] \end{aligned}$$

and

$$\begin{aligned} G : X' * Y' &\to X * Y \\ [(x', y', t)] &\mapsto [(g_1(x'), g_2(y'), t)]. \end{aligned}$$

These maps are well-defined since they are constant on respective equivalence classes. Since $X * Y$ is an identification space $X \sqcup X \times Y \times I \sqcup Y/(x, y, 1) \sim x, (x, y, 0) \sim y)$, then by lemma 1.4, to define a homotopy $(X * Y) \times I \to X * Y$ it suffices to define a homotopy $H : (X \sqcup X \times Y \times I \sqcup Y) \times I \to X * Y$ which is constant on equivalence classes. On X we use the homotopy $g_1 \circ f_1 \simeq \mathrm{id}_X$, on $X \times Y \times \{0\}$ we use the same homotopy, on $X \times Y \times I$ we use the constant identity map, and on $X \times Y \times \{1\}$ and Y we use the homotopy $g_2 \circ f_2 \simeq \mathrm{id}_Y$. Since these maps are constant on equivalence classes, they combine to give a well-defined homotopy $X * Y \times I \to X * Y$ from $G \circ F$ to id_{X*Y}. An analogous construction going the other way gives a well-defined homotopy $X' * Y' \times I \to X' * Y'$ from $F \circ G$ to $\mathrm{id}_{X'*Y'}$. Hence $X * Y \simeq X' * Y'$. □

Prop 3.39 (Homology of a join)**.** *Let X and Y be topological spaces. Then $\tilde{H}_{k+1}(X * Y) \cong \bigoplus_{i+j=k} \tilde{H}_i(X) \oplus \tilde{H}_j(Y) \oplus \bigoplus_{i+j=k-1} \mathrm{Tor}_1^{\mathbb{Z}}(\tilde{H}_i(X), \tilde{H}_j(Y))$.*

Proof. Let A be the image of $X \sqcup X \times Y \times [0, \frac{3}{4})$ in $X * Y$ and let B be the image of $Y \sqcup X \times Y \times (\frac{1}{4}, 1]$ in $X * Y$. Then A is an open subset of $X * Y$, since $X \sqcup X \times Y \times [0, \frac{3}{4})$ is a saturated open in $X \sqcup X \times Y \times I \sqcup Y$ under the quotient map identifying $(x, y, 0) \sim x$ (note that $q^{-1}(q(X \sqcup X \times Y \times [0, \frac{3}{4})) = X \sqcup X \times Y \times [0, \frac{3}{4})$, and I inherits the subspace topology from $\mathbb{R}$. Then since q is a quotient map, the image of a saturated open is open.) Similarly B is an open subset of $X * Y$, and we see that $A \cup B$ covers $X * Y$. The intersection $A \cap B = X \times Y \times (\frac{1}{4}, \frac{3}{4})$ is homeomorphic to $X \times Y$. Now A deformation retracts to X and B deformation retracts to Y, so the Mayer-Vietoris sequence yields a long exact sequence in homology

$$\cdots \to H_n(A) \oplus H_n(B) \xrightarrow{(\iota_A)_* - (\iota_B)_*} H_n(X * Y) \xrightarrow{\partial} H_{n-1}(A \cap B) \xrightarrow{(j_A)_*, (j_B)_*} H_{n-1}(A) \oplus H_{n-1}(B) \to \cdots$$

where $\iota_A : A \hookrightarrow X * Y$ is the inclusion and $j_A : A \cap B \hookrightarrow A$ is the inclusion. We claim that the inclusions ι_A and ι_B are nullhomotopic, so that their induced maps on homology are trivial, and the long exact sequence above splits into short exact sequences

$$0 \to H_{n+1}(X * Y) \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to 0$$

for every n. Note that any point in $Y \cong X \times Y \times \{1\}/\sim \subset X * Y$ cones off A, so that the inclusion of A into $X * Y$ is nullhomotopic. Similarly, any point in $X \cong X \times Y \times \{0\} \subset X * Y$ cones off B, so the inclusion of B into $X * Y$ is nullhomotopic, as desired.

Finally, we claim that the short exact sequence above splits. Note that the maps $(j_A)_*, (j_B)_*$ are induced by maps homotopic to projections $X \times Y \to X$ and $X \times Y \to Y$. We also have a map $f : H_n(A) \oplus H_{n-1}(B) \to H_n(A \cap B)$ induced by the map sending $[x] \in A$ and $[y] \in B$ to $[(x, y, \frac{1}{2})] \in A \cap B$. The composition $(j_A)_* \oplus (j_B)_* \circ f$ is the identity on $H_n(A) \oplus H_n(B)$, giving a splitting of the short exact sequence. Replacing $A \cap B$ with the homeomorphic $X \times Y$ and A, B with the homotopy equivalent X, Y, we have

$$H_n(X \times Y) \cong H_{n+1}(X * Y) \oplus H_n(X) \oplus H_n(Y)$$

Then by Kunneth for a product 1.73, we know that $H_n(X \times Y) \cong \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \oplus \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(X), H_j(Y))$. We see then that $H_n(X) \oplus H_n(Y)$ corresponds to the $H_n(X) \otimes H_0(Y) \oplus H_0(X) \otimes H_n(Y)$ summands in the Kunneth formula, so $H_{n+1}(X * Y)$ must be isomorphic to the remaining summands. (Taking reduced homology makes the $i = 0$ and $j = 0$ summands disappear, which is what we want.) □

Remark 3.40. The homology of a join looks like the Kunneth theorem for the homology of a *product*, except we go up a degree.

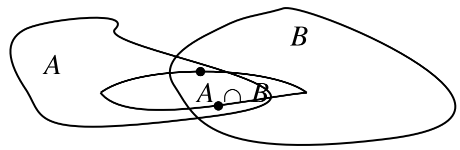
Prop 3.41 (Connectivity of a join)**.** *For $i = 1, \dots, p$, let X_i be an n_i-connected topological space. Then the join $X_1 * X_2 * \cdots * X_p$ is $(\sum_{i=1}^{p}(n_i + 2)) - 2$-connected.*

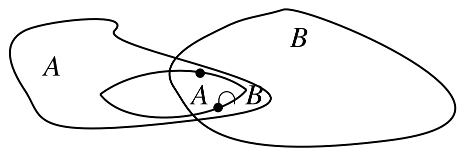
Proof. Notice that it suffices to prove this for $p = 2$: for if we can show that $X_1 * X_2$ is $(n_1 + 2) + (n_2 + 2) - 2 = n_1 + n_2 + 2$-connected, and by induction we have that $X_1 * X_2 * \cdots * X_{p-1}$ is $(\sum_{i=1}^{p-1}(n_i + 2)) - 2$-connected, then $X_1 * \cdots * X_p$ is

$$(\sum_{i=1}^{p-1} n_i + 2) - 2 + n_p + 2 = (\sum_{i=1}^{p-1} n_i + 2) + n_p = (\sum_{i=1}^{p} n_i + 2) - 2$$

connected.

- If $X = \varnothing$, then $X * Y = Y$, and the connectivity of $X * Y$ is $n_Y = n_X + n_Y + 2$, since the empty set is (-2)-connected.
- If $X, Y \neq \varnothing$, then both X and Y are (-1)-connected. Notice that $X * Y$ is path-connected, since every point of X is connected to every point of Y by a path through $X \times Y \times I$. Hence $X * Y$ is $n_X + n_Y + 2 = 0$-connected.
- if $X \neq \varnothing$ and Y is connected, we need to show that $X * Y$ is $n_X + n_Y + 2 = -1 + 0 + 2 = 1$-connected (simply-connected). Choose a basepoint $[(x, y, t)]$ and a loop in $X * Y$ based at $[(x, y, t)]$. We claim that any loop in $X * Y$ is homotopic to a loop in A (or B, depending on where the basepoint is). If this claim is shown, then the map induced by the inclusion $\iota_* : \pi_1(A) \twoheadrightarrow \pi_1(X * Y)$ is surjective. Since this map is also 0 (as we have seen that the inclusion of A into $X * Y$ is nullhomotopic), then we can conclude that $\pi_1(X * Y) = 0$. We do this by showing that any path in B which has endpoints in $A \cap B$ is homotopic (rel endpoints) to a path in $A \cap B$, so that a loop in $X * Y$ becomes a loop entirely in A:





Let γ be a path (lying in B) between (x, y, s) and $(x', y', s') \in A \cap B$. Since Y is path-connected, there exists a path between y and y'. Also $p_X(\gamma)$ is a path between x and x' in X, and we have a path between s and s' in I. Let ω be this new path (which lies in $A \cap B$). Then $p_X(\gamma) = p_X(\omega)$, and since p_X is a homotopy equivalence between X and A, then $\gamma \simeq \omega$? need to finish this. confused.

- Now suppose $n_X, n_Y \geqslant 0$. Then $X * Y$ is simply-connected. By Hurewicz 1.21, we know that $\tilde{H}_i(X) = 0$ for $i \leqslant n_X$ and $\tilde{H}_i(Y) = 0$ for $j \leqslant n_Y$. Then by prop 3.39, we obtain that $\tilde{H}_k(X * Y) = 0$ for $k \leqslant n_X + n_Y + 2$. Since $X * Y$ is simply-connected, the Hurewicz theorem gives that $\pi_k(X * Y) = 0$ for $k \leqslant n_X + n_Y + 2$.

□

3.3.1 Joins of simplicial complexes

Definition 3.42 (Join of ASC)**.** Let X, Y be ASC with vertex sets $V(X), V(Y)$. The join of X and Y is an abstract simplicial complex $X * Y$ with vertex sets $V(X * Y) = V(X) \cup V(Y)$ and simplices $S(X * Y) = S(X) \cup S(Y) \cup \{\sigma \cup \tau : \sigma \in S(X), \tau \in S(Y)\}$.

Lemma 3.43. *$X * Y$ is an abstract simplicial complex if X and Y are.*

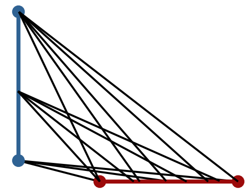
Proof. It suffices to show that if $\gamma \subseteq \sigma \cup \tau$, then $\gamma \in S(X * Y)$. Since $\gamma \subseteq \sigma \cup \tau$, then γ is a union of a face of σ and a face of τ. Since these faces are contained in $S(X)$ and $S(Y)$ respectively, then $\gamma \in S(X * Y)$. □

Prop 3.44. *Let X, Y be ASC. Then $|X * Y| \cong |X| * |Y|$.*

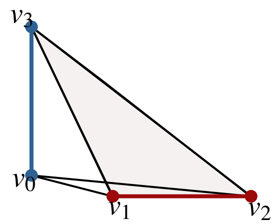
Proof. Say $x \in |X * Y|$ lies in the interior of a simplex $|\sigma \cup \tau| = |\{v_0, \ldots, v_p, w_0, \ldots, w_\ell\}|$ (so we express $x = \sum_{i=0}^{p} t_i v_i + \sum_{j=0}^{\ell} s_j w_j$). Since $|\sigma| \cong \Delta^k \subset \mathbb{R}^{k+1}$ and $|\tau| \cong \Delta^\ell \subset \mathbb{R}^{\ell+1}$, we naturally embed $|\sigma|$ in $\mathbb{R}^{k+\ell+2}$ as the first $k+1$ coordinates and $|\tau|$ as the next $\ell + 1$ coordinates, and send x to the point $\sum_{i=0}^{p} t_i v_i + \sum_{j=0}^{\ell} s_j w_j \in \mathbb{R}^{k+\ell+2}$. This defines a map

$$\begin{aligned} \varphi : |X * Y| &\to |X| * |Y| \\ |\sigma \cup \tau| &\mapsto |\sigma| * |\tau|. \end{aligned}$$

We check that this map is a homeomorphism. The geometric realization $|X| * |Y|$ has a canonical simplicial structure given as follows: A $(k + \ell + 1)$-simplex in $|X| * |Y|$ is the join of a k-simplex in $|X|$ and an ℓ-simplex in $|Y|$. (For example, if $|X| = D^1$ and $|Y| = D^1$ with the typical simplicial structure, we can give the join $|X| * |Y|$



a simplicial structure as a tetrahedron:



We see then that the 3-simplex $\{v_0, v_1, v_2, v_3\}$ is the join of $\{v_0, v_3\}$ (a 1-simplex in X) and $\{v_1, v_2\}$ (a 1-simplex in Y). We can then define the inverse of φ as a map ψ which sends $|\sigma| * |\tau| \to |\sigma \cup \tau|$. This simplicial structure on $|X| * |Y|$ forces our maps φ and ψ to be simplicial, and hence continuous. $\square$

Example 3.45. Let P be a poset and $\Delta(P)$ its order complex.

(a) For $x \in P$,
$$\mathrm{Lk}_{\Delta(P)}(x) = \Delta(P_{<x}) * \Delta(P_{>x}).$$
Recall that $P_{<x}$ consists of elements $a \in P$ such that $a < x$. Suppose that $\tau \in \mathrm{Lk}_{\Delta(P)}(x)$. Then $x \notin \tau$, but $\tau \cup \{x\} \in \Delta(P)$. Hence τ consists of vertices which belong to either $P_{<x}$ or $P_{>x}$, and these vertices form a chain in P (and also form a chain when x is included). In particular, the vertices in $P_{<x}$ form a chain (and hence a simplex in $\Delta(P_{<x})$) and the vertices in $P_{>x}$ form a chain (and hence a simplex in $\Delta(P_{>x})$). This means that τ is a union of a simplex in $\Delta(P_{<x})$ and in $\Delta(P_{>x})$, so $\tau \in \Delta(P_{<x}) * \Delta(P_{>x})$. Completely analogous reasoning shows that if $\tau \in \Delta(P_{<x}) * \Delta(P_{>x})$, then $\tau \in \mathrm{Lk}_{\Delta(P)}(x)$.

(b) If $\sigma \in \Delta(P)$ corresponds to the chain $x_0 < \cdots < x_p$, then
$$\mathrm{Lk}_{\Delta(P)}(\sigma) = \Delta(P_{<x_0}) * \Delta(P_{(x_0,x_1)}) * \cdots * \Delta(P_{(x_{p-1},x_p)}) * \Delta(P_{>x_p}).$$
If $\tau \in \mathrm{Lk}_{\Delta(P)}(\sigma)$, then $\tau \cap \sigma = \varnothing$ and $\tau \cup \sigma \in \Delta(P)$. Hence τ consists of vertices in $P_{<x_0} \cup P_{(x_0,x_1)} \cup \cdots \cup P_{(x_{p-1},x_p)} \cup P_{>x_p}$ which form a chain in P, and thus these vertices form chains when restricted to their respective posets $P_{<x_0}, P_{(x_0,x_1)}$, and so on. Hence τ is a union of simplices in $\Delta(P_{<x_0}) * \cdots * \Delta(P_{>x_p})$. The other implication is also clear.

Prop 3.46 (Star and link)**.** *Let X be an ASC and let $\sigma \in S(X)$. Then* $\mathrm{Star}_X(\sigma) = \sigma * \mathrm{Lk}_X(\sigma)$.

Proof. Suppose $\tau \in \sigma * \mathrm{Lk}_X(\sigma)$. Then τ is a union of a simplex τ_1 in σ and a simplex τ_2 in $\mathrm{Lk}_X(\sigma)$. Since $\tau_2 \cup \sigma \in S(X)$, and $\tau_1 \subseteq \sigma$, then $\tau \cup \sigma \in S(X)$, and we conclude that $\tau \in \mathrm{Star}_X(\sigma)$. Now suppose $\tau = \{v_0, \ldots, v_p\} \in \mathrm{Star}_X(\sigma)$. Let $v_0, \ldots, v_\ell$ be the vertices of τ contained in σ. Then $\tau \cup \sigma = \{v_{\ell+1}, \ldots, v_p\} \cup \sigma \in S(X)$, and since $v_{\ell+1}, \ldots, v_p$ do not belong to σ, then $\{v_{\ell+1}, \ldots, v_p\} \in \mathrm{Lk}_X(\sigma)$. Hence τ is the union of $\{v_0, \ldots, v_\ell\}$ (a simplex in σ) and $\{v_{\ell+1}, \ldots, v_p\}$ (a simplex in $\mathrm{Lk}_X(\sigma)$), so $\tau \in \sigma * \mathrm{Lk}_X(\sigma)$. $\square$

3.4 Barycentric subdivision

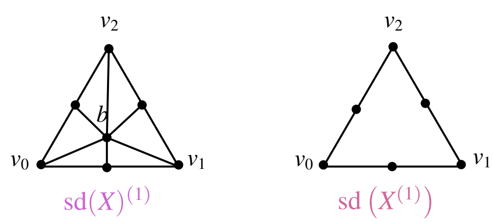
Definition 3.47 (Subdivision)**.** Let X be a Δ-complex. A subdivision of X is a new Δ-complex structure on X' (the underlying topological space of X) such that each simplex of X' is contained in a simplex of X, and every simplex of X is a finite union of simplices of X'. As topological spaces, X and X' agree.

Definition 3.48 (Barycentric subdivision)**.** We introduce a new vertex at the barycenter (center of mass) of each simplex, and take the coarsest associated subdivision. Each n-simplex is replaced by $(n+1)!$ many n-simplices. Given an n-simplex Δ^n_α on vertices $v_0, \ldots, v_n$, define the barycentric subdivision inductively:

- The subdivision of a 0-simplex v_0 is simply the vertex v_0.
- The barycentric subdivision of an n-simplex $\{v_0, \ldots, v_n\}$ is constructed by placing a new vertex b in the barycenter, and subdividing into a union of n-simplices $\{b, w_0, \ldots, w_{n-1}\}$ where $\{w_0, \ldots, w_{n-1}\}$ is an $(n-1)$-simplex in the barycentric subdivision of a face $\{v_0, \ldots, \hat{v}_i, \ldots, v_n\}$.

Prop 3.49. *The construction of* $\mathrm{sd}(X)$ *yields a valid* Δ*-complex structure, homeomorphic to* X*, and is a subdivision of* X *in the sense of definition 3.47.*

Example 3.50. If X is a Δ-complex, the barycentric subdivision of $X^{(k)}$ is *not* the k-skeleton of $\mathrm{sd}(X)$. For example, if $X = \Delta^2 = \{v_0, v_1, v_2\}$, then



Definition 3.51 (Barycentric subdivision of ASC)**.** Let X be an ASC. The barycentric subdivision $\mathrm{sd}(X)$ of X is an ASC defined as: $V(\mathrm{sd}(X))$ is the set of simplices $S(X)$ of X. A collection of simplices $\{\sigma_0, \ldots, \sigma_p\}$ spans a p-simplex of $\mathrm{sd}(X)$ precisely when they form a chain under inclusion.

Prop 3.52. *Let* X *be a simplicial complex, and let* $S = (S, \subseteq)$ *be the poset of simplices of* X*, ordered by inclusion. Then the realization of the order complex* $|S|$ *is canonically isomorphic to the barycentric subdivision of* X*.*

Proof. We get the isomorphism inductively on dimension. Suppose $F_0 \subset F_1 \subset \cdots \subset F_n$ is a chain of length $n+1$ of simplices of X (hence forms an n-simplex $\{F_0, \ldots, F_n\}$ in $|S|$). Write $F_n = \{v_0, \ldots, v_\ell\} \in S(X)$ for $\ell \geqslant n$. Consider the barycentric subdivision of F_n. We can label vertices in the barycentric subdivision by the vertices in X which they divide; for instance, the new vertices in the subdivision of the 1-skeleton of $\{v_0, \ldots, v_\ell\}$ are labeled by every $\binom{\ell+1}{2}$ 2-vertex combination of $\{v_0, \ldots, v_\ell\}$ (so these new vertices are labeled $\{v_0, v_1\}, \{v_0, v_2\}$, and so on), and the new 1-simplices look like $\{v_i, \{v_i, v_j\}\}$. The new vertices in the subdivision of the 2-skeleton are labeled by $\{v_i, v_j, v_k\}$ for a 2-simplex originally given by the vertices v_i, v_j, v_k, the new 1-simplices look like $\{v_i, \{v_i, v_j, v_k\}\}$ or $\{\{v_i, v_j\}, \{v_i, v_j, v_k\}\}$, and the new 2-simplices are labeled by $\{v_i, \{v_i, v_j\}, \{v_i, v_j, v_k\}\}$. Proceeding in this way, we see that vertices in $\mathrm{sd}(F_n) \subset \mathrm{sd}(X)$ are labeled by all tuples of length between 1 and $\ell+1$ in $v_0, \ldots, v_\ell$, and simplices are labeled by chains of inclusions of tuples. We thus have a natural n-simplex in $\mathrm{sd}(X)$ labeled by $\{F_0, F_1, F_2, \ldots, F_\ell\}$ (but where each F_i is replaced by the vertices which make it up). $\square$

Prop 3.53 (Induced map on barycentric subdivision)**.** *Let* $f : X \to Y$ *be a simplicial map of abstract simplicial complexes. Then* f *induces a simplicial map* $f_* : \mathrm{sd}(X) \to \mathrm{sd}(Y)$ *on the barycentric subdivisions of* X *and* Y*.*

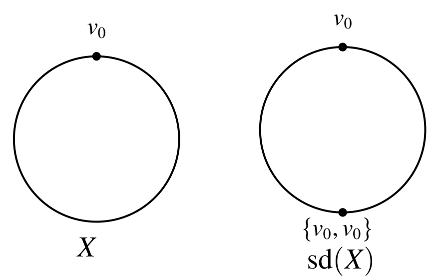
Proof. Let $\sigma = \{s_0, \dots, s_p\}$ be a simplex in $\mathrm{sd}(X)$, with each $s_i \in S(X)$. Then $f(s_i)$ is a simplex in Y since f is simplicial, and certainly $s_i \subset s_j$ implies $f(s_i) \subseteq f(s_j)$. Hence $f(\sigma) = \{f(s_0), \dots, f(s_p)\}$ is a simplex in $\mathrm{sd}(Y)$. ☐

Lemma 3.54. *If X is an ASC, then $|\mathrm{sd}(X)| \cong \mathrm{sd}(|X|)$.*

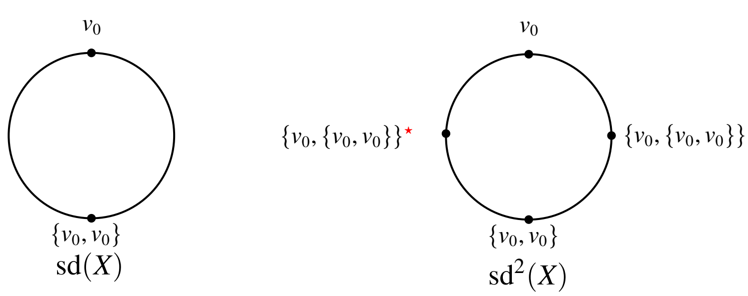
Prop 3.55 (Refining Δ to a simplicial complex)**.** *(a) Let X be a Δ-complex. Then its barycentric subdivision $\mathrm{sd}(X)$ has the property that every n-simplex has $n+1$ distinct vertices.*

(b) Let X be a Δ-complex with the property that any n-simplex has $n+1$ distinct vertices. Then $\mathrm{sd}(X)$ is a simplicial complex.

Proof. (a) Recall that if X is a Δ-complex, its barycentric subdivision is a Δ-complex constructed by inductively gluing in, in place of each n-simplex, a barycentrically subdivided n-simplex. Note that the new n-simplices are comprised of one original vertex from X, and the remaining n vertices come from the inductive barycentric subdivisions of the simplices of X. In particular, an n-simplex in $\mathrm{sd}(X)$ has vertices $\{s_0, \dots, s_n\}$, where each s_i is a simplex in X (with repeat vertices allowed) of a certain dimension, and the s_i form a proper chain $s_0 \subset \cdots \subset s_n$ (thus are distinct).



(b) Now suppose X is a Δ-complex with the property that any n-simplex has $n+1$ distinct vertices. We wish to show that every $n+1$ vertices in $\mathrm{sd}(X)$ determine at most one n-simplex. If X has more than one simplex determined by the same set of vertices, label the vertices of the barycentric subdivision of each such simplex with a decoration to distinguish them from the vertices of the barycentric subdivision of another simplex. Then every n-simplex in $\mathrm{sd}(X)$ must have a vertex given by the barycenter of an n-simplex of X, and any two n-simplices of $\mathrm{sd}(X)$ sharing this barycenter vertex must each have a vertex not shared by the other, since the original n-simplex in X was comprised of $n+1$ *distinct* vertices. Hence $n+1$ vertices in $\mathrm{sd}(X)$ will uniquely determine an n-simplex.



☐

Corollary 3.56. *If X is a Δ-complex, $\mathrm{sd}^2(X)$ is a simplicial complex.*

Lemma 3.57. *If K is a simplicial complex and L is a subcomplex, then* $\mathrm{sd}(L)$ *is full in* $\mathrm{sd}(K)$.

Proof. Suppose $s_0, \ldots, s_p$ are vertices in $\mathrm{sd}(L)$ (simplices in L) such that $\{s_0, \ldots, s_p\} \in S(\mathrm{sd}(K))$. Then $s_0 \subset \cdots \subset s_p$ is a chain of simplices under inclusion in K, and since each simplex actually belongs to L, we conclude that $\{s_0, \ldots, s_p\} \in S(\mathrm{sd}(L))$. □

3.5 Simplicial group actions

Definition 3.58 (Simplicial G-complex)**.** Suppose G is a discrete group, and that G acts on K simplicially, that is, each homeomorphism $g : K \to K$ is a simplicial map (if $\{v_0, \ldots, v_n\}$ is a simplex, then $\{gv_0, \ldots, gv_n\}$ is a simplex). Then K is a simplicial G-complex.

Definition 3.59 (Condition (S))**.** For any $g \in G$ and simplex σ of X (a simplicial G-complex), g leaves $\sigma \cap g\sigma$ pointwise fixed.

Definition 3.60 (Condition (S'))**.** If v and gv belong to the same simplex, then $v = gv$.

Lemma 3.61. *Condition (S) and (S') are equivalent.*

Proof. Suppose condition (S) 3.59 holds, and let v, gv belong to a simplex σ. Then $v, g^{-1}v$ belong to a simplex $g^{-1}\sigma$. So $v \in \sigma \cap g^{-1}\sigma$, and we know that g^{-1} leaves this intersection pointwise fixed by assumption, so $g^{-1}v = v$. Hence $v = gv$.

On the other hand, suppose condition (S') holds 3.60, and suppose $v \in \sigma \cap g\sigma$. Then there exists some $w \in \sigma$ such that $v = gw$, so both w and gw belong to σ. By condition (S'), we know that $w = gw = v$, so $gv = gw = w = v$, and thus g leaves $\sigma \cap g\sigma$ pointwise fixed. □

Lemma 3.62. *[Barycentric subdivision satisfies condition (S)] If X is a simplicial G-complex, the induced action on the barycentric subdivision X' satisfies condition (S). 3.59*

Proof. We'll show that $X' = \mathrm{sd}(X)$ satisfies condition (S') 3.60. Let σ be a vertex of $\mathrm{sd}(X)$ (and hence a simplex $\{v_0, \ldots, v_p\}$ in X), and suppose that $\sigma, g\sigma$ belong to the same simplex in $\mathrm{sd}(X)$. Since simplices in $\mathrm{sd}(X)$ are chains of simplices in X under inclusion, then as simplices of X, it must be the case that either $\sigma \subseteq g\sigma$ or $g\sigma \subseteq \sigma$. The former can only happen if the two simplices are equal, in which case we are done. If the simplices are not equal, then $gv_i = gv_j$ for some $v_i \neq v_j \in \sigma$, which means that g is not a homeomorphism from $X \to X$ (not injective!), a contradiction. □

Definition 3.63 (Condition (R))**.** We say that condition (R) is satisfied for a simplicial G-complex X if whenever $g_0, \ldots, g_n$ are elements of G and $\{v_0, \ldots, v_n\}, \{g_0v_0, \ldots, g_nv_n\}$ are both simplices of X, there exists a single element $g \in G$ such that $gv_i = g_iv_i$ for all i.

Definition 3.64 (Regular G-complex)**.** Say a simplicial G-complex is regular if condition (R) 3.63 holds for all subgroups $H \leqslant G$.

Prop 3.65. *If X is a simplicial G-complex satisfying condition (S) 3.62, then its barycentric subdivision* $\mathrm{sd}(X)$ *satisfies condition (R) 3.63.*

Proof. Suppose X satisfies condition (S). We show that $\mathrm{sd}(X)$ satisfies condition (R) by induction on the size of simplices. The claim holds trivially for 0-dimensional simplices. Now suppose the claim holds for simplices of dimension $n-1$. Let $\sigma = \{s_0, \ldots, s_n\}$ be a simplex of dimension n, ordered so that $s_0 \subset s_1 \subset \cdots \subset s_n$ forms a chain of simplices in X. Suppose $\{g_0s_0, \ldots, g_ns_n\}$ is also a simplex of $\mathrm{sd}(X)$. Then the faces $\{s_0, \ldots, s_{n-1}\}$

and $\{g_0 s_0, \ldots, g_{n-1}s_{n-1}\}$ are both simplices of $\mathrm{sd}(X)$, so by induction there exists $g \in G$ such that $gs_i = g_i s_i$ for $0 \leqslant i \leqslant n-1$. Since $\mathrm{sd}(X)$ is a simplicial G-complex (as X is), then

$$g^{-1} \cdot \{g_0 s_0, \ldots, g_{n-1}s_{n-1}, g_n s_n\} = \{s_0, \ldots, s_{n-1}, g^{-1}g_n s_n\}$$

is a simplex of $\mathrm{sd}(X)$. Note that $s_0, \ldots, s_{n-1} \subseteq s_n$ and $s_0, \ldots, s_{n-1} \subseteq g^{-1}g_n s_n$, so $s_0, \ldots, s_{n-1}$ all belong to $s_n \cap g^{-1}g_n s_n$. By condition (S), we know that $g^{-1}g_n$ fixes this intersection pointwise, so $g^{-1}g_n s_i = s_i$ (pointwise) for all $0 \leqslant i \leqslant n-1$. We therefore have that

$$\begin{aligned} g^{-1} \cdot \{g_0 s_0, \ldots, g_{n-1}s_{n-1}, g_n s_n\} = \{s_0, \ldots, s_{n-1}, g^{-1}g_n s_n\} &= \{g^{-1}g_n s_0, \ldots, g^{-1}g_n s_n\} \\ &= g^{-1}g_n \cdot \{s_0, \ldots, s_n\} \\ \implies \{g_0 s_0, \ldots, g_n s_n\} = \{g_n s_0, \ldots, g_n s_n\}. \end{aligned}$$

□

Corollary 3.66. *By lemma 3.65 and prop 3.62, a simplicial G-action on X induces a regular G-action on* $\mathrm{sd}^2(X)$.

Definition 3.67. Let X be an ASC with vertex set $V(X)$ and simplices $S(X)$. Suppose X admits a simplicial action by a group G. Define the quotient $\underline{X/G}$ to be an ASC with:

(i) A vertex $v^* \in V(X/G)$ is a G-orbit of vertices in $V(X)/G$.

(ii) A collection of vertices $\{v_0^*, \ldots, v_p^*\}$ span a simplex in $S(X/G)$ if at least one choice of representatives $\{v_0, \ldots, v_p\}$ of the *orbits* span a simplex in $S(X)$. The simplex $\{v_0, \ldots, v_p\}$ of X is called the simplex *over* the simplex $\{v_0^*, \ldots, v_p^*\}$ of X/G.

Lemma 3.68. *Let X be an ASC with a simplicial action by a group G. Verify that the ASC defined on X/G is in fact a valid abstract simplicial structure.*

Proof. Need to show that if $\tau \subseteq \sigma$ and $\sigma \in S(X/G)$ then $\tau \in S(X/G)$. Write $\sigma = \{v_0^*, \ldots, v_p^*\}$. Then there exists some choice of representatives of the orbits $v_0, \ldots, v_p$ such that $\{v_0, \ldots, v_p\} \in S(X)$. But then every subset of this is a simplex, so every subset of σ is a simplex in X/G. □

Remark 3.69. There is a natural simplicial map $X \to X/G, v \mapsto v^*$ giving rise to a simplicial map $|X| \to |X/G|$ on the geometric realizations. This map is G-equivariant with respect to the *trivial* G-action on $|X/G|$ (i.e., $g \cdot v^* = v^* \forall v^* \in X/G$). By the universal property of the quotient, the map factors through the orbit space $|X|/G \to |X/G|$ via continuous maps

$$\begin{tikzcd} {|X|} \arrow[d] \arrow[dr] & \\ {|X|/G} \arrow[r, dashed] & {|X/G|}. \end{tikzcd}$$

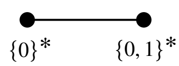
For a simplex $\{v_0, \ldots, v_p\}$ of X and a point $\sum_{i=0}^{p} t_i v_i$ in the simplex, consider the G-orbit of this point in $|X|/G$. The map $|X|/G \to |X/G|$ is defined as

$$G \cdot \sum_{i=1}^{p} t_i v_i \mapsto \sum_{i=1}^{p} t_i v_i^*.$$

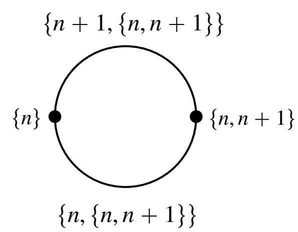
Importantly, the map $|X|/G \to |X/G|$ need not be a homeomorphism or a homotopy equivalence!!

Example 3.70. To see that $|X|/G \to |X/G|$ need not be a homotopy equivalence, consider the group $G = \mathbb{Z}$ acting on the simplicial set R, which has vertices $V(R) = \{n \in \mathbb{Z}\}$ and simplices $S(R)$ consisting of all vertices and edges of the form $\{n, n+1\}$. The element $g_m \in \mathbb{Z}$ acts on $V(R)$ via $n \mapsto m+n$. This action descends to a translation action on the geometric realization $|R|$. The abstract simplicial complex R/G has just one vertex $\{0^*\}$ since G acts transitively on $V(R)$. Hence $S(R/G) = \{0^*\}$. This shows that $|R/G| = *$. On the other hand, $|R|/G \simeq S^1$, with a vertex $[n]$ and a 1-simplex $[n, n+1]$ with both ends attached to $[n]$. Clearly $\varphi : |R|/G \to |R/G|$ must send all of $|R|/G$ to the single point $\{0^*\}$, so φ is not a homotopy equivalence.

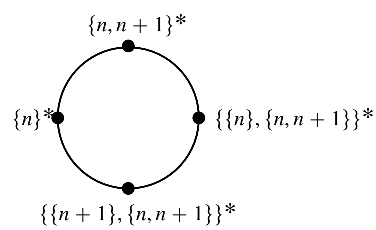
If we subdivide R (take order complex of poset of simplices of R under inclusion), then now $V(\mathrm{sd}(R)) = \{\{n\} : n \in \mathbb{Z}, \{n, n+1\} : n \in \mathbb{Z}\}$. Then $|\,\mathrm{sd}(R)/G|$ has two vertices ($\{0\}^*$ and $\{0,1\}^*$) and one 1-simplex (joining them), so looks like



while $|\,\mathrm{sd}(R)|/G$ looks like a subdivided S^1:



since the simplices $\{n+1, \{n, n+1\}\}$ and $\{n, \{n, n+1\}\}$ do not lie in the same G-orbit (whereas in $|\,\mathrm{sd}(R)/G|$, these both correspond to the edge $\{0^*, \{0,1\}^*\}$). But if we subdivide $\mathrm{sd}(R)$ (to form $\mathrm{sd}^2(R)$), suddenly $|\,\mathrm{sd}^2(R)/G|$ and $|\,\mathrm{sd}^2(R)|/G$ are homeomorphic! The former looks like



while the latter looks the same.

This phenomenon happens in general:

Theorem 3.71. *Let X be a regular G-complex and $H \leqslant G$ a subgroup, with X/H the quotient complex. Then $|X|/H$ has a natural simplicial structure, such that the map $|X|/H \to |X/H|$ is a simplicial isomorphism.*

Proof. Let X be a regular G-complex and let $\sigma = \{v_0^*, \dots, v_p^*\}$ be a simplex of X/G. The simplices of X lying over σ form a G-orbit of p-simplices: if $\tilde{\sigma} = \{v_0, \dots, v_p\} \in S(X)$ and $\tilde{\tau} = \{w_0, \dots, w_p\} \in S(X)$ are two simplices lying over σ, then there exists $g_0, \dots, g_p \in G$ such that $g_0 v_0 = w_0, \dots, g_p v_p = w_p$. Since X is a regular G-complex, there exists a $g \in G$ such that $g v_i = w_i$ for all i, and hence $\tilde{\sigma}$ and $\tilde{\tau}$ lie in the same G-orbit. Now the map $|X|/G \xrightarrow{\varphi} |X/G|$ is bijective: it is clearly surjective, since if $x = \sum_{i=0}^p t_i v_i^*$ (where $\sigma = \{v_0^*, \dots, v_p^*\}$ is a simplex in X/G), then by definition there exists a simplex $\tilde{\sigma} = \{v_0, \dots, v_p\} \in S(X)$ lying over σ, and φ sends the G-orbit $G \cdot \sum_{i=1}^p t_i v_i$ to x. In the case where X is a regular G-complex, we see that φ is also injective: if

$$\varphi(G \cdot \{v_0, \dots, v_p\} = \varphi(G \cdot \{w_0, \dots, w_\ell\})$$

then $\{v_0^*, \dots, v_p^*\} = \{w_0^*, \dots, w_\ell^*\}$. But $\{v_0, \dots, v_p\}$ and $\{w_0, \dots, w_\ell\}$ both lie over this simplex, so by the above argument they must lie in the same G-orbit.

Finally we verify that the topology on $|X|/G$ agrees with the weak topology on $|X/G|$. Recall that in the weak topology, a subset $S \subseteq |X/G|$ is closed if and only if $S \cap |\sigma|$ is closed for each simplex $\sigma \in X/G$. The topology on $|X|/G$ is that induced by the quotient map $q : |X| \to |X|/G$: a subset $S \subseteq |X|/G$ is closed if and only if its preimage $q^{-1}(S)$ is closed. We wish to show that if $S \subseteq |X/G|$ is closed, then $\varphi^{-1}(S)$ is closed in $|X|/G$, and if $T \subseteq |X|/G$ is closed, then $\varphi(T) \subseteq |X/G|$ is closed. Recall that we had a simplicial map $|X| \to |X/G|$, which we call ρ. This map factors as

$$\begin{tikzcd} |X| \arrow[d, "q"'] \arrow[dr, "\rho"] & \\ |X|/G \arrow[r, dashed, "\varphi"'] & |X/G|. \end{tikzcd}$$

If $S \subseteq |X/G|$ is closed, then $\varphi^{-1}(S)$ is closed in $|X|/G$ if and only if $q^{-1}(\varphi^{-1}(S))$ is closed in $|X|$, which holds if and only if $\rho^{-1}(S)$ is closed in $|X|$. But ρ is simplicial, and hence continuous, so $\rho^{-1}(S)$ is indeed closed in $|X|$. On the other hand, if $T \subseteq |X|/G$ is closed, we wish to show that $\varphi(T) \cap |\sigma|$ is closed in $|\sigma|$ for all $\sigma \in X/G$. Let $\sigma \in X/G$, and let $\tilde{\sigma} \in X$ be a simplex lying over σ. Since T is closed in $|X|/G$, we know that $q^{-1}(T) \cap |\tilde{\sigma}|$ is closed in $|\tilde{\sigma}|$. Then $\varphi(T) \cap |\sigma| = \varphi(q)$ finish this! □

Corollary 3.72. *By corollary 3.66, if X is a G-complex, $\mathrm{sd}^2(X)$ is a regular G-complex. Hence $|\,\mathrm{sd}^2(X)/G|$ is simplicially isomorphic to $|\,\mathrm{sd}^2(X)|/G$.*

Prop 3.73. *Suppose a simplicial action of G on X satisfies condition (S). Then $|X^G| \cong |X|^G$, where X^G is the subcomplex of simplices fixed pointwise by G, and $|X|^G$ is the subspace of $|X|$ fixed pointwise by the induced action of G.*

Proof. Define a map

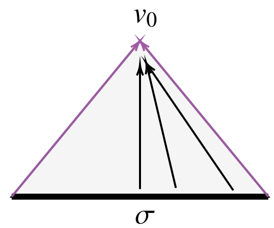
$$\begin{aligned} \varphi : |X^G| &\to |X|^G \\ x &\mapsto x \end{aligned}$$

in the obvious way: if $x \in |X^G|$ lies in a simplex fixed pointwise by G, then it certainly lies in a subspace of $|X|$ fixed pointwise by G. Injectivity is clear, and it remains to show that every $x \in |X|^G$ lies in a simplex of X fixed pointwise by G. Thus suppose $x \in |X|^G$ lies in a simplex $\sigma = \{v_0, \dots, v_p\}$ of X. Write $x = \sum_{i=0}^p t_i v_i$. Since $gx = x$ for all $g \in G$, then $\sum_{i=0}^p t_i g v_i = \sum_{i=0}^p t_i v_i$ for all $g \in G$. We wish to show that this forces $gv_i = v_i$ for every i. Note that since g fixes x, and the action by g is continuous, then $g\sigma = \sigma$ setwise. By condition (S), we know that g fixes the intersection $\{v_0, \dots, v_p\} \cap g\{v_0, \dots, v_p\}$ pointwise, but this intersection is just σ itself, so $\sigma \in |X^G|$, and hence $x \in |X^G|$. □

3.6 Cones and near cones

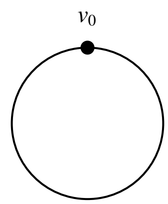
Definition 3.74 (Cone point)**.** Let X be an ASC. A vertex $v_0 \in X$ is a cone point of X if for every simplex $\sigma \in S(X)$, $\sigma \cup \{v_0\} \in S(X)$.

Example 3.75. If X has a cone point v_0, then X deformation retracts to v_0 (and hence is contractible). Note that we can do this retraction one simplex at a time: if $\sigma \in S(X)$, then $\sigma \cup \{v_0\} \in S(X)$, so one can push each point in σ to v_0 via the straight line homotopy in $\sigma \cup \{v_0\}$. This homotopy restricts to a straight line homotopy on the faces of σ, and is continuous on X since it is continuous on each simplex (weak topology).



Definition 3.76 (Cone point for generalized simplicial complex)**.** Let X be a generalized simplicial complex. Call v_0 a vertex in X a cone point if for any simplex $\sigma \in X$, either v_0 is a vertex of σ or $v_0 * \sigma$ is a simplex of X.

Example 3.77. A generalized simplicial complex with cone point v_0 is not necessarily contractible. Consider X the S^1:



Then v_0 is a vertex of every simplex in X, so is a cone point. But S^1 is not contractible.

Definition 3.78 (Near cone)**.** Let X be an abstract simplicial complex with vertex set $V(X)$, and let $v_0 \in V(X)$ be a distinguished vertex. Call X a near cone if for every simplex σ, if $v_0 \notin \sigma$, then for any vertex $w \in \sigma$ the set $(\sigma \backslash \{w\}) \cup \{v_0\}$ is a simplex of X.

Definition 3.79 ($B_{v_0}(X)$)**.** For a near cone X and distinguished vertex v_0, let

$$B_{v_0}(X) := \{\sigma \in S(X) : \sigma \cup \{v_0\} \notin S(X)\}$$

for the set of simplices of X not contained in $\mathrm{Star}_X(v_0)$. Hence X is constructed by gluing each simplex in $B_{v_0}(X)$ to $\mathrm{Star}_X(v_0)$ along its entire boundary, since if $\tau \in B_{v_0}(X)$, then $\tau \cup \{v_0\} \notin S(X)$, but $(\tau \backslash \{w\}) \cup \{v_0\} \in S(X)$, and hence $\tau \backslash \{w\} \in \mathrm{Star}_X(v_0)$ for every vertex $w \in \tau$. (And $\bigcup_{w \in \tau} (\tau \backslash \{w\}) = \partial \tau$).

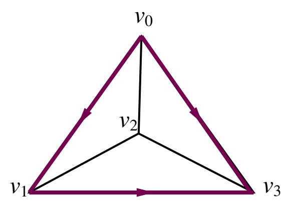
Lemma 3.80. *If X is a near-cone with distinguished vertex v_0, then*

$$X \simeq \bigvee_{\sigma \in B_{v_0}(X), \dim(\sigma) = p} S^p.$$

Proof. Since $\mathrm{Star}_X(v_0)$ is contractible, then if X is a near cone, we obtain (by lemma 3.88) that $X \simeq \bigvee_{\sigma \in B_{v_0}(X), \dim(\sigma)=p} S^p$. In particular, we can compute the Betti numbers as $\mathrm{rk}(\tilde{H}_p(X)) = \#\{\sigma \in B_{v_0}(X) : \dim(\sigma) = p\}$. □

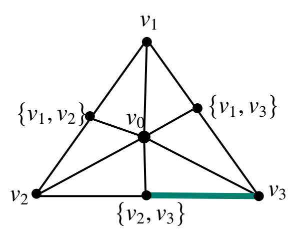
Example 3.81 ($\Delta^{n,k}$ is a near-cone)**.** Recall that the k-skeleton of an n-simplex $\Delta^{n,k} = (\Delta^n)^{(k)}$ is either contractible or homotopy equivalent to a wedge of k-spheres, by corollary 1.33. We can compute exactly how many k-spheres are in the wedge using the concept of a near-cone. Let v_0 be any vertex in $\Delta^{n,k}$. We claim that only k-dimensional simplices in $\Delta^{n,k}$ belong to $B_{v_0}(X)$: Since $\Delta^{n,k}$ consists of *every* k-dimensional simplex in Δ^n, and every $k+1$ vertices in Δ^n form a k-simplex in Δ^n, then if we choose any vertices $w_0, \ldots, w_\ell$ (for $\ell < k$) forming a simplex τ, it must be the case that $\{v_0, w_0, \ldots, w_\ell\}$ is a face of a k-simplex containing v_0, and hence belongs to the star of v_0. On the other hand, if $\tau = \{w_0, \ldots, w_k\} \in \Delta^{n,k}$ does not contain v_0 as a vertex, then because $\{w_0, \ldots, w_k\} \backslash \{w\}$ for any vertex $w \in \tau$ is a $(k-1)$-simplex, then $(\tau \backslash \{w\}) \cup \{v_0\}$ must be a simplex in $\Delta^{n,k}$. In particular, since we have $\binom{n}{k+1}$ such simplices, then $\Delta^{n,k} \simeq \bigvee_{\binom{n}{k+1}} S^k$.

We can describe an explicit basis for $H_k(\Delta^{n,k})$ of simplicial k-cycles. Since we have an explicit homotopy equivalence $f : \Delta^{n,k} \to \bigvee_{\binom{n}{k+1}} S^k$, this map induces an isomorphism on homology $f_* : H_k(\Delta^{n,k}) \to H_k(\bigvee_{\binom{n}{k+1}} S^k) \cong \bigoplus_{\binom{n}{k+1}} \mathbb{Z}$. One can lift each sphere in the wedge (corresponding to k-simplices not containing v_0) to the corresponding k-cycle as follows: if $\tau \in B_{v_0}(X)$, the corresponding simplicial k-cycle is given by τ and all the $(\tau \backslash \{w\}) \cup \{v_0\}$ for each vertex w. For instance, $\Delta^{3,1}$ is homotopy equivalent to a wedge of $\binom{3}{2} = 3$ circles S^1. The simplicial 1-cycle corresponding to $\tau = \{v_1, v_3\}$ is given by $\{v_0, v_1\} + \tau - \{v_0, v_3\}$.

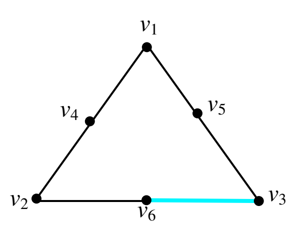


Example 3.82. Let $n \geqslant 0$ and $0 \leqslant k \leqslant n$.

(a) Let $X = (\mathrm{sd}(\Delta^n))^{(k)}$. Then X is a near-cone: let v_0 be the barycenter of a subdivided Δ^n, and let τ be a k-simplex in $\mathrm{sd}(\Delta^n)$. Every simplex in $\mathrm{sd}(\Delta^n)$ is contained in at least one n-simplex, and every n-simplex in $\mathrm{sd}(\Delta^n)$ has v_0 (the barycenter) as a vertex. If $k = n$ we are done, as every n-simplex contains v_0. If $k < n$ and τ does not contain v_0, then removing a vertex w and adjoining v_0 yields another k-simplex in $\mathrm{sd}(\Delta^n)$, since k-simplices in $\mathrm{sd}(\Delta^n)$ are given by chains $s_0 \subset \cdots \subset s_k$ of simplices in Δ^n, as well as by $\{v_0\} \cup \{s_0, \ldots, s_{k-1}\}$ where $s_0 \subset \cdots \subset s_{k-1}$ is a chain of simplices in Δ^n.



(b) On the other hand, $\mathrm{sd}(\Delta^{n,k})$ is not in general a near-cone. Taking $n = 2$ and $k = 1$, we see that if any vertex were to be distinguished, then (by symmetry) all would be distinguished. Without loss of generality, then, take v_1 to be the distinguished vertex. Then the edge $\{v_6, v_3\}$ is not in $\mathrm{Star}_X(v_1)$, but removing (either) vertex and adjoining $\{v_1\}$ does not yield a valid simplex.



3.7 Shellability

Definition 3.83 (Shellability)**.** Let X be a *generalized simplicial complex*. Say that a finite complex X is shellable if we can order the maximal faces (facets) $F_1, \ldots, F_\ell$ of X such that $F_j \cap (\bigcup_{i=1}^{j-1} F_i) \subseteq \partial F_j$ is a pure subcomplex of dimension $\dim F_j - 1$, for $j \geqslant 2$.

Call $F_1, \ldots, F_\ell$ a shelling order on the facets of X.

Definition 3.84 (Spanning simplices). Given a shelling order $F_1, \dots, F_\ell$ on the facets of a generalized simplicial complex X, call F_j a <u>spanning simplex</u> if $F_j \cap (\bigcup_{i=1}^{j-1} F_i) = \partial F_j$, the entire boundary of F_j.

It turns out that we have some flexibility on shelling orders of shellable complexes. In particular:

Lemma 3.85. *If X is shellable, there exists a shelling order on the facets of X such that:*

(i) We can order the spanning simplices last, after all the non-spanning simplices;

(ii) The spanning simplices may occur in any order.

Proof. Suppose $F_1, \dots, F_\ell$ is a shelling order on the facets of X. Let F_t be the first spanning simplex to appear. We claim that we can move it backwards without messing up the shellability property, that is,

$$F_1, \dots, F_{t-1}, F_{t+1}, F_t, F_{t+2}, \dots, F_\ell$$

is another shelling. Certainly $F_1, \dots, F_{t-1}$ still satisfies the shellability property. Let $\sigma \in F_{t+1} \cap (\bigcup_{i=1}^{t-1} F_i)$. If $\dim(\sigma) < \dim(F_{t+1}) - 1$, we need to show that there exists some $\tau \in F_{t+1} \cap (\bigcup_{i=1}^{t-1} F_i)$ with $\dim(\tau) = \dim(F_{t+1}) - 1$ and $\sigma \subset \tau$. Notice that σ also belongs to $F_{t+1} \cap (\bigcup_{i=1}^{t} F_i)$ and that σ is *not* maximal dimensional in ∂F_{t+1} by assumption, so by the fact that $F_1, \dots, F_t, F_{t+1}$ is part of a shelling order there must exist some $\tau \in F_{t+1} \cap (\bigcup_{i=1}^{t} F_i)$ with $\sigma \subseteq \tau$ and $\dim(\tau) = \dim(F_{t+1}) - 1$. If $\tau \in F_{t+1} \cap (\bigcup_{i=1}^{t-1} F_i)$ we are done. Otherwise we have $\tau \in F_{t+1} \cap F_t$, and since $\tau \neq F_t$ (else $F_t \subseteq F_{t+1}$, contradicting the maximality of the face F_t in X), then τ must be a simplex of ∂F_t. But then the fact that F_t is spanning implies that τ belongs to some prior face $F_1, \dots, F_{t-1}$, as desired.

Note that since $F_t \cap (\bigcup_{i=1}^{t-1} F_i) = \partial F_t$, if F_t gets pushed back its entire boundary still appears in the union of $F_1, \dots, F_{t-1}$, so F_t is still a spanning simplex. It is clear that by moving F_t back we also have not introduced any new spanning simplices. ◻

Corollary 3.86. *If X is shellable, there is a shelling order $F_1, \dots, F_\ell$ on its facets such that the spanning simplices occur last and are non-increasing in dimension.*

Shelling orders are nice because we can imagine them as a recipe for gluing on the maximal simplices of our generalized simplicial complex in a way that tells us the homotopy type of the complex. Any time we glue on a maximal simplex along a proper subcomplex of its boundary, we are effectively not gluing on anything at all, since we can contract this subcomplex to a point. (Imagine a disk being glued on to a piece of paper along only some of its boundary.) On the other hand, every time we glue on a maximal simplex along its *entire* boundary, we gain a sphere.

Lemma 3.87. *Let Δ^n denote an n-simplex, and let $A \subseteq \partial\Delta^n$ be a subcomplex that is pure of dimension $n-1$. If A is a proper subcomplex, then A is contractible.*

Proof. We know that A consists of a union of $(n-1)$-simplices, and since A is proper, there exists at least one $(n-1)$-simplex σ in $\partial\Delta^n$ not contained in A. Let v be the vertex opposite σ. We claim that A deformation retracts onto v: firstly, σ is the only $(n-1)$-simplex in Δ^n not containing v, so v belongs to every simplex in A. We can therefore contract every simplex in A to v, and note that this homotopy is continuous since its restriction to each simplex is. ◻

Lemma 3.88. *Let X be a regular generalized simplicial complex with $\pi_{n-1}(X) = 0$. Let Δ^n be an n-simplex, A a subcomplex of $\partial\Delta^n$ that is pure of dimension $n-1$, and $\phi : A \to X$ a simplicial embedding. Consider the space $\Delta^n \sqcup_\phi X$ obtained by gluing Δ^n to X along A. Then $\Delta^n \sqcup_\phi X$ is a generalized simplicial complex, and*

$$X \sqcup_\phi \Delta^n \simeq \begin{cases} X \vee S^n & A = \partial\Delta^n \\ X & A \subset \partial\Delta^n. \end{cases}$$

Proof. Recall that $\Delta^n \sqcup_\phi X = (\Delta^n \sqcup X)/(a \sim \phi(a), a \in A)$. The k-cells of $\Delta^n \sqcup_\phi X$ look like compositions $\phi_\alpha^k : \Delta^k \to X \hookrightarrow X \sqcup \Delta^n \to \Delta^n \sqcup_\phi X$ (with restrictions to $\partial\Delta^k$ coming from the $(k-1)$-cells of X composed with the quotient map), along with cells $\psi_i^k : \Delta_i^k \hookrightarrow A \xrightarrow{\phi} X \to \Delta^n \sqcup_\phi X$ induced by ϕ.

If $A = \partial\Delta^n$, then viewing $\phi : \partial\Delta^n \to X$ as an element of $\pi_{n-1}(X)$, the fact that $\pi_{n-1}(X) = 0$ implies that ϕ is nullhomotopic. By the corollary following Theorem 1.5, we obtain $\Delta^n \sqcup_\phi X \simeq X \vee S^n$. On the other hand, if A is a proper subcomplex of $\partial\Delta^n$, then A must be contractible by lemma 3.87, so $\Delta^n \sqcup_\phi X \simeq X$ by the other corollary following Theorem 1.5. $\square$

Theorem 3.89. *Let X be a shellable generalized simplicial complex. Then*

$$X \simeq \bigvee_{F \text{ spanning}} S^{\dim F}.$$

Proof. Let $F_1, \dots, F_\ell$ be a shelling order on the facets of X, such that the spanning simplices are shelled last and are non-increasing in dimension, possible by Corollary 3.86. Since F_1 is a simplex, it is contractible and satisfies the conditions for X of Lemma 3.88. Now let $A = F_2 \cap F_1 \subseteq \partial F_2$. If F_2 is spanning, then $A = \partial F_2 \cong \partial\Delta^{\dim F_2}$ simplicially embeds into F_1 (via, say, a map ϕ), and the generalized simplicial complex $\Delta^{\dim F_2} \sqcup_\phi F_1$ is homotopy equivalent to $S^{\dim F_2} \vee F_1$ via Lemma 3.88 (which in turn is homotopy equivalent to $S^{\dim F_2}$, since $F_1 \simeq *$). If F_2 is not spanning, then $F_2 \sqcup_\phi F_1 \simeq F_1 \simeq *$ via the same lemma. Continuing to construct X in this way, we see that X is homotopy equivalent to a wedge of spheres, one for each spanning simplex of X and in the dimension of that spanning simplex. $\square$

Example 3.90. Let $X = \Delta^{n,k} = (\Delta^n)^{(k)}$ be the k-skeleton of an n-simplex. Then $\Delta^{n,k}$ is shellable: let the vertices of Δ^n be given by $\{1, \dots, n+1\}$. Then k-simplices are given by $(k+1)$-element words on the letters $\{1, \dots, n+1\}$, ordered lexicographically (increasing order on $[n+1]$). We claim that the lexicographic order on all such $(k+1)$-element words gives a shelling order on $\Delta^{n,k}$. We have two claims:

Claim 3.91. *Once we've shelled all the faces that start with a 1, all the k-tuples have shown up in $F_1, \dots, F_{\binom{n}{k}}$. So $F_{\binom{n}{k}+\ell} \cap \bigcup_{i=1}^{\binom{n}{k}} F_i$ contains the entire boundary of $F_{\binom{n}{k}+\ell}$ for $1 \leqslant \ell \leqslant \binom{n+1}{k+1} - \binom{n}{k}$. We therefore have $\binom{n+1}{k+1} - \binom{n}{k} = \binom{n}{k+1}$ spanning simplices.*

Claim 3.92. *$F_1, \dots, F_{\binom{n}{k}}$ are not spanning simplices, but they do form a shelling order.*

In these simplices, the last k-element subset (everything after the 1) appears for the first time in that face. So the intersection of F_i with the union of the previous faces cannot be the whole boundary (a $(k-1)$-face is always missing). Suppose $\sigma \in F_i \cap F_j$ for $i < j$ and $2 \leqslant j \leqslant \binom{n}{k}$, with $|\sigma| < k$. We wish to show that there exists a face F_ℓ with $\ell < j$ such that $\sigma \in F_\ell$ and $|F_\ell \cap F_j| = k$. Firstly since every face $F_1, \dots, F_{\binom{n}{k}}$ starts with a 1, we may assume without loss of generality that σ contains 1. If one can decrease one of the numbers in F_j to get a valid simplex still (that is, if there is room to decrease numbers without repeating vertices) then the simplex with the decreased number must precede F_j. Thus we can look at $\sigma \in F_i \cap F_j$ and ask: can any of the numbers in F_j and not in σ be decreased? Certainly at least one can, else σ would not have appeared in a previous facet F_i. Choosing this vertex and decreasing it gives a previous face F_ℓ containing σ and only differing from F_j by one vertex, as desired.

By claim 3.91, we have that $\Delta^{n,k} \simeq \bigvee_{\binom{n}{k+1}} S^k$. This confirms the result we obtained in example 3.81.

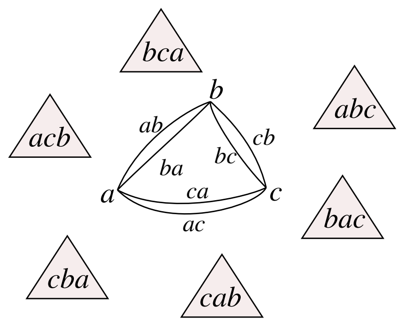
3.8 Important examples

3.8.1 Complex of injective words

Let $n \in \mathbb{N}$ and let $[n] = \{1, \ldots, n\}$. The *complex of injective words* $(I_n)_\bullet$ is a semisimplicial set with p-simplices the ordered $(p+1)$-letter words consisting of distinct letters from $[n]$. The $(p+1)$-simplices are attached to p-simplices via order-preserving face maps

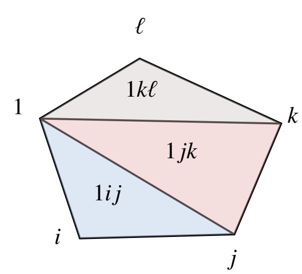
$$d_i(a_0 \cdots a_p) = a_0 \cdots \hat{a}_i \cdots a_p.$$

The boundary map $\partial : C_{p+1} \to C_p$ is given by $\partial = \sum_{i=0}^{p} (-1)^i d_i$.

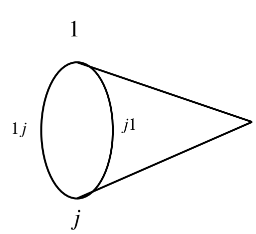


Prop 3.93. *For $n \geqslant 3$, $||(I_n)_\bullet||$ is simply-connected.*

Proof. Since $\pi_1((I_n)) = \pi_1((I_n)^{(2)})$, it suffices to show that the 2-skeleton of I_n is simply-connected. Let $f : (S^1, s_0) \to ((I_n)^{(2)}, \{1\})$ be a map based at the vertex 1. By cellular approximation 1.38, we may assume that f is cellular, so $\operatorname{Im} f$ lies in the 1-skeleton of I_n. If the loop f consists of 1-simplices with distinct vertex sets, then we can homotope f across the 2-cells in turn



Thus we only need to consider loops of the form $1j \to j1$ for $j \neq 1$. Since $n \geqslant 3$, there exists a vertex $k \neq 1, j$. Then both $1jk$ and $j1k$ are 2-simplices in I_n, and when glued along their boundaries they form a 2-disk



whose boundary is our loop. We can therefore nullhomotope this loop over the 2-disk. □

Prop 3.94. $||I_n||$ *is pure* $(n-1)$*-dimensional and shellable, with spanning simplices in bijection with derangements of* $[n]$*. Consequently*

$$I_n \simeq \bigvee_{!n} S^{n-2}.$$

Proof. By Theorem 3.89, the homotopy equivalence to a wedge of spheres follows once we show that I_n is shellable and has $!n$ many spanning simplices. Consider the lexicographic order on length n injective words of $[n]$. We claim that this is a shelling order on I_n, which means we need to show that $F_j \cap (\bigcup_{i=1}^{j-1} F_i)$ is a pure and maximal dimensional subcomplex of ∂F_j. This is equivalent to showing that if $\sigma \in F_i \cap F_j$ with $i < j$ and $|\sigma| \leqslant n-2$, then there exists some $k < j$ such that $\sigma \in F_k \cap F_j$ and $|F_k \cap F_j| = n-1$. Suppose such a σ exists. We consider the following claim:

Claim 3.95. *There exists an element* $x \in F_j \backslash \sigma$ *such that* x *is less than a predecessor or greater than its immediate successor.*

Suppose not. Then for all $x \in F_j \backslash \sigma$, x is greater than all of its predecessors and less than its immediate successor. Since x is greater than all of its predecessors, we cannot move x forward in F_j and get a lesser word (lexicographically) containing σ. Similarly, since x is less than its immediate successor, we cannot move x backwards (any number of positions) to get a lesser word lexicographically containing σ. Thus F_j must be the first word to contain σ in our order. But this contradicts $\sigma \in F_i$ with $i < j$.

Suppose $x \in F_j \backslash \sigma$ is less than a predecessor. Moving x to be in front of this predecessor and maintaining the order of the remaining letters gives a lesser word lexicographically which contains σ and which intersects F_j in an ordered $(n-1)$-word, as desired. Similarly, if x is greater than its immediate successor, performing the swap with the immediate successor achieves the same goal. This proves that the lexicographic order gives a shelling order.

It remains to recognize the spanning simplices (those for which $F_j \cap (\bigcup_{i=1}^{j-1} F_i) = \partial F_j$) as being in bijection with derangements of $[n]$. Two ways to do this: one which has the advantage of explicitly identifying the bijection, but the disadvantage that this bijection is not S_n-equivariant. The other just uses the fact that I_n being $(n-1)$-spherical means we can easily compute its Euler characteristic. Let's do this:

$$\sum_{i=0}^{n-1} (-1)^i \dim C_i(I_n) = \chi(I_n) = \sum_{i=0}^{n-1} (-1)^i \dim H_i(I_n) = 1 + (-1)^{n-1}\ell$$

where ℓ is the number of $(n-1)$-spheres in the homotopy type of I_n. Hence

$$\begin{aligned}
\sum_{i=0}^{n-1} (-1)^i \binom{n}{i+1} (i+1)! = 1 + (-1)^{n-1}\ell \implies \ell &= (-1)^{n-1} \cdot \left(-1 + \sum_{i=0}^{n-1} (-1)^i \binom{n}{i+1} (i+1)!\right) \\
&= (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1+i} \binom{n}{i+1} (i+1)! \\
&= (-1)^n + \sum_{i=0}^{n-1} (-1)^{n-1+i} \frac{n!}{(n-1-i)!} \\
&= (-1)^n + n! \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = !n.
\end{aligned}$$

This shows that $I_n \simeq \bigvee_{!n} S^{n-2}$. $\square$

Corollary 3.96. *The k-skeleton $I_n^{(k)}$ is $(k-1)$-connected.*

Proof. Can give a shelling (lexicographic order on $(k+1)$-element subsets of $[n]$). Or we can use the fact that $\pi_i(X) \cong \pi_i(X^{(i+1)})$ for every $i \geqslant 0$, so $\pi_i(I_n^{(k)}) \cong \pi_i(I_n) = 0$ for $i < k$. □

3.8.2 Complex of partial bases

Definition 3.97 (Partial basis)**.** Let R be a PID, and let M be a finite-rank free R-module. A set $\{b_1, \ldots, b_k\}$ in M is called a partial basis if it is a subset of a basis for M, equivalently, if it is the basis of a direct summand of M.

Lemma 3.98. *A vector $v \in R^n$ is called primitive if its entries generate R. A single vector forms a partial basis if and only if it is primitive.*

Proof. Suppose $v \in R^n$ forms a partial basis. Then there exist $v_2, \ldots, v_n \in R^n$ such that $v, v_2, \ldots, v_n$ form a basis for R^n. Let A be the matrix with columns $v, v_2, \ldots, v_n$, and let $T = A^{-1}$. Then $TA = I_n$, and we have (for T_i the i^{th} column of T),

$$[v]_1 T_1 + \cdots + [v]_n T_n = e_1.$$

In particular this means there exists $r_1, \ldots, r_n \in R$ such that $r_1[v]_1 + \cdots + r_n[v]_n = 1$, so the entries of v generate R, and v is primitive.

On the other hand, if the entries of v generate R, then one can row reduce the column vector v to look like e_i. Clearly this spans a direct summand of R^n, and this summand is invariant under invertible linear transformations (in the sense that $e_i \oplus W \cong R^n$ implies $Ae_i \oplus AW \cong R^n$ for A invertible), so the original vector v must have spanned a direct summand of R^n.

□

Definition 3.99 (Partial basis complex)**.** Let $M \cong R^n$ be a finite-rank free R-module. The partial basis complex $\mathrm{PB}(M)$ has vertices the primitive vectors of R^n, and $\{v_0, \ldots, v_p\}$ span a p-simplex in $\mathrm{PB}(M)$ whenever $v_0, \ldots, v_p$ form a partial basis for M.

Theorem 3.100 (Maazen's Theorem)**.** *When R is Euclidean ($\implies$ PID), $\mathrm{PB}_n(R)$ is Cohen-Macaulay of dimension $n-1$.*

Proof. The general strategy is to prove the theorem inductively, and identify links in $\mathrm{PB}_n(R)$ as other partial basis complexes. We fix the following notation:

$$\mathrm{PB}_n^m := \mathrm{Lk}_{\mathrm{PB}_{n+m}}(\{e_1, \ldots, e_m\})$$

where $e_1, \ldots, e_m$ are standard basis vectors in R^{n+m}. Hence $\mathrm{PB}_n^0 := \mathrm{PB}_n$. Note that PB_n^m is a subcomplex of PB_{n+m} consisting of $\tau = \{v_0, \ldots, v_p\}$ such that $\tau \cup \{e_1, \ldots, e_m\}$ is a simplex in PB_{n+m}, that is, $v_0, \ldots, v_p, e_1, \ldots, e_m$ form a partial basis for R^{n+m} (this encodes $\tau \cap \{e_1, \ldots, e_m\} = \varnothing$). So $\mathrm{Lk}_{\mathrm{PB}_{n+m}}(\{e_1, \ldots, e_m\})$ is a complex of partial bases of direct complements to $R^m = Re_1 \oplus \cdots \oplus Re_m \subseteq R^{n+m}$.

We use a few facts:

1. $\mathrm{PB}_0^m = \varnothing$: $\mathrm{PB}_0^m = \mathrm{Lk}_{\mathrm{PB}_m}(\{e_1, \ldots, e_m\}) = \varnothing$ since $R^m = \langle e_1, \ldots, e_m \rangle$.
2. PB_n^m has dimension $n-1$ for all $m \geqslant 0$. Certainly $\dim(\mathrm{PB}_n^m) \leqslant n-1$ since a basis of R^{n+m} must have $\leqslant n+m$ vectors, and hence to be in the link of $\{e_1, \ldots, e_m\}$ the simplex τ must have $\leqslant n$ vertices (and so be $\leqslant (n-1)$-dimensional). We also know that such a simplex exists: $\tau = \{e_{m+1}, \ldots, e_{n+m}\}$.

In particular this shows that $\dim(\mathrm{PB}_n) = \dim(\mathrm{PB}_n^0) = n-1$. We now want to show that PB_m^n is $(n-2)$-connected (since $\dim(\mathrm{PB}_n^m) = n-1$), and that for every $\sigma \in \mathrm{PB}_n^m$, $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$ is $n - \dim(\sigma) - 3$-connected. When $m = 0$ this will give Maazen's result.

Strat: "Link arguments"

1. Assume by induction on n that for some $n \geqslant 0$, $\mathrm{PB}_{n'}^{m'}$ is CM of dimension $n'-1$ for all $n' < n$ and for all $m' \geqslant 0$. We have this for $n = 1$, since $\mathrm{PB}_0^m = \varnothing$ for all $m \geqslant 0$.
2. We wish to show that PB_n^m is $(n-2)$-connected. Let $\phi : S^p \to \mathrm{PB}_n^m$ for $0 \leqslant p \leqslant n-2$. By the simplicial approximation theorem, we may assume ϕ is simplicial, and since S^p is compact, we can force (?) ϕ to be simplicial with respect to a finite simplicial structure on S^p.

   We'll define a "badness function" $R = R_\phi$ on the vertices x of S^p to $\mathbb{Z}_{\geqslant 0}$ that we view as a measure of the "badness" of the image $\phi(x)$. Define R so that, if we can homotope ϕ to reduce R to 0 at every vertex, the resulting map is nullhomotopic. We do this by pushing the image of ϕ into its link without introducing new bad vertices.

3. Understanding links in PB_n^m: Given a simplex $\sigma \in \mathrm{PB}_n^m$, we have an isomorphism of simplicial complexes $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma) \cong \mathrm{PB}_{n-\dim(\sigma)-1}^{m+\dim(\sigma)+1}$. Why? Recall that $\mathrm{PB}_n^m = \mathrm{Lk}_{\mathrm{PB}_{n+m}}(\{e_1, \dots, e_m\})$, so by lemma 3.14, we have

   $$\mathrm{Lk}_{\mathrm{Lk}_{\mathrm{PB}_{n+m}}(\{e_1,\dots,e_m\})}(\sigma) = \mathrm{Lk}_{\mathrm{PB}_{n+m}}(\sigma \cup \{e_1, \dots, e_m\}).$$

   Let $\sigma = \{v_0, \dots, v_p\}$ be a p-dimensional simplex, with $p \leqslant n-1$. Since $v_0, \dots, v_p, e_1, \dots, e_m$ forms a partial basis for R^{n+m}, and $e_1, \dots, e_m, \dots, e_{m+p+1}$ forms a partial basis for R^{n+m}, there exists a change of basis isomorphism $F : R^{n+m} \to R^{n+m}$ sending $e_1, \dots, e_m, v_0, \dots, v_p$ to $e_1, \dots, e_{m+p+1}$.This induces a simplicial isomorphism on the simplicial complexes. Without loss of generality, then, we may assume that $\sigma = \{e_{m+1}, \dots, e_{m+p+1}\}$. Hence $\mathrm{Lk}_{\mathrm{PB}_{n+m}}(\sigma \cup \{e_1, \dots, e_m\}) \cong \mathrm{Lk}_{\mathrm{PB}_{n+m}}(\{e_1, \dots, e_{m+p+1}\}) \cong \mathrm{PB}_{n-p-1}^{m+p+1}$. By induction, this is CM of dimension $n-p-2$, and is $(n - \dim(\sigma) - 3)$-connected, as desired.
4. It remains to show that PB_n^m is $(n-2)$-connected. Let $\phi : S^p \to \mathrm{PB}_n^m$ be map from a p-sphere (WLOG, simplicial with respect to finite simplicial structure on S^p). Nontrivial result from PL topology allows for S^p to be a combinatorial p-sphere, which means that its links are spheres. Define the following function on the vertices of PB_n^m, recalling that by lemma 3.98, a vertex in PB_n^m is a primitive vector in R^{n+m} (its entries generate R).

   $$\begin{aligned} F : \{\text{vertices of } \mathrm{PB}_n^m\} &\to \mathbb{Z}_{\geqslant 0} \\ v &\mapsto |(m+n)^{th} \text{ coordinate of } v| \end{aligned}$$

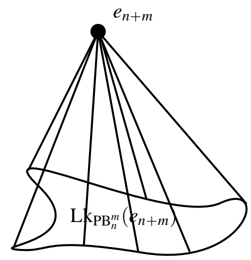
   where $|\cdot|$ is the Euclidean norm on R. Now we can define, associated to ϕ, the badness function $R = R_\phi$:

   $$\begin{aligned} R_\phi : \{\text{vertices of } S^p\} &\to \mathbb{Z}_{\geqslant 0} \\ x &\mapsto F(\phi(x)) \end{aligned}$$

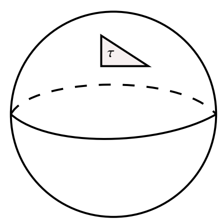
   Let $R^{\max} = \max_{x \in S^p} R(x)$. We want to homotope ϕ so that $R^{\max} = 0$.

   Why? What happens if $R^{\max} = 0$? Suppose this is the case, that is, $R(x) = 0$ for all vertices $x \in S^p$. We claim that $\operatorname{Im}\phi \in \mathrm{Lk}_{\mathrm{PB}_n^m}(\{e_{n+m}\})$. Since $R(x) = 0$ for all vertices $x \in S^p$, then (by definition) $F(\phi(x)) = 0$ for all vertices $x \in S^p$, which means that $\phi(x)$ has last coordinate 0 for all $x \in S^p$. (Because ϕ is simplicial, if $x_0, \dots, x_\ell$ are the finitely many vertices in S^p, then $\phi(x_i)$ is a simplex (vertex) in PB_n^m.) In particular, we see that $\langle\phi(x_i)\rangle \cap \langle e_{n+m}\rangle = \varnothing$ for any i. So if $\{x_0, \dots, x_q\}$ form a simplex in S^p, then $\{\phi(x_0), \dots, \phi(x_q)\}$ form a simplex in PB_n^m. Assuming without loss of generality that we have gotten rid of repeated vertices, the vectors $e_1, \dots, e_m, \phi(x_0), \dots, \phi(x_q)$ form a partial basis of R^{n+m}, and hence a basis for a direct summand of R^n in the complement of $\langle e_{n+m}\rangle$. Adjoining e_{n+m} to the partial basis then preserves the fact of being a partial basis, so that $\{\phi(x_0), \dots, \phi(x_q)\} \in \mathrm{Lk}_{\mathrm{PB}_n^m}(\{e_{n+m}\})$. (Really just needed that $\{\phi(x_0), \dots, \phi(x_q)\} \in \mathrm{Star}_{\mathrm{PB}_n^m}(e_{n+m})$. The point is that there exists a *single* vertex $\{e_{n+m}\}$ for which every simplex in $\phi(S^p)$ is in its link– not a different vertex for each simplex!)

Thus ϕ is homotopic to the constant map at e_{n+m} (we push ϕ through $\mathrm{Star}_{\mathrm{PB}_n^m}(e_{n+m}) = e_{n+m} * \mathrm{Lk}_{\mathrm{PB}_n^m}(e_{n+m})$).



5. Now suppose $R^{\max} = N > 0$. Let τ be a maximal dimensional simplex in S^p having the property that $R(x) = N$ for all $x \in \tau$.



We want to homotope ϕ off of $\phi(\tau)$ and into its link, in a way that reduces the R-values of ϕ and doesn't add new bad vertices.

Definition 3.101 (Subcomplexes $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$)**.** Fix $N \geqslant 0$. For a subcomplex X of PB_n^m, we use $X^{<N}$ to denote the subcomplex of X spanned by vertices $v \in X$ satisfying $F(v) < N$.

Claim 3.102. *If σ is a simplex in PB_n^m and w is a vertex of σ such that $F(w) = N > 0$, we claim there exists a simplicial retraction*

$$\pi = \pi_{\sigma,w} : \mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma) \to \mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}.$$

We construct π simplicially: first we define it on vertices, and then check that for every collection of vertices in $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$ spanning a simplex, the image vertices span a simplex in $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$. On a vertex $v = \begin{bmatrix} [v]_0 \\ \vdots \\ [v]_{n+m} \end{bmatrix}$, apply the Euclidean algorithm to write $[v]_{n+m} = q[w]_{n+m} + r = qN + r$ where $q, r \in R$ and $|r| < N$. Then define $\pi(v) = v - qw$, so that now $F(\pi(v)) < N$.

First we check that π is well-defined on vertices: if $\sigma = \{w_1, \dots, w_k\}$ and $v \in \mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$ (so $e_1, \dots, e_m, w_1, \dots, w_k, v$ form a partial basis for R^{n+m}), we need to show that $e_1, \dots, e_m, w_1, \dots, w_k, v - qw$ forms a partial basis for R^{n+m} (where $w \in \{w_1, \dots, w_k\}$ is chosen so that $F(w) = N$, and q is chosen as above). Recall that forming a partial basis for R^{n+m} is equivalent to forming a basis for a direct summand W of R^{n+m}. Let $e_1, \ldots, e_m, w_1, \ldots, w_k, v$ be a basis for W. Then for every $x \in W$, there exist unique $\alpha_i, \beta_j, \gamma \in R$ such that

$$x = \sum_{i=1}^{m} \alpha_i e_i + \sum_{j=1}^{k} \beta_j w_j + \gamma v.$$

Assuming $w = w_k$, x can also be expressed uniquely as

$$x = \sum_{i=1}^{m} \alpha_i e_i + \sum_{j=1}^{k-1} \beta_j w_j + (\beta_k + \gamma q) w_k + \gamma(v - qw).$$

Hence $e_1, \ldots, e_m, w_1, \ldots, w_k, v - qw$ is a basis for W. We also need to check that π can be extended over simplices, that is, if $\{v_1, \ldots, v_\ell\} \in \mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$, then for $w \in \{w_1, \ldots, w_k\}$ with $F(w) = N$ and integers $q_1, \ldots, q_\ell$ such that $F(v_1 - q_1 w), \ldots, F(v_\ell - q_\ell w) < N$, the set $e_1, \ldots, e_m, w_1, \ldots, w_k, v_1 - q_1 w, \ldots, v_\ell - q_\ell w$ is a partial basis for R^{n+m} (in particular, the vectors $v_i - q_i w$ are primitive). Again, this follows quickly from the same reasoning as above: if we express $x \in W (= \langle e_1, \ldots, e_m, w_1, \ldots, w_k, v_1, \ldots, v_\ell \rangle)$ uniquely as

$$x = \sum_{i=1}^{m} \alpha_i e_i + \sum_{j=1}^{k} \beta_j w_k + \sum_{s=1}^{\ell} \gamma_s,$$

then we can also express x uniquely as

$$x = \sum_{i=1}^{m} \alpha_i e_i + \sum_{j=1}^{k-1} \beta_j w_k + (\beta_k + \sum_{s=1}^{\ell} \gamma_s q_s) w_k + \sum_{s=1}^{\ell} \gamma_s (v_s - qw).$$

This shows that π is a well-defined simplicial map. We note also that π is the identity on $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$, as then $q_i = 0$ for all i. This proves that π is a retraction.

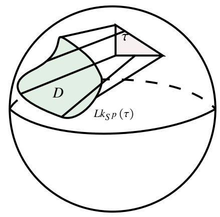
6. Since π is a simplicial retraction, if σ is a simplex in PB_n^m and $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)$ is $(n - \dim(\sigma) - 3)$-connected (which holds by our inductive hypothesis), then $\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N}$ must also be $(n - \dim(\sigma) - 3)$-connected (π induces a surjection $\pi_* : \pi_p(\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)) \to \pi_p(\mathrm{Lk}_{\mathrm{PB}_n^m}(\sigma)^{<N})$).

7. Call a simplex in S^p "bad" if every vertex has maximally-bad R-value N. Let τ be a bad simplex of maximal dimension in S^p. Suppose $\dim(\tau) = k$, and $\dim(\phi(\tau)) = \ell \leqslant k$. Since we assumed S^p is a combinatorial p-sphere, its links look like spheres: $\mathrm{Lk}_{S^p}(\tau) \cong S^{p-\dim(\tau)-1}$. Since τ was chosen to be maximal dimensional, we know that $R(x) < N$ for all $x \in \mathrm{Lk}_{S^p}(\tau)$ (else $\{x\} \cup \tau \in S^p$ would be a larger-dimensional simplex with $R(x) = N$ for all vertices, a contradiction). Hence $\phi(x) \notin \phi(\tau)$ (as $F(\phi(v)) = N$ for all $v \in \tau$, and $F(\phi(x)) < N$). As ϕ is assumed to be simplicial, we see that

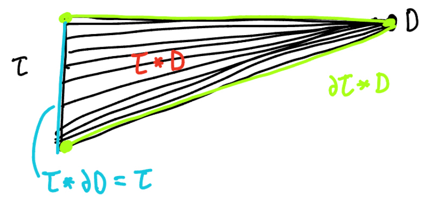
$$\phi(\mathrm{Lk}_{S^p}(\tau)) \subseteq \mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\tau))^{<N}.$$

8. Since $\ell \leqslant k$, then $-k \leqslant -\ell$. As we assumed $p \leqslant n - 2$, then $p - k - 1 \leqslant n - 3 - \ell$. Recall that $\mathrm{Lk}_{S^p}(\tau) \cong S^{p-\dim(\tau)-1} \cong S^{p-k-1}$, and hence the restriction of ϕ to $\mathrm{Lk}_{S^p}(\tau)$ can be viewed as a map $S^{p-k-1} \to \mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\tau))^{<N}$. But $\mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\tau))^{<N}$ is $(n - 3 - \ell)$-connected, and $p - k - 1 \leqslant n - 3 - \ell$, so the restriction of ϕ to $\mathrm{Lk}_{S^p}(\tau)$ is nullhomotopic. By results in PL topology (analogous to the ones we have for maps from topological spheres), there exists a *combinatorial* $(p - k)$-disk D and a simplicial map $\psi : D \to \mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\tau))^{<N}$ such that $\partial D = \mathrm{Lk}_{S^p}(\tau)$ and $\psi|_{\mathrm{Lk}_{S^p}(\tau)} = \phi|_{\mathrm{Lk}_{S^p}(\tau)}$.

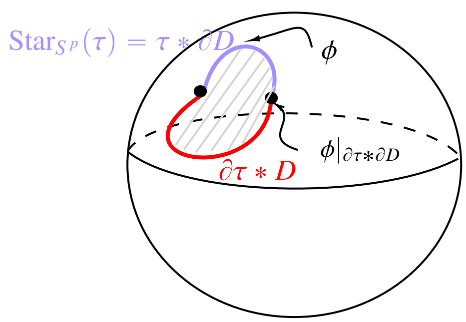
Claim 3.103. *$\phi|_\tau * \psi : \tau * D \to \mathrm{PB}_n^m$ defines a homotopy from $\phi|_{\tau * \mathrm{Lk}_{S^p}(\tau)}$ to $\phi|_{\partial\tau} * \psi$.*



Note that $\partial(\tau * D) = (\partial\tau * D) \cup (\tau * \partial D)$. For example, see



On one end of this boundary the map $\phi|_\tau * \psi$ restricts to $\phi|_{\partial\tau} * \psi$, and on the other end the map restricts to $\phi|_\tau * \psi|_{\partial D} = \phi|_\tau * \phi|_{\mathrm{Lk}_{S^p}(\tau)} = \phi|_{\mathrm{Star}_{S^p}(\tau)}$. Since τ is a k-simplex and $D \cong D^{p-k}$, $\tau * D$ is homeomorphic to a p-disk, and hence we can homotope the two maps on the different parts of the boundary through this disk. Note that the homotopy is continuous on $\tau * D$, since on the area where the two boundary components agree $(\partial\tau * \partial D)$, both maps are just $\phi|_{\partial\tau * \mathrm{Lk}_{S^p}(\tau)}$. Hence we can extend the homotopy to fix ϕ on the rest of S^p pointwise:



Crucially, while S^p might still have k-dimensional bad simplices, at least on the subcomplex $\mathrm{Star}_{S^p}(\tau)$ we have now reduced the dimension of the maximally bad simplices, since every simplex in $\partial\tau * D$ is the join (union) of a simplex in $\partial\tau$ and a simplex in D, and ϕ maps D to $\mathrm{Lk}_{\mathrm{PB}_n^m}(\phi(\tau))^{<N}$ (hence the only bad simplices in $\partial\tau * D$ are the faces of τ, which have strictly smaller dimension!).

9. If $\dim\tau = 0$, then $\partial\tau = \varnothing$, so this procedure reduced the R-value on τ. By iterating this procedure, we can homotope ϕ to have R-value strictly less than N on every vertex in the disk $\tau * \mathrm{Lk}_{S^p}(\tau)$. Repeating this procedure lowers $R^{\max}$. By induction we can reduce $R^{\max}$ to 0.

□

Remark 3.104. The high-connectivity of this complex is used to prove the *Ash-Rudolph theorem*, that $\mathrm{St}_n(F)$ (for F the field of fractions of R) is generated by integral apartment classes. (If $F^n = L_1 \oplus \cdots \oplus L_n$, and $S(L_1, L_2, \ldots, L_n)$ is the full subcomplex of $T_n(F)$ on the vertices consisting of direct sums of proper subsets of $\{L_1, \ldots, L_n\}$ (thus forming an apartment class), then we say that this apartment class is *integral* if $(L_1 \cap R^n) \oplus (L_2 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n$.

3.8.3 Tits building (of type A)

Definition 3.105 ($T(V)$)**.** Let V be a finite dimensional vector space over a field k. The Tits building $T(V)$ (of type A) is the order complex of the poset of proper, nonzero subspaces of V, ordered by inclusion.

Remark 3.106. The longest possible flag of simplices in $T_n(\mathbb{Q})$ ($= T(\mathbb{Q}^n)$) has length $n-1$, so $T_n(\mathbb{Q})$ is $(n-2)$-dimensional.

Definition 3.107 (Apartments)**.** Fixing a basis of $\mathbb{Q}^n$, we have collections of simplices (apartment) corresponding to all possible flags of subspaces spanned by some combination of our chosen basis vectors.

Example 3.108. For instance, if we fix $n=3$ and the standard basis $\langle e_1, e_2, e_3\rangle$, then the apartment corresponding to $\langle e_1, e_2, e_3\rangle$ is

$$\begin{tikzcd}
 & & \langle e_1\rangle \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \langle e_1, e_2\rangle \arrow[dl, dash] & & \langle e_1, e_3\rangle \arrow[dr, dash] & \\
\langle e_2\rangle \arrow[rr, dash] & & \langle e_2, e_3\rangle \arrow[rr, dash] & & \langle e_3\rangle.
\end{tikzcd}$$

Notice that this is the subdivided boundary of a 2-simplex. In general, an apartment in $T_n(\mathbb{Q})$ looks like the subdivided boundary of an $(n-1)$-simplex (and hence is homeomorphic to an $(n-2)$-sphere).

Theorem 3.109 (Solomon-Tits)**.** *If V is an n-dimensional vector space, then $T(V)$ is $(n-2)$-dimensional and $(n-3)$-connected. In particular, $T(V) \cong \bigvee S^{n-2}$, with infinitely many spheres in the wedge.*

Proof. Idea is to build up the Tits building in $n-1$ steps. We first verify that the theorem holds when $\dim V = 1$ (so $V \cong k$, the underlying field). In this case we need to show that $T(V)$ is (-2)-connected. Since V has no proper, nonzero subspaces, we conclude that $T(V) = \varnothing$, which is (-2)-connected.

Let V be a k-vector space of dimension $n \geqslant 2$, and assume by induction that the theorem holds for vector spaces of dimension $1, 2, \ldots, n-1$. Fix a line (1-dimensional subspace) $L_0 \subseteq V$, and define a filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_{n-1} = T(V)$ as follows:

> $X_0 = \mathrm{Star}_{T(V)}(L_0)$.
>
> X_1 is the full subcomplex on X_0 and all lines $L \subset V$ not in X_0 (in this case, all lines not equal to L_0, since X_0 only contains higher-dimensional simplices, by virtue of how the Tits building is defined–two distinct lines cannot form a simplex in $T(V)$, need subspace containment).
>
> X_2 is the full subcomplex on X_1 and all planes $P \subset V$ not in X_0, that is, all planes not containing L_0. Note that there are no new edges between the new vertices in $X_2 \backslash X_1$, because we only have edges between vertices of differing dimensions (as subspaces of V).

$\vdots$

> X_{n-1} is the full subcomplex on X_{n-1} and all $(n-1)$-dimensional subspaces of V not contained in X_0 (so not containing L_0), which equals $T(V)$.

Schematic: Start with a subcomplex $\mathrm{Star}_{T(V)}(L_0)$, and then slowly add in the remaining simplices dimension by dimension.

Let $W \in X_i \backslash X_{i-1}$ ($i = 1, \ldots, n-2$) be a vertex, that is, a subspace W of dimension i not containing L_0. Then $\mathrm{Lk}_{X_i}(W)$ consists of all ℓ-dimensional subspaces of W (with $1 \leqslant \ell \leqslant i-1$) as well as the vertices $\{L_0 \oplus W \oplus \tilde{W}\}$ for $\tilde{W}$ of dimension between 0 and $n-2-i$. (Note that such a vertex necessarily belongs to X_i because it belongs to $X_0 = \mathrm{Star}_{T(V)}(L_0)$.) This is a cone with cone point $\{L_0 \oplus W\}$: any vertex of the form $\{Q\}$ for $Q \subset W$ is necessarily a subspace of $L_0 \oplus W$, and hence $\{Q, L_0 \oplus W\}$ is a simplex in $\mathrm{Lk}_{X_i}(W)$, and $L_0 \oplus W$ is a subspace of $L_0 \oplus W \oplus \tilde{W}$, so the two vertices form an edge in $\mathrm{Lk}_{X_i}(W)$. We conclude that $\mathrm{Lk}_{X_i}(W)$ is contractible.

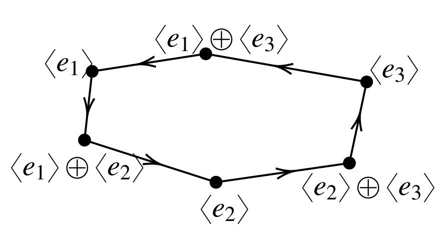
Since $X_0 = \mathrm{Star}_{T(V)}(L_0)$ is a cone (with vertex L_0), it is contractible. By a repeated application of 3.33, we conclude that $X_1, \ldots, X_{n-2}$ are also contractible. Now let $W \in X_{n-1} \backslash X_{n-2} = T(V) \backslash X_{n-2}$. Since W is maximal dimensional as a proper subspace of V, this time we have that $\mathrm{Lk}_{T(V)}(W)$ consists only of proper subspaces of W. Hence $\mathrm{Lk}_{T(V)}(W) \cong T(W)$. Since $\dim W = n-1$, by induction $T(W)$ is $(n-4)$-connected. By 3.33 again, we have that $T(V)$ is $(n-3)$-connected.

By 1.32, $T(V)$ is either contractible or homotopy equivalent to a wedge of spheres. But $T(V)$ is not contractible (by 3.112, $\mathrm{St}_n(T_n(F))$ is generated by integral apartment classes, of which $T_n(F)$ has many), so must be homotopy equivalent to a wedge of spheres. □

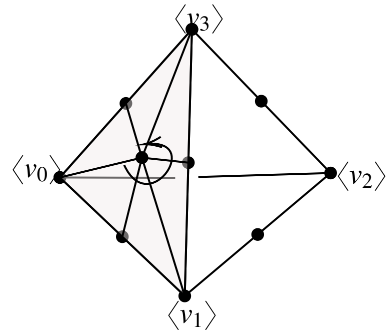
Remark 3.110 (Alternate proof using nerves). Let $T(V)$ be covered by $T(V)_{\geqslant L}$ where L ranges over the set P of lines in V. We claim that a nonempty finite intersection $\bigcap_{i=1}^m T(V)_{\geqslant L_i}$ is contractible: note that if $L_1 \oplus \cdots \oplus L_m$ is a proper subspace of V, and $\sigma = \{W_0, \ldots, W_p\}$ is a simplex in $\bigcap_{i=1}^m T(V)_{\geqslant L_i}$, then $L_i \subset W_j$ for every $1 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant p$. In particular, since the lines $L_1, \ldots, L_m$ are distinct, then each W_j contains $L_1 \oplus \cdots \oplus L_m$, which means that $\sigma \cup \{L_1 \oplus \cdots \oplus L_m\}$ is a simplex in $\bigcap_{i=1}^m T(V)_{\geqslant L_i}$. In particular $\bigcap_{i=1}^m T(V)_{\geqslant L_i}$ is a cone with cone point $L_1 \oplus \cdots \oplus L_m$, so is contractible. By the nerve lemma 3.21, we conclude that $T(V) \simeq \mathcal{N}(T(V))$. If $s \leqslant n-1$, then any collection of lines $\{L_1, \ldots, L_s\}$ corresponds to a simplex $\{T(V)_{\geqslant L_1}, \ldots, T(V)_{\geqslant L_s}\}$ in $\mathcal{N}(T(V))$, since the intersection $\bigcap_{i=1}^s T(V)_{\geqslant L_i}$ contains $L_1 \oplus \cdots \oplus L_s$. Thus the nerve of $T(V)$ is isomorphic to the full $(n-2)$-skeleton of the (infinite) simplex whose vertices are elements in P, and we know that such a skeleton is $(n-3)$-connected. Thus $T(V)$ (being homotopy equivalent to its nerve) is $(n-3)$-connected (and being of dimension $n-2$, must be $(n-2)$-spherical).

Definition 3.111 (Integral apartments)**.** If an apartment spanned by the lines $L_1 \oplus \cdots \oplus L_n = \mathbb{Q}^n$ satisfies $(L_1 \cap \mathbb{Z}^n) \oplus \cdots \oplus (L_n \cap \mathbb{Z}^n) = \mathbb{Z}^n$, we call this an <u>integral apartment</u>.

Visually: an apartment looks like the full subcomplex on direct sums of proper subsets of $\{L_1, \ldots, L_n\}$ (where $L_1 \oplus \cdots \oplus L_n = F^n$). We can orient the apartment by first choosing an ordering of $L_1, \ldots, L_n$ and then inducing an orientation via



Similarly,



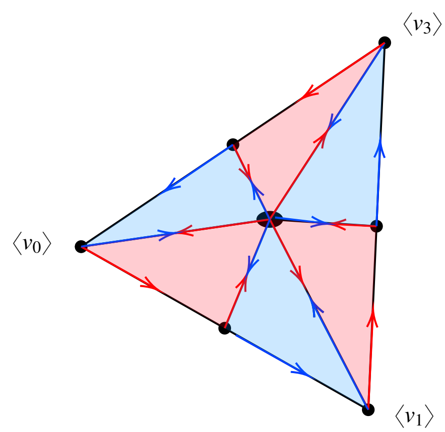
(The induced orientation on the boundary of the usual orientation of n-simplices.) Then just as orientations of simplices are well-defined up to even permutations of the vertices, the orientation of the apartment is well-defined up to even permutations of a choice of ordering of $L_1 < \cdots < L_n$. Notice that in the first example, viewing this subdivided boundary as a 1-chain in $T_3(\mathbb{Q})$, we have

$$0 \to C_1(T_3(\mathbb{Q})) \to C_0(T_3(\mathbb{Q})) \to \cdots$$

$$\partial_1([\langle e_1\rangle, \langle e_1, e_2\rangle] + [\langle e_1, e_2\rangle, \langle e_2\rangle] + [\langle e_2\rangle, \langle e_2, e_3\rangle] + [\langle e_2, e_3\rangle, \langle e_3\rangle] + [\langle e_3\rangle, \langle e_1, e_3\rangle] + [\langle e_1, e_3\rangle, \langle e_1\rangle]) = 0.$$

So this apartment corresponds to a cycle in $C_1(T_3(\mathbb{Q}))$, and thus a generator of $\tilde{H}_1(T_3(\mathbb{Q})) = \mathrm{St}_3(\mathbb{Q})$.

In the second example, we orient each 2-simplex so that it respects the natural orientation on the boundary. That is, if we focus on the 2-simplices in the subdivision of $\{v_0, v_1, v_3\}$, we get:



Theorem 3.112 (Ash-Rudolph [8])**.** *When R is Euclidean, $\mathrm{St}_n(F)$ is generated by integral apartment classes for $n \geqslant 2$.*

Proof. Let B be the poset of proper nonzero summands of F^n under inclusion, and let A be the poset of proper partial bases of R^n under inclusion. Then $|B| = T_n(F)$ and $|A|$ is $\mathrm{sd}((\mathrm{PB}_n(R))^{(n-2)})$ (subdivision of the $(n-2)$-skeleton of the order complex). We have a poset map

$$f : A \to B$$
$$\{v_0, \ldots, v_p\} \mapsto \mathrm{span}_F(v_0, \ldots, v_p).$$

which we claim satisfies the conditions of Quillen's lemma 3.18. Firstly, it is clear that f is increasing. We need to check that $|B| = T_n(F)$ is CM of dimension $n-2$. We already know (Solomon-Tits, 3.109) that $|B|$ is $(n-2)$-spherical, so it remains to show that $\mathrm{Lk}_{T_n(F)}(\sigma)$ is $(n-4-\dim(\sigma))$-connected for every simplex

$\sigma \in T_n(F)$. Write $\sigma = V_0 \subset \cdots \subset V_p \subset F^n$ for the chain of proper nonzero summands of F^n defining σ. Recall by 3.45, we have

$$\mathrm{Lk}_{T_n(F)}(\sigma) = B_{<V_0} * B_{(V_0,V_1)} * \cdots * B_{(V_{p-1},V_p)} * B_{>V_p}.$$

Now $B_{<V_0} \cong T(V_0)$, which is $(\dim(V_0) - 2)$-spherical (and hence $(\dim(V_0) - 3)$-connected). Similarly, $B_{(V_0,V_1)} \cong T(V_1/V_0)$, which is $(\dim(V_1) - \dim(V_0) - 2)$-spherical (and hence $(\dim(V_1) - \dim(V_0) - 3)$ connected), and so on. By 3.41 and the description above, we obtain that $\mathrm{Lk}_{T_n(F)}(\sigma)$ is

$$(\sum_{i=0}^{p+1} \dim(V_i) - \dim(V_{i-1}) - 3 + 2) - 2$$

connected, where $V_{-1} = \{0\}$ and $V_{p+1} = F^n$. This simplifies to $\dim(F_n) - (p+2) - 2 = n - 4 - p = n - 4 - \dim(\sigma)$, as desired. Now if $V \in F^n$ is a proper summand, then $f_{\leqslant V}$ consists of the partial bases of R^n whose F-span lies in V. This is just the poset of partial bases of $V \cap R^n \subset R^n$ (a rank $\dim_F(V)$-direct summand of R^n) under inclusion. By Maazen's theorem 3.100, $|f_{\leqslant V}|$ is CM of dimension $\dim(V) - 1 = \mathrm{ht}(V)$. By 3.18, we have that $f_* : \tilde{H}_{n-2}(|A|) \to \tilde{H}_{n-2}(|B|) = \mathrm{St}_n(F)$ is a surjection.

It remains to show that the image $f_*(\tilde{H}_{n-2}(|A|))$ is generated by integral apartment classes. Recall that the $(n-1)$-simplices in $\mathrm{PB}_n(R)$ correspond to bases of R^n, and their subdivided boundaries correspond to integral apartments in $T_n(F)$. Consider the pair $(\mathrm{PB}_n(R)), |A|)$, where we view $|A|$ as the codimension 1 skeleton of $\mathrm{PB}_n(R)$. (In reality, $|A|$ sits inside the subdivision of the codimension 1 skeleton of $\mathrm{PB}_n(R)$, but up to homeomorphism this does not matter.) The long exact sequence in homology yields an exact sequence

$$H_{n-1}(\mathrm{PB}_n(R), |A|) \to \tilde{H}_{n-2}(|A|) \to \tilde{H}_{n-2}(\mathrm{PB}_n(R)).$$

But $\mathrm{PB}_n(R)$ is $(n-2)$-connected by 3.100, so the Hurewicz theorem 1.21 tells us that $\tilde{H}_{n-2}(\mathrm{PB}_n(R)) = 0$. Thus $H_{n-1}(\mathrm{PB}_n(R), |A|) \to \tilde{H}_{n-2}(|A|)$ is a surjection. Recall: we have a simplicial chain complex

$$0 \to C_{n-1}(\mathrm{PB}_n(R))/C_{n-1}(|A|) \to C_{n-2}(\mathrm{PB}_n(R))/C_{n-2}(|A|) \to \cdots$$

Since $C_{n-1}(|A|) = 0$ (as $|A|$ is $(n-2)$-dimensional), then $H_{n-1}(\mathrm{PB}_n(R), |A|)$ is represented by elements in $C_{n-1}(\mathrm{PB}_n(R))$ whose boundaries lie in $C_{n-2}(|A|)$. But this holds for all simplices in $C_{n-1}(\mathrm{PB}_n(R))$, and these simplices correspond to integral apartment classes. (The connecting homomorphism sends an $(n-1)$-chain to its boundary in $|A|$.) Thus $\mathrm{St}_n(F)$ is generated by integral apartment classes. □

Remark 3.113. The Euclidean part here really comes into play in the proof of Maazen's theorem, which uses the fact that R is Euclidean to run the link argument.

3.8.4 Subtleties in the choice of rings

Crucially, the proof of theorem 3.112 used the fact that if R is a Dedekind domain, then being a partial basis of R^n whose span lies in V makes you a partial basis of V. This fact is not true in general!! This is recorded in Church-Farb-Putman as Lemma 2.2:

Lemma 3.114 (CFP Lemma 2.2). *Let $\mathcal{O}$ be a Dedekind domain with field of fractions K and let M be a finitely generated projective $\mathcal{O}$-module of rank $n \geqslant 1$.*

(a) A submodule $U \subset M$ is a direct summand of M if and only if M/U is torsion-free.

(b) If U and U' are summands of M and $U \subset U'$ then U is a summand of U'.

(c) *The assignment $U \mapsto U \otimes_O K$ defines a bijection*

$$\{\textit{direct summands of } M\} \leftrightarrow \{K - \textit{subspaces of } M \otimes_O K\}$$

with inverse $V \mapsto V \cap M$.

Sketch in the case of $O = R$ a PID. Proof motivated by [3] Lemma 2.5. We wish to show that if V is a subspace of F^n then $V \cap R^n$ is a direct summand of R^n. Since V is a subspace of F^n, we can express it as $\ker \varphi$ for $\varphi : F^n \to F^n$ a linear transformation. Then $V \cap R^n = \varphi|_{R^n}$, so we have a short exact sequence

$$0 \to V \cap R^n \to R^n \to \varphi(R^n) \to 0.$$

But $\varphi(R^n)$ is torsion-free (being an R-submodule of F^n), and finitely generated (being the image of a finitely generated free R-module) over a PID, so by the structure theorem it must be free. Hence the short exact sequence splits, so $V \cap R^n$ is a direct summand of R^n, as desired. □

Definition 3.115 (Dedekind domain)**.** A Dedekind domain is either a field or an integral domain which is integrally closed, Noetherian, and of Krull dimension 1 (nonzero primes are maximal).

Example 3.116 (Examples of Dedekind domains)**.** Every PID is a Dedekind domain. Algebraic integers O_K in a number field K are Dedekind domains. Euclidean domains are PIDs, so are also Dedekind domains.

Definition 3.117 (Rank of a finitely generated projective module)**.** Let M be a finitely generated projective R-module. The rank of M is given by $\mathrm{rk}(M) := \dim_K(M \otimes_R K)$ for K the field of fractions of R.

Remark 3.118. The Tits building of a PID R is the same as the Tits building of its field of fractions $\mathrm{Frac}(R)$. In particular $T_n(\mathbb{Z})$ is naturally isomorphic to $T_n(\mathbb{Q})$. Is the same true of the partial basis complexes?

Note that the Tits building of a PID R is the same as the Tits building of its field of fractions $\mathrm{Frac}(R)$, since nonzero proper summands of R^n are in bijection with nonzero proper subspaces of F^n ($F = \mathrm{Frac}(R)$) when R is a Dedekind domain (and PIDs are Dedekind domains) 3.114. The partial basis complex $\mathrm{PB}_n(R)$ has p-simplices the partial bases $\{v_0, \dots, v_p\}$ of R^n (i.e., if $p+1$ vectors $v_0, \dots, v_p$ span a $(p+1)$-rank direct summand of R^n), while $\mathrm{PB}_n(F)$ has p-simplices the partial bases $\{v_0, \dots, v_p\}$ of F^n. Do we have a correspondence between bases of R^n and bases of F^n? Certainly if $v_0, \dots, v_p$ span a rank $p+1$ summand of R^n, then their F-span is a $(p+1)$-dimensional subspace of F^n (as they must be linearly independent), and hence they form a partial basis of F^n. But going the other way is difficult: if $v_1, \dots, v_p$ form a basis for a p-dimensional subspace of F^n, it is *not* true that their R-analogues (obtained by clearing denominators) form a basis for a rank p free summand of R^n. We *do* know that $\langle v_1, \dots, v_p \rangle \cap R^n$ is a rank p free summand of R^n, and so there exists an R-basis $w_1, \dots, w_p$ of it. But much is unclear: how do we actually obtain these basis vectors w_i? How do these vectors relate to the v_i? Is there a *canonical* way to relate partial bases of F^n and partial bases of R^n? My guess is no.

4 Cohomology of arithmetic groups

4.1 Summary of main results

1. Ash-Rudolph prove that $\mathrm{St}_n(\mathbb{Z})$ (actually, $\mathrm{St}_n(O)$ for O Euclidean (which implies PID)) is generated by *integral* apartment classes. That is, if $L_1, \ldots, L_n$ are lines in K^n (1-dimensional subspaces such that $L_1 \oplus \cdots \oplus L_n = K^n$, where $K = \mathrm{Frac}(O)$) such that $O^n = (L_1 \cap O^n) \oplus \cdots \oplus (L_n \cap O^n)$, and $[S(L_1, \ldots, L_n)]$ is the image under the pushforward of the inclusion into $T_n(K)$ of a chosen generator (chosen orientation for the $(n-2)$-sphere determined by $S(L_1, \ldots, L_n)$ in the Tits building $T_n(O)$). This "integral generation" helps prove that the top cohomology of $\mathrm{SL}_n(\mathbb{Z})$ with $\mathbb{Q}$ coefficients vanishes.

2. If O_K is the ring of integers in $K = \mathbb{Q}(\sqrt{d})$ for $d = -19, -43, -67, -163$ (quadratic imaginary number rings which are PIDs but <u>not</u> Euclidean. In particular, their class number is 1), Miller-Patzt-Wilson-Yasaki show that $\mathrm{St}_n(K)$ is *not* generated by integral apartment classes. In fact, they construct nonvanishing cohomology classes in the top rational cohomology of $\mathrm{SL}_n(O_K)$ for some of these rings! Church-Farb-Putman proved that the top rational cohomology of $\mathrm{SL}_n(O_K)$ does not vanish if the class number of O_K is greater than 1, and Weinberger proved that the Generalized Riemann Hypothesis implies that every number ring either has class number great than 1, is Euclidean, or is quadratic imaginary. In particular, assuming GRH, MPWY show that for a number field K, $\mathrm{St}_n(K)$ is generated by integral apartment classes if and only if O_K is Euclidean.

4.2 Principal congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$

$\mathrm{SL}_n \mathbb{Z}$ has torsion-elements, so $\mathrm{cd}(\mathrm{SL}_n \mathbb{Z}) = \infty$, and its cohomology with $\mathbb{Z}$-coefficients is nonzero in infinitely many degrees. We claim that $\mathrm{SL}_n \mathbb{Z}$ has finite-index torsion-free subgroups, and by Serre's theorem all of these subgroups have the same cohomological dimension.

Lemma 4.1 (Brown II.4 Ex 3 [2])**.** *Fix $n \geqslant 1$, and let $N \geqslant 2$. Let Γ_N be the kernel of the canonical map*

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$$

(i.e., $\Gamma_N = \{g \in \mathrm{SL}_n(\mathbb{Z}) : g \equiv I_n \mod N\}$.) This group is called the principal congruence subgroup of level N, and it is a finite-index torsion-free subgroup of $\mathrm{SL}_n(\mathbb{Z})$ for $N \geqslant 3$.

Proof. Since $\mathrm{SL}_n(\mathbb{Z}/N\mathbb{Z})$ is finite, the first isomorphism theorem guarantees that Γ_N has finite index in $\mathrm{SL}_n(\mathbb{Z})$. It remains to show that Γ_N is torsion-free for $N \geqslant 3$. Suppose $A \in \mathrm{SL}_n(\mathbb{Z})$ is such that $A \equiv I_n \pmod{p}$ for some prime p. If $A \neq I_n$, we claim that there exists a unique positive integer $d = d(A)$ such that $A \equiv I_n \pmod{p^d}$ and $A \not\equiv I_n \mod p^{d+1}$: certainly there exists some positive integer d for which $A \equiv I_n \mod p^d$, since this holds for $d = 1$. If $A \equiv I_n \mod p^d$ for all d, then $A - I_n$ has entries divisible by p^d for all d, which means $A - I_n = 0$, which means $A = I_n$, a contradiction. Thus there exists some $d = d(A)$ as desired.

Note that if $q \neq p$ is a prime, then $d(A^q) = d(A)$: for we can express $A = I_n + p^d B$ for some $B \not\equiv 0 \mod p$, and then

$$A^q = (I_n + p^d B)^q = I_n + qp^d B + \binom{q}{2} p^{2d} B^2 + \cdots + qp^{(q-1)d} B^{q-1} + p^{qd} B^q$$

which is clearly equivalent to 0 mod p^d, but is not equivalent to 0 mod p^{d+1} since $p \nmid (qB + \binom{q}{2} p^d + \cdots + p^{(q-1)d} B^q)$, as $p \neq q$ (a prime) and p divides the remaining terms.

On the other hand, if p is odd or if $d(A) \geqslant 2$, then $d(A^p) = d(A) + 1$:

$$\begin{aligned} A^p = (I_n + p^d B)^p &= I_n + pp^d B + \binom{p}{2} p^{2d} B^2 + \cdots + pp^{(p-1)d} B^{p-1} + p^{pd} B^p \\ &= I_n + p^{d+1} B + \binom{p}{2} p^{2d} B^2 + \cdots + p^{(p-1)d+1} B^{p-1} + p^{pd} B^p \end{aligned}$$

so $A^p \equiv I_n \mod p^{d+1}$, but not mod p^{d+2} (as p^{d+2} divides every term after I_n but p^{d+1}).

We wish to show that the only $A \in \Gamma_N$ with finite order is the identity, when $N \geqslant 3$. Let $A \neq I_n \in \Gamma_N$. If there exists some k such that $A^k = I_n$, then there exists some prime q for which a power of A is the identity, so without loss of generality we may assume A has prime order q. Now fix a prime p dividing N. Since $A \in \Gamma_N$ means $A \equiv I_n \mod N$, then we must also have $A \equiv I_n \mod p$. Then $d(A)$ is a well-defined positive integer, and we know that $d(A^q) = d(A)$ if $q \neq p$ is a prime. But $A^q = I_n$, so $d(A^q)$ does not exist! On the other hand, since $N \geqslant 3$, then $p \geqslant 3$, so if $q = p$, we have $d(A^q) = d(A^p) = d(A) + 1$, again a contradiction. Hence Γ_N has no nonidentity elements of finite order. ◻

Prop 4.2. *Γ_N has type* FP.

Proof. This follows from the next section. Since there exists a contractible space X on which Γ_N acts freely and cocompactly, the chain complex $C_\bullet(X; \mathbb{Z})$ is a resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}G$-modules. We know that a symmetric space $\overline{X}$ exists that $\mathrm{SL}_n(\mathbb{Z})$ acts properly discontinuously and cocompactly on. Now Γ_N being torsion-free means that stabilizers of points in X must be trivial, so Γ_N acts freely on $\overline{X}$. ◻

4.3 Borel-Serre duality

Mostly taken from [8].

Let $X_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. ($\mathrm{SO}(n)$ is a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$.) This space is homeomorphic to the space of positive-definite symmetric real $n \times n$ matrices by sending the coset of a matrix $A \in \mathrm{SL}_n(\mathbb{R})$ to AA^T. We can compute the dimension of this space: $\dim_{\mathbb{R}}(X_n) = \dim_{\mathbb{R}}(\mathrm{SL}_n(\mathbb{R})) - \dim_{\mathbb{R}}(\mathrm{SO}(n)) = (n^2 - 1) - (\frac{n^2-n}{2}) = \frac{n^2+n}{2} - 1 = \frac{n(n+1)}{2} - 1$. (Recall: the tangent space to a vector space at the identity (which is also a real Lie group) has the same manifold dimension, so it suffices to compute the dimension of $T_I(\mathrm{SO}(n))$. This is the Lie algebra consisting of $n \times n$ skew-symmetric matrices, so its dimension as a real manifold is $\frac{n^2-n}{2}$.)

$\mathrm{SL}_n(\mathbb{Z})$ acts on X_n by multiplication on the left. This action is properly discontinuous (since $\mathrm{SL}_n(\mathbb{Z})$ is discrete?) but not free. But it has finite stabilizers since it acts properly (for every compact $K \subset X_n$, the set $\{\gamma \in \mathrm{SL}_n(\mathbb{Z}) : \gamma \cdot K \cap K \neq \varnothing\}$ is finite). For instance, the stabilizer of the coset of the identity is $\mathrm{SL}_n(\mathbb{Z}) \cap \mathrm{SO}(n)$, which must be finite since there are only finitely many unit vectors in $\mathbb{Z}^n$. Note that X_n is contractible, since we can contract the space of positive-definite symmetric real $n \times n$ matrices via the homotopy $H_t(P) = tI_n + (1-t)P$ (easy to see: at every $0 \leqslant t \leqslant 1$, $tI_n + (1-t)P$ is positive-definite symmetric). By 2.79, we know that $H^*(X_n/\Gamma; \mathbb{Q}) \cong H^*(\Gamma; \mathbb{Q})$ for any subgroup $\Gamma \leqslant \mathrm{SL}_n(\mathbb{Z})$. If Γ is torsion-free, then this isomorphism also holds integrally.

4.3.1 Bordification

X_n is not compact, so we cannot apply Poincaré-Lefschetz duality to it. The goal is to replace X_n with a "partial compactification" manifold $\overline{X_n}$ on which Γ still acts properly-discontinuously and with finite stabilizers. We also want to recognize $\partial\overline{X_n}$ as a familiar space ($T_n(\mathbb{Q})$, see 3.105). Borel and Serre construct

this bordification so that $\Gamma\backslash\overline{X_n}$ is a compact manifold with boundary whenever Γ is a finite-index subgroup. They also do it in such a way that the $\mathrm{SL}_n(\mathbb{Z})$ action on X_n extends to $\overline{X_n}$, and that on the boundary $\partial\overline{X_n}$ this action is the action of $\mathrm{SL}_n(\mathbb{Z})$ on $T_n(\mathbb{Q})$. They *also* do it in such a way that $X_n = \overline{X_n}\backslash\partial\overline{X_n}$ embeds into $\overline{X_n}$ as homotopy equivalence (so $\overline{X_n}$ is still contractible), and $\Gamma\backslash\overline{X_n}$ is the compactification of $\Gamma\backslash X_n$.
This means that

$$H^*(\Gamma;\mathbb{Q}) \cong H^*(X_n/\Gamma;\mathbb{Q}) \cong H^*(\overline{X_n}/\Gamma;\mathbb{Q}).$$

We look at the case $n = 2$ first. Then $X_2 = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2)$ is homeomorphic to the upper half plane $\mathbb{H}$: Send a coset of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $z = \frac{ai+b}{ci+d} = \frac{i+ac+bd}{c^2+d^2}$. Note that the coset of the identity is sent to i (and actually the only matrices which get sent to i are those belonging to $\mathrm{SO}(2)$). How does $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathbb{H}$? If $z = x + iy \in \mathbb{H}$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

implies that $Az = (ax+by) + (cx+dy)i$. Now on $\mathbb{H}$ we can draw horoballs: circles which touch the real line at reduced rationals $\frac{a}{b}$, with diameter $1/b^2$. A horizontal line at $z = i$ corresponds to the "horoball touching infinity". These horoballs touch but do not intersect. We can then slightly shrink the horoballs but keep their attachment points the same (at the reduced rationals $\frac{a}{b}$), remove the interiors of these balls from $\mathbb{H}$, and get a connected, contractible manifold with boundary.

The boundary of $\overline{X_2}$ is a union of open intervals indexed by the rationals, along with the interval given by the ∞ horoball, so the boundary is homotopy equivalent to $\mathbb{Q} \cup \{\infty\}$. We can identify this with the lines through $\mathbb{Q}^2$ by sending

$$q \in \mathbb{Q} \to \begin{pmatrix} q \\ 1 \end{pmatrix}$$
$$\infty \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This shows $\partial\overline{X_2} \simeq T_2(\mathbb{Q})$.

Now since $\mathrm{SL}_n(\mathbb{Z})$ acts properly discontinuously and cocompactly on $\overline{X_n}$, then any finite-index subgroup $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$ acts freely and cocompactly on $\overline{X_n}$. We thus have, by 2.142, that

$$H^k(\Gamma;\mathbb{Z}[\Gamma]) \cong H^k_C(\overline{X_n};\mathbb{Z}).$$

Poincaré-Lefschetz duality (1.79) then implies that $H^k_C(\overline{X_n};\mathbb{Z})$ is isomorphic to $H_{d-k}(\overline{X_n},\partial\overline{X_n};\mathbb{Z})$. Since $\overline{X_n}$ is contractible, the long exact sequence of the pair $(\overline{X_n},\partial\overline{X_n})$ yields $H_{d-k}(\overline{X_n},\partial\overline{X_n};\mathbb{Q}) \cong \tilde{H}_{d-k-1}(\partial\overline{X_n};\mathbb{Q})$. Finally, since $\partial\overline{X_n} \simeq T_n(\mathbb{Q})$, then we know $\tilde{H}_{d-k-1}(\partial\overline{X_n};\mathbb{Q}) \neq 0$ only when $d-k-1 = n-2$, or $k = d-n+1$. Since X_n is $(\frac{n(n+1)}{2} - 1)$-dimensional, then

$$H^k(\Gamma;\mathbb{Q}[\Gamma]) \neq 0 \iff k = \frac{n(n+1)-2n}{2} = \frac{n(n-1)}{2} = \binom{n}{2}.$$

This proves:

Prop 4.3 (vcd of $\mathrm{SL}_n(\mathbb{Z})$)**.** *The virtual cohomological dimension of* $\mathrm{SL}_n(\mathbb{Z})$ *is* $\binom{n}{2}$.

The above also proves:

Prop 4.4 (Virtual dualizing module of $\mathrm{SL}_n(\mathbb{Z})$)**.** *Let Γ be a finite-index torsion-free subgroup of $\mathrm{SL}_n(\mathbb{Z})$ of type* FP. *The dualizing module $D = H^{\binom{n}{2}}(\Gamma; \mathbb{Z}\Gamma)$ of Γ is given by the Steinberg module $\tilde{H}_{n-2}(T_n(\mathbb{Q}); \mathbb{Z})$.*

Proof. We know

$$H^{\binom{n}{2}}(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}_{n-2}(T_n(\mathbb{Q}); \mathbb{Z})$$

which is free abelian (as $T_n(\mathbb{Q}) \simeq \bigvee S^{n-2}$, by 3.109). By prop 2.144, we know that $D = \tilde{H}_{n-2}(T_n(\mathbb{Q}); \mathbb{Z})$ is a virtual dualizing module for $\mathrm{SL}_n(\mathbb{Z})$. □

Corollary 4.5. *Since $\mathrm{SL}_n(\mathbb{Z})$ is a virtual duality group with $D = \mathrm{St}_n(\mathbb{Q}) = \tilde{H}_{n-2}(T_n(\mathbb{Q}); \mathbb{Z})$, it is a rational duality group with $D = \mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q}$.*

Proof. See 2.146. □

Bieri-Eckmann duality yields $H^{\binom{n}{2}-i}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}) = H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q}) \otimes \mathbb{Q})$.

4.4 Lee-Szczarba: codimension 0 $\mathbb{Q}$-cohomology of $\mathrm{SL}_n(\mathbb{Z})$

Theorem 4.6 (Lee-Szczarba)**.** *If R is Euclidean, then $H^{\mathrm{vcd}}(\mathrm{SL}_n(R); \mathbb{Q}) = 0$.*

Since $\mathrm{SL}_n(R)$ is a duality group with dualizing module $\mathrm{St}_n(F)$ (for F the fraction field of R), we know

$$H^{\mathrm{vcd}}(\mathrm{SL}_n(R); \mathbb{Q}) \cong H_0(\mathrm{SL}_n(R); \mathbb{Q} \otimes \mathrm{St}_n(F)) \cong (\mathbb{Q} \otimes \mathrm{St}_n(F))_{\mathrm{SL}_n(R)}$$

where the last isomorphism follows from 2.39. It remains to show that $(\mathbb{Q} \otimes \mathrm{St}_n(F))_{\mathrm{SL}_n(R)} = 0$. Recall that by Ash-Rudolph 3.112, we know that $\mathrm{St}_n(F)$ is generated by integral apartment classes in F^n (that is, the image of those apartments $S(L_1, \ldots, L_n)$ where each L_i is a line in F^n, $L_1 \oplus \cdots \oplus L_n = F^n$, and $(L_1 \cap R^n) \oplus \cdots \oplus (L_n \cap R^n) = R^n$). Hence it suffices to show that the $\mathrm{SL}_n(R)$-coinvariants of the integral apartment classes of $\mathrm{St}_n(F)$ vanish (note that $\mathrm{SL}_n(R)$ acts trivially on $\mathbb{Q}$).

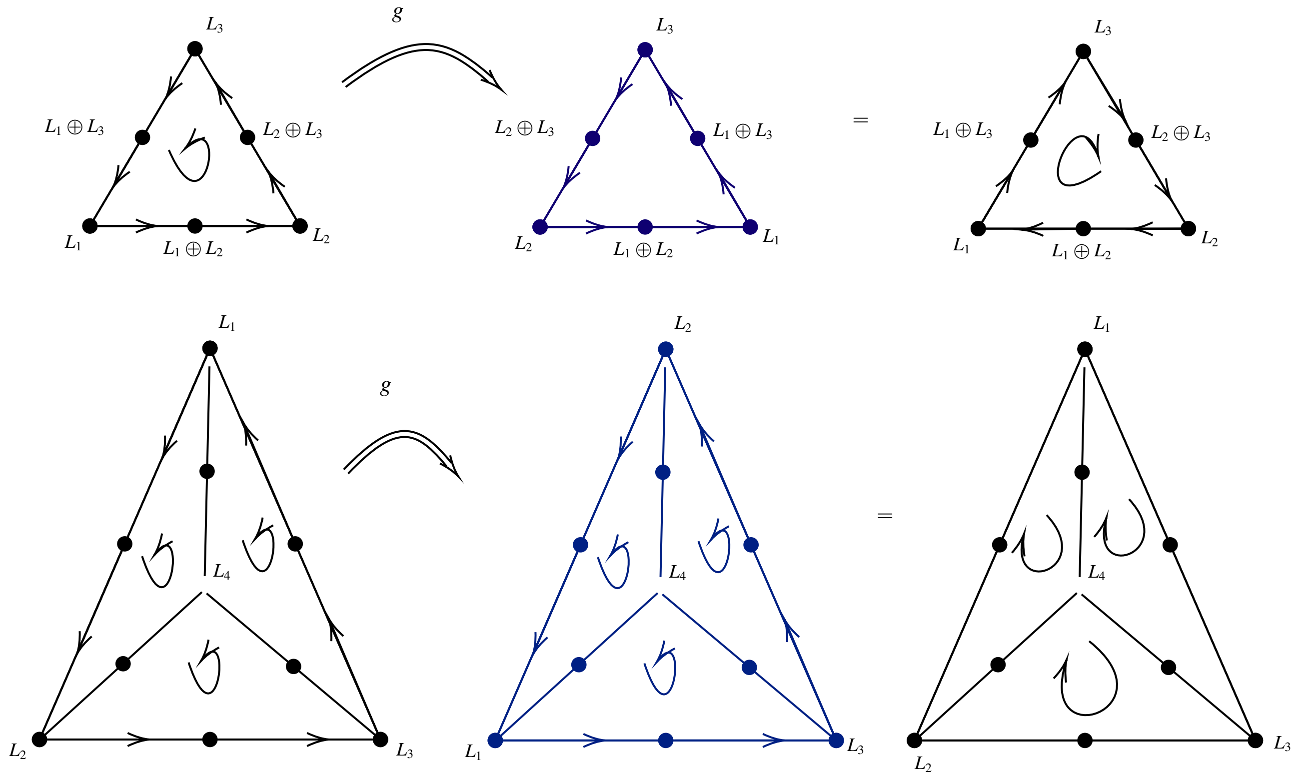
Let $S(L_1, \ldots, L_n)$ be an integral apartment, and let $x = [S(L_1, \ldots, L_n)] \in \mathrm{St}_n(F)$. We claim that there exists some $g \in \mathrm{SL}_n(R)$ such that $g \cdot x = -x$, so that $(\mathbb{Q} \otimes \mathrm{St}_n(F))_{\mathrm{SL}_n(R)} = 0$. For this we claim that there exists some matrix $g \in \mathrm{SL}_n(R)$ such that $g \cdot L_1 = L_2$, $g \cdot L_2 = L_1$, and $g \cdot L_i = L_i$ for $i \geqslant 3$: Note that each line L_i is the F-span of a vector $w_i \in F^n$. Since $S(L_1, \ldots, L_n)$ is an *integral* apartment, we can choose a vector $v_i \in R^n$ whose F-span generates the same line as w_i, and such that $v_1, \ldots, v_n$ forms a basis of R^n. Then the matrix

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

is invertible ($\det(A) \in R^\times$). Swapping two columns multiplies the determinant by -1, and scaling a column by -1 multiplies the determinant by -1, so the matrix B with v_1 and v_2 swapped, and v_2 negated, is invertible (and has the same determinant as A). Then $g = BA^{-1} \in \mathrm{GL}_n(R)$, and we claim that it's actually in $\mathrm{SL}_n(R)$: since the determinant is multiplicative, we have $\det(g) = \det(B)\det(A^{-1}) = \det(A)\det(A)^{-1} = 1$. So

$$\begin{bmatrix} | & & | \\ gv_1 & \cdots & gv_n \\ | & & | \end{bmatrix} = gA = B = \begin{bmatrix} | & | & & | \\ -v_2 & v_1 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

as desired. Now g stabilizes (but reverses the orientation) of each integral apartment in $T_n(\mathbb{Q})$. For instance, in the cases where $n = 3$ or $n = 4$, the action of g sends



Hence $g \cdot [S(L_1, \ldots, L_n)] = -[S(L_1, \ldots, L_n)]$, so $(\mathrm{St}_n(F))_{\mathrm{SL}_n(R)} = 0$.

Remark 4.7. The Euclidean part here really comes into play in the proof of Maazen's theorem, which uses the fact that R is Euclidean to run the link argument.

4.5 Church-Putman [3]: codimension 1 $\mathbb{Q}$-cohomology of $\mathrm{SL}_n(\mathbb{Z})$.

4.5.1 Outline

The purpose of this paper is to show that $H^{\binom{n}{2}-1}(\mathrm{SL}_n \mathbb{Z}; \mathbb{Q}) = H^{\binom{n}{2}-1}(\mathrm{GL}_n \mathbb{Z}; \mathbb{Q}) = 0$, or more generally, that $H^{\binom{n}{2}-1}(\mathrm{SL}_n \mathbb{Z}; V_\lambda) = H^{\binom{n}{2}-1}(\mathrm{GL}_n \mathbb{Z}; V_\lambda) = 0$ for V_λ the rational representation of $\mathrm{GL}_n \mathbb{Q}$ of highest weight λ, with $n \geqslant 3 + ||\lambda||$. General strategy:

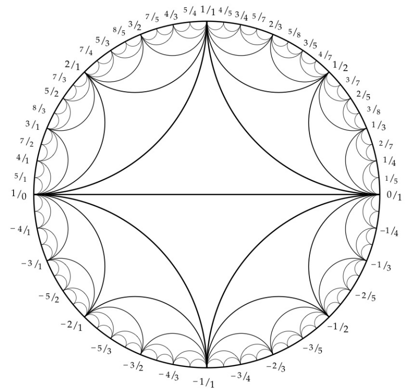
1. Since $\mathrm{SL}_n \mathbb{Z}$ is a virtual duality group with dualizing module $D = H^{\binom{n}{2}}(\mathrm{SL}_n \mathbb{Z}; \mathbb{Z}\,\mathrm{SL}_n \mathbb{Z}) = \mathrm{St}_n(\mathbb{Q}) = \tilde{H}_{n-2}(T_n(\mathbb{Q}); \mathbb{Z})$, Bieri-Eckmann duality yields

$$H^{\binom{n}{2}-i}(\mathrm{SL}_n \mathbb{Z}; \mathbb{Q}) \cong H_i(\mathrm{SL}_n \mathbb{Z}; \mathrm{St}_n(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

   Thus to show vanishing of cohomology in codimension 1 it suffices to show vanishing of homology in dimension 1 with coefficients twisted by the Steinberg module.

2. Let $\mathrm{St}_n^{\mathbb{Q}} := \mathrm{St}_n(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$. Want to resolve $\mathrm{St}_n^{\mathbb{Q}}$ by flat $\mathrm{SL}_n \mathbb{Z}$-modules $\cdots \to F_1 \to F_0 \to \mathrm{St}_n^{\mathbb{Q}}$ such that $F_1 \otimes_{\mathrm{SL}_{\mathbb{Z}}} \mathbb{Q} = 0$. This will allow us to conclude that $H_1(\mathrm{SL}_n \mathbb{Z}; \mathrm{St}_n^{\mathbb{Q}}) = 0$, as desired.

3. Manin's presentation of $\mathrm{St}_2(\mathbb{Q})$: Use the fact that $\mathcal{T}_2(\mathbb{Q})$ can be identified with the 0-skeleton of $\mathcal{B}_2$, the 1-dimensional and connected complex of partial frames for $\mathbb{Z}^2$. This gives $\mathrm{St}_2 = \tilde{H}_0(\mathcal{T}_2(\mathbb{Q}); \mathbb{Z}) = \tilde{C}_0(\mathcal{T}_2(\mathbb{Q}); \mathbb{Z})$. Use fact that $\mathcal{B}_2$ is the classical Farey graph, under the identification of vertices $(a, b)^{\pm}$ (line in $\mathbb{Z}^2$, the two-element set $\{(a, b), (-a, -b)\}$) with $\frac{a}{b} \in \mathbb{Q} \cup \{\infty\}$.



Since $C_2(\mathcal{B}_2;\mathbb{Z}) = 0$, have that $H_1(\mathcal{B}_2;\mathbb{Z}) = \ker(C_1(\mathcal{B}_2;\mathbb{Z}) \to \tilde{C}_0(\mathcal{B}_2;\mathbb{Z}))$, and hence a short exact sequence

$$0 \to H_1(\mathcal{B}_2;\mathbb{Z}) \to C_1(\mathcal{B}_2;\mathbb{Z}) \to \mathrm{St}_2(\mathbb{Q}) \to 0$$

giving a presentation of $\mathrm{St}_2(\mathbb{Q})$. We want a generating set for $H_1(\mathcal{B}_2;\mathbb{Z})$. Consider $\tilde{\mathcal{B}}_2$, the complex $\mathcal{B}_2$ with 2-simplices attached to triples of vertices $\{v_1^{\pm}, v_2^{\pm}, v_3^{\pm}\}$. Since the edges must correspond to (frames corresponding to) bases of $\mathbb{Z}^2$, then $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$ are all bases of $\mathbb{Z}^2$. We must have a relation $\pm v_1 \pm v_2 \pm v_3 = 0$ for some choice of signs: Express $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, and $v_3 = \alpha v_1 + \beta v_2$. Then $\pm 1 = \det\left(\begin{pmatrix} a & \alpha a+\beta c \\ b & \alpha b+\beta d \end{pmatrix}\right) = \beta(ad - bc) = \pm\beta$ since $ad - bc = \pm 1$. This shows that $\beta = \pm 1$, and similarly we can deduce that $\alpha = \pm 1$. Without loss of generality (by negating and reordering) $v_3 = v_1 + v_2$. Allegedly $\tilde{\mathcal{B}}_2$ is simply-connected, so $H_1(\tilde{\mathcal{B}}_2) = 0$, and hence

$$C_2(\tilde{\mathcal{B}}_2) \xrightarrow{\partial_2} C_1(\tilde{\mathcal{B}}_2) \xrightarrow{\partial_1} C_0(\tilde{\mathcal{B}}_2)$$

is exact. The map ∂_1 is the same as that for $\mathcal{B}_2$, so $H_1(\mathcal{B}_2;\mathbb{Z}) = \ker\partial_1 = \mathrm{Im}\,\partial_2$.

4. Using the presentation for $\mathrm{St}_2(\mathbb{Q})$, argue that there exists an analogous presentation for $\mathrm{St}_n(\mathbb{Q})$ with analogous relations: $\mathrm{St}_n(\mathbb{Q})$ is the abelian group with generators $[v_1, \dots, v_n]$, one for each basis $\{v_1, \dots, v_n\}$ of $\mathbb{Z}^n$, subject to the relations

$$\begin{aligned} &R1 : [v_1, \dots, v_n] = [v_1, v_1 + v_2, v_3, \dots, v_n] + [v_1 + v_2, v_2, v_3, \dots, v_n]. \\ &R2 : [\pm v_1, \dots, \pm v_n] = [v_1, \dots, v_n] \quad \text{for any choice of signs.} \\ &R3 : [v_{\sigma(1)}, \dots, v_{\sigma(n)}] = (-1)^{\sigma}[v_1, \dots, v_n] \quad \text{for } \sigma \in S_n. \end{aligned}$$

Do this by defining the complex of *partial augmented frames* $\mathcal{BA}_n$ whose p-simplices consist of $(p+1)$-element partial augmented frames for $\mathbb{Z}^n$, or equivalently, which form frames or augmented frames for a rank $(p+1)$ direct summand V of $\mathbb{Z}^n$. The point of this complex is to get a better connectivity result than the partial basis complex of Maazen 3.100, which was used by Church-Farb-Putman to prove Ash-Rudolph 3.112, giving a generating set (but not a presentation!!) for $\mathrm{St}_n(\mathbb{Q})$.

Define a subcomplex $\mathcal{BA}_n'$ consisting of those simplices $\sigma = \{v_1^{\pm}, \dots, v_k^{\pm}\} \in \mathcal{BA}_n$ such that $k < n$. Hence the only simplices excluded from $\mathcal{BA}_n'$ are the frames and augmented frames for $\mathbb{Z}^n$. Three claims are in order:

Claim 4.8. *The abelian group described by the presentation above coincides with $H_{n-1}(\mathcal{BA}_n, \mathcal{BA}_n';\mathbb{Z})$.*

Claim 4.9. $H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \cong \tilde{H}_{n-2}(\mathcal{BA}'_n; \mathbb{Z})$.

Claim 4.10. $\tilde{H}_{n-2}(\mathcal{BA}'_n; \mathbb{Z}) \cong \mathrm{St}_n(\mathbb{Q})$.

If the three claims are shown, then $\mathrm{St}_n(\mathbb{Q})$ is precisely the abelian group described by the presentation. Postpone the proof of these claims until later. Will highlight a general "link argument" / badness argument.

5. Since $\mathrm{St}_n(\mathbb{Q}) \cong H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) = \operatorname{coker}(C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}))$ by virtue of $C_{n-2}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) = 0$, we have an exact sequence

$$C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \xrightarrow{\partial_n} C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \to \mathrm{St}_n(\mathbb{Q}) \to 0.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, we have that

$$C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\partial_n} C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{St}_n(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

is still exact, and hence we have a presentation

$$C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q}) \to C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q}) \to \mathrm{St}_n^{\mathbb{Q}}.$$

If we can show that $C_i(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q})$ is a flat $\mathbb{Z}\,\mathrm{SL}_n(\mathbb{Z})$-module for $i = n, n-1$, and if we can show that tensoring $C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q})$ with $\mathbb{Q}$ (or V_λ) over $\mathbb{Z}\,\mathrm{SL}_n(\mathbb{Z})$ is trivial, then we can conclude that $H_1(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n^{\mathbb{Q}}) = 0$ by the definition of group homology. The fact that $C_i(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q})$ is a flat $\mathbb{Z}\,\mathrm{SL}_n(\mathbb{Z})$-module follows from the fact that the setwise stabilizer G_σ of every simplex $\sigma \in \mathcal{BA}_n^{(i)} \backslash \mathcal{BA}_n'^{(i)}$ is finite (as this stabilizer is a subgroup of the signed permutation group $\pm S_n$, which is finite), and that $G = \mathrm{SL}_n \mathbb{Z}$ acts simplicially on $\mathcal{BA}_n$ and preserves the subcomplex $\mathcal{BA}'_n$. We can therefore express

$$C_i(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q}) \cong \bigoplus_{\sigma \in (\mathcal{BA}_n^{(i)} \backslash \mathcal{BA}_n'^{(i)}) / \mathrm{SL}_n(\mathbb{Z})} M_\sigma$$

where $M_\sigma = \{[g(\sigma)] : g \in \mathrm{SL}_n(\mathbb{Z})\}$. Letting $\mathbb{Q}_\sigma$ be the orientation module of G_σ ($g \in G_\sigma$ acts by ± 1 depending on whether it preserves or reverses orientation of σ), we have

$$M_\sigma \cong \bigoplus_{g \in G/G_\sigma} g \cdot Q_\sigma \cong \mathbb{Q}G \otimes_{\mathbb{Q}G_\sigma} \mathbb{Q}_\sigma = \operatorname{Ind}_{G_\sigma}^{G} \mathbb{Q}_\sigma.$$

But $\mathbb{Q}_\sigma$ is an irreducible representation of G_σ, hence a direct summand of $\mathbb{Q}G_\sigma$, so M_σ is a direct summand of $\mathbb{Q}G$, and is therefore $\mathbb{Z}G$-flat.

We still need to show that $C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q}) \otimes_{\mathbb{Z}\,\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q} = 0$. Let σ be an augmented frame $\sigma = \{v_0^{\pm}, \ldots, v_n^{\pm}\}$ for $\mathbb{Z}^n$, since this is precisely the type of n-simplex in $\mathcal{BA}_n \backslash \mathcal{BA}'_n$. A bit technical: there exists an element (they construct it) φ in the setwise stabilizer of σ such that $\varphi([\sigma]) = -[\sigma]$. Since $\mathbb{Q}$ is a trivial $\mathbb{Z}\,\mathrm{SL}_n(\mathbb{Z})$-module, φ acts trivially on any $q \in \mathbb{Q}$, and hence in $C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Q}) \otimes_{\mathbb{Z}\,\mathrm{SL}_n(\mathbb{Z})} \mathbb{Q}$ we have

$$[\sigma] \otimes q = [\sigma] \otimes \varphi \cdot q = \varphi \cdot [\sigma] \otimes q = -[\sigma] \otimes q$$

which implies that $[\sigma] \otimes q = 0$.

4.5.2 Expanding with more detail

We need to show the three claims in point 4 above.

1. Claim 1: The abelian group described by the presentation above coincides with $H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z})$.

   Proof. Since $C_{n-2}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) = 0$, we have that $H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) = \operatorname{coker}(C_n(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \to C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}))$. The simplices in $C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z})$ are n-frames for $\mathbb{Z}^n$, which are specified by an (unordered) basis $\{v_1, \ldots, v_n\}$. Multiplying the vectors by ± 1 does not change the frame and neither does permuting the vectors, but the permutation does change the orientation of the simplex corresponding to $\{v_1^{\pm}, \ldots, v_n^{\pm}\}$. Hence $C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n)$ is generated by formal symbols $\langle\!\langle v_1, \ldots, v_n \rangle\!\rangle$ for bases $v_1, \ldots, v_n$ of $\mathbb{Z}^n$ subject to the relations $\langle\!\langle v_1, \ldots, v_n \rangle\!\rangle = \langle\!\langle \pm v_1, \ldots, \pm v_n \rangle\!\rangle$ and $\langle\!\langle v_{\sigma(1)}, \ldots, v_{\sigma(n)} \rangle\!\rangle = (-1)^{\sigma} \cdot \langle\!\langle v_1, \ldots, v_n \rangle\!\rangle$ for $\sigma \in S_n$. Similarly, simplices in $C_n(\mathcal{BA}_n, \mathcal{BA}'_n)$ are those augmented n-frames $\{v_0^{\pm}, \ldots, v_n^{\pm}\}$ with (without loss of generality) $v_1, \ldots, v_n$ a basis for $\mathbb{Z}^n$ and $v_0 = v_1 + v_2$. The subset $\{v_0, \ldots, \hat{v}_i, \ldots, v_n\}$ spans a proper direct summand of $\mathbb{Z}^n$ for $i \geqslant 3$ and hence lies in $\mathcal{BA}'_n$, so the portion of the boundary consisting of such subsets is killed in $C_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n)$. The image of ∂_n then consists of $\langle\!\langle v_1, v_2, \ldots, v_n \rangle\!\rangle - \langle\!\langle v_1 + v_2, v_2, v_3, \ldots, v_n \rangle\!\rangle + \langle\!\langle v_1 + v_2, v_1, v_3, \ldots, v_n \rangle\!\rangle$. Hence $H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z})$ has precisely the presentation stated for $\mathrm{St}_n(\mathbb{Q})$. □

2. Claim 2: $H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n; \mathbb{Z}) \cong \tilde{H}_{n-2}(\mathcal{BA}'_n; \mathbb{Z})$.

   Proof. If one knows that $\mathcal{BA}_n$ is Cohen-Macaulay of dimension n, then $\tilde{H}_i(\mathcal{BA}_n) = 0$ for $i < n$. The long exact sequence of the pair $(\mathcal{BA}_n, \mathcal{BA}'_n)$ yields

   $$0 \to H_{n-1}(\mathcal{BA}_n, \mathcal{BA}'_n) \to \tilde{H}_{n-2}(\mathcal{BA}'_n) \to 0$$

   as desired. But we still need to show that $\mathcal{BA}_n$ is CM of dimension n (postponed to later). *Note:* The complex of partial bases $\mathrm{PB}_n(\mathbb{Z})$ is CM of dimension $n-1$ (so $(n-2)$-connected 3.100! Not $n-1$!). □

3. Claim 3: $\tilde{H}_{n-2}(\mathcal{BA}'_n; \mathbb{Z}) \cong \mathrm{St}_n(\mathbb{Q})$.

   Proof. Let $\mathbb{T}_n$ be the poset of proper nonzero direct summands of $\mathbb{Z}^n$ under inclusion. This is isomorphic to the poset of proper nonzero $\mathbb{Q}$-subspaces of $\mathbb{Z}^n$, so $|\mathbb{T}_n| \cong |\mathcal{T}_n|$. Have a poset map $F : \mathcal{P}(\mathcal{BA}'_n) \to \mathbb{T}_n$ defined by $\{v_1^{\pm}, \ldots, v_k^{\pm}\} \mapsto \mathrm{span}_{\mathbb{Z}}(v_1^{\pm}, \ldots, v_k^{\pm})$. Invoke Quillen's acyclic lemma for posets 3.18. Use fact that $\mathbb{T}_n$ is Cohen-Macaulay of dimension $n-2$, and verify that for all proper nonzero direct summands V of $\mathbb{Z}^n$, the fiber poset $F_{\leqslant V}$ is $\dim(V)$-spherical (by showing that it is itself homeomorphic to $\mathcal{BA}_{\mathrm{rk}(V)}$, and using assumption that $\mathcal{BA}_n$ is CM of dimension n. Since $\dim V = \mathrm{ht}(V) + 1$, then Quillen's lemma $\implies$ F induces an iso on $\tilde{H}_{n-2}$. □

4. Bulk of the proof: showing $\mathcal{BA}_n$ is CM of dimension n. Idea: define a complex $\mathcal{BA}_n^m$ to be the full subcomplex of $\mathrm{Lk}_{\mathcal{BA}_{n+m}}(\{e_1^{\pm}, \ldots, e_m^{\pm}\})$ spanned by vertices $v^{\pm} \in \mathrm{Lk}_{\mathcal{BA}_{n+m}}(\{e_1^{\pm}, \ldots, e_m^{\pm}\})$ such that $v \notin \mathrm{span}_{\mathbb{Z}}(e_1, \ldots, e_m)$. Claim that $\mathcal{BA}_n^m$ is CM of dimension n for $n \geqslant 1$ and $m \geqslant 0$ with $n + m \geqslant 2$. Show base case: if $n = 1$, show that $\mathcal{BA}_1^m$ is a connected graph (Cayley graph of $\mathbb{Z}^m$ with respect to generating set $\{e_1, \ldots, e_m\}$). Assume that $\mathcal{BA}_{n'}^{m'}$ is CM of dimension n' for $1 \leqslant n' < n$ and $m' \geqslant 0$ and $m' + n' \geqslant 2$. Describe the links of different simplices in $\mathcal{BA}_n^m$, see that these links are isomorphic to $\mathcal{BA}_{n'}^{m'}$ for various $n' < n$ and with $n' + m' = n + m$, or to cones of such complexes. Obtain corresponding connectivity results for links. But still need to show that $\mathcal{BA}_n^m$ itself is $(n-1)$-connected!

5. Show that $\mathcal{BA}_n^m$ is $(n-1)$-connected using a *badness* argument. We let $\varphi : S^p \to \mathcal{BA}_n^m$ be a simplicial map, with S^p being a combinatorial p-sphere, $0 \leqslant p \leqslant n-1$. WTS φ is homotopic to a constant map. Let $F : \mathbb{Z}^{m+n} \to \mathbb{Z}$ be the linear map taking v to its e_{m+n}-coordinate. For a vertex $v^\pm \in \mathcal{BA}_n^m$, define $r(v^\pm) = |F(v)|$. Let $R(\varphi) = \max\{r(\varphi(x)) : x \text{ vertex in } S^p\}$. See that if $R = 0$, then $\varphi(S^p) \subseteq \mathrm{Star}_{\mathcal{BA}_n^m}(e_{m+n}^\pm)$, so φ is homotopic to the constant map at e_{m+n}. Goal is thus to show that φ can be homotoped to decrease $R(\varphi)$. Repeating this process until $R(\varphi) = 0$ gives the result.

6. Consider certain conditions on φ. Let $\star$ be a condition on a simplex σ of S^p. Consider a maximal dimensional simplex σ in S^p satisfying $\star$. The precise (unstated) condition of $\star$ and the maximality of σ ensures that φ takes $\mathrm{Lk}_{S^p}(\sigma)$ into $\mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma))^{<R}$. By description of links, have that $\mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma)) \cong \mathrm{B}_{n-\dim(\varphi(\sigma))}^{m+\dim(\varphi(\sigma))}$, which is $(n-\dim(\varphi(\sigma))-2)$-connected. If we can show that $\mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma))^{<R}$ is a retract of $\mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma))$ (postponed), then $\mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma))^{<R}$ is $(n-\dim(\varphi(\sigma))-2)$-connected. Now $\mathrm{Lk}_{S^p}(\sigma)$ is a combinatorial $(p-\dim(\sigma)-1)$-sphere. Obtain that $\varphi|_{\mathrm{Lk}_{S^p}(\sigma)}$ is nullhomotopic inside $\mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma))^{<R}$. By Zeeman's relative simplicial approximation theorem 1.40, deduce that there exists a combinatorial $(p-k)$-ball B with $\partial B \cong \mathrm{Lk}_{S^p}(\sigma)$ and a simplicial map $\psi : B \to \mathrm{Lk}_{\mathcal{BA}_n^m}(\varphi(\sigma))^{<R}$ such that $\psi|_{\partial B} = \varphi|_{\mathrm{Lk}_{S^p}(\sigma)}$ (so we're *extending* φ). Then note that $\partial(\sigma * B) = \partial\sigma * B \cup \sigma * \partial B$. On $\partial\sigma * B$ we have the map

$$\varphi|_{\partial\sigma} * \psi : \partial\sigma * B \to \mathcal{BA}_n^m$$

and on $\sigma * \partial B$ we have

$$\varphi|_{\sigma} * \psi|_{\partial B} = \varphi : \mathrm{Star}_{S^p}(\sigma) \to \mathcal{BA}_n^m.$$

Since $\sigma * B$ forms a combinatorial $(p+1)$-ball, we can homotope $\varphi|_{\mathrm{Star}_{S^p}(\sigma)}$ across this $(p+1)$-ball to $\varphi|_{\partial\sigma} * \psi$. Now we've gotten rid of the maximal dimensional simplex σ! Since simplices in $\partial\sigma * B$ come as unions of simplices in $\partial\sigma$ (smaller dimension) and B (which do not satisfy $\star$, since the condition $\star$ requires that at least one vertex has $r(v^\pm) = R$, and we know $\psi(B) \subset (\mathcal{BA}_n^m)^{<R}$), our homotopy does not introduce new bad simplices.

4.6 Kupers-Miller-Patzt-Wilson: On the Generalized Bykovskiĭ Presentation of Steinberg Modules [7]

Overview: the purpose of this paper is to give vanishing results in the codimension 1 rational cohomology of $\mathrm{SL}_n(\mathcal{O}_K)$ (and $\mathrm{GL}_n(\mathcal{O}_K)$) for two Euclidean number rings $\mathcal{O}_K$ (namely: $\mathcal{O}_K = \mathbb{Z}[i]$ (Gaussian integers) and $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ (Eisenstein integers)). Note that the codimension 0 vanishing result follows from Lee-Sczcarba, since that paper gives codimension 0 vanishing for any Euclidean domain. Also note that in general, the ring of integers of a number field does *not* have to be Euclidean. There are still codimension 0 vanishing results for $\mathrm{SL}_n(\mathcal{O}_K)$ in some of these cases (See Church-Farb-Putman, Integrality in the Steinberg module and the top-dimensional cohomology of $\mathrm{SL}_n(\mathcal{O}_K)$), by showing that the Steinberg module $\mathrm{St}_n(K) = \tilde{H}_{n-2}(\mathcal{T}_n(F); \mathbb{Z})$ is integrally generated.

The strategy of the paper is to generalize Bykovskiĭ's presentation of $\mathrm{St}_n(\mathbb{Q})$ to give presentations of $\mathrm{St}_n(K)$ for these number fields K. The methods are very similar/ slight modifications of Church-Putman's paper [3], where now instead of using these highly-connected "complexes of partial augmented frames" for $\mathbb{Z}^n$, they use the same complexes but for $\mathcal{O}^n$! That is, $\mathcal{BA}_{n+m}(\mathcal{O})$ has $p-1$-simplices $\{v_1, \ldots, v_p\}$ where each v_i is a *line* in $\mathcal{O}^{n+m}$ (and thus there exists some primitive vector $\vec{v_i}$, unique up to multiplication by a unit in $\mathcal{O}$, which spans the line– think $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ vs. $\begin{bmatrix} 4 \\ 6 \end{bmatrix}$) and $v_1 \oplus \cdots \oplus v_p$ is isomorphic to a direct summand of $\mathcal{O}^n$, and also p-simplices $\{v_0, \dots, v_p\}$ where $v_1, \dots, v_p$ satisfy the same condition as before, but $v_0 = u_i v_i + u_j v_j$ for some units $u_i, u_j \in \mathcal{O}^\times$ (*augmented*). Can then define

$$\mathcal{BA}_n^m(\mathcal{O}) = \mathrm{Lk}_{\mathcal{BA}_{n+m}(\mathcal{O})}(e_1, \dots, e_m)$$

and under the added condition that if $v \in \mathcal{BA}_n^m(\mathcal{O})$ then $v \notin \mathrm{span}_{\mathcal{O}}(e_1, \dots, e_m)$. They prove high-connectivity of these complexes, namely, that $\mathcal{BA}_n^m(\mathcal{O})$ is Cohen-Macaulay of dimension n whenever $n \geqslant 1$ and $n+m \geqslant 2$. Several of the lemmas used to prove this theorem involve specific properties of $\mathcal{O}_K$ being the Eisenstein or Gaussian integers:

1. In the base case that $\mathcal{BA}_1^m$ is 0-connected, they use the fact that this complex is a graph which is connected if and only if $\mathcal{O}$ is additively generated by units. Fortunately both $\mathbb{Z}[i]$ and $\mathbb{Z}[\rho]$ are additively generated by units.

2.

The limitations of this paper: show that their strategy provably does not work for other Euclidean number rings, and don't know if it can be adapted for...?

4.6.1 Embeddings of a number field

Definition 4.11 (Number field)**.** A number field L is a finite extension of $\mathbb{Q}$.

Definition 4.12 (Automorphism group)**.** If L/F is a field extension, we have $\mathrm{Aut}(L/F)$ the automorphisms of L which fix F.

Definition 4.13 (Real versus complex embedding)**.** Let L be a number field. For an automorphism σ of $L/\mathbb{Q}$, we have either

1. $\mathrm{Im}\,\sigma \subset \mathbb{R}$ (real embedding)

2. $\mathrm{Im}\,\sigma \subset \mathbb{C} \backslash \mathbb{R}$. (complex embedding)

Let r_1 denote the number of real embeddings and let $2r_2$ denote the number of complex embeddings (since complex embeddings come in conjugate pairs, we get a factor of 2 in front).

Borel and Serre proved that the virtual cohomological dimension of $\mathrm{SL}_n(\mathcal{O}_K)$ (where K is a number field and $\mathcal{O}_K$ is the ring of integers of K, that is, those elements of K which satisfy monic integral polynomials) is

$$v_n = \frac{r_1}{2}((n+1)n - 2) + r_2(n^2 - 1) - n + 1.$$

In particular, when $L = \mathbb{Q}$ (so $r_1 = 1$ and $2r_2 = 0$), we see that $v_n = \frac{1}{2}((n+1)n) = \binom{n}{2}$, which is what we know to be the virtual cohomological dimension of $\mathrm{SL}_n(\mathbb{Z})$. Recall 2.111

$$H^i(G; -) = 0 \quad \forall i > \mathrm{vcd}(G)$$

but that it is possible for

$$H^{\mathrm{vcd}(\mathrm{SL}_n(\mathcal{O}_K))}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q})$$

to vanish or not. Both of the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ are Euclidean, so by Lee-Sczcarba 4.6 we know that $H^{\mathrm{vcd}(\mathrm{SL}_n(\mathcal{O}_K))}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q}) = 0$. But we don't know that this is the best vanishing range! That is, it is still possible that $H^{\mathrm{vcd}-i}(\mathrm{SL}_n(\mathcal{O}_K); \mathbb{Q}) = 0$ for $i \geqslant 1$.

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