# MODULAR PRINCIPAL SERIES REPRESENTATION OF $\mathrm{GL}_2$ OVER FINITE RINGS

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ABSTRACT. Given any prime $p \geq 3$, $r \in \mathbb{N}$, and character $\chi$ on the Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$, we construct a Jordan-Hölder series for the modulo $p$ reduction of the principal series representation of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$. As a corollary we provide the semisimplifications of all characteristic $p$ principal series representations of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$, and explain a process to compute such semisimplifications in small cases by the means of Brauer characters, verifying the computation from the constructed Jordan-Hölder series.

## 1. Introduction

One often wishes to determine how the irreducible representations of a group "fit together" in the composition of some other representation of concern. Under sufficiently nice conditions this problem is solved: Given a finite group $G$ and a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over a field of characteristic not dividing the order of $G$, the classical Maschke's theorem guarantees that $\rho$ is *completely reducible*, meaning it can be uniquely expressed as a direct sum of irreducible representations of the group $G$, up to isomorphism. However, when $V$ is over a field of characteristic $p$ and $p$ divides the order of the group, Maschke's theorem no longer holds, requiring a different method of determining how the irreducible modular representations of a finite group $G$ make up our representation of interest.

This may be done through investigating Jordan-Hölder series of the representation, which are filtrations

$$0 \subset V_1 \subset \cdots \subset V_d = V$$

of subrepresentations with inclusions being proper and maximal, so that each composition factor $V_{i+1}/V_i$ is isomorphic to an irreducible representation of $G$. The Jordan-Hölder Theorem states that such composition series need not be unique, but that the *set* of composition factors of a representation, known as the irreducible constituents, is unique. We can then define

$$V^{\mathrm{ss}} := \bigoplus_{i=0}^{d-1} V_{i+1}/V_i \tag{1}$$

to be the *semisimplification* of $V$. Since each quotient $V_{i+1}/V_i$ is isomorphic to an irreducible representation of $G$, we have

$$V^{\mathrm{ss}} = \bigoplus_{j} \rho_j^{d_j} \tag{2}$$

where $\rho_j$ is an irreducible representation of $G$ and $d_j$ is its multiplicity in the semisimplification of $V$. This gives us a way to view modular representations in analogy to how Maschke's theorem decomposes ordinary characteristic 0 representations. A consequence of the Jordan-Hölder theorem is that $V^{\mathrm{ss}}$ is unique up to rearrangement of factors in the direct sum, so $V^{\mathrm{ss}}$ is unique up to isomorphism.

Fixing a prime $p \geq 3$, we consider the non-archimedean local field $L = \mathbb{F}_p((t))$. The ring of integers $\mathcal{O}_L$ is given by $\mathbb{F}_p[[t]]$ and consists of all formal power series in $t$ with coefficients in $\mathbb{F}_p$, with a unique maximal ideal generated by $t$. For any $r \in \mathbb{N}$ we consider the general linear group
$GL_2(\mathbb{F}_p[t]/(t^r))$, which we henceforth denote by $G_r$.

The choice of $L = \mathbb{F}_p((t))$ puts us in the equal characteristic setting, where $L$ has the same characteristic as its residue field $\mathbb{F}_p$. For work done in the mixed characteristic setting, see the appendix in [4].

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Given the finite group $G_r$, let $B_r \leq G_r$ denote the *Borel subgroup* of $G_r$ consisting of $2 \times 2$ upper triangular invertible matrices with entries in $\mathbb{F}_p[t]/(t^r)$. Fixing a field $E$ of characteristic 0 whose residue field $k_E = \mathcal{O}_E/(\varpi_E)$ is of characteristic $p$, let $\chi_1, \chi_2 : (\mathbb{F}_p[t]/(t^r))^\times \to E^\times$ be group homomorphisms, and define

$$\begin{aligned} \chi : B_r &\to E^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto \chi_1(a)\chi_2(d). \end{aligned}$$

The *principal series representation* of $G_r$ associated to $\chi$ is the induced representation $\mathrm{Ind}_{B_r}^{G_r}(\chi)$, a vector space

$$\mathrm{Ind}_{B_r}^{G_r}(\chi) := \{f : G_r \to E \mid f(bg) = \chi(b)f(g) \quad \forall g \in G_r, b \in B_r\} \tag{3}$$

with a $G_r$-action given by

$$\begin{aligned} \vartheta_\chi : G_r &\to \mathrm{GL}(\mathrm{Ind}_{B_r}^{G_r}(\chi)) \\ (x \cdot f)(g) &= f(gx) \end{aligned} \tag{4}$$

for all $x, g \in G_r, f \in \mathrm{Ind}_{B_r}^{G_r}(\chi)$. This paper explores the modulo $p$ reduction of the principal series representation, where $\chi$ now maps to $k_E = \mathcal{O}_E/(\varpi_E) \cong \overline{\mathbb{F}_p}$ and where all maps $f \in \mathrm{Ind}_{B_r}^{G_r}(\chi)$ have codomain $k_E$. Hereafter we abuse notation and write $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ to mean the principal series representation after reduction modulo $p$. Hence $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ is a characteristic $p$ vector space of dimension $[G_r : B_r] \cdot \dim(\chi) = (p+1)p^{r-1}$, with a $G_r$-action still given by (4).

As the $r = 1$ case is well-studied, the main result of the paper is an inductive construction of a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ which terminates in $\mathrm{Ind}_{B_1}^{G_1}(\chi)$.

**Proposition 1.1.** *Let $p \geq 3$ be a prime, let $r \in \mathbb{N}_{\geq 2}$, and let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character. There exists a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ given by*

$$0 \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \cdots \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \mathrm{Ind}_{B_r}^{G_r}(\chi), \tag{5}$$

*where $I_r^{r-1} := \{\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d \end{smallmatrix}\right] \in G_r : c \in \mathbb{F}_p\}$, $\sigma := \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$, and $\sigma^{(k)}$ is an $I_r^{r-1}$-invariant $k$-dimensional subspace of $\sigma$.*

In §3 we give a precise description of the $k$-dimensional subspaces $\sigma^{(k)}$ and use their construction to prove the main result, shown in §4:

**Theorem 1.1.** *For the $I_r^{r-1}$-invariant $k$-dimensional subspaces $\sigma^{(k)}$ satisfying Prop 1.1, we have*

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r} \mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a_0}{d_0})^k) \tag{6}$$

*for $0 \leq k \leq p-1$, where $\chi \cdot (\frac{a_0}{d_0})^k$ is the character $\chi \cdot (\frac{a_0}{d_0})^k : B_r \to \overline{\mathbb{F}_p}^\times$ mapping* $\left[\begin{smallmatrix} a_{r-1}t^{r-1}+\cdots+a_0 & b \\ 0 & d_{r-1}t^{r-1}+\cdots+d_0 \end{smallmatrix}\right] \mapsto \chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) \cdot (a_0 d_0^{-1})^k$.

Theorem 1.1 implies that the filtration in Prop 1.1 may be refined inductively to a filtration in terms of $\mathrm{Ind}_{B_1}^{G_1}(\psi)$ for varying characters $\psi$, which may be then further refined to a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ using the known Jordan-Hölder series for $\mathrm{Ind}_{B_1}^{G_1}(\psi)$.

In §2 we provide preliminaries, and in §5 we give a corollary of the main theorem regarding semisimplification numbers. Finally, since determining the semisimplification of a given representation can be done without a Jordan-Hölder series via a computational process using Brauer characters, we compute a small example using this method in §6, and show that the semisimplification matches with what is deduced from our main theorem.

## 2. Preliminaries

2.1. **Basic Representation Theory.** We begin by providing key definitions from representation theory.

**Definition 2.1.** (Modular representation of a finite group) A *characteristic p representation* of a finite group $G$ is a group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

where $V$ is a finite-dimensional vector space over a field of characteristic $p$ and $\mathrm{GL}(V)$ is the general linear group of $V$. Equivalently we may define a representation of a finite group as a group action of $G$ on a vector space $V$, such that $g \cdot v = \rho(g)(v)$.

**Remark 2.2.** Although a representation of a group $G$ is specified by both a vector space $V$ and a group homomorphism $\rho$, we will often refer to the vector space $V$ as the representation of $G$, keeping in mind that $V$ is equipped with a $G$-action.

**Definition 2.3.** (Subrepresentations) Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and consider a subspace $W \leq V$. We say $W$ is a *subrepresentation* of $V$ if

$$\rho(g)(w) \in W$$

for all $g \in G, w \in W$.

**Definition 2.4.** (Irreducible representation) A representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if its only subrepresentations are the zero subspace and the whole vector space $V$. Otherwise we say $V$ is *reducible.*

2.2. **Maschke's Theorem and its Converse.**

**Proposition 2.5.** *(Maschke's Theorem) Let $G$ be a finite group and let $\mathbb{F}$ be a field of characteristic zero or of positive characteristic not dividing $|G|$. If $V$ is a finite-dimensional representation of $G$ over $\mathbb{F}$ and $U$ is any subrepresentation of $V$, then $V$ has a subrepresentation $W$ such that $V = U \oplus W$.*

Maschke's theorem implies that every finite-dimensional representation $V$ of a finite group $G$ over a field whose characteristic does not divide the order of the group can be expressed uniquely as a direct sum of irreducible representations. A partial converse of Maschke's theorem holds as well: if $G$ is a finite group and $V$ is a representation over a field $\mathbb{F}$ whose order *does* divide $|G|$, then $V$ may not be completely reducible. That is, it is possible for there to exist some subrepresentation $U$ of $V$ which has no complement subrepresentation $W$ in $V$.

For an example of Maschke's Theorem failing when the characteristic of $\mathbb{F}$ divides $|G|$, consider:

**Example 2.6.** Let $G = \mathbb{Z}/p\mathbb{Z} = g$ and let $V = \overline{\mathbb{F}_p}^2$ over $\overline{\mathbb{F}_p}$. Define an action of $G$ on $V$ via $g \cdot e_1 = e_1$ and $g \cdot e_2 = e_1 + e_2$. Note that this is indeed a representation, as $\rho(0) = \rho(p \cdot g) = \rho(g)^p = \left[\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ since the characteristic of the underlying field is $p$. Notice that $e_1$ is stable under the action of $G$ and that $e_1$ is isomorphic to the trivial representation. We claim that there does not exist $V'$ a subrepresentation of $V$ such that $V = e_1 \oplus V'$. For, if there was, then $V/e_1 \cong V'$. But $V/e_1$ is isomorphic to $\overline{e_2}$, which, according to the action of $G$ on $V$, is isomorphic to the trivial representation, as

$$g \cdot \overline{e_2} = \overline{e_1 + e_2} = \overline{e_2}.$$

This implies that $V$ is isomorphic to the direct sum of two copies of the trivial representation, and hence that the fixed subspace of $V$, denoted $V^G$, is two-dimensional. But $V^G$ is one-dimensional: if $\alpha_1 e_1 + \alpha_2 e_2 \in V^G$, then $g \cdot (\alpha_1 e_1 + \alpha_2 e_2) = \alpha_1 e_1 + \alpha_2(e_1 + e_2) = \alpha_1 e_1 + \alpha_2 e_2$ implies that $\alpha_2 = 0$ and hence that $V^G = e_1$.

The key to this example is that the defined action of $G$ on $V$ fails to be a representation when the characteristic of the field underlying $V$ is not divisible by $p$.

## 3. Constructing $I_r^{r-1}$-invariant subspaces.

3.1. **Characters of $B_r$.** It is known ([1]) that every character $\chi : B_1 \to \overline{\mathbb{F}_p}^\times$ is of the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto a^\ell (ad)^s$$

for some $0 \le \ell, s \le p-2$. An analogue holds in the general $B_r$ case, in the sense that every character $\chi : B_r \to \overline{\mathbb{F}_p}^{\times}$ is of the form

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^{\ell}(a_0 d_0)^s$$

for some $0 \le \ell, s \le p-2$, and hence only depends on the constant terms $a_0, d_0$ belonging to $\mathbb{F}_p^{\times}$.

**Lemma 3.1.** *Every character $\chi_i : (\mathbb{F}_p[t]/(t^r))^{\times} \to \overline{\mathbb{F}_p}^{\times}$ is completely determined by where it maps the constant terms belonging to $\mathbb{F}_p^{\times}$. That is, $\chi_i(a_0 + a_1 t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$.*

*Proof.* We first show that $\chi_i : (\mathbb{F}_p[t]/(t^r))^{\times} \to \overline{\mathbb{F}_p}^{\times}$ must always map an element of the form $1 + a_1 t + \cdots + a_{r-1}t^{r-1}$ to 1. By applying the monomial identity $(x+y)^p = x^p + y^p$ in the field $\mathbb{F}_p$ inductively, we obtain $(1 + a_1 t + \cdots + a_{r-1}t^{r-1})^p = 1 + a_1 t^p + \cdots + a_{r-1}t^{p(r-1)}$. Choosing the minimal $k \in \mathbb{N}$ such that $p^k \ge r$ gives

$$\begin{aligned} (1 + a_1 t + \cdots + a_{r-1}t^{r-1})^{p^k} &= 1 + a_1 t^{p^k} + \cdots + a_{r-1}t^{p^k(r-1)} \\ &= 1 \end{aligned}$$

Thus $\chi_i(1 + a_1 t + \cdots + a_{r-1}t^{r-1})$ must have order dividing $p^k$ in $\overline{\mathbb{F}_p}^{\times}$. But no elements in $\overline{\mathbb{F}_p}^{\times}$ have order $p^{\ell}$ for any $1 \le \ell \le k$, since $\overline{\mathbb{F}_p}^{\times} = \bigcup_{k\in\mathbb{N}} \mathbb{F}_{p^k}^{\times}$. Hence $\chi_i(1 + a_1 t + \cdots + a_{r-1}t^{r-1})$ has order 1, making it the identity element of $\overline{\mathbb{F}_p}^{\times}$.

Now $\chi_i(a_0 + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0 \cdot (1 + \frac{a_1}{a_0}t + \cdots + \frac{a_{r-1}}{a_0})) = \chi_i(a_0)\chi_i(1 + \frac{a_1}{a_0}t + \cdots + \frac{a_{r-1}}{a_0}) = \chi_i(a_0)$, completing the proof. ∎

**Lemma 3.2.** *Every multiplicative map $\chi : B_r \to \overline{\mathbb{F}_p}^{\times}$ is of the form*

$$\chi : B_r \to (\mathbb{F}_p[t]/(t^r))^{\times}$$

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^{\ell}(a_0 d_0)^s$$

*for some $0 \le \ell, s \le p-2$.*

*Proof.* We first show that any matrix $\left[\begin{smallmatrix} 1+\cdots+a_{r-1}t^{r-1} & b \\ 0 & 1+\cdots+d_{r-1}t^{r-1} \end{smallmatrix}\right]$ must get mapped to 1 in $\mathbb{F}_p^{\times}$ under any multiplicative map $\chi$. Notice that

$$\begin{bmatrix} 1 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & 1 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}^p = \begin{bmatrix} 1 + \cdots & pb(1 + \cdots) \\ 0 & 1 + \cdots \end{bmatrix}$$

and since $pb \equiv 0$ in $\mathbb{F}_p$, we must have that

$$\chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix})^p = \chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}^p) = \chi(\begin{bmatrix} 1 + \cdots & 0 \\ 0 & 1 + \cdots \end{bmatrix}).$$

Because any multiplicative map on a diagonal matrix in $G_r$ must be the product of two multiplicative maps on each entry in the diagonal, and since such diagonal elements belong to $(\mathbb{F}_p[t]/(t^r))^{\times}$, each of the two multiplicative maps must be of the form in Lemma 3.1. In particular this shows that $\chi(\left[\begin{smallmatrix} 1+\cdots & b \\ 0 & 1+\cdots \end{smallmatrix}\right]) = 1$.

Now any matrix $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in B_r$ can be expressed as

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix}$$

so $\chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right])$. But a multiplicative map on a diagonal matrix is again just the product of multiplicative maps on its diagonal entries, implying that $\chi = \chi_1 \times \chi_2$ where each $\chi_i$ is a map as in Lemma

3.1. In particular, since Lemma 3.1 shows that $\chi_i(a_0 + a_1 t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$ for an element $a_0 + \cdots + a_{r-1}t^{r-1} \in (\mathbb{F}_p[t]/(t^r))^\times$, then we conclude

$$\chi(\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}) = \chi_1(a_0) \cdot \chi_2(d_0).$$

But both $a_0$ and $d_0$ belong to $\mathbb{F}_p^\times$, a cyclic group of order $p-1$, and hence $\chi_1(a_0)$ and $\chi_2(d_0)$ must be $(p-1)^{st}$ roots of unity in $\overline{\mathbb{F}_p}^\times$. Since all $p-1$ such roots of unity lie in $\mathbb{F}_p^\times \subset \overline{\mathbb{F}_p}^\times$, then both $\chi_1$ and $\chi_2$ map into $\mathbb{F}_p^\times$, which is cyclic of order $p-1$. This implies that $\chi_1(a_0) = a_0^m$ for some $0 \le m \le p-2$ and $\chi_2(d_0) = d_0^s$ for some $0 \le s \le p-2$. Alternatively, we can express $a_0^m \cdot d_0^s$ as $a_0^\ell (a_0 d_0)^s$ where $\ell = m - s \mod p$. □

**Remark 3.3.** In this paper we abuse notation and write $\frac{a}{d} : B_r \to \overline{\mathbb{F}_p}^\times$ to mean the map $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mapsto a_0 d_0^{-1} = a_0 d_0^{p-2}$, since the lemmas above guarantee that any character $\chi : \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \to \overline{\mathbb{F}_p}^\times$ is of the form $a_0^\ell (a_0 d_0)^s$.

3.2. **Induction from Borel subgroup.** Let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character. For $r \ge 2$, we define the Iwahori subgroup

$$I_r^{r-1} := \{\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in G_r\} \tag{7}$$

to be the invertible matrices in $G_r$ whose $(2,1)$-entry have no terms of the form $c_k t^k$ for $0 \le k \le r-2$. Equivalently, we may define $I_r^{r-1}$ to be the preimage of $B_{r-1}$ under the surjective homomorphism

$$\begin{aligned} \pi : G_r &\twoheadrightarrow G_{r-1} \\ t^{r-1} &\mapsto 0. \end{aligned} \tag{8}$$

Let $\sigma := \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$. As $\dim(\sigma) = [I_r^{r-1} : B_r] = p$, we fix a basis $\{\delta_0, \ldots, \delta_{p-1}\}$ of $\sigma$ by setting

$$\begin{aligned} \delta_j &: I_r^{r-1} \to \overline{\mathbb{F}_p}^\times \\ \delta_j(i) &= \mathbb{1}_{B_r x_j} \cdot \chi(i x_j^{-1}) \end{aligned} \tag{9}$$

where $B_r x_j := B_r \left[\begin{smallmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{smallmatrix}\right]$ and $\mathbb{1}$ is the indicator function. As each of these $p$ functions has support on a distinct right coset of $B_r$ in $I_r^{r-1}$, they are linearly independent. If $bi \in B_r x_j$, we have

$$\delta_j(bi) = \chi(bi x_j^{-1}) = \chi(b)\delta_j(i)$$

and if $bi \notin B_r x_j$, then $i \notin B_r x_j$, and

$$\delta_j(bi) = 0 = \chi(b)\delta_j(i),$$

which shows that these functions belong to $\sigma$. We note that by composition of induction, constructing a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ is equivalent to constructing a Jordan-Hölder series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma)$. Thus one may initially construct a Jordan-Hölder series for $\sigma$ and "induce up" to get a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$, which can then be further refined to a full composition series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$. Since this is the approach we take in Theorem 1.1, we first construct a Jordan-Hölder series for $\sigma$.

**Proposition 3.4.** *For every $0 \le k \le p$ there exists a $k$-dimensional $I_r^{r-1}$-invariant subspace $\sigma^{(k)}$ of $\sigma$, such that*

$$0 \subset \sigma^{(1)} \subset \cdots \sigma^{(p-1)} \subset \sigma$$

*is a Jordan-Hölder series for $\sigma$.*

The cases of $k = 0$ and $k = p$ are trivial. For each $1 \le k \le p-1$, we construct a $k$-dimensional subspace of $\sigma$ denoted $\sigma^{(k)}$:

$$\sigma^{(k)} := \sum_{j=0}^{p-1} \binom{j}{j} \delta_j, \sum_{j=0}^{p-2} \binom{j+1}{j} \delta_j, \ldots, \sum_{j=0}^{p-k} \binom{j+k-1}{j} \delta_j \tag{10}$$

Setting $S_\ell := \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j$ allows us to express $\sigma^{(k)} = S_1, \ldots, S_k$. From the construction of $\sigma^{(k)}$ it is clear that we get a filtration of subspaces. To see that the vectors $\{S_\ell : 1 \le \ell \le k\}$ are linearly independent and hence form a basis for $\sigma^{(k)}$, we notice that if we express each sum as a tuple in the basis $\{\delta_0, \dots, \delta_{p-1}\}$, then putting the $k$ $p$-tuples into a $p \times k$ matrix gives

$$A = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{k-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{k}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{p-2}{p-2} & \binom{p-1}{p-2} & 0 & \cdots & 0 \\ \binom{p-1}{p-1} & 0 & 0 & \cdots & 0 \end{bmatrix}_{p \times k}. \tag{11}$$

We verify that the columns $\{\vec{v_1}, \dots, \vec{v_k}\}$ are linearly independent by noting that if

$$a_1\vec{v_1} + \cdots + a_k\vec{v_k} = 0$$

then in particular $a_1\binom{p-1}{p-1} = 0$, implying that $a_1 = 0$. Then since $a_1\binom{p-2}{p-2} + a_2\binom{p-1}{p-2} = 0$, we deduce that $a_2 = 0$. The fact that $A_{ij} = 0$ for $j \geq p - i + 2$ allows us to inductively deduce that $a_i = 0$ for $1 \leq i \leq k$.

To see that $\sigma^{(k)}$ is $I_r^{r-1}$-invariant and therefore a subrepresentation of $\sigma$, we check that it is invariant under every generator of $I_r^{r-1}$. By the Iwahori factorization of $I_r^{r-1}$, any matrix $\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d \end{smallmatrix}\right] \in I_r^{r-1}$ is expressible as

$$\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1} + d \end{bmatrix} \cdot \begin{bmatrix} 1 & ba^{-1} \\ 0 & 1 \end{bmatrix}$$

which allows us to conclude that

$$I_r^{r-1} = \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \tag{12}$$

for $0 \leq k \leq r-1$ and $a, d \in (\mathbb{F}_p[t]/(t^r))^\times$. In order to determine how $I_r^{r-1}$ acts on each subspace $\sigma^{(k)}$, we first observe how each generator of $I_r^{r-1}$ in (12) acts on an ordinary basis vector $\delta_j$ of $\sigma$.

**Lemma 3.5.** *Let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character of $B_r$ and let $\sigma = \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$. Let $\{\delta_0, \dots, \delta_{p-1}\}$ be the ordered basis of $\sigma$ given in* (9)*. Then the generators of $I_r^{r-1}$ act on each $\delta_j$ via*

$$\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j = \delta_j \tag{13}$$

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j = \delta_{j-1} \tag{14}$$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \cdot \delta_{\frac{d_0}{a_0}j} \tag{15}$$

*where all indices $j$ are taken modulo $p$.*

*Proof.* We have that

$$(\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j)(i) \neq 0 \iff \delta_j(i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) \neq 0$$

by definition of the $G_r$ action on $\sigma$. But

$$\delta_j(i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) \neq 0 \iff i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \iff i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -t^k \\ 0 & 1 \end{bmatrix} \iff i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix}$$

and thus $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]\cdot\delta_j$ has support on $B_r x_j$. Now suppose $i\in B_r x_j$, so $i=b\cdot\left[\begin{smallmatrix}1 & 0\\ jt^{r-1} & 1\end{smallmatrix}\right]$ for some $b\in B_r$. Then

$$\begin{aligned}(\begin{bmatrix}1 & t^k\\ 0 & 1\end{bmatrix}\cdot\delta_j)(i)=\delta_j(b\begin{bmatrix}1 & 0\\ jt^{r-1} & 1\end{bmatrix}\cdot\begin{bmatrix}1 & t^k\\ 0 & 1\end{bmatrix})=\delta_j(b\begin{bmatrix}1 & t^k\\ jt^{r-1} & jt^{r-1+k}+1\end{bmatrix})&=\chi(b\begin{bmatrix}1 & t^k\\ jt^{r-1} & jt^{r-1+k}+1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ -jt^{r-1} & 1\end{bmatrix})\\ &=\chi(b\begin{bmatrix}1-jt^{r+k-1} & t^k\\ -j^2t^{2r-2+k} & jt^{r-1+k}+1\end{bmatrix})\\ &=\chi(b)\chi(\begin{bmatrix}1-jt^{r+k-1} & t^k\\ 0 & 1+jt^{r-1+k}\end{bmatrix})\\ &=\delta_j(i)\end{aligned}$$

since $\chi(\left[\begin{smallmatrix}1+\cdots & b\\ 0 & 1+\cdots\end{smallmatrix}\right])=1$ by the proof of Lemma 3.2. Hence $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]\cdot\delta_j=\delta_j$. A similar argument shows that $\left[\begin{smallmatrix}1 & 0\\ t^{r-1} & 1\end{smallmatrix}\right]\cdot\delta_j$ has support on $B_r x_{j-1}$, and if $i=bx_{j-1}$ for some $b\in B_r x_{j-1}$, then

$$(\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\delta_j)(b\begin{bmatrix}1 & 0\\ (j-1)t^{r-1} & 1\end{bmatrix})=\delta_j(b\begin{bmatrix}1 & 0\\ (j-1)t^{r-1} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix})=\delta_j(b\begin{bmatrix}1 & 0\\ jt^{r-1} & 1\end{bmatrix})=\chi(b)=\delta_{j-1}(i),$$

allowing us to conclude $\left[\begin{smallmatrix}1 & 0\\ t^{r-1} & 1\end{smallmatrix}\right]\cdot\delta_j=\delta_{j-1}$. Finally, an analogous computation shows that $\left[\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right]\cdot\delta_j$ has support on $B_r x_{\frac{d_0}{a_0}j}$, so we suppose $i=b\left[\begin{smallmatrix}1 & 0\\ \frac{d_0}{a_0}jt^{r-1} & 1\end{smallmatrix}\right]$ for some $b\in B_r$, and find that

$$(\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}\cdot\delta_j)(i)=\delta_j(b\begin{bmatrix}1 & 0\\ \frac{d_0}{a_0}jt^{r-1} & 1\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix})=\delta_j(b\begin{bmatrix}a & 0\\ d_0jt^{r-1} & d\end{bmatrix})=\chi(b\begin{bmatrix}a & 0\\ d_0jt^{r-1} & d\end{bmatrix}\begin{bmatrix}1 & 0\\ -jt^{r-1} & 1\end{bmatrix})=\chi(b)\chi(\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix})$$

whereas

$$\delta_{\frac{d_0}{a_0}j}(b\begin{bmatrix}1 & 0\\ \frac{d_0}{a_0}jt^{r-1} & 1\end{bmatrix})=\chi(b)$$

by definition, which shows that $\left[\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right]\cdot\delta_j=\chi(\left[\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right])\delta_{\frac{d_0}{a_0}j}$ as desired.

∎

Recall that we wish to show $\sigma^{(k)}$ is $I_r^{r-1}$-invariant. Since $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]$ acts trivially on each $\delta_j$, then certainly $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]\cdot S_\ell=S_\ell$ for each $1\le\ell\le k$. The actions by the other generators are more involved, so we provide them as lemmas.

**Lemma 3.6.**

$$\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot S_\ell=\sum_{m=1}^{\ell}S_m \tag{16}$$

*so that if the basis vectors of $\sigma^{(k)}$ are ordered, then acting on each basis vector by $\left[\begin{smallmatrix}1 & 0\\ t^{r-1} & 1\end{smallmatrix}\right]$ yields a sum of the vector being acted on and the preceding basis vectors, thus remaining in $\sigma^{(k)}$.*

*Proof.* We prove (16) by induction on $\ell$: when $\ell=1$, we have

$$\begin{aligned}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-1}\binom{j}{j}\delta_j&=\sum_{j=0}^{p-1}\binom{j}{j}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\delta_j\\ &=\sum_{j=0}^{p-1}\binom{j}{j}\delta_{j-1}\\ &=\sum_{j=0}^{p-1}\binom{j}{j}\delta_j\end{aligned}$$

so that the base case holds. Now suppose (16) holds for some $\ell\in\mathbb{N},\ell<k$. We wish to show the claim holds for $\ell+1$. By the binomial coefficient recurrence relation $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ (where $\binom{n-1}{k-1}=0$ whenever

$k < 1$), and by the fact that we can express $\sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j}\delta_j = \sum_{j=0}^{p-\ell} \binom{j+\ell}{j}\delta_j$ since the coefficient $\binom{p}{p-\ell}$ of $\delta_{p-\ell}$ is zero mod $p$, we get

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j}\delta_j = \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell}{j}\delta_j$$

$$= \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \left( \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j}\delta_j + \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j-1}\delta_j \right). \tag{17}$$

Our inductive hypothesis guarantees that

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j}\delta_j = \sum_{m=0}^{\ell} \sum_{j=0}^{p-m} \binom{j+m-1}{j}\delta_j, \tag{18}$$

while

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j-1}\delta_j = \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j-1}\delta_{j-1}$$

$$= \sum_{j=1}^{p-\ell} \binom{j+\ell-1}{j-1}\delta_{j-1}$$

$$= \sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j}\delta_j \tag{19}$$

since the coefficient $\binom{j+\ell-1}{j-1} = 0$ for $j = 0$, by convention. Hence from (17), (18) and (19), we conclude that

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j}\delta_j = \sum_{m=1}^{\ell+1} \sum_{j=0}^{p-m} \binom{j+m-1}{j}\delta_j \tag{20}$$

$$\implies \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot S_{\ell+1} = \sum_{m=1}^{\ell+1} S_m \tag{21}$$

confirming $\sigma^{(k)}$ is indeed invariant under $\left[\begin{smallmatrix} 1 & 0 \\ t^{r-1} & 1 \end{smallmatrix}\right]$. □

It remains to show that $\sigma^{(k)}$ is invariant under $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. As in the $\left[\begin{smallmatrix} 1 & 0 \\ t^{r-1} & 1 \end{smallmatrix}\right]$ case, we show that acting on $S_\ell \in \sigma^{(k)}$ by $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ yields an $\overline{\mathbb{F}_p}$-linear combination of $S_m \in \sigma^{(k)}$ for $m \leq \ell$, and hence belongs to $\sigma^{(k)}$. Explicitly, we claim:

**Lemma 3.7.** *Given $a, d \in \mathbb{F}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times$, let $\alpha_i := \binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}}$, where $ad^{-1}$ is a representative in $\mathbb{N}$ of the equivalence class $ad^{-1}$ in $\mathbb{Z}/p\mathbb{Z}$. Then*

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_\ell = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \sum_{m=1}^{\ell} c_m S_m \tag{22}$$

*where each $c_m$ is given by $\sum_{i=1}^{m} (-1)^{i+1} \binom{m-1}{i-1}\alpha_i$.*

*Proof.* We first ensure that the $\alpha_i$ are well-defined up to mod $p$, such that they give the same binomial coefficient mod $p$ regardless of the choice of $ad^{-1}$ in $\mathbb{N}$. It suffices to show that, given a representative of $ad^{-1} \in \mathbb{N}$,

$$\binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}} \equiv \binom{(p-i)(ad^{-1}+pk)+\ell-1}{(p-i)(ad^{-1}+pk)} \mod p \tag{23}$$

for $k \in \mathbb{N}$. Let the base $p$ expansion of $(p-i)ad^{-1}+\ell-1$ be given by $a_r p^r + \cdots + a_1 p + a_0$. Since $\ell - 1 \leq p-2$, the base $p$ expansion of $\ell - 1$ is given by $0p^r + \cdots + 0p + \ell - 1$, so by Lucas' theorem we have

$$\begin{aligned}\binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}} = \binom{(p-i)ad^{-1}+\ell-1}{\ell-1} &\equiv \binom{a_r}{0}\cdots\binom{a_0}{\ell-1} \mod p\\ &\equiv \binom{a_0}{\ell-1} \mod p.\end{aligned}$$

Thus it suffices to show that $(p-i)ad^{-1}+\ell-1$ and $(p-i)(ad^{-1}+pk)+\ell-1$ have the same constant term in their base $p$ expansions. This follows quickly from the fact that their difference is given by $pk(p-i)$, which is a multiple of $p$ and so has no constant term in its base $p$ expansion. We conclude that $\alpha_i$ is independent of the choice of $ad^{-1} \in \mathbb{N}$. In particular we may always take the canonical representative.

By the action of $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right]$ on each $\delta_j$, we have

$$\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix} \cdot S_\ell = \begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j}\delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}) \sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_{\frac{d}{a}j}. \tag{24}$$

For $0 \leq n \leq p-1$, we see that $\delta_n$ appears in the right hand sum of (24) with a coefficient of $\chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$ (where $\frac{a}{d}$ is shorthand for the *representative in* $\mathbb{N}$ of $ad^{-1}$), and since $\delta_n$ appears in each vector $S_m = \sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j$ with a coefficient of $\binom{n+m-1}{n}$ for the respective $1 \leq m \leq \ell$, it suffices to verify

$$c_1\binom{n}{n} + c_2\binom{n+1}{n} + \cdots + c_\ell\binom{n+\ell-1}{n} \equiv \binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$$

for the proposed coefficients $c_1, \dots, c_\ell$. That is, we wish to show

$$\sum_{r=1}^{\ell}\binom{n+r-1}{n}\sum_{i=1}^{r}(-1)^{i+1}\binom{r-1}{i-1}\alpha_i = \alpha_{p-n}. \tag{25}$$

Counting how often each $\alpha_r$ appears in the left hand side of (25) allows us to express

$$\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r = \sum_{r=1}^{\ell}(-1)^{r+1}\left(\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}\right)\alpha_r \tag{26}$$

such that our goal is to show

$$\sum_{r=1}^{\ell}(-1)^{r+1}\left(\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}\right)\alpha_r = \alpha_{p-n}. \tag{27}$$

When $n=0$, we need to show that $\sum_{r=1}^{\ell}\binom{r-1}{0}c_r = \alpha_p = \binom{\ell-1}{0} = 1$. By (26) we know that

$$\sum_{r=1}^{\ell}c_r = \sum_{r=1}^{\ell}(-1)^{r+1}\sum_{j=r-1}^{\ell-1}\binom{j}{0}\binom{j}{r-1}\alpha_r = \sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\alpha_r.$$

Writing

$$\alpha_1 = \binom{(p-1)\frac{a}{d}+\ell-1}{(p-1)\frac{a}{d}} = \frac{1}{(\ell-1)!}(\ell-1-\frac{a}{d})\cdots(1-\frac{a}{d})$$

and letting the variable $x$ stand in for $\frac{a}{d}$, we have that

$$\alpha_1 = \frac{1}{(\ell-1)!}(a_{\ell-1}x^{\ell-1} + a_{\ell-2}x^{\ell-2} + \cdots + a_1 x + (\ell-1)!)$$

for some coefficients $a_{\ell-1}, \dots, a_1$. Then

$$\alpha_r = \frac{1}{(\ell-1)!}((-1)^{\ell-1}r^{\ell-1}x^{\ell-1} + \cdots + a_1 r x + (\ell-1)!)$$

so that the constant term of $\sum_{r=1}^{\ell} c_r$, when viewed as a polynomial in $x = \frac{a}{d}$, is given by

$$\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{(\ell-1)!}{(\ell-1)!} = (-1)\sum_{r=1}^{\ell}(-1)^r\binom{\ell}{r} = (-1)\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r} - (-1) = 1$$

using $\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r} = 0$. On the other hand, the coefficient of $x^m$ in the polynomial $\sum_{r=1}^{\ell} c_r$ for $1 \le m \le \ell - 1$ is given by

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r}\frac{a_m}{(\ell-1)!} = \frac{-a_m}{(\ell-1)!}\sum_{r=0}^{\ell}(-1)^r r^m\binom{\ell}{r} = 0$$

due to the combinatorial sum identity $\sum_{r=0}^{\ell}(-1)^r r^m\binom{\ell}{r} = 0$ given in [7]. We conclude that $\sum_{r=1}^{\ell} c_r = 1 = \alpha_p$.

To prove $\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r = \alpha_{p-n}$ for $1 \le n \le p-1$, we compare the coefficient of $x^m$ in both expressions. Since the coefficient of $x^m$ in $\alpha_r$ is given by $\frac{a_m}{(\ell-1)!}r^m$, then from (26) we deduce that the coefficient of $x^m$ in $\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r$ must be

$$\sum_{r=1}^{\ell}(-1)^{r+1}\frac{a_m}{(\ell-1)!}r^m\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}.$$

On the other hand, the coefficient of $x^m$ in $\alpha_{p-n}$ is given by $(-n)^m\frac{a_m}{(\ell-1)!}$, so it suffices to prove

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1} = (-n)^m. \tag{28}$$

Because $\binom{j}{r-1} = 0$ whenever $j < r-1$, we can express the left hand side of (28) as

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=0}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}. \tag{29}$$

Identity 3.155 in [6] tells us that $\sum_{k=0}^{s-1}\binom{k}{n}\binom{k+m}{m} = \binom{s}{n}\binom{s+m}{m}\frac{s-n}{m+n+1}$, which allows us to express (29) as

$$\begin{aligned}
\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=0}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1} &= \sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r-1}\binom{\ell+n}{n}\frac{\ell-r+1}{r+n} \\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r-1}\frac{\ell-r+1}{r+n} \\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}r^m\cdot r\binom{\ell}{r}\frac{1}{r+n} \\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{r^{m+1}}{r+n}. \qquad (30)
\end{aligned}$$

Finally, identity 1.47 in [6] shows that $\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{k^j}{x+k} = (-1)^j\frac{x^{j-1}}{\binom{x+\ell}{\ell}}$, and therefore (30) becomes

$$\begin{aligned}
\binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{r^{m+1}}{r+n} &= \binom{\ell+n}{n}(-1)\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r}\frac{r^{m+1}}{r+n} \\
&= \binom{\ell+n}{n}(-1)(-1)^{m+1}\frac{n^m}{\binom{n+\ell}{\ell}} \\
&= (-1)^m n^m \\
&= (-n)^m \qquad (31)
\end{aligned}$$

as desired. This proves that there exist $c_1,\dots,c_\ell\in\mathbb{Z}$ such that

$$\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_{\frac{d}{a}j}=\sum_{m=1}^{\ell}c_m\sum_{j=0}^{p-m}\binom{j-m+1}{j}\delta_j \tag{32}$$

which means that there exist $c_1,\dots,c_\ell\in\mathbb{Z}$ such that

$$\begin{bmatrix}a&0\\0&d\end{bmatrix}\cdot S_\ell=\sum_{m=1}^{\ell}\chi(\begin{bmatrix}a&0\\0&d\end{bmatrix})c_mS_m. \tag{33}$$

Since the left hand side is given mod $p$, we may reduce the right hand side mod $p$ to conclude that there exist $c_1,\dots,c_\ell\in\overline{\mathbb{F}_p}$ such that (33) holds. Because this holds for all $1\le\ell\le k$, we have that $\sigma^{(k)}$ is invariant under action by $\left[\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right]$. This lemma also concludes the proof of the proposition. □

## 4. Proof of main theorem.

4.1. **Inducing up to a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$.** Proposition 3.4 gives us a length $p$ Jordan-Hölder series

$$0\subset\sigma^{(1)}\subset\cdots\subset\sigma^{(p-1)}\subset\sigma.$$

Since each $\sigma^{(k)}$ is a subrepresentation of $\sigma$ which is itself a representation of $I_r^{r-1}$, then inducing each $\sigma^{(k)}$ to $G_r$ gives a filtration

$$0\subset\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)})\subset\cdots\subset\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)})\subset\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma).$$

In order to refine this filtration to a composition series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma)=\mathrm{Ind}_{B_r}^{G_r}(\chi)$, we note that it suffices to find a composition series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})$ which begins with $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ for each $0\le k\le p-1$. But this is equivalent to finding a composition series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ and then lifting the subrepresentations under the projection map $q:\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})\to\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$. Furthermore, since

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})\cong\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$$

then we only need consider composition series of $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$ in order to answer our original question.

We claim that $\sigma^{(k+1)}/\sigma^{(k)}$ is equivalent to $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ as one-dimensional $I_r^{r-1}$ representations, where $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ refers to the inflation to $I_r^{r-1}$ of the character sending $\left[\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right]\mapsto\chi(\left[\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right])\cdot(\frac{a}{d})^k\in\overline{\mathbb{F}_p}^\times$. To prove this equivalence it suffices to show that $I_r^{r-1}$ acts on $\sigma^{(k+1)}/\sigma^{(k)}$ via multiplication by $\chi\cdot(\frac{a}{d})^k$. Again we show this claim only for the three types of generators of $I_r^{r-1}$.

**Lemma 4.1.** *The generators $\left[\begin{smallmatrix}1&t^\ell\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1&0\\t^{r-1}&1\end{smallmatrix}\right]$ act trivially on $\sigma^{(k+1)}/\sigma^{(k)}$ for $0\le\ell\le r-1$ and $0\le k\le p-1$.*

*Proof.* Notice $\sigma^{(k+1)}/\sigma^{(k)}=\overline{S_{k+1}}$. Since $\left[\begin{smallmatrix}1&t^\ell\\0&1\end{smallmatrix}\right]$ acts trivially on each $\delta_j$, then clearly $\left[\begin{smallmatrix}1&t^\ell\\0&1\end{smallmatrix}\right]$ acts trivially on $\overline{S_{k+1}}$. On the other hand, by the proof of Lemma 3.6, we know that

$$\begin{aligned}\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\overline{S_{k+1}}&=\overline{\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot S_{k+1}} &(34)\\&=\overline{\sum_{m=1}^{k+1}S_m}\\&=\overline{S_{k+1}} &(35)\end{aligned}$$

where (35) follows from the fact that $\overline{S_i}=0\in\sigma^{(k+1)}/\sigma^{(k)}$ for $1\le i\le k$. This proves that $\left[\begin{smallmatrix}1&0\\t^{r-1}&1\end{smallmatrix}\right]$ acts trivially on $\sigma^{(k+1)}/\sigma^{(k)}$. □

**Lemma 4.2.** *The generator $\left[\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right]$ acts on $\sigma^{(k+1)}/\sigma^{(k)}$ via scaling by $\chi(\left[\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right])\cdot(\frac{a}{d})^k$.*

*Proof.* By Lemma 3.7 we have that

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \overline{S_{k+1}} = \overline{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_{k+1}} = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \sum_{m=1}^{k+1} c_m \overline{S_m}$$

and since $\overline{S_m} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$ for $1 \leq m \leq k$, then in $\sigma^{(k+1)}/\sigma^{(k)}$ we have

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \overline{S_{k+1}} = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) c_{k+1} \overline{S_{k+1}}.$$

To prove our claim it suffices to show that $c_{k+1} = (\frac{a}{d})^k$. Recall that by Lemma 3.7, we have

$$c_{k+1} = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} \alpha_i$$

where here $\alpha_i = \binom{(p-i)\frac{a}{d}+k}{(p-i)\frac{a}{d}} = \frac{(k-i\frac{a}{d})\cdots(1-i\frac{a}{d})}{k!}$. In particular, since we may write out $\alpha_1 = \frac{(k-x)\cdots(1-x)}{k!} = \frac{1}{k!}((-1)^k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + k!)$ where $x = \frac{a}{d}$, then we have that $\alpha_i = \frac{1}{k!}((-1)^k i^k x^k + a_{k-1} i^{k-1} x^{k-1} + \cdots + a_1 i x + k!)$ for $1 \leq i \leq k+1$. Since the coefficient of $x^m$ in $\alpha_i$ is given by $\frac{a_m}{k!} \cdot i^m$, then the coefficient of $x^m$ in the expression of $c_{k+1}$ is given by

$$\sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} \frac{a_m}{k!} i^m = \frac{a_m}{k!} \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m. \tag{36}$$

Since we wish to show that $c_{k+1} = x^k = (\frac{a}{d})^k$, it suffices to show that (36) is zero whenever $0 \leq m \leq k-1$ and is 1 whenever $m = k$. When $m = 0$, we have that $a_0 = k!$, so $\frac{a_0}{k!} \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^0 = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0$, as desired. On the other hand, the identity

$$\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} i^m = 0 \tag{37}$$

holds for $1 \leq m \leq k$ (see [7], #3 in 0.154), and since $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$, we deduce from (37) that

$$\sum_{i=0}^{k+1} (-1)^i \binom{k}{i} i^m + \sum_{i=0}^{k+1} (-1)^i \binom{k}{i-1} i^m = 0$$

which implies that

$$\sum_{i=0}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m = \sum_{i=0}^{k+1} (-1)^i \binom{k}{i} i^m = \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^m$$

since $\binom{k}{k+1} = 0$ by convention. Now $\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^m = 0$ for $0 \leq m \leq k-1$ by the identity in (37), so $\sum_{i=0}^{k+1} (-1)^{i+1} \binom{k}{i-1} j^m = 0$ for $0 \leq m \leq k-1$. When $m > 0$ we have that $0^m = 0$, so we conclude $\sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m = 0$ for $0 \leq m \leq k-1$ as desired. On the other hand, identity #4 in §0.154 of [7] gives

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k = (-1)^k k!\,, \tag{38}$$

which in combination with (37) and the fact that $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$ gives

$$\begin{aligned} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} j^k &= \sum_{j=0}^{k+1} (-1)^j \binom{k}{j} j^k + \sum_{j=0}^{k+1} (-1)^j \binom{k}{j-1} j^k \\ \implies \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k}{j-1} j^k &= (-1)^k k! \end{aligned}$$

which is precisely what we wished to show. Hence the coefficient of $x^m$ in $c_{k+1}$ is $\frac{a_m}{k!}\cdot 0 = 0$ for $0 \le m \le k-1$ while the coefficient of $x^k$ is $\frac{(-1)^k}{k!}\cdot(-1)^k k! = (-1)^{2k} = 1$, completing the proof that $c_{k+1} = (\frac{a}{d})^k$, and therefore that $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right]\cdot\overline{S_{k+1}} = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\cdot(\frac{a}{d})^k\overline{S_{k+1}}$. $\square$

Recall we wish to show that $\sigma^{(k+1)}/\sigma^{(k)}$ is equivalent to $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ as $I_r^{r-1}$ representations. Let $T : \overline{S_{k+1}} \to \mathbb{F}_p$ be the isomorphism sending $\overline{S_{k+1}} \mapsto 1$. For all $\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d\end{smallmatrix}\right] \in I_r^{r-1}$, we have

$$\begin{aligned} T\left(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix}\cdot\overline{S_{k+1}}\right) &= T\left(\begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\begin{bmatrix} 1 & ba^{-1} \\ 0 & 1\end{bmatrix}\cdot\overline{S_{k+1}}\right) \\ &= T\left(\begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\cdot\overline{S_{k+1}}\right). \end{aligned} \tag{39}$$

Now $(\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right]\cdot\delta_j)(i) \neq 0$ if and only if $i\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right] \in B_r\left[\begin{smallmatrix} 1 & 0 \\ jt^{r-1} & 1\end{smallmatrix}\right]$, which holds if and only if $i \in B_r\left[\begin{smallmatrix} 1 & 0 \\ jt^{r-1} & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right]^{-1} = B_r\left[\begin{smallmatrix} 1 & 0 \\ \frac{d}{a}jt^{r-1} & 1\end{smallmatrix}\right]$. A similar argument as the one for $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right]\cdot\delta_j = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\delta_{\frac{d}{a}j}$ reveals that $\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right]\cdot\delta_j = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\delta_{\frac{d}{a}j}$, and therefore Lemma 4.2 applies to (39) to give $\chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])(\frac{a}{d})^k\cdot T(\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}) = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])(\frac{a}{d})^k$. On the other hand, we have that

$$\begin{aligned} \mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix})(T(S_k)) &= (\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(T(S_k)) \\ &= \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(\frac{a}{d})^k \end{aligned} \tag{40}$$

which shows that $T\circ\sigma^{(k+1)}/\sigma^{(k)}(\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d\end{smallmatrix}\right]) = \mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)(\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d\end{smallmatrix}\right])\circ T$, and hence that $\sigma^{(k+1)}/\sigma^{(k)}$ and $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ are isomorphic as $I_r^{r-1}$-representations.

Now because the diagram

$$\begin{array}{ccc} I_r^{r-1} & \xrightarrow{t^{r-1}\mapsto 0} & B_{r-1} \\ \downarrow & & \downarrow \\ G_r & \xrightarrow{t^{r-1}\mapsto 0} & G_{r-1} \end{array}$$

commutes, we have by commutativity of inflation and induction that $\mathrm{Ind}_{I_r^{r-1}}^{G_r}\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$. But this implies $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$, completing the proof of Theorem 1.1.

4.2. **A remark on the inductive construction.** Theorem 1.1 tells us what the successive quotients in the filtration given in (5) look like, but it doesn't explicitly tell us what the Jordan-Hölder series of $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ looks like. Fortunately, we just proceed inductively: once we know that

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$$

then we can set out to find a Jordan-Hölder series of $\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$ (using the same process as in our original problem) and then "piece it in" between $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ and $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})$ in the filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$. Since the literature already contains the Jordan-Hölder series for the mod $p$ principal series representations of $\mathrm{Ind}_{B_1}^{G_1}(\chi)$, we have all the parts necessary to complete the original filtration to a full Jordan-Hölder series.

## 5. Semisimplifications

From Theorem 1.1 we deduce that

$$\begin{aligned} (\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} &= (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^{p-1}))^{ss} \\ &= (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \end{aligned} \tag{41}$$

where inflations to $G_r$ are always implicitly assumed. In particular, we see that $(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}$ appears twice in the direct sum of (41), while $(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}$ appears once in the direct sum for every $1 \leq k \leq p-2$. Hence we may express

$$(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} = ((\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss})^2 \oplus \bigoplus_{k=1}^{p-2}(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}. \tag{42}$$

Since the semisimplifications of $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ are known for all characters $\chi : B(\mathrm{GL}_2(\mathbb{F}_p)) \to \overline{\mathbb{F}_p}^{\times}$ (Lemma 2.2 in [3]), it is desirable to express (42) explicitly in terms of $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ for various $\chi$. We claim that we may continue simplifying (42) inductively to obtain:

**Corollary 5.1.** *For a prime $p$, $(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}}$.*

*Proof.* We prove the corollary by induction on $r$. When $r = 1$, the claim is that

$$(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^0+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^0-1}{p-1}}$$

which is easily seen to be true when one simplifies the exponents on the right hand side of the equality. Suppose the claim in the proposition holds for some $r \in \mathbb{N}$. We wish to show it holds for $r+1$. As a corollary of Theorem 1.1, we have that

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = ((\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss})^2 \oplus \bigoplus_{k=1}^{p-2}(\mathrm{Ind}_{B_r}^{G_r}(\chi\cdot(\frac{a}{d})^k))^{ss}.$$

Utilizing the inductive hypothesis on $(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss}$ and on each $(\mathrm{Ind}_{B_r}^{G_r}(\chi\cdot(\frac{a}{d})^k))^{ss}$ gives

$$\begin{aligned}(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = &\left(((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}}\right)^2 \\ &\oplus \left(\bigoplus_{k=1}^{p-2}\left[((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{m\neq k}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^m))^{ss})^{\frac{p^{r-1}-1}{p-1}}\right]\right).\end{aligned} \tag{43}$$

Counting how many times $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ appears in the direct sum of (43) yields that $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ appears

$$2(\frac{p^{r-1}+p-2}{p-1}) + (p-2)\frac{p^{r-1}-1}{p-1} = \frac{p^r+p-2}{p-1}$$

times, whereas counting how many times $(\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^n))^{ss}$ appears in (43) for a given $1 \leq n \leq p-2$ yields that $(\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^n))^{ss}$ appears

$$2(\frac{p^{r-1}-1}{p-1}) + \frac{p^{r-1}+p-2}{p-1} + (p-3)\frac{p^{r-1}-1}{p-1} = \frac{p^r-1}{p-1}$$

times. Therefore

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^r+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^r-1}{p-1}}. \tag{44}$$

proving the inductive claim. □

A complete semisimplification expresses the given representation as a direct sum of its unique set of composition factors, which are each irreducible representations. Hence giving the semisimplification of $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ requires knowing the irreducible characteristic $p$ representations of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$.

5.1. **Classifying Modular Irreps of** $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$**.** We claim that every irreducible characteristic $p$ representation of $G_r$ is of the form $\rho \circ \pi$, where $\pi$ is the surjective homomorphism

$$\pi : \mathrm{GL}_2(\mathbb{F}_p[t]/(t^r)) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_p)$$

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \tag{45}$$

and $\rho$ is an irreducible characteristic $p$ representation of $\mathrm{GL}_2(\mathbb{F}_p)$. To prove this fact we need the following two known lemmas, which then establish the result as an immediate corollary.

**Lemma 5.2.** *Let $G$ be a finite group and let $H \trianglelefteq G$ be a $p$-group. If $V$ is an irreducible characteristic $p$ representation of $G$, then $V^H = V$, that is, $H$ acts trivially on all elements of $V$.*

In particular Lemma 5.2 tells us that if $G$ is a finite group, $H \trianglelefteq G$ is a $p$-group, and $V$ is an irreducible characteristic $p$ representation of $G$, then $V$ must be the direct sum of trivial representations on $H$. We claim that this implies $V$ factors through $G/H$.

**Lemma 5.3.** *A representation of a finite group $G$ is trivial on a normal subgroup $H$ if and only if it factors through $G/H$.*

The preceding lemmas allow us to prove the claim established at the beginning of this section:

**Proposition 5.4.** *Any irreducible modular representation of* $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ *is the inflation of an irreducible modular representation of* $\mathrm{GL}_2(\mathbb{F}_p)$.

*Proof.* The surjective homomorphism $\pi$ in (45) gives us $H = \ker \pi \trianglelefteq G_r$. We claim that $H$ is a $p$-group: Notice that $G_1 = \mathrm{GL}_2(\mathbb{F}_p)$ may be viewed as a subgroup of $G_r$, as it respects multiplication in $G_r$. Since the matrix

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}$$

belongs to $\ker \pi$ if and only if $a_0 = d_0 = 1, b_0 = c_0 = 0$, and $a_i, b_i, c_i, d_i \in \mathbb{F}_p$ for $1 \leq i \leq r-1$, then $|\ker \pi| = |\mathbb{F}_p|^{4(r-1)} = p^{4(r-1)}$. Hence by Lemma 5.2 any irreducible modular representation of $G_r$ must be trivial on $H$. But by Lemma 5.3, we know that a representation of $G_r$ is trivial on a normal subgroup $H$ if and only if it factors through $G_r/H$. Since $G_r/H \cong \mathrm{GL}_2(\mathbb{F}_p)$, then every irreducible characteristic $p$ representation $\tilde{\rho}$ of $G_r$ must be of the form $\rho \circ \pi$ where $\pi$ is the map given in (45) and $\rho$ is an irreducible characteristic $p$ representation of $\mathrm{GL}_2(\mathbb{F}_p)$. $\square$

Fortunately the irreducible characteristic $p$ representations of $\mathrm{GL}_2(\mathbb{F}_p)$ are fully classified (see [1] or [8] for the proofs). Given $0 \leq n \leq p-1$ and $0 \leq \ell \leq p-2$, let $P_n$ be the $\overline{\mathbb{F}_p}$ span of the basis $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$. Define

$$\rho_{n,\ell} : \mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}(P_n) \tag{46}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot P(x,y) = P(ax+cy, bx+dy) \cdot \left(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{\ell}.$$

Then $\{\rho_{n,\ell}\}$ gives a complete set of irreducible characteristic $p$ representations of $\mathrm{GL}_2(\mathbb{F}_p)$ up to equivalence. Hence every irreducible characteristic $p$ representation of $G_r$ is given by $\tilde{\rho}_{n,\ell} = \rho_{n,\ell} \circ \pi$, where $\pi$ is as in (45).

5.2. **Semisimplification of** $\mathrm{Ind}_{B_r}^{G_r}(\chi)$**.** Recall that any multiplicative map $\chi : B_1 \to \overline{\mathbb{F}_p}^{\times}$ is of the form $\chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = a^r(ad)^s$, where $0 \leq r, s \leq p-2$. From [3] we know that if $r = 0$, then $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \rho_{0,s} \oplus \rho_{p-1,s}$, where $\rho_{p-1,s}$ may be recognized as the twisted Steinberg representation. On the other hand, if $r \neq 0$, then $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \rho_{p-1-r,r+s} \oplus \rho_{r,s}$. In particular this tells us that $(\mathrm{Inf}_{G_1}^{G_r} \mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \tilde{\rho}_{0,s} \oplus \tilde{\rho}_{p-1,s}$ or $(\mathrm{Inf}_{G_1}^{G_r} \mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \tilde{\rho}_{p-1-r,r+s} \oplus \tilde{\rho}_{r,s}$ depending on $\chi$. In combination with Corollary 5.1 this fact allows us to explicitly give the semisimplification of $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ for any character $\chi$.

## 6. Appendix: Computing Semisimplifications via Brauer Characters

Brauer pioneered modular representation theory largely to better understand the relationships between characteristic $p$ representations and ordinary character theory. A key development in this theory is the invention of *Brauer characters*, which assign to particular elements of a group $G$ a value in a field of characteristic 0 dependent on a characteristic $p$ representation. The utility of such characters in our problem comes from their ability to solve for the semisimplification numbers given in Corollary 5.1 without requiring any knowledge about the Jordan-Hölder series itself.

To compute the Brauer character of a representation we outline a process described in greater generality in [5] and [9]. Let $m$ be the least common multiple of the orders of $p$-regular elements of $G$ (elements of $G$ with order coprime to $p$). Let $\rho$ be an irreducible characteristic $p$ representation of $G$. For any $g \in G$ a $p$-regular element, $\rho(g)$ must have order dividing $|g|$ in $\overline{\mathbb{F}_p}^\times$, and hence has order dividing $m$. In particular this means that the eigenvalues of $\rho(g)$ are all powers of $m^{th}$ roots of unity in $\overline{\mathbb{F}_p}^\times$.

Writing $\zeta_m$ for a primitive $m^{th}$ root of unity in $\overline{\mathbb{F}_p}^\times$ allows us to express the eigenvalues of $\rho(g)$ as $\zeta_m^{m_1}, \dots, \zeta_m^{m_k}$, where $k$ is the dimension of the representation $\rho$, and $0 \leq m_i \leq m-1$. We fix a bijection between the $m^{th}$ roots of unity in $\overline{\mathbb{F}_p}^\times$ and the $m^{th}$ roots of unity in $\mathbb{C}$ by mapping $\zeta_m \mapsto \omega_m = e^{\frac{2\pi i}{m}}$. The *Brauer character* of $\rho$ evaluated at $g$ is given by $\theta_\rho(g) = \sum_{i=1}^k \omega_m^{m_i}$. Notice that since elements of a $p$-regular conjugacy class have the same eigenvalues, Brauer characters are constant on $p$-regular conjugacy classes.

Fix a field $E$ of characteristic 0 whose residue field is of characteristic $p$. Given an ordinary representation $\psi : G \to \mathrm{GL}(V)$ with associated character $\chi : G \to E^\times$, we have that the *mod p reduction* $\overline{\chi} : G \to k_E^\times$ of $\chi$ may be expressed as a non-negative integer linear combination of the irreducible Brauer characters of $G$ [2]. This means that for all $p$-regular $g \in G$,

$$\overline{\chi}(g) = \sum_{\rho \text{ modular irreps of } G} d_\rho \theta_\rho(g) \tag{47}$$

where $d_\rho \in \mathbb{Z}_{\geq 0}$. We call these $d_\rho$ the *decomposition numbers* of $\overline{\psi}$ as they give the multiplicity of the irreducible representation $\rho$ in the semisimplification of $\overline{\psi}$.

We wish to compute the semisimplification of $\vartheta_\chi = \mathrm{Ind}_{B_r}^{G_r}(\chi)$ via Brauer characters. As Brauer characters are only defined on $p$-regular conjugacy classes, we determine these conjugacy classes for $G_r$. The conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_p)$ are well-known, and the $p$-regular conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ for $r \in \mathbb{N}$ have representatives in $\mathrm{GL}_2(\mathbb{F}_p)$. For primes $p$, we have the $p$-regular conjugacy classes:

(1) $\left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{F}_p^\times\right\}$. There are $|\mathbb{F}_p^\times| = p-1$ such conjugacy classes.
(2) $\left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{F}_p^\times\right\}$. Swapping the position of $a$ and $b$ yields conjugate matrices, but a different pair of $(a, b)$ yields a non-conjugate matrix. Hence we have $\binom{p-1}{2}$ such conjugacy classes.
(3) $\left\{\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}\right\}$ where $D$ is not a square in $\mathbb{F}_p$, and $\alpha + \beta\sqrt{D}$ is a characteristic root of a matrix in $\mathrm{GL}_2(\mathbb{F}_p)$ with $\beta \neq 0$. The matrices $\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}$ and $\begin{bmatrix} \alpha & -D\beta \\ -\beta & \alpha \end{bmatrix}$ are conjugate, so we only need consider $\beta \in \{1, \dots, \frac{p-1}{2}\}$.

None of the matrices of type (3) above are conjugate to an upper triangular matrix in $\mathrm{GL}_2(\mathbb{F}_p)$, else their eigenvalues would lie in $\mathbb{F}_p$. We see that this must hold in the larger group $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ as well: if any of the matrices of type (3) were conjugate in $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ to an upper triangular matrix, then their eigenvalues would have to lie in $(\mathbb{F}_p[t]/(t^r))^\times \cap (\mathbb{F}_p[\sqrt{D}])^\times = \mathbb{F}_p^\times$, and thus in particular must be of type (1) or (2). This contradicts the fact that (1), (2), and (3) give distinct conjugacy class types.

The character of the representation $\vartheta_\chi$, denoted by $\theta_\chi$, has a formula due to Mackey:

$$\theta_\chi(g) = \sum_{\substack{x_i \in B_r\backslash G_r \\ x_i g x_i^{-1} \in B}} \chi(\chi_i g \chi_i^{-1}). \tag{48}$$

We use this formula to compute the character on our $p$-regular conjugacy classes. For each conjugacy class of type $[\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}]$, we have that $x_i[\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}]x_i^{-1} = [\begin{smallmatrix} a & 0 \\ 0 & a\end{smallmatrix}] \in B$, so using (48) we obtain

$$\theta_\chi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) = |B_r\backslash G_r| \cdot \chi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) \tag{49}$$

$$= p^{r-1}(p+1)\chi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}). \tag{50}$$

We now suppose $g = [\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]$ where $a, b \in \mathbb{F}_p^\times$ and $a \neq b$. If $x_j$ is the coset representative for $B_r$ in the set of right cosets of $B_r$ (that is, $x_j$ is the identity matrix), then $x_j[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_j^{-1} \in B$ trivially. Now suppose $x_j = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta\end{bmatrix}$ where $\gamma \neq 0$ (so that $x_j \notin B_r$). We wish to determine when $x_j[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_j^{-1} \in B$. Note

$$\begin{aligned} x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} \\ &= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \\ &= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a\delta & -a\beta \\ -b\gamma & b\alpha \end{bmatrix} \\ &= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} a\alpha\delta - b\beta\gamma & -a\alpha\beta + b\beta\alpha \\ a\gamma\delta - b\delta\gamma & -a\beta\gamma + b\delta\alpha \end{bmatrix} \end{aligned} \tag{51}$$

so that $x_j[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_j^{-1} \in B$ if and only if $a\delta\gamma - b\delta\gamma = 0$, that is, if and only if $(a-b)\delta\gamma = 0$. Since $a \neq b$ and $a, b \in \mathbb{F}_p^\times$, then $a - b \in \mathbb{F}_p^\times$, and thus we must have $\delta\gamma = 0$. But we assumed that $\gamma \neq 0$, so we must have $\delta = 0$. From (51) we see then that if $x_j \notin B_r$ and $x_j[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_j^{-1} \in B$, then

$$x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} = \frac{-1}{\beta\gamma} \begin{bmatrix} -b\beta\gamma & (b-a)\alpha\beta \\ 0 & -a\beta\gamma \end{bmatrix} = \begin{bmatrix} b & \frac{(a-b)\alpha}{\gamma} \\ 0 & a \end{bmatrix} \tag{52}$$

so that $\chi(x_j[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_j^{-1}) = \chi([\begin{smallmatrix} b & 0 \\ 0 & a\end{smallmatrix}])$. To see that no other coset representative $x_\ell$ gives $x_\ell[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_\ell^{-1} \in B$, suppose such an $x_\ell$ did exist with $Bx_j \neq Bx_\ell$. Let $x_j = \begin{bmatrix} \alpha & \beta \\ \gamma & 0\end{bmatrix}$, where $\gamma \neq 0$ so that $x_j \notin B$. Let $x_\ell = [\begin{smallmatrix} x & y \\ z & w\end{smallmatrix}]$. Then

$$\begin{aligned} Bx_j \neq Bx_\ell &\iff x_\ell x_j^{-1} \notin B \\ &\iff \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}^{-1} \notin B \\ &\iff \frac{-1}{\beta\gamma} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 0 & -\beta \\ -\gamma & \alpha \end{bmatrix} \notin B \\ &\iff \frac{-1}{\beta\gamma} \begin{bmatrix} -y\gamma & -x\beta + y\alpha \\ -w\gamma & -z\beta + w\alpha \end{bmatrix} \notin B \\ &\iff w \neq 0. \end{aligned}$$

Recall from our computation above that $x_\ell[\begin{smallmatrix} a & 0 \\ 0 & b\end{smallmatrix}]x_\ell^{-1} \in B$ if and only if $(x_\ell)_{22} = 0$, that is, if and only if $w = 0$. This contradiction allows us to conclude that

$$\theta_\chi(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}) = \chi(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}) + \chi(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}). \tag{53}$$

Finally, if $g = \left[\begin{smallmatrix} \alpha & D\beta \\ \beta & \alpha \end{smallmatrix}\right]$ is a matrix as in type (3), then we know that $g$ has no upper triangular conjugates. Thus

$$\theta_\chi(\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}) = 0. \tag{54}$$

To illustrate how to obtain the semisimplification numbers from the above computation, we fix $p = 3$ and $\chi$ to be the trivial character. From the above computation, we have the following table for representatives of the 3-regular conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_3[t]/(t^r))$:

| | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ |
|---|---|---|---|---|---|---|
| $\theta_\chi(g)$ | $p^{r-1}(p+1)$ | $p^{r-1}(p+1)\cdot\chi(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix})$ | $\chi(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}) + \chi(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix})$ | 0 | 0 | 0 |

We wish to solve for the $d_\rho$ in (47), which requires us to know how $\theta_\rho$ evaluates on $g$ for each conjugacy class and for each $\rho$ an irreducible modular representation of $G_r$. An omitted computation yields the following Brauer characters:

| | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ | $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$ |
|---|---|---|---|---|---|---|
| $\theta_{0,0}$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\theta_{0,1}$ | 1 | 1 | -1 | 1 | -1 | -1 |
| $\theta_{1,0}$ | 2 | -2 | 0 | 0 | $-i\sqrt{2}$ | $i\sqrt{2}$ |
| $\theta_{1,1}$ | 2 | -2 | 0 | 0 | $i\sqrt{2}$ | $-i\sqrt{2}$ |
| $\theta_{2,0}$ | 3 | 3 | 1 | -1 | -1 | -1 |
| $\theta_{2,1}$ | 3 | 3 | -1 | -1 | 1 | 1 |

TABLE 1. Brauer character table, $p = 3$

where $\theta_{n,\ell}$ is the Brauer character corresponding to $\tilde{\rho}_{n,\ell}$. Recall that any character $\overline{\chi} : B_1 \to \overline{\mathbb{F}_p}^\times$ is of the form $\chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = a^r(ad)^s$ where $0 \le r, s \le p-2$, and hence when $p = 3$ we have only three choices: $r = s = 0$ (yielding the trivial character triv), $r = 0, s = 1$ (det), and $r = 1$, $s = 0$ (alt).

Solving a system of equations according to (47) for every choice of character $\chi$ in the $p = 3$ case yields:

| $(\chi_1, \chi_2)$ | $\mathrm{d}_{0,0}$ | $\mathrm{d}_{0,1}$ | $\mathrm{d}_{1,0}$ | $\mathrm{d}_{1,1}$ | $\mathrm{d}_{2,0}$ | $\mathrm{d}_{2,1}$ |
|---|---|---|---|---|---|---|
| triv | $\frac{3^{r-1}+1}{2}$ | $\frac{3^{r-1}-1}{2}$ | 0 | 0 | $\frac{3^{r-1}+1}{2}$ | $\frac{3^{r-1}-1}{2}$ |
| det | $\frac{3^{r-1}-1}{2}$ | $\frac{3^{r-1}+1}{2}$ | 0 | 0 | $\frac{3^{r-1}-1}{2}$ | $\frac{3^{r-1}+1}{2}$ |
| alt | 0 | 0 | $3^{r-1}$ | $3^{r-1}$ | 0 | 0 |

TABLE 2

We verify that this aligns with our numbers in Corollary 5.1. For simplicity we take $\chi = $ triv, noticing that by Table 2, we have

$$(\mathrm{Ind}_{B_r}^{G_r}(\mathrm{triv}))^{ss} = \tilde{\rho}_{0,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{0,1}^{\frac{3^{r-1}-1}{2}} \oplus \tilde{\rho}_{2,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{2,1}^{\frac{3^{r-1}-1}{2}}.$$

On the other hand, by Corollary 5.1 we have that

$$(\mathrm{Ind}_{B_r}^{G_r}(\mathrm{triv}))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}))^{ss})^{\frac{3^{r-1}+1}{2}} \oplus ((\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}\cdot\frac{a}{d}))^{ss})^{\frac{3^{r-1}-1}{2}}$$

Now $(\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}))^{ss} = \rho_{0,0} \oplus \rho_{2,0}$, and since $\frac{a}{d} = ad^{-1} = ad$ in $\overline{\mathbb{F}_3}$, then $(\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv} \cdot \frac{a}{d}))^{ss} = \rho_{0,1} \oplus \rho_{2,1}$. Hence

$$(\mathrm{Ind}_{B_r}^{G_r}(\mathrm{triv}))^{ss} = \tilde{\rho}_{0,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{2,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{0,1}^{\frac{3^{r-1}-1}{2}} \oplus \tilde{\rho}_{2,1}^{\frac{3^{r-1}-1}{2}}$$

which is precisely what we deduced from Table 2. A similar computation verifies the other two cases of $\chi$.

For larger primes computing the Brauer table is much more computationally intensive, so we resort to the semisimplification numbers which resulted from the Jordan-Hölder series.

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