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Hatcher Algebraic Topology

1 Chapter 0: Some Underlying Geometric Notions

1.1 Summary sections

Every few pages I'll try to summarize what I read.

We start with the equivalence problem in mathematics. In the case of algebraic topology, the equivalence problem concerns when 2 spaces (resp., 2 maps) are homotopy equivalent (resp, homotopic). BUT WHAT DOES THAT MEAN??‼

Suppose we have that a space X deformation retracts onto its subspace A. Let $f_t : X \to X$ be the family of maps of the deformation retraction. Let $i : A \to X$ be the inclusion map and let $r : X \to A$ be the retraction map (f_1). Notice that $r \circ i = \mathrm{id}$ (EQUALS THE IDENTITY ON A) since we know that $r|_A$ is the identity on A (by definition of a retraction) and thus first including A into X, then applying r will just give us the identity. On the other hand, $i \circ r$ is NOT simply equal to the identity, because while r maps all points in X to A, it does NOT act as the identity on points in $X \setminus A$. Points in $X \setminus A$ will still be mapped to points in A, and then applying the inclusion map doesn't change anything. It does however mean that $i \circ r : X \to X$ is not the identity. HOWEVER, $i \circ r$ is HOMOTOPIC to the identity map of X. Why??? Consider $F(t, x) = f_t(x)$, our homotopy. Notice that $F(0, x) = f_0(x) = \mathrm{id}_X$ by definition of a homotopy. On the other hand, $F(1, x) = f_1(x) = i \circ r$ since $i \circ r = r$. Therefore $i \circ r \simeq \mathrm{id}_X$. Since $r : X \to A$ and there exists a map $i : A \to X$ such that $r \circ i \simeq \mathrm{id}$ and $i \circ r \simeq \mathrm{id}$, then X and A are homotopy equivalent.

1.2 Homotopy and Homotopy Type

Basic idea: homotopy is a broader sense of homeomorphism for thinking of two spaces as "equivalent." Example given in the beginning: consider a bold letter inside of a block outline of that letter. We can consider sliding the points on the block outline inside via straight lines such that at time 0 the points on the block outline remain on the block outline, and at time 1 the points on the block outline are now at a point on the bold letter inside. Actually for every time t between 0 and 1 we can consider the map $f_t : X \to X$ (where X is the space enclosed by the block outline) such that f_0 is the identity map, $f_1(X) = A$ describes the final locations of the points in X (which is a subspace A, in our example the bold letter which is a subspace of the block letter), and such that f_t restricted to A is always A, for all t (the points in the space we're trying to map to never move!!). Each $f_t(x)$ gives the position of some point $x \in X$ at time t. These maps give rise to the following definition:

Definition 1. A deformation retraction of a space X onto a subspace A is a family of maps $f_t : X \to X, t \in I$ such that $f_0 = \text{id}$ (the identity map), $f_1(X) = A$, and f_t restricted to A is the identity for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X, (x,t) \mapsto f_t(x)$ is continuous.

Definition 2. Let $f : X \to Y$ be a continuous map between spaces, and consider the quotient space $(X \times I) \sqcup Y / \sim$ where the points $(x, 0)$ are identified with $f(x) \in Y$. This space is called the mapping cylinder M_f.

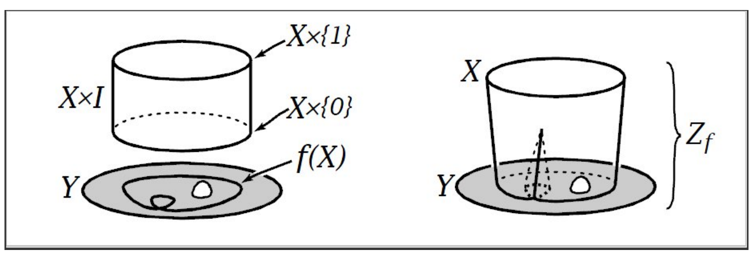

Figure 1: Picture courtesy of Hatcher Algebraic Topology

Let $\pi : (X \times I) \sqcup Y \to M_f$ be the quotient map. Then we can equip M_f with a quotient topology, where open sets V in M_f are precisely those such that $\pi^{-1}(V)$ is open in $(X \times I) \sqcup Y$.

Example 1. Let $X = S^1$, let $i : S^1 \to \mathbb{C}$ be the inclusion map. Then we can visualize the mapping cylinder as a complex plane with a cylinder literally stuck on top.

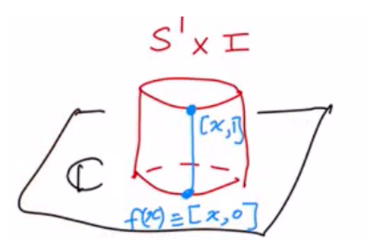

Figure 2: Professor Cooper's drawing

Lemma 1.1. A mapping cylinder M_f deformation retracts to Y.

Proof. Essentially we wish to squash the cylinder flat in a continuous way. We have two kinds of points in M_f. We can take the equivalence class of a point in $X \times I$. Such a point is an ordered pair $(x, s) \in X \times I$. The other kind of point in the mapping cylinder is the equivalence class of a point $y \in Y$.

We wish to define a homotopy from the identity map on the mapping cylinder to the map r. We need to define the time t maps for each of our possible points in M_f. We define $f_t([(x, s)]) = [(x, ts)]$. Thus when $t = 1$, we have the identity map, and when $t = 0$, we have $[(x, 0)] = f(x) \in Y$. On the other hand, $f_t([y]) = [y]$, so that f_t is always the identity on $[y] \in M_f$. We see then that $f_1 = \mathrm{id}_{M_f}$, $f_0 = r$ (the retraction map $r : (X \times I) \sqcup Y \to Y$), and the restriction of f_t to Y is the identity for all t.

All that's left to check is that F is continuous. We need to use the definition of quotient topology. Let $V \subseteq Y$ be open. We need to check that $F^{-1}(V)$ is open in M_f. Well $F^{-1}(V)$ is open in M_f if and only if $\pi^{-1}(F^{-1}(V))$ is open in $(X \times I) \sqcup Y$, where π is the quotient map.FINISH THIS LATER!! $\square$

Definition 3. A homotopy is a family of maps $f_t : X \to Y$, $t \in I$ such that the map $F : X \times I \to Y$, $F(x, t) = f_t(x)$ is continuous.

What does it mean to be continuous in 2 variables? If we change x a little bit and change t a little bit, the image point moves a small amount. Equivalently, a map is continuous iff preimage of open sets is open. So take an open set V in Y, consider its preimage $F^{-1}(V)$ in $X \times I$. We know by the product topology that a set will be open in $X \times I$ if it is a (arbitrary union of finite intersections of sets) of the form $p^{-1}(V_1)$ where V_1 is an open set in X and $p : X \times I \to X$ is the projection, or of the form $q^{-1}(V_2)$ where V_2 is an open set in I and $q : X \times I \to I$ is the projection map.

Definition 4. Two maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy between them. That is, we say $f_0 \cong f_1$ if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.

Lemma 1.2. Homotopy is an equivalence relation.

Proof. Using $F(x, t) = f(x)$ we see that $f \simeq f$, so $\simeq$ is reflexive. To see that $\simeq$ is symmetric, let $f \simeq g$. Then there exists $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Define $G(x, t) = F(x, 1 - t)$. Then $G(x, 0) = F(x, 1) = g(x)$ and $G(x, 1) = F(x, 0) = f(x)$, which shows that $g \simeq f$. Finally, to see that $\simeq$ is transitive, suppose $f \simeq g$ via $F(x, t)$ and $g \simeq h$ via $G(x, t)$. Define $H(x, t)$ by $H(x, t) = F(x, 2t), 0 \leq t \leq \frac{1}{2}$, and $H(x, t) = G(x, 1 - 2t), \frac{1}{2} \leq t \leq 1$. (My instinct was to define $H(x, t) = (1 - t)F(x, t) + tG(x, t)$. HOWEVER, because we are mapping between arbitrary topological spaces, we don't necessarily have a well-defined vector space structure, so it's unclear what addition would even mean.)
Then $H(x, 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 1) = h(x)$, and it is well-defined at $\frac{1}{2}$. We also need to check that the map is continuous at $t = \frac{1}{2}$. This is done via the pasting lemma (see below). Essentially, because we know that $F(x, t)$ is continuous at $t = \frac{1}{2}$ (by definition of homotopy) and the map $(x, t) \mapsto (x, 2t)$ is continuous, then the composition of the maps,

$F(x, 2t)$ is continuous on $X \times [0, \frac{1}{2}]$. Similarly, $G(x, 1-2t)$ is continuous on $X \times [\frac{1}{2}, 1]$. Finally, since $F(x, 2t)$ and $G(x, 1-2t)$ agree on $X \times \{\frac{1}{2}\} = (X \times [0, \frac{1}{2}]) \cap (X \times [\frac{1}{2}, 1])$, and since $X \times [0, \frac{1}{2}], X \times [\frac{1}{2}, 1]$ are both closed sets, the pasting lemma gives us that H is continuous at $X \times \frac{1}{2}$. □

Lemma 1.3. put pasting lemma here

Definition 5. A function is nullhomotopic if homotopic to a constant map.

Example 2. Given $f, g : X \to \mathbb{R}^n$ (NOTE: NEED TO MAP INTO EUCLIDEAN SPACE!!), the straight line homotopy is $F : X \times I \to \mathbb{R}^n$, $F(x,t) = tf(x)+(1-t)g(x)$. Then $F(x, 0) = g(x)$ and $F(x, 1) = f(x)$. So $f \simeq g$. WHAT! Any two maps from a space into Euclidean space are HOMOTOPIC! So really everything is equivalent to a constant map!! But this only works because $\mathbb{R}^n$ is a vector space.

Example 3. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane. Given $n \in \mathbb{Z}$, define $f_n : S^1 \to S^1$ by $f_n(z) = z^n$. So $(\cos\theta, \sin\theta) \mapsto (\cos n\theta, \sin n\theta)$. So the circle gets mapped around itself n times (either forwards of backgrounds depending on if n is pos or neg). If $n = 0$, we get the constant map $(\cos\theta, \sin\theta) \mapsto (1, 0)$. Later we will show that $f_m \simeq f_n \iff m = n$. Later we will also show that any continuous map from the circle to the circle is homotopic to f_n for some n.

Definition 6. A retraction of X onto A is a map $r : X \to X$ such that $r(X) = A$ and $r|_A = \text{id}$.

Remark 1.4. We can think of deformation retraction as a homotopy from the identity map to the retraction map of X onto A. This is because a deformation retraction of X onto A is a family of maps (like a homotopy!) $f_t : X \to X$ such that $f_0 = \text{id}$ and $f_1 = r$, the retraction of X onto A. We define the family in the deformation retraction to be continuous, and thus it really is a homotopy between the two maps f_0 and f_1.

Remark 1.5. If a space deformation retracts onto a point, then that space must be path-connected (think about why– what a deformation retraction onto a point implies). But not every path-connected space contains a deformation retraction onto a point! Consider block letters with a hole in the middle (like the letter A). These are path-connected, but cannot deformation retract onto a point.

Example 4. Let $r : S^1 \to \{1\}$. Then r is a retraction (notice $r(S^1) = \{1\}$ and $r(\{1\}) = \{1\}$) but does NOT come from a deformation retraction. WHY? Later we will see that a deformation retraction is a homotopy equivalence, so if we could deformation retract a circle onto a point, we would have that S^1 is homotopy equivalent to a point... The proof this is not possible requires a topological invariant which we don't have right now.

Example 5. let $r : \mathbb{C} \setminus \{0\} \to S^1$ by $r(xe^{i\theta}) = e^{i\theta}$ for $x > 0$. Then r is a retraction (if $z \in S^1$, then $z = e^{i\theta}$ for some θ, so it gets mapped to itself. We can get a homotopy from the identity map on $\mathbb{C} \setminus \{0\}$ (punctured complex plane) to r via $F(xe^{i\theta}, t) = ((1-t)x + t)e^{i\theta}$. When $t = 0$ we get $F(xe^{i\theta}, 0) = xe^{i\theta}$. When $t = 1$, we get $F(xe^{i\theta}, 1) = e^{i\theta} = r(e^{i\theta}$. When $x = 1$ (i..e, when $xe^{i\theta}$ is on the unit circle), we get that $F(e^{i\theta}, t) = e^{i\theta}$, so indeed $f_t|_{S^1}$ is the identity for all $t \in I$, and F is a deformation retraction. BUT! F is NOT a straight line homotopy... notice that we only have a straight line homotopy type situation in the radial direction.

Example 6. Let X be a circle with a line sticking out of it. Let A be the subspace corresponding to the circle. Then we can deformation retract X into A by pushing the line inside to a point.

Example 7. There does NOT exist a retraction $r : [0,1] \to \{0,1\}$. Why? Such a retraction would have to be continuous. But a continuous map maps connected spaces to connected spaces– while $[0,1]$ is connected, $\{0,1\}$ is disconnected.

Definition 7. If a homotopy whose restriction to a subspace A of X gives the identity map regardless of t, we call it a homotopy relative to A. In this case, we have $F(a,t) = a$ for all $t \in I$.

Definition 8. If $f : X \to Y$ is a cts map and $g : Y \to X$ is such that $g \circ f \simeq \mathrm{id}_X$, we say that g is the homotopy inverse of f. A map $f : X \to Y$ is called a homotopy equivalence if there is a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}$ and $g \circ f \simeq \mathrm{id}$. So essentially a map is a homotopy equivalence if it has some "inverse" under homotopy. If X and Y have a homotopy equivalence between them, they have the same homotopy type.

Definition 9. A space is contractible if it is homotopy equivalent to a point.

Lemma 1.6. A deformation retraction is a homotopy equivalence (that is, if a space X deformation retracts onto a subspace A, then X and A are homotopy equivalent).

Proof. Suppose $A \subseteq X$ and $r : X \to A$ is a deformation retraction. Then there exists a homotopy $F : X \times U \to X$ such that $f_0 = \mathrm{id}_X$, $f_1 = r$, and for all t, $f_t|_A = \mathrm{id}_A$. Let $i : A \to X$ be the inclusion map. Then $r \circ i = \mathrm{id}_A$ and $i \circ r = r = f_1 \simeq f_0 = \mathrm{id}_X$. So $r \circ i \simeq \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_X$, which means i and r are homotopy inverses of each other, which makes r a homotopy equivalence. $\square$

Example 8. A tree (from graph theory) is contractible. Just take each edge and squish it into a point, each time reducing the number of edges, until you're left with just an interval, which (via the constant map) can be deformation retracted to a point, which would make the original tree homotopy equivalent to the point (via the lemma).

Claim: X and Y are homotopy equivalent (have the same homotopy type) iff there exists a third space Z containing both X and Y as deformation retracts.

Proof. Suppose X and Y are homotopy equivalent under $f : X \to Y$. Let $Z = M_f$. We wish to show that M_f deformation retracts to both X and Y. We showed above that M_f deformation retracts to Y. To see that it deformation retracts to X, we can define FINISH THIS LATER $\square$

1.3 Cell Complexes

1.3.1 Quotient Topology

We first need to review some stuff about quotient topologies.

A quotient construction gives us a way to construct topologies from old topologies. For instance, if we take a square and identify opposite edges, we get a torus. Identifying points is the same as giving an equivalence relation on a space.

Thus: given a space X and an equivalence relation $\sim$, the quotient set $X/\sim$ inherits a topology. What is that topology? Let $q : X \to X/\sim$ be a map. The quotient topology is the finest topology (aka largest) on $X/\sim$ for which q is continuous. Explicitly, a set $U \subseteq X/\sim$ is open in the quotient topology iff $q^{-1}(U)$ is open in X.

Claim: This is a topology.

Proof. Since $q^{-1}(\emptyset) = \emptyset$ which is open in X, then $\emptyset$ is open in $X/\sim$. Similarly, $q^{-1}(X/\sim)$ is open in X since q is continuous, so $X/\sim$ is open in $X/\sim$. Finally, since the preimage of a union (resp. intersection) is a union (resp. intersection) of preimages, then unions of open sets are open in $X/\sim$ and finite intersections are open in $X/\sim$. $\square$

1.3.2 CW Complexes

A CW complex is a space built out of smaller spaces iteratively by a process of attaching cells.

Anything homeomorphic to the disk $D^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$ is a k-cell. D^1 is an interval. We need to attach the cell to the existing space such that the boundary of the k-cell is GLUED to the space.

Definition 10. Attaching a k-cell to a space X. Let D^k be a k-cell. We write $X \sqcup D^k$. We need a continuous map φ from the boundary of the k-cell to X which we can use to identify the boundary of the k-cell with certain points of X. Explicitly, we define $\sim$ by $z \sim \varphi(z)$ for $z \in \partial D^k$. Then attaching a k-cell means taking the space $(X \sqcup D^k)/\sim$.

The map φ is really important!! It could completely change what the resulting space looks like. For example, if we let X be two disjoint points and D^1 be the interval $[-1, 1]$, then we can consider two different ways of attaching the boundary of D^1 to X. The first way is by taking φ to be the identity map, and then we have just an interval attached between the points. The other way is by taking φ to be a constant map, sending both boundary points of D^1 to one of the two points of X, and thus the resulting space would be one point with a circle attached to it (the circle would be the interval D^1 with both endpoints attached to the point) next to a disjoint point of X. Then clearly these two spaces are quite different because one of them