Contents

Trying to understand wtf is complex analysis... Hoping this will serve both as a final review and as a cheat sheet to refer back to for complex analysis part II, electric boogaloo.

1 Chapter 1: The Complex Plane

1.1 Basic Topology of the Complex Numbers, Sequences and Series

If $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$, then $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$. Similarly, assuming $z_2 \neq 0$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$.

We say that a sequence $\{z_n\} \in \mathbb{C}$ converges to $z \in \mathbb{C}$ if $|z_n - z| \to 0$ (since $\{|z_n - z|\}$ is a sequence in $\mathbb{R}$). Since $|\operatorname{Re} z|, |\operatorname{Im} z| \leq |z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$, then $z_n \to z$ if and only if $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z$.

Because we're working in $\mathbb{C} \cong \mathbb{R}^2$, then $\{z_n\}$ is Cauchy iff convergent.

For testing when a series converges: sometimes trying to see if it absolutely converges is easier (because imaginary terms might just become real, and we can use what we know about the convergence of real series)– we know absolute convergence implies convergence.

Definition 1. *An open connected set will be called a region.*

1.2 Continuous Functions

Remark 1.1. *The sum, product, and quotient (with nonzero denominator) of continuous functions are continuous.*

Definition 2. *A sequence of functions $\{f_n\}$ converges to f uniformly in D if for each $\varepsilon > 0$, there is an $N > 0$ such that $n > N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in D$.*

Theorem 1.2. *The uniform limit of continuous functions is continuous.*

The proof is just to look at the real and imaginary parts of the sequence and notice that this holds for the real case.

Theorem 1.3. *Suppose f_k is continuous in D. If $|f_k(z)| \leq M_k$ throughout D and if $\sum_{k=1}^{\infty} M_k$ converges,then $\sum_{k=1}^{\infty} f_k(z)$ converges to a function f which is continuous.*

Theorem 1.4. *Partial Derivatives are Zero means Function is Constant. Suppose $u(x, y)$ has partial derivatives u_x and u_y that vanish at every point of a region D. Then u is constant in D.*

The proof uses polygonal paths and the mean value theorem in one variable to show the values between successive endpoints must remain the same.

1.3 Stereographic Projection: The Points at Infinity

Let $\Sigma = \{(x, y, z) : x^2 + y^2 + (z - \frac{1}{2})^2 = \frac{1}{4}^2\}$. Then if we consider the point $(0, 0, 1)$ at the top of the sphere, then any ray between $(0, 0, 1)$ and a point on the sphere will intersect $\mathbb{C}$ at precisely one point, and thus we have an identification of $\Sigma \setminus (0, 0, 1) \cong \mathbb{C}$. Let $a + ib$ be the complex number that results from intersecting the complex plane with the line from $(0, 0, 1)$ through $(x, y, z) \in \Sigma$. Then we have

$$a = \frac{x}{1-z}, \quad b = \frac{y}{1-z}$$

or, solving in the other direction,

$$x = \frac{a}{a^2+b^2+1}, \quad y = \frac{b}{a^2+b^2+1}, \quad z = \frac{a^2+b^2}{a^2+b^2+1}$$

Also, if we let $\{(x_k, y_k, z_k)\}$ be a sequence of points which converge to $(0, 0, 1)$, we see that we get a corresponding sequence (a_k, b_k) such that $|(a_k, b_k)| \to \infty$. So we have an identification of $\Sigma \cong \mathbb{C} \cup \infty$. We define neighborhoods of ∞ in $\mathbb{C}$ to correspond to spherical neighborhoods of $(0, 0, 1)$ in Σ.

2 Chapter 2: Functions of the Complex Variable z

For the most part we can think of functions of z as multivariable functions of x and y. But there are times when this thinking is a bit flawed. For instance, we could have functions of x and y separately which are not functions of $z = x + iy$. For instance, the function $f(x, y) = x^2 + y^2 - 2ixy$ is expressible in terms of x and y (clearly), but *not* in terms of z. Notice that we would need to have a z^2 term, but $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$, which is not equal to our desired expression. We therefore need to think about functions of z slightly differently. This motivates our notion of an analytic function. In essence, these will be the functions that are explicit expressions of z.

Remark 2.1. *Notice that because we want an analytic function to be a direct expression of z, this will motivate our later equivalent definition that $\frac{\partial f}{\partial \bar{z}} = 0$ in order for f to be analytic.*

2.1 Let's Start with Polynomials

Since analytic functions are sort of generalizations of polynomials, it helps to first examine what happens in the case of polynomials.

Definition 3. *A complex polynomial $P(z)$ is defined as a finite sum $\sum_{k=0}^{m} \alpha_k z^k$ where $\alpha_i \in \mathbb{C}$.*

Example 1. *x^2+y^2-2ixy is a polynomial in (x, y), but it is not an analytic/ complex polynomial in z.*

Suppose that the example above was a polynomial in z. Then there exists some $m \in \mathbb{N}$ and $\alpha_i \in \mathbb{C}$ such that $x^2 + y^2 - 2ixy = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$. Therefore

$$x^2 + y^2 - 2ixy = \alpha_0 + \alpha_1(x + iy) + \alpha_2(x + iy)^2 + \cdots + \alpha_m(x + iy)^m$$

If we set $y = 0$, we get $x^2 = \alpha_0 + \alpha_1 x + \cdots + \alpha_m x^m$ which implies that $\alpha_i = 0$ for all $i \neq 2$, and $\alpha_2 = 1$. Thus $x^2 + y^2 - 2ixy = (x + iy)^2$. But $(x+iy)^2 = x^2 - y^2 + 2ixy$, which does not equal $x^2 + y^2 - 2ixy$. This gives a contradiction, so $x^2 + y^2 - 2ixy$ is *not* a complex polynomial.

Proposition 1. *A polynomial $P(x, y)$ is analytic if and only if $P_y = iP_x$.*

Suppose $P(x, y)$ is analytic. Then $P(x, y) = \alpha_0 + \alpha_1(x+iy) + \cdots + \alpha_m(x+iy)^m$. We therefore have that $P_x = \alpha_1 + 2\alpha_2(x + iy) + \cdots + m\alpha_m(x + iy)^{m-1}$ whereas $P_y = i\alpha_1 + 2i\alpha_2(x + iy) + \cdots + im\alpha_m(x + iy)^{m-1}$. Thus $P_y = iP_x$.

Suppose $P_y = iP_x$. We wish to show that $P(x, y)$ is analytic. Let the n^{th} term of $P(x, y)$ be given by $Q(x, y) = \alpha_0 x^n + \alpha_1 x^{n-1} y + \cdots + \alpha_n y^n$. Then since $P_y = iP_x$, in particular this must hold for the terms of the n^{th} degree. We therefore have that $Q_y = iQ_x$, such that

$$\alpha_1 x^{n-1} + 2\alpha_2 x^{n-2} y + \cdots + n a_n y^{n-1} = i(n\alpha_0 x^{n-1} + (n-1)\alpha_1 x^{n-2} y + \cdots + \alpha_{n-1} y^{n-1})$$

this implies that $\alpha_1 = in\alpha_0 = i\binom{n}{1}\alpha_0$, and in general that $\alpha_k = i^k\binom{n}{k}\alpha_0$. In sum, this gives us that $Q(x, y) = \sum_{k=0}^{n} \alpha_k x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k = (x+iy)^n$. Thus $Q(x, y)$ is analytic, and since this holds for the n^{th} power term for any n, we conclude that $P(x, y)$ is analytic.

Remark 2.2. *Notice that if we wrote $P(x, y) = u(x, y) + iv(x, y)$, then $P_x = u_x + iv_x$, $P_y = u_y + iv_y$, and $P_y = iP_x$ implies that $u_y = -v_x$ and $v_y = u_x$. These are called the Cauchy-Riemann Equations.*

Definition 4. *A complex valued function $f(z)$ is said to be complex differentiable at z if*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, irregardless of how h approaches 0 (since h is complex, it has infinitely many choices for how to approach 0).

Definition 5. *A complex valued function $f : U \to \mathbb{C}$ is said to be holomorphic if it is complex differentiable at every point in its domain U.*

Basic properties hold for complex differentiable functions which we would expect to hold from calculus. For instance, if both f and g are complex differentiable at z, then $(f + g)'(z) = f'(z) + g'(z)$, while $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$. Similarly, if $g(z) \neq 0$, we get that $(\frac{f}{g})'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$. In addition, if f is a complex polynomial, that is, $f(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$, then, as expected for polynomials, we have $f'(z) = \alpha_1 + \cdots + m\alpha_m z^{m-1}$.

2.2 Generalizing Polynomials: Power Series

We can now consider a larger class of functions: those given by convergent power series. Let $f(z) = \sum c_k z^k$.

Theorem 2.3 (Cauchy-Hadamard)**.** *Suppose $\lim \sqrt[k]{|c_k|} = L$.*

(1) If $L = 0$, $\sum c_k z^k$ converges for all z.

(2) If $L = \infty$, $\sum c_k z^k$ converges for $z = 0$ only.

(iii) If $0 < L < \infty$, set $R = \frac{1}{L}$. Then $c_k z^k$ converges for $|z| < R$ and diverges for $|z| > R$.

Remark 2.4. *A power series is continuous throughout its radius of convergence.*

Notice that if $f(z)$ converges for $|z| < R$, then it also converges for $|z| < R - \delta$, and we get that

$$\sum_{k=0}^{\infty} |c_k z^k| \leq \sum_{k=0}^{\infty} |c_k|(R-\delta)^k$$

Remark 2.5. *The sum of two convergent power series converges to the sum of the individual limits inside of their shared radius of convergence. The Cauchy product of convergent power series also converges inside of a particular radius.*

Just like in the real case, power series in z are differentiable, and their derivative is just the term by term derivative. The resultant power series converges in the same radius of convergence. We summarize this below:

Theorem 2.6. *Suppose $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < R$. Then $f'(z)$ exists and equals $\sum_{n=0}^{\infty} n c_n z^{n-1}$ throughout $|z| < R$.*

See page 26 of Bak and Neumann for the proof. The essential idea is that we want to show that $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_{n=1}^{\infty} n c_n z^{n-1}$ by showing that $\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n c_n z^{n-1}$ as a power series will go to zero as $|h| \to 0$.

Example 2. *To illustrate, consider $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. We expect $\frac{d}{dz} e^z = e^z$, and indeed, we have that $\frac{d}{dz} e^z = \sum_{k=0}^{\infty} k \frac{z^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{z^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ as desired.*

Theorem 2.6 actually shows that $f(z)$ is infinitely differentiable, and that the radius of convergence for all of its derivatives is the same as the original radius of convergence.

In addition, by a simple computation (taking derivatives of the power series and comparing coefficients), we see that $c_n = \frac{f^{(n)}(0)}{n!}$ for all n. This implies that if $f(z)$ is zero throughout a neighborhood of the origin, then all of its derivatives on a neighborhood of the origin must be zero, which implies that the function is constant, and in fact that it is constantly zero on that neighborhood. This leads us to a big theorem that forms a basis for a lot of cool principles in complex analysis: the *uniqueness theorem.*

Theorem 2.7. *Uniqueness Theorem for Power Series Suppose $\sum_{n=0}^{\infty} c_n z^n$ is zero at all points of a nonzero sequence $\{z_k\}$ which converges to zero. Then the power series is identically zero.*

We use limits to prove this, as well as factoring zeroes. We also use an inductive argument. This leads to the following corollary: If a power series equals zero at all the points of a set with an accumulation point at the origin, the power series is identically zero. This follows pretty immediately from the uniqueness theorem: if we have an accumulation point at zero, then we have a sequence of points approaching the origin, such that our power series is 0 at all of the points in our sequence. This also gives us an adapted version of the Uniqueness Theorem for Power Series, which is much more telling of where it gets its name:

Theorem 2.8. *If $\sum a_n z^n$ and $\sum b_n z^n$ converge and agree on a set of points with an accumulation point at the origin, then $a_n = b_n$ for all n.*

We see that we must have $a_n - b_n = 0$ for all n, and the claim follows.

2.3 Interesting/Important Exercises from BN, Worthy of Typing

3 Chapter 3: Analytic Functions

The last chapter focused on polynomials and power series. We now wish to extend these results to more general functions. For instance, we know that complex polynomials satisfy $P_y = iP_x$, and indeed this holds for any complex differentiable (holomorphic) function, that is, if f is holomorphic, then $f_y = if_x$. The proof this follows by showing that the limit of the difference quotient must be the same for all $h \to 0$, whether we approach 0 via the x-axis or via the y-axis.

Remark 3.1. *We claimed above that if f is holomorphic, then f satisfies the Cauchy-Riemann equations. It is very important to note that the converse is not true.*

But we do have a partial converse to the statement above, which is that if f_y and f_x both exist, *and* f_y and f_x are both continuous at a neighborhood of z, *then* f is complex differentiable at z. The proof uses mean value theorem and is quite gross: **maybe review this?**

Definition 6. *A complex-valued function f is analytic at z if f is differentiable in a neighborhood of z. Similarly, f is analytic on a set S if f is differentiable at all points of some open set containing S.*

Definition 7. *A complex-valued function which is differentiable everywhere is called an entire function.*

General properties of differentiable complex functions hold as expected, such as sums of differentiable functions are differentiable, and so are compositions of differentiable functions. Notice that in general we do *not* have that inverses of differentiable functions are differentiable. *But* there are some conditions we may impose under which we can state that a complex-valued function has a differentiable inverse. The proof and conditions are basically just the inverse function theorem:

Proposition 2. *Inverse Function Theorem for Holomorphic Functions. Suppose g is a local inverse of f at z_0 and g is continuous at z_0. Then if f is differentiable at $g(z_0)$, and if $f'(g(z_0)) \neq 0$, then g is differentiable at z_0 and $g'(z_0) = \frac{1}{f'(g(z_0))}$.*

The proof is actually really chill. It just uses the difference quotient $\frac{g(z)-g(z_0)}{z-z_0}$. Since g is the local inverse of f at z_0, then we know that about a neighborhood of z_0, $f(g(z)) - f(g(z_0)) = z - z_0$. Therefore we can express

$$\frac{g(z)-g(z_0)}{z-z_0} = \frac{g(z)-g(z_0)}{f(g(z))-f(g(z_0))} = \frac{1}{\frac{f(g(z))-f(g(z_0))}{g(z)-g(z_0)}}$$

Then as $z \to z_0$, we know that $g(z) \to g(z_0)$ (this follows from continuity of g at z_0), so the difference quotient as $z \to z_0$ gives $g'(z_0) = \frac{1}{f'(g(z_0))}$.

We now introduce some propositions which follow from the fact that Cauchy-Riemann equations must be satisfied whenever f is analytic.

Proposition 3. *If $f = u + iv$ is analytic in a region D and $u(x, y)$ is constant, then f is constant.*

If $u(x, y)$ is constant, then $u_x = u_y = 0$. Then by Cauchy-Riemann equations, we also get that $v_x = v_y = 0$, so both u and v are constant, so f is constant.

Proposition 4. *If f is analytic in a region D and if $|f|$ is constant there, then f is constant.*

Recall that $|f| = |u + iv| = \sqrt{u^2 + v^2}$. If $|f| = 0$, then $u = v = 0$, and f is constant. If $|f| \neq 0$, then we can use Cauchy-Riemann equations to deduce that $u_x = u_y = v_x = v_y = 0$, and again f is constant.

3.1 Extending Functions We Already Know: Exponential, Trig Functions

We now wish to expand the definitions of the exponential, and of $\sin$ and $\cos$ past the real axis to the entire complex plane.

Definition 8. *Complex Exponential Let $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$.*

Notice that like the real exponential (actually, because of the real exponential) we have that $e^z \neq 0$ for any $z \in \mathbb{C}$, since in order for this to happen we must have that $e^x = 0$ (which never happens) or $\cos(y) + i\sin(y) = 0$, which would require $\cos(y)$ and $\sin(y)$ to be simulateneously zero, which never happens. Unlike the real exponential, however, the complex exponential is surjective onto the puncutred complex plane, that is, any complex value other than 0 can be achieved. This is because e^{iy} can achieve any number on the unit circle, and then e^x can scale by as large or as small of a factor as we would like (other than be zero). Also, unlike the injectivity of the real exponential map, we have that $e^z = a$ has infinitely many solutions, since if $z = \theta$ is a solution, then so is $z = \theta + 2\pi k$ for all $k \in \mathbb{Z}$.

Definition 9. *We define $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$.*

Then we can verify that the usual properties still hold, even in the complex plane. That is, $\sin'(z) = \cos(z)$, $\sin^2(z) + \cos^2(z) = 1$, and $\sin(2z) = 2\sin(z)\cos(z)$. Also, notice that unlike the real $\sin$ and $\cos$, the complex $\sin$ and $\cos$ are *not* bounded in modulus. This is because $|e^{iz}|$ is not bounded in modulus.

3.2 Interesting/Important Exercises from BN, Worthy of Typing

4 Chapter 4: Line Integrals and Entire Functions

We showed in chapter 2 that an everywhere convergent power series is everywhere complex differentiable, and thus that everywhere convergent power series are actually entire functions. We now wish to prove the converse of that result: entire functions (that is, functions which are complex differentiable everywhere) can always be represented as an everywhere convergent power series (that is, power series with an infinite radius of convergence). As a corollary we deduce that entire functions are infinitely differentiable everywhere (since everywhere convergent power series are infinitely differentiable everywhere).

4.1 Line Integrals

Most of the integrals we will be evaluating are called line integrals, and they can be found via

$$\int_C f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$$

where C is a curve parametrized by $z(t), a \le t \le b$. Notice that just like in the real case, in order to define this integral we require f to be continuous. If f is analytic, then f is continuous, and the line integral is well-defined.

Although C was parametrized by $z(t)$ over $[a, b]$, we could have traced out the same curve using a different parametrization over a different subset of $\mathbb{R}$. Since we want the line integral of $f(z)$ over C to be the same no matter the parametrization (so long as we have "equivalent" parametrizations), we first need to define what an equivalent parametrization would be, and then prove that the integrals are the same over the two parametrizations.

Definition 10. *Two curves $C_1 : z(t), \quad a \le t \le b, C_2 : w(t), \quad c \le t \le d$ are said to be equivalent if there exists an injective, continuously differentiable mapping $\lambda(t) : [c, d] \to [a, b]$ such that $\lambda(c) = a$, $\lambda(d) = b$, $\lambda'(t) > 0$, and $w(t) = z(\lambda(t))$.*

Proof of the equivalence of the integrals over equivalent parametrizations involves change of variable formula for the real case.

Since our line integral behaves mostly as expected, then in particular it satisfies the triangle inequality for integrals, that is, if $G(t)$ is continuous and complex-valued, then $\left|\int_a^b G(t)\,dt\right| \le \int_a^b |G(t)|\,dt$. We also have something called the ML-formula, which is actually super useful (i.e., you should learn it!!!):

Proposition 5. *ML formula. Suppose C is a smooth curve of length L, f is continuous on C, and $|f| \le M$ throughout C. Then $\left|\int_C f(z)\,dz\right| \le ML$.*

We also have statements about the integrals of a sequence of functions which converge to f uniformly. Let $\{f_n\}$ be a sequence of continuous functions which converge uniformly to some function f over a smooth curve C. Then the integral of f over C is the limit of the integrals of f_n over C. Recall that uniform convergence of $\{f_n\} \to f$ means that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that for all $z \in \mathbb{C}$, for all $n \ge N$, we have $|f_n(z) - f(z)| < \varepsilon$. We state the proposition:

Proposition 6. *Suppose $\{f_n\}$ is a sequence of continuous functions and $f_n \to f$ uniformly on the smooth curve C. Then*

$$\lim_{n\to\infty} \int_C f_n(z)\,dz = \int_C f(z)$$

The proof is simple, and uses the uniform convergence + the ML-formula.

We also have an analogue to the Fundamental Theorem of Calculus, which states that if f is the derivative of an analytic function F on C, then $\int_C f(z)\,dz = F(z(b)) - F(z(a))$. This is pretty much as expected, and should be used whenever we know the "antiderivative."

We now consider one of the big principles of this class: The Closed Curve Theorem for Entire Functions.

4.2 Closed Curve Theorem for Entire Functions

We begin with stating something called the Integral Theorem, which will prelude Cauchy's Integral Theorem (which is a better case of the closed curve theorem for entire functions).

Theorem 4.1. *Entire functions have entire antiderivatives. If f is an entire function, then f is everywhere the derivative of an analytic function, that is, there exists an F analytic such that $F'(z) = f(z)$ for all $z \in \mathbb{C}$.*

The proof involves defining $F(z) = \int_0^z f(\zeta)d\zeta$, similarly to how we define antiderivatives in the real case. Since $z \in \mathbb{C}$, then the integral from $\int_0^z$ tells us to first integrate $\int_0^{\Re z}$ (that is, along the x-axis) and then to integral along $\int_{\Re z}^z$ (which will be along a vertical line). We then prove that for this $F(z)$, we have that $F'(z) = f(z)$, so that it is truly the antiderivative.

Theorem 4.2. *Closed Curve Theorem for Entire Functions. Let f be an entire function, and C be any smooth closed curve. Then $\int_C f(z)\,dz = 0$.*

By 4.2, we know that if f is entire, then it has an antiderivative, and the claim follows very quickly. Notice however that we don't need that f is entire. So long as our function has an antiderivative which is analytic on some neighborhood, and our smooth closed curve lies within that neighborhood of analyticity, then we are set, the integral will still be zero.

4.3 Interesting/Important Exercises from BN, Worthy of Typing

5 Chapter 5: Properties of Entire Functions

5.1 Integral of the Difference Quotient (and its Consequences)

There were a few times in this class when we wanted to know properties of the integral of the "derivative" of an entire function f at a specific point, that is, if we defined a function $g(z)$ to give the difference quotient $\frac{f(z)-f(a)}{z-a}$ when $z \neq a$ and $f'(a)$ when $z = a$. Notice that this definition makes g continuous. We then determined that g satisfies the closed curve theorem and the integral theorem (although a priori we don't know that g is entire), that is, it has an analytic antiderivative, and its integral over a smooth, simple closed curve is zero. The way we showed this was by proving that the rectangle theorem holds for $g(z)$ (that is, integral of $g(z)$ over any rectangle is zero), and then because g is continuous, the proofs of the integral theorem (analytic antiderivative) and the closed curve theorem follow **(did they really just use continuity?)**

We now state one of the main results of the course:

Theorem 5.1. *Cauchy Integral Formula. Suppose that f is entire, $a \in \mathbb{C}$, and the curve C is given by $Re^{i\theta}$, $0 \leq \theta \leq 2\pi$, with $R > |a|$. Then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$.*

Intuitively, this means we can recover information about the value of an analytic function inside of an enclosed region given that we know its value on the boundary. The proof is actually really not too bad. It uses the fact that we showed above $\frac{f(z)-f(a)}{z-a}$ is analytic and has integral equal to 0 over a smooth closed curve C not containing a, and then we calculate that $\int_C \frac{1}{z-a}\, dz$ has integral equal to $2\pi i$. This is done by expanding $\frac{1}{z-a} = \frac{1}{z(1-\frac{a}{z})}$ as a power series.

Actually, while the textbook takes $C = Re^{i\theta}$ to be a circle in the complex plane, it doesn't need to be, so long as it is piecewise smooth and closed. We also don't need f to be entire, so long as it is analytic on $U \subseteq \mathbb{C}$ open. The proof of this (which we did in class, October 19th) required us to draw a circle of radius ε around our point a that is properly contained within the boundary C, and then we used the fact that the region enclosed by C is compact to utilize Stokes' Theorem.

Remark 5.2. *We can also extend this integral formula to give us the derivatives of f at a point $a \in \mathbb{C}$.*

Theorem 5.3. *Extended Cauchy Integral Formula. Let $U \subseteq \mathbb{C}$ be open. Let $f : U \to \mathbb{C}$ be C' and complex differentiable. Then $f \in C^\infty$ and $f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$.*

Theorem 5.4. *Taylor Expansion of an Entire Functions. If f is entire, it has a power series representation as $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$.*

The proof comes from first using the Cauchy integral formula to show that actually $f(z)$ *can* be written as a power series, with coefficients that (a priori) depend on a, some complex number on which we chose a radius $R > |a|$ for the curve over which we integrate f (in the Cauchy integral formula bit). But actually since we know that if $f(z) = \sum_{k=0}^{\infty} c_k z^k$, and f is entire, then it is complex differentiable everywhere, and we get that the coefficients must be exactly what we expect (i.e., the ones that show up in the Taylor expansion). Notice also that we never prove that $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$ is a convergent power series, but it is implicit in the fact that it equals $f(z)$, and that we showed $f(z)$ is equal to a convergent power series. As a corollary we have that entire functions are infinitely differentiable everywhere, since they are equal to everywhere convergent power series, and power series are infinitely differentiable.

Again, we didn't need entire here, so long as $f : U \to \mathbb{C}$ is complex differentiable, and $U \subseteq \mathbb{C}$ is open. Then we get that $f(z)$ is equal to a convergent power series in a disk of radius $r < d(a, dU)$, where $d(a, dU)$ is the distance between a and the nearest boundary point of U. The proof of both of these versions of the Taylor expansion utilized the fact that we had uniform convergence of $\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta-a)^{n+1}}$ throughout $D(a, r)$, so we exchange the infinite series and the integral in the proof (where the integral comes from Cauchy's integral formula).

Remark 5.5. *Notice that in the power series definition, we can actually replace $f^{(k)}(0)$ with $f^{(k)}(a)$ and z^k with $(z-a)^k$, for any $a \in \mathbb{C}$ (if f is entire).*

We now remark on a statement made in the beginning of this section regarding the "entireness" of g given that $g(z) = \frac{f(z)-f(a)}{z-a}$ when $z \neq a$ and $g(z) = f'(a)$ when $z = a$. Notice that if f is entire, and g is defined as above, then when $z \neq a$, we have that $g(z) = g(a) + g'(a)(z - a) + \cdots = f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$. Since g is equal to an everywhere convergent power series, then g is entire (notice that at $z = a$, the above still holds).

5.2 Louisville Theorem

We now state a result that is perhaps the first surprising result about entire complex functions.

Theorem 5.6. *Louisville's Theorem. A bounded entire function is constant.*

The proof involves multiple major results: we take $a, b \in \mathbb{C}$ to be two complex numbers and consider a circle of radius R about the origin, enclosing both a and b. We then use the Cauchy integral formula for entire functions, and the fact that f is bounded, and the ML-formula to get a bound on $|f(b) - f(a)|$ which has an R in the denominator. Since we may choose R to be as large as possible (f is entire), then we see that $|f(b) - f(a)| \to 0$ as $R \to \infty$. Therefore f is constant.

Theorem 5.7. *Extended Louisville Theorem If f is entire and if, for some integer $k \geq 0$, there exist positive constants A and B such that $|f(z)| \leq A + B|z|^k$, then f is a polynomial of degree at most k.*

The proof follows by induction on k and by considering $g(z)$ to be the entire function of the difference quotient from before, i.e., by considering $g(z) = \frac{f(z)-f(0)}{z}$ when $z \neq 0$, and $g(z) = f'(0)$ when $z = 0$.

5.3 Fundamental Theorem of Algebra

The proof that $\mathbb{C}$ is algebraically closed (that is, every nonconstant polynomial of degree n with coefficients in $\mathbb{C}$ has n roots in $\mathbb{C}$) is really simple if we consider complex functions. The way we prove this is by showing that every nonconstant polynomial in $\mathbb{C}$ has a zero in $\mathbb{C}$, and then if we factor out that zero, we get another polynomial in $\mathbb{C}$ which ought to have a zero in $\mathbb{C}$, and so on.

Theorem 5.8. *Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.*

Suppose $P(z)$ is a nonconstant polynomial. If $P(z)$ does not have a zero in $\mathbb{C}$, then $f(z) = \frac{1}{P(z)}$ is an entire function. Notice that as $z \to \infty$, we have that $\frac{1}{P(z)} \to 0$, which means that $f(z)$ is an entire, bounded function. Hence $f(z)$ must be constant (Louisville's Theorem), contradicting the fact that $P(z)$ is nonconstant.

5.4 Gauss-Lucas Theorem

Theorem 5.9. *The zeroes of the derivative of any polynomial lie within the convex hull of the zeroes of the polynomial.*

5.5 Interesting/Important Exercises from BN, Worthy of Typing

6 Chapter 6: Properties of Analytic Functions

In the last chapter we stated several theorems that hold for entire functions. Turns out analyticity is all we needed (for the statements to hold in open disks within the domain of analyticity). I'll just state some of the theorems that still hold for analytic functions (not necessarily entire):

(1) Existence of Analytic Antiderivatives: If f is analytic on $D(\alpha, r)$ and $a \in D(\alpha, r)$, then there exist functions F and G analytic in $D(\alpha, r)$ such that $F'(z) = f(z)$, and $G'(z) = \frac{f(z)-f(a)}{z-a}$. That is, analytic functions on a disk have analytic antiderivatives.

(2) Closed Curve Theorem for Analytic Functions: If $f(z)$ and $\frac{f(z)-f(a)}{z-a}$ are as above (analytic on $D(\alpha, r)$, where $a \notin D(\alpha, r)$) and C is a smooth, closed curve in $D(\alpha, r)$, then $\int_C f(z)\, dz = \int_C \frac{f(z)-f(a)}{z-a}\, dz = 0$.

(3) Cauchy Integral Formula: Suppose f is analytic in $D(\alpha, r)$, $0 < \rho < r$, and $|a - \alpha| < \rho$. Then $f(a) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z-a}$, where $C_\rho = \alpha + \rho e^{i\theta}$.

(4) Power Series Representation for Functions Analytic in a Disc: If f is analytic in $D(\alpha, r)$ there exist constants c_k such that $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D(\alpha, r)$. So if f is analytic on some disk, there exists a power series representation of f which is convergent on that disk. In fact, we have $c_k = \frac{f^{(k)}(\alpha)}{k!} = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz$.

(5) Analyticity of the Difference Quotient: If f is analytic at α, so is the function $g(z) = \frac{f(z)-f(a)}{z-a}$ when $z \neq a$, $g(z) = f'(a)$ when $z = a$.

We now extend some of the results from the power series section to analytic functions (since we just showed that analytic functions are locally power series).

6.1 Uniqueness Theorem, Mean Value Theorem, Maximum/Minimum Modulus Theorem

Theorem 6.1. *Uniqueness Theorem for Analytic Functions. Suppose that f is analytic in a region D and that $f(z_n) = 0$ where $\{z_n\}$ is a sequence of distinct points and $z_n \to z_0 \in D$.Then $f \equiv 0$ in D.*

For the proof we actually assume that D is connected (perhaps we assume that D is a disk? So it is connected). Let z_0 be as in the hypothesis. Then f can be expressed as a power series about the point z_0 for some radius of convergence, and by the uniqueness theorem of power series from chapter 2, we have that the function $f(z)$ must be identically zero on that disk. Then we partition D into two sets, one consisting of the accumulation points of zero and one consisting of the complement points. We show that both sets are open, and since D is connected, then the second set must be empty, such that every point in D is an accumulation point of zeroes, and we have that $f(z) \equiv 0$ identically on D by the continuity of $f(z)$.

Remark 6.2. *Consequence of Uniqueness Theorem: If two functions f and g, analytic in a region D, agree at a set of points with an accumulation point in D, then $f \equiv g$ through D.*

Theorem 6.3. *If f is entire and if $f(z) \to \infty$ as $z \to \infty$, then f is a polynomial.*

The proof is pretty cool. We show that $f(z) \to \infty$ as $z \to \infty$ implies $f(z)$ has only finite many zeroes, (else the set of zeroes of f would have an accumulation point in a disk, and f would be identically zero, contradicting $f(z) \to \infty$). Then dividing by this set of zeroes gives an entire function, and we utilize some other results we've proven (such as being bounded by $A + |z|^n$ implies being a polynomial of degree at most n) to get that $f(z)$ is actually a constant times the product of $(z - a_i)$, where a_i is a zero of f.

Theorem 6.4. *Mean Value Theorem. If f is analytic in D and $\alpha \in D$, then $f(\alpha)$ is equal to the mean value of f taken around the boundary of any disc centered at α and contained in D. That is,*

$$f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha + re^{i\theta}) d\theta$$

when $D(\alpha, r) \subset D$.

The proof is just a reformulation of the Cauchy Integral Formula.

Theorem 6.5. *Maximum Modulus Theorem. A non-constant analytic function in a region D does not have any interior maximum points: For each $z \in D$ and $\delta > 0$, there exists some $\omega \in D(z, \delta) \cap D$, such that $|f(\omega)| > |f(z)|$.*

From the mean value theorem, we know that for some $r > 0$ such that $D(z, r) \subseteq D$:

$$|f(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta \right| \leq \max_\theta |f(z + re^{i\theta})|$$

by the ML-formula. Letting $|f(\omega)| = \max_\theta |f(z + re^{i\theta})|$ gives $|f(z)| \leq |f(\omega)|$. To see that we get strict inequality, notice that if we have equality, then $|f(z)| = \max_\theta |f(z + re^{i\theta})|$ implies that $|f(z)|$ is constant on $D(z, r)$. But this implies that f is constant on $D(z, r)$, and by the uniqueness theorem, we get that f must be constant on all of D.

We have a very similar analogue to the maximum modulus theorem by consider a function $\frac{1}{f}$, where f is analytic on a region D.

Theorem 6.6. *Minimum Modulus Theorem. If f is a non-constant analytic function in a region D, then no point $z \in D$ can be a relative minimum of f unless $f(z) = 0$.*

If $f(z) \neq 0$, then we take $g = \frac{1}{f}$. If z were a local minimum, then $\frac{1}{f(z)}$ would be a local maximum value for $\frac{1}{f}$, and thus $\frac{1}{f}$ would have to be constant. But this implies f is constant.

7 Chapter 7: Further (Advanced) Properties of Analytic Functions

7.1 Open Mapping Theorem

We notice that as a nice consequence of the uniqueness theorem, if f is analytic and constant on any region in its domain, then f must be constant in its entire domain on analyticity. Similarly, if f is analytic and $|f|$ is constant on any region in its domain, then f must be constant in its entire domain on analyticity. Hence if f is analytic and nonconstant, we cannot have that f maps an open set into a point (a point is a region, hence will imply f is constant) nor into a circular arc (since $|f|$ is constant there). This motivates the following theorem:

Theorem 7.1. *Open Mapping Theorem. The image of an open set under a nonconstant analytic mapping is an open set.*

The proof in the book lowkey sucks, but Putinar had a good proof in his video on November 8th. We assume that $f : U \to \mathbb{C}$ is non-constant and analytic, and that $a \in U$. We then consider two cases: $f'(a) = 0$ and $f'(a) \neq 0$. If $f'(a) \neq 0$, then the inverse function theorem guarantees us that there exists V open containing a such that $W = f(V)$ is open, and $f|_V : V \to W$ is bijective with an analytic inverse. On the other hand, if $f'(a) = 0$, then we consider the power series of f (since f is analytic). Then we know there exists some $r > 0$ such that $f(z) = (z-a)^m h(z)$, with $h(z)$ analytic in $|z-a| < r$ and such that $h(a) \neq 0$. We claim the m^{th} root of $h(z)$ exists, such that we write $f(z) = ((z-a)h(z)^{\frac{1}{m}})^m$. Letting $f_1(z) = (z-a)h(z)^{\frac{1}{m}}$, we see that $h_1(z)$ is analytic, $f_1(a) = 0$, and $f_1'(a) \neq 0$ (since the power series of $f_1(z)$ starts with $c_0^{\frac{1}{m}}(z-a)$, which means that $f'(a) = c_0^{\frac{1}{m}}$ which is nonzero). Then we get that $f_1(z)$ is an open map via the inverse function theorem, and therefore $f_1(z)^m$ is also an open mapping (we had as an example that z^d is an open mapping). Therefore f is an open mapping.

7.2 Schwarz Lemma

Theorem 7.2. *Schwarz Lemma. Suppose that f is analytic in the unit disc, that $|f| \leq 1$ there and that $f(0) = 0$. Then*

i. $|f(z)| \leq |z|$

ii. $|f'(0)| \leq 1$.

with equality in either of the above if and only if $f(z) = e^{i\theta}z$.

This follows by applying the Maximum-Modulus Theorem to the function

$$g(z) = \begin{cases} \frac{f(z)}{z} & 0 < |z| < 1 \\ f'(0) & z = 0. \end{cases}$$

As a corollary, we have something EXTREMELY USEFUL!!!! BLASCHKE PRODUCT!!

Remark 7.3. *If $f : D \to D$ is analytic (where D is the unit disk) and $a \in D$ with $f(a) = 0$, then $|f(z)| \leq |\frac{z-a}{1-\bar{a}z}|$, for $|z| < 1$.*

Let $a \in D$ with $f(a) = 0$. Then $w = \frac{z-a}{1-\overline{a}z}$ is an invertible map which takes D to D biholomorphically, and maps the boundary of D to the boundary of D. It is not difficult to see that $|\frac{z-a}{1-\overline{a}z}| = 1$ for $|z| = 1$. Then we can express $z = \frac{w+a}{1+\overline{a}w}$, and we see that we have an invertible map $f \circ w^{-1}$ which maps D to D and which has $f(w^{-1}(0)) = 0$, so by Schwarz Lemma, we get $|f \circ w^{-1}(z)| \leq |z|$ and $|f(z)| \leq |w(z)|$ on $|z| \leq 1$ as desired.

Actually, we proved (Schur's Algorithm) that if $f : D \to D$ is analytic, and f maps the unit disc into the unit disc, then f is a composition of linear fractional transforms.

7.3 Morera's Theorem

We now have something that is essentially a converse of the rectangle theorem (which we needed to prove the closed curve theorem):

Theorem 7.4. *Morera's Theorem. Let f be a continuous function on an open set D. If*

$$\int_\Gamma f(z)\, dz = 0$$

whenever Γ is the boundary of a closed rectangle in D, then f is analytic on D.

This theorem is typically used to prove the analyticity of functions which are given in integral form. The textbook gives $f(z) = \int_0^\infty \frac{e^{zt}}{t+1}$ as an example.

7.4 Schwarz Reflection Principle

Theorem 7.5. *Schwarz Reflection Principle. Suppose f is C-analytic in a region D that is contained in either the upper or lower half plane and whose boundary contains a segment L on the real axis, and suppose f is real for real z. Then we can define an analytic "extension" g of f to the region $D \cup L \cup D^*$ that is symmetric with respect to the real axis by setting*

$$g(z) = \begin{cases} f(z) & z \in D \cup L \\ \overline{f(\overline{z})} & z \in D^* \end{cases}$$

where $D^* = \{z : z^* \in D\}$.

Remark 7.6. *Corollary to Schwarz Reflection Principle: If f is analytic in a region symmetric with respect to the real axis and if f is real for real z, then $f(z) = \overline{f(\overline{z})}$*

8 Chapter 8: Simply Connected Domains & Logarithmic Integral

I'm gonna deviate from the text a bit here.

Definition 11. *A region D is simply connected if any two curves with the same endpoints can be smoothly morphed into each other. That is, a region D is simply connected if any two paths with the same endpoints are homotopic.*

If D is convex, then D is simply connected. We now use simply-connectedness to define the logarithmic integral:

Theorem 8.1. *Let $U \subseteq \mathbb{C}$ be simply connected and let $f : U \to \mathbb{C} \setminus \{0\}$ be analytic, never vanishing. Then there exists $g : U \to \mathbb{C}$ analytic such that $f = e^g$.*

The proof is kind of strange so I will omit it, but it utilizes the analyticity of $\frac{f'}{f}$. We actually want for $g' = \frac{f'}{f}$, so we define $g(z) = C \int_a^z \frac{f'(\omega)}{f(\omega)} d\omega$, analytic, with $g(a) = 0$.

As a remark, note that if $f(z) = z$, then z never vanishes on a simply connected set U which does not include the origin. Thus there exists an analytic function $g(z)$ such that $e^{g(z)} = z$, but such a function $g(z)$ is NOT unique. For instance, $e^{g(z)+2\pi ik} = z$.

8.1 Main Branch of $\ln(z)$

Consider $U = \mathbb{C} \setminus (-\infty, 0]$ simply connected. Then there exists a function $h(z)$ such that $e^{h(z)} = z$. Then $h(z) = \ln |z| + i\theta + 2\pi in(z)$, for $n(z) \in \mathbb{Z}$. Since h is analytic with respect to z, then $n(z)$ is analytic with respect to z, but it takes values in $\mathbb{Z}$, which means it must be constant. So we define a choice convention $h(z) = \ln(z) = \ln(|z|) + i\theta$, where $\ln(1) = 0$, $-\pi \leq \theta \leq \pi$. So we have extended the logarithm from calculus.

Finally, notice here that $\ln(z) = \int_1^z \frac{d\omega}{\omega}$, and if we parametrize ω, then we get that $\ln(z) = \int_\gamma \frac{\omega'(t)}{\omega(t)} dt$.

Given $f : U \to \mathbb{C}$ analytic, non-constant. Let $\Omega \subset U$ be a compact subset with a piecewise smooth boundary where $f(\zeta) \neq 0$ for all $\zeta \in \partial\Omega$. We claim:

Theorem 8.2.

$$|V(f)| = |\{\lambda \in \Omega : f(\lambda) = 0\}| = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta)} d\zeta$$

where the integral on the RHS is the logarithmic integral.

For the proof: Let $V(f) = \lambda_1, \ldots, \lambda_m$, where each λ_i is a distinct zero of a certain multiplicity n_i. We can express $f(z) = (z - z_j)^{n_j} g_j(z)$ where g_j is analytic, $g_j(z) \neq 0$ for $|z - \lambda_j| < \varepsilon$. We can choose ε small enough such that $\overline{D(\lambda_j, \varepsilon)} \cap \overline{D(\lambda_k, \varepsilon)} = \emptyset$ for $j \neq k$. Now we can express

$$\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{m} \int_{|z-\lambda_j|=\varepsilon} \frac{f'(z)}{f(z)} dz$$

Now by Leibniz rule, we have $f'(z) = n_j(z - \lambda_j)^{n_j - 1} g_j(z) + (z - \lambda_j)^{n_j} g_j'(z)$. Therefore $\frac{f'(z)}{f(z)} = \frac{n_j}{z - \lambda_j} + \frac{g_j'(z)}{g_j(z)}$. Now because $\frac{g_j'(z)}{g_j(z)}$ is analytic on $|z - \lambda_j| = \varepsilon$ (because $g_j(z) \neq 0$ on this curve by assumption), we have by the closed curve theorem that $\int_{|z-\lambda_j|=\varepsilon} \frac{g_j'(z)}{g_j(z)} = 0$. So we have that

$$\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{m} n_j \int_{|z-\lambda_j|=\varepsilon} \frac{1}{z-\lambda_j} dz$$

Now since λ_j is inside the region enclosed by $|z-\lambda_j| = \varepsilon$, then $\int_{|z-\lambda_j|} = 2\pi i$, and we get

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\sum_{j=1}^{m} n_j 2\pi i = \sum_{j=1}^{m} n_j = |V(f)|$$

Remark 8.3. *From Theorem 8.1 we see that* $\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta)} d\zeta$ *is always an integer.*

8.2 Argument Principle and Rouché's Theorem

Consider $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz$ for f analytic inside of a simply connected domain, and γ is a smooth (or piecewise smooth) closed curve enclosing that domain. Then if we define $g(z) = \int_a^z \frac{f'(z)}{f(z)} dz$ where $g(a) = 0$, then $g'(z) = \frac{f'(z)}{f(z)} dz$ which implies that $g(z) = \ln(f(z)) + i \operatorname{Arg} f(z)$. Therefore if we parametrize γ by $z(t), 0 \le t \le 1$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}(\ln(f(z(1))) - \ln(f(z(0))))$$

Then since $z(0) = z(1)$, we get that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} i(\operatorname{Arg} f(z(1)) - \operatorname{Arg} f(z(0))) \\
&= \frac{1}{2\pi} \operatorname{Arg} f(z(1)) - \operatorname{Arg} f(z(0)) \\
&= \frac{1}{2\pi} Var_\gamma f(z)
\end{aligned}$$

Let's see an application of this: Suppose we have a polynomial $P(z) = z^4 + z^3 + 4z^2 + 2z + 3$. We wish to see how many zeroes $P(z)$ has in the first quadrant of the complex plane. Denote by U the first quadrant. Then

$$|V_U(P)| = \frac{1}{2\pi} Var_\gamma P$$

If we let γ be the curve given by $Re^{i\theta}$, $0 \le \theta \le \frac{\pi}{2}$, then $P(z(\theta)) = R^4 e^{4i\theta}(1 + \frac{1}{Re^{i\theta}} + \cdots + \frac{3}{R^4 e^{4i\theta}})$. Then as $R \to \infty$ (looking at whole first quadrant), we get that the variance of $P(z)$ should be 4θ, from $0 \le \theta \le \frac{\pi}{2}$ gives 2π. So $|V_U(P)| = \frac{1}{2\pi}(2\pi) = 1$.

We can also apply the above example to find solutions of transcendental equations... or at least determine where these solutions must lie...

Theorem 8.4. *Rouché's Theorem If f and g are analytic in an open set, $f, g : U \to \mathbb{C}$, $\Omega \subset U$ is compact, and $|f(\zeta)| > |g(\zeta)|$, $\zeta \in \partial\Omega$, then $|V_\Omega(f)| = |V_\Omega(f+g)$.*

We proved this using the fact that $t \mapsto \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(\zeta)+tg'(\zeta)}{f(\zeta)+tg(\zeta)} d\zeta \in \mathbb{Z}$ is analytic, and the integral has a never vanishing denominator for $0 \leq t \leq 1$. Then since this is analytic, and the right hand side gives the number of zeroes of $f + tg$ (logarithmic derivative), then we have an integer-valued analytic function, which must be constant. Therefore plugging in $t = 0$ and $t = 1$ gives us the same value, and we are done.

We get the following corollary (not entirely sure how we proved it though):

Remark 8.5. *If $f, g : U \to \mathbb{C}$ are analytic, $\Omega \subseteq U$ compact, $f(\zeta) \neq 0$, $\zeta \in \partial\Omega$. Then $\sum_{\lambda \in V(f)} g(\lambda) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta)} g(\zeta) d\zeta$.*

9 Chapter 9: Isolated Singularities of an Analytic Function

9.1 Classifying Isolated Singularities

Definition 12. *f is said to have an isolated singularity at z_0 if f is analytic in a deleted neighborhood D of z_0 but is not analytic at z_0. (Note: this requires f to also be discontinuous at z_0).*

We now distinguish between the types of isolated singularities:

Definition 13. *Removable Singularity. An isolated singularity z_0 is called removable if there exists a function g, analytic at z_0 and such that $f(z) = g(z)$ for all z in some deleted neighborhood of z_0.*

Definition 14. *Polar Singularity. An isolated singularity z_0 is called polar if $C_{-n} = 0$ for all $n \geq N$ large. So if we only have finitely many terms in the negative power part of the Laurent expansion of f.*

Definition 15. *Essential Singularity. An isolated singularity z_0 is called polar if we have infinitely many negative power terms in the Laurent expansion of f, that is, if $C_{-n} \neq 0$ for infinitely many n.*

We now wish to find sufficient conditions for when a singularity falls into one of the categories:

Theorem 9.1. *If f has an isolated singularity at z_0 and $\lim_{z \to z_0}(z - z_0)f(z) = 0$, the singularity is removable.*

The proof follows by letting $h(z) = (z - z_0)f(z)$ when $z \neq z_0$ and $h(z) = 0$ when $z = z_0$. Then h is analytic on a deleted disk of z_0 and continuous at z_0, which (by a theorem in chapter 7 about functions which are analytic on a neighborhood set minus finitely many points and are continuous at those points implies that the function is analytic on the whole neighborhood) means h is analytic on the whole disc surrounding z_0. Then $\frac{h(z)}{z - z_0}$ is analytic at z_0 (since $h(z_0) = 0$) and is equal to $f(z)$ everywhere except z_0 on the disc, so we have a removable singularity.

Theorem 9.2. *f has a removable singularity at z_0 if and only if f is bounded on a deleted neighborhood of z_0.*

If f has a removable singularity at z_0, then there exists an F analytic at z_0 such that $F(z) = f(z)$ for all z in the deleted neighborhood of z_0. If we express f in its power series form centered at z_0, then $\lim_{z\to z_0} f(z) = c_0$ which is finite, hence bounded. Now suppose f is bounded on a deleted neighborhood of z_0. We wish to show that $\lim_{z\to z_0}(z - z_0)f(z) = 0$, such that the singularity is removable by Thm 9.1. Well f bounded implies $|f| \le M$ so $\lim_{z\to z_0} |z - z_0||f(z)| \le \lim_{z\to z_0} |z - z_0|M = 0$, as desired.

Theorem 9.3. *If f has a removable singularity at z_0, then the Laurent expansion of f has only positive power terms.*

The proof follows from the fact that there must exist some analytic function F which is equal to f everywhere on a deleted disk, and analytic functions have convergent Taylor series expansions.

We now wish to find sufficient conditions for when a singularity is polar:

Theorem 9.4. *A singularity z_0 of f is polar $\iff$ $\lim_{z\to z_0} f(z) = \infty$.*

Recall that if z_0 of f is polar, then the Laurent expansion of f about z_0 has finitely many negative power terms. But as $z \to z_0$, these terms approach infinity. On the other hand, if $f \to \infty$ as $z \to z_0$, then there exists a deleted disc about z_0 such that $f(z) \neq 0$. Then $\frac{1}{f(z)}$ is analytic in this deleted disc, so if $f(z) \to \infty$, we must have $\frac{1}{f(z)} \to 0$ as $z \to z_0$. Let $\frac{1}{f(z)}$ be written in its power series expansion about z_0, which is convergent in some neighborhood of z_0. If we factor out a zero of order p (so we factor out $(z - z_0)^p$, then we get a $f(z) = \frac{1}{z^p} \cdot \frac{1}{\gamma_p + \gamma_{p+1}(z-z_0)+\cdots} = \frac{1}{z^p}(d_0 + d_1(z - z_0) + \cdots)$, which is convergent and has finitely many terms with a negative power of $z - z_0$, thus a polar singularity.

Theorem 9.5. *If f has an essential singularity at z_0 and if D is a deleted neighborhood of z_0, then the range $R = \{f(z) : z \in D\}$ is dense in the complex plane.*

The proof follows by assuming that R was not dense in $\mathbb{C}$, and then showing that this implies $f(z)$ must either have a polar singularity or a removable singularity. Therefore by the contrapositive, an essential singularity implies that R is dense in $\mathbb{C}$.

Although we've been talking about Laurent expansions, we haven't formally introduced when they're convergent, and when they're even used. Recall that an annulus $A_{r,R}$ is the set of points $z \in \mathbb{C}$ such that $r < |z| < R$. This motivates the following theorem:

Theorem 9.6. *$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ is convergent in the domain $D = \{z : R_1 < |z| < R_2\}$, where $R_2 = 1/\limsup \sqrt[k]{|a_k|}$ and $R_1 = \limsup \sqrt[k]{|a_{-k}|}$. Also f is analytic in its radius of convergence.*

The proof follows pretty trivially. For R_2, notice that $\sum_{k=-\infty}^{-1} a_k z^k = \sum_{k=1}^{\infty} a_{-k} z^{-k} = \sum_{k=1}^{\infty} a_{-k}(\frac{1}{z})^k$ which converges when $\left|\frac{1}{z}\right| < \frac{1}{R_1}$, or when $|z| > R_1$.

Theorem 9.7. *Analytic Functions and Laurent Expansions. If f is analytic in the annulus A : $R_1 < |z| < R_2$, then f has a Laurent expansion, $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ in A.*

We consider $R_1 < r_1 < r_2 < R_2$. Then f is analytic on $\overline{A_{r_1,r_2}}$. Let $z \in A_{r_1,r_2}$. By Cauchy's integral formula, we get

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=r_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{|\zeta|=r_1} \frac{f(\zeta)}{\zeta - z}d\zeta$$

Since $|z| < r_2$, we have that $\frac{1}{\zeta_2 - z} = \frac{1}{\zeta_2} \cdot \frac{1}{1-\frac{z}{\zeta_2}}$, and $\frac{1}{1-\frac{z}{\zeta_2}}$ may be expressed as a convergent geometric series. Similarly, since $|z| > r_1$, we have that $\frac{1}{\zeta_1 - z} = \frac{1}{z} \cdot \frac{-1}{1-\frac{\zeta_1}{z}}$ can be expressed as $\sum_{k=0}^{\infty}(-1)\frac{\zeta_1^k}{z^{k+1}}$. At the end, this gives us $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ where $a_k = \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(z)}{z^{k+1}}dz$ where $R_1 < r < R_2$.

10 Chapter 10: The Residue Theorem

10.1 Winding Numbers and the Cauchy Residue Theorem

Recall from chapter 9 that if z_0 is an isolated singularity of a function f which is analytic on $D^*(z_0, \varepsilon)$, then we have a Laurent expansion about z_0 given by $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ where $c_k = \frac{1}{2\pi i}\int_{|z|=\varepsilon}\frac{f(z)}{z^{k+1}}dz$. So in particular, we have that $c_{-1} = \frac{1}{2\pi i}\int_{|z|=\varepsilon} f(z)dz$.

Definition 16. *If $f(z) = \sum_{k=-\infty}^{\infty} c_k(z - z_0)^k$ in a deleted neighborhood of z_0, C_{-1} is called the residue of f at z_0. We use the notation C_{-1} = Res(f, z_0).*

Theorem 10.1. *If f has a simple pole at z_0, that is, if $f(z) = \frac{A(z)}{B(z)}$ where A and B are analytic at z_0, $A(z_0) \neq 0$ and B has a simple zero at z_0, then $C_{-1} = \lim_{z\to z_0}(z - z_0)f(z) = \frac{A(z_0)}{B'(z_0)}$*

The proof is quite simple. Since f has a simple pole at z_0, the Laurent expansion of f about z_0 will only have one negative power term, which is $c_1(z - z_0)^{-1}$. Then the rest follows by multiplying $f(z)$ by $(z - z_0)$ and taking limits.

Remark 10.2. *If f has a pole of order k at z_0, then $C_{-1} = \frac{1}{(k-1)!}\frac{d^{(k-1)}}{dz^{k-1}}(z - z_0)^k f(z)$ evaluated at z_0. Follow a very similar process as in Theorem 10.1.*

Definition 17. *Suppose that γ is a closed curve and that $a \notin \gamma$. Then $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is called the winding number of γ around a.*

Remark 10.3. *If γ circles the point a k times, that is, if $\gamma(\theta) = a + re^{i\theta}, 0 \leq \theta \leq 2k\pi$—then $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = k$, which explains the terminology "winding number."*

Theorem 10.4. *For any closed curve γ and $a \notin \gamma, n(\gamma, a)$ is an integer.*

When the curve is SIMPLE, the proof essentially follows from the fact that the logarithmic integral always gives $2\pi i$ times the number of zeroes of a function f. Let $f(z) = z - a$. Then $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz = 1$ when a is inside the region enclosed by γ and 0 when a is outside the region enclosed by γ. When the curve is not simple, we have a slightly different proof.... uses stuff from chapter 8, I don't really care...

Theorem 10.5. *Suppose f is analytic in a simply connected domain D except for isolated singularities at $z_1, z_2, \ldots, z_m$. Let γ be a closed curve not intersecting any of the singularities. Then*

$$\frac{1}{2\pi i}\int_{\partial\Omega} f = \sum_k n(\gamma, z_k) Res_{z_k} f$$

Definition 18. *We say f is meromorphic in a domain D if f is analytic there except at isolated poles.*

Theorem 10.6. *Suppose γ is a regular closed curve. If f is meromorphic inside and on γ and contains no zeroes or poles on γ, and if Z = number of zeroes of f inside γ (a zero of order k being counted k times), P = number of poles of f inside γ (again with multiplicity), then*

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = Z - P$$