Cohomology of odd symplectic groups

February 10, 2026

1 Rational cohomology of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$

We begin by computing the rational cohomology of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$. Insert algebro-geometric interpretation.

Definition 1.1 (Symplectic group)**.** Let $J_{2n} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{pmatrix}$ represent the standard symplectic form ω on $\mathbb{R}^{2n}$ (equipped with the symplectic basis $e_1, f_1, \ldots, e_n, f_n$). That is, ω is bilinear, skew-symmetric, and

$$\omega(e_i, f_j) = \delta_{ij}, \quad \omega(e_i, e_j) = 0, \quad \omega(f_i, f_j) = 0 \quad \forall i, j.$$

Then $\mathrm{Sp}_{2n}(\mathbb{Z}) = \{A \in \mathrm{GL}_{2n}(\mathbb{Z}) : A^T J A = J\}$. Thus $\mathrm{Sp}_{2n}(\mathbb{Z})$ consists of matrices in $\mathrm{GL}_{2n}(\mathbb{Z})$ that preserve ω.

Remark 1.2. There is only one alternating bilinear form on $\mathbb{R}^{2n}$ up to a change of basis.

Definition 1.3 (Odd symplectic group)**.** Consider a symplectic basis $e_0, f_0, \ldots, e_n, f_n$ for $\mathbb{R}^{2n+2}$, reordered to $e_0, e_1, f_1, \ldots, e_n, f_n, f_0$. Let $\tilde{J} = \begin{pmatrix} & & 1 \\ & J_{2n} & \\ -1 & & \end{pmatrix}$. Then $\mathrm{Sp}_{2n+2}(\mathbb{Z}) = \{A \in \mathrm{GL}_{2n+2}(\mathbb{Z}) : A^T \tilde{J} A = \tilde{J}\}$. Define the *odd symplectic group* $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ to be the subset of $\mathrm{Sp}_{2n+2}(\mathbb{Z})$ fixing f_0:

$$\mathrm{Sp}_{2n+1}(\mathbb{Z}) = \mathrm{Stab}_{\mathrm{Sp}_{2n+2}(\mathbb{Z})}(f_0).$$

Lemma 1.4 ([1], §1)**.** *$\mathrm{Sp}_{2n+1}(\mathbb{Z}) \cong H \rtimes \mathrm{Sp}_{2n}(\mathbb{Z})$ for $n \geqslant 1$, where $H = \{\begin{pmatrix} 1 & & \\ \vec{v} & I_{2n} & \\ \alpha & -\vec{v}^T J_{2n} & 1 \end{pmatrix} : \vec{v} \in \mathbb{Z}^{2n}, \alpha \in \mathbb{Z}\}$ is a Heisenberg group.*

Proof. Let $A \in \mathrm{Sp}_{2n+1}(\mathbb{Z})$. Since $Af_0 = f_0$ and A preserves the form prescribed by $\tilde{J}$, A decomposes as

$$A = \left(\begin{array}{c|c|c} 1 & \vec{w} & 0 \\ \hline \vec{v} & B & 0 \\ \hline \alpha & \vec{z} & 1 \end{array}\right)$$

for $\vec{v}, \vec{w}, \vec{z} \in \mathbb{Z}^{2n}$, $\alpha \in \mathbb{Z}$, and $B \in \mathrm{GL}_{2n}(\mathbb{Z})$. Furthermore,

$$A^T \tilde{J} A = \tilde{J} \implies \left(\begin{array}{c|c|c} 1 & \vec{v}^T & \alpha \\ \hline \vec{w}^T & B^T & \vec{z}^T \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & J & 0 \\ \hline -1 & 0 & 0 \end{array}\right) \cdot \left(\begin{array}{c|c|c} 1 & \vec{w} & 0 \\ \hline \vec{v} & B & 0 \\ \hline \alpha & \vec{z} & 1 \end{array}\right) = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & J & 0 \\ \hline -1 & 0 & 0 \end{array}\right)$$

$$\implies \left(\begin{array}{c|c|c} \vec{v}^T J \vec{v} & -\alpha\vec{w} + \vec{v}^T J B + \vec{z} & 1 \\ \hline B^T J \vec{v} + \alpha \vec{w}^T - \vec{z}^T & B^T J B & \vec{w}^T \\ \hline -1 & -\vec{w} & 0 \end{array}\right) = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & J & 0 \\ \hline -1 & 0 & 0 \end{array}\right)$$

so that $B \in \mathrm{Sp}_{2n}(\mathbb{Z})$, $\vec{w} = 0$, and $\vec{z} = -\vec{v}^T JB$. As it is always true that $\vec{v}^T J\vec{v} = 0$, we obtain no restrictions on $\vec{v}$ and α. The projection

$$\mathrm{Sp}_{2n+1}(\mathbb{Z}) \xrightarrow{\theta} \mathrm{Sp}_{2n}(\mathbb{Z})$$
$$\begin{pmatrix} 1 & & \\ \vec{v} & B & \\ \alpha & -\vec{v}^T JB & 1 \end{pmatrix} \mapsto B$$

is a group homomorphism, for $\begin{pmatrix} 1 & & \\ v_1 & B_1 & \\ \alpha_1 & -v_1^T JB_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & \\ v_2 & B_2 & \\ \alpha_2 & -v_2^T JB_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ v_1+B_1v_2 & B_1B_2 & \\ \alpha_1+\alpha_2-v_1^T JB_1v_2 & -v_1^T JB_1B_2-v_2^T JB_2 & 1 \end{pmatrix}$.
One obtains a section of θ via the embedding $B \mapsto \begin{pmatrix} 1 & & \\ & B & \\ & & 1 \end{pmatrix}$. Clearly $\ker\theta$ is the Heisenberg group H. Since θ has a section, we obtain a split short exact sequence of groups

$$1 \to H \to \mathrm{Sp}_{2n+1}(\mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z}) \to 1.$$

□

Understanding H and its relationship to $\mathrm{Sp}_{2n}(\mathbb{Z})$ will be vital to computing the group cohomology of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$. The Lyndon-Hochschild-Serre spectral sequence and the semidirect product structure of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ yields this immediate consequence:

Corollary 1.5 (S.). *The abelianization of* $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ *is trivial for* $n \geqslant 3$.

Proof. By [8] Lemma A.2, $H_1(\mathrm{Sp}_{2n}(\mathbb{Z})) = 0$ for $n \geqslant 3$. Now $H_1(\mathrm{Sp}_{2n+1}(\mathbb{Z})) \cong H_1(\mathrm{Sp}_{2n}(\mathbb{Z})) \times (H_1(H))_{\mathrm{Sp}_{2n}(\mathbb{Z})} = \mathbb{Z}^{2n}_{\mathrm{Sp}_{2n}(\mathbb{Z})}$ for $n \geqslant 3$. Let $\vec{v} \in \mathbb{Z}^{2n}$ and let $A = -I_{2n} \in \mathrm{Sp}_{2n}(\mathbb{Z})$. Then $A \cdot \vec{v} = -\vec{v}$, so that $\mathbb{Z}^{2n}_{\mathrm{Sp}_{2n}(\mathbb{Z})} = 0$. □

In fact, can use Lemma A.2 to describe the abelianization of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ for all n.
Our decompositions of H aids in showing that H is actually a Poincaré duality group. Recall the following theorem in Bieri [2] on short exact sequences of duality groups.

Lemma 1.6 ([2] Thm 9.10)**.** *Let* $1 \to N \to G \to Q \to 1$ *be a short exact sequence of groups with* N *and* Q *duality groups of dimension* n *and* q*, respectively, over* R*. Then* G *is a duality group of dimension* $n+q$ *over* R*, with dualizing module*

$$H^{n+q}(G; RG) \cong H^q(Q; RQ) \otimes_R H^n(N; RN)$$

such that G *acts diagonally on the right hand side.*

Lemma 1.7. *H is an orientable Poincaré duality group of dimension* $2n+1$.

Proof. Note that H contains a central subgroup $\Gamma = \{\begin{pmatrix} 1 & & \\ 0 & I_{2n} & \\ \alpha & 0 & 1 \end{pmatrix} : \alpha \in \mathbb{Z}\}$ isomorphic to $\mathbb{Z}$, such that the quotient H/Γ is isomorphic to $\mathbb{Z}^{2n}$: For one has that $\Gamma v \cdot \Gamma w = \Gamma(v+w)$, since

$$\begin{pmatrix} 1 & & \\ \vec{v} & I_{2n} & \\ \alpha & -\vec{v}^T J & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ \vec{w} & I_{2n} & \\ \beta & -\vec{w}^T J & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 0 & I_{2n} & \\ \alpha+\beta-\vec{v}^T J\vec{w} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ \vec{v}+\vec{w} & I_{2n} & \\ 0 & -(\vec{v}+\vec{w})^T J & 1 \end{pmatrix}.$$

Since $\Gamma \cong \mathbb{Z}$ and $H/\Gamma \cong \mathbb{Z}^{2n}$ are Poincaré duality groups of dimensions 1 and $2n$, respectively, applying Lemma 1.6 to the short exact sequence of groups

$$1 \to \Gamma \to H \to H/\Gamma \to 1$$

yields that H is a duality group of dimension $2n+1$, with dualizing module

$$H^{2n+1}(H; \mathbb{Z}H) \cong H^{2n}(H/\Gamma; \mathbb{Z}H/\Gamma) \otimes H^1(\Gamma; \mathbb{Z}\Gamma) \cong \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}.$$

Finally, by Bieri [2] Prop 2.7, we know that if Γ is of type FP_∞, then H is of type FP_∞ if and only if H/Γ is. Since both Γ and H/Γ are Poincaré duality groups, they must be of type FP_∞, so Bieri's proposition lets us conclude that H is as well. Orientability follows from the fact that there exists an orientable closed $K(H,1)$-manifold... see Example 1 in Brown §VIII.10. "Still more generally, if G is an arbitrary Lie group with only finitely many connected components and Γ is a discrete, co-compact, torsion-free subgroup, then Γ is a Poincare duality group and is orientable if G is connected (cf. example 4 of §9" □

Prop 1.8. $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ *is a* $(n+1)^2$*-dimensional virtual duality group with* $\mathbb{Q}$*-dualizing module* $\mathbb{Z}\otimes \mathrm{St}_n^\omega(\mathbb{Q})\otimes\mathbb{Q}$*.*

Proof. Since H and $\mathrm{Sp}_{2n}(\mathbb{Z})$ are virtual duality groups of dimensions $2n+1$ and n^2, respectively, they are rational duality groups of the same dimension. By Lemmas 1.4, 1.6, and 1.7, we conclude that $\mathrm{cd}_{\mathbb{Q}}(\mathrm{Sp}_{2n+1}(\mathbb{Z})) = n^2+2n+1 = (n+1)^2$.

Since $\mathrm{Sp}_{2n+1}(\mathbb{Z}) \cong H \rtimes \mathrm{Sp}_{2n}(\mathbb{Z})$ by Lemma 1.4, and $\mathrm{Sp}_{2n}(\mathbb{Z})$ contains a finite-index duality subgroup K by work of Borel-Serre (cite), the semidirect product $H\rtimes K$ is a finite-index subgroup of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ (does this require some justification?) which is a duality group. The latter fact follows from Lemma 1.6 again, using the short exact sequence of groups

$$1\to H\to H\rtimes K\to K\to 1$$

of which H and K are duality groups. The rational dualizing module D of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ is then

$$\begin{aligned} D = H^{(n+1)^2}(\mathrm{Sp}_{2n+1};\mathbb{Q}\,\mathrm{Sp}_{2n+1}) \cong H^{(n+1)^2}(\mathrm{Sp}_{2n+1};\mathbb{Z}\,\mathrm{Sp}_{2n+1})\otimes\mathbb{Q} &\cong H^{2n+1}(H;\mathbb{Z}H)\otimes H^{n^2}(\mathrm{Sp}_{2n};\mathbb{Z}\,\mathrm{Sp}_{2n})\otimes\mathbb{Q}\\ &\cong (\mathbb{Z}\otimes\mathrm{St}_n^\omega(\mathbb{Q}))\otimes\mathbb{Q}\end{aligned}$$

where $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ acts diagonally on both tensor products. (prop 2.146 in prelim for some iso justifications). □

Remark 1.9. One ought to compute how $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ explicitly acts on its dualizing module D. Note that every element $g\in\mathrm{Sp}_{2n+1}(\mathbb{Z})$ can be decomposed as

$$\underbrace{\begin{bmatrix} 1 & & \\ \vec{v} & A & \\ \alpha & -\vec{v}^T JA & 1\end{bmatrix}}_{g} = \underbrace{\begin{bmatrix} 1 & & \\ \vec{v} & I & \\ \alpha & -\vec{v}^T J & 1\end{bmatrix}}_{h}\cdot\underbrace{\begin{bmatrix} 1 & & \\ 0 & A & \\ 1 & -0 & 1\end{bmatrix}}_{A}$$

so that, for instance, the right (?) action of g on $H^{2n+1}(H;\mathbb{Z}H)$ is given by the composite of the right actions

$$g^*: H^{2n+1}(H;\mathbb{Z}H)\xrightarrow{h^*} H^{2n+1}(H;\mathbb{Z}H)\xrightarrow{A^*} H^{2n+1}(H;\mathbb{Z}H),$$

and similarly for the right action of g on $H^{n^2}(\mathrm{Sp}_{2n}(\mathbb{Z});\mathbb{Z}[\mathrm{Sp}_{2n}(\mathbb{Z})])\cong\mathrm{St}_n^\omega(\mathbb{Q})$. Thus it is enough to compute how H and $\mathrm{Sp}_{2n}(\mathbb{Z})$ act on their own and each other's dualizing modules. Most cases are immediate: Since $\mathrm{Sp}_{2n}(\mathbb{Z})\cong\mathrm{Sp}_{2n+1}(\mathbb{Z})/H$, we get that H acts trivially on $\mathrm{Sp}_{2n}(\mathbb{Z})$, inducing a trivial action of H on $\mathrm{St}_n^\omega(\mathbb{Q})$. We conclude that $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ acts on $\mathrm{St}_n^\omega(\mathbb{Q})$ as $\mathrm{Sp}_{2n}(\mathbb{Z})$, identifying $\begin{bmatrix} 1 & & \\ \vec{v} & A & \\ \alpha & -\vec{v}^T JA & 1\end{bmatrix}$ with its restriction to A. some confusion: we're acting on the right, but the papers I've seen for symplectic Steinberg act on the left...

Since H is an orientable Poincaré duality group by Lemma 1.7, it acts trivially on its dualizing module $H^{2n+1}(H;\mathbb{Z}H)$. The action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $H^{2n+1}(H;\mathbb{Z}H)$ is also trivial, although the proof is more involved.

Lemma 1.10. $\mathrm{Sp}_{2n}(\mathbb{Z})$ *acts trivially on* $H^{2n+1}(H;\mathbb{Z}H)$*.*

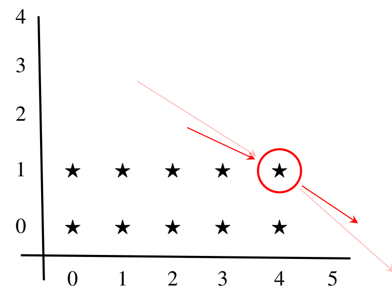

Figure 1: $E_2^{p,q}$ when $n = 2$. The (co)domains of all subsequent differentials into (and out of) $E_2^{2n,1}$ are 0.

Proof. The Lyndon-Hochschild-Serre (LHS) cohomological spectral sequence associated to this central extension for H satisfies $E_2^{p,q} = H^p(H/\Gamma; H^q(\Gamma; \mathbb{Z}H))$, where $\mathbb{Z}H$ is a left Γ-module by left multiplication of matrices. Since $H^p(H/\Gamma, -) = 0$ for $p > 2n$ and $H^q(\Gamma; -) = 0$ for $q > 1$, the $p + q = 2n + 1$ diagonal has $E_2^{2n,1} = H^{2n}(\mathbb{Z}^{2n}; H^1(\Gamma; \mathbb{Z}H))$ as its only (potentially) nonzero group. The differentials d_r are all zero for $r \geqslant 2$, which means $E_2^{2n,1} = E_\infty^{2n,1}$.
Let c_A be the automorphism of H (and abusing notation, of H/Γ) associated to conjugation by $A \in \mathrm{Sp}_{2n}(\mathbb{Z})$. As Γ is central in H, we have a commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \Gamma \arrow[r] \arrow[d, "c_A = \mathrm{id}"] & H \arrow[r] \arrow[d, "c_A"] & H/\Gamma \arrow[r] \arrow[d, "c_A"] & 1 \\ 1 \arrow[r] & \Gamma \arrow[r] & H \arrow[r] & H/\Gamma \arrow[r] & 1 \end{tikzcd}$$

inducing fibration maps

$$\begin{tikzcd} \mathrm{BH} \arrow[r, "p"] \arrow[d, "B_{c_A}"] & B(H/\Gamma) \arrow[d, "B_{c_A}"] \\ \mathrm{BH} \arrow[r, "p"'] & \mathrm{B}(H/\Gamma) \end{tikzcd}$$

on the classifying spaces. (This follows, for instance, from functoriality of the bar construction for discrete groups.) Naturality of the cohomological Serre spectral sequence with local coefficients yields a morphism of spectral sequences equivalent to

$$\begin{tikzcd} H^p(H/\Gamma; H^q(\Gamma; \mathbb{Z}H)) \arrow[r, Rightarrow] & H^{p+q}(H; \mathbb{Z}H) \\ H^p(H/\Gamma; H^q(\Gamma; \mathbb{Z}H)) \arrow[r, Rightarrow] \arrow[u, "H^p(c_A)"] & H^{p+q}(H; \mathbb{Z}H). \arrow[u, "H^{p+q}(c_A)"] \end{tikzcd}$$

in the Lyndon-Hochschild-Serre setting ([9], Theorem 24.5.2). Let $0 = F^{2n+1} \subset F^{2n} \subset \cdots \subset F^0 = H^{2n+1}(H; \mathbb{Z}H)$ be a filtration of the relevant limiting object. Since $F^p/F^{p+1} \cong E_\infty^{p,2n+1-p}$, and $E_\infty^{p,2n+1-p} = 0$ unless $p = 2n$ (in which case $E_\infty^{2n,1} = E_2^{2n,1}$), the filtration collapses to $0 \subset F^{2n} = F^0 = H^{2n+1}(H; \mathbb{Z}H)$. The presence of a morphism of spectral sequences tells us that the induced map

$$H^{2n+1}(H; \mathbb{Z}H) \xrightarrow{H^{2n+1}(c_A)} H^{2n+1}(H; \mathbb{Z}H)$$

restricts to $c_{A*}^\infty : E_\infty^{2n,1} \to E_\infty^{2n,1}$ on the quotient $E_\infty^{2n,1} \cong F^{2n}$, where $c_{A*}^\infty = c_{A*}^2$ is the map $H^{2n}(c_A)$. Hence the abutment isomorphism $H^{2n}(\mathbb{Z}^{2n}, H^1(\Gamma; \mathbb{Z}H)) \xrightarrow{\cong} H^{2n+1}(H; \mathbb{Z}H)$ respects the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$.

We now compute the $\mathrm{Sp}_{2n}(\mathbb{Z})$ action on $H^{2n}(\mathbb{Z}^{2n}; H^1(\Gamma;\mathbb{Z}H))$. Note that $H^1(\Gamma;\mathbb{Z}H) \cong (\mathbb{Z}H)_\Gamma$ inherits a $\mathbb{Z}^{2n}$-module structure from H acting on Γ by conjugation. For a basis element $h \in \mathbb{Z}H$ representing $h = \begin{bmatrix} 1 & & \\ v & I & \\ \alpha & -v^TJ & 1 \end{bmatrix}$, left multiplication by the generator Z of Γ has the effect of adding 1 to α. Thus elements of $(\mathbb{Z}H)_\Gamma \cong \mathbb{Z}H/(Z-1)\mathbb{Z}H$ are completely determined by $\vec{v} \in \mathbb{Z}^{2n}$, and $H^1(\Gamma;\mathbb{Z}H) \cong \mathbb{Z}[H/\Gamma] \cong \mathbb{Z}[\mathbb{Z}^{2n}]$. For $B \in H$, the conjugation action c_B of B on Γ induces a map $H^1(c_B) : \mathbb{Z}[\mathbb{Z}^{2n}] \to \mathbb{Z}[\mathbb{Z}^{2n}]$ given by left multiplication by B. As Γ acts trivially ([4] Ch.III.8), we can view this as a well-defined action by $\overline{B} \in \mathbb{Z}^{2n}$. Thus $\mathbb{Z}^{2n}$ acts on $H^1(\Gamma;\mathbb{Z}H)$ by left multiplication.

We have reduced the problem to computing the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $H^{2n}(\mathbb{Z}^{2n};\mathbb{Z}[\mathbb{Z}^{2n}])$, where $\mathbb{Z}[\mathbb{Z}^{2n}]$ is a left $\mathbb{Z}^{2n}$-module via matrix multiplication. To avoid algebraic technicalities it would be convenient to use the $\mathbb{Z}^{2n}$-module isomorphism

$$H^{2n}(\mathbb{Z}^{2n};\mathbb{Z}[\mathbb{Z}^{2n}]) \cong H^{2n}_c(\mathbb{R}^{2n};\mathbb{Z})$$

([4] Ch.VIII Prop 7.5), which exists since $\mathbb{R}^{2n}$ is a contractible free $\mathbb{Z}^{2n}$-complex with $\mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong T^{2n}$ compact. Recall that the left action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n}$ is given by translation. The above isomorphism is induced by the isomorphism on cochains

$$\begin{gathered} \mathrm{Hom}_{\mathbb{Z}^{2n}}(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}[\mathbb{Z}^{2n}]) \xrightarrow{\Phi} \mathrm{Hom}_c(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}) \\ F \mapsto F_e \end{gathered}$$

where $F(\sigma) = \sum_{g\in\mathbb{Z}^{2n}} F_g(\sigma)\cdot g$, the maps $F_g : C_\bullet(\mathbb{R}^{2n}) \to \mathbb{Z}$ satisfy $F_g(\sigma) = 0$ for all but finitely many $g \in \mathbb{Z}^{2n}$, and $F_g(\sigma) = F_e(g^{-1}\cdot\sigma)$ ([4] Ch.VIII Lemma 7.4). We claim that this isomorphism is equivariant with respect to the $\mathrm{Sp}_{2n}(\mathbb{Z})$ action on $\mathbb{Z}^{2n}$. Let $A \in \mathrm{Sp}_{2n}(\mathbb{Z})$ and let $c_A : \mathbb{Z}^{2n} \to \mathbb{Z}^{2n}$ be the map on $\mathbb{Z}^{2n} \cong H/\Gamma$ induced by conjugation with $A \in \mathrm{Sp}_{2n}(\mathbb{Z})$. Explicitly, this map takes $\vec{v} \in \mathbb{Z}^{2n}$ to the vector $A\vec{v} \in \mathbb{Z}^{2n}$, and induces a canonical map on the group rings $(c_A)_+ : \mathbb{Z}[\mathbb{Z}^{2n}] \to \mathbb{Z}[\mathbb{Z}^{2n}]$ in the same manner. We also have a continuous and $\mathbb{Z}^{2n}$-equivariant map $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\vec{x} \mapsto A\vec{x}$, since

$$f(v\cdot\vec{x}) = A(\vec{v}+\vec{x}) = A\vec{v} + A\vec{x} = c_A(\vec{v})\cdot A\vec{x} = c_A(v)\cdot f(\vec{x}).$$

This induces a chain map $f_\# : C_\bullet(\mathbb{R}^{2n}) \to C_\bullet(\mathbb{R}^{2n})$ which is also $\mathbb{Z}^{2n}$-equivariant:

$$f_\#(v\cdot\sigma) = (x \mapsto f(v\cdot\sigma(x))) = (x \mapsto c_A(v)\cdot f(\sigma(x))) = c_A(v)\cdot f_\#(\sigma)$$

for $v \in \mathbb{Z}^{2n}, \sigma : \Delta^\bullet \to \mathbb{R}^{2n}$, and $x \in \Delta^\bullet$. Note also that f is a proper map, as $f^{-1}(K) = A^{-1}K$ is compact for any $K \subset \mathbb{R}^{2n}$ compact (being the continuous image of a compact set). Thus f^* sends a compactly supported cochain to a compactly supported cochain, and we have a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}^{2n}}(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}[\mathbb{Z}^{2n}]) & \xrightarrow{\cong} & \mathrm{Hom}_c(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}) \\ {\scriptstyle F\mapsto(c_A)_+^{-1}\circ F\circ f_\#}\big\uparrow & & \big\uparrow{\scriptstyle f^*} \\ \mathrm{Hom}_{\mathbb{Z}^{2n}}(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}[\mathbb{Z}^{2n}]) & \xrightarrow[F\mapsto F_e]{\cong} & \mathrm{Hom}_c(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}) \end{array}$$

which we claim commutes, establishing naturality of the isomorphism Φ. Let $F \in \mathrm{Hom}_{\mathbb{Z}^{2n}}(C_\bullet(\mathbb{R}^{2n}),\mathbb{Z}[\mathbb{Z}^{2n}])$ be expressed as $F(\sigma) = \sum_{g\in\mathbb{Z}^{2n}} F_g(\sigma)\cdot g$. Then

$$((c_A)_+^{-1}\circ F\circ f_\#)(\sigma) = (c_A)_+^{-1}\Big(\sum_{g\in\mathbb{Z}^{2n}} F_g(f_\#(\sigma))\cdot g\Big) = \sum_{g\in\mathbb{Z}^{2n}} F_g(f_\#(\sigma))\cdot A^{-1}g = \sum_{\gamma=A^{-1}g\in\mathbb{Z}^{2n}} F_{A\gamma}(f_\#(\sigma))\cdot\gamma.$$

We see that $((c_A)_+^{-1} \circ F \circ f_\#)_e(\sigma) = F_{A\cdot e}(f_\#(\sigma)) = F_e(f_\#(\sigma))$ since $e = \vec{0} \in \mathbb{Z}^{2n}$, and $A\vec{0} = \vec{0}$. On the other hand, $f^*(F_e)(\sigma) = F_e(f_\#(\sigma))$, so the diagram commutes.

Hence the automorphism c_A on $H^{2n}(\mathbb{Z}^{2n}; \mathbb{Z}[\mathbb{Z}^{2n}])$ can be computed via the pullback $f^* : H_c^{2n}(\mathbb{R}^{2n}; \mathbb{Z}) \to H_c^{2n}(\mathbb{R}^{2n}; \mathbb{Z})$. Recall that $H_c^{2n}(\mathbb{R}^{2n}; \mathbb{Z})$ is generated by a volume form $e_1 \wedge \cdots \wedge e_{2n}$, and the pullback sends this generator to $f(e_1) \wedge \cdots \wedge f(e_n) = Ae_1 \wedge \cdots \wedge Ae_n = \det(A) \cdot e_1 \wedge \cdots \wedge e_n$. idk if this is correct lol. add more details here!!!. We conclude that the action of $\mathrm{Sp}_{2n}(\mathbb{Z})$ on $H^{2n+1}(H; \mathbb{Z}H)$ is via the determinant, and since $\det(A) = 1$ for all $A \in \mathrm{Sp}_{2n}(\mathbb{Z})$, this action is trivial. □

Two easy corollaries are obtained regarding the high-degree rational cohomology of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$.

Corollary 1.11 (Codimension 0 vanishing)**.** $H^{(n+1)^2}(\mathrm{Sp}_{2n+1}(\mathbb{Z}); \mathbb{Q}) = 0$.

Proof. By Bieri-Eckmann duality, we know

$$H^{(n+1)^2}(\mathrm{Sp}_{2n+1}(\mathbb{Z}); \mathbb{Q}) \cong H_0(\mathrm{Sp}_{2n+1}(\mathbb{Z}); H^{2n+1}(H; \mathbb{Z}H) \otimes \mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q}) \cong (\mathbb{Z} \otimes \mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q})_{\mathrm{Sp}_{2n+1}(\mathbb{Z})}$$

where $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ acts diagonally on $(H^{2n+1}(H; \mathbb{Z}H) \otimes \mathrm{St}_n^\omega(\mathbb{Q})) \otimes \mathbb{Q}$. By remark 1.9 and lemma 1.10, $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ acts trivially on $H^{2n+1}(H; \mathbb{Z}H)$, and on $\mathrm{St}_n^\omega(\mathbb{Q})$ acts as $\mathrm{Sp}_{2n}(\mathbb{Z})$ does. By [6] (find exact citation), $\mathrm{St}_n^\omega(\mathbb{Q})$ is generated by integral symplectic apartment classes (define these). Given such an apartment class $\sigma = [v_1, w_1, \ldots, v_n, w_n]$, let $A_\sigma \in \mathrm{Sp}_{2n}(\mathbb{Z})$ be the matrix which "symplectically" swaps the first two columns, in the sense that

$$A_\sigma \cdot [v_1, w_1, \ldots, v_n, w_n] = [-w_1, v_1, \ldots, v_n, w_n].$$

This reverses the orientation of the apartment class, so that $A_\sigma \cdot \sigma = -\sigma$. Identifying A_σ with $\begin{bmatrix} 1 & \\ & A_\sigma & \\ & & 1 \end{bmatrix} \in \mathrm{Sp}_{2n+1}(\mathbb{Z})$, we see that $A_\sigma \cdot (n \otimes \sigma \otimes q) = -n \otimes \sigma \otimes q$, so that $(H^{2n+1}(H; \mathbb{Z}H) \otimes \mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q})_{\mathrm{Sp}_{2n+1}(\mathbb{Z})} = 0$. □

should give some background on apartment classes. in particular these two bases represent the same apartment class by work of Gunnels. See BPS [5]pages 83-84.

Corollary 1.12 (Codimension 1 vanishing)**.** $H^{(n+1)^2-1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}); \mathbb{Q}) = 0$.

Proof. Since $H^{(n+1)^2-1}(\mathrm{Sp}_{2n+1}(\mathbb{Z}); \mathbb{Q}) \cong H_1(\mathrm{Sp}_{2n+1}(\mathbb{Z}); \mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q})$ by Borel-Serre duality, it suffices to show this latter group is trivial. Recall that the LHS spectral sequence gives, for a G-module M and a short exact sequence of groups $1 \to H \to G \to G/H \to 1$, a five term exact sequence

$$H_2(G; M) \xrightarrow{\text{coinf}} H_2(G/H; M_H) \xrightarrow{\text{transgression}} (H_1(H; M))_{G/H} \xrightarrow{\text{cor}} H_1(G; M) \xrightarrow{\text{coinf}} H_1(G/H; M_H) \to 0.$$

Let $1 \to H \to \mathrm{Sp}_{2n+1}(\mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z}) \to 1$ be our short exact sequence from above, and let M be the $\mathrm{Sp}_{2n+1}(\mathbb{Z})$-module $\mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q}$, where $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ acts on $\mathrm{St}_n^\omega(\mathbb{Q})$ by its restriction to $\mathrm{Sp}_{2n}(\mathbb{Z})$, and acts trivially on $\mathbb{Q}$. Let $\mathrm{St}_{n,\mathbb{Q}}^\omega$ denote $\mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q}$. The five term exact sequence becomes

$$H_2(\mathrm{Sp}_{2n+1}; \mathrm{St}_{n,\mathbb{Q}}^\omega) \to H_2(\mathrm{Sp}_{2n}; (\mathrm{St}_{n,\mathbb{Q}}^\omega)_H) \to (H_1(H; \mathrm{St}_{n,\mathbb{Q}}^\omega))_{\mathrm{Sp}_{2n}} \to H_1(\mathrm{Sp}_{2n+1}; \mathrm{St}_{n,\mathbb{Q}}^\omega) \to H_1(\mathrm{Sp}_{2n}; (\mathrm{St}_{n,\mathbb{Q}}^\omega)_H) \to 0.$$

Recall that H acts trivially on $\mathrm{St}_{n,\mathbb{Q}}^\omega$, so the right-most term is $H_1(\mathrm{Sp}_{2n}; (\mathrm{St}_{n,\mathbb{Q}}^\omega)_H) = H_1(\mathrm{Sp}_{2n}; \mathrm{St}_{n,\mathbb{Q}}^\omega) = 0$ by Theorem A of [5]. It is now enough to show that $(H_1(H; \mathrm{St}_{n,\mathbb{Q}}^\omega))_{\mathrm{Sp}_{2n}(\mathbb{Z})} = 0$. As $\mathrm{St}_{n,\mathbb{Q}}^\omega$ is a trivial H-module, the universal coefficient theorem gives an (NEED TO VERIFY) $\mathrm{Sp}_{2n}(\mathbb{Z})$-equivariant isomorphism

$$H_1(H; \mathrm{St}_{n,\mathbb{Q}}^\omega) \cong H_1(H) \otimes \mathrm{St}_{n,\mathbb{Q}}^\omega$$

where $\mathrm{Sp}_{2n}(\mathbb{Z})$ acts diagonally on the right-hand side. The commutator of H is also its center, so $H_1(H) = \mathrm{Ab}(H) = H/[H,H] = H/\Gamma \cong \mathbb{Z}^{2n}$. Since $\mathrm{Sp}_{2n}(\mathbb{Z})$ fixes Γ, the isomorphism $H_1(H) \cong \mathbb{Z}^{2n}$ is $\mathrm{Sp}_{2n}(\mathbb{Z})$-equivariant, where $\mathrm{Sp}_{2n}(\mathbb{Z})$ acts on $\mathbb{Z}^{2n}$ by left matrix multiplication. It is now clear that the $\mathrm{Sp}_{2n}(\mathbb{Z})$-coinvariants of $H_1(H; \mathrm{St}_{n,\mathbb{Q}}^\omega)$ vanish: let $\sigma = [v_1, w_1, \ldots, v_n, w_n]$ be an apartment class in $\mathrm{St}_n^\omega(\mathbb{Q})$, and let $A_\sigma = -I_{2n} \in \mathrm{Sp}_{2n}(\mathbb{Z})$. Then $A_\sigma \cdot \sigma = \sigma$ (again, use BPS pages 83-84 to justify this using Gunnels), so for an element $\vec{v} \otimes \sigma \otimes q \in H_1(H; \mathrm{St}_{n,\mathbb{Q}}^\omega)$, we have $A_\sigma \cdot (\vec{v} \otimes \sigma \otimes q) = -\vec{v} \otimes \sigma \otimes q$. □

2 Conjecture: Double symplectic Tits building?

In Miller-Patzt-Putman [11], they prove that sufficiently high connectivity of the double Tits building $\mathcal{T}^2(\mathbb{Z}^n)$ will give a sufficiently long partial resolution of $\mathrm{St}(\mathrm{GL}_n;\mathbb{F})$ or $\mathrm{St}(\mathrm{SL}_n;\mathbb{F})$ for $\mathbb{F}$ a commutative ring from which one can then inductively calculate $H_i(\mathrm{GL}_n(\mathbb{Z});\mathrm{St}(\mathrm{GL}_n;\mathbb{F}))$ or $H_i(\mathrm{SL}_n(\mathbb{Z});\mathrm{St}(\mathrm{SL}_n;\mathbb{F}))$. See the proof of Theorem C in their paper. In particular, they show that $\mathcal{T}^2(\mathbb{Z}^n)$ is n-connected for $n \geqslant 4$, which is sufficient for showing that $H_i(\mathrm{SL}_n(\mathbb{Z});\mathrm{St}_n(\mathbb{Q})\otimes\mathbb{Q}) = 0$ for $i \leqslant 2$ for $n \geqslant 4$, as was proven by several people in the past!

Is someone considering an analogue for symplectic groups? Would need to define a double symplectic Tits building $\mathcal{T}^2_\omega(\mathbb{Z}^{2n})$ and show that its connectivity would imply vanishing of homology. Then it suffices to argue that this connectivity actually holds (hard). Maybe clear that it would hold in small cases? Analogous to the proof of theorem D in [11] or something?

Ok the proof of Thm C looks very difficult!!!

3 Principal congruence subgroups

Definition 3.1 (Level m principal congruence subgroup)**.** Let $\mathrm{Sp}_{2n+1}(\mathbb{Z})[m]$ (or $\mathrm{Sp}_{2n+1}[m]$ if the coefficients $\mathbb{Z}$ are assumed) be the *level m principal congruence subgroup* of Sp_{2n+1}, that is,

$$\mathrm{Sp}_{2n+1}[m] = \ker(\mathrm{Sp}_{2n+1}(\mathbb{Z}) \xrightarrow{\bmod m} \mathrm{Sp}_{2n+1}(\mathbb{Z}/m\mathbb{Z})).$$

Equivalently, $\mathrm{Sp}_{2n+1}[m] = \{B \in \mathrm{Sp}_{2n+1}(\mathbb{Z}) : B \equiv I_{2n+2} \mod m\}$.

Remark 3.2. For general m, the mod m map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$ is only surjective when $n = 2, 3$, since there exist matrices A that are invertible over $\mathbb{Z}/m$ ($\det A \in (\mathbb{Z}/m)^\times$) which do not have lifts invertible over $\mathbb{Z}$ (which requires $\det A = \pm 1$). For Sp_{2n+1}, the mod m is surjective by [3] Prop 2.6.

The semidirect product structure of Sp_{2n+1} carries through to its principal congruence subgroups: For H the Heisenberg subgroup of Sp_{2n+1}, let $H[m]$ be the kernel of $H \hookrightarrow \mathrm{Sp}_{2n+1}(\mathbb{Z}) \xrightarrow{\bmod m} \mathrm{Sp}_{2n+1}(\mathbb{Z}/m)$. It is clear that the projection $\pi : \mathrm{Sp}_{2n+1}[m] \to \mathrm{Sp}_{2n}[m]$, $\begin{pmatrix} 1 & & \\ \vec{v} & A & \\ \alpha & -\vec{v}^T JA & 1 \end{pmatrix} \mapsto A$ has kernel $H[m]$, and a section $\sigma : \mathrm{Sp}_{2n}[m] \to \mathrm{Sp}_{2n+1}[m], A \mapsto \begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix}$. Hence the short exact sequence

$$1 \to H[m] \to \mathrm{Sp}_{2n+1}[m] \to \mathrm{Sp}_{2n}[m] \to 1 \tag{1}$$

splits. Note that $H[m]$ is a finite-index subgroup of the orientable Poincaré duality group H ($H/H[m]$ is isomorphic to a subgroup of $\mathrm{GL}_{2n+2}(\mathbb{Z}/m)$, a finite group), and so is also a Poincaré duality group.

The same arguments used to compute the top rational cohomology of $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ can now be used to compute the top rational cohomology of $\mathrm{Sp}_{2n+1}(\mathbb{Z})[m]$. In particular, if we consider the LHS spectral sequence associated to our short exact sequence in (1), we obtain

$$E_2^{p,q} = H^p(\mathrm{Sp}_{2n}[m]; H^q(H[m];\mathbb{Q})) \implies H^{p+q}(\mathrm{Sp}_{2n+1}[m];\mathbb{Q}).$$

As $\mathrm{cd}(\mathrm{Sp}_{2n}[m]) = n^2$ and $\mathrm{cd}(H[m]) = 2n+1$, $E_2^{p,q} = 0$ for $p > n^2$ and $q > 2n+1$. Thus the $p+q = (n+1)^2$ diagonal on the E_2 page vanishes except possibly at $E_2^{n^2,2n+1} = H^{n^2}(\mathrm{Sp}_{2n}[m]; H^{2n+1}(H[m];\mathbb{Q}))$. Every differential d_r for $r \geqslant 2$ originating or ending at $E_2^{n^2,2n+1}$ must have either source or target 0, so that $E_2^{n^2,2n+1} \cong E_\infty^{n^2,2n+1}$. As we have no extension problems, this latter group is isomorphic to $H^{(n+1)^2}(\mathrm{Sp}_{2n+1}[m];\mathbb{Q})$.

Sicne $\mathrm{Sp}_{2n}[m]$ is an n^2-dimensional duality group with dualizing module $\mathrm{St}_n^\omega(\mathbb{Q})$, we have

$$E_2^{n^2,2n+1} = H^{n^2}(\mathrm{Sp}_{2n}[m]; H^{2n+1}(H[m];\mathbb{Q})) \cong (\mathrm{St}_n^\omega(\mathbb{Q}) \otimes H^{2n+1}(H[m];\mathbb{Q}))_{\mathrm{Sp}_{2n}[m]} \tag{2}$$

where $\mathrm{Sp}_{2n}[m]$ acts diagonally on the tensor product. The center of $H[m]$ is the subgroup of matrices $\Gamma[m] = \{\begin{pmatrix} 1 & & \\ \vec{0} & I & \\ \alpha & \vec{0} & 1 \end{pmatrix} : \alpha \in p\mathbb{Z}\} \cong p\mathbb{Z} \cong \mathbb{Z}$, and the quotient $H[m]/\Gamma[m]$ can similarly be seen to be $(p\mathbb{Z})^{2n} \cong \mathbb{Z}^{2n}$. As in the proof of Lemma 1.10, the LHS spectral sequence associated to the ($\mathrm{Sp}_{2n}[m]$-equivariant) central extension $1 \to \Gamma[m] \to H[m] \to H[m]/\Gamma[m] \to 1$ allows us to conclude that $H^{2n+1}(H[m];\mathbb{Q}) \cong H^{2n}(\mathbb{Z}^{2n};\mathbb{Q}) \cong \mathbb{Q}$ as $\mathrm{Sp}_{2n}[m]$-modules. Since $\mathrm{Sp}_{2n}(\mathbb{Z})$ acts by the determinant (and hence trivially) on $H^{2n}(\mathbb{Z}^{2n};\mathbb{Q})$, the same holds for the restriction to $\mathrm{Sp}_{2n}[m]$. Thus (2) can be written as

$$H^{(n+1)^2}(\mathrm{Sp}_{2n+1}[m];\mathbb{Q}) \cong (\mathrm{St}_n^\omega(\mathbb{Q}) \otimes \mathbb{Q})_{\mathrm{Sp}_{2n}[m]}.$$

But the latter group is isomorphic to $H^{n^2}(\mathrm{Sp}_{2n}[m];\mathbb{Q})$ by Borel-Serre duality. The following is a direct corollary of CITECapivolla-Searle

Prop 3.3. *Let $p \geqslant 3$ be prime. Then $H^{(n+1)^2}(\mathrm{Sp}_{2n+1}[p];\mathbb{Q}) \twoheadrightarrow \tilde{H}_{n-1}(\mathcal{T}_n^\omega(\mathbb{Q})/\mathrm{Sp}_{2n}[p])$.*

Proof. By the above discussion, $H^{(n+1)^2}(\mathrm{Sp}_{2n+1}[p];\mathbb{Q}) \cong H^{n^2}(\mathrm{Sp}_{2n}[p];\mathbb{Q})$. The latter groups surjects on $\tilde{H}_{n-1}(\mathcal{T}_n^\omega(\mathbb{Q})/\mathrm{Sp}_{2n}[p])$ by CITE. ◻

Work in progress by Urshita CITEshows that when $m = 3$, we have $H^{n^2}(\mathrm{Sp}_{2n}[3];\mathbb{Q}) \cong \mathrm{St}_n^\omega(\mathbb{F}_3)$. As an immediate corollary, we obtain

Corollary 3.4. $H^{(n+1)^2}(\mathrm{Sp}_{2n+1}[3];\mathbb{Q}) \cong \mathrm{St}_n^\omega(\mathbb{F}_3)$.

4 Braid groups and their congruence subgroups

5 Surjectivity of odd symplectic representation

This is known actually!! See [13] section 2.3! Paper by Church proves this apparently!

Only need to consider transvections (see Farb-Margalit thm 6.1).

6 Key player #1: bidecorated surfaces

We start with a category of bidecorated surfaces $\mathcal{M}_2$, defined in Harr-Vistrup-Wahl [7]:

Definition 6.1 (Bidecorated surfaces)**.** A bidecorated surface S is a surface with two marked intervals I_0 and I_1 in its boundary (can lie on the same or different boundary components).

Definition 6.2 (The category $\mathcal{M}_2$)**.** Objects in $\mathcal{M}_2$ are bidecorated surfaces. Morphisms in $\mathcal{M}_2$ are mapping classes, i.e., isotopy classes of homeomorphisms. (Recall: an isotopy is a homotopy of homeomorphisms which is a homeomorphism at each time t.) Obtain a *monoidal structure* by identifying the marked intervals of surfaces in pairs.

Example 6.3 (Bidecorated disk D)**.** Taking sums of the unit disk D with itself, we obtain a surface of any genus with either 1 or 2 boundary components:

$$
\begin{aligned}
D^{\#2g+1} &= S_{g,1} \\
D^{\#2g+2} &= S_{g,2}
\end{aligned}
$$

To obtain a surface with genus g, b boundary components, and s punctures ($S^s_{g,b}$), we take $S\#D^{\#2g}$ for $S = S^s_{0,b}$ (genus 0 surface with b boundary components and s punctures). In particular, note that $D = S_{0,1}$, $D\#D = S_{0,2}$, and $D\#D\#D = S_{1,1}$:

Get a special sequence of bidecorated surfaces when we take sums of the disk, as follows: Let $X_1 = X = D^2 \subset \mathbb{C}^2$ be the unit disk, and embed two marked intervals

$$
\begin{aligned}
I_0 = \iota^0_1(t) &= e^{i(\frac{\pi}{4}+t\frac{\pi}{2})} \quad 0 \leqslant t \leqslant 1 \\
I_1 = \iota^1_1(t) &= e^{i(\frac{5\pi}{4}+t\frac{\pi}{2})} \quad 0 \leqslant t \leqslant 1.
\end{aligned}
$$

Denote by $\overline{\iota^i_1} : I \to X_1$ the reversed map $t \mapsto \iota^i_1(1-t)$. Construct $X_{m+1} = X_1^{\#m+1}$ inductively via

$$
X_{m+1} = \frac{X_m \sqcup X_1}{\iota^i_m(t) \sim \overline{\iota^i_1(t)}} \qquad \frac{1}{2} \leqslant t \leqslant 1,\ \ i = 0, 1.
$$

Define the new marked intervals by $\iota^i_{m+1}(t) = \begin{cases} \iota^i_m(t) & t \leqslant \frac{1}{2} \\ \iota^i_1(t) & \text{else.} \end{cases}$ (That is, to construct X_{m+1}, we take our space X_m and attach the *first half* of I_0 in X_1 backwards onto the *second half* of I_0 in X_m, and similarly attach the *first half* of I_1 in X_1 backwards onto the *second half* of I_1 in X_m. The new marked intervals in X_{m+1} consist of the first half interval of X_m and the second half interval of X_1, for each of I_0, I_1.)

Note: every *other* time, the marked intervals will lie in different boundary components of X_{m+1}.

Lemma 6.4 (Lemma 3.1 in [7])**.** *Let $m \geqslant 1$. Then $X_m \cong S_{g,b}$ is a surface of genus g and b boundary components, where*

$$
(g, b) = \begin{cases} (\frac{m}{2} - 1, 2) & m \text{ is even} \\ (\frac{m-1}{2}, 1) & m \text{ is odd} \end{cases}
$$

Proof. Each X_m is a connected surface for each m, since $X_1 = D$ is a disk and each X_m is obtained by successively attaching strips. Adding a strip decreases the Euler characteristic by 1 (adding a 2-cell and two 1-cells, so changes Euler characteristic by $-2 + 1 = -1$), and $\chi(X_1) = 1$, so $\chi(X_m) = 1 - m$. By the classification of surfaces, it suffices to compute the number of boundary components ($\chi(S^s_{g,b}) = 2 - 2g - (b + s)$). When we glue a disk onto a surface whose two intervals lie on the same boundary component, then the new marked intervals now lie on different boundary components (and vice versa). We never create boundary components without marked intervals. ◻

7 Key player #2: formed spaces with boundary

We start with a category $\mathcal{F}_\partial$ of *formed spaces with boundary*, defined in Sierra-Wahl [12].

Definition 7.1 (Even symplectic groups, hyperbolic space)**.** Let R be a commutative ring. Define $\mathrm{Sp}_{2n}(R) = \mathrm{Aut}(\mathcal{H}^{\oplus n})$, where $\mathcal{H} = (R^2, \lambda_{\mathcal{H}})$ is the hyperbolic space of R^2 equipped with a *non-degenerate, alternating* form $\lambda_{\mathcal{H}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Definition 7.2 (Formed space)**.** A pair (M, λ) with M and R-module and λ an alternating form is called a formed space.

Traditionally we study homological stability of symplectic groups by taking a direct sum with $\mathcal{H}$ in the category of formed spaces. But then the slope of the stable range is not ideal! Want the stability to happen *slower*, so we introduce *formed spaces with boundary* and consequently the *odd symplectic groups*. This includes the additional data of a linear map $\partial : M \to R$ which we think of as a boundary, and yields a rank 1 stabilization map $\#X$ for $X = (R, 0, \mathrm{id})$ of $\mathcal{F}_\partial$.

We'll get an identification $\mathrm{Sp}_{2n}(R) \cong \mathrm{Aut}(X^{\#2n+1})$ while the intermediate odd symplectic group $\mathrm{Sp}_{2n-1}(R) = \mathrm{Aut}(X^{\#2n})$ identifies with a parabolic subgroup in $\mathrm{Sp}_{2n}(R)$ (that stabilizing a unimodular vector in R^{2n}).

Definition 7.3 (Category of formed spaces with boundary)**.** The objects of $\mathcal{F}_\partial$ are triples (M, λ, ∂) where M is an R-module, λ is an alternating bilinear form ($\lambda \in (\bigwedge^2 M)^\vee$), and $\partial : M \to R$ is a linear map ($\partial \in M^\vee$). Morphisms are structure-preserving module maps ($\varphi \in \mathrm{Aut}(M) = \mathrm{Aut}(M, \lambda, \partial)$ must be an R-module automorphism of M, satisfy $\lambda(m, m') = \lambda(\varphi(m), \varphi(m'))$, and $\partial(m) = \partial(\varphi(m))$).

Define a monoidal structure as follows:

$$(M_1, \lambda_1, \partial_1)\#(M_2, \lambda_2, \partial_2) = (M_1 \oplus M_2, \lambda_1\#\lambda_2, \partial_1 + \partial_2)$$

where $\lambda_1\#\lambda_2$ is given by the matrix $\begin{pmatrix} \lambda_1 & \partial_1^T\partial_2 \\ -\partial_2^T\partial_1 & \lambda_2 \end{pmatrix}$.

Where does this monoidal structure come from? Well, we have a monoidal functor $\mathcal{M}_2 \to \mathcal{F}_\partial$ sending

$$(S, I_0, I_1) \to (H_1(S, I_0 \cup I_1), \lambda, \partial)$$

where λ is the intersection form on $H_1(S, I_0 \cup I_1)$, described as follows: notice that to a bidecorated surface (S, I_0, I_1) we can associate a new surface S^+ by gluing on a strip whose ends are attached to the intervals I_0, I_1. If H is the attached handle, the long exact sequence of the pair (S^+, H) (and the fact that $H_1(S^+, H) \cong$

$H_1(S, I_0 \cup I_1))$ yields that $H_1(S^+) \cong H_1(S, I_0 \cup I_1)$. We can define an intersection pairing on $H_1(S^+)$ using Poincaré duality and the cup product. Since S^+ is a surface (2-dimensional), we have

$$H_1(S^+; R) \cong H^1(S^+, \partial S^+; R) \qquad \text{(Poincaré duality)}$$

$$H^1(S^+, \partial S^+; R) \wedge H^1(S^+, \partial S^+; R) \xrightarrow{\lambda} H^2(S^+, \partial S^+; R) \cong H_0(S^+; R) \cong R. \qquad \text{(cup product+ Poincaré duality).}$$

The boundary map $\partial : H_1(S, I_0 \cup I_1) \to R$ comes from the reduced long exact sequence of the pair $(S, I_0 \cup I_1)$, since $\tilde{H}_0(I_0 \cup I_1) \cong R\langle b_1 - b_0 \rangle$ for b_0, b_1 the midpoints of I_0, I_1 respectively.

Another note about S^+: Since (S, I_0, I_1) is oriented data, the surface S^+ inherits an orientation. If we represent classes in $M = H_1(S, I_0 \cup I_1)$ using arcs and circles, then we get representatives in $H_1(S^+)$ by closing off arcs (which necessarily go from boundary I_0 to boundary I_1) using the core $I \times \{\frac{1}{2}\}$ in the handle.

Morphisms in $\mathcal{M}_2$ are mapping classes of homeomorphisms that preserve the boundary parameterizations (note: we only get a morphism between S and S' in $\mathcal{M}_2$ if the two are homeomorphic to begin with. There's actually more structure in $\mathcal{M}_2$ than what I've described, but we don't have to worry about it right now. In particular we see that $\mathcal{M}_2$ is a monoidal *groupoid*, as all morphisms are invertible). Morphisms in $\mathcal{F}_\partial$ are structure-preserving module maps. The monoidal functor sends a mapping class $\phi : S \to S$ to its induced map on first homology $\phi_* : H_1(S, I_0 \cup I_1) \to H_1(S, I_0 \cup I_1)$. Note that this is well-defined on relative homology because the mapping classes must preserve the boundary components of S, so $I_0 \cup I_1$ maps to $I_0 \cup I_1$. Since mapping classes preserve the intersection form on $H_1(S)$– READ ABOUT THIS IN CHAPTER 6 OF FARB-MARGALIT TONIGHT‼, we see that the image of a mapping class under the functor preserves λ. Why must it also preserve ∂?

We claimed that this functor is monoidal, so we need to check that

$$\begin{aligned}(S_1 \# S_2, I_0, I_1) &\mapsto (H_1(S_1, I_0^1 \cup I_1^1), \lambda_1, \partial_1) \# (H_1(S_2, I_0^2 \cup I_1^2), \lambda_2, \partial_2) \\ &= (H_1(S_1, I_0^1 \cup I_1^1) \oplus H_1(S_2, I_0^2 \cup I_1^2), \lambda_1 \# \lambda_2, \partial_1 + \partial_2).\end{aligned}$$

1. $H_1(S_1 \# S_2, I_0 \cup I_1) \cong H_1(S_1, I_0^1 \cup I_1^1) \oplus H_1(S_2, I_0^2 \cup I_0^2)$.

   Proof. Recall that $H_1(S_1 \# S_2, I_0 \cup I_1) \cong H_1((S_1 \# S_2)^+)$. So it suffices to show that $H_1((S_1 \# S_2)^+) \cong H_1(S_1^+) \oplus H_1(S_2^+)$. Indeed, if we decompose the handle H as $H = H_\ell \cup_C H_r$ for $H = I \times I$, $H_\ell = I \times [0, \frac{1}{2}]$, $H_r = I \times [\frac{1}{2}, 1]$, and $C = I \times \{\frac{1}{2}\}$ (the core), then we see that

   $$(S_1 \# S_2)^+ = (S_1 \cup H_\ell) \cup_C (S_2 \cup H_r) = S_1^+ \cup_{H_r^1 \sim H_\ell^2} S_2^+.$$

   Since $S_1^+ \cap S_2^+ \neq \emptyset$, the reduced version of the Mayer-Vietoris sequence yields

   $$H_1(H_r^1) = 0 \to H_1(S_1^+) \oplus H_1(S_2^+) \xrightarrow{\cong} H_1((S_1 \# S_2)^+) \to \tilde{H}_0(H_r^1) = 0$$

   where the isomorphism is induced by inclusions. □

2. The intersection form of $(S_1 \# S_2)^+$ is given by $\begin{pmatrix} \lambda_1 & \partial_1^T \partial_2 \\ -\partial_2 \partial_1 & \lambda_2 \end{pmatrix}$.

   Proof. Let $\alpha_1, \ldots, \alpha_m, \rho_1$ be a basis for $H_1(S_1, I_0^1 \cup I_1^1)$, where the α_i are simple closed curves with trivial boundary (think: in S_g, we have $2g$ such curves α_i corresponding to each hole (genus), as pictured below), and ρ_1 is an arc from b_0 to b_1. Similarly let $\alpha'_1, \ldots, \alpha'_n, \rho_2$ be a basis for $H_1(S_2, I_0^2 \cup I_1^2)$. The identification $H_1(S_i, I_0^i \cup I_1^i) \cong H_1(S_i^+)$ sends each basis element α_j to itself, and maps ρ_i to $\rho_i \cup C$ (completes the arc to a loop using the core of the handle).

   In these bases, only the last elements have a potential intersection in $(S_1 \# S_2)^+$. So on all of the other generators, the intersection pairing of the sum is correct (and there the boundaries are trivial, so the extra ∂ factors contribute nothing). Note that by our definition of $\lambda_1 \# \lambda_2$, we should have

$$(\lambda_1 \# \lambda_2)(\rho_1 + 0, 0 + \rho_2) = \lambda_1(\rho_1, 0) + \lambda_2(0, \rho_2) + \partial_1(\rho_1)\partial_2(\rho_2) - \partial_1(0)\partial_2(0) = 1$$

   since both $\partial_1(\rho_1)$ and $\partial_2(\rho_2)$ give the generator of $\tilde{H}_0(I_0 \cup I_1) \cong R$. Indeed, taking transverse representatives of $\rho_1 \cup C$ and $\rho_2 \cup C$, we see that the two curves intersect once in $(S_1 \# S_2)^+$ in an orientation-preserving way, so the intersection pairing yields 1, as desired. □

3. Since the (relative) Mayer-Vietoris sequence and the long exact sequence of the pair are both natural, we have a commutative diagram

$$\begin{tikzcd} {} \arrow[r] & H_1(S_1, I_0^1 \cup I_1^1) \oplus H_1(S_2, I_0^2 \cup I_1^2) \arrow[r, "\cong"'] \arrow[d, "\partial_1+\partial_2"] & H_1(S_1 \# S_2, I_0 \cup I_1) \arrow[d, "\partial"] \\ {} \arrow[r] & \tilde{H}_0(I_0^1 \cup I_1^1) \oplus \tilde{H}_0(I_0^2 \cup I_1^2) \arrow[r] & \tilde{H}_0(I_0 \cup I_1) \end{tikzcd}$$

   which shows that $\partial = \partial_1 + \partial_2$.

7.1 Important subcategory generated by $X = (R, 0, \mathrm{id})$.

Note that the image of (D, I_0, I_1) under the monoidal functor is $(R, 0, \mathrm{id})$, since $H_1(D^2, I_0 \cup I_1)$ is generated by (the unique isotopy class of) the arc ρ from b_0 to b_1, which intersects trivially with itself, and satisfies $\partial(\rho) = b_1 - b_0$, which generates $\tilde{H}_0(I_0 \cup I_1)$. Similarly, the surface D^+ is a cylinder, and the generator $H_1(D^+)$ is represented by the middle circle of the cylinder.

Under this identification, the standard basis of $X^{\# n} = (R^n, \lambda, \partial)$ corresponds to a collection of arcs ρ_i going from I_0 to I_1, each through one of the disks composing $D^{\# n}$. Or we can get another basis by setting $\alpha_i = \rho_i \cdot \rho_{i+1}^{-1}$

(or in homological notation, $[\rho_i]-[\rho_{i+1}]$) for $i=1,\dots,n-1$, plus the arc ρ_n.

Remark 7.4. The formed space $(H_1(S, I_0\cup I_1),\lambda,\partial)$ contains $(H_1(S),\lambda|_{H_1(S)},0)$ as a subspace, where $H_1(S)$ identifies with $\ker\partial$ (generated by the curves which have a trivial boundary! i.e. those curves which just go around the genus, don't connect the intervals in the boundary component), and where $\lambda|_{H_1(S)}$ is the standard intersection pairing on $H_1(S)$.

Prop 7.5 (Prop 2.16 [12]). *There are isomorphisms of formed spaces with boundary*

$$\begin{aligned} X^{\#2g} &\cong (\mathcal{H}^{\oplus g},\partial=\lambda(e_g',-)) = (R^{2g},\lambda_{\mathcal{H}^{\oplus g}},\partial=\lambda(e_g',-)) \\ X^{\#2g+1} &\cong (\mathcal{H}^{\oplus g},0)\#X = (R^{2g}\oplus R,\lambda_{\mathcal{H}^{\oplus g}}\oplus 0,\partial=pr_R) \end{aligned}$$

where $e_1',f_1',\dots,e_g',f_g'$ is a symplectic basis for R^{2g}.

Proof. Slightly different proof than [12], motivated instead by the proof in [10]. We start with the case $g=1$, letting e_1,e_2 be the standard basis of R^2. From the description of $X\#X$ we obtain

$$X\#X=(R^2,\lambda=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\partial=\begin{pmatrix}1&1\end{pmatrix}).$$

Let $e_1'=e_1-e_2$, $f_1'=e_2$. In this new basis, we have that λ still looks like $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, so that e_1',f_1' form a symplectic basis. On the other hand, the boundary map ∂ now looks like $\partial=\begin{pmatrix}0&1\end{pmatrix}$, which means we can identify it with $\lambda(e_1',-)$.

Under the standard basis e_1,e_2,e_3 for R^3, we see that

$$X^{\#3}=(R^3,\lambda=\begin{pmatrix}0&1&1\\-1&0&1\\-1&-1&0\end{pmatrix},\partial=\begin{pmatrix}1&1&1\end{pmatrix}).$$

Under a change of basis $e_1'=e_1-e_2$, $f_1'=e_2-e_3$, $e_2'=e_1-e_2+e_3$, we see that λ becomes $\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}=\lambda_{\mathcal{H}}\oplus 0$. The boundary now becomes $\partial=\begin{pmatrix}0&0&1\end{pmatrix}$, which is just the projection onto e_2'. In particular, we see that $\mathcal{H}$ is identified with $\ker\partial$. UNDERSTAND WHAT IT HAPPENING TOPOLOGICALLY TO OUR GENERATING CURVES UNDER THIS CHANGE OF BASIS. ALSO WHERE IS THIS CHANGE OF BASIS COMING FROM??

Since $X^{\#4}=X^{\#3}\#X=(\mathcal{H},0)\#X\#X\cong(\mathcal{H},0)\#X^{\#2}$, we have a basis e_1',f_1',e_2',f_2' for R^4 such that $\lambda=\mathcal{H}^{\oplus 2}$ and $\partial=\lambda(e_2',-)$.

Since $X^{\#5}=X^{\#3}\#X^{\#2}\cong(\mathcal{H},0)\#X\#X^{\#2}\cong(\mathcal{H},0)\#X^{\#3}$, we have a basis e_1',f_1' for $(\mathcal{H},0)$ and a basis e_2',f_2',e_3' for $X^{\#3}$ such that $\lambda=\lambda_{\mathcal{H}^{\oplus 2}}\oplus 0$ and $\partial=pr_{e_3'}$. The rest continues inductively. ∎

Lemma 7.6 (Prop 2.17 in [12]).

$$\begin{aligned} \mathrm{Aut}(X^{\#2g}) &\cong \mathrm{Stab}_{\mathrm{Sp}_{2g}(R)}(e_g') =: \mathrm{Sp}_{2g-1}(R) \\ \mathrm{Aut}(X^{\#2g+1}) &\cong \mathrm{Sp}_{2g}(R). \end{aligned}$$

Proof. Automorphisms φ of $X^{\#2g}$ must preserve the alternating bilinear form $\lambda = \mathcal{H}^{\oplus g}$, and hence lie in the subgroup $\mathrm{Sp}_{2g}(R)$. Furthermore, they must preserve the boundary map:

$$\begin{aligned}\lambda(\varphi(e'_g), \varphi(x)) = \lambda(e'_g, x) = \partial(x) = \partial(\varphi(x)) = \lambda(e'_g, \varphi(x)) \ \forall x \in R^{2g} \\ \implies \lambda(\varphi(e'_g) - e'_g, \varphi(x)) = 0 \ \forall x \in R^{2g}.\end{aligned}$$

Since λ is nondegenerate on R^{2g}, this equality holds if and only if $\varphi(e'_g) = e'_g$. Hence $\mathrm{Aut}(X^{\#2g}) \cong \mathrm{Stab}_{\mathrm{Sp}_{2g}(R)}(e'_g) = \mathrm{Sp}_{2g-1}(R)$.

On the other hand, $\mathrm{Aut}(X^{\#2g+1}) \cong \mathrm{Aut}((\mathcal{H}^{\oplus g}, 0)\#X)$. Let $\lambda^\vee : M \to M^\vee$ be defined by $m \mapsto \lambda(m, -)$. Then $\partial^{-1}(1) \cap \ker \lambda^\vee$ is the single vector e'_{g+1} generating the X summand. This vector must be preserved by any automorphism, since if $\varphi \in \mathrm{Aut}(X^{\#2g+1})$, then $\lambda(e'_{g+1}, -) = \lambda(\varphi(e'_{g+1}), -)$ and $\partial(\varphi(e'_{g+1})) = \partial(e'_{g+1}) = 1$, so that $\varphi(e'_{g+1})$ also lies in $\partial^{-1}(1) \cap \ker \lambda^\vee = \{e'_{g+1}\}$. Moreover, $\mathcal{H}^{\oplus g} \cong M/\langle e'_{g+1}\rangle$ must also be preserved by any automorphism: the boundary map is just projection onto the last factor e'_{g+1}, giving an identification of $\mathcal{H}^{\oplus g}$ with $\ker \partial$ (and $\ker \partial$ must be preserved by automorphisms). Hence $\mathrm{Aut}(X^{\#2g+1}) \subseteq \mathrm{Aut}(\mathcal{H}^{\oplus g}) = \mathrm{Sp}_{2g}(R)$. For the other inclusion, if we start with an automorphism of $(\mathcal{H}^{\oplus g}, 0)$, then to get an automorphism of $X^{\#2g+1}$ we must fix the vector e'_{g+1} which generates X. □

8 Defining representation

Definition 8.1 (Defining representation, Def 3.5 in [10])**.** The defining representation of Sp_n is the homomorphism

$$\mathrm{Sp}_n \to \mathrm{GL}(\ker \partial) = \mathrm{GL}_n .$$

Note that when $n = 2g$ is even, we have $\mathrm{Sp}_{2g}(R) \cong \mathrm{Aut}(X^{\#2g+1}) \cong \mathrm{Aut}((\mathcal{H}^{\oplus g}, 0)\#X)$, so the defining representation is just the restriction of $T \in \mathrm{Sp}_{2g}(R)$ to $\mathcal{H}^{\oplus g}$ (as $\ker \partial$ identifies with $\mathcal{H}^{\oplus g}$). In matrix form (under the basis $e'_1, f'_1, \ldots, e'_g, f'_g, e'_{g+1}$, where the first $2g$ vectors span a symplectic subspace), we have that T is given by a $2g \times 2g$-block in $\mathrm{Sp}_{2g}(R)$ and a 1 in the bottom right corner, so as a map of matrices this is just the restriction of T to its upper left $2g \times 2g$ block. In particular, this map is injective, since $\varphi \in \mathrm{Aut}((\mathcal{H}^{\oplus g}, 0)\#X)$ must stabilize the X summand, and so is completely determined by its action on $\mathcal{H}^{\oplus g}$.

When $n = 2g + 1$ is odd, we have that $\mathrm{Sp}_{2g+1}(R) \cong \mathrm{Aut}(X^{\#2g+2})$. The boundary map ∂ is identified with the unimodular vector e'_{g+1}, so $\ker \partial = \langle e'_1, f'_1, \ldots, e'_{g+1}\rangle$. Reorder this basis to $e'_{g+1}, e'_1, \ldots, e'_g, f'_1, \ldots, f'_g, f'_{g+1}$. The defining representation thus sends $T \in \mathrm{Sp}_{2g-1}(R)$ to its restriction on $\ker \partial$ (in matrix form, we just forget the last row and column). Note that the image of this defining representation consists of invertible $(2g-1) \times (2g-1)$ matrices which preserve the (maximal rank, rk ker $= 1$) symplectic form on $e'_1, f'_1, \ldots, e'_{g-1}, f'_{g-1}$, which is precisely Proctor's group $\mathrm{Sp}^P_{2g-1}(R)$. is it clear that the image gives us the entire Proctor's group? Or do we just land in a subgroup? Need to check surjectivity. The defining representation is *not* injective: I haven't checked this carefully, but seems like you could have one matrix $A \in \mathrm{Sp}_{2g-1}(R)$ which sends f'_j to f'_j and one B which sends f'_j to $f'_j - e'_i$ for some i, for instance. In fact, can we just send f'_j to itself plus any linear combination of guys in $\ker \partial$?

9 Preliminaries

10 $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ is a rational duality group

Definition 10.1 (Variant of Proctor)**.** Proctor '88 [cite] defines $\mathrm{Sp}_{2n+1}(\mathbb{C})$ as the subgroup of matrices of $\mathrm{GL}_{2n+1}(\mathbb{C})$ which preserve a maximal rank skew-form on $\mathbb{C}^{n+1}$. In other words, we have a skew-form ω :

$\mathbb{C}^{2n+1} \times \mathbb{C}^{2n+1} \to \mathbb{C}$ and a basis $e_0, e_1, \overline{e_1}, \ldots, e_n, \overline{e_n}$ such that $\{e_i, \overline{e_i} : 1 \leqslant i \leqslant n\}$ pairs "symplectically" and $e_0 \in \ker \omega$. (Note: for ω to be maximal rank, need $\ker \omega$ to have dimension 1, so spanned by a single basis vector. This is because we have no vector for e_0 to pair with when the number of basis vectors is odd!) Replacing $\mathbb{C}$ with $\mathbb{Z}$, we can think of defining $\mathrm{Sp}^P_{2n+1}(\mathbb{Z})$ as the subgroup of matrices of $\mathrm{GL}_{2n+1}(\mathbb{Z})$ which preserve such a maximal rank skew-form on $\mathbb{Z}^{2n+1}$. Proctor lets $\mathrm{SSp}_{2n+1}(\mathbb{Z})$ be those matrices which fix the vector e_0. Other people (Gelfand and Zelinsky?) would call this the intermediate odd symplectic group $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$.

We can decompose $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ in terms of the standard $\mathrm{Sp}_{2n}(\mathbb{Z})$. Note: we can view $e_1, \overline{e_1}, \ldots, e_n, \overline{e_n}$ as a symplectic basis for $\mathbb{Z}^{2n}$ embedded inside $\mathbb{Z}^{2n+1}$ consisting of those matrices with 0^{th} coordinate equal to 0. The elements of $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ look like

$$\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z}) = \{\begin{pmatrix} 1 & \ell \\ 0 & A \end{pmatrix} : \ell \in \mathbb{Z}^{2n},\, A \in \mathrm{Sp}_{2n}(\mathbb{Z})\}.$$

The symplectic group $\mathrm{Sp}_{2n}(\mathbb{Z})$ embeds into $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ via $\ell = \vec{0}$. Let U be the subgroup of $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ given by the matrices with $A = I$:

$$U = \{\begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_n \\ & I_n & & \end{pmatrix} : \alpha_i \in \mathbb{Z}\}.$$

One can check (via multiplication of matrices) that U is isomorphic to the additive group $\mathbb{Z}^{2n}$. Can check that $U \trianglelefteq \widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ (NEED TO CHECK EXPLICITLY!) and that $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})/U \cong \mathrm{Sp}_{2n}(\mathbb{Z})$, viewed as a subgroup of $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ by setting $\ell = \vec{0}$, so the intermediate odd symplectic group $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ is the semidirect product $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z}) \cong U \rtimes \mathrm{Sp}_{2n}(\mathbb{Z})$. In other words, we have a split short exact sequence of groups

$$1 \to U \to \widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z}) \to 1. \tag{3}$$

Prop 10.2. *$\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$ is a rational duality group of dimension $n^2 + 2n$, with dualizing module $D = \mathrm{St}^{\omega}_n(\mathbb{Z})$.*

Proof. By Theorem 1.6 and the short exact sequence 3, it suffices to show that U and $\mathrm{Sp}_{2n}(\mathbb{Z})$ are rational duality groups. [In fact, we'll try to argue that they're virtual duality groups, which is even stronger!] Since $U \cong \mathbb{Z}^{2n}$, and $\mathbb{Z}^{2n}$ has the $(2n)$-fold torus $\mathbb{T}^{2n}$ as a classifying space, we see that U is a Poincaré duality group (and hence a virtual duality group, and hence a duality group over $\mathbb{Q}$). Since $\mathrm{Sp}_{2n}(\mathbb{Z})$ is a virtual duality group of dimension n^2 with virtual dualizing module $\mathrm{St}^{\omega}_n(\mathbb{Z})$, the conclusion now follows by Theorem 1.6. □

11 Computing cohomology of $\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})$

Prop 11.1. *Let $\nu_n := \mathrm{cd}_{\mathbb{Q}}(\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z})) = n^2 + 2n$ (vcd?). Then $H^{\nu_n}(\widetilde{\mathrm{Sp}}_{2n+1}(\mathbb{Z}); \mathbb{Q}) = 0$ for $n \geqslant 1$. might not be able to conclude for $\mathrm{Sp}_3(\mathbb{Z})$ since we don't have vanishing of top degree cohom of $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$?*

12 Relating $\mathrm{Sp}_{2n+1}(\mathbb{Z})$ and $\mathrm{Sp}_{2n+2}(\mathbb{Z})$.

Following Mihai '06 §3.6. Embed $\mathbb{Z}^{2n+1}$ as a "hyperplane" in $\mathbb{Z}^{2n+2}$ (take last coordinate to be zero in all our $\mathbb{Z}^{2n+1}$ vectors). We can extend ω (degenerate skew-form on $\mathbb{Z}^{2n+1}$) to a *symplectic* form $\tilde{\omega}$ on $\mathbb{Z}^{2n+2}$ (by adding a basis element $\overline{e_0}$ of $\mathbb{Z}^{2n+2}$ and defining $\tilde{\omega}$ such that it pairs $e_0, \overline{e_0}$ symplectically). Then $\mathrm{Sp}_{2n+2}(\mathbb{Z})$ will be the symplectic group fixing $\tilde{\omega}$. Let P be the parabolic subgroup of $\mathrm{Sp}_{2n+2}(\mathbb{Z})$ preserving the line $R = \ker \omega = \langle e_0 \rangle$.

Claim 12.1. *P also preserves the orthogonal complement $E = \langle e_0, e_1, \overline{e_1} \dots, e_{2n}, \overline{e_{2n}} \rangle$ of R.*

Proof. (Note: $\overline{e_0}$ does not lie in the orthogonal complement, since $\langle e_0, \overline{e_0} \rangle = 1$.) Suppose there exists $e \in E$ such that $P(e) \notin E$. Then $\mathbb{Z}^{2n+2} = \langle \overline{e_0} \rangle \oplus E$, so we can express $P(e) = \alpha \overline{e_0} + \beta f$ for some $\alpha \neq 0$ and $f \in E$. But then $0 = \tilde{\omega}(e_0, e) = \tilde{\omega}(Pe_0, Pe) = \tilde{\omega}(Pe_0, \alpha\overline{e_0} + \beta f) \neq 0$ (since $Pe_0 \neq 0$, as P is invertible). $\Rightarrow\Leftarrow$ □

If $g \in P$, then $g|_E$ preserves ω, since $\tilde{\omega}|_E = \omega$. Thus any $g|_E \in \mathrm{Sp}_{2n+1}(\mathbb{Z})$ for $g \in P$.

Prop 12.2 (Mihai '06 Prop 3.7 [cite])**.** *The morphism $P \twoheadrightarrow \mathrm{Sp}_{2n+1}(\mathbb{Z})$ given by $g \mapsto g|_E$ is surjective.*

Proof. Note: Mihai proves this for $\mathbb{C}$, not for $\mathbb{Z}$. Will the proof work over $\mathbb{Z}$ instead? □

Mihai '06
Gelfand and Zelevinski '84
Shtepin' 93

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