MODULAR PRINCIPAL SERIES REPRESENTATION OF GL_2 OVER FINITE RINGS

GAHL SHEMY

Abstract. Given any prime $p \geq 3$, $r \in \mathbb{N}$, and character χ on the Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$, we construct a Jordan-Hölder series for the modulo p reduction of the principal series representation of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$. As a corollary we provide the semisimplifications of all characteristic p principal series representations of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$, and explain a process to compute such semisimplifications in small cases by the means of Brauer characters, verifying the computation from the constructed Jordan-Hölder series.

1. Introduction

In representation theory one often wishes to determine how the irreducible representations of a group "fit together" in the composition of some other representation of concern. Under sufficiently nice conditions this problem is completed solved: Given a finite group G and a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over a field of characteristic not dividing the order of G, the classical Maschke's theorem guarantees that ρ is *completely reducible*, meaning it can be uniquely expressed as a direct sum of irreducible representations of the group G, up to isomorphism. Maschke's theorem no longer holds when V is over a field of characteristic p and p divides the order of the group, so a different method is required in order to determine how the irreducible modular representations of a finite group G make up some other representation of interest. This may be done through investigating Jordan-Hölder series of the representation, which are filtrations

$$0 \subset V_1 \subset \cdots \subset V_d = V$$

of subrepresentations with inclusions being proper and maximal, so that each composition factor V_{i+1}/V_i is isomorphic to an irreducible representation of G. The Jordan-Hölder Theorem states that such composition series need not be unique, but that the *set* of composition factors of a representation, known as the irreducible constituents, is unique. We can then define

$$V^{\mathrm{ss}} := \bigoplus_{i=0}^{d-1} V_{i+1}/V_i \tag{1}$$

to be the *semisimplification* of V. Since each quotient V_{i+1}/V_i is isomorphic to an irreducible representation of G, we have

$$V^{\mathrm{ss}} = \bigoplus_{j} \rho_j^{d_j} \tag{2}$$

where ρ_j is an irreducible representation of G and d_j is its multiplicity in the semisimplification of V. A consequence of the Jordan-Hölder theorem is that V^{ss} is unique up to rearrangement of factors in the direct sum, so V^{ss} is unique up to isomorphism.

Fixing a prime $p \geq 3$, we consider the non-archimedean local field $L = \mathbb{F}_p((t))$. The ring of integers $\mathcal{O}_L$ is given by $\mathbb{F}_p[[t]]$ and consists of all formal power series in t with coefficients in $\mathbb{F}_p$, with a unique maximal ideal generated by t. For any $r \in \mathbb{N}$ we consider the general linear group $GL_2(\mathbb{F}_p[t]/(t^r))$, which we henceforth denote by G_r.

The choice of $L = \mathbb{F}_p((t))$ puts us in the equal characteristic setting, where L has the same characteristic as its residue field $\mathbb{F}_p$. For work done in the mixed characteristic setting, see the appendix in [4].

Date: April 11, 2023.

Key words and phrases. rep theory.

Given the finite group G_r, let $B_r \leq G_r$ denote the *Borel subgroup* of G_r consisting of 2×2 upper triangular invertible matrices with entries in $\mathbb{F}_p[t]/(t^r)$. Fixing a field E of characteristic 0 whose residue field $k_E = \mathcal{O}_E/(\varpi_E)$ is of characteristic p, let $\chi_1, \chi_2 : (\mathbb{F}_p[t]/(t^r))^\times \to E^\times$ be group homomorphisms, and define

$$\begin{aligned} \chi : B_r &\to E^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto \chi_1(a)\chi_2(d). \end{aligned}$$

The *principal series representation* of G_r associated to χ is the induced representation $\mathrm{Ind}_{B_r}^{G_r}(\chi)$, a vector space

$$\mathrm{Ind}_{B_r}^{G_r}(\chi) := \{f : G_r \to E \mid f(bg) = \chi(b)f(g) \quad \forall g \in G_r, b \in B_r\} \tag{3}$$

with a G_r-action given by

$$\begin{aligned} \vartheta_\chi : G_r &\to \mathrm{GL}(\mathrm{Ind}_{B_r}^{G_r}(\chi)) \\ (x \cdot f)(g) &= f(gx) \end{aligned} \tag{4}$$

for all $x, g \in G_r, f \in \mathrm{Ind}_{B_r}^{G_r}(\chi)$. This paper explores the modulo p reduction of the principal series representation, where χ now maps to $k_E = \mathcal{O}_E/(\varpi_E) \cong \overline{\mathbb{F}_p}$ and where all maps $f \in \mathrm{Ind}_{B_r}^{G_r}(\chi)$ have codomain k_E. Hereafter we abuse notation and write $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ to mean the principal series representation after reduction modulo p. Hence $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ is a characteristic p vector space of dimension $[G_r : B_r] \cdot \dim(\chi) = (p+1)p^{r-1}$, with a G_r-action still given by (4).

As the $r = 1$ case is well-studied, the main result of the paper is an inductive construction of a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ which terminates in $\mathrm{Ind}_{B_1}^{G_1}(\chi)$.

Proposition 1.1. *Let $p \geq 3$ be a prime, let $r \in \mathbb{N}_{\geq 2}$, and let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character. There exists a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ given by*

$$0 \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \cdots \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \mathrm{Ind}_{B_r}^{G_r}(\chi), \tag{5}$$

where $I_r^{r-1} := \{\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d \end{smallmatrix}\right] \in G_r : c \in \mathbb{F}_p\}$, $\sigma := \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$, and $\sigma^{(k)}$ is an I_r^{r-1}-invariant k-dimensional subspace of σ.

In §3 we give a precise description of the k-dimensional subspaces $\sigma^{(k)}$ and use their construction to prove the main result, shown in §4:

Theorem 1.1. *For the I_r^{r-1}-invariant k-dimensional subspaces $\sigma^{(k)}$ satisfying Prop 1.1, we have*

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r} \mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a_0}{d_0})^k) \tag{6}$$

for $0 \leq k \leq p-1$, where $\chi \cdot (\frac{a_0}{d_0})^k$ is the character $\chi \cdot (\frac{a_0}{d_0})^k : B_r \to \overline{\mathbb{F}_p}^\times$ mapping $\left[\begin{smallmatrix} a_{r-1}t^{r-1}+\cdots+a_0 & b \\ 0 & d_{r-1}t^{r-1}+\cdots+d_0 \end{smallmatrix}\right] \mapsto \chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) \cdot (a_0 d_0^{-1})^k$.

Theorem 1.1 implies that the filtration in Prop 1.1 may be refined inductively to a filtration in terms of $\mathrm{Ind}_{B_1}^{G_1}(\psi)$ for varying characters ψ, which may be then further refined to a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ using the known Jordan-Hölder series for $\mathrm{Ind}_{B_1}^{G_1}(\psi)$.

In §2 we provide preliminaries, and in §5 we give a corollary of the main theorem regarding semisimplification numbers. Finally, since determining the semisimplification of a given representation can be done without a Jordan-Hölder series via a computational process using Brauer characters, we compute a small example using this method in §6, and show that the semisimplification matches with what is deduced from our main theorem.

2. Preliminaries

2.1. **Basic Representation Theory.** We begin by providing key definitions from representation theory.

Definition 2.1. (Modular representation of a finite group) A *characteristic p representation* of a finite group G is a group homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

where V is a finite-dimensional vector space over a field of characteristic p and $\mathrm{GL}(V)$ is the general linear group of V. Equivalently we may define a representation of a finite group as a group action of G on a vector space V, such that $g \cdot v = \rho(g)(v)$.

Remark 2.2. Although a representation of a group G is specified by both a vector space V and a group homomorphism ρ, we will often refer to the vector space V as the representation of G, keeping in mind that V is equipped with a G-action.

Definition 2.3. (Subrepresentations) Let $\rho : G \to \mathrm{GL}(V)$ be a representation, and consider a subspace $W \leq V$. We say W is a *subrepresentation* of V if

$$\rho(g)(w) \in W$$

for all $g \in G, w \in W$.

Definition 2.4. (Irreducible representation) A representation $\rho : G \to \mathrm{GL}(V)$ is *irreducible* if its only subrepresentations are the zero subspace and the whole vector space V. Otherwise we say V is *reducible.*

2.2. **Maschke's Theorem and its Converse.**

Proposition 2.5. *(Maschke's Theorem) Let G be a finite group and let $\mathbb{F}$ be a field of characteristic zero or of positive characteristic not dividing $|G|$. If V is a finite-dimensional representation of G over $\mathbb{F}$ and U is any subrepresentation of V, then V has a subrepresentation W such that $V = U \oplus W$.*

Maschke's theorem implies that every finite-dimensional representation V of a finite group G over a field whose characteristic does not divide the order of the group can be expressed uniquely as a direct sum of irreducible representations. A partial converse of Maschke's theorem holds as well: if G is a finite group and V is a representation over a field $\mathbb{F}$ whose order *does* divide $|G|$, then V may not be completely reducible. That is, it is possible for there to exist some subrepresentation U of V which has no complement subrepresentation W in V.

For an example of Maschke's Theorem failing when the characteristic of $\mathbb{F}$ divides $|G|$, consider:

Example 2.6. Let $G = \mathbb{Z}/p\mathbb{Z} = g$ and let $V = \overline{\mathbb{F}_p}^2$ over $\overline{\mathbb{F}_p}$. Define an action of G on V via $g \cdot e_1 = e_1$ and $g \cdot e_2 = e_1 + e_2$. Note that this is indeed a representation, as $\rho(0) = \rho(p \cdot g) = \rho(g)^p = \left[\begin{smallmatrix} 1 & p \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ since the characteristic of the underlying field is p. Notice that e_1 is stable under the action of G and that e_1 is isomorphic to the trivial representation. We claim that there does not exist V' a subrepresentation of V such that $V = e_1 \oplus V'$. For, if there was, then $V/e_1 \cong V'$. But V/e_1 is isomorphic to $\overline{e_2}$, which, according to the action of G on V, is isomorphic to the trivial representation, as

$$g \cdot \overline{e_2} = \overline{e_1 + e_2} = \overline{e_2}.$$

This implies that V is isomorphic to the direct sum of two copies of the trivial representation, and hence that the fixed subspace of V, denoted V^G, is two-dimensional. But V^G is one-dimensional: if $\alpha_1 e_1 + \alpha_2 e_2 \in V^G$, then $g \cdot (\alpha_1 e_1 + \alpha_2 e_2) = \alpha_1 e_1 + \alpha_2 (e_1 + e_2) = \alpha_1 e_1 + \alpha_2 e_2$ implies that $\alpha_2 = 0$ and hence that $V^G = e_1$.

The key to this example is that the defined action of G on V fails to be a representation when the characteristic of the field underlying V is not divisible by p.

3. Constructing I_r^{r-1}-invariant subspaces.

3.1. **Characters of B_r.** It is known ([1]) that every character $\chi : B_1 \to \overline{\mathbb{F}_p}^\times$ is of the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto a^\ell (ad)^s$$

for some $0 \le \ell, s \le p-2$. An analogue holds in the general B_r case, in the sense that every character $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ is of the form

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^\ell (a_0 d_0)^s$$

for some $0 \le \ell, s \le p-2$, and hence only depends on the constant terms a_0, d_0 belonging to $\mathbb{F}_p^\times$.

Lemma 3.1. *Every character $\chi_i : (\mathbb{F}_p[t]/(t^r))^\times \to \overline{\mathbb{F}_p}^\times$ is completely determined by where it maps the constant terms belonging to $\mathbb{F}_p^\times$. That is, $\chi_i(a_0 + a_1 t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$.*

Proof. We first show that $\chi_i : (\mathbb{F}_p[t]/(t^r))^\times \to \overline{\mathbb{F}_p}^\times$ must always map an element of the form $1 + a_1 t + \cdots + a_{r-1}t^{r-1}$ to 1. By applying the monomial identity $(x+y)^p = x^p + y^p$ in the field $\mathbb{F}_p$ inductively, we obtain $(1 + a_1 t + \cdots + a_{r-1}t^{r-1})^p = 1 + a_1 t^p + \cdots + a_{r-1}t^{p(r-1)}$. Choosing the minimal $k \in \mathbb{N}$ such that $p^k \ge r$ gives

$$\begin{aligned} (1 + a_1 t + \cdots + a_{r-1}t^{r-1})^{p^k} &= 1 + a_1 t^{p^k} + \cdots + a_{r-1}t^{p^k(r-1)} \\ &= 1 \end{aligned}$$

Thus $\chi_i(1 + a_1 t + \cdots + a_{r-1}t^{r-1})$ must have order dividing p^k in $\overline{\mathbb{F}_p}^\times$. But no elements in $\overline{\mathbb{F}_p}^\times$ have order p^ℓ for any $1 \le \ell \le k$, since $\overline{\mathbb{F}_p}^\times = \bigcup_{k\in\mathbb{N}} \mathbb{F}_{p^k}^\times$. Hence $\chi_i(1 + a_1 t + \cdots + a_{r-1}t^{r-1})$ has order 1, making it the identity element of $\overline{\mathbb{F}_p}^\times$.

Now $\chi_i(a_0 + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0 \cdot (1 + \frac{a_1}{a_0}t + \cdots + \frac{a_{r-1}}{a_0})) = \chi_i(a_0)\chi_i(1 + \frac{a_1}{a_0}t + \cdots + \frac{a_{r-1}}{a_0}) = \chi_i(a_0)$, completing the proof. $\square$

Lemma 3.2. *Every multiplicative map $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ is of the form*

$$\chi : B_r \to (\mathbb{F}_p[t]/(t^r))^\times$$

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^\ell (a_0 d_0)^s$$

for some $0 \le \ell, s \le p-2$.

Proof. We first show that any matrix $\left[\begin{smallmatrix} 1+\cdots+a_{r-1}t^{r-1} & b \\ 0 & 1+\cdots+d_{r-1}t^{r-1} \end{smallmatrix}\right]$ must get mapped to 1 in $\mathbb{F}_p^\times$ under any multiplicative map χ. Notice that

$$\begin{bmatrix} 1 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & 1 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}^p = \begin{bmatrix} 1 + \cdots & pb(1 + \cdots) \\ 0 & 1 + \cdots \end{bmatrix}$$

and since $pb \equiv 0$ in $\mathbb{F}_p$, we must have that

$$\chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix})^p = \chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}^p) = \chi(\begin{bmatrix} 1 + \cdots & 0 \\ 0 & 1 + \cdots \end{bmatrix}).$$

Because any multiplicative map on a diagonal matrix in G_r must be the product of two multiplicative maps on each entry in the diagonal, and since such diagonal elements belong to $(\mathbb{F}_p[t]/(t^r))^\times$, each of the two multiplicative maps must be of the form in Lemma 3.1. In particular this shows that $\chi(\left[\begin{smallmatrix} 1+\cdots & b \\ 0 & 1+\cdots \end{smallmatrix}\right]) = 1$.

Now any matrix $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in B_r$ can be expressed as

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix}$$

so $\chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right])$. But a multiplicative map on a diagonal matrix is again just the product of multiplicative maps on its diagonal entries, implying that $\chi = \chi_1 \times \chi_2$ where each χ_i is a map as in Lemma

3.1. In particular, since Lemma 3.1 shows that $\chi_i(a_0 + a_1 t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$ for an element $a_0 + \cdots + a_{r-1}t^{r-1} \in (\mathbb{F}_p[t]/(t^r))^\times$, then we conclude

$$\chi(\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}) = \chi_1(a_0) \cdot \chi_2(d_0).$$

But both a_0 and d_0 belong to $\mathbb{F}_p^\times$, a cyclic group of order $p-1$, and hence $\chi_1(a_0)$ and $\chi_2(d_0)$ must be $(p-1)^{st}$ roots of unity in $\overline{\mathbb{F}_p}^\times$. Since all $p-1$ such roots of unity lie in $\mathbb{F}_p^\times \subset \overline{\mathbb{F}_p}^\times$, then both χ_1 and χ_2 map into $\mathbb{F}_p^\times$, which is cyclic of order $p-1$. This implies that $\chi_1(a_0) = a_0^m$ for some $0 \le m \le p-2$ and $\chi_2(d_0) = d_0^s$ for some $0 \le s \le p-2$. Alternatively, we can express $a_0^m \cdot d_0^s$ as $a_0^\ell (a_0 d_0)^s$ where $\ell = m - s \mod p$. □

Remark 3.3. In this paper we abuse notation and write $\frac{a}{d} : B_r \to \overline{\mathbb{F}_p}^\times$ to mean the map $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mapsto a_0 d_0^{-1} = a_0 d_0^{p-2}$, since the lemmas above guarantee that any character $\chi : \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \to \overline{\mathbb{F}_p}^\times$ is of the form $a_0^\ell (a_0 d_0)^s$.

3.2. **Induction from Borel subgroup.** Let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character. For $r \ge 2$, we define the Iwahori subgroup

$$I_r^{r-1} := \{\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in G_r\} \tag{7}$$

to be the invertible matrices in G_r whose $(2,1)$-entry have no terms of the form $c_k t^k$ for $0 \le k \le r-2$. Equivalently, we may define I_r^{r-1} to be the preimage of B_{r-1} under the surjective homomorphism

$$\begin{aligned} \pi : G_r &\twoheadrightarrow G_{r-1} \\ t^{r-1} &\mapsto 0. \end{aligned} \tag{8}$$

Let $\sigma := \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$. As $\dim(\sigma) = [I_r^{r-1} : B_r] = p$, we fix a basis $\{\delta_0, \ldots, \delta_{p-1}\}$ of σ by setting

$$\begin{aligned} \delta_j &: I_r^{r-1} \to \overline{\mathbb{F}_p}^\times \\ \delta_j(i) &= \mathbb{1}_{B_r x_j} \cdot \chi(i x_j^{-1}) \end{aligned} \tag{9}$$

where $B_r x_j := B_r \left[\begin{smallmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{smallmatrix}\right]$ and $\mathbb{1}$ is the indicator function. As each of these p functions has support on a distinct right coset of B_r in I_r^{r-1}, they are linearly independent. If $bi \in B_r x_j$, we have

$$\delta_j(bi) = \chi(bi x_j^{-1}) = \chi(b)\delta_j(i)$$

and if $bi \notin B_r x_j$, then $i \notin B_r x_j$, and

$$\delta_j(bi) = 0 = \chi(b)\delta_j(i),$$

which shows that these functions belong to σ. We note that by composition of induction, constructing a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ is equivalent to constructing a Jordan-Hölder series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma)$. Thus one may initially construct a Jordan-Hölder series for σ and "induce up" to get a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$, which can then be further refined to a full composition series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$. Since this is the approach we take in Theorem 1.1, we first construct a Jordan-Hölder series for σ.

Proposition 3.4. *For every $0 \le k \le p$ there exists a k-dimensional I_r^{r-1}-invariant subspace $\sigma^{(k)}$ of σ, such that*

$$0 \subset \sigma^{(1)} \subset \cdots \sigma^{(p-1)} \subset \sigma$$

is a Jordan-Hölder series for σ.

The cases of $k = 0$ and $k = p$ are trivial. For each $1 \le k \le p-1$, we construct a k-dimensional subspace of σ denoted $\sigma^{(k)}$:

$$\sigma^{(k)} := \sum_{j=0}^{p-1} \binom{j}{j} \delta_j, \sum_{j=0}^{p-2} \binom{j+1}{j} \delta_j, \ldots, \sum_{j=0}^{p-k} \binom{j+k-1}{j} \delta_j \tag{10}$$

Setting $S_\ell := \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j$ allows us to express $\sigma^{(k)} = S_1, \ldots, S_k$. From the construction of $\sigma^{(k)}$ it is clear that we get a filtration of subspaces. To see that the vectors $\{S_\ell : 1 \le \ell \le k\}$ are linearly independent and hence form a basis for $\sigma^{(k)}$, we notice that if we express each sum as a tuple in the basis $\{\delta_0, \ldots, \delta_{p-1}\}$, then putting the k p-tuples into a $p \times k$ matrix gives

$$A = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{k-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{k}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{p-2}{p-2} & \binom{p-1}{p-2} & 0 & \cdots & 0 \\ \binom{p-1}{p-1} & 0 & 0 & \cdots & 0 \end{bmatrix}_{p \times k}. \tag{11}$$

We verify that the columns $\{\vec{v_1}, \ldots, \vec{v_k}\}$ are linearly independent by noting that if

$$a_1\vec{v_1} + \cdots + a_k\vec{v_k} = 0$$

then in particular $a_1\binom{p-1}{p-1} = 0$, implying that $a_1 = 0$. Then since $a_1\binom{p-2}{p-2} + a_2\binom{p-1}{p-2} = 0$, we deduce that $a_2 = 0$. The fact that $A_{ij} = 0$ for $j \geq p - i + 2$ allows us to inductively deduce that $a_i = 0$ for $1 \leq i \leq k$.

To see that $\sigma^{(k)}$ is I_r^{r-1}-invariant and therefore a subrepresentation of σ, we check that it is invariant under every generator of I_r^{r-1}. By the Iwahori factorization of I_r^{r-1}, any matrix $\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d \end{smallmatrix}\right] \in I_r^{r-1}$ is expressible as

$$\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1} + d \end{bmatrix} \cdot \begin{bmatrix} 1 & ba^{-1} \\ 0 & 1 \end{bmatrix}$$

which allows us to conclude that

$$I_r^{r-1} = \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \tag{12}$$

for $0 \leq k \leq r-1$ and $a, d \in (\mathbb{F}_p[t]/(t^r))^\times$. In order to determine how I_r^{r-1} acts on each subspace $\sigma^{(k)}$, we first observe how each generator of I_r^{r-1} in (12) acts on an ordinary basis vector δ_j of σ.

Lemma 3.5. *Let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character of B_r and let $\sigma = \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$. Let $\{\delta_0, \ldots, \delta_{p-1}\}$ be the ordered basis of σ given in* (9)*. Then the generators of I_r^{r-1} act on each δ_j via*

$$\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j = \delta_j \tag{13}$$

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j = \delta_{j-1} \tag{14}$$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \cdot \delta_{\frac{d_0}{a_0}j} \tag{15}$$

where all indices j are taken modulo p.

Proof. We have that

$$(\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j)(i) \neq 0 \iff \delta_j(i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) \neq 0$$

by definition of the G_r action on σ. But

$$\delta_j(i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) \neq 0 \iff i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \iff i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -t^k \\ 0 & 1 \end{bmatrix} \iff i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix}$$

and thus $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]\cdot\delta_j$ has support on $B_r x_j$. Now suppose $i\in B_r x_j$, so $i=b\cdot\left[\begin{smallmatrix}1 & 0\\ jt^{r-1} & 1\end{smallmatrix}\right]$ for some $b\in B_r$. Then

$$\begin{aligned}(\begin{bmatrix}1 & t^k\\ 0 & 1\end{bmatrix}\cdot\delta_j)(i)=\delta_j(b\begin{bmatrix}1 & 0\\ jt^{r-1} & 1\end{bmatrix}\cdot\begin{bmatrix}1 & t^k\\ 0 & 1\end{bmatrix})=\delta_j(b\begin{bmatrix}1 & t^k\\ jt^{r-1} & jt^{r-1+k}+1\end{bmatrix})&=\chi(b\begin{bmatrix}1 & t^k\\ jt^{r-1} & jt^{r-1+k}+1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\ -jt^{r-1} & 1\end{bmatrix})\\ &=\chi(b\begin{bmatrix}1-jt^{r+k-1} & t^k\\ -j^2t^{2r-2+k} & jt^{r-1+k}+1\end{bmatrix})\\ &=\chi(b)\chi(\begin{bmatrix}1-jt^{r+k-1} & t^k\\ 0 & 1+jt^{r-1+k}\end{bmatrix})\\ &=\delta_j(i)\end{aligned}$$

since $\chi(\left[\begin{smallmatrix}1+\cdots & b\\ 0 & 1+\cdots\end{smallmatrix}\right])=1$ by the proof of Lemma 3.2. Hence $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]\cdot\delta_j=\delta_j$. A similar argument shows that $\left[\begin{smallmatrix}1 & 0\\ t^{r-1} & 1\end{smallmatrix}\right]\cdot\delta_j$ has support on $B_r x_{j-1}$, and if $i=bx_{j-1}$ for some $b\in B_r x_{j-1}$, then

$$(\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\delta_j)(b\begin{bmatrix}1 & 0\\ (j-1)t^{r-1} & 1\end{bmatrix})=\delta_j(b\begin{bmatrix}1 & 0\\ (j-1)t^{r-1} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix})=\delta_j(b\begin{bmatrix}1 & 0\\ jt^{r-1} & 1\end{bmatrix})=\chi(b)=\delta_{j-1}(i),$$

allowing us to conclude $\left[\begin{smallmatrix}1 & 0\\ t^{r-1} & 1\end{smallmatrix}\right]\cdot\delta_j=\delta_{j-1}$. Finally, an analogous computation shows that $\left[\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right]\cdot\delta_j$ has support on $B_r x_{\frac{d_0}{a_0}j}$, so we suppose $i=b\left[\begin{smallmatrix}1 & 0\\ \frac{d_0}{a_0}jt^{r-1} & 1\end{smallmatrix}\right]$ for some $b\in B_r$, and find that

$$(\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}\cdot\delta_j)(i)=\delta_j(b\begin{bmatrix}1 & 0\\ \frac{d_0}{a_0}jt^{r-1} & 1\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix})=\delta_j(b\begin{bmatrix}a & 0\\ d_0jt^{r-1} & d\end{bmatrix})=\chi(b\begin{bmatrix}a & 0\\ d_0jt^{r-1} & d\end{bmatrix}\begin{bmatrix}1 & 0\\ -jt^{r-1} & 1\end{bmatrix})=\chi(b)\chi(\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix})$$

whereas

$$\delta_{\frac{d_0}{a_0}j}(b\begin{bmatrix}1 & 0\\ \frac{d_0}{a_0}jt^{r-1} & 1\end{bmatrix})=\chi(b)$$

by definition, which shows that $\left[\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right]\cdot\delta_j=\chi(\left[\begin{smallmatrix}a & 0\\ 0 & d\end{smallmatrix}\right])\delta_{\frac{d_0}{a_0}j}$ as desired.

□

Recall that we wish to show $\sigma^{(k)}$ is I_r^{r-1}-invariant. Since $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]$ acts trivially on each δ_j, then certainly $\left[\begin{smallmatrix}1 & t^k\\ 0 & 1\end{smallmatrix}\right]\cdot S_\ell=S_\ell$ for each $1\le\ell\le k$. The actions by the other generators are more involved, so we provide them as lemmas.

Lemma 3.6.

$$\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot S_\ell=\sum_{m=1}^{\ell}S_m \tag{16}$$

so that if the basis vectors of $\sigma^{(k)}$ are ordered, then acting on each basis vector by $\left[\begin{smallmatrix}1 & 0\\ t^{r-1} & 1\end{smallmatrix}\right]$ yields a sum of the vector being acted on and the preceding basis vectors, thus remaining in $\sigma^{(k)}$.

Proof. We prove (16) by induction on ℓ: when $\ell=1$, we have

$$\begin{aligned}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-1}\binom{j}{j}\delta_j&=\sum_{j=0}^{p-1}\binom{j}{j}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\delta_j\\ &=\sum_{j=0}^{p-1}\binom{j}{j}\delta_{j-1}\\ &=\sum_{j=0}^{p-1}\binom{j}{j}\delta_j\end{aligned}$$

so that the base case holds. Now suppose (16) holds for some $\ell\in\mathbb{N},\ell<k$. We wish to show the claim holds for $\ell+1$. By the binomial coefficient recurrence relation $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ (where $\binom{n-1}{k-1}=0$ whenever

$k<1$), and by the fact that we can express $\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j=\sum_{j=0}^{p-\ell}\binom{j+\ell}{j}\delta_j$ since the coefficient $\binom{p}{p-\ell}$ of $\delta_{p-\ell}$ is zero mod p, we get

$$\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j=\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell}{j}\delta_j$$

$$=\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\left(\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j+\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_j\right).\tag{17}$$

Our inductive hypothesis guarantees that

$$\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j=\sum_{m=0}^{\ell}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j,\tag{18}$$

while

$$\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_j=\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_{j-1}$$

$$=\sum_{j=1}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_{j-1}$$

$$=\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j\tag{19}$$

since the coefficient $\binom{j+\ell-1}{j-1}=0$ for $j=0$, by convention. Hence from (17), (18) and (19), we conclude that

$$\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j=\sum_{m=1}^{\ell+1}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j\tag{20}$$

$$\implies\begin{bmatrix}1&0\\t^{r-1}&1\end{bmatrix}\cdot S_{\ell+1}=\sum_{m=1}^{\ell+1}S_m\tag{21}$$

confirming $\sigma^{(k)}$ is indeed invariant under $\left[\begin{smallmatrix}1&0\\t^{r-1}&1\end{smallmatrix}\right]$. □

It remains to show that $\sigma^{(k)}$ is invariant under $\left[\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right]$. As in the $\left[\begin{smallmatrix}1&0\\t^{r-1}&1\end{smallmatrix}\right]$ case, we show that acting on $S_\ell\in\sigma^{(k)}$ by $\left[\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right]$ yields an $\overline{\mathbb{F}_p}$-linear combination of $S_m\in\sigma^{(k)}$ for $m\le\ell$, and hence belongs to $\sigma^{(k)}$. Explicitly, we claim:

Lemma 3.7. *Given $a,d\in\mathbb{F}_p^\times\cong(\mathbb{Z}/p\mathbb{Z})^\times$, let $\alpha_i:=\binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}}$, where ad^{-1} is a representative in $\mathbb{N}$ of the equivalence class ad^{-1} in $\mathbb{Z}/p\mathbb{Z}$. Then*

$$\begin{bmatrix}a&0\\0&d\end{bmatrix}\cdot S_\ell=\chi(\begin{bmatrix}a&0\\0&d\end{bmatrix})\sum_{m=1}^{\ell}c_mS_m\tag{22}$$

where each c_m is given by $\sum_{i=1}^{m}(-1)^{i+1}\binom{m-1}{i-1}\alpha_i$.

Proof. We first ensure that the α_i are well-defined up to mod p, such that they give the same binomial coefficient mod p regardless of the choice of ad^{-1} in $\mathbb{N}$. It suffices to show that, given a representative of $ad^{-1}\in\mathbb{N}$,

$$\binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}}\equiv\binom{(p-i)(ad^{-1}+pk)+\ell-1}{(p-i)(ad^{-1}+pk)}\mod p\tag{23}$$

for $k \in \mathbb{N}$. Let the base p expansion of $(p-i)ad^{-1}+\ell-1$ be given by $a_r p^r + \cdots + a_1 p + a_0$. Since $\ell-1 \leq p-2$, the base p expansion of $\ell-1$ is given by $0p^r + \cdots + 0p + \ell - 1$, so by Lucas' theorem we have

$$\begin{aligned}\binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}} = \binom{(p-i)ad^{-1}+\ell-1}{\ell-1} &\equiv \binom{a_r}{0}\cdots\binom{a_0}{\ell-1} \mod p \\ &\equiv \binom{a_0}{\ell-1} \mod p.\end{aligned}$$

Thus it suffices to show that $(p-i)ad^{-1}+\ell-1$ and $(p-i)(ad^{-1}+pk)+\ell-1$ have the same constant term in their base p expansions. This follows quickly from the fact that their difference is given by $pk(p-i)$, which is a multiple of p and so has no constant term in its base p expansion. We conclude that α_i is independent of the choice of $ad^{-1} \in \mathbb{N}$. In particular we may always take the canonical representative.

By the action of $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right]$ on each δ_j, we have

$$\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix} \cdot S_\ell = \begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j}\delta_j = \chi\left(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\right) \sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_{\frac{d}{a}j}. \tag{24}$$

For $0 \leq n \leq p-1$, we see that δ_n appears in the right hand sum of (24) with a coefficient of $\chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$ (where $\frac{a}{d}$ is shorthand for the *representative in* $\mathbb{N}$ of ad^{-1}), and since δ_n appears in each vector $S_m = \sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j$ with a coefficient of $\binom{n+m-1}{n}$ for the respective $1 \leq m \leq \ell$, it suffices to verify

$$c_1\binom{n}{n} + c_2\binom{n+1}{n} + \cdots + c_\ell\binom{n+\ell-1}{n} \equiv \binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$$

for the proposed coefficients $c_1, \ldots, c_\ell$. That is, we wish to show

$$\sum_{r=1}^{\ell}\binom{n+r-1}{n}\sum_{i=1}^{r}(-1)^{i+1}\binom{r-1}{i-1}\alpha_i = \alpha_{p-n}. \tag{25}$$

Counting how often each α_r appears in the left hand side of (25) allows us to express

$$\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r = \sum_{r=1}^{\ell}(-1)^{r+1}\left(\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}\right)\alpha_r \tag{26}$$

such that our goal is to show

$$\sum_{r=1}^{\ell}(-1)^{r+1}\left(\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}\right)\alpha_r = \alpha_{p-n}. \tag{27}$$

When $n=0$, we need to show that $\sum_{r=1}^{\ell}\binom{r-1}{0}c_r = \alpha_p = \binom{\ell-1}{0} = 1$. By (26) we know that

$$\sum_{r=1}^{\ell} c_r = \sum_{r=1}^{\ell}(-1)^{r+1}\sum_{j=r-1}^{\ell-1}\binom{j}{0}\binom{j}{r-1}\alpha_r = \sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\alpha_r.$$

Writing

$$\alpha_1 = \binom{(p-1)\frac{a}{d}+\ell-1}{(p-1)\frac{a}{d}} = \frac{1}{(\ell-1)!}(\ell-1-\frac{a}{d})\cdots(1-\frac{a}{d})$$

and letting the variable x stand in for $\frac{a}{d}$, we have that

$$\alpha_1 = \frac{1}{(\ell-1)!}(a_{\ell-1}x^{\ell-1} + a_{\ell-2}x^{\ell-2} + \cdots + a_1 x + (\ell-1)!)$$

for some coefficients $a_{\ell-1}, \ldots, a_1$. Then

$$\alpha_r = \frac{1}{(\ell-1)!}((-1)^{\ell-1}r^{\ell-1}x^{\ell-1} + \cdots + a_1 r x + (\ell-1)!)$$

so that the constant term of $\sum_{r=1}^{\ell} c_r$, when viewed as a polynomial in $x = \frac{a}{d}$, is given by

$$\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{(\ell-1)!}{(\ell-1)!} = (-1)\sum_{r=1}^{\ell}(-1)^r\binom{\ell}{r} = (-1)\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r} - (-1) = 1$$

using $\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r} = 0$. On the other hand, the coefficient of x^m in the polynomial $\sum_{r=1}^{\ell} c_r$ for $1 \le m \le \ell-1$ is given by

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r}\frac{a_m}{(\ell-1)!} = \frac{-a_m}{(\ell-1)!}\sum_{r=0}^{\ell}(-1)^r r^m\binom{\ell}{r} = 0$$

due to the combinatorial sum identity $\sum_{r=0}^{\ell}(-1)^r r^m\binom{\ell}{r} = 0$ given in [7]. We conclude that $\sum_{r=1}^{\ell} c_r = 1 = \alpha_p$.

To prove $\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r = \alpha_{p-n}$ for $1 \le n \le p-1$, we compare the coefficient of x^m in both expressions. Since the coefficient of x^m in α_r is given by $\frac{a_m}{(\ell-1)!}r^m$, then from (26) we deduce that the coefficient of x^m in $\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r$ must be

$$\sum_{r=1}^{\ell}(-1)^{r+1}\frac{a_m}{(\ell-1)!}r^m\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}.$$

On the other hand, the coefficient of x^m in α_{p-n} is given by $(-n)^m\frac{a_m}{(\ell-1)!}$, so it suffices to prove

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1} = (-n)^m. \tag{28}$$

Because $\binom{j}{r-1} = 0$ whenever $j < r-1$, we can express the left hand side of (28) as

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=0}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}. \tag{29}$$

Identity 3.155 in [6] tells us that $\sum_{k=0}^{s-1}\binom{k}{n}\binom{k+m}{m} = \binom{s}{n}\binom{s+m}{m}\frac{s-n}{m+n+1}$, which allows us to express (29) as

$$\begin{aligned}
\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=0}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1} &= \sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r-1}\binom{\ell+n}{n}\frac{\ell-r+1}{r+n}\\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r-1}\frac{\ell-r+1}{r+n}\\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}r^m\cdot r\binom{\ell}{r}\frac{1}{r+n}\\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{r^{m+1}}{r+n}. \qquad (30)
\end{aligned}$$

Finally, identity 1.47 in [6] shows that $\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{k^j}{x+k} = (-1)^j\frac{x^{j-1}}{\binom{x+\ell}{\ell}}$, and therefore (30) becomes

$$\begin{aligned}
\binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{r^{m+1}}{r+n} &= \binom{\ell+n}{n}(-1)\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r}\frac{r^{m+1}}{r+n}\\
&= \binom{\ell+n}{n}(-1)(-1)^{m+1}\frac{n^m}{\binom{n+\ell}{\ell}}\\
&= (-1)^m n^m\\
&= (-n)^m \qquad (31)
\end{aligned}$$

as desired. This proves that there exist $c_1, \ldots, c_\ell \in \mathbb{Z}$ such that

$$\sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_{\frac{d}{a}j} = \sum_{m=1}^{\ell} c_m \sum_{j=0}^{p-m} \binom{j-m+1}{j} \delta_j \tag{32}$$

which means that there exist $c_1, \ldots, c_\ell \in \mathbb{Z}$ such that

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_\ell = \sum_{m=1}^{\ell} \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) c_m S_m. \tag{33}$$

Since the left hand side is given mod p, we may reduce the right hand side mod p to conclude that there exist $c_1, \ldots, c_\ell \in \overline{\mathbb{F}_p}$ such that (33) holds. Because this holds for all $1 \le \ell \le k$, we have that $\sigma^{(k)}$ is invariant under action by $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. This lemma also concludes the proof of the proposition. □

4. Proof of main theorem.

4.1. **Inducing up to a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$.** Proposition 3.4 gives us a length p Jordan-Hölder series

$$0 \subset \sigma^{(1)} \subset \cdots \subset \sigma^{(p-1)} \subset \sigma.$$

Since each $\sigma^{(k)}$ is a subrepresentation of σ which is itself a representation of I_r^{r-1}, then inducing each $\sigma^{(k)}$ to G_r gives a filtration

$$0 \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \cdots \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma).$$

In order to refine this filtration to a composition series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \mathrm{Ind}_{B_r}^{G_r}(\chi)$, we note that it suffices to find a composition series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})$ which begins with $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ for each $0 \le k \le p-1$. But this is equivalent to finding a composition series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ and then lifting the subrepresentations under the projection map $q : \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) \to \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$. Furthermore, since

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$$

then we only need consider composition series of $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$ in order to answer our original question.

We claim that $\sigma^{(k+1)}/\sigma^{(k)}$ is equivalent to $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)$ as one-dimensional I_r^{r-1} representations, where $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)$ refers to the inflation to I_r^{r-1} of the character sending $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mapsto \chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) \cdot (\frac{a}{d})^k \in \overline{\mathbb{F}_p}^\times$. To prove this equivalence it suffices to show that I_r^{r-1} acts on $\sigma^{(k+1)}/\sigma^{(k)}$ via multiplication by $\chi \cdot (\frac{a}{d})^k$. Again we show this claim only for the three types of generators of I_r^{r-1}.

Lemma 4.1. *The generators $\left[\begin{smallmatrix} 1 & t^\ell \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ t^{r-1} & 1 \end{smallmatrix}\right]$ act trivially on $\sigma^{(k+1)}/\sigma^{(k)}$ for $0 \le \ell \le r-1$ and $0 \le k \le p-1$.*

Proof. Notice $\sigma^{(k+1)}/\sigma^{(k)} = \overline{S_{k+1}}$. Since $\left[\begin{smallmatrix} 1 & t^\ell \\ 0 & 1 \end{smallmatrix}\right]$ acts trivially on each δ_j, then clearly $\left[\begin{smallmatrix} 1 & t^\ell \\ 0 & 1 \end{smallmatrix}\right]$ acts trivially on $\overline{S_{k+1}}$. On the other hand, by the proof of Lemma 3.6, we know that

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \overline{S_{k+1}} = \overline{\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot S_{k+1}} \tag{34}$$

$$= \overline{\sum_{m=1}^{k+1} S_m}$$

$$= \overline{S_{k+1}} \tag{35}$$

where (35) follows from the fact that $\overline{S_i} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$ for $1 \le i \le k$. This proves that $\left[\begin{smallmatrix} 1 & 0 \\ t^{r-1} & 1 \end{smallmatrix}\right]$ acts trivially on $\sigma^{(k+1)}/\sigma^{(k)}$. □

Lemma 4.2. *The generator $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ acts on $\sigma^{(k+1)}/\sigma^{(k)}$ via scaling by $\chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]) \cdot (\frac{a}{d})^k$.*

Proof. By Lemma 3.7 we have that

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \overline{S_{k+1}} = \overline{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_{k+1}} = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \sum_{m=1}^{k+1} c_m \overline{S_m}$$

and since $\overline{S_m} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$ for $1 \leq m \leq k$, then in $\sigma^{(k+1)}/\sigma^{(k)}$ we have

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \overline{S_{k+1}} = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) c_{k+1} \overline{S_{k+1}}.$$

To prove our claim it suffices to show that $c_{k+1} = (\frac{a}{d})^k$. Recall that by Lemma 3.7, we have

$$c_{k+1} = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} \alpha_i$$

where here $\alpha_i = \binom{(p-i)\frac{a}{d}+k}{(p-i)\frac{a}{d}} = \frac{(k-i\frac{a}{d})\cdots(1-i\frac{a}{d})}{k!}$. In particular, since we may write out $\alpha_1 = \frac{(k-x)\cdots(1-x)}{k!} = \frac{1}{k!}((-1)^k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + k!)$ where $x = \frac{a}{d}$, then we have that $\alpha_i = \frac{1}{k!}((-1)^k i^k x^k + a_{k-1} i^{k-1} x^{k-1} + \cdots + a_1 i x + k!)$ for $1 \leq i \leq k+1$. Since the coefficient of x^m in α_i is given by $\frac{a_m}{k!} \cdot i^m$, then the coefficient of x^m in the expression of c_{k+1} is given by

$$\sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} \frac{a_m}{k!} i^m = \frac{a_m}{k!} \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m. \tag{36}$$

Since we wish to show that $c_{k+1} = x^k = (\frac{a}{d})^k$, it suffices to show that (36) is zero whenever $0 \leq m \leq k-1$ and is 1 whenever $m = k$. When $m = 0$, we have that $a_0 = k!$, so $\frac{a_0}{k!} \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^0 = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0$, as desired. On the other hand, the identity

$$\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} i^m = 0 \tag{37}$$

holds for $1 \leq m \leq k$ (see [7], #3 in 0.154), and since $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$, we deduce from (37) that

$$\sum_{i=0}^{k+1} (-1)^i \binom{k}{i} i^m + \sum_{i=0}^{k+1} (-1)^i \binom{k}{i-1} i^m = 0$$

which implies that

$$\sum_{i=0}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m = \sum_{i=0}^{k+1} (-1)^i \binom{k}{i} i^m = \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^m$$

since $\binom{k}{k+1} = 0$ by convention. Now $\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^m = 0$ for $0 \leq m \leq k-1$ by the identity in (37), so $\sum_{i=0}^{k+1} (-1)^{i+1} \binom{k}{i-1} j^m = 0$ for $0 \leq m \leq k-1$. When $m > 0$ we have that $0^m = 0$, so we conclude $\sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m = 0$ for $0 \leq m \leq k-1$ as desired. On the other hand, identity #4 in §0.154 of [7] gives

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} j^k = (-1)^k k!\,, \tag{38}$$

which in combination with (37) and the fact that $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$ gives

$$\begin{aligned} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} j^k &= \sum_{j=0}^{k+1} (-1)^j \binom{k}{j} j^k + \sum_{j=0}^{k+1} (-1)^j \binom{k}{j-1} j^k \\ \implies \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k}{j-1} j^k &= (-1)^k k! \end{aligned}$$

which is precisely what we wished to show. Hence the coefficient of x^m in c_{k+1} is $\frac{a_m}{k!}\cdot 0 = 0$ for $0 \le m \le k-1$ while the coefficient of x^k is $\frac{(-1)^k}{k!}\cdot(-1)^k k! = (-1)^{2k} = 1$, completing the proof that $c_{k+1} = (\frac{a}{d})^k$, and therefore that $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right]\cdot\overline{S_{k+1}} = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\cdot(\frac{a}{d})^k\overline{S_{k+1}}$. ☐

Recall we wish to show that $\sigma^{(k+1)}/\sigma^{(k)}$ is equivalent to $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ as I_r^{r-1} representations. Let $T : \overline{S_{k+1}} \to \mathbb{F}_p$ be the isomorphism sending $\overline{S_{k+1}} \mapsto 1$. For all $\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d\end{smallmatrix}\right] \in I_r^{r-1}$, we have

$$\begin{aligned} T\left(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix}\cdot\overline{S_{k+1}}\right) &= T\left(\begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\begin{bmatrix} 1 & ba^{-1} \\ 0 & 1\end{bmatrix}\cdot\overline{S_{k+1}}\right) \\ &= T\left(\begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\cdot\overline{S_{k+1}}\right). \end{aligned} \tag{39}$$

Now $(\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right]\cdot\delta_j)(i) \neq 0$ if and only if $i\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right] \in B_r\left[\begin{smallmatrix} 1 & 0 \\ jt^{r-1} & 1\end{smallmatrix}\right]$, which holds if and only if $i \in B_r\left[\begin{smallmatrix} 1 & 0 \\ jt^{r-1} & 1\end{smallmatrix}\right]\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right]^{-1} = B_r\left[\begin{smallmatrix} 1 & 0 \\ \frac{d}{a}jt^{r-1} & 1\end{smallmatrix}\right]$. A similar argument as the one for $\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right]\cdot\delta_j = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\delta_{\frac{d}{a}j}$ reveals that $\left[\begin{smallmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{smallmatrix}\right]\cdot\delta_j = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])\delta_{\frac{d}{a}j}$, and therefore Lemma 4.2 applies to (39) to give $\chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])(\frac{a}{d})^k\cdot T(\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}) = \chi(\left[\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right])(\frac{a}{d})^k$. On the other hand, we have that

$$\begin{aligned} \mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix})(T(S_k)) &= (\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(T(S_k)) \\ &= \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(\frac{a}{d})^k \end{aligned} \tag{40}$$

which shows that $T\circ\sigma^{(k+1)}/\sigma^{(k)}(\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d\end{smallmatrix}\right]) = \mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)(\left[\begin{smallmatrix} a & b \\ ct^{r-1} & d\end{smallmatrix}\right])\circ T$, and hence that $\sigma^{(k+1)}/\sigma^{(k)}$ and $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ are isomorphic as I_r^{r-1}-representations.

Now because the diagram

$$\begin{array}{ccc} I_r^{r-1} & \xrightarrow{t^{r-1}\mapsto 0} & B_{r-1} \\ \big\downarrow & & \big\downarrow \\ G_r & \xrightarrow{t^{r-1}\mapsto 0} & G_{r-1} \end{array}$$

commutes, we have by commutativity of inflation and induction that $\mathrm{Ind}_{I_r^{r-1}}^{G_r}\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$. But this implies $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$, completing the proof of Theorem 1.1.

4.2. **A remark on the inductive construction.** Theorem 1.1 tells us what the successive quotients in the filtration given in (5) look like, but it doesn't explicitly tell us what the Jordan-Hölder series of $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ looks like. Fortunately, we just proceed inductively: once we know that

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$$

then we can set out to find a Jordan-Hölder series of $\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$ (using the same process as in our original problem) and then "piece it in" between $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ and $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})$ in the filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$. Since the literature already contains the Jordan-Hölder series for the mod p principal series representations of $\mathrm{Ind}_{B_1}^{G_1}(\chi)$, we have all the parts necessary to complete the original filtration to a full Jordan-Hölder series.

5. Semisimplifications

From Theorem 1.1 we deduce that

$$\begin{aligned} (\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} &= (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^{p-1}))^{ss} \\ &= (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}\oplus\cdots\oplus(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \end{aligned} \tag{41}$$

where inflations to G_r are always implicitly assumed. In particular, we see that $(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}$ appears twice in the direct sum of (41), while $(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}$ appears once in the direct sum for every $1\le k\le p-2$. Hence we may express

$$(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss}=((\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss})^2\oplus\bigoplus_{k=1}^{p-2}(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k))^{ss}. \tag{42}$$

Since the semisimplifications of $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ are known for all characters $\chi: B(\mathrm{GL}_2(\mathbb{F}_p))\to\overline{\mathbb{F}_p}^{\times}$ (Lemma 2.2 in [3]), it is desirable to express (42) explicitly in terms of $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ for various χ. We claim that we may continue simplifying (42) inductively to obtain:

Corollary 5.1. *For a prime p, $(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss}=((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}}\oplus\bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}}$.*

Proof. We prove the corollary by induction on r. When $r=1$, the claim is that

$$(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}=((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^0+p-2}{p-1}}\oplus\bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^0-1}{p-1}}$$

which is easily seen to be true when one simplifies the exponents on the right hand side of the equality. Suppose the claim in the proposition holds for some $r\in\mathbb{N}$. We wish to show it holds for $r+1$. As a corollary of Theorem 1.1, we have that

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss}=((\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss})^2\oplus\bigoplus_{k=1}^{p-2}(\mathrm{Ind}_{B_r}^{G_r}(\chi\cdot(\frac{a}{d})^k))^{ss}.$$

Utilizing the inductive hypothesis on $(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss}$ and on each $(\mathrm{Ind}_{B_r}^{G_r}(\chi\cdot(\frac{a}{d})^k))^{ss}$ gives

$$\begin{aligned}(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss}=&\left(((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}}\oplus\bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}}\right)^2\\&\oplus\left(\bigoplus_{k=1}^{p-2}\left[((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}+p-2}{p-1}}\oplus\bigoplus_{m\neq k}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^m))^{ss})^{\frac{p^{r-1}-1}{p-1}}\right]\right).\end{aligned}\tag{43}$$

Counting how many times $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ appears in the direct sum of (43) yields that $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ appears

$$2(\frac{p^{r-1}+p-2}{p-1})+(p-2)\frac{p^{r-1}-1}{p-1}=\frac{p^r+p-2}{p-1}$$

times, whereas counting how many times $(\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^n))^{ss}$ appears in (43) for a given $1\le n\le p-2$ yields that $(\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^n))^{ss}$ appears

$$2(\frac{p^{r-1}-1}{p-1})+\frac{p^{r-1}+p-2}{p-1}+(p-3)\frac{p^{r-1}-1}{p-1}=\frac{p^r-1}{p-1}$$

times. Therefore

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss}=((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^r+p-2}{p-1}}\oplus\bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi\cdot(\frac{a}{d})^k))^{ss})^{\frac{p^r-1}{p-1}}. \tag{44}$$

proving the inductive claim. ☐

A complete semisimplification expresses the given representation as a direct sum of its unique set of composition factors, which are each irreducible representations. Hence giving the semisimplification of $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ requires knowing the irreducible characteristic p representations of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$.

5.1. **Classifying Modular Irreps of** $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$**.** We claim that every irreducible characteristic p representation of G_r is of the form $\rho \circ \pi$, where π is the surjective homomorphism

$$\pi : \mathrm{GL}_2(\mathbb{F}_p[t]/(t^r)) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_p)$$

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \tag{45}$$

and ρ is an irreducible characteristic p representation of $\mathrm{GL}_2(\mathbb{F}_p)$. To prove this fact we need the following two known lemmas, which then establish the result as an immediate corollary.

Lemma 5.2. *Let G be a finite group and let $H \trianglelefteq G$ be a p-group. If V is an irreducible characteristic p representation of G, then $V^H = V$, that is, H acts trivially on all elements of V.*

In particular Lemma 5.2 tells us that if G is a finite group, $H \trianglelefteq G$ is a p-group, and V is an irreducible characteristic p representation of G, then V must be the direct sum of trivial representations on H. We claim that this implies V factors through G/H.

Lemma 5.3. *A representation of a finite group G is trivial on a normal subgroup H if and only if it factors through G/H.*

The preceding lemmas allow us to prove the claim established at the beginning of this section:

Proposition 5.4. *Any irreducible modular representation of* $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ *is the inflation of an irreducible modular representation of* $\mathrm{GL}_2(\mathbb{F}_p)$*.*

Proof. The surjective homomorphism π in (45) gives us $H = \ker \pi \trianglelefteq G_r$. We claim that H is a p-group: Notice that $G_1 = \mathrm{GL}_2(\mathbb{F}_p)$ may be viewed as a subgroup of G_r, as it respects multiplication in G_r. Since the matrix

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}$$

belongs to $\ker \pi$ if and only if $a_0 = d_0 = 1, b_0 = c_0 = 0$, and $a_i, b_i, c_i, d_i \in \mathbb{F}_p$ for $1 \leq i \leq r-1$, then $|\ker \pi| = |\mathbb{F}_p|^{4(r-1)} = p^{4(r-1)}$. Hence by Lemma 5.2 any irreducible modular representation of G_r must be trivial on H. But by Lemma 5.3, we know that a representation of G_r is trivial on a normal subgroup H if and only if it factors through G_r/H. Since $G_r/H \cong \mathrm{GL}_2(\mathbb{F}_p)$, then every irreducible characteristic p representation $\tilde{\rho}$ of G_r must be of the form $\rho \circ \pi$ where π is the map given in (45) and ρ is an irreducible characteristic p representation of $\mathrm{GL}_2(\mathbb{F}_p)$. $\square$

Fortunately the irreducible characteristic p representations of $\mathrm{GL}_2(\mathbb{F}_p)$ are fully classified (see [1] or [8] for the proofs). Given $0 \leq n \leq p-1$ and $0 \leq \ell \leq p-2$, let P_n be the $\overline{\mathbb{F}_p}$ span of the basis $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$. Define

$$\rho_{n,\ell} : \mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}(P_n) \tag{46}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot P(x,y) = P(ax+cy, bx+dy) \cdot \left(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^{\ell}.$$

Then $\{\rho_{n,\ell}\}$ gives a complete set of irreducible characteristic p representations of $\mathrm{GL}_2(\mathbb{F}_p)$ up to equivalence. Hence every irreducible characteristic p representation of G_r is given by $\tilde{\rho}_{n,\ell} = \rho_{n,\ell} \circ \pi$, where π is as in (45).

5.2. **Semisimplification of** $\mathrm{Ind}_{B_r}^{G_r}(\chi)$**.** Recall that any multiplicative map $\chi : B_1 \to \overline{\mathbb{F}_p}^{\times}$ is of the form $\chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = a^r(ad)^s$, where $0 \leq r, s \leq p-2$. From [3] we know that if $r = 0$, then $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \rho_{0,s} \oplus \rho_{p-1,s}$, where $\rho_{p-1,s}$ may be recognized as the twisted Steinberg representation. On the other hand, if $r \neq 0$, then $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \rho_{p-1-r,r+s} \oplus \rho_{r,s}$. In particular this tells us that $(\mathrm{Inf}_{G_1}^{G_r} \mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \tilde{\rho}_{0,s} \oplus \tilde{\rho}_{p-1,s}$ or $(\mathrm{Inf}_{G_1}^{G_r} \mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \tilde{\rho}_{p-1-r,r+s} \oplus \tilde{\rho}_{r,s}$ depending on χ. In combination with Corollary 5.1 this fact allows us to explicitly give the semisimplification of $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ for any character χ.

6. Computing Semisimplifications via Brauer Characters

Richard Brauer pioneered modular representation theory largely to better understand the relationships between characteristic p representations and ordinary character theory. A key development in this theory is the invention of *Brauer characters*, which assign to particular elements of a group G a value in a field of characteristic 0 dependent on a characteristic p representation. The utility of such characters in our problem comes from their ability to solve for the semisimplification numbers given in Corollary 5.1 without requiring any knowledge about the Jordan-Hölder series itself.

To compute the Brauer character of a representation we outline a process described in greater generality in [5] and [9]. Let m be the least common multiple of the orders of p-regular elements of G, which are those elements of G that have order coprime to p. Let ρ be an irreducible characteristic p representation of G. For any $g \in G$ a p-regular element, $\rho(g)$ must have order dividing $|g|$ in $\overline{\mathbb{F}_p}^\times$, and hence has order dividing m. In particular this tells us that the eigenvalues of $\rho(g)$ are all powers of m^{th} roots of unity in $\overline{\mathbb{F}_p}^\times$, so writing ζ_m for a primitive m^{th} root of unity in $\overline{\mathbb{F}_p}^\times$ allows us to express the eigenvalues of $\rho(g)$ as $\zeta_m^{m_1}, \dots, \zeta_m^{m_k}$, where k is the dimension of the representation ρ. We fix a bijection between the m^{th} roots of unity in $\overline{\mathbb{F}_p}^\times$ and the m^{th} roots of unity in $\mathbb{C}$ by mapping $\zeta_m \mapsto \omega_m = e^{\frac{2\pi i}{m}}$. Then the Brauer character of ρ evaluated at g is given by $\theta_\rho(g) = \sum_{i=1}^k \omega_m^{m_i}$. Notice that since elements of a p-regular conjugacy class have the same eigenvalues, then Brauer characters must be constant on p-regular conjugacy classes.

Fixing a field E of characteristic 0 whose residue field is of characteristic p, it is a big theorem of Brauer and Nesbitt [2] that given an ordinary representation $\psi : G \to \mathrm{GL}(V)$ with associated character $\chi : G \to E^\times$, we have that the *mod p reduction* $\overline{\chi} : G \to k_E^\times$ of χ may be expressed as a non-negative integer linear combination of the irreducible Brauer characters of G. This means that for all p-regular $g \in G$,

$$\overline{\chi}(g) = \sum_{\rho \text{ modular irreps of } G} d_\rho \theta_\rho(g) \tag{47}$$

where each d_ρ belongs to $\mathbb{Z}_{\geq 0}$. Furthermore these d_ρ are called the *decomposition numbers* of $\overline{\psi}$ as they give the multiplicity of the irreducible representation ρ in the semisimplification of $\overline{\psi}$.

We wish to compute the semisimplification of $\vartheta_\chi = \mathrm{Ind}_{B_r}^{G_r}(\chi)$ via Brauer characters. Since Brauer characters are only defined on p-regular conjugacy classes, we determine these conjugacy classes for G_r. Fortunately the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_p)$ are well-known, and the p-regular conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ for $r \in \mathbb{N}$ have representatives in $\mathrm{GL}_2(\mathbb{F}_p)$. Hence for general primes p, we have the following p-regular conjugacy classes:

(1) $\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{F}_p^\times\}$. We have $|\mathbb{F}_p^\times| = p-1$ such conjugacy classes.

(2) $\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\} : a, b \in \mathbb{F}_p^\times$. Swapping the position of a and b yields conjugate matrices, but a different pair of (a,b) yields a non-conjugate matrix. Hence we have $\binom{p-1}{2}$ such conjugacy classes.

(3) $\{\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}\}$ where D is not a square in $\mathbb{F}_p$, and $\alpha + \beta\sqrt{D}$ is a characteristic root of a matrix in $\mathrm{GL}_2(\mathbb{F}_p)$ with $\beta \neq 0$. The matrices $\{\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}\}$ and $\{\begin{bmatrix} \alpha & -D\beta \\ -\beta & \alpha \end{bmatrix}\}$ are conjugate, so we only need consider $\beta \in \{1, \dots, \frac{p-1}{2}\}$.

None of the matrices of type (3) above are conjugate to an upper triangular matrix in $\mathrm{GL}_2(\mathbb{F}_p)$ (else their eigenvalues would lie in $\mathbb{F}_p$). We see that this must hold in the larger group $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ as well: if any of the matrices of type (3) were conjugate in $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ to an upper triangular matrix, then their eigenvalues would have to lie in $(\mathbb{F}_p[t]/(t^r))^\times$. But their eigenvalues also lie in $(\mathbb{F}_p[\sqrt{D}])^\times$, and $(\mathbb{F}_p[\sqrt{D}])^\times \cap (\mathbb{F}_p[t]/(t^r))^\times = \mathbb{F}_p^\times$. Hence if the matrices of type (3) were conjugate to an upper triangular matrix in $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$, then they must have eigenvalues in $\mathbb{F}_p^\times$, and thus in particular must be of type (1) or (2). This contradicts the fact that (1), (2), and (3) give distinct conjugacy class types.

The character of the representation ϑ_χ, which we denote θ_χ, has a nice formula due to Mackey:

$$\theta_\chi(g) = \sum_{\substack{x_i \in B_r \backslash G_r \\ x_i g x_i^{-1} \in B}} \chi(\chi_i g \chi_i^{-1}) \tag{48}$$

We use this formula to compute the character on our p-regular conjugacy classes. For each conjugacy class of type $\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]$, we have that $x_i \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] x_i^{-1} = \left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right] \in B$ since scalar matrices belong to the center of G_r, and thus using (48) we get

$$\theta_\chi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) = |B_r \backslash G_r| \cdot \chi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) \tag{49}$$

$$= p^{r-1}(p+1)\chi(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}) \tag{50}$$

We now suppose $g = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$ where $a, b \in \mathbb{F}_p^\times$ and $a \neq b$. If x_j is the coset representative for B_r in the set of right cosets of B_r (that is, x_j is the identity matrix), then $x_j \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_j^{-1} \in B$ trivially. Now suppose $x_j = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ where $\gamma \neq 0$ (so that $x_j \notin B_r$). We wish to determine when $x_j \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_j^{-1} \in B$. Note

$$\begin{aligned}
x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} \\
&= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \\
&= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a\delta & -a\beta \\ -b\gamma & b\alpha \end{bmatrix} \\
&= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} a\alpha\delta - b\beta\gamma & -a\alpha\beta + b\beta\alpha \\ a\gamma\delta - b\delta\gamma & -a\beta\gamma + b\delta\alpha \end{bmatrix} \qquad (51)
\end{aligned}$$

so that $x_j \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_j^{-1} \in B$ if and only if $a\delta\gamma - b\delta\gamma = 0$, that is, if and only if $(a-b)\delta\gamma = 0$. Since $a \neq b$ and $a, b \in \mathbb{F}_p^\times$, then $a - b \in \mathbb{F}_p^\times$, and thus we must have $\delta\gamma = 0$. But we assumed in the beginning that $\gamma \neq 0$, so we must have $\delta = 0$. From (51) we see then that if $x_j \notin B_r$ and $x_j \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_j^{-1} \in B$, then

$$x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} = \frac{-1}{\beta\gamma} \begin{bmatrix} -b\beta\gamma & (b-a)\alpha\beta \\ 0 & -a\beta\gamma \end{bmatrix} = \begin{bmatrix} b & \frac{(a-b)\alpha}{\gamma} \\ 0 & a \end{bmatrix} \tag{52}$$

so that $\chi(x_j \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_j^{-1}) = \chi(\left[\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right])$. To see that no other coset representative x_ℓ gives $x_\ell \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_\ell^{-1} \in B$, suppose such an x_ℓ did exist with $Bx_j \neq Bx_\ell$. Let $x_j = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & 0 \end{smallmatrix}\right]$, where $\gamma \neq 0$ so that $x_j \notin B$. Let $x_\ell = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$. Then

$$\begin{aligned}
Bx_j \neq Bx_\ell &\iff x_\ell x_j^{-1} \notin B \\
&\iff \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}^{-1} \notin B \\
&\iff \frac{-1}{\beta\gamma} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 0 & -\beta \\ -\gamma & \alpha \end{bmatrix} \notin B \\
&\iff \frac{-1}{\beta\gamma} \begin{bmatrix} -y\gamma & -x\beta + y\alpha \\ -w\gamma & -z\beta + w\alpha \end{bmatrix} \notin B \\
&\iff w \neq 0
\end{aligned}$$

But recall from our computation above that $x_\ell \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] x_\ell^{-1} \in B$ if and only if $(x_\ell)_{22} = 0$, that is, if and only if $w = 0$. This contradiction allows us to conclude that

$$\theta_\chi(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}) = \chi(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}) + \chi(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}). \tag{53}$$

Finally, if $g = \left[\begin{smallmatrix} \alpha & D\beta \\ \beta & \alpha \end{smallmatrix}\right]$ is a matrix as in type (3), then we already know from an earlier discussion that g has no upper triangular conjugates. Thus

$$\theta_\chi(\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}) = 0 \tag{54}$$

which completes our computation for the character of the principal series representation on the p-regular conjugacy classes of G_r.

To illustrate how to obtain the semisimplification numbers from the above computation, we fix $p = 3$ and χ to be the trivial character. From the above computation, we have as a result of Mackey's formula the following table for representatives of the 3-regular conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_3[t]/(t^r))$:

	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$
$\theta_\chi(g)$	$p^{r-1}(p+1)$	$p^{r-1}(p+1) \cdot \chi(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix})$	$\chi(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}) + \chi(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix})$	0	0	0

TABLE 1

We wish to solve for the d_ρ in (47), which requires us to know how θ_ρ evaluates on g for each conjugacy class and for each ρ an irreducible modular representation of G_r. An omitted computation yields the following Brauer characters:

	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$
$\theta_{0,0}$	1	1	1	1	1	1
$\theta_{0,1}$	1	1	-1	1	-1	-1
$\theta_{1,0}$	2	-2	0	0	$-i\sqrt{2}$	$i\sqrt{2}$
$\theta_{1,1}$	2	-2	0	0	$i\sqrt{2}$	$-i\sqrt{2}$
$\theta_{2,0}$	3	3	1	-1	-1	-1
$\theta_{2,1}$	3	3	-1	-1	1	1

TABLE 2. Brauer character table, $p = 3$

where $\theta_{n,\ell}$ is the Brauer character corresponding to $\tilde{\rho}_{n,\ell}$. Recall that any character $\overline{\chi} : B_1 \to \overline{\mathbb{F}_p}^\times$ is of the form $\chi(\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]) = a^r(ad)^s$ where $0 \le r, s \le p-2$, and hence when $p = 3$ we have only three choices: $r = s = 0$ (yielding the trivial character, which we call triv), $r = 0, s = 1$ (which we call det), and $r = 1$, $s = 0$ (which we call alt). Notice that if $r = s = 1$ then χ picks out the $(2,2)$-entry of $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$, but in Table 1 we see that for this χ, θ_χ will take the same values as when $\chi = $ alt.

Solving a system of equations according to (47) for every choice of character χ in the $p = 3$ case yields:

(χ_1, χ_2)	$\mathrm{d}_{0,0}$	$\mathrm{d}_{0,1}$	$\mathrm{d}_{1,0}$	$\mathrm{d}_{1,1}$	$\mathrm{d}_{2,0}$	$\mathrm{d}_{2,1}$
triv	$\frac{3^{r-1}+1}{2}$	$\frac{3^{r-1}-1}{2}$	0	0	$\frac{3^{r-1}+1}{2}$	$\frac{3^{r-1}-1}{2}$
det	$\frac{3^{r-1}-1}{2}$	$\frac{3^{r-1}+1}{2}$	0	0	$\frac{3^{r-1}-1}{2}$	$\frac{3^{r-1}+1}{2}$
alt	0	0	3^{r-1}	3^{r-1}	0	0

TABLE 3

We verify that this aligns with our numbers in Corollary 5.1. For simplicity we take $\chi = $ triv, noticing that by Table 3, we have

$$(\mathrm{Ind}_{B_r}^{G_r}(\mathrm{triv}))^{ss} = \tilde{\rho}_{0,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{0,1}^{\frac{3^{r-1}-1}{2}} \oplus \tilde{\rho}_{2,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{2,1}^{\frac{3^{r-1}-1}{2}}.$$

On the other hand, by Corollary 5.1 we have that

$$(\mathrm{Ind}_{B_r}^{G_r}(\mathrm{triv}))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}))^{ss})^{\frac{3^{r-1}+1}{2}} \oplus ((\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}\cdot\frac{a}{d}))^{ss})^{\frac{3^{r-1}-1}{2}}$$

Now $(\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}))^{ss} = \rho_{0,0} \oplus \rho_{2,0}$, and since $\frac{a}{d} = ad^{-1} = ad$ in $\overline{\mathbb{F}_3}$, then $(\mathrm{Ind}_{B_1}^{G_1}(\mathrm{triv}\cdot\frac{a}{d}))^{ss} = \rho_{0,1} \oplus \rho_{2,1}$. Hence

$$(\mathrm{Ind}_{B_r}^{G_r}(\mathrm{triv}))^{ss} = \tilde{\rho}_{0,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{2,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{0,1}^{\frac{3^{r-1}-1}{2}} \oplus \tilde{\rho}_{2,1}^{\frac{3^{r-1}-1}{2}}$$

which is precisely what we deduced from Table 3. A similar computation verifies the other two cases of χ.

For larger primes computing the Brauer table is much more computationally intensive, so we resort to the semisimplification numbers which resulted from the Jordan-Hölder series.

Acknowledgements

The author would like to thank professors Charlotte Chan and Karol Koziol for both posing the problem and providing tremendous guidance throughout the research. The author would also like to acknowledge Andy Gordon for answering questions, the University of Michigan for running the REU, and the NSF for providing funding.

References

[1] Laurent Berger and Sandra Rozensztajn, *Modular representations of gl2(fp).*
[2] R. Brauer and C. Nesbitt, *On the modular characters of groups*, Annals of Mathematics **42** (1941), no. 2, 556–590.
[3] C. Breuil, V. Paskunas, and American Mathematical Society, *Towards a modulo p langlands correspondence for gl2*, Memoirs of the American Mathematical Society, American Mathematical Society, 2011.
[4] Christophe Breuil and Fred Diamond, *Formes modulaires de hilbert modulo et valeurs d'extensions entre caracteres galoisens.*
[5] Daniel Bump, *Brauer characters.*
[6] Henry W. Gould, *Combinatorial identities: A standardized et of tables, listing 500 binomial coefficient summations*, West Virginia University, 1972.
[7] Gradstejn Izrail, Ryzik Iosif M., and Daniel Zwillinger, *Table of integrals, series, and products*, 2015.
[8] Niels Ketelaars, *The irreducible mod p representations of gl2(fp)*, Jun 2021.
[9] Peter Webb, *University of minnesota*, Feb 2016.

Univ. of California- Santa Barbara
Email address: `gahlshemy@ucsb.edu`