

# MODULAR PRINCIPAL SERIES REPRESENTATION OF $\mathrm{GL}_2$ OVER FINITE RINGS

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ABSTRACT. Given any prime  $p \geq 3$ ,  $r \in \mathbb{N}$ , and character  $\chi$  on the Borel subgroup of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ , we construct a Jordan-Hölder series for the mod  $p$  reduction of the principal series representation of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ . As a corollary we provide the semisimplifications of all characteristic  $p$  principal series representations of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ , and explain a process to compute such semisimplifications in small cases by the means of Brauer characters.

## 1. INTRODUCTION

Since irreducible representations form the building blocks of representation theory, we often want to determine how the irreducible representations of a group “fit together” in the composition of some other representation of concern. Under sufficiently nice conditions this problem is solved: Given a finite group  $G$  and a finite-dimensional representation  $\rho : G \rightarrow \mathrm{GL}(V)$  over a field of characteristic not dividing the order of  $G$ , the classical Maschke’s theorem guarantees that  $\rho$  is *completely reducible*. This means that  $\rho$  can be uniquely expressed as a direct sum of irreducible representations of the group  $G$ , up to isomorphism. However, when  $V$  is over a field of characteristic  $p$  and  $p$  divides the order of the group, Maschke’s theorem no longer holds, thus requiring a different method of determining how the irreducible modular representations of a finite group  $G$  make up our representation of interest.

One approach is through investigating Jordan-Hölder series of the representation. These are filtrations

$$0 \subset V_1 \subset \cdots \subset V_d = V$$

of subrepresentations with inclusions being proper and maximal, so that each composition factor  $V_{i+1}/V_i$  is isomorphic to an irreducible representation of  $G$ . The Jordan-Hölder Theorem states that such composition series need not be unique, but that the *set* of composition factors of a representation, known as the *irreducible constituents*, is unique. We can then define

$$(1) \quad V^{\mathrm{ss}} := \bigoplus_{i=0}^{d-1} V_{i+1}/V_i$$

to be the *semisimplification* of  $V$ . Since each quotient  $V_{i+1}/V_i$  is isomorphic to an irreducible representation of  $G$ , we have

$$(2) \quad V^{\mathrm{ss}} = \bigoplus_j \rho_j^{d_j}$$

where  $\rho_j$  is an irreducible representation of  $G$  and  $d_j$  is its multiplicity in the semisimplification of  $V$ . This gives us a way to view modular representations in analogy to how Maschke’s theorem decomposes ordinary characteristic 0 representations. A consequence of the Jordan-Hölder theorem is that  $V^{\mathrm{ss}}$  is unique up to rearrangement of factors in the direct sum, so  $V^{\mathrm{ss}}$  is unique up to isomorphism.

Fixing a prime  $p \geq 3$ , we consider the non-archimedean local field  $L = \mathbb{F}_p((t))$ . The ring of integers  $\mathcal{O}_L$  is given by  $\mathbb{F}_p[[t]]$  and consists of all formal power series in  $t$  with coefficients in  $\mathbb{F}_p$ , with a unique maximal ideal generated by  $t$ . For any  $r \in \mathbb{N}$  we consider the general linear group  $G_r := \mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ .

The choice of  $L = \mathbb{F}_p((t))$  puts us in the equal characteristic setting, where  $L$  has the same characteristic as its residue field  $\mathbb{F}_p$ . For work done in the mixed characteristic setting, see the appendix in [4].

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Given the finite group  $G_r$ , let  $B_r \leq G_r$  be the *Borel subgroup* of  $G_r$  consisting of  $2 \times 2$  upper triangular invertible matrices with entries in  $\mathbb{F}_p[t]/(t^r)$ . Fixing a field  $E$  of characteristic 0 whose residue field  $k_E = \mathcal{O}_E/(\varpi_E)$  is of characteristic  $p$ , let  $\chi_1, \chi_2 : (\mathbb{F}_p[t]/(t^r))^\times \rightarrow E^\times$  be group homomorphisms, and define

$$\begin{aligned} \chi : B_r &\rightarrow E^\times \\ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} &\mapsto \chi_1(a)\chi_2(d). \end{aligned}$$

The *principal series representation* of  $G_r$  associated to  $\chi$  is the induced representation  $\text{Ind}_{B_r}^{G_r}(\chi)$ , a vector space

$$(3) \quad \text{Ind}_{B_r}^{G_r}(\chi) := \{f : G_r \rightarrow E \mid f(bg) = \chi(b)f(g) \quad \forall g \in G_r, b \in B_r\}$$

with a  $G_r$ -action given by

$$(4) \quad \begin{aligned} \vartheta_\chi : G_r &\rightarrow \text{GL}(\text{Ind}_{B_r}^{G_r}(\chi)) \\ (x \cdot f)(g) &= f(gx) \end{aligned}$$

for all  $x, g \in G_r, f \in \text{Ind}_{B_r}^{G_r}(\chi)$ . This paper explores the modulo  $p$  reduction of the principal series representation, where  $\chi$  now maps to  $k_E = \mathcal{O}_E/(\varpi_E) \cong \overline{\mathbb{F}}_p$  and where all maps  $f \in \text{Ind}_{B_r}^{G_r}(\chi)$  have codomain  $k_E$ . Hereafter we abuse notation and write  $\text{Ind}_{B_r}^{G_r}(\chi)$  to mean the principal series representation after reduction modulo  $p$ . Hence  $\text{Ind}_{B_r}^{G_r}(\chi)$  is a characteristic  $p$  vector space of dimension  $[G_r : B_r] \cdot \dim(\chi) = (p+1)p^{r-1}$ , with a  $G_r$ -action still given by (4).

As the  $r = 1$  case is well-studied, the main result of this paper is an inductive construction of a Jordan-Hölder series for  $\text{Ind}_{B_r}^{G_r}(\chi)$  which terminates in  $\text{Ind}_{B_1}^{G_1}(\chi)$ .

**Proposition 1.1.** *Let  $p \geq 3$  be a prime, let  $r \in \mathbb{N}_{\geq 2}$ , and let  $\chi : B_r \rightarrow \overline{\mathbb{F}}_p^\times$  be a character. There exists a filtration for  $\text{Ind}_{B_r}^{G_r}(\chi)$  given by*

$$(5) \quad 0 \subset \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \cdots \subset \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \text{Ind}_{B_r}^{G_r}(\chi),$$

where  $I_r^{r-1} := \{ \begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in G_r : c \in \mathbb{F}_p \}$ ,  $\sigma := \text{Ind}_{B_r}^{I_r^{r-1}}(\chi)$ , and  $\sigma^{(k)}$  is an  $I_r^{r-1}$ -invariant  $k$ -dimensional subspace of  $\sigma$ .

In §3 we give a precise description of the  $k$ -dimensional subspaces  $\sigma^{(k)}$  and use their construction to prove the main result, shown in §4:

**Theorem 1.1.** *For the  $I_r^{r-1}$ -invariant  $k$ -dimensional subspaces  $\sigma^{(k)}$  satisfying Prop 1.1, we have*

$$(6) \quad \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \text{Inf}_{G_{r-1}}^{G_r} \text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a_0}{d_0})^k)$$

for  $0 \leq k \leq p-1$ , where  $\chi \cdot (\frac{a_0}{d_0})^k$  is the character  $\chi \cdot (\frac{a_0}{d_0})^k : B_r \rightarrow \overline{\mathbb{F}}_p^\times$  mapping  $\begin{bmatrix} a_{r-1}t^{r-1} + \cdots + a_0 & b \\ 0 & d_{r-1}t^{r-1} + \cdots + d_0 \end{bmatrix} \mapsto \chi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) \cdot (a_0d_0^{-1})^k$ .

Theorem 1.1 implies that the filtration in Prop 1.1 may be refined inductively to a filtration in terms of  $\text{Ind}_{B_1}^{G_1}(\psi)$  for varying characters  $\psi$ , which may be then further refined to a Jordan-Hölder series for  $\text{Ind}_{B_r}^{G_r}(\chi)$  using the known Jordan-Hölder series for  $\text{Ind}_{B_1}^{G_1}(\psi)$ .

In §2 we provide preliminaries, and in §5 we give a corollary of the main theorem regarding semisimplification numbers. Finally, since determining the semisimplification of a given representation can be done without a Jordan-Hölder series via a computational process using Brauer characters, we compute a small example using this method in §6, and show that the semisimplification matches with what is deduced from our main theorem.

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## 2. PRELIMINARIES

**2.1. Basic Representation Theory.** We begin by providing key definitions from representation theory.

**Definition 2.1.** (Modular representation of a finite group) A *characteristic  $p$  representation* of a finite group  $G$  is a group homomorphism

$$\rho : G \rightarrow GL(V)$$

where  $V$  is a finite-dimensional vector space over a field of characteristic  $p$  and  $GL(V)$  is the general linear group of  $V$ . Equivalently we may define a representation of a finite group as a group action of  $G$  on a vector space  $V$ , such that  $g \cdot v = \rho(g)(v)$ .

**Remark 2.2.** Although a representation of a group  $G$  is specified by both a vector space  $V$  and a group homomorphism  $\rho$ , we will often refer to the vector space  $V$  as the representation of  $G$ , keeping in mind that  $V$  is equipped with a  $G$ -action.

**Definition 2.3.** (Subrepresentations) Let  $\rho : G \rightarrow GL(V)$  be a representation, and consider a subspace  $W \subseteq V$ . We say  $W$  is a *subrepresentation* of  $V$  if

$$\rho(g)(w) \in W$$

for all  $g \in G, w \in W$ .

**Definition 2.4.** (Irreducible representation) A representation  $\rho : G \rightarrow GL(V)$  is *irreducible* if its only subrepresentations are the zero subspace and the whole vector space  $V$ . Otherwise we say  $V$  is *reducible*.

## 2.2. Maschke's Theorem and its Converse.

**Proposition 2.5.** (*Maschke's Theorem*) Let  $G$  be a finite group and let  $\mathbb{F}$  be a field of characteristic zero or of positive characteristic not dividing  $|G|$ . If  $V$  is a finite-dimensional representation of  $G$  over  $\mathbb{F}$  and  $U$  is any subrepresentation of  $V$ , then  $V$  has a subrepresentation  $W$  such that  $V = U \oplus W$ .

Maschke's theorem implies that every finite-dimensional representation  $V$  of a finite group  $G$  over a field whose characteristic does not divide the order of the group can be expressed uniquely as a direct sum of irreducible representations. A partial converse of Maschke's theorem holds as well: if  $G$  is a finite group and  $V$  is a representation over a field  $\mathbb{F}$  whose order *does* divide  $|G|$ , then  $V$  may not be completely reducible. That is, it is possible for there to exist some subrepresentation  $U$  of  $V$  which has no complement subrepresentation  $W$  in  $V$ .

For an example of Maschke's Theorem failing when the characteristic of  $\mathbb{F}$  divides  $|G|$ , consider:

**Example 2.6.** Let  $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle$  and let  $V = \overline{\mathbb{F}_p}^2$  over  $\overline{\mathbb{F}_p}$ . Define an action of  $G$  on  $V$  via  $g \cdot e_1 = e_1$  and  $g \cdot e_2 = e_1 + e_2$ . Note that this is indeed a representation, as  $\rho(0) = \rho(p \cdot g) = \rho(g)^p = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  since the characteristic of the underlying field is  $p$ . Notice that  $\langle e_1 \rangle$  is stable under the action of  $G$  and that  $\langle e_1 \rangle$  is isomorphic to the trivial representation. We claim that there does not exist  $V'$  a subrepresentation of  $V$  such that  $V = \langle e_1 \rangle \oplus V'$ . For, if there was, then  $V/\langle e_1 \rangle \cong V'$ . But  $V/\langle e_1 \rangle$  is isomorphic to  $\langle \overline{e_2} \rangle$ , which, according to the action of  $G$  on  $V$ , is isomorphic to the trivial representation, as

$$g \cdot \overline{e_2} = \overline{e_1} + \overline{e_2} = \overline{e_2}.$$

This implies that  $V$  is isomorphic to the direct sum of two copies of the trivial representation, and hence that the fixed subspace of  $V$ , denoted  $V^G$ , is two-dimensional. But  $V^G$  is one-dimensional: if  $\alpha_1 e_1 + \alpha_2 e_2 \in V^G$ , then  $g \cdot (\alpha_1 e_1 + \alpha_2 e_2) = \alpha_1 e_1 + \alpha_2 (e_1 + e_2) = \alpha_1 e_1 + \alpha_2 e_2$  implies that  $\alpha_2 = 0$  and hence that  $V^G = \langle e_1 \rangle$ .

The key to this example is that the defined action of  $G$  on  $V$  fails to be a representation when the characteristic of the field underlying  $V$  is not divisible by  $p$ .

3. CONSTRUCTING  $I_r^{-1}$ -INVARIANT SUBSPACES.

**3.1. Characters of  $B_r$ .** It is known ([1]) that every character  $\chi : B_1 \rightarrow \overline{\mathbb{F}_p}^\times$  is of the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto a^\ell (ad)^s$$

for some  $0 \leq \ell, s \leq p-2$ . An analogue holds in the general  $B_r$  case, in the sense that every character  $\chi : B_r \rightarrow \overline{\mathbb{F}_p}^\times$  is of the form

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^\ell (a_0 d_0)^s$$

for some  $0 \leq \ell, s \leq p-2$ , and hence only depends on the constant terms  $a_0, d_0$  belonging to  $\mathbb{F}_p^\times$ .

**Lemma 3.1.** *Every character  $\chi_i : (\mathbb{F}_p[t]/(t^r))^\times \rightarrow \overline{\mathbb{F}_p}^\times$  is completely determined by where it maps the constant terms belonging to  $\mathbb{F}_p^\times$ . That is,  $\chi_i(a_0 + a_1t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$ .*

*Proof.* We first show that  $\chi_i : (\mathbb{F}_p[t]/(t^r))^\times \rightarrow \overline{\mathbb{F}_p}^\times$  must always map an element of the form  $1 + a_1t + \cdots + a_{r-1}t^{r-1}$  to 1. By applying the monomial identity  $(x+y)^p = x^p + y^p$  in the field  $\mathbb{F}_p$  inductively, we obtain  $(1 + a_1t + \cdots + a_{r-1}t^{r-1})^p = 1 + a_1t^p + \cdots + a_{r-1}t^{p(r-1)}$ . Choosing the minimal  $k \in \mathbb{N}$  such that  $p^k \geq r$  gives

$$\begin{aligned} (1 + a_1t + \cdots + a_{r-1}t^{r-1})^{p^k} &= 1 + a_1t^{p^k} + \cdots + a_{r-1}t^{p^k(r-1)} \\ &= 1 \end{aligned}$$

Thus  $\chi_i(1 + a_1t + \cdots + a_{r-1}t^{r-1})$  must have order dividing  $p^k$  in  $\overline{\mathbb{F}_p}^\times$ . But no elements in  $\overline{\mathbb{F}_p}^\times$  have order  $p^\ell$  for any  $1 \leq \ell \leq k$ , since  $\overline{\mathbb{F}_p}^\times = \bigcup_{k \in \mathbb{N}} \mathbb{F}_{p^k}^\times$ . Hence  $\chi_i(1 + a_1t + \cdots + a_{r-1}t^{r-1})$  has order 1, making it the identity element of  $\overline{\mathbb{F}_p}^\times$ .

Now  $\chi_i(a_0 + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0 \cdot (1 + \frac{a_1}{a_0}t + \cdots + \frac{a_{r-1}}{a_0}t^{r-1})) = \chi_i(a_0)\chi_i(1 + \frac{a_1}{a_0}t + \cdots + \frac{a_{r-1}}{a_0}t^{r-1}) = \chi_i(a_0)$ , completing the proof.  $\square$

**Lemma 3.2.** *Every multiplicative map  $\chi : B_r \rightarrow \overline{\mathbb{F}_p}^\times$  is of the form*

$$\begin{aligned} \chi : B_r &\rightarrow (\mathbb{F}_p[t]/(t^r))^\times \\ \begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} &\mapsto a_0^\ell (a_0 d_0)^s \end{aligned}$$

for some  $0 \leq \ell, s \leq p-2$ .

*Proof.* We first show that any matrix  $\begin{bmatrix} 1 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & 1 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}$  must get mapped to 1 in  $\overline{\mathbb{F}_p}^\times$  under any multiplicative map  $\chi$ . Notice that

$$\begin{bmatrix} 1 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & 1 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}^p = \begin{bmatrix} 1 + \cdots & pb(1 + \cdots) \\ 0 & 1 + \cdots \end{bmatrix}$$

and since  $pb \equiv 0$  in  $\mathbb{F}_p$ , we must have that

$$\chi\left(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}\right)^p = \chi\left(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}\right)^p = \chi\left(\begin{bmatrix} 1 + \cdots & 0 \\ 0 & 1 + \cdots \end{bmatrix}\right).$$

Because any multiplicative map on a diagonal matrix in  $G_r$  must be the product of two multiplicative maps on each entry in the diagonal, and since such diagonal elements belong to  $(\mathbb{F}_p[t]/(t^r))^\times$ , each of the two multiplicative maps must be of the form in Lemma 3.1. In particular this shows that  $\chi\left(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}\right) = 1$ .

Now any matrix  $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in B_r$  can be expressed as

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix}$$

so  $\chi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = \chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right)$ . But a multiplicative map on a diagonal matrix is again just the product of multiplicative maps on its diagonal entries, implying that  $\chi = \chi_1 \times \chi_2$  where each  $\chi_i$  is a map as in Lemma 3.1. In particular, since Lemma 3.1 shows that  $\chi_i(a_0 + a_1t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$  for an element  $a_0 + \cdots + a_{r-1}t^{r-1} \in (\mathbb{F}_p[t]/(t^r))^\times$ , then we conclude

$$\chi\left(\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}\right) = \chi_1(a_0) \cdot \chi_2(d_0).$$

But both  $a_0$  and  $d_0$  belong to  $\mathbb{F}_p^\times$ , a cyclic group of order  $p-1$ , and hence  $\chi_1(a_0)$  and  $\chi_2(d_0)$  must be  $(p-1)^{st}$  roots of unity in  $\overline{\mathbb{F}_p}^\times$ . Since all  $p-1$  such roots of unity lie in  $\mathbb{F}_p^\times \subset \overline{\mathbb{F}_p}^\times$ , then both  $\chi_1$  and  $\chi_2$  map into  $\mathbb{F}_p^\times$ , which is cyclic of order  $p-1$ . This implies that  $\chi_1(a_0) = a_0^m$  for some  $0 \leq m \leq p-2$  and  $\chi_2(d_0) = d_0^s$  for some  $0 \leq s \leq p-2$ . Alternatively, we can express  $a_0^m \cdot d_0^s$  as  $a_0^\ell (a_0 d_0)^s$  where  $\ell = m - s \pmod{p}$ .  $\square$

**Remark 3.3.** In this paper we abuse notation and write  $\frac{a}{d} : B_r \rightarrow \overline{\mathbb{F}_p}^\times$  to mean the map  $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto a_0 d_0^{-1} = a_0 d_0^{p-2}$ , since the lemmas above guarantee that any character  $\chi : \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \rightarrow \overline{\mathbb{F}_p}^\times$  is of the form  $a_0^\ell (a_0 d_0)^s$ .

**3.2. Induction from Borel subgroup.** Let  $\chi : B_r \rightarrow \overline{\mathbb{F}_p}^\times$  be a character. For  $r \geq 2$ , we define the Iwahori subgroup

$$(7) \quad I_r^{r-1} := \left\{ \begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in G_r \right\}$$

to be the invertible matrices in  $G_r$  whose  $(2,1)$ -entry have no terms of the form  $c_k t^k$  for  $0 \leq k \leq r-2$ . Equivalently, we may define  $I_r^{r-1}$  to be the preimage of  $B_{r-1}$  under the surjective homomorphism

$$(8) \quad \begin{aligned} \pi : G_r &\twoheadrightarrow G_{r-1} \\ t^{r-1} &\mapsto 0. \end{aligned}$$

Let  $\sigma := \text{Ind}_{B_r}^{I_r^{r-1}}(\chi)$ . As  $\dim(\sigma) = [I_r^{r-1} : B_r] = p$ , we fix a basis  $\{\delta_0, \dots, \delta_{p-1}\}$  of  $\sigma$  by setting

$$(9) \quad \begin{aligned} \delta_j &: I_r^{r-1} \rightarrow \overline{\mathbb{F}_p}^\times \\ \delta_j(i) &= \mathbb{1}_{B_r x_j} \cdot \chi(i x_j^{-1}) \end{aligned}$$

where  $B_r x_j := B_r \begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix}$  and  $\mathbb{1}$  is the indicator function. As each of these  $p$  functions has support on a distinct right coset of  $B_r$  in  $I_r^{r-1}$ , they are linearly independent. If  $bi \in B_r x_j$ , we have

$$\delta_j(bi) = \chi(bi x_j^{-1}) = \chi(b) \delta_j(i)$$

and if  $bi \notin B_r x_j$ , then  $i \notin B_r x_j$ , and

$$\delta_j(bi) = 0 = \chi(b) \delta_j(i),$$

which shows that these functions belong to  $\sigma$ . We note that by composition of induction, constructing a Jordan-Hölder series for  $\text{Ind}_{B_r}^{G_r}(\chi)$  is equivalent to constructing a Jordan-Hölder series for  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma)$ . Thus one may initially construct a Jordan-Hölder series for  $\sigma$  and “induce up” to get a filtration for  $\text{Ind}_{B_r}^{G_r}(\chi)$ , which can then be further refined to a full composition series for  $\text{Ind}_{B_r}^{G_r}(\chi)$ . Since this is the approach we take in Theorem 1.1, we first construct a Jordan-Hölder series for  $\sigma$ .

**Proposition 3.4.** *For every  $0 \leq k \leq p$  there exists a  $k$ -dimensional  $I_r^{r-1}$ -invariant subspace  $\sigma^{(k)}$  of  $\sigma$ , such that*

$$0 \subset \sigma^{(1)} \subset \cdots \subset \sigma^{(p-1)} \subset \sigma$$

*is a Jordan-Hölder series for  $\sigma$ .*

The cases of  $k = 0$  and  $k = p$  are trivial. For each  $1 \leq k \leq p-1$ , we construct a  $k$ -dimensional subspace of  $\sigma$  denoted  $\sigma^{(k)}$ :

$$(10) \quad \sigma^{(k)} := \left\langle \sum_{j=0}^{p-1} \binom{j}{j} \delta_j, \sum_{j=0}^{p-2} \binom{j+1}{j} \delta_j, \dots, \sum_{j=0}^{p-k} \binom{j+k-1}{j} \delta_j \right\rangle$$

Setting  $S_\ell := \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j$  allows us to express  $\sigma^{(k)} = \langle S_1, \dots, S_k \rangle$ . From the construction of  $\sigma^{(k)}$  it is clear that we get a filtration of subspaces. To see that the vectors  $\{S_\ell : 1 \leq \ell \leq k\}$  are linearly independent and so form a basis for  $\sigma^{(k)}$ , we notice that if we express each sum as a tuple in the basis  $\delta_0, \dots, \delta_{p-1}$ , then putting the  $k$   $p$ -tuples into a  $p \times k$  matrix gives

$$(11) \quad A = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{k-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{k}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{p-2}{p-2} & \binom{p-1}{p-2} & 0 & \cdots & 0 \\ \binom{p-1}{p-1} & 0 & 0 & \cdots & 0 \end{bmatrix}_{p \times k}.$$

We verify that the columns  $\{\vec{v}_1, \dots, \vec{v}_k\}$  are linearly independent by noting that if

$$a_1 \vec{v}_1 + \cdots + a_k \vec{v}_k = 0$$

then in particular  $a_1 \binom{p-1}{p-1} = 0$ , implying that  $a_1 = 0$ . Then since  $a_1 \binom{p-2}{p-2} + a_2 \binom{p-1}{p-2} = 0$ , we deduce that  $a_2 = 0$ . The fact that  $A_{ij} = 0$  for  $j \geq p - i + 2$  allows us to inductively deduce that  $a_i = 0$  for  $1 \leq i \leq k$ .

To see that  $\sigma^{(k)}$  is  $I_r^{r-1}$ -invariant and therefore a subrepresentation of  $\sigma$ , we check that it is invariant under every generator of  $I_r^{r-1}$ . By the Iwahori factorization of  $I_r^{r-1}$ , any matrix  $\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in I_r^{r-1}$  is expressible as

$$\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1} + d \end{bmatrix} \cdot \begin{bmatrix} 1 & ba^{-1} \\ 0 & 1 \end{bmatrix}$$

which allows us to conclude that

$$(12) \quad I_r^{r-1} = \left\langle \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\rangle$$

for  $0 \leq k \leq r-1$  and  $a, d \in (\mathbb{F}_p[t]/(t^r))^\times$ . In order to determine how  $I_r^{r-1}$  acts on each subspace  $\sigma^{(k)}$ , we first observe how each generator of  $I_r^{r-1}$  in (12) acts on an ordinary basis vector  $\delta_j$  of  $\sigma$ .

**Lemma 3.5.** *Let  $\chi : B_r \rightarrow \overline{\mathbb{F}_p}^\times$  be a character of  $B_r$  and let  $\sigma = \text{Ind}_{B_r}^{I_r^{r-1}}(\chi)$ . Let  $\delta_0, \dots, \delta_{p-1}$  be the ordered basis of  $\sigma$  given in (9). Then the generators of  $I_r^{r-1}$  act on each  $\delta_j$  via*

$$(13) \quad \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j = \delta_j$$

$$(14) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j = \delta_{j-1}$$

$$(15) \quad \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j = \chi \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) \cdot \delta_{\frac{d_0}{a_0}j}$$

where all indices  $j$  are taken modulo  $p$ .

*Proof.* We have that

$$\left( \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j \right)(i) \neq 0 \iff \delta_j \left( i \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \right) \neq 0$$

by definition of the  $G_r$  action on  $\sigma$ . But

$$\delta_j \left( i \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \right) \neq 0 \iff i \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \in B_r \begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \iff i \in B_r \begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -t^k \\ 0 & 1 \end{bmatrix} \iff i \in B_r \begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix}$$

and thus  $\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j$  has support on  $B_r x_j$ . Now suppose  $i \in B_r x_j$ , so  $i = b \cdot \begin{bmatrix} 1 & t^k \\ j t^{r-1} & 1 \end{bmatrix}$  for some  $b \in B_r$ . Then

$$\begin{aligned} \left( \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j \right)(i) &= \delta_j(b \begin{bmatrix} 1 & 0 \\ j t^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) = \delta_j(b \begin{bmatrix} 1 & t^k \\ j t^{r-1} & j t^{r-1+k} + 1 \end{bmatrix}) = \chi(b \begin{bmatrix} 1 & t^k \\ j t^{r-1} & j t^{r-1+k} + 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -j t^{r-1} & 1 \end{bmatrix}) \\ &= \chi(b \begin{bmatrix} 1 - j t^{r+k-1} & t^k \\ -j^2 t^{2r-2+k} & j t^{r-1+k} + 1 \end{bmatrix}) \\ &= \chi(b) \chi \left( \begin{bmatrix} 1 - j t^{r+k-1} & t^k \\ 0 & 1 + j t^{r-1+k} \end{bmatrix} \right) \\ &= \delta_j(i) \end{aligned}$$

since  $\chi(\begin{bmatrix} 1 & \dots & b \\ 0 & 1 & \dots \end{bmatrix}) = 1$  by the proof of Lemma 3.2. Hence  $\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j = \delta_j$ . A similar argument shows that  $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j$  has support on  $B_r x_{j-1}$ , and if  $i = b x_{j-1}$  for some  $b \in B_r x_{j-1}$ , then

$$\left( \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j \right)(b \begin{bmatrix} 1 & 0 \\ (j-1)t^{r-1} & 1 \end{bmatrix}) = \delta_j(b \begin{bmatrix} 1 & 0 \\ (j-1)t^{r-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}) = \delta_j(b \begin{bmatrix} 1 & 0 \\ j t^{r-1} & 1 \end{bmatrix}) = \chi(b) = \delta_{j-1}(i),$$

allowing us to conclude  $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j = \delta_{j-1}$ . Finally, an analogous computation shows that  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j$  has support on  $B_r x_{\frac{d_0}{a_0} j}$ , so we suppose  $i = b \begin{bmatrix} 1 & 0 \\ \frac{d_0}{a_0} j t^{r-1} & 1 \end{bmatrix}$  for some  $b \in B_r$ , and find that

$$\left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j \right)(i) = \delta_j(b \begin{bmatrix} 1 & 0 \\ \frac{d_0}{a_0} j t^{r-1} & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) = \delta_j(b \begin{bmatrix} a & 0 \\ d_0 j t^{r-1} & d \end{bmatrix}) = \chi(b \begin{bmatrix} a & 0 \\ d_0 j t^{r-1} & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -j t^{r-1} & 1 \end{bmatrix}) = \chi(b) \chi \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right)$$

whereas

$$\delta_{\frac{d_0}{a_0} j}(b \begin{bmatrix} 1 & 0 \\ \frac{d_0}{a_0} j t^{r-1} & 1 \end{bmatrix}) = \chi(b)$$

by definition, which shows that  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \delta_{\frac{d_0}{a_0} j}$  as desired.  $\square$

Recall that we wish to show  $\sigma^{(k)}$  is  $I_r^{r-1}$ -invariant. Since  $\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}$  acts trivially on each  $\delta_j$ , then  $\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot S_\ell = S_\ell$  for each  $1 \leq \ell \leq k$ . The actions by the other generators are more involved.

**Lemma 3.6.**

$$(16) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot S_\ell = \sum_{m=1}^{\ell} S_m$$

so that if the basis vectors of  $\sigma^{(k)}$  are ordered, then acting on each basis vector by  $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}$  yields a sum of the vector being acted on and the preceding basis vectors, thus remaining in  $\sigma^{(k)}$ .

*Proof.* We prove (16) by induction on  $\ell$ : when  $\ell = 1$ , we have

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-1} \binom{j}{j} \delta_j = \sum_{j=0}^{p-1} \binom{j}{j} \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j = \sum_{j=0}^{p-1} \binom{j}{j} \delta_{j-1} = \sum_{j=0}^{p-1} \binom{j}{j} \delta_j$$

so that the base case holds. Now suppose (16) holds for some  $\ell \in \mathbb{N}$ ,  $\ell < k$ . We wish to show the claim holds for  $\ell + 1$ . By the binomial coefficient recurrence relation  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$  (where  $\binom{n-1}{k-1} = 0$  whenever  $k < 1$ ), and by the fact that we can express  $\sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j} \delta_j = \sum_{j=0}^{p-\ell} \binom{j+\ell}{j} \delta_j$  since the coefficient  $\binom{p}{p-\ell}$  of  $\delta_{p-\ell}$  is zero mod  $p$ , we get

$$(17) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j} \delta_j = \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell}{j} \delta_j = \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \left( \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j + \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j-1} \delta_j \right).$$

Our inductive hypothesis guarantees that

$$(18) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j = \sum_{m=0}^{\ell} \sum_{j=0}^{p-m} \binom{j+m-1}{j} \delta_j,$$

while

$$(19) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j-1} \delta_j = \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j-1} \delta_{j-1} = \sum_{j=1}^{p-\ell} \binom{j+\ell-1}{j-1} \delta_{j-1} = \sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j} \delta_j$$

since the coefficient  $\binom{j+\ell-1}{j-1} = 0$  for  $j = 0$ , by convention. Hence from (17), (18) and (19), we conclude that

$$(20) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-(\ell+1)} \binom{j+\ell}{j} \delta_j = \sum_{m=1}^{\ell+1} \sum_{j=0}^{p-m} \binom{j+m-1}{j} \delta_j$$

$$(21) \quad \implies \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot S_{\ell+1} = \sum_{m=1}^{\ell+1} S_m.$$

This confirms that  $\sigma^{(k)}$  is invariant under the action by  $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}$ .  $\square$

It remains to show that  $\sigma^{(k)}$  is invariant under the action by  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ . As in the  $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}$  case, we show that acting on  $S_\ell \in \sigma^{(k)}$  by  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$  yields an  $\mathbb{F}_p$ -linear combination of  $S_m \in \sigma^{(k)}$  for  $m \leq \ell$ .

**Lemma 3.7.** *Given  $a, d \in \mathbb{F}_p^\times \cong (\mathbb{Z}/p\mathbb{Z})^\times$ , let  $\alpha_i := \binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}}$ , where  $ad^{-1}$  is a representative in  $\mathbb{N}$  of the equivalence class  $ad^{-1}$  in  $\mathbb{Z}/p\mathbb{Z}$ . Then*

$$(22) \quad \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_\ell = \chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \sum_{m=1}^{\ell} c_m S_m$$

where each  $c_m$  is given by  $\sum_{i=1}^m (-1)^{i+1} \binom{m-1}{i-1} \alpha_i$ .

*Proof.* We first ensure that the  $\alpha_i$  are well-defined up to mod  $p$ , such that they give the same binomial coefficient mod  $p$  regardless of the choice of  $ad^{-1}$  in  $\mathbb{N}$ . It suffices to show that, given a representative of  $ad^{-1} \in \mathbb{N}$ ,

$$(23) \quad \binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}} \equiv \binom{(p-i)(ad^{-1}+pk)+\ell-1}{(p-i)(ad^{-1}+pk)} \pmod{p}$$

for  $k \in \mathbb{N}$ . Let the base  $p$  expansion of  $(p-i)ad^{-1}+\ell-1$  be given by  $a_r p^r + \dots + a_1 p + a_0$ . Since  $\ell-1 \leq p-2$ , the base  $p$  expansion of  $\ell-1$  is given by  $0p^r + \dots + 0p + \ell-1$ , so by Lucas' theorem we have

$$\begin{aligned} \binom{(p-i)ad^{-1}+\ell-1}{(p-i)ad^{-1}} &= \binom{(p-i)ad^{-1}+\ell-1}{\ell-1} \equiv \binom{a_r}{0} \cdots \binom{a_0}{\ell-1} \pmod{p} \\ &\equiv \binom{a_0}{\ell-1} \pmod{p}. \end{aligned}$$

Thus it suffices to show that  $(p-i)ad^{-1}+\ell-1$  and  $(p-i)(ad^{-1}+pk)+\ell-1$  have the same constant term in their base  $p$  expansions. This follows quickly from the fact that their difference is given by  $pk(p-i)$ , which is a multiple of  $p$  and so has no constant term in its base  $p$  expansion. We conclude that  $\alpha_i$  is independent of the choice of  $ad^{-1} \in \mathbb{N}$ . In particular we may always take the canonical representative.

By the action of  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$  on each  $\delta_j$ , we have

$$(24) \quad \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_\ell = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j = \chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_{\frac{a}{d}j}.$$

For  $0 \leq n \leq p-1$ , we see that  $\delta_n$  appears in the right hand sum of (24) with a coefficient of  $\chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$  (where  $\frac{a}{d}$  is shorthand for the representative in  $\mathbb{N}$  of  $ad^{-1}$ ), and since  $\delta_n$  appears in each vector  $S_m = \sum_{j=0}^{p-m} \binom{j+m-1}{j} \delta_j$  with a coefficient of  $\binom{n+m-1}{n}$  for the respective  $1 \leq m \leq \ell$ , it suffices to verify

$$c_1 \binom{n}{n} + c_2 \binom{n+1}{n} + \dots + c_\ell \binom{n+\ell-1}{n} \equiv \binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$$

for the proposed coefficients  $c_1, \dots, c_\ell$ . That is, we wish to show

$$(25) \quad \sum_{r=1}^{\ell} \binom{n+r-1}{n} \sum_{i=1}^r (-1)^{i+1} \binom{r-1}{i-1} \alpha_i = \alpha_{p-n}.$$

Counting how often each  $\alpha_r$  appears in the left hand side of (25) allows us to express

$$(26) \quad \sum_{r=1}^{\ell} \binom{n+r-1}{n} c_r = \sum_{r=1}^{\ell} (-1)^{r+1} \left( \sum_{j=r-1}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1} \right) \alpha_r$$

such that our goal is to show

$$(27) \quad \sum_{r=1}^{\ell} (-1)^{r+1} \left( \sum_{j=r-1}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1} \right) \alpha_r = \alpha_{p-n}.$$

When  $n = 0$ , we need to show that  $\sum_{r=1}^{\ell} \binom{r-1}{0} c_r = \alpha_p = \binom{\ell-1}{0} = 1$ . By (26) we know that

$$\sum_{r=1}^{\ell} c_r = \sum_{r=1}^{\ell} (-1)^{r+1} \sum_{j=r-1}^{\ell-1} \binom{j}{0} \binom{j}{r-1} \alpha_r = \sum_{r=1}^{\ell} (-1)^{r+1} \binom{\ell}{r} \alpha_r.$$

Writing

$$\alpha_1 = \binom{(p-1)\frac{a}{d} + \ell - 1}{(p-1)\frac{a}{d}} = \frac{1}{(\ell-1)!} (\ell-1 - \frac{a}{d}) \cdots (1 - \frac{a}{d})$$

and letting the variable  $x$  stand in for  $\frac{a}{d}$ , we have that

$$\alpha_1 = \frac{1}{(\ell-1)!} (a_{\ell-1} x^{\ell-1} + a_{\ell-2} x^{\ell-2} + \cdots + a_1 x + (\ell-1)!)$$

for some coefficients  $a_{\ell-1}, \dots, a_1$ . Then

$$\alpha_r = \frac{1}{(\ell-1)!} ((-1)^{\ell-1} r^{\ell-1} x^{\ell-1} + \cdots + a_1 r x + (\ell-1)!)$$

so that the constant term of  $\sum_{r=1}^{\ell} c_r$ , when viewed as a polynomial in  $x = \frac{a}{d}$ , is given by

$$\sum_{r=1}^{\ell} (-1)^{r+1} \binom{\ell}{r} \frac{(\ell-1)!}{(\ell-1)!} = (-1) \sum_{r=1}^{\ell} (-1)^r \binom{\ell}{r} = (-1) \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} - (-1) = 1$$

using  $\sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} = 0$ . On the other hand, the coefficient of  $x^m$  in the polynomial  $\sum_{r=1}^{\ell} c_r$  for  $1 \leq m \leq \ell-1$  is given by

$$\sum_{r=1}^{\ell} (-1)^{r+1} r^m \binom{\ell}{r} \frac{a_m}{(\ell-1)!} = \frac{-a_m}{(\ell-1)!} \sum_{r=0}^{\ell} (-1)^r r^m \binom{\ell}{r} = 0$$

due to the combinatorial sum identity  $\sum_{r=0}^{\ell} (-1)^r r^m \binom{\ell}{r} = 0$  given in [7]. We conclude that  $\sum_{r=1}^{\ell} c_r = 1 = \alpha_p$ .

To prove  $\sum_{r=1}^{\ell} \binom{n+r-1}{n} c_r = \alpha_{p-n}$  for  $1 \leq n \leq p-1$ , we compare the coefficient of  $x^m$  in both expressions. Since the coefficient of  $x^m$  in  $\alpha_r$  is given by  $\frac{a_m}{(\ell-1)!} r^m$ , then from (26) we deduce that the coefficient of  $x^m$  in  $\sum_{r=1}^{\ell} \binom{n+r-1}{n} c_r$  must be

$$\sum_{r=1}^{\ell} (-1)^{r+1} \frac{a_m}{(\ell-1)!} r^m \sum_{j=r-1}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1}.$$

On the other hand, the coefficient of  $x^m$  in  $\alpha_{p-n}$  is given by  $(-n)^m \frac{a_m}{(\ell-1)!}$ , so it suffices to prove

$$(28) \quad \sum_{r=1}^{\ell} (-1)^{r+1} r^m \sum_{j=r-1}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1} = (-n)^m.$$

Because  $\binom{j}{r-1} = 0$  whenever  $j < r-1$ , we can express the left hand side of (28) as

$$(29) \quad \sum_{r=1}^{\ell} (-1)^{r+1} r^m \sum_{j=0}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1}.$$

Identity 3.155 in [6] tells us that  $\sum_{k=0}^{s-1} \binom{k}{n} \binom{k+m}{m} = \binom{s}{n} \binom{s+m}{m} \frac{s-n}{m+n+1}$ , which allows us to express (29) as

$$(30) \quad \begin{aligned} \sum_{r=1}^{\ell} (-1)^{r+1} r^m \sum_{j=0}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1} &= \sum_{r=1}^{\ell} (-1)^{r+1} r^m \binom{\ell}{r-1} \binom{\ell+n}{n} \frac{\ell-r+1}{r+n} \\ &= \binom{\ell+n}{n} \sum_{r=1}^{\ell} (-1)^{r+1} r^m \binom{\ell}{r-1} \frac{\ell-r+1}{r+n} \\ &= \binom{\ell+n}{n} \sum_{r=1}^{\ell} (-1)^{r+1} r^m \cdot r \binom{\ell}{r} \frac{1}{r+n} \\ &= \binom{\ell+n}{n} \sum_{r=1}^{\ell} (-1)^{r+1} \binom{\ell}{r} \frac{r^{m+1}}{r+n}. \end{aligned}$$

Finally, identity 1.47 in [6] shows that  $\sum_{k=0}^{\ell} (-1)^k \binom{\ell}{k} \frac{k^j}{x+k} = (-1)^j \frac{x^{j-1}}{\binom{x+\ell}{\ell}}$ , and therefore (30) becomes

$$(31) \quad \begin{aligned} \binom{\ell+n}{n} \sum_{r=1}^{\ell} (-1)^{r+1} \binom{\ell}{r} \frac{r^{m+1}}{r+n} &= \binom{\ell+n}{n} (-1) \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} \frac{r^{m+1}}{r+n} \\ &= \binom{\ell+n}{n} (-1) (-1)^{m+1} \frac{n^m}{\binom{n+\ell}{\ell}} = (-1)^m n^m = (-n)^m \end{aligned}$$

as desired. This proves that there exist  $c_1, \dots, c_{\ell} \in \mathbb{Z}$  such that

$$(32) \quad \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_{\frac{d}{a}j} = \sum_{m=1}^{\ell} c_m \sum_{j=0}^{p-m} \binom{j-m+1}{j} \delta_j$$

which means that there exist  $c_1, \dots, c_{\ell} \in \mathbb{Z}$  such that

$$(33) \quad \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_{\ell} = \sum_{m=1}^{\ell} \chi \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right) c_m S_m.$$

Since the left hand side is given mod  $p$ , we may reduce the right hand side mod  $p$  to conclude that there exist  $c_1, \dots, c_{\ell} \in \overline{\mathbb{F}_p}$  such that (33) holds. Because this holds for all  $1 \leq \ell \leq k$ , we have that  $\sigma^{(k)}$  is invariant under the action by  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ . This lemma also concludes the proof of the proposition.  $\square$

#### 4. PROOF OF MAIN THEOREM.

**4.1. Inducing up to a filtration for  $\text{Ind}_{B_r}^{G_r}(\chi)$ .** Proposition 3.4 gives us a length  $p$  Jordan-Hölder series

$$0 \subset \sigma^{(1)} \subset \dots \subset \sigma^{(p-1)} \subset \sigma.$$

Since each  $\sigma^{(k)}$  is a subrepresentation of  $\sigma$  which is itself a representation of  $I_r^{r-1}$ , then inducing each  $\sigma^{(k)}$  to  $G_r$  gives a filtration

$$0 \subset \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \dots \subset \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma).$$

In order to refine this filtration to a composition series for  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \text{Ind}_{B_r}^{G_r}(\chi)$ , we note that it suffices to find a composition series for  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})$  which begins with  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$  for each  $0 \leq k \leq p-1$ . This is equivalent to finding a composition series for  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$  and then lifting the subrepresentations under the projection map  $q : \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) \rightarrow \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ . Furthermore, since

$$\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)} / \sigma^{(k)})$$

then we only need consider composition series of  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$  in order to answer our original question.

We claim that  $\sigma^{(k+1)}/\sigma^{(k)}$  is equivalent to  $\text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)$  as one-dimensional  $I_r^{r-1}$  representations, where  $\text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)$  refers to the inflation to  $I_r^{r-1}$  of the character sending  $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \chi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) \cdot (\frac{a}{d})^k \in \overline{\mathbb{F}_p}^\times$ . To prove this equivalence it suffices to show that  $I_r^{r-1}$  acts on  $\sigma^{(k+1)}/\sigma^{(k)}$  via multiplication by  $\chi \cdot (\frac{a}{d})^k$ . Again we show this claim only for the three types of generators of  $I_r^{r-1}$ .

**Lemma 4.1.** *The generators  $\begin{bmatrix} 1 & t^\ell \\ 0 & 1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}$  act trivially on  $\sigma^{(k+1)}/\sigma^{(k)}$  for  $0 \leq \ell \leq r-1$  and  $0 \leq k \leq p-1$ .*

*Proof.* Notice  $\sigma^{(k+1)}/\sigma^{(k)} = \langle \overline{S_{k+1}} \rangle$ . Since  $\begin{bmatrix} 1 & t^\ell \\ 0 & 1 \end{bmatrix}$  acts trivially on each  $\delta_j$ , then  $\begin{bmatrix} 1 & t^\ell \\ 0 & 1 \end{bmatrix}$  acts trivially on  $\overline{S_{k+1}}$ . On the other hand, by the proof of Lemma 3.6, we know that

$$(34) \quad \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \overline{S_{k+1}} = \overline{\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot S_{k+1}} = \overline{\sum_{m=1}^{k+1} S_m} = \overline{S_{k+1}}$$

where (34) follows from the fact that  $\overline{S_i} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$  for  $1 \leq i \leq k$ .  $\square$

**Lemma 4.2.** *The generator  $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$  acts on  $\sigma^{(k+1)}/\sigma^{(k)}$  via scaling by  $\chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \cdot (\frac{a}{d})^k$ .*

*Proof.* By Lemma 3.7 we have that

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \overline{S_{k+1}} = \overline{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot S_{k+1}} = \chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) \sum_{m=1}^{k+1} c_m \overline{S_m}$$

and since  $\overline{S_m} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$  for  $1 \leq m \leq k$ , then in  $\sigma^{(k+1)}/\sigma^{(k)}$  we have

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \overline{S_{k+1}} = \chi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) c_{k+1} \overline{S_{k+1}}.$$

To prove our claim it suffices to show that  $c_{k+1} = (\frac{a}{d})^k$ . Recall that by Lemma 3.7, we have

$$c_{k+1} = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} \alpha_i$$

where here  $\alpha_i = \binom{(p-i)\frac{a}{d}+k}{(p-i)\frac{a}{d}} = \frac{(k-i\frac{a}{d}) \cdots (1-i\frac{a}{d})}{k!}$ . In particular, since we may write  $\alpha_1 = \frac{(k-x) \cdots (1-x)}{k!} = \frac{1}{k!}((-1)^k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + k!)$  where  $x = \frac{a}{d}$ , then we have that  $\alpha_i = \frac{1}{k!}((-1)^k i^k x^k + a_{k-1} i^{k-1} x^{k-1} + \cdots + a_1 i x + k!)$  for  $1 \leq i \leq k+1$ . Since the coefficient of  $x^m$  in  $\alpha_i$  is given by  $\frac{a_m}{k!} \cdot i^m$ , then the coefficient of  $x^m$  in the expression of  $c_{k+1}$  is given by

$$(35) \quad \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} \frac{a_m}{k!} i^m = \frac{a_m}{k!} \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m.$$

Since we wish to show that  $c_{k+1} = x^k = (\frac{a}{d})^k$ , it suffices to show that (35) is zero whenever  $0 \leq m \leq k-1$  and is 1 whenever  $m = k$ . When  $m = 0$ , we have that  $a_0 = k!$ , so  $\frac{a_0}{k!} \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^0 = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} = \sum_{i=0}^k (-1)^i \binom{k}{i} = 0$ , as desired. On the other hand, the identity

$$(36) \quad \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} i^m = 0$$

holds for  $1 \leq m \leq k$  (see [7], #3 in 0.154), and since  $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$ , we deduce from (36) that

$$\sum_{i=0}^{k+1} (-1)^i \binom{k}{i} i^m + \sum_{i=0}^{k+1} (-1)^i \binom{k}{i-1} i^m = 0$$

which implies that

$$\sum_{i=0}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m = \sum_{i=0}^{k+1} (-1)^i \binom{k}{i} i^m = \sum_{i=0}^k (-1)^i \binom{k}{i} i^m$$

since  $\binom{k}{k+1} = 0$  by convention. Now  $\sum_{i=0}^k (-1)^i \binom{k}{i} i^m = 0$  for  $0 \leq m \leq k-1$  by the identity in (36), so  $\sum_{i=0}^{k+1} (-1)^{i+1} \binom{k}{i-1} j^m = 0$  for  $0 \leq m \leq k-1$ . When  $m > 0$  we have that  $0^m = 0$ , so we conclude  $\sum_{i=1}^{k+1} (-1)^{i+1} \binom{k}{i-1} i^m = 0$  for  $0 \leq m \leq k-1$  as desired. On the other hand, identity #4 in §0.154 of [7] gives

$$(37) \quad \sum_{j=0}^k (-1)^j \binom{k}{j} j^k = (-1)^k k!,$$

which in combination with (36) and the fact that  $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$  gives

$$\begin{aligned} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} j^k &= \sum_{j=0}^{k+1} (-1)^j \binom{k}{j} j^k + \sum_{j=0}^{k+1} (-1)^j \binom{k}{j-1} j^k \\ \implies \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k}{j-1} j^k &= (-1)^k k!. \end{aligned}$$

This is precisely what we wished to show. Hence the coefficient of  $x^m$  in  $c_{k+1}$  is  $\frac{a_m}{k!} \cdot 0 = 0$  for  $0 \leq m \leq k-1$  while the coefficient of  $x^k$  is  $\frac{(-1)^k}{k!} \cdot (-1)^k k! = (-1)^{2k} = 1$ , completing the proof that  $c_{k+1} = (\frac{a}{d})^k$ , and therefore that  $[\frac{a}{0} \frac{0}{d}] \cdot \overline{S_{k+1}} = \chi([\frac{a}{0} \frac{0}{d}]) \cdot (\frac{a}{d})^k \overline{S_{k+1}}$ .  $\square$

Recall we want to show that  $\sigma^{(k+1)}/\sigma^{(k)}$  is equivalent to  $\text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)$  as  $I_r^{r-1}$  representations. Let  $T : \langle \overline{S_{k+1}} \rangle \rightarrow \mathbb{F}_p$  be the isomorphism sending  $\overline{S_{k+1}} \mapsto 1$ . For all  $[\frac{a}{ct^{r-1}} \frac{b}{d}] \in I_r^{r-1}$ , we have

$$(38) \quad \begin{aligned} T\left(\left[\begin{array}{cc} a & b \\ ct^{r-1} & d \end{array}\right] \cdot \overline{S_{k+1}}\right) &= T\left(\left[\begin{array}{cc} 1 & 0 \\ ca^{-1}t^{r-1} & 1 \end{array}\right] \left[\begin{array}{cc} a & 0 \\ 0 & -ca^{-1}bt^{r-1} + d \end{array}\right] \left[\begin{array}{cc} 1 & ba^{-1} \\ 0 & 1 \end{array}\right] \cdot \overline{S_{k+1}}\right) \\ &= T\left(\left[\begin{array}{cc} 1 & 0 \\ ca^{-1}t^{r-1} & 1 \end{array}\right] \left[\begin{array}{cc} a & 0 \\ 0 & -ca^{-1}bt^{r-1} + d \end{array}\right] \cdot \overline{S_{k+1}}\right). \end{aligned}$$

Now  $([\frac{a}{0} \frac{0}{-ca^{-1}bt^{r-1} + d}] \cdot \delta_j)(i) \neq 0$  if and only if  $i[\frac{a}{0} \frac{0}{-ca^{-1}bt^{r-1} + d}] \in B_r[\frac{1}{jt^{r-1}} \frac{0}{1}]$ , which holds if and only if  $i \in B_r[\frac{1}{jt^{r-1}} \frac{0}{1}][\frac{a}{0} \frac{0}{-ca^{-1}bt^{r-1} + d}]^{-1} = B_r[\frac{a}{jt^{r-1}} \frac{0}{1}]$ . A similar argument as the one for  $[\frac{a}{0} \frac{0}{d}] \cdot \delta_j = \chi([\frac{a}{0} \frac{0}{d}]) \delta_{\frac{a}{d}j}$  reveals that  $[\frac{a}{0} \frac{0}{-ca^{-1}bt^{r-1} + d}] \cdot \delta_j = \chi([\frac{a}{0} \frac{0}{d}]) \delta_{\frac{a}{d}j}$ , and therefore Lemma 4.2 applies to (38) to give  $\chi([\frac{a}{0} \frac{0}{d}]) (\frac{a}{d})^k \cdot T(\sum_{j=0}^{p-k} \binom{j+k-1}{j} \overline{\delta_j}) = \chi([\frac{a}{0} \frac{0}{d}]) (\frac{a}{d})^k$ . On the other hand, we have that

$$(39) \quad \text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k) \left( \left[\begin{array}{cc} a & b \\ ct^{r-1} & d \end{array}\right] (T(S_k)) \right) = (\chi \cdot (\frac{a}{d})^k) \left( \left[\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right] (T(S_k)) \right) = \chi \left( \left[\begin{array}{cc} a & 0 \\ 0 & d \end{array}\right] \right) (\frac{a}{d})^k$$

which shows that  $T \circ \sigma^{(k+1)}/\sigma^{(k)}([\frac{a}{ct^{r-1}} \frac{b}{d}]) = \text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)([\frac{a}{ct^{r-1}} \frac{b}{d}]) \circ T$ , and hence that  $\sigma^{(k+1)}/\sigma^{(k)}$  and  $\text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k)$  are isomorphic as  $I_r^{r-1}$ -representations.

Now because the diagram

$$\begin{array}{ccc} I_r^{r-1} & \xrightarrow{t^{r-1} \mapsto 0} & B_{r-1} \\ \downarrow & & \downarrow \\ G_r & \xrightarrow{t^{r-1} \mapsto 0} & G_{r-1} \end{array}$$

commutes, the commutativity of inflation and induction gives  $\text{Ind}_{I_r^{r-1}}^{G_r} \text{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi \cdot (\frac{a}{d})^k) \cong \text{Inf}_{G_{r-1}}^{G_r} \text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k)$ . This implies  $\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \text{Inf}_{G_{r-1}}^{G_r} \text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k)$ , completing the proof of Theorem 1.1.

**4.2. A remark on the inductive construction.** Theorem 1.1 tells us what the successive quotients in the filtration given in (5) look like, but it doesn't explicitly tell us what the Jordan-Hölder series of  $\text{Ind}_{B_r}^{G_r}(\chi)$  looks like. Fortunately, we just proceed inductively: once we know that

$$\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \text{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \text{Inf}_{G_{r-1}}^{G_r} \text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k)$$

then we can set out to find a Jordan-Hölder series of  $\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k)$  (using the same process as in our original problem) and then “piece it in” between  $\text{Ind}_{B_{r-1}}^{G_r}(\sigma^{(k)})$  and  $\text{Ind}_{B_{r-1}}^{G_r}(\sigma^{(k+1)})$  in the filtration for  $\text{Ind}_{B_r}^{G_r}(\chi)$ . Since the literature already contains the Jordan-Hölder series for the mod  $p$  principal series representations of  $\text{Ind}_{B_1}^{G_1}(\chi)$ , we have all the parts necessary to complete the original filtration to a full Jordan-Hölder series.

## 5. SEMISIMPLIFICATIONS

From Theorem 1.1 we deduce that

$$(40) \quad (\text{Ind}_{B_r}^{G_r}(\chi))^{ss} = (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \oplus \cdots \oplus (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss} \oplus \cdots \oplus (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^{p-1}))^{ss} \\ = (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \oplus \cdots \oplus (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss} \oplus \cdots \oplus (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}$$

where inflations to  $G_r$  are always implicitly assumed. In particular, we see that  $(\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}$  appears twice in the direct sum of (40), while  $(\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss}$  appears once in the direct sum for every  $1 \leq k \leq p-2$ . Hence we may express

$$(41) \quad (\text{Ind}_{B_r}^{G_r}(\chi))^{ss} = ((\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss})^2 \oplus \bigoplus_{k=1}^{p-2} (\text{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss}.$$

Since the semisimplifications of  $\text{Ind}_{B_1}^{G_1}(\chi)$  are known for all characters  $\chi : B(GL_2(\mathbb{F}_p)) \rightarrow \overline{\mathbb{F}_p}^\times$  (Lemma 2.2 in [3]), it is desirable to express (41) explicitly in terms of  $(\text{Ind}_{B_1}^{G_1}(\chi))^{ss}$  for various  $\chi$ . We claim that we may continue simplifying (41) inductively to obtain:

**Corollary 5.1.** *For a prime  $p$ ,  $(\text{Ind}_{B_r}^{G_r}(\chi))^{ss} = ((\text{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2} ((\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}}$ .*

*Proof.* We prove the corollary by induction on  $r$ . When  $r = 1$ , the claim is that

$$(\text{Ind}_{B_1}^{G_1}(\chi))^{ss} = ((\text{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^0+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2} ((\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^0-1}{p-1}}$$

which is easily seen to be true when one simplifies the exponents on the right hand side of the equality. Suppose the claim in the proposition holds for some  $r \in \mathbb{N}$ . We wish to show it holds for  $r+1$ . As a corollary of Theorem 1.1, we have that

$$(\text{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = ((\text{Ind}_{B_r}^{G_r}(\chi))^{ss})^2 \oplus \bigoplus_{k=1}^{p-2} (\text{Ind}_{B_r}^{G_r}(\chi \cdot (\frac{a}{d})^k))^{ss}.$$

Utilizing the inductive hypothesis on  $(\text{Ind}_{B_r}^{G_r}(\chi))^{ss}$  and on each  $(\text{Ind}_{B_r}^{G_r}(\chi \cdot (\frac{a}{d})^k))^{ss}$  gives

$$(42) \quad (\text{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = \left( ((\text{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2} ((\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}} \right)^2 \\ \oplus \left( \bigoplus_{k=1}^{p-2} \left[ ((\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{m \neq k} ((\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^m))^{ss})^{\frac{p^{r-1}-1}{p-1}} \right] \right).$$

Counting how many times  $(\text{Ind}_{B_1}^{G_1}(\chi))^{ss}$  appears in the direct sum of (42) yields that  $(\text{Ind}_{B_1}^{G_1}(\chi))^{ss}$  appears

$$2\left(\frac{p^{r-1}+p-2}{p-1}\right) + (p-2)\frac{p^{r-1}-1}{p-1} = \frac{p^r+p-2}{p-1}$$

times, whereas counting how many times  $(\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^n))^{ss}$  appears in (42) for a given  $1 \leq n \leq p-2$  yields that  $(\text{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^n))^{ss}$  appears

$$2\left(\frac{p^{r-1}-1}{p-1}\right) + \frac{p^{r-1}+p-2}{p-1} + (p-3)\frac{p^{r-1}-1}{p-1} = \frac{p^r-1}{p-1}$$

times. Therefore

$$(43) \quad (\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^r+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2} ((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^r-1}{p-1}}.$$

proving the inductive claim.  $\square$

A complete semisimplification expresses the given representation as a direct sum of its unique set of composition factors, which are each irreducible representations. Hence giving the semisimplification of  $\mathrm{Ind}_{B_r}^{G_r}(\chi)$  requires knowing the irreducible characteristic  $p$  representations of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ .

**5.1. Classifying Modular Irreps of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$ .** We claim that every irreducible characteristic  $p$  representation of  $G_r$  is of the form  $\rho \circ \pi$ , where  $\pi$  is the surjective homomorphism

$$(44) \quad \begin{aligned} \pi : \mathrm{GL}_2(\mathbb{F}_p[t]/(t^r)) &\rightarrow \mathrm{GL}_2(\mathbb{F}_p) \\ \begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} &\mapsto \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \end{aligned}$$

and  $\rho$  is an irreducible characteristic  $p$  representation of  $\mathrm{GL}_2(\mathbb{F}_p)$ . To prove this fact we need the following two known lemmas, which then establish the result as an immediate corollary.

**Lemma 5.2.** *Let  $G$  be a finite group and let  $H \trianglelefteq G$  be a  $p$ -group. If  $V$  is an irreducible characteristic  $p$  representation of  $G$ , then  $V^H = V$ , that is,  $H$  acts trivially on all elements of  $V$ .*

In particular Lemma 5.2 tells us that if  $G$  is a finite group,  $H \trianglelefteq G$  is a  $p$ -group, and  $V$  is an irreducible characteristic  $p$  representation of  $G$ , then  $V$  must be the direct sum of trivial representations on  $H$ . We claim that this implies  $V$  factors through  $G/H$ .

**Lemma 5.3.** *A representation of a finite group  $G$  is trivial on a normal subgroup  $H$  if and only if it factors through  $G/H$ .*

The preceding lemmas allow us to prove the claim established at the beginning of this section:

**Proposition 5.4.** *Any irreducible modular representation of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$  is the inflation of an irreducible modular representation of  $\mathrm{GL}_2(\mathbb{F}_p)$ .*

*Proof.* The surjective homomorphism  $\pi$  in (44) gives us  $H = \ker \pi \leq G_r$ . We claim that  $H$  is a  $p$ -group: Notice that  $G_1 = \mathrm{GL}_2(\mathbb{F}_p)$  may be viewed as a subgroup of  $G_r$ , as it respects multiplication in  $G_r$ . Since the matrix

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}$$

belongs to  $\ker \pi$  if and only if  $a_0 = d_0 = 1, b_0 = c_0 = 0$ , and  $a_i, b_i, c_i, d_i \in \mathbb{F}_p$  for  $1 \leq i \leq r-1$ , then  $|\ker \pi| = |\mathbb{F}_p|^{4(r-1)} = p^{4(r-1)}$ . Hence by Lemma 5.2 any irreducible modular representation of  $G_r$  must be trivial on  $H$ . But by Lemma 5.3, we know that a representation of  $G_r$  is trivial on a normal subgroup  $H$  if and only if it factors through  $G_r/H$ . Since  $G_r/H \cong \mathrm{GL}_2(\mathbb{F}_p)$ , then every irreducible characteristic  $p$  representation  $\tilde{\rho}$  of  $G_r$  must be of the form  $\rho \circ \pi$  where  $\pi$  is the map given in (44) and  $\rho$  is an irreducible characteristic  $p$  representation of  $\mathrm{GL}_2(\mathbb{F}_p)$ .  $\square$

Fortunately the irreducible characteristic  $p$  representations of  $\mathrm{GL}_2(\mathbb{F}_p)$  are fully classified (see [1] or [8] for the proofs). Given  $0 \leq n \leq p-1$  and  $0 \leq \ell \leq p-2$ , let  $P_n$  be the  $\overline{\mathbb{F}}_p$  span of the basis  $\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\}$ . Define

$$(45) \quad \begin{aligned} \rho_{n,\ell} : \mathrm{GL}_2(\mathbb{F}_p) &\rightarrow \mathrm{GL}(P_n) \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot P(x,y) &= P(ax+cy, bx+dy) \cdot \left( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)^\ell. \end{aligned}$$

Then  $\{\rho_{n,\ell}\}$  gives a complete set of irreducible characteristic  $p$  representations of  $\mathrm{GL}_2(\mathbb{F}_p)$  up to equivalence. Hence every irreducible characteristic  $p$  representation of  $G_r$  is given by  $\tilde{\rho}_{n,\ell} = \rho_{n,\ell} \circ \pi$ , where  $\pi$  is as in (44).

**5.2. Semisimplification of  $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ .** Recall that any multiplicative map  $\chi : B_1 \rightarrow \overline{\mathbb{F}_p}^\times$  is of the form  $\chi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = a^r(ad)^s$ , where  $0 \leq r, s \leq p-2$ . From [3] we know that if  $r = 0$ , then  $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \rho_{0,s} \oplus \rho_{p-1,s}$ , where  $\rho_{p-1,s}$  may be recognized as the twisted Steinberg representation. On the other hand, if  $r \neq 0$ , then  $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \rho_{p-1-r,r+s} \oplus \rho_{r,s}$ . In particular this tells us that  $(\mathrm{Inf}_{G_1}^{G_r} \mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \tilde{\rho}_{0,s} \oplus \tilde{\rho}_{p-1,s}$  or  $(\mathrm{Inf}_{G_1}^{G_r} \mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = \tilde{\rho}_{p-1-r,r+s} \oplus \tilde{\rho}_{r,s}$  depending on  $\chi$ . In combination with Corollary 5.1 this fact allows us to explicitly give the semisimplification of  $\mathrm{Ind}_{B_r}^{G_r}(\chi)$  for any character  $\chi$ .

## 6. APPENDIX: COMPUTING SEMISIMPLIFICATIONS VIA BRAUER CHARACTERS

Brauer pioneered modular representation theory to better understand the relationships between characteristic  $p$  representations and ordinary character theory. A key development in this theory is the invention of *Brauer characters*, which assign to particular elements of a group  $G$  a value in a field of characteristic 0 dependent on a characteristic  $p$  representation. The utility of such characters in our problem comes from their ability to solve for the semisimplification numbers given in Corollary 5.1 without requiring any knowledge about the Jordan-Hölder series itself.

To compute the Brauer character of a representation we outline a process described in greater generality in [5] and [9]. Let  $m$  be the least common multiple of the orders of  $p$ -regular elements of  $G$  (elements of  $G$  with order coprime to  $p$ ). Let  $\rho$  be an irreducible characteristic  $p$  representation of  $G$ . For any  $g \in G$  a  $p$ -regular element,  $\rho(g)$  must have order dividing  $|g|$  in  $\overline{\mathbb{F}_p}^\times$ , and hence has order dividing  $m$ . In particular this means that the eigenvalues of  $\rho(g)$  are all powers of  $m^{\mathrm{th}}$  roots of unity in  $\overline{\mathbb{F}_p}^\times$ .

Writing  $\zeta_m$  for a primitive  $m^{\mathrm{th}}$  root of unity in  $\overline{\mathbb{F}_p}^\times$  allows us to express the eigenvalues of  $\rho(g)$  as  $\zeta_m^{m_1}, \dots, \zeta_m^{m_k}$ , where  $k$  is the dimension of the representation  $\rho$ , and  $0 \leq m_i \leq m-1$ . We fix a bijection between the  $m^{\mathrm{th}}$  roots of unity in  $\overline{\mathbb{F}_p}^\times$  and the  $m^{\mathrm{th}}$  roots of unity in  $\mathbb{C}$  by mapping  $\zeta_m \mapsto \omega_m = e^{\frac{2\pi i}{m}}$ . The Brauer character of  $\rho$  evaluated at  $g$  is given by  $\theta_\rho(g) = \sum_{i=1}^k \omega_m^{m_i}$ . Notice that since elements of a  $p$ -regular conjugacy class have the same eigenvalues, Brauer characters are constant on  $p$ -regular conjugacy classes.

Fix a field  $E$  of characteristic 0 whose residue field is of characteristic  $p$ . Given an ordinary representation  $\psi : G \rightarrow \mathrm{GL}(V)$  with associated character  $\chi : G \rightarrow E^\times$ , we have that the *mod  $p$  reduction*  $\bar{\chi} : G \rightarrow k_E^\times$  of  $\chi$  may be expressed as a non-negative integer linear combination of the irreducible Brauer characters of  $G$  [2]. This means that for all  $p$ -regular  $g \in G$ ,

$$(46) \quad \bar{\chi}(g) = \sum_{\rho \text{ modular irreps of } G} d_\rho \theta_\rho(g)$$

where  $d_\rho \in \mathbb{Z}_{\geq 0}$ . We call these  $d_\rho$  the *decomposition numbers* of  $\bar{\psi}$  as they give the multiplicity of the irreducible representation  $\rho$  in the semisimplification of  $\bar{\psi}$ .

We wish to compute the semisimplification of  $\vartheta_\chi = \mathrm{Ind}_{B_r}^{G_r}(\chi)$  via Brauer characters. As Brauer characters are only defined on  $p$ -regular conjugacy classes, we determine these conjugacy classes for  $G_r$ . The conjugacy classes of  $\mathrm{GL}_2(\mathbb{F}_p)$  are well-known, and the  $p$ -regular conjugacy classes of  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$  for  $r \in \mathbb{N}$  have representatives in  $\mathrm{GL}_2(\mathbb{F}_p)$ . For primes  $p$ , we have the  $p$ -regular conjugacy classes:

- (1)  $\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{F}_p^\times \right\}$ . There are  $|\mathbb{F}_p^\times| = p-1$  such conjugacy classes.
- (2)  $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{F}_p^\times \right\}$ . Swapping the position of  $a$  and  $b$  yields conjugate matrices, but a different pair of  $(a, b)$  yields a non-conjugate matrix. Hence we have  $\binom{p-1}{2}$  such conjugacy classes.
- (3)  $\left\{ \begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix} \right\}$  where  $D$  is not a square in  $\mathbb{F}_p$ , and  $\alpha + \beta\sqrt{D}$  is a characteristic root of a matrix in  $\mathrm{GL}_2(\mathbb{F}_p)$  with  $\beta \neq 0$ . The matrices  $\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}$  and  $\begin{bmatrix} \alpha & -D\beta \\ -\beta & \alpha \end{bmatrix}$  are conjugate, so we only need consider  $\beta \in \{1, \dots, \frac{p-1}{2}\}$ .

None of the matrices of type (3) above are conjugate to an upper triangular matrix in  $\mathrm{GL}_2(\mathbb{F}_p)$ , else their eigenvalues would lie in  $\mathbb{F}_p$ . We see that this must hold in the larger group  $\mathrm{GL}_2(\mathbb{F}_p[t]/(t^r))$  as well: if

any of the matrices of type (3) were conjugate in  $GL_2(\mathbb{F}_p[t]/(t^r))$  to an upper triangular matrix, then their eigenvalues would have to lie in  $(\mathbb{F}_p[t]/(t^r))^\times \cap (\mathbb{F}_p[\sqrt{D}])^\times = \mathbb{F}_p^\times$ , and thus in particular must be of type (1) or (2). This contradicts the fact that (1), (2), and (3) give distinct conjugacy class types.

The character of the representation  $\vartheta_\chi$ , denoted by  $\theta_\chi$ , has a formula due to Mackey:

$$(47) \quad \theta_\chi(g) = \sum_{\substack{x_i \in B_r \setminus G_r \\ x_i g x_i^{-1} \in B}} \chi(\chi_i g \chi_i^{-1}).$$

We use this formula to compute the character on our  $p$ -regular conjugacy classes. For each conjugacy class of type  $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ , we have that  $x_i \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} x_i^{-1} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \in B$ , so using (47) we obtain

$$(48) \quad \theta_\chi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = |B_r \setminus G_r| \cdot \chi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right) = p^{r-1}(p+1)\chi\left(\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right).$$

We now suppose  $g = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  where  $a, b \in \mathbb{F}_p^\times$  and  $a \neq b$ . If  $x_j$  is the coset representative for  $B_r$  in the set of right cosets of  $B_r$  (that is,  $x_j$  is the identity matrix), then  $x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} \in B$  trivially. Now suppose  $x_j = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$  where  $\gamma \neq 0$  (so that  $x_j \notin B_r$ ). We wish to determine when  $x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} \in B$ . Note

$$(49) \quad \begin{aligned} x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \\ &= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a\delta & -a\beta \\ -b\gamma & b\alpha \end{bmatrix} \\ &= \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} a\alpha\delta - b\beta\gamma & -a\alpha\beta + b\beta\alpha \\ a\gamma\delta - b\delta\gamma & -a\beta\gamma + b\delta\alpha \end{bmatrix} \end{aligned}$$

so that  $x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} \in B$  if and only if  $a\delta\gamma - b\delta\gamma = 0$ , that is, if and only if  $(a-b)\delta\gamma = 0$ . Since  $a \neq b$  and  $a, b \in \mathbb{F}_p^\times$ , then  $a-b \in \mathbb{F}_p^\times$ , and thus we must have  $\delta\gamma = 0$ . But we assumed that  $\gamma \neq 0$ , so we must have  $\delta = 0$ . From (49) we see then that if  $x_j \notin B_r$  and  $x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} \in B$ , then

$$(50) \quad x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1} = \frac{-1}{\beta\gamma} \begin{bmatrix} -b\beta\gamma & (b-a)\alpha\beta \\ 0 & -a\beta\gamma \end{bmatrix} = \begin{bmatrix} b & \frac{(a-b)\alpha}{\gamma} \\ 0 & a \end{bmatrix}$$

so that  $\chi(x_j \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_j^{-1}) = \chi(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix})$ . To see that no other coset representative  $x_\ell$  gives  $x_\ell \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_\ell^{-1} \in B$ , suppose such an  $x_\ell$  did exist with  $Bx_j \neq Bx_\ell$ . Let  $x_j = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$ , where  $\gamma \neq 0$  so that  $x_j \notin B$ . Let  $x_\ell = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ . Then

$$\begin{aligned} Bx_j \neq Bx_\ell &\iff x_\ell x_j^{-1} \notin B \iff \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}^{-1} \notin B \\ &\iff \frac{-1}{\beta\gamma} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 0 & -\beta \\ -\gamma & \alpha \end{bmatrix} \notin B \\ &\iff \frac{-1}{\beta\gamma} \begin{bmatrix} -y\gamma & -x\beta + y\alpha \\ -w\gamma & -z\beta + w\alpha \end{bmatrix} \notin B \\ &\iff w \neq 0. \end{aligned}$$

Recall from our computation above that  $x_\ell \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} x_\ell^{-1} \in B$  if and only if  $(x_\ell)_{22} = 0$ , that is, if and only if  $w = 0$ . This contradiction allows us to conclude that

$$(51) \quad \theta_\chi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = \chi\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) + \chi\left(\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}\right).$$

Finally, if  $g = \begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}$  is a matrix of type (3), then we know that  $g$  has no upper triangular conjugates. Thus

$$(52) \quad \theta_\chi\left(\begin{bmatrix} \alpha & D\beta \\ \beta & \alpha \end{bmatrix}\right) = 0.$$

To illustrate how to obtain the semisimplification numbers from the above computation, we fix  $p = 3$  and  $\chi$  to be the trivial character. From the above computation, we have the following table for representatives of the 3-regular conjugacy classes of  $GL_2(\mathbb{F}_3[t]/(t^r))$ :

$$\theta_\chi(g) \left| \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ p^{r-1}(p+1) \end{array} \right| \begin{array}{c} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ p^{r-1}(p+1) \cdot \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \end{array} \left| \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ \chi\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) + \chi\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) \end{array} \right| \begin{array}{c} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \\ 0 \end{array} \left| \begin{array}{c} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \\ 0 \end{array} \right| \begin{array}{c} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \\ 0 \end{array}$$

We wish to solve for the  $d_\rho$  in (46), which requires us to know how  $\theta_\rho$  evaluates on  $g$  for each conjugacy class and for each  $\rho$  an irreducible modular representation of  $G_r$ . An omitted computation yields the following Brauer characters:

	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}$
$\theta_{0,0}$	1	1	1	1	1	1
$\theta_{0,1}$	1	1	-1	1	-1	-1
$\theta_{1,0}$	2	-2	0	0	$-i\sqrt{2}$	$i\sqrt{2}$
$\theta_{1,1}$	2	-2	0	0	$i\sqrt{2}$	$-i\sqrt{2}$
$\theta_{2,0}$	3	3	1	-1	-1	-1
$\theta_{2,1}$	3	3	-1	-1	1	1

TABLE 1. Brauer character table,  $p = 3$ 

where  $\theta_{n,\ell}$  is the Brauer character corresponding to  $\tilde{\rho}_{n,\ell}$ . Recall that any character  $\bar{\chi} : B_1 \rightarrow \overline{\mathbb{F}_p}^\times$  is of the form  $\chi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) = a^r(ad)^s$  where  $0 \leq r, s \leq p-2$ , and hence when  $p = 3$  we have only three choices:  $r = s = 0$  (yielding the trivial character  $\text{triv}$ ),  $r = 0, s = 1$  ( $\det$ ), and  $r = 1, s = 0$  ( $\text{alt}$ ).

Solving a system of equations according to (46) for every choice of character  $\chi$  in the  $p = 3$  case yields:

$(\chi_1, \chi_2)$	$d_{0,0}$	$d_{0,1}$	$d_{1,0}$	$d_{1,1}$	$d_{2,0}$	$d_{2,1}$
$\text{triv}$	$\frac{3^{r-1}+1}{2}$	$\frac{3^{r-1}-1}{2}$	0	0	$\frac{3^{r-1}+1}{2}$	$\frac{3^{r-1}-1}{2}$
$\det$	$\frac{3^{r-1}-1}{2}$	$\frac{3^{r-1}+1}{2}$	0	0	$\frac{3^{r-1}-1}{2}$	$\frac{3^{r-1}+1}{2}$
$\text{alt}$	0	0	$3^{r-1}$	$3^{r-1}$	0	0

TABLE 2

We verify that this aligns with our numbers in Corollary 5.1. For simplicity we take  $\chi = \text{triv}$ , noticing that by Table 2, we have

$$(\text{Ind}_{B_r}^{G_r}(\text{triv}))^{ss} = \tilde{\rho}_{0,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{0,1}^{\frac{3^{r-1}-1}{2}} \oplus \tilde{\rho}_{2,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{2,1}^{\frac{3^{r-1}-1}{2}}.$$

On the other hand, by Corollary 5.1 we have that

$$(\text{Ind}_{B_r}^{G_r}(\text{triv}))^{ss} = ((\text{Ind}_{B_1}^{G_1}(\text{triv}))^{ss})^{\frac{3^{r-1}+1}{2}} \oplus ((\text{Ind}_{B_1}^{G_1}(\text{triv} \cdot \frac{a}{d}))^{ss})^{\frac{3^{r-1}-1}{2}}$$

Now  $(\text{Ind}_{B_1}^{G_1}(\text{triv}))^{ss} = \rho_{0,0} \oplus \rho_{2,0}$ , and since  $\frac{a}{d} = ad^{-1} = ad$  in  $\overline{\mathbb{F}_3}$ , then  $(\text{Ind}_{B_1}^{G_1}(\text{triv} \cdot \frac{a}{d}))^{ss} = \rho_{0,1} \oplus \rho_{2,1}$ . Hence

$$(\text{Ind}_{B_r}^{G_r}(\text{triv}))^{ss} = \tilde{\rho}_{0,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{2,0}^{\frac{3^{r-1}+1}{2}} \oplus \tilde{\rho}_{0,1}^{\frac{3^{r-1}-1}{2}} \oplus \tilde{\rho}_{2,1}^{\frac{3^{r-1}-1}{2}}$$

which is precisely what we deduced from Table 2. A similar computation verifies the other two cases of  $\chi$ .

For larger primes computing the Brauer table is much more computationally intensive, so we resort to the semisimplification numbers which resulted from the Jordan-Hölder series.

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