Chapter 0: Useful Theorems from Linear Algebra

Theorem 0.1. *Every matrix is similar to an upper triangular matrix. Every nilpotent matrix is similar to an upper triangular matrix with zeros on the diagonal.*

Theorem 0.2. *Each $A \in M_n(\mathbb{C})$ has a unique decomposition as $A = S + N$ where S is diagonalizable, N is nilpotent, and $SN = NS$.*

1 Chapter 1: Introduction to Matrix Lie Groups

1.1 Definitions and Examples

Although we won't really need the definition of a real Lie group as of right now, we recall it just for completeness:

Definition 1. *A (real)* ***Lie group*** *G is a smooth manifold equipped with a group structure such that the group operation $G \times G \to G$ and the inverse map $g \mapsto g^{-1}$ are both smooth.*

More important for this course will be the definition of a *matrix Lie group*.

Definition 2. *A* ***matrix Lie group*** *is a closed subgroup of $GL_n(\mathbb{C})$, where $n \in \mathbb{N}$.*

Recall that $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) : A \text{ is invertible}\}$. To be closed in $GL_n(\mathbb{C})$ means to be closed as a subset under the subset topology of $GL_n(\mathbb{C})$, where $GL_n(\mathbb{C})$ is equipped with the usual topology of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. We can therefore come up with a better, working definition of a matrix Lie group, which directly involves the requirement that the subgroup be *closed.*

Definition 3 (Improved Def 2)**.** *A subgroup $G \leq GL_n(\mathbb{C})$ is a* ***matrix Lie group*** *if $A_m \in G$, $A = \lim_{m\to\infty} A_m$ implies $A \in G$ or $A \notin GL_n(\mathbb{C})$.*

Definition 3 essentially tells us that we only care about those points in $GL_n(\mathbb{C})$ which are limits of sequences in G– it is these points which we require to also belong to G. If there is some other point in $M_n(\mathbb{C}) \setminus GL_n(\mathbb{C})$ that is the limit of a sequence in G, then that limit certainly won't be in G (since it's not in $GL_n(\mathbb{C})$), but that's ok, we only care about those limit points which belong to $GL_n(\mathbb{C})$.

Example 1.1. *The Special Linear Group over $\mathbb{C}$, denoted $SL_n(\mathbb{C})$, is the group of $n \times n$ invertible matrices in $\mathbb{C}$ with determinant equal to 1.*

We verify that $SL_n(\mathbb{C})$ is a matrix Lie group. First notice that it is indeed a group since determinants are multiplicative and since $\det(A^{-1}) = (\det(A))^{-1} = 1$ when $A \in SL_n(\mathbb{C})$. Next we verify that $SL_n(\mathbb{C})$ is closed via the limit definition: Let $\{A_m\}$ be a sequence in $SL_n(\mathbb{C})$ and let $A = \lim_{m\to\infty} A_m$. Then

$$\begin{aligned}
\det(A) &= \det(\lim_{m\to\infty} A_m) && (1)\\
&= \lim_{m\to\infty} \det(A_m) && (2)\\
&= \lim_{m\to\infty} (1) && (3)\\
&= 1 && (4)
\end{aligned}$$

where (2) follows from the fact that the determinant function is continuous. Hence $A \in SL_n(\mathbb{C})$, so $SL_n(\mathbb{C})$ is closed in $GL_n(\mathbb{C})$.

Example 1.2. *The Special Linear Group over $\mathbb{R}$ is the group $SL_n(\mathbb{R}) = SL_n(\mathbb{C}) \cap M_n(\mathbb{R})$.*

Example 1.3. $GL_n(\mathbb{Q}) = GL_n(\mathbb{C}) \cap M_n(\mathbb{Q})$

Example 1.4. *The orthogonal group O(n) is the set of $n \times n$ real invertible orthogonal matrices, i.e., $A \in GL_n(\mathbb{R})$ such that $A^T A = I_n$ (equivalently, such that $A^{-1} = A^T$).*

The orthogonal group $O(n)$ is a matrix lie group: verifying that $O(n)$ is a group is not difficult, so we just verify that $O(n)$ is closed under taking limits. Let $\{A_m\} \in O(n)$ such that $A = \lim_{m\to\infty} A_m$. Then $A_m^T A_m = I$ for all m, so

$$\begin{aligned} I &= A_m^T A_m \quad \forall m \\ \implies \lim_{m\to\infty} I &= \lim_{m\to\infty} A_m^T A_m \\ \implies I = \lim_{m\to\infty} A_m^T &\cdot \lim_{m\to\infty} A_m \\ \implies I &= A^T A \\ \implies A &\in O(n) \end{aligned}$$

Hence $O(n)$ is a matrix lie group. Besides defining $O(n)$ as the set of orthogonal matrices, we could consider more generally a non-degenerate symmetric bilinear form on $\mathbb{R}^n$, and define the orthogonal group to be the set of matrices which preserve that form. This is because we have a one-to-one correspondence between non-degenerate symmetric bilinear forms on $\mathbb{R}^n$ and invertible matrices whose transpose is equal to itself. If we fix a basis $e_1, \ldots, e_n \in \mathbb{R}^n$, we see this correspondence as

$$\{\text{non-degenerate symmetric bilinear forms on } \mathbb{R}^n\} \leftrightarrow \{Q \in M_n(\mathbb{R}) : Q^T = Q,\ \det Q \neq 0\}$$

$$\text{via}$$

$$\begin{aligned} \langle \cdot, \cdot \rangle &\mapsto (\langle e_k, e_k \rangle)_{jk} \\ Q &\mapsto \langle x, y \rangle = x^T Q y \end{aligned}$$

We can thus define a general orthogonal group G to be the group of matrices $A \in GL_n(\mathbb{R})$ such that $\langle Ax, Ay \rangle = \langle x, y \rangle$, that is, the set of $n \times n$ invertible matrices over $\mathbb{R}$ that preserve the non-degenerate symmetric bilinear form. Because $\langle Ax, Ay \rangle = (Ax)^T Ay = x^T A^T Q A y$ and $\langle x, y \rangle = x^T Q y$ for some $Q \in GL_n(\mathbb{R})$ such that $Q^T = Q$, we must have that $x^T A^T Q A y = x^T Q y$ which implies that $A^T Q A = Q$. Notice that when $Q = I$, this just gives our standard orthogonal group $O(n)$. On the other hand, when $Q = \begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$ for $a + b = n$, the corresponding orthogonal group is called the orthogonal group of signature a,b and is denoted $O(a, b)$.

If we instead consider non-degenerate Hermitian forms (that is, maps $\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ which satisfy $\langle x, y \rangle = \overline{\langle y, x \rangle}$, are linear in the second component, conjugate linear in the first component, and non-degenerate), then to each such form we associate a Hermitian (self-adjoint) matrix as follows:

$$\{\text{non-degenerate sesquilinear Hermitian forms on } \mathbb{C}^n\} \leftrightarrow \{Q \in M_n(\mathbb{C}) : Q^* = Q,\ \det Q \neq 0\}$$

via

$$\langle \cdot, \cdot \rangle \mapsto (\langle e_k, e_k \rangle)_{jk}$$
$$Q \mapsto \langle x, y \rangle = x^* Q y$$

where $e_1, \ldots, e_n$ is a fixed basis of $\mathbb{C}^n$ viewed as an n-dimensional vector space over $\mathbb{C}$. We can then define a general unitary group $G = \{A \in GL_n(\mathbb{C}) : \langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{C}^n\}$, or equivalently, $G = \{A \in GL_n(\mathbb{C}) : A^* Q A = Q\}$ for the Hermitian matrix $Q = (\langle e_j, e_k \rangle)_{jk}$.

Example 1.5. *In the above example, if we consider the standard dot product in $\mathbb{C}^n$, then $Q = I$ and we get that G is the unitary group U(n) consisting of those invertible complex $n \times n$ matrices such that $A^* A = I$.*

Example 1.6. *When $Q = \begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$, we have that G becomes $U(a, b)$, the unitary group of signature a,b.*

We can similarly define a symplectic form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ as follows: $w(x, y) = x_1 y_{n+1} + \cdots + x_n y_{2n} - x_{n+1} y_1 - \cdots - x_{2n} y_n$. Notice that this form is symplectic because $w(x, y) = -w(y, x)$, the form is bilinear, and $w(x, y) = 0$ for all $y \in \mathbb{R}^{2n}$ implies $x = 0$. To this form we can associate the skew symmetric matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ so that $w(x, y) = x^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} y$.

Example 1.7. *Let $Sp_n(\mathbb{R}) = \{A \in GL_{2n}(\mathbb{R})" w(Ax, Ay) = w(x, y) \quad \forall x, y \in \mathbb{R}^n\}$, or equivalently, $Sp_n(\mathbb{R}) = \{A \in GL_{2n}(\mathbb{R}) : A^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\}$. This group is a matrix lie group called the symplectic group.*

Do we need to define the Heisenberg group?

Finally, we notice that many everyday vector spaces are actually matrix lie groups under the proper identification.

Example 1.8. *All three sets $\mathbb{R}^\times, \mathbb{C}^\times, S^1$ form groups under multiplication. $\mathbb{R}^\times$ is precisely $GL_1(\mathbb{R})$, which is a closed subgroup of $GL_1(\mathbb{C})$, thus is a matrix Lie group. $\mathbb{C}^\times$ is precisely $GL_1(\mathbb{C})$, and is therefore a matrix Lie group. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is isomorphic to $U(1)$, since $U(1)$ consists of those $z \in \mathbb{C}$ such that $\overline{z} \cdot z = 1$, which implies $|z|^2 = 1$ and thus that $|z| = 1$.*

Example 1.9. *The sets $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$, and $\mathbb{C}^n$ all form groups under addition. Notice that $\mathbb{R}$ is isomorphic to $\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}\}$ which is a closed subgroup of $GL_2(\mathbb{C})$. Similarly, $\mathbb{R}^n$ is isomorphic to $\{\begin{pmatrix} I_n & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R}^n\}$ which is a closed subgroup of $GL_{n+1}(\mathbb{C})$*

Non-example of a matrix Lie group

It's important we also see an example of a group which is *not* a matrix Lie group. Consider the set of matrices

$$G = \{\begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} : t \in \mathbb{R}\}$$

where $a \in \mathbb{R} \setminus \mathbb{Q}$. Through a bit of work (proof not shown), it can be seen that the closure of this set in $GL_n(\mathbb{C})$ is the set

$$\overline{G} = \{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} : \theta, \phi \in \mathbb{R}\}$$

which does not equal G, and hence G is not closed in $GL_n(\mathbb{C})$, which means G is not a matrix Lie group.

1.2 Topology of Matrix Lie Groups

We recall first that a matrix lie group is a *closed* subgroup of $GL_n(\mathbb{C})$, which is a subgroup of $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$, and thus may be equipped with the subset topology. Recall that under the subset topology, G is closed in $GL_n(\mathbb{C})$ if $G = GL_n(\mathbb{C}) \cap F$ for some set F closed in $M_n(\mathbb{C})$. We equip $\mathbb{C}^{n^2}$ with the product topology, or the topology induced by a norm (using open balls, ϵ-δ type beat) for several different norms (we'll probably use the Frobenius norm, aka L^2). We can also just consider a set $R \subseteq M_n(\mathbb{C})$ to be closed in $M_n(\mathbb{C})$ if for all sequences $\{A_m\} \in R$ with $\lim_{m\to\infty} A_m = A$, we have that $A \in R$.

Definition 4. *A **homeomorphism** is a bijective map which is continuous and which has a continuous inverse.*

Proposition 1.1. *If $G \leq GL_n(\mathbb{C})$, then the group operations and the map of taking inverses are homeomorphisms. Similarly, the maps $\ell_A : G \to G$ and $r_A : G \to G$ defined by left and right (respectively) multiplication by a matrix $A \in G$ are homeomorphisms.*

We can now define several topological properties and state whether or not they apply to our matrix Lie group examples.

Definition 5. *A topological space is **compact** if every open cover has a finite subcover. A **subset of** $\mathbb{R}^n$ **is compact** if it is closed and bounded.*

Since matrix Lie groups are closed subgroups of $GL_n(\mathbb{C})$ (and thus subgroups of $GL_{2n}(\mathbb{R})$), then Heine-Borel applies and we just need to check closed and bounded. However, notice that we need to check closed in $GL_n(\mathbb{C})$, not in $M_n(\mathbb{C})$. So for all $\{A_m\} \in G$, if $\lim_{n\to\infty} A_m = A$, either $A \in G$ or $A \notin GL_n(\mathbb{C})$ implies that G is closed.

Definition 6. *A topological space is **connected** if it cannot be written as the disjoint union of two nonempty open sets.*

For Lie groups, connectedness is equivalent to path-connectedness, which is the definition we'll really use whenever we consider whether or not a matrix Lie group is connected.

Definition 7. *A topological space is **path-connected** if for all $x, y \in X$, there exists a continuous map $f : [0, 1] \to X$ with $f(0) = x, f(1) = y$.*

If a topological space is not connected, we measure "how connected" it is by considering the connected component of an element of the space.

Definition 8. *The **connected component** of $x \in X$ is the (unique) largest connected subset containing x.*

Notice that if x, y are connected by a continuous path in X, then x, y belong to the same connected component, and if U, V are connected subsets of X, then $U \cap V \neq$ implies that $U \cup V$ is connected (that is, distinct connected components are disjoint).

We give a special name to the connected component of a group G containing the identity; we call this the identity component of G and denote it by G_0.

Definition 9. *Roughly, we say that a space X is simply connected if for any two paths starting at a and ending at b (so forming a continuous loop), where $a, b \in X$, we can continuously deform one path into another.*

For each of the groups we defined in Section 1.1, we can now verify certain topological properties, which are summarized in the table below. Hyperlinked are rough sketches of proofs.

Matrix Lie Group	**Compact?**	**Connected?**
$GL_n(\mathbb{R})$	no	no
$GL_n(\mathbb{C})$	no	yes
$SL_n(\mathbb{R})$	no for $n \geq 2$	yes
$SL_n(\mathbb{C})$	no for $n \geq 2$	yes
$U(n)$	yes	yes
$SU(n)$	yes	yes
$U(a, b)$	only if $a = 0$ or $b = 0$	
$Sp_n(\mathbb{R})$	no	yes
$O(n)$	yes	no
$SO(n)$	yes	yes
$O(a, b)$	only if $a = 0$ or $b = 0$	
$GL_n(\mathbb{R})^+$	no	yes

Table 1: Topological Properties of Matrix Lie Groups