Chapter 1

Network Description

This document describes the network of a receiver, by multiport networks.

1.1 Multiport Networks

In the following a 2-port network will be analyzed. Later this can be easily extendet to a multiport network.
Figure 1.1 shows a 2-port network. The twoport network can be represented by its impedance matrix $\mathbf{Z}$. The elements Z_{ij} of the matrix are defined in the following way:

$$Z_{ij} = \frac{V_i}{I_j}, \quad \text{for the currents} \quad I_l = 0, \quad l \neq j. \tag{1.1}$$

Therefore the 2-port's input/output relations can be written as

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{Z} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{1.2}$$

For a multiport network, elements V_1, V_2, I_1 and I_2 become vectors, and the elements $Z_{ij}, \quad i, j \in \{1, 2\}$ become submatrices. Looking from the left into the

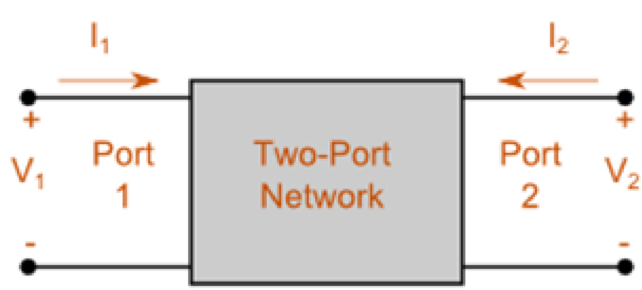

Figure 1.1: A twoport network ("www.antennamagnus.com").

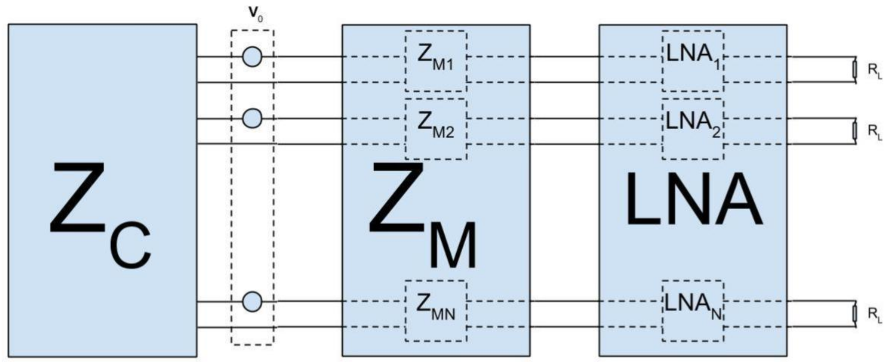

Figure 1.2: The receiver, with the coupling network $\mathbf{Z}_C$, the SP-matching network $\mathbf{Z}_M$, the LNA, and loads attached to the LNA.

network with a load R_L attached to port two, the equivalent input impedance becomes

$$\mathbf{Z}_{in_1} = \mathbf{Z}_{11} - \mathbf{Z}_{21} \cdot (\mathbf{Z}_{22} + R_L \cdot \mathbf{I})^{-1} \cdot \mathbf{Z}_{12}. \tag{1.3}$$

With no load attached on port two, the equivalent input impedance becomes

$$\mathbf{Z}_{in_1} = \mathbf{Z}_{11}. \tag{1.4}$$

Last, having port one short-circuited and looking from port two into the network, the equivalent network impedance becomes

$$\mathbf{Z}_{in_2} = \mathbf{Z}_{22} - \mathbf{Z}_{12} \cdot (\mathbf{Z}_{11})^{-1} \cdot \mathbf{Z}_{21}. \tag{1.5}$$

For reciprocal (multiport-)networks (RLC-Networks, containing only passive elements), $\mathbf{Z}_{12} = \mathbf{Z}_{21}^T$.

1.2 Receiver Blocks

The receiver (as shown in Figure 1.2) consists of three blocks. From left to right:

1. the coupling network ($\mathbf{Z}_C$),
2. the matching network ($\mathbf{Z}_M$), and
3. the low-noise-amplifier (**LNA**).

All these blocks can be described in multiport networks. In the following each block will be discussed.

1.2.1 The Coupling Network

The coupling network comes from the fact, that compact antenna arrays are used. The strength of the coupling between two antennas depends on the spacing between the antennas. In the following a unform spacing of the antennas is assumed. As the effect of coupling from one antenna to another is the same like the reverse, the impedance matrix $\mathbf{Z}_C$ becomes symmetric.

1.2.2 The Matching Network

Behind the each receiving antenna a matching network placed, to improve the performance of the receiver. For complexity and bandwith reasons ("WCNC - Paper"), single-port (SP) matching is assumed. The matching network has the form of

$$\mathbf{Z}_M = \begin{bmatrix} \mathbf{Z}_{M11} & \mathbf{Z}_{M12} \\ \mathbf{Z}_{M21} & \mathbf{Z}_{M22} \end{bmatrix}. \tag{1.6}$$

For a matching network to be lossles it must be pure imaginary and symmetric[**?**]. Because we assume SP matching the submatrices become diagonal, with the additional property of $\mathbf{Z}_{M12} = \mathbf{Z}_{M21}^T \implies \mathbf{Z}_{M12} = \mathbf{Z}_{M21}$.

1.2.3 The Low-Noise-Amplifier

In the LNA-block the received signal after the matching network gets amplified. As in the matchig network the LNA can be represented in the following way

$$\mathbf{LNA} = \begin{bmatrix} \mathbf{c} & \mathbf{d} \\ \mathbf{e} & \mathbf{g} \end{bmatrix}. \tag{1.7}$$

As each branch has it's own LNA, the submatrices $\mathbf{c}, \mathbf{e}$ and $\mathbf{g}$ are again diagonal. Additionally, matrix $\mathbf{d}$ is an all-zeros matrix if the unilateral assumtion (*the input of the LNA is not affected of the output of the LNA*) is applied.

1.3 Transfer Function of the Receiver

The main interest lies in the transfer function of the input voltages $\mathbf{v}_0$ to the voltage measured at the loads (in Figure 1.2) $\mathbf{v}_L$. In the following the transferfunction over each block of the receiver will be derived. We use the termination: $\mathbf{v}_i$ is the voltage on the left of the block i (i.e. v_M is the voltage on the left ports of the matching network). Additionally, we see the left ports as input ports and the right ports as output ports of each block.
For the derivation of the transferfunction we need four equivalent impedance matrices, namely

1. $\mathbf{Z}_{eqM_1}$, the impedance matrix looking from the left into the matching network,
2. $\mathbf{Z}_{eqM_2}$, the impedance matrix looking from the right into the matching network,
3. $\mathbf{Z}_{eqLNA_1}$, the impedance matrix looking from the left into the LNA, and
4. $\mathbf{Z}_{eqLNA_2}$, the impedance matrix looking from the right into the LNA.

To calculate $\mathbf{Z}_{eqM_1}$ and $\mathbf{Z}_{eqLNA_1}$, we use Equation (1.3) and get the following

$$\mathbf{Z}_{eqM_1} = \mathbf{Z}_{M11} - \mathbf{Z}_{M21} \cdot (\mathbf{Z}_{M22} + \mathbf{Z}_{eqLNA_1})^{-1} \cdot \mathbf{Z}_{M12}, \tag{1.8}$$

$$\mathbf{Z}_{eqLNA_1} = \mathbf{c} - \mathbf{e} \cdot (R_L \mathbf{I}_{N_r} + \mathbf{g})^{-1} \cdot \mathbf{d} = \mathbf{c}. \tag{1.9}$$

As we step through the receiver blocks from left to right, we always have the parallel voltages applied to each receiver block on the left. Therefore the equivalent input impedances $\mathbf{Z}_{eqM_2}$ and $\mathbf{Z}_{eqLNA_2}$ can be derived using Equation (1.4) and lead to the following

$$\mathbf{Z}_{eqM_2} = \mathbf{Z}_{M22} - \mathbf{Z}_{M12} \cdot (\mathbf{Z}_{M11})^{-1} \cdot \mathbf{Z}_{M21}, \tag{1.10}$$

$$\mathbf{Z}_{eqLNA_2} = \mathbf{g} - \mathbf{d} \cdot (\mathbf{c})^{-1} \cdot \mathbf{e} = \mathbf{g}. \tag{1.11}$$

To get the parallel input voltage on the left of the matching network, we use the principle of a voltage divider as

$$\mathbf{v_M} = \mathbf{Z}_{eqM_1} \cdot (\mathbf{Z}_{eqM_1} + \mathbf{Z}_C)^{-1} \cdot \mathbf{v_0} \tag{1.12}$$

To transfer the voltages from the left ports to the right ports of each block we proceed for each block in the following:

1. Calculate the input currents,
2. transfer the input currents to the output voltages in series, and
3. calculate the output voltages in parallel, by a voltage divider.

For the matching network, it follows:

$$\begin{aligned}
\mathbf{i_M} &= \mathbf{Z}_{M11}^{-1} \cdot \mathbf{v_M}, && (1.13)\\
\mathbf{v}_{LNA_{series}} &= \mathbf{Z}_{M12} \cdot \mathbf{i_M} = \mathbf{Z}_{M12}\mathbf{Z}_{M11}^{-1} \cdot \mathbf{v_M}, && (1.14)\\
\mathbf{v}_{LNA} &= \mathbf{Z}_{eqLNA_1}(\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_{eqM_2})^{-1} \cdot \mathbf{v}_{LNA_{series}} && (1.15)\\
&= \mathbf{Z}_{eqLNA_1}(\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_{eqM_2})^{-1}\mathbf{Z}_{M12}\mathbf{Z}_{M11}^{-1} \cdot \mathbf{v_M},
\end{aligned}$$

and equivalent for the LNA block

$$\mathbf{v_L} = R_L\mathbf{I}_{N_R}(R_L\mathbf{I}_{N_R} + \mathbf{Z}_{LNA_2})^{-1}\mathbf{Z}_{LNA12}\mathbf{Z}_{LNA11}^{-1} \cdot \mathbf{v}_{LNA}. \tag{1.16}$$

Therefore we have three transfer functions to characterize our system. Denoting the transfer function form voltage $\mathbf{v}_j$ to voltage $\mathbf{v}_i$ as $\mathbf{H}_{i,j}$ (i.e. $\mathbf{v}_i = \mathbf{H}_{i,j} \cdot \mathbf{v}_j$), it follows

$$\begin{aligned}
\mathbf{H}_{M,0'} &= \mathbf{Z}_{eqM_1} \cdot (\mathbf{Z}_{eqM_1} + \mathbf{Z}_C)^{-}1, && (1.17)\\
\mathbf{H}_{LNA,M} &= \mathbf{Z}_{eqLNA_1}(\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_{eqM_2})^{-1}\mathbf{Z}_{M12}\mathbf{Z}_{M11}^{-1}, \quad \text{and} && (1.18)\\
\mathbf{H}_{L,LNA} &= R_L\mathbf{I}_{N_R}(R_L\mathbf{I}_{N_R} + \mathbf{Z}_{eqLNA_2})^{-1}\mathbf{Z}_{LNA12}\mathbf{Z}_{LNA11}^{-1}. && (1.19)
\end{aligned}$$

And the overall transferfunction

$$\mathbf{H}_{L,0'} = \mathbf{H}_{L,LNA} \cdot \mathbf{H}_{LNA,M} \cdot \mathbf{H}_{M,0'}. \tag{1.20}$$

1.4 Port Reduction

Passive antennas (e.g. relay antennas) can be modeled by connecting a load directly onto the coupling network as shown in Figure 1.3. In the following, $\mathbf{v}_C$ denotes the voltages on the ports between the coupling matrix $\mathbf{Z}_C$ and the voltage sources $\mathbf{v}_0$, the same for the currents. As we are not interested in the input/output-relation of these passive antennas, a port reduction is performed. For the port reduction, the coupling matrix $\mathbf{Z}_C$ can be represented by four submatrices

$$\mathbf{Z}_C = \begin{bmatrix} \mathbf{Z}_{OO} & \mathbf{Z}_{OL} \\ \mathbf{Z}_{LO} & \mathbf{Z}_{LL} \end{bmatrix}, \tag{1.21}$$

so that we get the system relations

$$\begin{bmatrix} \mathbf{v}_{CO} \\ \mathbf{v}_{CL} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{OO} & \mathbf{Z}_{OL} \\ \mathbf{Z}_{LO} & \mathbf{Z}_{LL} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{i}_{CO} \\ \mathbf{i}_{CL} \end{bmatrix}. \tag{1.22}$$

The index "O" denotes hereby the open circuit ports, the index "L" the ports with a load attached. We assume, that there are N_R antennas, whereby the first Ni

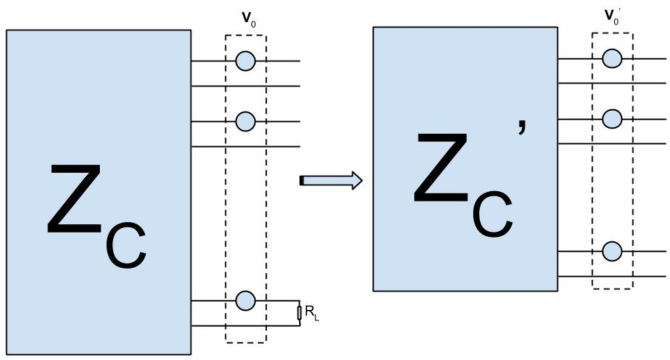

Figure 1.3: Port reduction on a network with one load connected to the last port.

antennas are active, and the later ones are passive (and therefore modeled by a load). $\mathbf{Z}_{pass}$ denotes in the following the $N_r - N_i$ loads representing the passive antennas. Therefore it is a diagonal matrix.
From Equation (**??**) and the property

$$\mathbf{v}_{CL} = \mathbf{v}_0[N_i + 1 : N_R] - \mathbf{Z}_{pass} \cdot \mathbf{i}_{CL} \quad \text{it follows,} \tag{1.23}$$

$$\begin{aligned} \mathbf{i}_{CL} = &-(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{Z}_{LO} \cdot \mathbf{i}_{CO} - \\ &(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1} \cdot \mathbf{v}_0[N_i + 1 : N_R] \end{aligned} \tag{1.24}$$

and therefore,

$$\begin{aligned} \mathbf{v}_{CO} = &(\mathbf{Z}_{OO} - \mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{Z}_{LO}) \cdot \mathbf{i}_{CO} - \\ &\mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1} \cdot \mathbf{v}_0[N_i + 1 : N_R]. \end{aligned} \tag{1.25}$$

With this port reduction we have the new coupling matrix and the new voltage input as

$$\mathbf{Z}_C^{'} = \mathbf{Z}_{OO} - \mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{Z}_{LO} \quad \text{and} \tag{1.26}$$

$$\mathbf{v}_0^{'} = \mathbf{H}_0 \cdot \mathbf{v_0} \tag{1.27}$$

$$\text{with} \quad \mathbf{H}_0 = \begin{bmatrix} \mathbf{I}_{N_i} & -\mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1} \end{bmatrix},$$

and so we can reduce the system shown in Figure 1.3, to the system shown in Figure 1.2.

1.5 Signal Covariance Matrix

To calculate the achievable sum rate of the systems, we need to derive the signal covariance matrix, defined as $\mathbb{E}[\mathbf{v}_L\mathbf{v}_L^H]$. As in all the transfer functions from $\mathbf{v}_0$ to $\mathbf{v}_L$ only impedance matrices appear, the covariance matrix becomes

$$\mathbf{K}_s = \mathbb{E}[\mathbf{v}_L\mathbf{v}_L^H] = \mathbf{H}_{L,0} \cdot \mathbb{E}[\mathbf{v}_0\mathbf{v}_0^H] \cdot \mathbf{H}_{L,0}^H, \tag{1.28}$$

where $\mathbf{H}_{L,0}$ is the transfer function including any port reduction.

Chapter 2

Noise Description

As mentioned in [**?**], there are four main noise sources in the receiver. In the following each noise source will be described and its transfer function towards the loads will be derived.

2.1 Antenna Noise

The antennas introduce two noise sources. The external noise $\mathbf{n}_{ext}$, collected from the radiation component of the antenna array and the noise generated by the losses in the antennas $\mathbf{n}_l$. From [**?**] it follows,

$$\mathbf{R}_{na} = \mathbb{E}[\mathbf{n}_{AR}\mathbf{n}_{AR}^H] = 4k_B B_W (T_{AE}\mathbb{R}\{\mathbf{Z}_{AR}\} + T_{AL}\mathbf{R}_{AR}), \tag{2.1}$$

with k_B the Boltzmann constant and B_W the bandwidth.

2.1.1 Transfer Function of the Antenna Noise

As the antenna noise is picked up by the antennas in the same way as the signal, the transfer function remains the same as the one derived in Section 1.3 for the signal, e.g.

$$\begin{aligned} \mathbf{H}_{L,0} &= \mathbf{H}_{L,LNA} \cdot \mathbf{H}_{LNA,M} \cdot \mathbf{H}_{M,0}, \quad \text{including port reduction} \\ \mathbf{H}_{L,0} &= \mathbf{H}_{L,LNA} \cdot \mathbf{H}_{LNA,M} \cdot \mathbf{H}_{M,0} \cdot \mathbf{H}_0. \end{aligned} \tag{2.2}$$

2.2 LNA Noise

The LNA introduces the third noise source. From the discussion in [**?**], the noise of the LNA is modeled by a series of voltages and parallel currents at the input of the LNA. The noise sources have the following statistical properties,

$$\begin{aligned} \mathbb{E}[\mathbf{i}_{LNA}\mathbf{i}_{LNA}^H] &= \beta \mathbf{I}_{N_r}, \\ \mathbb{E}[\mathbf{v}_{LNA}\mathbf{v}_{LNA}^H] &= \beta R_n^2 \mathbf{I}_{N_r}, \quad \text{and} \\ \mathbb{E}[\mathbf{v}_{LNA}\mathbf{i}_{LNA}^H] &= \rho\beta R_n \mathbf{I}_{N_r}, \end{aligned} \tag{2.3}$$

with ρ and β as correlation coefficients.

2.2.1 Transfer Function of the LNA Noise

As written above, we have two noise sources, the serial voltage sources and the parallel current sources. First we will transfer the current source into a voltage source in series as we then can use the same transfer function for the voltage and the transferred current sources. To do so, we need the equivalent input impedances looking from the left into the LNA block and from the right into the matching network. The equivalent input impedance for the LNA network $\mathbf{Z}_{eqLNA_1}$ was already derived in (**??**). To get the equivalent input impedance for the matching network looking from the right into it we use Equation (1.3) and get

$$\mathbf{Z}_R = \mathbf{Z}_{M22} - \mathbf{Z}_{M12} \cdot (\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1} \cdot \mathbf{Z}_{M21}. \tag{2.4}$$

Transferring the current source into a series of voltages gives us

$$\mathbf{v}_{LNA_c} = -\mathbf{Z}_R \mathbf{i}_{LNA}, \tag{2.5}$$

and therefore a transfer function of

$$\begin{aligned} \mathbf{H}_{L,LNA_v} &= \mathbf{H}_{L,LNA} \cdot \mathbf{Z}_{eqLNA_1} (\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_R)^{-1}, \quad \text{for the series voltages and} \\ \mathbf{H}_{L,LNA_c} &= -\mathbf{H}_{L,LNA} \cdot \mathbf{Z}_{eqLNA_1} (\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_R)^{-1} \mathbf{Z}_R, \end{aligned} \tag{2.6}$$

for the LNA noise currents.

2.3 Downstream Noise

The last noise source is the downstream noise, generated by all the circuitry after the LNA [**?**] and modeled by voltage sources $\tilde{\mathbf{v}}_n$ in series to the loads. With the statistical property

$$\mathbb{E}[\tilde{\mathbf{v}}_n \tilde{\mathbf{v}}_n^H] = \psi \mathbf{I}_{N_r}. \tag{2.7}$$

2.3.1 Transfer Function of the Downstream Noise

For the transfer function of the downstream noise we need a simple voltage divider of the loads and the equivalent input impedance $\mathbf{Z}_{eqLNA_2}$ looking from the right into the LNA block. The equivalent input impedance was calculated in (**??**). Therefore the transfer function becomes

$$\mathbf{H}_{L,n} = R_L \mathbf{I}_{N_r} (R_L \mathbf{I}_{N_r} + \mathbf{Z}_{eqLNA_2})^{-1}. \tag{2.8}$$

2.4 Noise Covariance Matrix

As in Chapter **??**, we will derive the noise covariance matrix. As we have multiple noise sources, first we will describe the noise output $\mathbf{u}_L$ by all noise sources, i.e.:

$$\mathbf{u}_L = \mathbf{H}_{L,0} \cdot \mathbf{n}_{AR} + \mathbf{H}_{L,LNA_v} \cdot \mathbf{v}_{LNA} + \mathbf{H}_{L,LNA_c} \cdot \mathbf{i}_{LNA} + \mathbf{H}_{L,n} \cdot \tilde{\mathbf{v}}_n. \tag{2.9}$$

Because all the noise sources are uncorrelated, except for the LNA noise sources (Equation (**??**)), the noice covariance matrix will have six terms of the sum:

$$\begin{aligned} \mathbf{K}_n = \mathbb{E}[\mathbf{u}_L \mathbf{u}_L^H] =& \mathbf{H}_{L,0} \mathbb{E}[\mathbf{n}_{AR} \mathbf{n}_{AR}^H] \mathbf{H}_{L,0}^H + \mathbf{H}_{L,LNA_v} \mathbb{E}[\mathbf{v}_{LNA} \mathbf{v}_{LNA}^H] \mathbf{H}_{L,LNA_v}^H - \\ & \mathbf{H}_{L,LNA_v} \mathbb{E}[\mathbf{v}_{LNA} \mathbf{i}_{LNA}^H] \mathbf{H}_{L,LNA_c}^H - \mathbf{H}_{L,LNA_c} \mathbb{E}[\mathbf{i}_{LNA} \mathbf{v}_{LNA}^H] \mathbf{H}_{L,LNA_v}^H + \\ & \mathbf{H}_{L,LNA_c} \mathbb{E}[\mathbf{i}_{LNA} \mathbf{i}_{LNA}^H] \mathbf{H}_{L,LNA_c}^H + \mathbf{H}_{L,n} \mathbb{E}[\tilde{\mathbf{v}}_n \tilde{\mathbf{v}}_n^H] \mathbf{H}_{L,n}^H. \end{aligned} \tag{2.10}$$

As -$\mathbf{H}_{L,LNA_v} \cdot \mathbf{Z}_R = \mathbf{H}_{L,LNA_c}$ and using (**??**), (**??**) and (**??**), the covariance matrix reduces to

$$\begin{aligned}
\mathbf{K}_n &= \mathbf{H}_{L,0}\mathbf{R}_{na}\mathbf{H}_{L,0}^H + \mathbf{H}_{L,LNA_v}\beta R_N^2 \mathbf{I}_{N_r}\mathbf{H}_{L,LNA_v}^H + \\
&\quad \mathbf{H}_{L,LNA_v}\rho\beta R_N \mathbf{I}_{N_r}\mathbf{Z}_R^H\mathbf{H}_{L,LNA_v}^H + \mathbf{H}_{L,LNA_v}\mathbf{Z}_R\rho^*\beta R_N \mathbf{I}_{N_r}^H\mathbf{H}_{L,LNA_v}^H + \\
&\quad \mathbf{H}_{L,LNA_v}\mathbf{Z}_R\beta\mathbf{I}_{N_r}\mathbf{Z}_R^H\mathbf{H}_{L,LNA_v}^H + \mathbf{H}_{L,n}\psi\mathbf{I}_{N_r}\mathbf{H}_{L,n}^H \\
&= \mathbf{H}_{L,0}\mathbf{R}_{na}\mathbf{H}_{L,0}^H + && \textit{Antenna noise} \\
&\quad \mathbf{H}_{L,LNA_v}\beta(R_N^2\mathbf{I}_{N_r} + 2R_N\mathbb{R}\{\rho^*\mathbf{Z}_R\} + \mathbf{Z}_R\mathbf{Z}_R^H)\mathbf{H}_{L,LNA_v}^H + && \textit{LNA noise} \\
&\quad \mathbf{H}_{L,n}\psi\mathbf{I}_{N_r}\mathbf{H}_{L,n}^H. && \textit{Downstream noise}
\end{aligned} \tag{2.11}$$

Chapter 3

Analytical Gradient

In the following the derivation of the gradients will be given. In the first section the signal gradients will be derived, in the later the noise gradients. The gradients are derived to improve the achievable sum rate function of the system. It is given by

$$\begin{aligned} r &= \log_2(\det(\mathbf{K}_s\mathbf{K}_n^{-1} + \mathbf{I}_{N_r})) \\ &= \log_2(\det(\mathbf{K}_s + \mathbf{K}_n^{-1})) - \log_2(\det(\mathbf{K}_n)), \end{aligned} \tag{3.1}$$

where $\mathbf{K}_s$ and $\mathbf{K}_n$ denote the signal and noise covariance matrices derived in (**??**) and (**??**).
We are interested in the gradient of the achievable sum rate with respect to the matching network and the passive antenna loads. Using the matrix relations from [**?**] the gradient becomes

$$\frac{\partial r}{\partial \mathbf{Z}_{l,ij}} = \frac{1}{\ln 2}\mathrm{Tr}\left((\mathbf{K}_s + \mathbf{K}_n)^{-1}\left(\frac{\partial \mathbf{K}_s}{\partial \mathbf{Z}_{l,ij}} + \frac{\partial \mathbf{K}_n}{\partial \mathbf{Z}_{l,ij}}\right) - \mathbf{K}_n^{-1}\frac{\partial \mathbf{K}_n}{\partial \mathbf{Z}_{l,ij}}\right), \tag{3.2}$$

with $l \in \{11, 12, 22, pass\}$ denoting the sub-matrices of the matching network or the passive antenna loads and i, j denoting the element in the i-th row and j-th column. In the following the gradient is split up in the signal and the noise part. The derivatives for each covariance matrix will be derived.

3.1 Signal Gradients

For the signal covariance matrix, we see from Equation (**??**), that we can take the derivative of each sub transfer function and place them together afterwards by the chain rule. First of all, we check which sub transfer functions are affected by the gradient. We see that for the matching network sub matrices, we need to take the derivative of the transfer functions $\mathbf{H}_{LNA,M}$ and $\mathbf{H}_{M,0}$. For the passive antenna loads $\mathbf{Z}_{pass}$ obviously $\mathbf{H}_0$ for the port reduction is affected. Additionally $\mathbf{H}_{M,0}$ is affected because it contains $\mathbf{Z}_{C'}$.
For the following $\mathbf{J}_{ij}$ denotes the single entry matrix, with respect to the i-th row and the j-th column. If follows for the port reduction transfer function

$$\frac{\partial \mathbf{H}_0}{\partial \mathbf{Z}_{pass,ij}} = \begin{bmatrix} \mathbf{0}_{N_i} & \mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{J}_{ij}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1} \end{bmatrix} \tag{3.3}$$

with $\mathbf{0}_{N_i}$ the N_i by N_i all-zeros matrix. For the voltage divider before the matching network it follows,

$$\frac{\partial \mathbf{H}_{M,0}}{\partial \mathbf{Z}_{11,ij}} = \mathbf{J}_{ij}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1} - \mathbf{Z}_{eqM1}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}\mathbf{J}_{ij}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}, \tag{3.4}$$

$$\text{with} \quad \frac{\partial \mathbf{Z}_{eqM1}}{\partial \mathbf{Z}_{M12}} = \mathbf{J}_{ij}^T(\mathbf{Z}_{M22} + \mathbf{Z}_{eqLNA1})^{-1}\mathbf{Z}_{M12} + \mathbf{Z}_{M12}^T(\mathbf{Z}_{M22} + \mathbf{Z}_{eqLNA1})^{-1}\mathbf{J}_{ij},$$

$$\frac{\partial \mathbf{H}_{M,0}}{\partial \mathbf{Z}_{12,ij}} = \frac{\partial \mathbf{Z}_{eqM1}}{\partial \mathbf{Z}_{M12}}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1} - \mathbf{Z}_{eqM1}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}\frac{\partial \mathbf{Z}_{eqM1}}{\partial \mathbf{Z}_{M12}}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}, \tag{3.5}$$

$$\text{with} \quad \frac{\partial \mathbf{Z}_{eqM1}}{\partial \mathbf{Z}_{M22}} = \mathbf{Z}_{M12}^T(\mathbf{Z}_{M22} + \mathbf{Z}_{eqLNA1})^{-1}\mathbf{J}_{ij}^T(\mathbf{Z}_{M22} + \mathbf{Z}_{eqLNA1})^{-1}\mathbf{Z}_{M12},$$

$$\frac{\partial \mathbf{H}_{M,0}}{\partial \mathbf{Z}_{22,ij}} = \frac{\partial \mathbf{Z}_{eqM1}}{\partial \mathbf{Z}_{M22}}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1} - \mathbf{Z}_{eqM1}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}\frac{\partial \mathbf{Z}_{eqM1}}{\partial \mathbf{Z}_{M22}}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}, \tag{3.6}$$

$$\text{and with} \quad \frac{\partial \mathbf{Z}_{C'}}{\partial \mathbf{Z}_{pass,ij}} = \mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{J}_{ij}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{Z}_{LO}$$

$$\frac{\partial \mathbf{H}_{M,0}}{\partial \mathbf{Z}_{pass,ij}} = -\mathbf{Z}_{eqM1}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}\frac{\partial \mathbf{Z}_{C'}}{\partial \mathbf{Z}_{pass,ij}}(\mathbf{Z}_{eqM1} + \mathbf{Z}_{C'})^{-1}. \tag{3.7}$$

Last, the transfer function over the matching network is affected as follows

$$\frac{\partial \mathbf{H}_{LNA,M}}{\partial \mathbf{Z}_{11,ij}} = -\mathbf{H}_{LNA,M}\mathbf{J}_{ij}\mathbf{Z}_{M11}^{-1}\mathbf{Z}_{M12}(\mathbf{Z}_{eqLNA1} + \mathbf{Z}_{eqM2})^{-1}\mathbf{Z}_{M12}\mathbf{Z}_{M11}^{-1} - \mathbf{H}_{LNA,M}\mathbf{J}_{ij}\mathbf{Z}_{M11}^{-1} \tag{3.8}$$

$$\begin{aligned}\frac{\partial \mathbf{H}_{LNA,M}}{\partial \mathbf{Z}_{12,ij}} = {} & \mathbf{Z}_{eqLNA1}(\mathbf{Z}_{eqLNA1} + \mathbf{Z}_{eqM2})^{-1}(-\mathbf{J}_{ij}^T\mathbf{Z}_{M11}^{-1}\mathbf{Z}_{M12} - \mathbf{Z}_{M12}^T\mathbf{Z}_{M11}^{-1}\mathbf{J}_{ij}) \cdot \\ & (\mathbf{Z}_{eqLNA1} + \mathbf{Z}_{eqM2})^{-1}\mathbf{Z}_{M12}\mathbf{Z}_{M11}^{-1} + \mathbf{Z}_{eqLNA1}(\mathbf{Z}_{eqLNA1} + \mathbf{Z}_{eqM2})^{-1}\mathbf{J}_{ij}\mathbf{Z}_{M11}^{-1},\end{aligned} \tag{3.9}$$

$$\frac{\partial \mathbf{H}_{LNA,M}}{\partial \mathbf{Z}_{22,ij}} = \mathbf{Z}_{eqLNA1}(\mathbf{Z}_{eqLNA1} + \mathbf{Z}_{eqM2})^{-1}\mathbf{J}_{ij}(\mathbf{Z}_{eqLNA1} + \mathbf{Z}_{eqM2})^{-1}\mathbf{Z}_{M12}\mathbf{Z}_{M11}^{-1} \tag{3.10}$$

With these derivations, we can apply the chain rule on the transfer function in (**??**) to get $\frac{\partial \mathbf{K}_s}{\partial \mathbf{Z}_{l,ij}}$.

3.2 Noise Gradients

As written in Section **??**, we have four noise sources. Because they sum up at the loads after the LNA, we can look at each noise transfer function seperately and analyze it on its own.

3.2.1 Antenna Noise Gradient

As the antenna noise is picked up at the same place as the signal, its gradient is also the same as the signal covariance gradient. The formulas from (**??**) to (**??**) apply in the same manner to the antenna noise transfer functions. Therefore they are omitted here. We only note, that the trasfer functions $\mathbf{H}_0$, $\mathbf{H}_{M,0}$ and $\mathbf{H}_{LNA,M}$ are affected in the derivation of $\frac{\partial \mathbf{K}_n}{\partial \mathbf{Z}_{l,ij}}$, with $l \in \{11, 12, 22, pass\}$.

3.2.2 LNA Noise Gradient

Looking at the transfer function of the two LNA noise sources, we see that neither $\mathbf{H}_{L,LNA}$ nor $\mathbf{Z}_{eqLNA_1}$ are affected by the derivations. Only $\mathbf{Z}_R$ and therefore $\mathbf{H}_{L,LNA_v}$ have to be taken into account. It follows,

$$\frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{11,ij}} = \mathbf{Z}_{M12}(\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1}\mathbf{J}_{ij}(\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1}, \tag{3.11}$$

$$\frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{12,ij}} = -\mathbf{J}_{ij}(\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1}\mathbf{Z}_{M21} - \mathbf{Z}_{M12}(\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1}\mathbf{J}_{ij}^T, \tag{3.12}$$

$$\frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{22,ij}} = \mathbf{J}_{ij}, \tag{3.13}$$

and with $\dfrac{\partial \mathbf{Z}_{C'}}{\partial \mathbf{Z}_{pass,ij}} = \mathbf{Z}_{OL}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{J}_{ij}(\mathbf{Z}_{pass} + \mathbf{Z}_{LL})^{-1}\mathbf{Z}_{LO}$,

$$\frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{pass,ij}} = \mathbf{Z}_{M12}(\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1}\frac{\partial \mathbf{Z}_{C'}}{\partial \mathbf{Z}_{pass,ij}}(\mathbf{Z}_{M11} + \mathbf{Z}_{C'})^{-1}\mathbf{Z}_{M21}, \quad \text{which results in} \tag{3.14}$$

$$\frac{\mathbf{H}_{L,LNA_v}}{\partial \mathbf{Z}_{l,ij}} = -\mathbf{H}_{L,LNA}\mathbf{Z}_{eqLNA_1}(\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_R)^{-1}\frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{l,ij}}(\mathbf{Z}_{eqLNA_1} + \mathbf{Z}_R)^{-1}. \tag{3.15}$$

3.2.3 Downstream Noise Gradient

Last, we see, that the transfer function of the downstream noise $\mathbf{H}_{L,\tilde{n}}$ is not affected by any of the derivations above.

3.2.4 Noise Gradient

We can now write the gradient of the noise covariance matrix as

$$\begin{aligned}
\frac{\partial \mathbf{K}_n}{\partial \mathbf{Z}_{l,ij}} = {} & \frac{\partial \mathbf{H}_{L,0}}{\partial \mathbf{Z}_{l,ij}}\mathbf{R}_{na}\mathbf{H}_{L,0}^H + \mathbf{H}_{L,0}\mathbf{R}_{na}\frac{\partial \mathbf{H}_{L,0}}{\partial \mathbf{Z}_{l,ij}}^H \\
& \frac{\partial \mathbf{H}_{L,LNA_v}}{\partial \mathbf{Z}_{l,ij}}\beta\left(R_N^2\mathbf{I}_{N_r} - 2R_N\mathbb{R}\{\rho^*\mathbf{Z}_R\} + \mathbf{Z}_R\mathbf{Z}_R^H\right)\mathbf{H}_{L,LNA_v}^H + \\
& \mathbf{H}_{L,LNA_v}\beta\left(2R_N\frac{\partial\mathbb{R}\{\rho^*\mathbf{Z}_R\}}{\partial \mathbf{Z}_{l,ij}} + \frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{l,ij}}\mathbf{Z}_R^H + \mathbf{Z}_R\left(\frac{\partial \mathbf{Z}_R}{\partial \mathbf{Z}_{l,ij}}\right)^H\right)\mathbf{H}_{L,LNA_v}^H + \\
& \mathbf{H}_{L,LNA_v}\beta\left(R_N^2\mathbf{I}_{N_r} - 2R_N\mathbb{R}\{\rho^*\mathbf{Z}_R\} + \mathbf{Z}_R\mathbf{Z}_R^H\right)\frac{\partial \mathbf{H}_{L,LNA_v}}{\partial \mathbf{Z}_{l,ij}}^H.
\end{aligned} \tag{3.16}$$