1 Coordinate Frames

1.1 Descriptions

1.1.1 Description of a position

Let A be a defined coordinate system in addition to the universe coordinate system and P a position vector. Then

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

is a point represented as a vector in the coordinate system $\{A\}$ (p_x, p_y and p_z indicate distances along the axes of $\{A\}$)

1.1.2 Description of an orientation

Let $\{A\}$ and $\{B\}$ be frames and
The rotation matrix which describes $\{B\}$ relative to $\{A\}$ is defined as:

$$^{A}_{B}R := {}^{B}_{A}R^{T} = {}^{B}_{A}R^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

1.1.3 Description of a frame

Let $\{A\}$ be a coordinate system, and
let $^{A}_{B}R$ a rotation matrix that describes $\{B\}$ relative to $\{A\}$, and
let $^{A}P_{BORG}$ be a vector that locates the origin of $\{B\}$ relative to $\{A\}$.
The frame $\{B\}$ is defined as:

$$\{B\} := \{ {}^{A}_{B}R, {}^{A}P_{BORG}\}$$

1.2 Mapping of frames

Let $\{A\}$ and $\{B\}$ be frames and
$^{A}_{B}R$ a rotation matrix that describes $\{B\}$ relative to $\{A\}$.
Let $^{A}P_{BORG}$ be a vector that locates the origin of $\{B\}$ relative to $\{A\}$.
Let ^{B}P be a vector that locates a point P relative to $\{B\}$.
Then this point relative to $\{A\}$ is defined as:

$$^{A}P := {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG}$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} ^{A}_{B}R & ^{A}P_{BORG} \\ \hline 0\;0\;0 & 1 \end{array}\right] \cdot \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix}$$

$$^{A}P = {}^{A}_{B}T \cdot {}^{B}P$$

1.3 Transformation arithmetic

1.3.1 Compound transformations

Let $^{A}_{B}T$ and $^{B}_{C}T$ be homogeneous transforms.
Then $^{A}_{C}T$ is defined as:

$$^{A}_{C}T := {}^{A}_{B}T \cdot {}^{B}_{C}T = \left[\begin{array}{c|c} ^{A}_{B}R \cdot {}^{B}_{C}R & ^{A}_{B}R \cdot {}^{B}P_{CORG} + {}^{A}P_{BORG} \\ \hline 0\;0\;0 & 1 \end{array}\right]$$

1.3.2 Inverting a transform

Let $^{A}_{B}T$ be a transform.
Then $^{B}_{A}T$ is defined as:

$$^{B}_{A}T := {}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} ^{A}_{B}R^{T} & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{BORG} \\ \hline 0\;0\;0 & 1 \end{array}\right]$$

2 Forward Kinematics

2.1 Joints

- Revolute/Rotational/Revolving/"wisting joint: $\Delta\theta$
- Prismatic/Linear joint: Δx
- Helical/Screw/Cylindrical joint: $\Delta x, \Delta\theta$
- Spherical joint: $\Delta\theta, \Delta\phi, \Delta\psi$
- Flat/Planar joint: $\Delta x, \Delta y, \Delta\theta$

2.2 Denavit-Hartenberg

2.2.1 Attach a frame to a link

Let link i be a link with two axes Axis i and Axis $i+1$ and the link length a_i.
The frame $\{i\}$ will be located on the link as follows:

The $\hat{Z}$-axis of frame $\{i\}$ is called $\hat{Z}_i$ and is coincident with the Axis i

The origin of frame $\{i\}$ is located where the a_i orthogonal intersects the Axis i

The $\hat{X}$-axis of frame $\{i\}$ is called $\hat{X}_i$ and points along a_i in the direction from Axis i to Axis $i+1$

The link twist α_i is measured in the right-hand sense about $\hat{X}_i$

The $\hat{Y}$-axis of frame $\{i\}$ is called $\hat{Y}_i$ and is formed by the right-hand rule

2.2.2 Link-Frame Attachment Procedure

1. Identify the joint axes and imagine (or draw) infinite lines along them. steps 2 through 5 below, consider two of these neighboring lines (at axes i and $i+1$).
2. Identify the common perpendicular between them, or point of intersection. At the point of intersection, or at the point where the common perpendicular meets the i-th axis, assign the link-frame origin.
3. Assign the $\hat{Z}_i$ axis pointing along the i-th joint axis.
4. Assign the $\hat{X}_i$ axis pointing along the common perpendicular, or, if the axes intersect, assign $\hat{X}_i$ to be normal to the plane containing the two axes.
5. Assign the $\hat{Y}_i$ axis to complete a right-hand coordinate system.
6. Assign $\{0\}$ to match $\{1\}$ when the first joint variable is zero. For $\{N\}$, choose an origin location and $\hat{X}_N$ direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

2.2.3 Denavit-Hartenberg notation

Let a_{i-1} be the link length from link $i-1$ to i, and
let α_i be the link twist from axis $i-1$ to i (right-hand sense), and
let d_i be the link offset from link $i-1$ to i, and
let θ_i be the joint angle of a rotation about the common axes of link $i-1$ and i (right-hand sense).
It follows:

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	d_1	θ_1
2	a_1	α_1	d_2	θ_2
		$\vdots$		
n	a_{n-1}	α_{n-1}	d_n	θ_n

2.2.4 Link parameters in terms of the link frames

$$a_i = \text{the distance from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured along } \hat{X}_i$$
$$\alpha_i = \text{the angle from } \hat{Z}_i \text{ to } \hat{Z}_{i+1} \text{ measured about } \hat{X}_i$$
$$d_i = \text{the distance from } \hat{X}_{i-1} \text{ to } \hat{X}_i \text{ measured along } \hat{Z}_i$$
$$\theta_i = \text{the angle from } \hat{X}_{i-1} \text{ to } \hat{X}_i \text{ measured about } \hat{Z}_i$$

2.3 Forward Kinematics

2.3.1 Link transformations

Let a_{i-1} be the link length from link $i-1$ to i, and
let α_i be the link twist from axis $i-1$ to i (right-hand sense), and
let d_i be the link offset from link $i-1$ to i, and
let θ_i be the joint angle of a rotation about the common axes of link $i-1$ and i (right-hand sense).
The transform that defines frame $\{i\}$ relative to frame $\{i-1\}$ is calculated as:

$$^{i-1}_{i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3.2 Concatenating link transformations

Let $^{i-1}_{i}T$ be the transform that defines frame $\{i\}$ relative to frame $\{i-1\}$ and let frame $\{0\}$ be the first frame and $\{N\}$ be the last frame.
$^{0}_{N}T$ is calculated as:

$$^{0}_{N}T = {^{0}_{1}T} \cdot {^{1}_{2}T} \cdot \ldots \cdot {^{N-1}_{N}T}$$

2.3.3 Forward Kinematics of a planar manipulator

Let be given a manipulator with n joints, and
Let all joints be parallel, and
let the joint variable of the i-joint be θ_i. It follows:

$$^{i-1}_{i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}_{n}R = \begin{bmatrix} \cos(\theta_1+\cdots+\theta_n) & -\sin(\theta_1+\cdots+\theta_n) & 0 \\ \sin(\theta_1+\cdots+\theta_n) & \cos(\theta_1+\cdots+\theta_n) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(The orientation of the frame $\{n\}$ can be described using only one angle $\theta = \theta_1 + \cdots + \theta_n$)

$$^{0}P_{nORG} = {^{0}P_{1ORG}} + {^{0}_{1}R} \cdot {^{1}P_{2ORG}} + {^{0}_{2}R} \cdot {^{2}P_{3ORG}} + \cdots + {^{0}_{n-1}R} \cdot {^{n-1}P_{nORG}} = {^{0}P_{1ORG}} + \sum_{i=2}^{n} {^{0}_{i-1}R} \cdot {^{i-1}P_{iORG}}$$

$$^{0}_{n+1}T = \begin{bmatrix} \cos(\theta_1+\cdots+\theta_n) & -\sin(\theta_1+\cdots+\theta_n) & 0 & a_1\cdot\cos(\theta_1) + a_2\cdot\cos(\theta_1+\theta_2) + \cdots + l_n\cdot\cos(\theta_1+\cdots+\theta_n) \\ \sin(\theta_1+\cdots+\theta_n) & \cos(\theta_1+\cdots+\theta_n) & 0 & a_1\cdot\sin(\theta_1) + a_2\cdot\sin(\theta_1+\theta_2) + \cdots + l_n\cdot\sin(\theta_1+\cdots+\theta_n) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.4 Frames With Standard Names

- Base Frame $\{B\}$
- Station Frame $\{S\}$
- Wrist Frame $\{W\}$
- Tool Frame $\{T\}$
- Goal Frame $\{G\}$

3 Jacobians: velocities and static forces

3.1 Linear Velocity

Let ${}^{B}Q$ be a position vector relative to frame $\{B\}$.
The velocity of Q relative to frame $\{B\}$ is computed as the derivative of ${}^{B}Q$ w.r.t. time and defined as:

$$ {}^{B}V_Q = \frac{d}{dt}\,{}^{B}Q $$

Let ${}^{U}CORG$ be the origin of frame $\{C\}$ relative to frame $\{U\}$
The velocity of ${}^{U}CORG$ relative to the universe frame $\{U\}$ is defined as:

$$ v_C = {}^{U}V_{CORG} $$

and the velocity vector of ${}^{U}CORG$ expressed int terms of frame $\{A\}$ (though differentiation done relative to $\{U\}$) is defined as:

$$ {}^{A}v_C = {}^{A}_{U}R \cdot v_C $$

3.2 Angular Velocity

Let $\{A\}$ and $\{B\}$ be two frames.
The angular velocity of a rotation of frame $\{B\}$ relative to $\{A\}$ is defined as:

$$ {}^{A}\Omega_B $$

, where the direction of ${}^{A}\Omega_B$ indicates the axis of rotation
and the magnitude of ${}^{A}\Omega_B$ indicates the speed of rotation

Let $\{C\}$ be a frame and $\{U\}$ be the universe frame
The angular velocity of a rotation of frame $\{C\}$ relative to $\{U\}$ is defined as:

$$ \omega_C = {}^{U}\Omega_C $$

and the angular velocity vector of $\{C\}$ relative to $\{U\}$ expressed in terms of frame $\{A\}$ is defined as:

$$ {}^{A}\omega_C $$

3.3 Transformation of velocities

Let $\{A\}$ and $\{B\}$ be two frames
Let ${}^{B}Q$ be a position vector relative to $\{B\}$ which may change with time
Let the location of $\{B\}$ relative to $\{A\}$ be described by a position vector ${}^{A}P_{BORG}$ which may change with time
Let ${}^{A}\Omega_B$ be a vector describing the rotational velocity of $\{B\}$ relative to $\{A\}$
Let ${}^{A}_{B}R$ a rotation matrix describing $\{B\}$ relative to $\{A\}$
The velocity of Q relative to $\{A\}$ is computed as:

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R \cdot {}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R \cdot {}^{B}Q $$

3.4 Velocity "Propagation" from link to link

3.4.1 Rotational joints

Let joint $i+1$ be rotational:

Let ${}^{i+1}_{i}R$ a rotation matrix describing $\{i\}$ relative to $\{i+1\}$
Let ${}^{i}\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
The angular velocity of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i + \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1} $$

, where $\dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$

Let ${}^{i+1}_{i}R$ a rotation matrix describing $\{i\}$ relative to $\{i+1\}$
Let ${}^{i}v_i$ be the linear velocity of the origin of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^{i}\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^{i}P_{i+1}$ be the origin of $\{i+1\}$ relative to $\{i\}$ The linear velocity of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) $$

3.4.2 Prismatic joints

Let joint $i+1$ be prismatic:

Let ${}^{i+1}_{i}R$ a rotation matrix describing $\{i\}$ relative to $\{i+1\}$
Let ${}^{i}\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
The angular velocity of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\omega_i $$

Let ${}^{i+1}_{i}R$ a rotation matrix describing $\{i\}$ relative to $\{i+1\}$
Let ${}^{i}v_i$ be the linear velocity of the origin of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^{i}\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^{i}P_{i+1}$ be the origin of $\{i+1\}$ relative to $\{i\}$ The linear velocity of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1} $$

3.5 Jacobians

3.5.1 Jacobian-Matrix

Let $p(\Theta) = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ be a position vector. The Jacobian is computed as:

$$ {}^{i}\mathcal{J}(\Theta) = \begin{bmatrix} \frac{\partial p_1}{\partial \theta_1} & \dots & \frac{\partial p_1}{\partial \theta_n} \\ \frac{\partial p_2}{\partial \theta_1} & \dots & \frac{\partial p_2}{\partial \theta_n} \\ \frac{\partial p_3}{\partial \theta_1} & \dots & \frac{\partial p_3}{\partial \theta_n} \\ {}^{0}Z_1 & \dots & {}^{0}Z_n \end{bmatrix} $$

, with $\Theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$

3.5.2 Relate Joint Velocities to Cartesian Velocities

$$ {}^{i}\nu = \begin{bmatrix} {}^{i}v \\ {}^{i}\omega \end{bmatrix} = {}^{i}\mathcal{J}(\Theta) \cdot \dot{\Theta} $$

$$ \dot{\Theta} = {}^{i}\mathcal{J}^{-1}(\Theta) \cdot {}^{i}\nu $$

, where ${}^{i}\nu$ is a vector of Cartesian velocities (here: of the end-effector frame relative to $\{B\}$): A linear velocity vector expressed in terms of $\{i\}$ and a rotational velocity vector expressed in $\{i\}$. and
$\Theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$

Dimension of the Jacobian Matrix:

- Number of rows $\hat{=}$ Number of degrees of freedom in the Cartesian space ($\times 2$ for velocity and angular velocity)
- Number of columns $\hat{=}$ Number of joints of the manipulator

3.5.3 Changing a Jacobian's frame of reference

Let ${}^{B}\mathcal{J}(\Theta)$ be a Jacobian written in frame $\{B\}$ (with ${}^{B}\nu = {}^{B}\mathcal{J}(\Theta)\dot{\Theta}$)
The Jacobian written in $\{A\}$ is computed as:

$$ {}^{A}\mathcal{J}(\Theta) = \left[\begin{array}{c|c} {}^{A}_{B}R & 0 \\ \hline 0 & {}^{A}_{B}R \end{array}\right] \cdot {}^{B}\mathcal{J}(\Theta) $$

3.6 Singularity

3.6.1 Categories

- **Workspace-boundary singularities** occur when the manipulator is fully stretched out or folded back on itself in such a way that the end-effector is at or very near the boundary of the workspace
- **Workspace interior singularities** occur away from the workspace boundary; they generally are caused by a lining up of two or more joint axes

3.6.2 Calculation

Let $\mathcal{J}(\Theta)$ be a Jacobian (with $v = \mathcal{J}(\Theta)\dot{\Theta}$)
The Jacobian is singular (a singularity of the mechanism exists) if:

$$ \det[\mathcal{J}(\Theta)] = 0 $$

3.7 Static Forces in Manipulators

Let ${}^{i+1}_{i}R$ be a rotation matrix describing $\{i\}$ relative to $\{i+1\}$ and
let ${}^{i+1}f_{i+1}$ be a force exerted on link $i+1$ by link i relative to link $i+1$
The force exerted on link i by link $i-1$ relative to link i is defined as:

$$ {}^{i}f_{i} = {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} $$

Let ${}^{i+1}_{i}R$ be a rotation matrix describing $\{i\}$ relative to $\{i+1\}$ and
let ${}^{i+1}n_{i+1}$ be a torque exerted on link $i+1$ by link i relative to link $i+1$ and
let ${}^{i}P_{i+1}$ be the origin of frame $\{i+1\}$ relative to $\{i\}$
The torque exerted on link i by link $i-1$ relative to link i is defined as:

$$ {}^{i}n_{i} = {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i} $$

3.7.1 Calculate the joint Torque required to maintain the static equilibrium

3.7.1.1 Rotational joints:

Let ${}^{i}n_{i}$ be the torque exerted on link i by link $i-1$ relative to link i and
let ${}^{i}\hat{Z}_{i}$ be the joint axis vector.
The joint torque required to maintain the static equilibrium is computed as:

$$ \tau_i = {}^{i}n_{i}^{T} \cdot {}^{i}\hat{Z}_{i} $$

3.7.1.2 Prismatic joints:

Let ${}^{i}f_{i}$ be the force exerted on link i by link $i-1$ relative to link i and
let ${}^{i}\hat{Z}_{i}$ be the joint axis vector.
The joint torque required to maintain the static equilibrium is computed as:

$$ \tau_i = {}^{i}f_{i}^{T} \cdot {}^{i}\hat{Z}_{i} $$

3.8 Jacobians in the Force Domain

Let $\mathcal{J}$ be a Jacobian (with $\nu = \mathcal{J}(\Theta)\dot{\Theta}$) expressed in terms of $\{0\}$ and
let $\mathcal{F}$ be a 6×1 Cartesian force-moment vector acting at the end-effector expressed in terms of $\{0\}$ ($\mathcal{F} = [F, N]^T$ and F is the force acting on the center of the mass of the link and N is the moment acting on the center of the mass of the link)
The vector of torques at the joints is calculated as:

$$\tau = {}^0\mathcal{J}^T \cdot {}^0\mathcal{F}$$

4 Dynamics

4.1 Acceleration of a rigid body

Let ${}^BV_Q$ the velocity of a point Q relative to $\{B\}$
The linear acceleration of Q relative to $\{B\}$ is defined as:

$${}^B\dot{V}_Q = \frac{d}{dt}\ {}^BV_Q$$

Let ${}^A\Omega_B$ be the angular velocity of a rotation of frame $\{B\}$ relative to $\{A\}$
The angular acceleration of $\{B\}$ relative to $\{A\}$ is defined as:

$${}^A\dot{\Omega}_B = \frac{d}{dt}\ {}^A\Omega_B$$

Let ${}^UAORG$ be the origin of frame $\{A\}$ relative to frame $\{U\}$
The acceleration of ${}^UAORG$ relative to the universe frame $\{U\}$ is defined as:

$$\dot{v}_A = {}^U\dot{V}_{AORG}$$

Let $\{A\}$ be a frame and $\{U\}$ be the universe frame
The angular acceleration of a rotation of frame $\{A\}$ relative to $\{U\}$ is defined as:

$$\dot{\omega}_A = {}^U\dot{\Omega}_A$$

4.1.1 Linear acceleration

MAYBE
Let ${}^BQ$ be a position vector relative to frame $\{B\}$ and
let ${}^B\dot{V}_Q$ be the linear acceleration of Q relative to $\{B\}$ and
let ${}^BV_Q$ be the linear velocity of Q relative to $\{B\}$ and
let ${}^A\dot{\Omega}_B$ be a vector describing the angular acceleration of $\{B\}$ relative to $\{A\}$ and
let ${}^A\Omega_B$ be a vector describing the angular velocity of $\{B\}$ relative to $\{A\}$
The linear acceleration of a position vector Q relative to $\{A\}$ is calculated as:

$${}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR \cdot {}^B\dot{V}_Q + 2 \cdot {}^A\Omega_B \times {}^A_BR \cdot {}^BV_Q + {}^A\dot{\Omega}_B \times {}^A_BR \cdot {}^BQ + {}^A\Omega_B \times ({}^A\Omega_B \times {}^A_BR \cdot {}^BQ)$$

in the case of ${}^BV_Q = {}^B\dot{V}_Q = 0$:

$${}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A\dot{\Omega}_B \times {}^A_BR \cdot {}^BQ + {}^A\Omega_B \times ({}^A\Omega_B \times {}^A_BR \cdot {}^BQ)$$

4.1.2 Angular acceleration

MAYBE
Let ${}^A\Omega_B$ be a vector describing the angular velocity of $\{B\}$ relative to $\{A\}$ and
let ${}^A\dot{\Omega}_B$ be the vector describing the angular acceleration of $\{B\}$ relative to $\{A\}$ and
let ${}^B\Omega_C$ be a vector describing the angular velocity of $\{C\}$ relative to $\{B\}$ and
let ${}^B\dot{\Omega}_C$ be the vector describing the angular acceleration of $\{C\}$ relative to $\{B\}$
The angular acceleration of a rotation of $\{C\}$ relative to $\{A\}$ is defined as:

$${}^A\dot{\Omega}_C = {}^A\dot{\Omega}_B + {}^A_BR \cdot {}^B\dot{\Omega}_C + {}^A\Omega_B \times {}^A_BR \cdot {}^B\Omega_C$$

4.2 Acceleration from link to link

4.2.1 Rotational joints

Let joint $i+1$ be rotational:

Let ${}^i\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\dot{\omega}_i$ be the angular acceleration vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
The angular acceleration of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^i\dot{\omega}_i + {}^{i+1}_{i}R \cdot {}^i\omega_i \times \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}$$

, where $\ddot{\theta}_{i+1} \cdot^{i+1} \hat{Z}_{i+1} = {}^{i+1}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$

Let ${}^i\dot{v}_i$ be the linear acceleration of the origin of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\dot{\omega}_i$ be the angular acceleration vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^iP_{i+1}$ be the origin of $\{i+1\}$ relative to $\{i\}$ The linear acceleration of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{i+1}) +^i \dot{v}_i)$$

4.2.2 Prismatic joints

Let joint $i+1$ be prismatic:

Let ${}^i\dot{\omega}_i$ be the angular acceleration vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
The angular acceleration of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^i\dot{\omega}_i$$

Let ${}^iv_i$ be the linear velocity of the origin of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\dot{\omega}_i$ be the angular acceleration vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^iP_{i+1}$ be the origin of $\{i+1\}$ relative to $\{i\}$ The linear velocity of link $i+1$ with respect to frame $\{i+1\}$ is calculated as:

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^i\dot{\omega}_i \times {}^iP_{i+1} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{i+1}) + {}^i\dot{v}_i) + 2 \cdot {}^{i+1}\omega_{i+1} \times \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}$$

4.3 Newton's Equation, Euler's Equation

4.3.1 Newton's Equation

MAYBE
Let m be the total mass of a body (e.g. a link) and
let $\dot{v}_C$ be the acceleration with which the center of mass is accelerating.
The force acting at the center of mass and causing this acceleration is defined as:

$$F = m\dot{v}_C$$

4.3.2 Euler's Equation

MAYBE
Let ω be the angular velocity with which a rigid body is rotating and
let $\dot{\omega}$ be the corresponding acceleration and
let ${}^CI$ be the inertia tensor of the body written in frame $\{C\}$, whose origin is located at the center of mass
The moment N, which must be acting on the body to cause this motion is defined as:

$$N = {}^CI\dot{\omega} + \omega \times {}^CI\omega$$

4.4 Iterative Newton-Euler Dynamic Formulation

4.4.1 Outward iterations to compute velocities and acceleration

Let $\{C_i\}$ be a frame attached to each link, having its origin located at the center of mass of the link and having the same orientation as the link frame $\{i\}$ and
let ${}^i\dot{v}_i$ be the linear acceleration of the origin of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^i\dot{\omega}_i$ be the angular acceleration vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
Let ${}^iP_{C_i}$ be the origin of $\{C_i\}$ relative to $\{i\}$.
The linear acceleration of the center of mass of each link is computed as:

$${}^i\dot{v}_{C_i} = {}^i\dot{\omega}_i \times {}^iP_{C_i} + {}^i\omega_i \times ({}^i\omega_i \times {}^iP_{C_i}) + {}^i\dot{v}_i$$

4.4.2 Force and Torque acting on a link

Let $\{C_i\}$ be a frame attached to each link, having its origin located at the center of mass of the link and having the same orientation as the link frame $\{i\}$ and
The force and torque acting at the center of mass of each link expressed in $\{i\}$ is computed as:

$${}^iF_i = m_i \cdot {}^i\dot{v}_{C_i}$$

$${}^iN_i = {}^{C_i}I_i \cdot {}^i\dot{\omega}_i + {}^i\omega_i \times {}^{C_i}I_i \cdot {}^i\omega_i$$

4.4.3 Inward iterations to compute forces and torques

let ${}^{i+1}f_{i+1}$ be a force exerted on link $i+1$ by link i relative to link $i+1$ and
let ${}^iF_i$ be the force acting at the center of mass and causing this acceleration relative to $\{i\}$.
The force exerted on link i by link $i-1$ relative to link i is defined as:

$${}^if_i = {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} + {}^iF_i$$

Let ${}^{i+1}n_{i+1}$ be a torque exerted on link $i+1$ by link i relative to link $i+1$ and
let ${}^iP_{i+1}$ be the origin of frame $\{i+1\}$ relative to $\{i\}$ and
let ${}^iP_{C_i}$ be the origin of frame $\{C_i\}$ relative to $\{i\}$ and
let $\{C_i\}$ be a frame attached to link $\{i\}$, having its origin located at the center of mass of the link and having the same orientation as the link frame $\{i\}$ and
let ${}^iN_i$ be the torque acting at the center of mass of link $\{i\}$ relative to $\{i\}$ and
let ${}^iF_i$ be the force acting at the center of mass relative to $\{i\}$.
The torque exerted on link i by link $i-1$ relative to link i is defined as:

$${}^in_i = {}^iN_i + {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^iP_{C_i} \times {}^iF_i + {}^iP_{i+1} \times {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1}$$

The required joint torques that will result in the net forces and torques being applied to each link are computed as:
Rotational joint:

$$\tau_i = {}^in_i^T \cdot {}^i\hat{Z}_i$$

Prismatic joint:

$$\tau_i = {}^if_i^T \cdot {}^i\hat{Z}_i$$

4.4.4 The iterative Newton-Euler dynamics algorithm

for i = 0 to n-1
do
    calculate: ${}^{i+1}\omega_{i+1}$, ${}^{i+1}\dot{\omega}_{i+1}$, ${}^{i+1}\dot{v}_{i+1}$, ${}^{i+1}\dot{v}_{C_{i+1}}$, ${}^{i+1}F_{i+1}$ and ${}^{i+1}N_{i+1}$
done

for i = n downto 1
do
    calculate: ${}^if_i$, ${}^in_i$ and τ_i
done

4.4.5 Inclusion of gravity forces in the dynamics algorithm

Set

$$^0\dot{v}_0 = G$$

, where G has the magnitude of the gravity vector but points in the opposite direction (usually G is positive).

4.5 The Structure of a Manipulator's Dynamic Equations

4.5.1 State-space Equation

Let $\tau = [\tau_1, \dots, \tau_n]$ be a vector of actuator torques and
let $M(\Theta)$ be the $n \times n$ mass matrix of the manipulator and
let $V(\Theta, \dot{\Theta})$ be an $n \times 1$ vector of centrifugal and Coriolis terms and
let $G(\Theta)$ be an $n \times 1$ vector of gravity terms.
The state-space equation is defined as:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

4.5.2 Configuration-space Equation

Let $\tau = [\tau_1, \dots, \tau_n]$ be a vector of actuator torques and
let $M(\Theta)$ be the $n \times n$ mass matrix of the manipulator and
let $B(\Theta)$ be a matrix of dimensions $n \times n(n-1)/2$ of Coriolis coefficient and
let $[\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1\dot{\theta}_2, \dot{\theta}_1\dot{\theta}_3, \dots, \dot{\theta}_{n-1}\dot{\theta}_n]^T$ with dimension $n(n-1)/2 \times 1$ and
let $C(\Theta)$ be an $n \times n$ matrix of centrifugal coefficients and
let $[\dot{\Theta}^2] = [\dot{\theta}_1^2, \dots, \dot{\theta}_n^2]^T$ of dimension $n \times 1$ and
let $G(\Theta)$ be an $n \times 1$ vector of gravity terms.
The configuration-space equation is defined as:

$$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$$

4.6 Lagrangian Formulation of Manipulator Dynamics

4.6.1 Kinematic energy

Let m_i be the mass of the link i and
let v_{C_i} be the linear velocity of the center of mass of link i and
Let $^i\omega_i$ be the angular velocity vector of link $\{i\}$ relative to $\{B\}$ expressed in terms of frame $\{i\}$
let ^{C_i}I be the inertia tensor of the body written in frame $\{C_i\}$, whose origin is located at the center of mass of link i.
The kinematic energy of the ith link is defined as:

$$k_i = \frac{1}{2} m_i \cdot v_{C_i}^T \cdot v_{C_i} + \frac{1}{2}\ {}^i\omega_i^T \cdot\ {}^{C_i}I_i \cdot\ {}^i\omega_i$$

The total kinematic energy of the manipulator is computed as:

$$k = \sum_{i=1}^{n} k_i$$

$$k(\Theta, \dot{\Theta}) = \frac{1}{2}\dot{\Theta}^T M(\Theta)\dot{\Theta}$$

, where $M(\Theta)$ is the $n \times n$ mass matrix of the manipulator and $\Theta = [\theta_1, \dots, \theta_n]^T$

4.6.2 Potential energy

Let m_i be the mass of the link i and
let ^0g be the 3×1 gravity vector (usually negative) and
let $^0P_{C_i}$ be the vector locating the center of mass of the ith link and
let u_{ref_i} be a constant chosen so that the minimum values of u_i is zero.
The potential energy of the ith link is defines as:

$$u_i = -m_i \cdot\ {}^0g^T \cdot\ {}^0P_{C_i} + u_{ref_i}$$

The total potential energy stored in the manipulator is computed as:

$$u = u(\Theta) = \sum_{i=1}^{n} u_i$$

4.6.3 Lagrangian formulation

Let $k(\Theta, \dot{\Theta})$ be the total kinematic energy of a manipulator and
let $u(\Theta)$ be the total potential energy of the manipulator.
The equations of motions for the manipulator are given by:

$$\tau_i = \frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}_i} - \frac{\partial k}{\partial \theta_i} + \frac{\partial u}{\partial \theta_i}$$

4.6.4 Lagrangian formulation algorithm

for i = 1 to n
do
    calculate: ${}^{i}\omega_i$, $v_{C_i} = \frac{d}{dt}\ {}^{0}P_{C_i}$, k_i, u_i
done
calculate: k, u
for i = 1 to n
do
    calculate: τ_i
done

5 Linear control of manipulators

5.1 Second-Order Linear Systems - DGLs

Let be given a spring-mass system with friction where
m be the mass of a block attached to the spring and
k be the stiffness of the spring and
b a coefficient of friction.
It follows:

$$m\ddot{x} + b\dot{x} + kx = 0$$

The corresponding characteristic equation is defined as:

$$ms^2 + bs + k = 0$$

with the roots (poles of the system)

$$s_{1,2} = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4mk}}{2m}$$

5.1.1 Real and Unequal Roots

- If the roots are real and negative → Overdamped (non-oscillatory exponential decay)
- If the roots are real and positive → Not BIBO stable (non-oscillatory exponential increase)

Let $b^2 > 4mk$
It follows:

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

, where c_1 and c_2 are constants which can be computed for any given set of initial conditions

5.1.2 Complex Roots

- If the roots are complex with negative real components → Underdamped (oscillatory decay)
- If the roots are complex with positive real components → Not BIBO stable (oscillatory increase)
- If the roots are purely imaginary → Undamped (oscillatory behavior without increase/decay)

Let $b^2 < 4mk$
It follows:

$$x(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

, where $s_{1,2} = \lambda \pm \mu i$

5.1.3 Real and Equal Roots

- If the roots are real, equal and negative → Critically damped (fastest non-oscillatory exponential decay)

Let $b^2 = 4mk$
It follows:

$$x(t) = c_1 e^{-\frac{b}{2m}t} + c_2 t e^{-\frac{b}{2m}t}$$

5.2 Control of Second-Order Systems

Let be given a spring-mass system with friction and an actuator where
m be the mass of a block attached to the spring and
k be the stiffness of the spring and
b a coefficient of friction and
f be a force applied by an actuator to the block
It follows:

$$m\ddot{x} + b\dot{x} + kx = f$$

Let x be the position of the block detected by a sensor and
let $\dot{x}$ be the velocity of the block detected by a sensor and
The force that should be applied by the actuator is computed as:

$$f = -k_p x - k_v \dot{x}$$

, where k_p is the position gain and k_v is the velocity gain

5.2.1 Closed-loop Stiffness

Let the closed-loop stiffness be $k' = k + k_p$ and
let $b' = b + k_v$.
It follows:

$$m\ddot{x} + b'\dot{x} + k'x = 0$$

- If b' or k' is negative → Unstable Control System
- For a critical damping: $b' = 2\sqrt{mk'}$

5.2.2 Computing k_v, k_p in a critically damping system

Let be given: m, k, b, ω_{res}.
k_v and k_p are determined using following equations:

$$b' = b + k_v$$
$$k' = k + k_p$$
$$b' = 2\sqrt{mk'}$$
$$\omega_n = \sqrt{\frac{k'}{m}}$$
$$\omega_n \leq \frac{1}{2}\omega_{res}$$

5.3 Control-Law Partitioning

- Partitioning of the controller into
  - a model-based portion and
  - a servo portion

5.3.1 Model-based portion

Let be given a spring-mass system with friction and an actuator where
f be the force applied by an actuator, and
m be the mass of a block attached to the spring, and
k be the stiffness of the spring, and
b a coefficient of friction
The model-based potion of the control is defined as:

$$f = \alpha f' + \beta$$

α and β are chosen as follows:

$$\alpha = m$$
$$\beta = b\dot{x} + kx$$

For a critical damping follows:

$$k_v = 2\sqrt{k_p}$$

, where $f' = \ddot{x} = -k_v\dot{x} - k_p x$

5.4 Trajectory-Following Control

5.4.1 Servo Error

Let the trajectory be given as a function of time with
$x_d(t)$ be the desired position of the block and
$\dot{x}_d(t)$ be the desired velocity of the block and
$\ddot{x}_d(t)$ be the desired acceleration of the block and
The servo error is defined as:

$$e = x_d - x$$
$$\dot{e} = \dot{x}_d - \dot{x}$$
$$\ddot{e} = \ddot{x}_d - \ddot{x}$$

5.4.2 Trajectory following Servo-control Law (Servo Portion)

The servo-control law that will cause trajectory following is defined as:

$$f' = \ddot{x}_d + k_v\dot{e} + k_p e$$

It follows:

$$\ddot{e} + k_v\dot{e} + k_p e = 0$$

5.5 Creating a PD controller

1. Determine forces that apply on objects
2. Derive equation of motion
   - single: $f = m\ddot{x} + b\dot{x} + kx$
   - multi: $f = M\ddot{x} + B\dot{x} + Kx$,
     where $f, \ddot{x}, \dot{x}, x$ are $n \times 1$ vectors and M, B, K are $n \times n$ matrices
3. model-based portion: $f = \alpha f' + \beta$,
   where $\alpha = M, f' = \ddot{x}$ and $\beta = B\dot{x} + Kx$
4. $E = x_d - x$
5. servo portion: $f' = \ddot{x}_d + K_v\dot{E} + K_p E \iff \ddot{E} + K_v\dot{E} + K_p E = 0$
6. $k_{vi} = 2\sqrt{k_{pi}}$

5.6 Disturbance Rejection

Let e be the servo error with its derivatives $\dot{e}$ and $\ddot{e}$
The disturbance force is computed as:

$$f_{dist} = m(\ddot{e} + k_v\dot{e} + k_p e)$$

If f_{dist} is bounded $\implies$ the solution of the differential equation $e(t)$ is also bounded
$\implies$ BIBO (bounded-input, bounded-output) stability

5.6.1 Steady-state Error

Let the disturbance force f_{dist} be constant.
The steady-state error is computed as:

$$e = \frac{f_{dist}}{k_p}$$

$\implies$ The higher the position gain k_p, the smaller will be the steady-state error.

5.6.2 PID control law (proportional, integral, derivative control law)

By adding an integral term to the control law to eliminate steady-state error.
It follows:

$$f' = \ddot{x}_d + k_v\dot{e} + k_p e + k_i \int e \text{ dt}$$

The disturbance force is computed as:

$$f_{dist} = \ddot{e} + k_v\dot{e} + k_p e + k_i \int e \text{ dt}$$

If $e(t) = 0$ for $t < 0$ it follows:

$$\dot{f}_{dist} = \dddot{e} + k_v\ddot{e} + k_p\dot{e} + k_i e$$

It follows that the steady-state error $e = 0$

6 Nonlinear control of manipulators

6.1 Control-Law Partitioning of a (MIMO) Multi-Input, Multi-Output Control System

6.1.1 Model-Based Portion

Let $F = [f_1, \ldots, f_n]^n$ be a vector of forces applied by n actuators.
The Model-based portion of the control is defined as:

$$F = \alpha F' + \beta$$

, where α is a $n \times n$ matrix and F' and β are $n \times 1$ vectors and the model-based portion of the control law is called a linearizing and decoupling control law.

6.1.2 Servo-Portion

Let $E = X_d - X$ a $n \times 1$ vector of the errors in position and
Let $\dot{E}$ a $n \times 1$ vector of the errors in velocity and
let X_d be a vector of the desired positions and
let X be a vector of the current positions.
The servo law is defined as:

$$F' = \ddot{X}_d + K_v\dot{E} + K_p E$$

, where K_v and K_p are $n \times n$ matrices, which are generally chosen to be diagonal with constant gains

6.2 The Control Problem for Manipulators

6.2.1 Manipulator's Dynamic Equation with friction

Let $\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$ be the dynamic equation of a manipulator and
let $F(\Theta, \dot{\Theta})$ be a model of friction dependent on the joint position Θ,
The dynamic equation with friction is defined as:

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$$

6.2.2 Partitioning of a Dynamics Equation with Friction

Let $\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$ be the dynamic equation of a manipulator with friction
The model-based portion is defined as:

$$\tau = \alpha\tau' + \beta$$

, where

$$\alpha = M(\Theta)$$
$$\beta = V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$$

The servo law is defined as:

$$\tau' = \ddot{\Theta}_d + K_v\dot{E} + K_pE$$

, where $E = \Theta_d - \Theta$

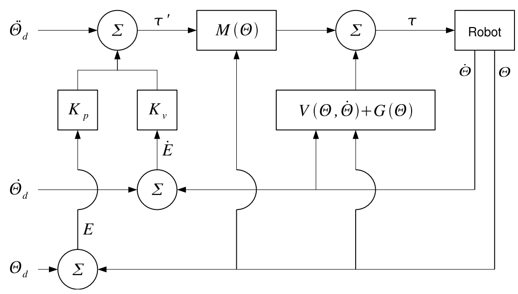
It follows:

$$\ddot{E} + K_v\dot{E} + K_pE = 0$$

and on a joint-by-joint basis:

$$\ddot{e}_i + k_{vi}\dot{e} + k_{pi}e = 0$$

, where this equation is possible because the vector equation is decoupled



6.3 Creating a nonlinear PD controller (Manipulator)

1. Compute τ (Dynamics)
2. $\tau = \alpha\tau' + \beta$,
   where $\alpha = M(\Theta), f' = \ddot{\theta}$ and $\beta = V(\Theta, \dot{\Theta}) + G(\Theta)$
3. $E = \theta_d - \theta$
4. $\tau' = \ddot{\theta}_d + K_v\dot{E} + K_pE \iff \ddot{E} + K_v\dot{E} + K_pE = 0$
5. $k_{vi} = 2\sqrt{k_{pi}}$ (critically damped)

6.4 Natural frequency

Let k_{pi} be a gain.
The natural frequency in the context of manipulator control is computed as:

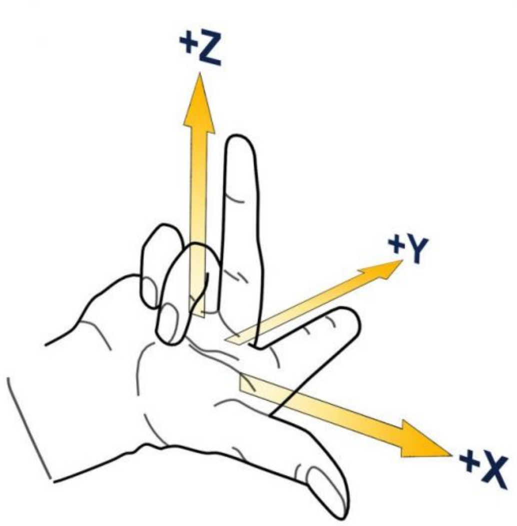
$$\omega_{ni} = \sqrt{k_{pi}}$$

7 Other Basics

7.1 Joint axis vectors

$$\hat{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{Y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{Z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

7.2 Right-Hand Coordinate System



7.3 Schematic notation

7.3.1 Parallel axes

Double hash marks on a simple schematic notation indicate that the axis are mutually parallel

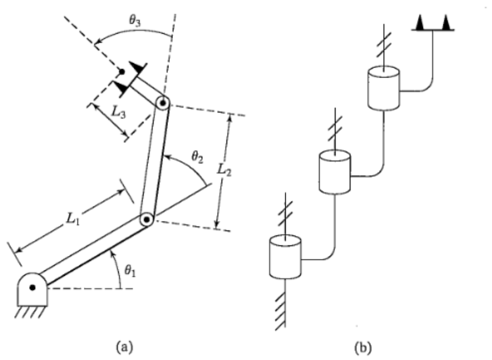


FIGURE 3.6: A three-link planar arm. On the right, we show the same manipulator by means of a simple schematic notation. Hash marks on the axes indicate that they are mutually parallel.

7.3.2 orthogonal axes

8 Mathematical Basics

8.1 Rotation Matrix

Rotation about x-axis: $R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

Rotation about y-axis: $R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

Rotation about z-axis: $R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Determinant: $\det R = +1$

8.2 Crossproduct

Let $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be two vectors.
The cross product is computed as:

$$a \times b = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

8.3 Trigonometric Functions

8.3.1 Arcustangens 2

$$\text{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y > 0 \\ \pm\pi & \text{if } x < 0 \text{ and } y = 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

8.3.2 Simplifications

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

8.3.3 Sine, Cosine, and Tangent table

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$\sin\theta = -\sin(\theta - 180°) = -\sin(-\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{(2)}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{(2)}}{2}$	$\frac{1}{2}$	0
$\cos\theta = -\cos(\theta - 180°) = \cos(-\theta)$	1	$\frac{\sqrt{(3)}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{(3)}}{2}$	-1
$\tan\theta = \tan(\theta - 180°) = -\tan(-\theta)$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

9 Notations

9.1 CoSinus

$$\cos\theta_i \equiv c\theta_i \equiv c_i \qquad \text{for } i \in \mathbb{N}$$
$$\sin\theta_i \equiv s\theta_i \equiv s_i \qquad \text{for } i \in \mathbb{N}$$

9.2 CoSinus2

$$c_{12} = c_1 c_2 - s_1 s_2 = \cos(\theta_1 + \theta_2)$$
$$s_{12} = c_1 s_2 + s_1 c_2 = \sin(\theta_1 + \theta_2)$$

9.3 CoSinus3

$$c_{123} = c_1 s_2 s_3 + s_1 s_2 c_3 - s_1 c_2 s_3 - c_2 c_3 c_4$$
$$s_{123} = s_1 c_2 c_3 + c_1 c_2 s_3 - c_1 s_2 c_3 - s_1 s_2 s_3$$

Note

This is a short summary of the lecture Robotics of the Technical University Munich. This lecture was presented by Burschka D. in the winter semester 2018/19. This summary was created by Gaida B. All provided information is without guarantee.

References

John J. Craig. *Introduction to Robotics Mechanics and Control*. Pearson Education, Inc. New York. 2005