Course Project
Part 2: Orbital Dynamics Simulation

Due Date: March 6, 2020 at 1:25pm in Akerman Hall 225
Last Updated: February 21, 2020

The purpose of this part of the project is to derive the equations of motion that describe the orbital dynamics of your chosen spacecraft and numerically simulate. The following are the detailed tasks for Part 2 of the project:

1. **Orbital Dynamics Equations of Motion**: Follow the three steps to success in particle dynamics to derive the orbital dynamics (translational equations of motion) of the satellite you chose in Part 1 of the project. Specifically, complete the following:

   (a) Using your result from Q. 3(a) on Part 1 of the project or starting from scratch, show that given

   $$\underrightarrow{r}^{sp} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}} \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix},$$

   you can obtain

   $$\underrightarrow{a}^{sp/a/a} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}} \begin{bmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{z}_a \end{bmatrix},$$

   where s is a particle representing the center of mass of your spacecraft, p is an unforced particle at the center of the body your spacecraft is orbiting about (e.g., Earth, Mars, Sun, etc.), and $\mathcal{F}_a$ is an inertial frame. You can draw a diagram to help explain or illustrate your reasoning.
   Hint: The position vector $\underrightarrow{r}^{sp}$ is resolved in $\mathcal{F}_a$, so the first two steps to success in kinematics are unnecessary.

   (b) Draw a free body diagram of your spacecraft (simply particle s for now) including a gravitational force of

   $$\underrightarrow{f}^{sg} = -\frac{\mu m_s}{\left\| \underrightarrow{r}^{sp} \right\|_2^3} \underrightarrow{r}^{sp} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}} \left(-\frac{\mu m_s}{r^3} \mathbf{r}_a^{sp} \right). \tag{1}$$

   The terms in (1) are:

   - m_s: mass of your spacecraft.
   - $\mu = Gm_p$: gravitational constant of the planetary body your spacecraft is orbiting about ($\mu = 3.986 \times 10^{14}$ m^3/s^2 for the Earth).
   - $G = 66.73 \times 10^{-12}$: university gravitational constant.
   - m_p: mass of the planetary body.
   - $r = \left\| \underrightarrow{r}^{sp} \right\|_2 = \sqrt{\mathbf{r}_a^{sp^{\mathsf{T}}} \mathbf{r}_a^{sp}}$.

   (c) Derive the equations of motion that describe the translational equations of motion of your spacecraft using Newton's 2nd Law.

   (d) Write your derived equations of motion in first-order form:

   $$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{r}_a^{sp} \\ \dot{\mathbf{r}}_a^{sp} \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \\ z_a \\ \dot{x}_a \\ \dot{y}_a \\ \dot{z}_a \end{bmatrix}.$$

(e) Write out the total energy of the spacecraft: $E_{sp/a} = T_{sp/a} + V_{sp}$, where

$$T_{sp/a} = \tfrac{1}{2} m_s \mathbf{v}_a^{sp/a^\mathsf{T}} \mathbf{v}_a^{sp/a},$$

$$V_{sp} = -\frac{\mu m_s}{r}.$$

(f) *Bonus!* Redo parts (b)-(e) with a J_2 perturbation force due to the oblateness (flatness at the poles) of the planetary body your spacecraft orbits.

To implement a J_2 perturbation force (see pgs. 156–164, Section 7.3, of [1]), simply replace the gravitational force with

$$\begin{aligned} \underset{\rightarrow}{f}^{sg} &= -\frac{\mu m_s}{\left\| \underset{\rightarrow}{r}^{sp} \right\|_2^3} \underset{\rightarrow}{r}^{sp} + \frac{3\mu m_s J_2 R_p^2}{2 \left\| \underset{\rightarrow}{r}^{sp} \right\|_2^5} \left(\left(5 \frac{\left(\underset{\rightarrow}{r}^{sp} \cdot \underset{\rightarrow}{a}^3 \right)^2}{\left\| \underset{\rightarrow}{r}^{sp} \right\|_2^2} - 1 \right) \underset{\rightarrow}{r}^{sp} - 2 \left(\underset{\rightarrow}{r}^{sp} \cdot \underset{\rightarrow}{a}^3 \right) \underset{\rightarrow}{a}^3 \right), \\ &= \underset{\rightarrow}{\mathcal{F}}_a^\mathsf{T} \left(-\frac{\mu m_s}{r^3} \mathbf{r}_a^{sp} + \frac{3\mu m_s J_2 R_p^2}{2 r^5} \left(\left(\frac{5}{r^2} \left(\boldsymbol{r}^{sp^\mathsf{T}} \mathbf{1}_3 \right)^2 - 1 \right) \mathbf{r}_a^{sp} - 2 \left(\mathbf{r}_a^{sp^\mathsf{T}} \mathbf{1}_3 \right) \mathbf{1}_3 \right) \right), \end{aligned}$$

where $J_2 = 1.08262645 \times 10^{-3}$ is the perturbation coefficient due to Earth's oblateness, R_p is the radius of the planetary body ($R_p = 6.3712 \times 10^6$ m for the Earth), and $\mathbf{1}_3^\mathsf{T} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. The first term in the gravitational force assumes the the planetary body your spacecraft orbits is perfectly spherical, while the second term corresponds to a J_2 perturbation in the gravitational force due to the oblateness (sphere flattened at the poles) of the planetary body.

If your spacecraft does not orbit the Earth, you can still include the J_2 perturbation due to Earth's oblateness in your gravitational force.

Summary of Deliverables for Question 1

- Submit your work detailing your solutions to parts (a)-(e).

2. **Numerical Simulation of Orbital Dynamics**: Numerically simulate the equations of motion you derived in Q. 1 in the following steps.

(a) Modify the sample `matlab` code provided for the pendulum example in class to numerically simulate your spacecraft's equations of motion. Specifically, you will need to modify the following files:

- `constants_struct.m`: Use this script to define your relevant physical and orbital parameters (e.g., m_s, μ, R_p, etc.).
- `ODEs.m`: Use this function to calculate $\mathbf{f}(\mathbf{x})$ at each instance in time. You will need to extract the relevant constants from the const structure.

- `main_compile.m`: Set initial conditions for simulation. The provided function `FindRandV.m` can help you find a suitable initial condition.
- `post_processing.m`: Calculate the total energy of the spacecraft at each instance in time using the expression for $E_{sw/a}$ you derived in Q.1(e).
- `plot_script.m`: Generate plots of x_a, y_a, z_a, $\dot{x}_a$, $\dot{y}_z$, $\dot{z}_a$, $E_{sw/a}$, and $\frac{E_{sw/a}-E_{sw/a}(0)}{E_{sw/a}(0)}$ versus time. Make sure to label the axes of your plots. You can also use the provided function `EarthPlot.m` to plot the orbit of your spacecraft about the Earth. The function `EarthPlot.m` requires time histories of x_a, y_a, and z_a, and R_p as inputs. You must also have `coast.mat` in the same directory as your code.

Submit your code for `constants_struct.m`, `main_compile.m`, `ODEs.m`, and `post_processing.m`.

(b) Run the simulation for at least one orbital period. You will need to determine suitable initial conditions for $\mathbf{x}^{\mathsf{T}}(0) = \begin{bmatrix} x_a(0) & y_a(0) & z_a(0) & \dot{x}_a(0) & \dot{y}_a(0) & \dot{z}_a(0) \end{bmatrix}$ given the orbit of your spacecraft that you specified in Part 1 of the project. You can use the provided function `FindRandV.m` or solve for these by hand on your own. To use `FindRandV.m`, you will need to provide the function the following orbital parameters: gravitational constant (μ), eccentricity (e), inclination (i), semimajor axis (a), argument of perigee (ω), longitude of the ascending node (Ω), time of perigee passage (t_0), and the time that you want to evaluate the position and velocity at ($t = 0$ s). It is perfectly acceptable to choose $\omega = \Omega = 0$ rad and $t_0 = 0$ s. You will also need to have the provided function `rot.m` in the same directory as your code.

Submit plots of x_a, y_a, z_a, $\dot{x}_a$, $\dot{y}_z$, $\dot{z}_a$, $E_{sw/a}$, and $\frac{E_{sw/a}-E_{sw/a}(0)}{E_{sw/a}(0)}$ versus time, as well as the plot generated by `EarthPlot.m` if you use it. Is the total energy constant (within a reasonably small tolerance) throughout your simulation? If not, this indicates that either a smaller integration tolerance is required or there is an error in your code.

If you completed the bonus in Q.1(f), do not include any J_2 perturbation force or gravitational forces due to other planetary bodies at this step in your simulation. If you included the J_2 term in your `ODEs.m` file, then set $J_2 = 0$ when running this simulation.

(c) *Bonus!* Run your simulation again with the J_2 perturbation force you calculated in Q.1(f). When calculating the total energy of the spacecraft, you will want to include the additional potential energy due to the J_2 perturbation force, given by

$$V_{sp}^{J_2} = \frac{\mu m_s}{r^3} J_2 R_p^2 \left(\frac{3}{2r^2} \left(\mathbf{r}_a^{sp^{\mathsf{T}}} \mathbf{1}_3 \right)^2 - \tfrac{1}{2} \right).$$

Provide the same plots as in Q.2(b). Again, the total energy should be constant within a reasonably small tolerance.

Summary of Deliverables for Question 2

- Submit your code for `constants_struct.m`, `main_compile.m`, `ODEs.m`, and `post_processing.m`.
- Submit plots of x_a, y_a, z_a, $\dot{x}_a$, $\dot{y}_z$, $\dot{z}_a$, $E_{sw/a}$, $\frac{E_{sw/a}-E_{sw/a}(0)}{E_{sw/a}(0)}$ versus time, as well as the plot generated by `EarthPlot.m` if you use it. You will provide a second set of the same plots if you choose to do the bonus (part (c)).

References

[1] A. H. J. de Ruiter, C. J. Damaren, and J. R. Forbes, *Spacecraft Dynamics and Control: An Introduction*. West Sussex, UK: John Wiley & Sons, Ltd., 2013.