Chapter 4: Nominal and Effective Interest Rates

Session 9-10-11

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Topics to Be Covered in Today's Lecture

- Section 4.1: Nominal and Effective Interest Rates statements
- Section 4.2: Effective Annual Interest Rates

Section 4.1
NOMINAL & EFFECTIVE RATES

- Review Simple Interest and Compound Interest (from Chapter 1)
- Compound Interest –
  - Interest computed on Interest
  - For a given interest period
- The time standard for interest computations – One Year
- One Year: Can be segmented into:
  - 365 days
  - 52 Weeks
  - 12 Months
  - One quarter: 3 months – 4 quarters/year
- Interest can be computed more frequently than one time a year

4.1 Common Compounding Frequencies

- Interest May be computed (compounded):
  - Annually – One time a year (at the end)
  - Every 6 months – 2 times a year (semi-annual)
  - Every quarter – 4 times a year (quarterly)
  - Every Month – 12 times a year (monthly)
  - Every Day – 365 times a year (daily)
  - …
  - Continuous – infinite number of compounding periods in a year.

4.1 Quotation of Interest Rates

- Interest rates can be quoted in more than one way.
- Example:
  - Interest equals “5% per 6-months”
  - Interest is “12%” (12% per what?)
  - Interest is 1% per month
  - “Interest is “12.5% per year, compounded monthly”
- Thus, one must “decipher” the various ways to state interest and to calculate.

4.1 Two Common Forms of Quotation

- Two types of interest quotation
  - 1. Quotation using a Nominal Interest Rate
  - 2. Quoting an Effective Periodic Interest Rate
- Nominal and Effective Interest rates are common in business, finance, and engineering economy
- Each type must be understood in order to solve various problems where interest is stated in various ways.

4.1 Notion of a Nominal Interest Rate

- A *Nominal* Interest Rate, r.
- Definition:

A Nominal Interest Rate, *r*,
is an interest Rate that does
not include
any consideration
of compounding

Nominal means, “in name only”,
not the real rate in
this case.

4.1 Quoting a Nominal Interest Rate

- Interest rates may be quoted (stated – communicated) in terms of a nominal rate.
- You will see there are two ways to quote an interest rate:
  - 1. Quote the Nominal rate
  - 2. Quote the true, effective rate.
- For now – we study the nominal quotation.

4.1 Definition of a Nominal Interest Rate

- Mathematically we have the following definition:

r = (interest rate per period)(No. of Periods)

Examples Follow…..

4.1 Examples – Nominal Interest Rates

- 1.5% per month for 24 months
  - **Same as: (1.5%)(24) = 36% per 24 months**
- 1.5% per month for 12 months
  - **Same as (1.5%)(12 months) = 18%/year**
- 1.5% per month for 6 months
  - **Same as: (1.5%)(6 months) = 9%/6 months or semiannual period**
- 1% per week for 1 year
  - **Same as: (1%)(52 weeks) = 52% per year**

4.1 Nominal Rates…..

- A nominal rate (so quoted) do not reference the frequency of compounding. They all have the format "*r%* per time period"
- Nominal rates can be misleading
- We need an alternative way to quote interest rates….
- The true *Effective Interest Rate* is then applied….

4.1 The Effective Interest Rate (EIR)

- When so quoted, an Effective interest rate is a true, periodic interest rate.
- It is a rate that applies for a stated period of time
- It is conventional to use the year as the time standard
- So, the EIR is often referred to as the Effective Annual Interest Rate (EAIR)

4.1 The EAIR

- Example:
  - "*12 per cent compounded monthly*"
- Pick this statement apart:
  - **12% is the nominal rate**
  - **"compounded monthly" conveys the frequency of the compounding throughout the year**
  - **This example: 12 compounding periods within a year.**

4.1 The EAIR and the Nominal Rate

- The EAIR adds to a nominal rate by informing the user of the frequency of compounding within a year.
- Notation:
- It is conventional to use the following notation for the EAIR
  - **“i_a” or,**
  - **“i”**
- **The EAIR is an extension of the nominal rate – “*r*”**

4.1 Focus on the Differences

- Nominal Rates:
  - Format: “**r**% per time period, t”
  - Ex: 5% per 6-months”
- Effective Interest Rates:
  - Format: “**r**% per time period, compounded ‘**m**’ times a year.
  - ‘**m**’ denotes or infers the number of times per year that interest is compounded.
  - Ex: 18% per year, compounded monthly

4.1 Which One to Use: "r" or "i"?

- Some problems may state only the nominal interest rate.
- **Remember: Always apply the Effective Interest Rate in solving problems.**
- Published interest tables, closed-form time value of money formula, and spreadsheet function assume that only Effective interest is applied in the computations.

4.1 Time Based Units Associated with Interest Rate Statements

- Time Period, ***t*** - **Interest rates are stated as % per time period.** **(*t*** usually in years)
- Compounding Period, **(CP) - Length of time between compounding operations.**
- Compounding Frequency-Let "m" represent the number of times that interest is computed (compounded) within time period "t".

4.1 Effective Rate per CP

- The Effective interest Rate per compounding period, CP is:

$$i_{\text{effective}} \text{ per CP} = \frac{r\%/\text{time period } t}{m \text{ compounding periods}/t}$$

4.1 Example:

- Given:

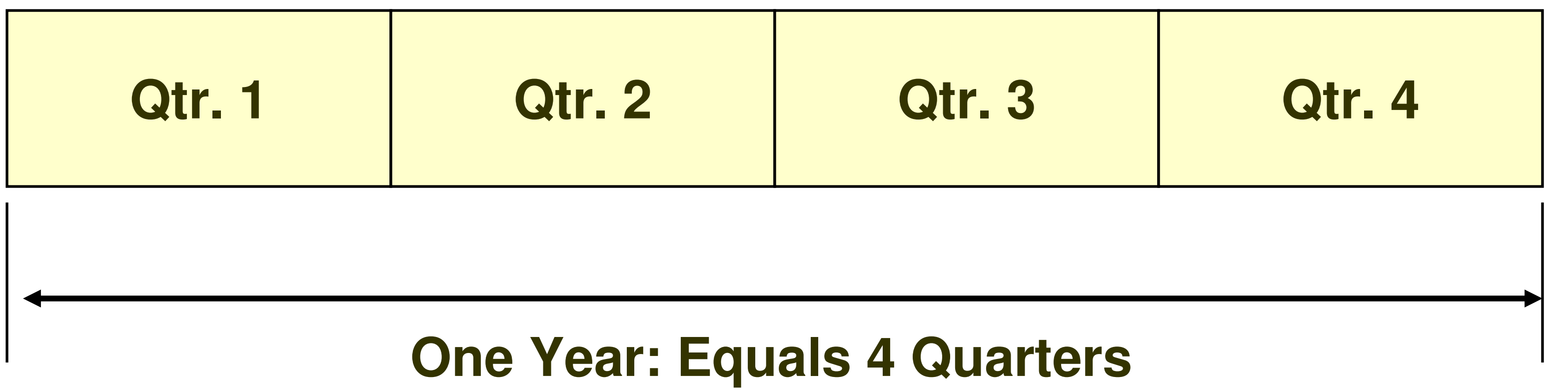
r = 9% per year, compounded monthly

Effective Monthly Rate:
0.09/12 = 0.0075 = 0.75%/month

Here, "m" counts months so, m = 12 compounding periods within a year.

4.1 Example 4.1

- Given, "9% per year, compounded quarterly"



CP equals a quarter (3 – months)

4.1 Example 4.1 (9%/yr: Compounded quarterly)

- Given, "9% per year, compounded quarterly"

Qtr. 1	Qtr. 2	Qtr. 3	Qtr. 4

What is the Effective Rate per Quarter?

- $i_{Qtr.} = 0.09/4 = 0.0225 = 2.25\%/\text{quarter}$
- **9% rate is a nominal rate;**
- **The 2.25% rate is a true effective monthly rate**

4.1 Example 4.1 (9%/yr: Compounded quarterly)

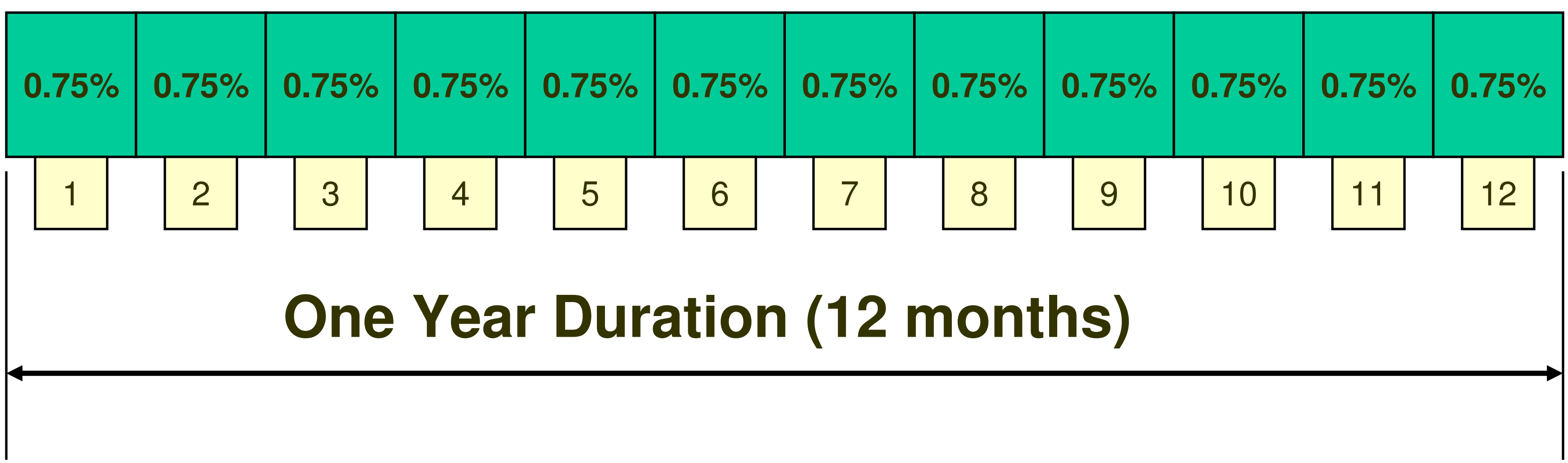
- Given, “9% per year, compounded quarterly”

Qtr. 1: 2.25%	**Qtr. 2:2.25%**	**Qtr. 3:2.25%**	**Qtr. 4:2.25%**

The effective rate (true rate) per quarter is 2.25% per quarter

4.1 Statement: 9% compounded monthly

- r = 9% (the nominal rate).
- "compounded monthly means "m" =12.
- The true (effective) monthly rate is:
  - **0.09/12 = 0.0075 = *0.75% per month***



4.1 Statement: 4.5% per 6 months – compounded weekly

- Nominal Rate: 4.5%.
- Time Period: 6 months.
- Compounded weekly:
  - Assume 52 weeks per year
  - 6-months then equal 52/2 = 26 weeks per 6 months
- The true, effective weekly rate is:
  - (0.045/26) = 0.00173 = 0.173% per week

4.1 Table 4.1

- It can be unclear as to whether a stated rate is a nominal rate or an effective rate.
- Three different statements of interest follow………

4.1 Varying Statements of Interest Rates

- "8% per year, compounded quarterly"
  - **Nominal rate is stated: 8%**
  - **Compounding Frequency is given**
    - **Compounded quarterly**
    - **True quarterly rate is 0.8/4 = 0.02 = <u>2% per quarter</u>**

Here, one must calculate the effective quarterly rate!

4.1 Effective Rate Stated

- "Effective rate = 8.243% per year, compounded quarterly:
  - No nominal rate given (must be calculated)
  - Compounding periods – m = 4
- No need to calculate the true effective rate!
  - **It is already given: 8.243% per year**!

4.1 Only the interest rate is stated

- "8% per year".
- Note:
  - No information on the frequency of compounding.
  - Must assume it is for one year!
  - "m" is assumed to equal "1".
- *Assume* that "8% per year" is a true, effective annual rate!
  - No other choice!

Section 4.2
Effective Annual Interest Rates

- Here, we show how to calculate true, effective, annual interest rates.
- We assume the year is the standard of measure for time.
- The year can be comprised of various numbers of compounding periods (within the year).

4.2 Typical Compounding Frequencies

- Given that one year is the standard:
  - **m = 1: compounded annually (end of the year);**
  - **m = 2: semi-annual compounding (every 6 months);**
  - **m = 4: quarterly compounding;**
  - **m = 12: monthly compounding;**
  - **m = 52: weekly compounding;**
  - **m = 365: daily compounding;**
- Could keep sub-dividing the year into smaller and smaller time periods.

4.2 Calculation of the EAIR

- EAIR – “the Effective Annual Interest Rate”.
- The EAIR is the true, annual rate given a frequency of compounding within the year.
- We need the following notation…….

4.2 EAIR Notation

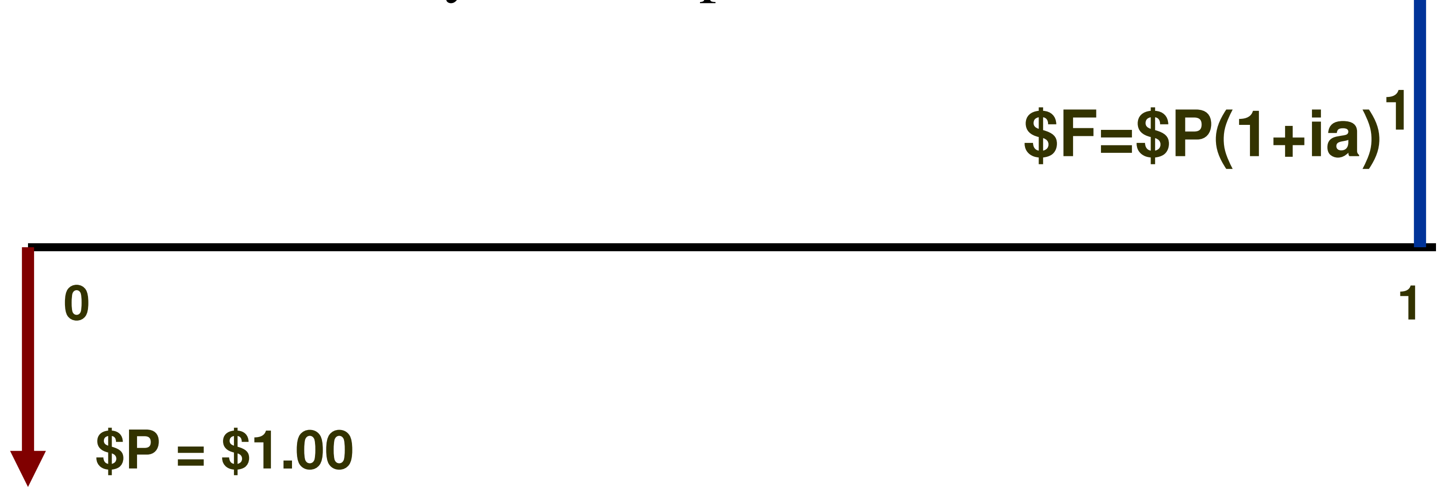
- r = the nominal interest rate per year.
- m = the number of compounding periods within the year.
- **i** = the effective interest rate per compounding period (r/m)
- i_a or i_e = the true, effective annual rate given the value of m.

4.2 Derivation of the EAIR relationship

- Assume $1 of principal at time t = 0.
- Conduct a period-by-period Future Worth calculation.
- Notation "problem".
  - **At times, "i" is used in place of "i_e" or "i_a".**
  - **So, "i" can also represent the true effective annual interest rate!**

4.2 Deriving the EAIR…

- Consider a one-year time period.

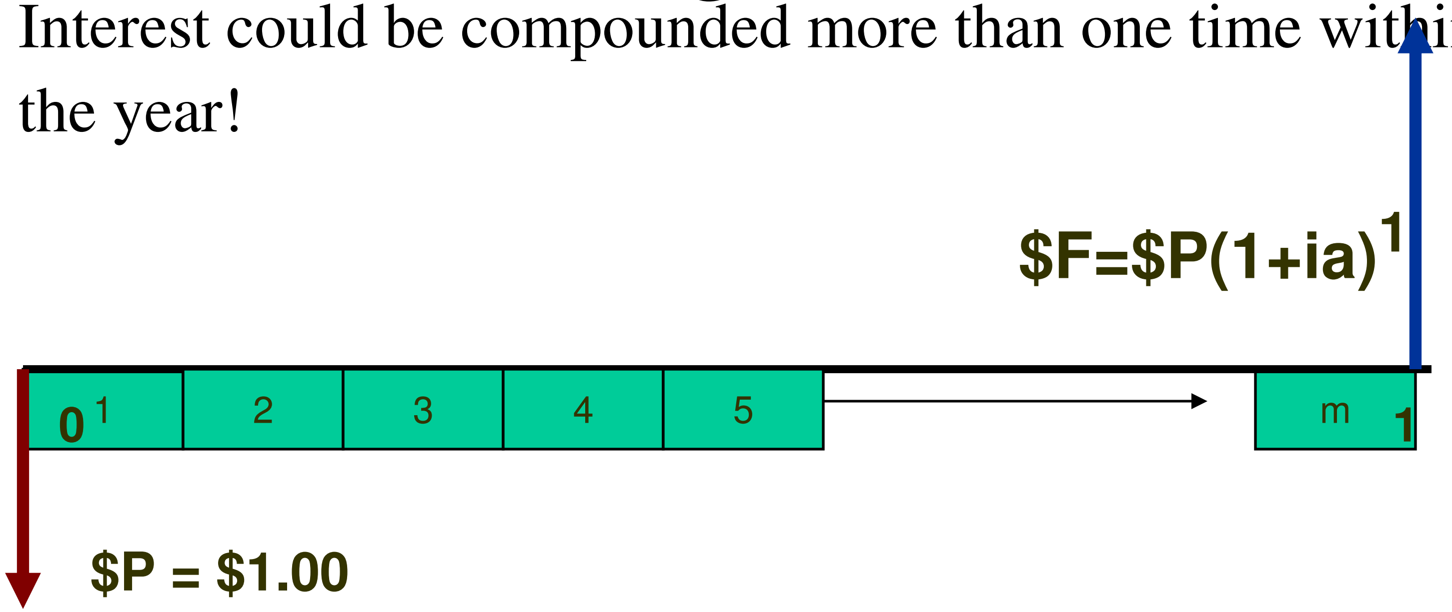


Invest $1 of principal at time t = 0 at interest rate i per year.

One year later, $F = P(1+ia)^1$

4.2 Deriving the EAIR…

- Interest could be compounded more than one time within the year!



Assume the one year is now divided into "m" compounding periods.

Replace "i" with "i_a" since m now > 1.

4.2 Rewriting….

- $F = P + P(i_a)$
- Now, the rate i per CP must be compounded through all "m" periods to obtain F_1
- Rewrite as:
  - $\mathbf{F = P + P(i_a) = P(1 + i_a)}$
  - $\mathbf{F = P(1 + i)^{\mathit{m}}}$

4.2 Two similar expressions for F

- We have two expressions for F;
- $\mathbf{F = P(1 + i_a);}$
- $F = P(1 + i)^m;$
- Equate the two expressions;
- $P(1 + i_a) = P(1 + i)^m;$
- $\cancel{P}(1 + i_a) = \cancel{P}(1 + i)^m;$

Solve i_a in terms of "i".

4.2 Expression for i_a

- Solving for i_a yields;

$$1 + i_a = (1+i)^m \qquad (1)$$

$$i_a = (1 + i)^m - 1 \qquad (2)$$

If we start with a nominal rate, "*r*" then….

4.2 The EAIR is……

- Given a nominal rate, “r”
- $i_{\text{Compounding period}} = r/m$;
- The EAIR is calculated as;

$$\textbf{EAIR} = (1 + r/m)^{m} - 1. \qquad (3)$$

Or, $i_{\text{Compounding period}} = (1 + i_a)^{1/m} - 1$

Then: Nominal rate – “r” = (*i*)(*m*) $\qquad$ **(4)**

4.2 Example: EAIR given a nominal rate.

- Given: interest is 8% per year compounded quarterly".
- What is the true annual interest rate?
- Calculate:

$EAIR = (1 + 0.08/4)^4 - 1$

$EAIR = (1.02)^4 - 1 = 0.0824 =$ 8.24%/year

4.2 Example: "18%/year, comp. monthly"

- **What is the true, effective annual interest rate?**

$r = 0.18/12 = 0.015 = 1.5\%$ per month.

1.5% per month is an effective monthly rate.

The effective annual rate is:

$(1 + 0.18/12)^{12} - 1 = 0.1956 =$ <u>19.56%</u>/year

4.2 Previous Example

- "18%, c.m. (compounded monthly)
- Note:
  - **Nominal Rate is 18%;**
  - **The true effective monthly rate is 1.5%/month;**
  - **The true effective annual rate is 19.56%/year**.
- One nominal rate creates 2 effective rates!
  - Periodic rate and an effective annual rate.

4.2 EAIR's for 18%

- m = 1
  - EAIR = $(1 + 0.18/1)^1 - 1 = \underline{0.18}$ (18%)
- m = 2 (semiannual compounding)
  - EAIR = $(1 + 0.18/2)^2 - 1 = \underline{18.810\%}$
- m = 4 (quarterly compounding)
  - EAIR = $(1 + 0.18/4)^4 - 1 = \underline{19.252\%}$
- m = 12 (monthly compounding)
  - EAIR = $(1 + 0.18/12)^{12} - 1 = \underline{19.562\%}$
- _m = 52 (weekly compounding)
  - EAIR = $(1 + 0.18/52)^{52} - 1 = \underline{19.684\%}$

4.2 Continuing for 18%.....

- m = 365 (daily compounding).
  - EAIR = $(1 + 0.18/365)^{365} - 1$ = <u>19.714%</u>
- m = 365(24) (hourly compounding).
  - EAIR = $(1 + 0.18/8760)^{8760} - 1$ = <u>19.72%</u>
- Could keep subdividing the year into smaller time periods.
- Note: There is an apparent limit as “m” gets larger and larger…called continuous compounding.

4.2 Example: 12% Nominal

	No. of Comp. Per.	EAIR (Decimal)	EAIR (per cent)
Annual	1	0.1200000	12.00000%
semi-annual	2	0.1236000	12.36000%
Quartertly	4	0.1255088	12.55088%
Bi-monthly	6	0.1261624	12.61624%
Monthly	12	0.1268250	12.68250%
Weekly	52	0.1273410	12.73410%
Daily	365	0.1274746	12.74746%
Hourly	8760	0.1274959	12.74959%
Minutes	525600	0.1274968	12.74968%
seconds	31536000	0.1274969	12.74969%

12% nominal for various compounding periods

Assignments and Announcements

- Assignments due at the beginning of next class:
  - Read Sections 4.2, 4.3, 4.4, 4.5,

Topics to Be Covered in Today's Lecture

- Section 4.3: Payment Period (PP)
- Section 4.4: Equivalence: Comparing PP to CP
- Section 4.5: Single Amounts: CP >= PP
- Section 4.6: Series Analysis – PP >= CP
- Section 4.7: Single Amounts/Series with PP < CP

Section 4.3: Payment Period (PP)

- Recall:
  - **CP is the “compounding period”**
- PP is now introduced:
  - **PP is the “payment period”**
- Why “CP” and “PP”?
  - **Often the frequency of depositing funds or making payments does not coincide with the frequency of compounding.**

4.3 Comparisons:

- Example 4.4

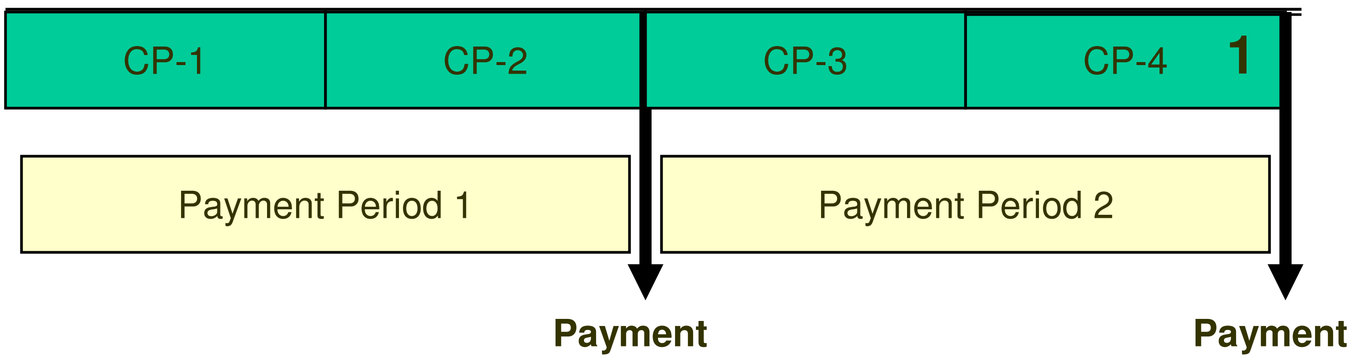
Three different interest charging plans. Payments are made on a loan every 6 months. Three interest plans are presented:

1. **9% per year, c.q. (compounded quarterly).**
2. **3% per quarter, (compounded quarterly).**
3. **8.8% per year, c.m. (compounded monthly)**

Which Plan has the lowest annual interest rate?

4.3 Comparing 3 Plans: Plan 1

- 9% per year, c.q.
- Payments made every 6 months.



9%, c.q. = 0.09/4 = 0.045 per 3 months = 2.25% per 3 months

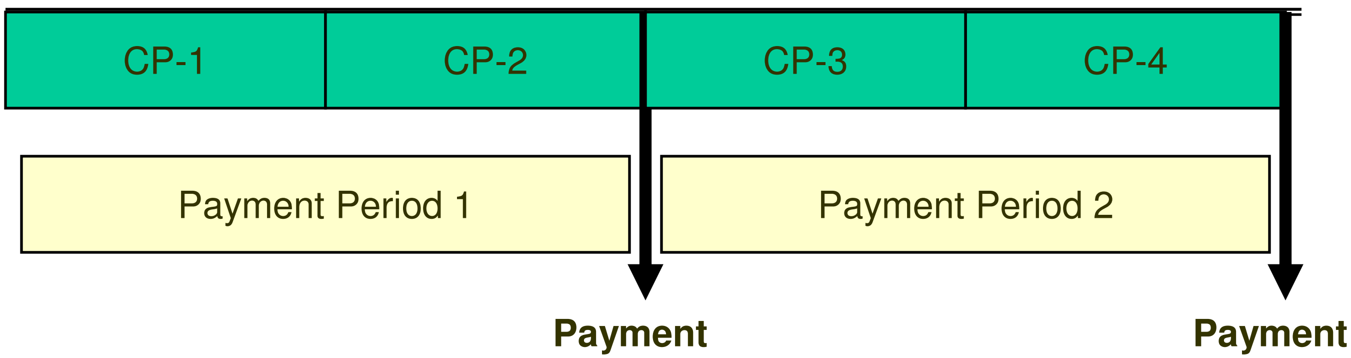
Rule: The interest rate must match the payment period!

4.3 The Matching Rule

- Again, the interest must be consistent with the payment period!
- We need a 6-month effective rate and then calculate the 1 year true, effective rate!
- To compare the 3 plans:
  - Compute the true, effective 6-month rate or,
  - Compute the true effective 1 year rate.
  - Then one can compare the 3 plans!

4.3 Comparing 3 Plans: Plan 1

- 9% per year, c.q. = 2.25%/quarter
- Payments made every 6 months.



True 6-month rate is:

$(1.0225)^2 - 1 = 0.0455 =$ <u>4.55% per 6-months</u>

EAIR $= (1.0225)^4 - 1 =$ <u>9.31% per year</u>

4.3 Plan 2

- 3% per quarter, c.q.
- Effective=3%/quarter
- Find the EIR for 6-months
- Calculate:
  - For a 6-month effective interest rate -
  - $(1.03)^2 - 1 = 0.0609 =$ **6.09% per 6-months**
  - Or, for a 1 year effective interest rate -
  - $(1.03)^4 - 1 =$ **12.55%/year**

4.3 Plan 3:" 8.8% per year, c.m."

- "r" = 8.8%
- "m" = 12
- Payments are twice a year
- 6-month nominal rate = 0.088/2 =4.4%/6-months ("r" = 0.044)
- $EIR_{6\text{-months}} = (1 + 0.044/6)^6 - 1 = (1.0073)^6 - 1 =$ 4.48%/ 6-months
- $EIR_{12\text{-months}} = (1 + 0.088/12)^{12} - 1 =$ 9.16%/year

4.3 Summarizing the 3 plans….

Plan No.	6-month	1-year
1	4.55%	9.31%
2	6.09%	12.55%
3	4.48%	9.16%

Plan 3 presents the lowest interest rate.

4.3 Can be confusing ???

- The 3 plans state interest differently.
- Difficult to determine the best plan by mere inspection.
- Each plan must be evaluated by:
  - **Calculating the true, effective 6-month rate**

  Or,
  - **Calculating the true, effective 12 month, (1 year) true, effective annual rate.**
  - **Then all 3 plans can be compared using the EIR or the EAIR.**

Section 4.4: Equivalence: Comparing PP to CP

- Reality:
  - PP and CP’s do not always match up;
  - May have monthly cash flows but…
  - Compounding period different than monthly.
- Savings Accounts – for example;
  - Monthly deposits with,
  - Quarterly interest earned or paid;
  - They don’t match!
- Make them match! (*by adjusting the interest period to match the payment period*.)

Situations

Situation	Text Reference
• PP = CP	Sections 4.5 and 4.6
• PP > CP	Sections 4.5 and 4.6
• PP < CP	Section 4.7

Section 4.5
Single Amounts: PP >= CP

Example1:

- "r" = 15%, c.m. (compounded monthly)
- Let P = $1500.00
- Find F at t = 2 years.
- 15% c.m. = 0.15/12 = 0.0125 = 1.25%/month.
- n = 2 years OR 24 months
- Work in months or in years

4.5 Single Amounts: PP >= CP

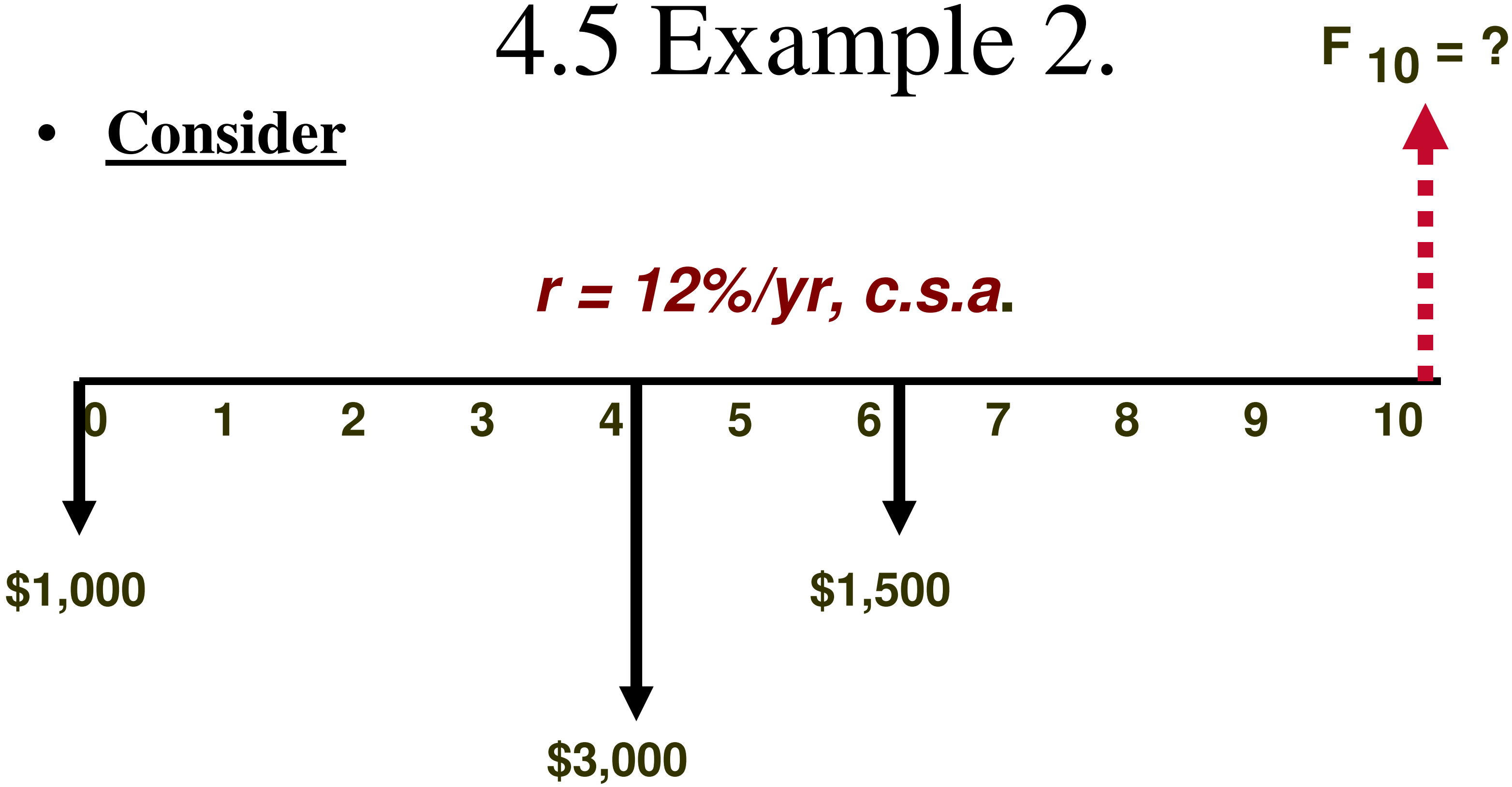
- Approach 1. (n relates to months)
- State:
  - **F_{24} = \$1,500(F/P,0.15/12,24);**
  - **i/month = 0.15/12 = 0.0125 (1.25%);**
  - **F_{24} = \$1,500(F/P,1.25%,24);**
  - **$F_{24} = \$1{,}500(1.0125)^{24} = \$1{,}500(1.3474)$;**
  - **F_{24} = \$2,021.03.**

4.5 Single Amounts: CP >= CP

- Approach 2. (n relates to years)
- State:
  - **F_2 = $1,500(F/P,i%,2);**
  - **Assume n = 2 (years) we need to apply an annual effective interest rate.**
  - **i/month =0.0125**
  - **EAIR = $(1.0125)^{12} - 1$ = 0.1608 (16.08%)**
  - **F_2 = $1,500(F/P,16.08%,2)**
  - **$F_2 = \$1,500(1.1608)^2$ = $2,021.19**
  - **Slight roundoff compared to approach 1**

4.5 Example 2.

- **Consider**

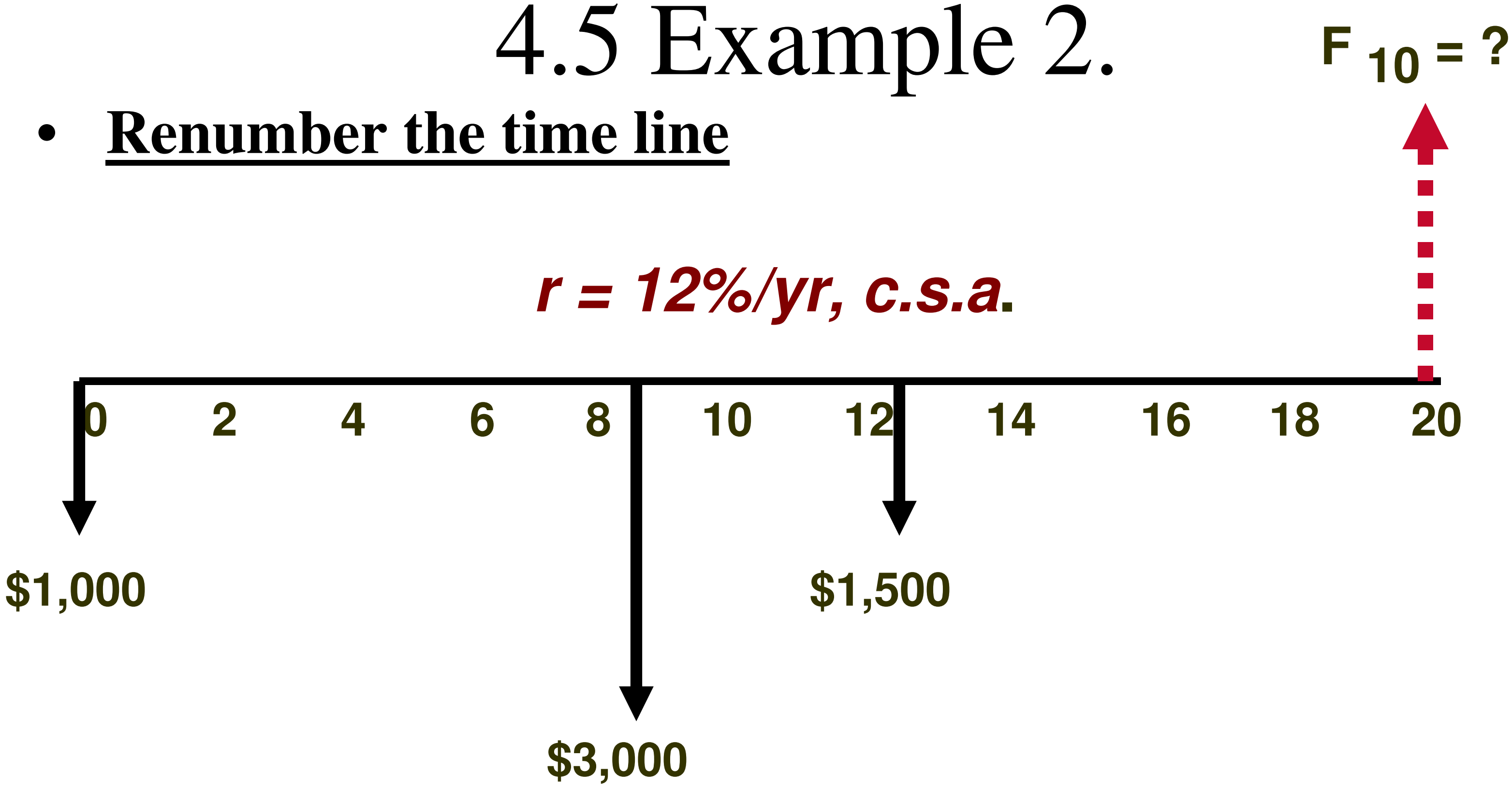


Suggest you work this in 6- month time frames

Count "n" in terms of "6-month" intervals

4.5 Example 2.

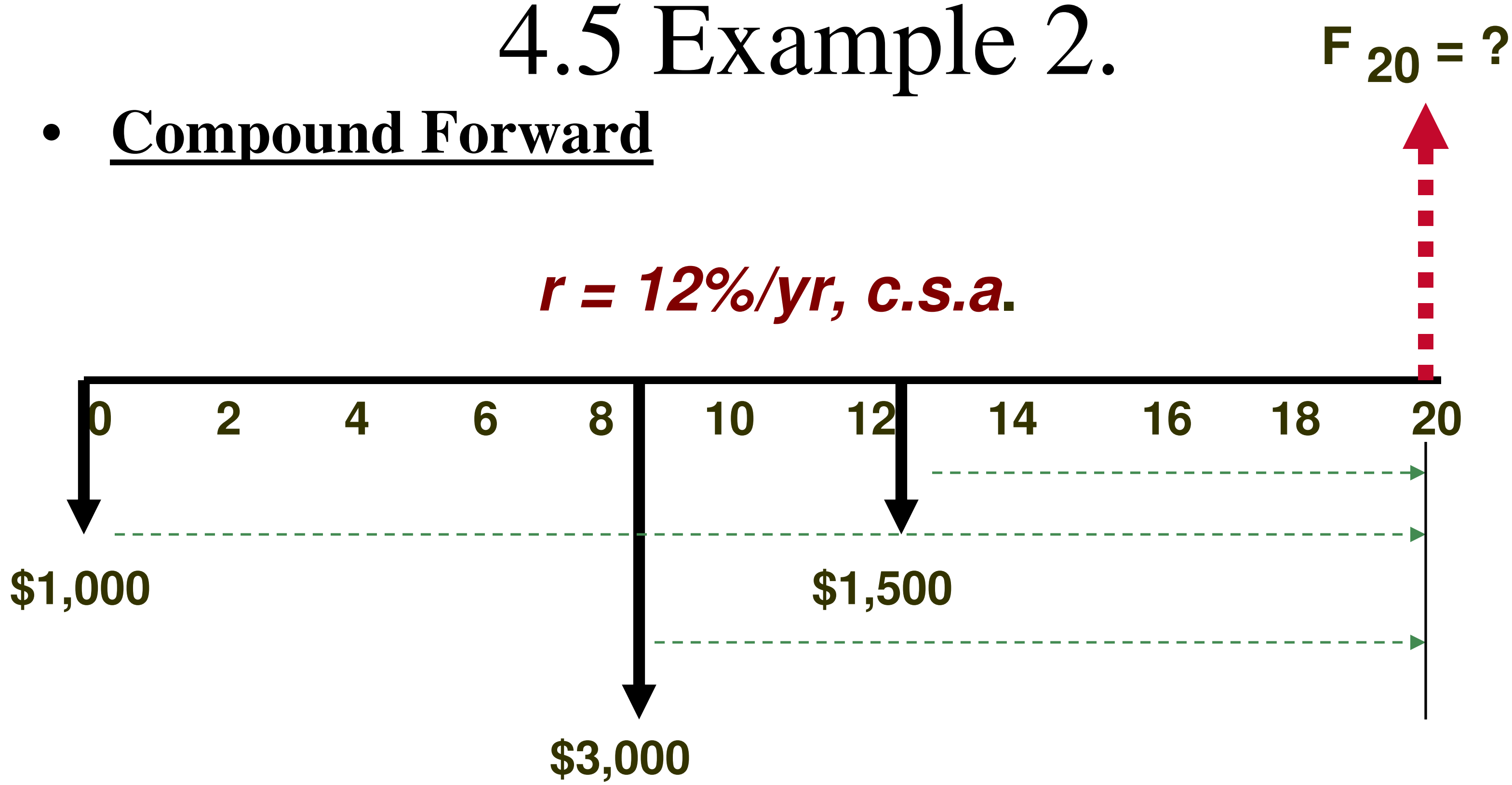
- **Renumber the time line**



i/6 months = 0.12/2 = 6%/6 months; n counts 6-month time periods

4.5 Example 2.

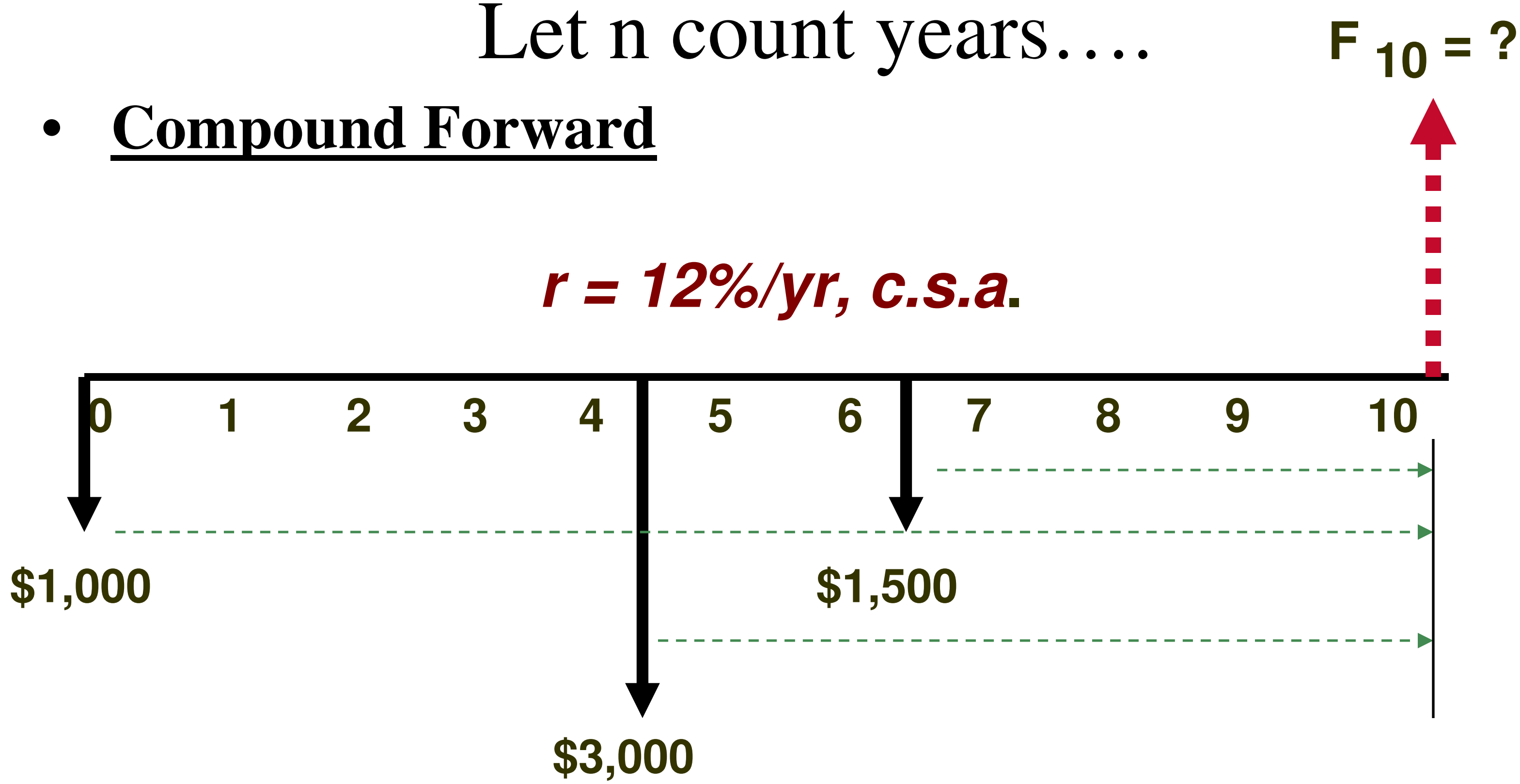
- **Compound Forward**



$$F_{20} = \$1,000(F/P,6\%,20) + \$3,000(F/P,6\%,12) + \$1,500(F/P,6\%,8) = \underline{\$11,634}$$

4.5 Example 2.
Let n count years….

- **Compound Forward**



IF n counts years, interest must be an annual rate.

EAIR = $(1.06)^2 - 1$ = 12.36%

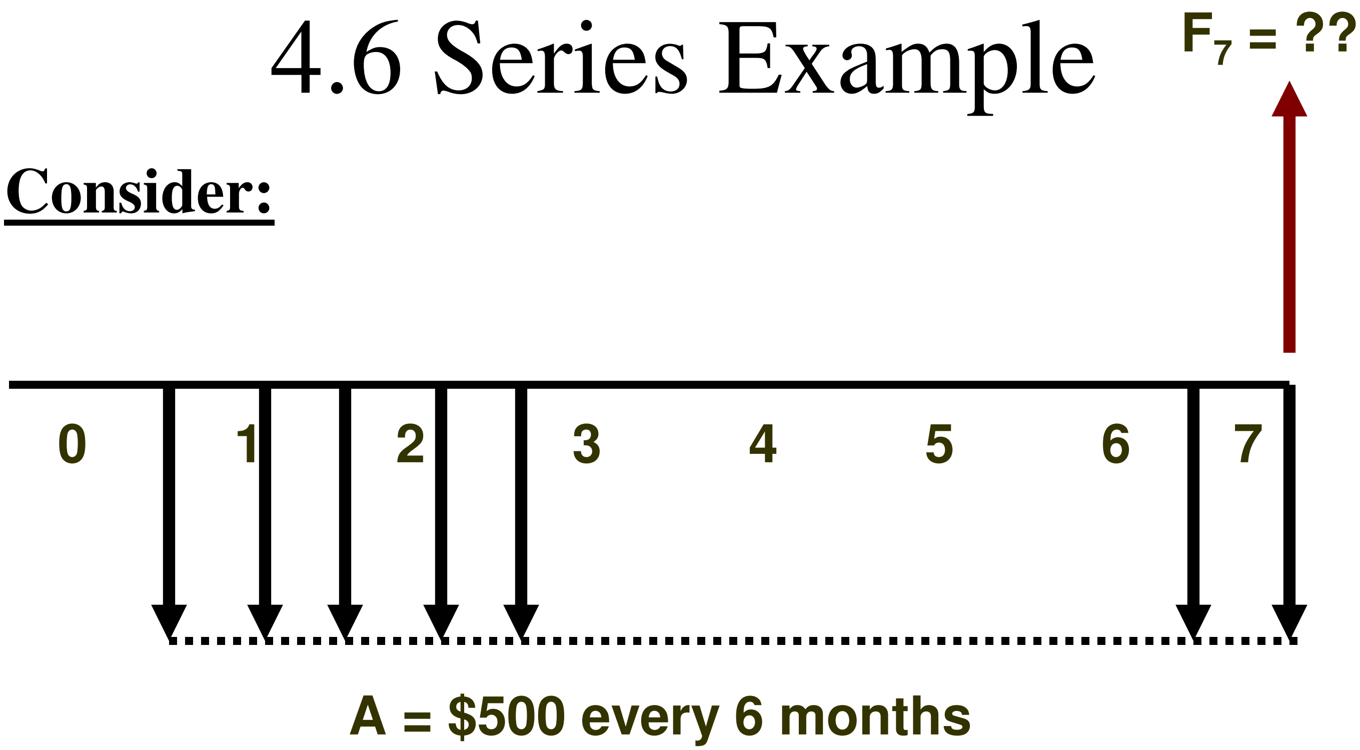
Compute the FV where n is years and i = 12.36%!

Section 4.6
Series Analysis – PP >= CP

- **Find the effective “i” per payment period.**
- **Determine “n” as the total number of payment periods.**
- **“n” will equal the number of cash flow periods in the particular series.**
- **Example follows…..**

4.6 Series Example

- **Consider:**



Find F_7 if "r" = 20%/yr, c.q. (PP > CP)

We need i per 6-months – effective.

$i_{6\text{-months}}$ = adjusting the nominal rate to fit.

4.6 Series Example

- **Adjusting the interest**
- **r = 20%, c.q.**
- **i/qtr. = 0.20/4 = 0.05 = 5%/qtr.**
- **2-qtrs in a 6-month period.**
- $i_{6\text{-months}} = (1.05)^2 - 1 =$ **<u>10.25%/6-months.</u>**
- **Now, the interest matches the payments.**
- $F_{\text{year } 7} = F_{\text{period } 14} = \$500(F/A, 10.25\%, 14)$
- **F = \$500(28.4891) = \$14,244.50**

4.6 This Example: Observations

- **Interest rate must match the frequency of the payments.**
- **In this example – we need effective interest per 6-months: Payments are every 6-months.**
- **The effective 6-month rate computed to equal 10.25% - un-tabulated rate.**
- **Calculate the F/A factor or interpolate.**
- **Or, use a spreadsheet that can quickly determine the correct factor!**

4.6 This Example: Observations

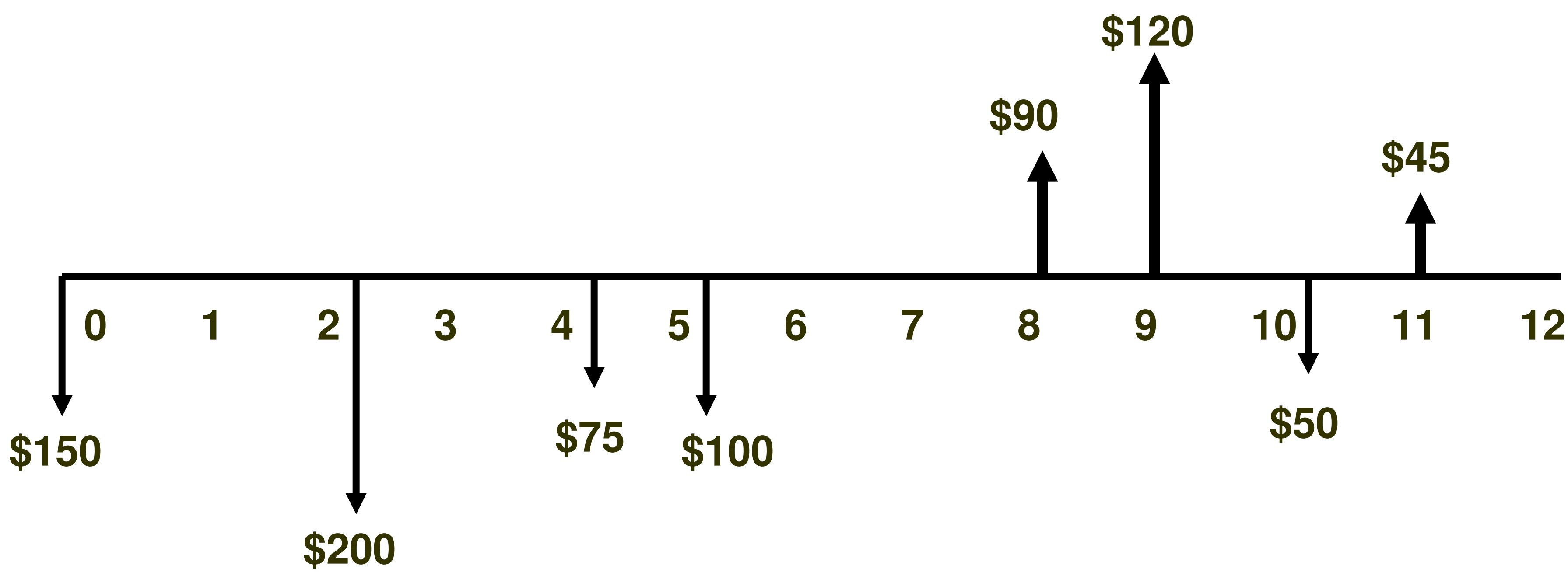
- **Do not attempt to adjust the payments to fit the interest rate!**
- **This is Wrong!**
- **At best a gross approximation – do not do it!**
- **This type of problem almost always results in an un-tabulated interest rate**
  - **You have to use your calculator to compute the factor or a spreadsheet model to achieve exact result.**

Section 4.7
Single Amounts/Series with PP < CP

- **This situation is different than the last.**
- **Here, PP is less than the compounding period (CP).**
- **Raises questions?**
- **Issue of <u>interperiod compounding</u>**
- **An example follows.**

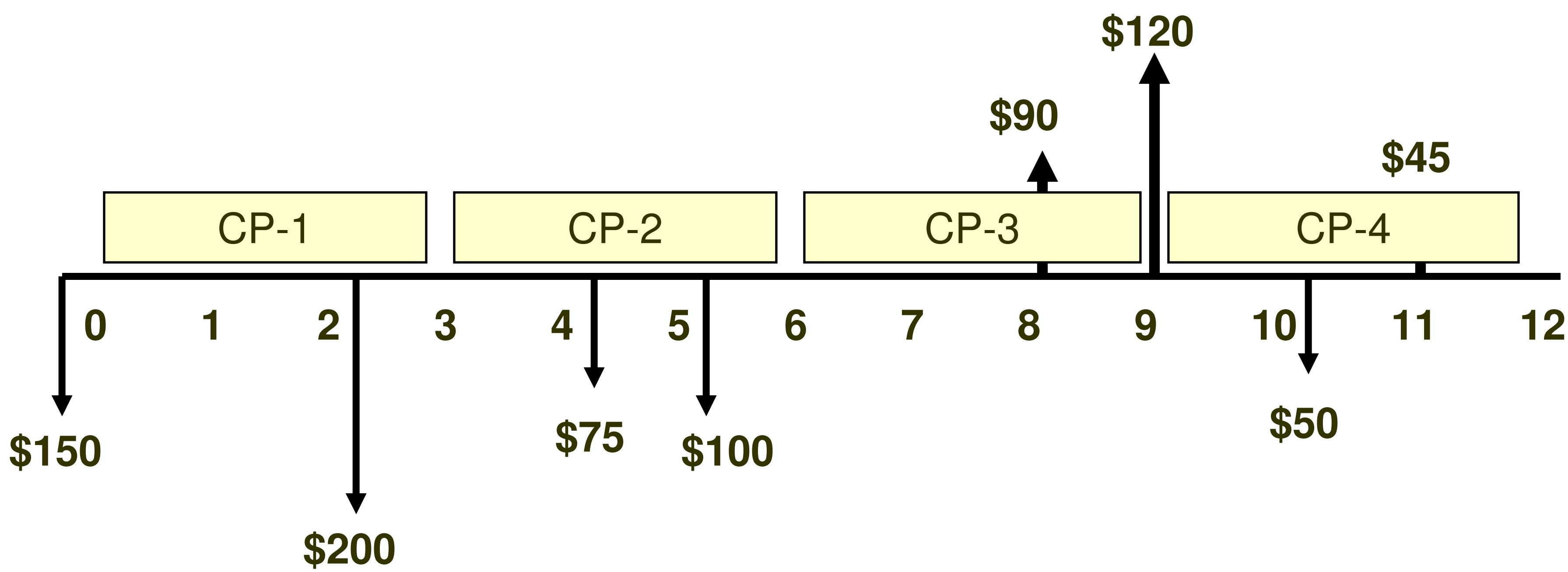
4.7 Interperiod Compounding Issues

- **Consider a one-year cash flow situation.**
- **Payments are made at end of a given month.**
- **Interest rate is "r = 12%/yr, c.q."**



4.7 Interperiod Compounding

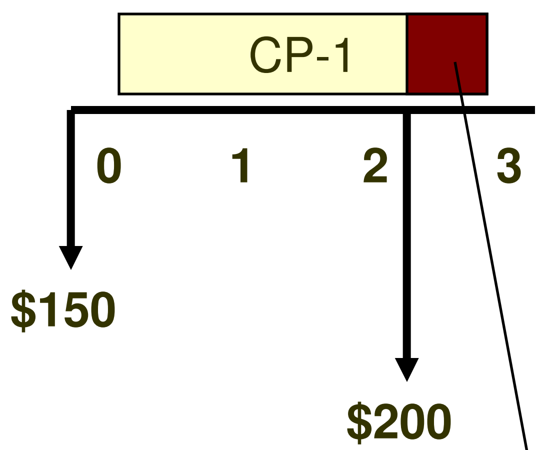
- **r =12%/yr. c.q.**



Note where some of the cash flow amounts fall with respect to the compounding periods!

4.7 Take the first $200 cash flow

- **Will any interest be earned/owed on the $200 since interest is compounded at the end of each quarter?**



The $200 is at the end of month 2 and will it earn interest for one month to go to the end of the first compounding period?

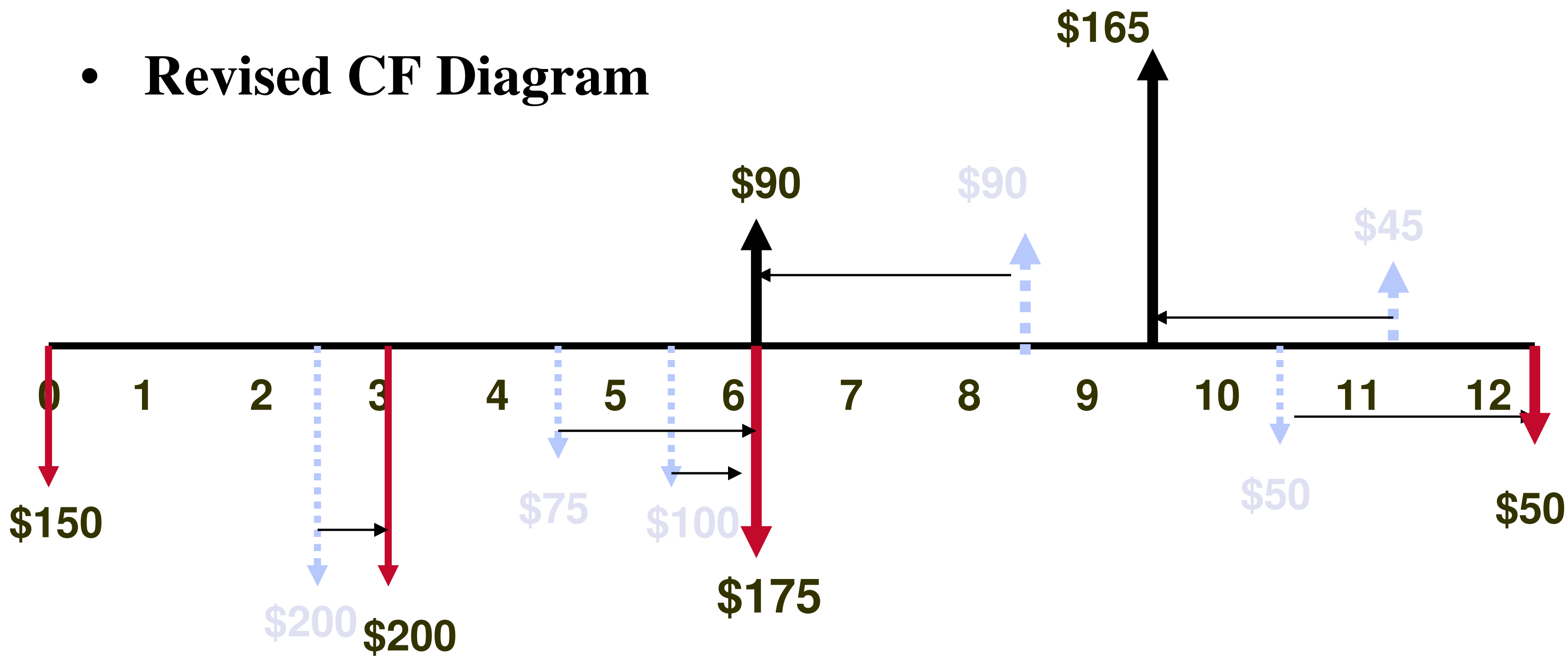
The last month of the first compounding period.
Is this an interest-earning period?

4.7 Interperiod Issues

- **The $200 occurs 1 month before the end of compounding period 1.**
- **Will interest be earned or charged on that $200 for the one month?**
- **If not then the revised cash flow diagram for all of the cash flows should look like…..**

4.7 No interperiod compounding

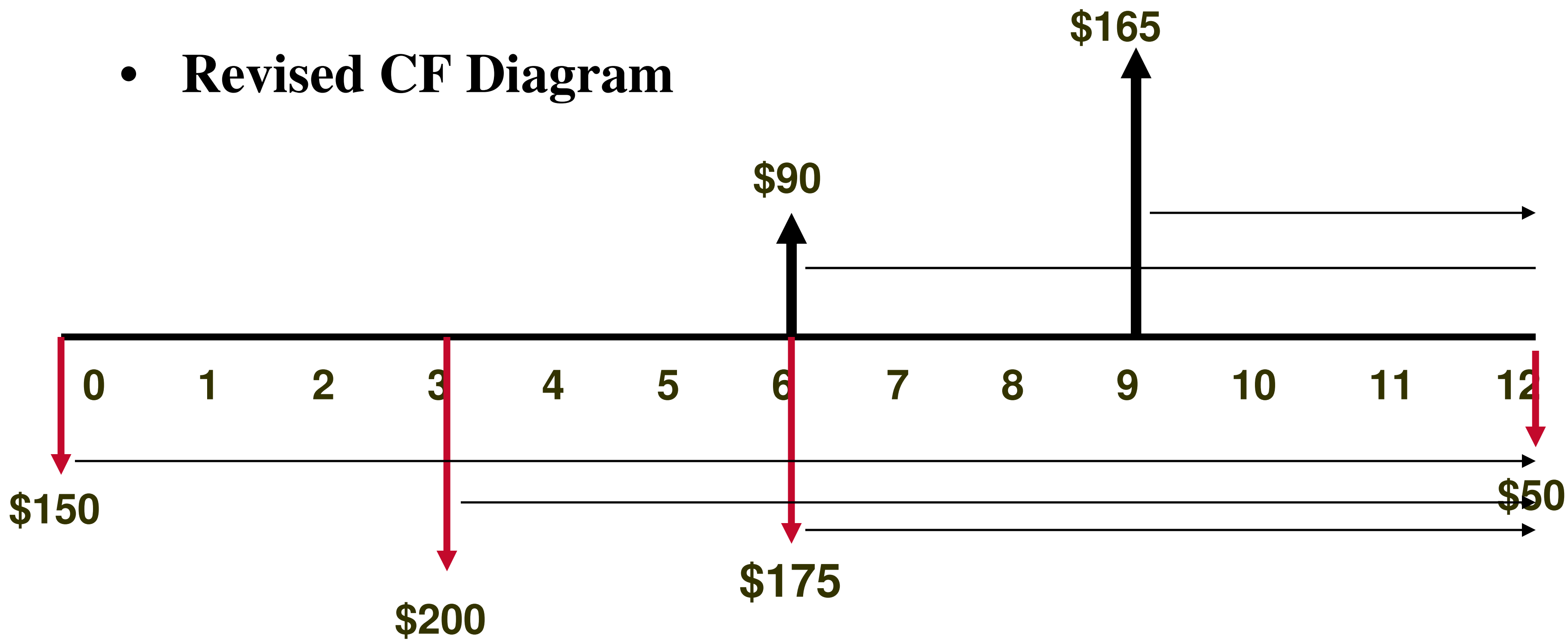
- **Revised CF Diagram**



All negative CF's move to the end of their respective quarters and all positive CF's move to the beginning of their respective quarters.

4.7 No interperiod compounding

- **Revised CF Diagram**



Now, determine the future worth of this revised series using the F/P factor on each cash flow.

4.7 Final Results: No interperiod Comp.

- **With the revised CF compute the future worth.**

“r” = 12%/year, compounded quarterly

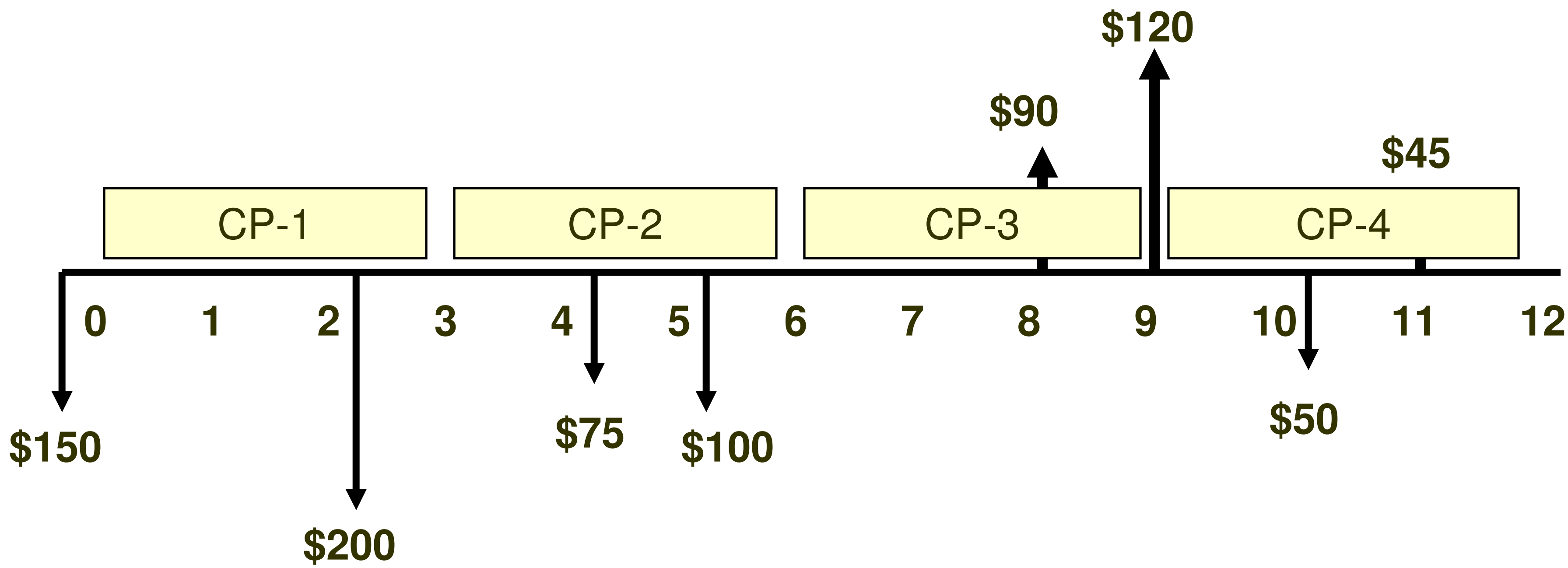
“i” = 0.12/4 = 0.03 = <u>3% per quarter</u>

F_{12} = [-150(F/P3%,4) – 200(F/P,3%,3) + (-175 +90)(F/P,3%,2) + 165(F/P,3%,1) – 50]

= <u>$-357.59</u>

4.7 Interperiod Compounding

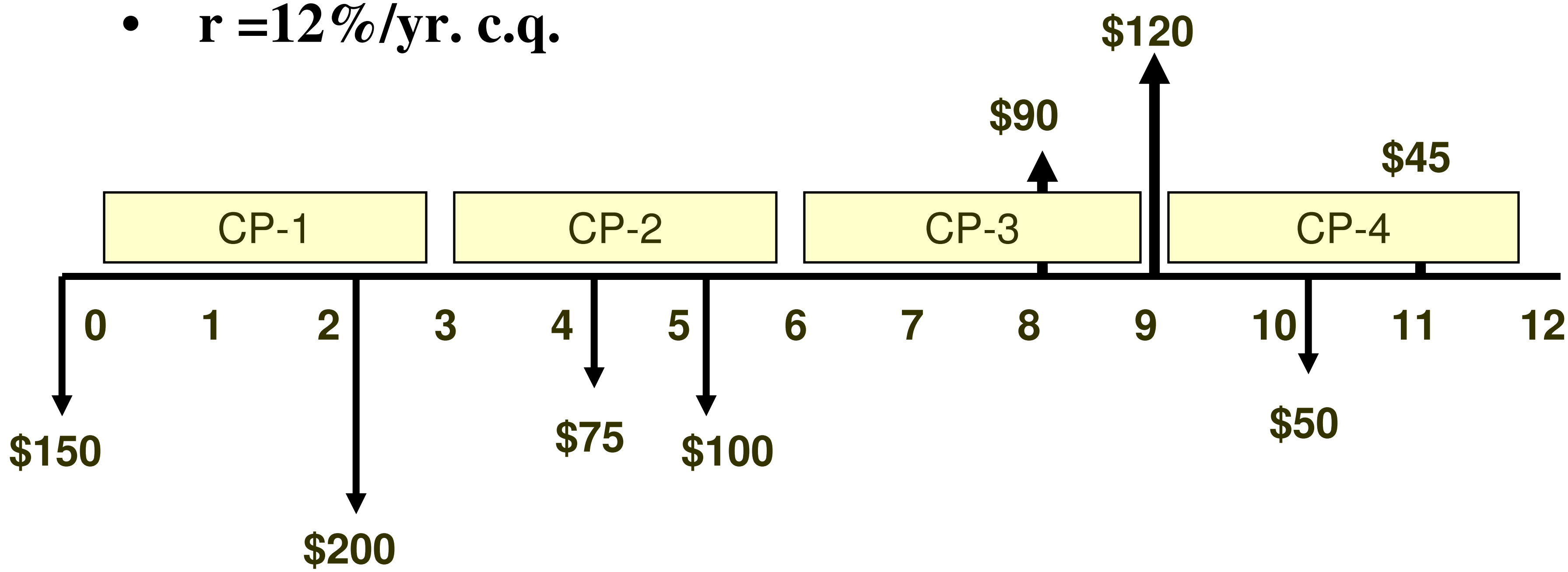
- **r =12%/yr. c.q.**



The cash flows are not moved and equivalent P, F, or A values are determined using the effective interest rate per payment period

4.7 Interperiod Compounding

- **r =12%/yr. c.q.**



If the inter-period compounding is earned, then we should compute the effective interest rate per compounding period.

i=3% is the effective quarterly rate

$I_{monthly}=(1+i)^{1/3}-1=(1.03)^{1/3}-1=0.99\%$

Topics to Be Covered in Today's Lecture

- Section 4.8: Continuous Compounding
- Section 4.9: Interest Rates that vary over time

Section 4.8 Continuous Compounding

- **Recall:**
  - **EAIR = i = $(1 + r/m)^m - 1$**
  - **What happens if we let m approach infinity?**
  - **That means an infinite number of compounding periods within a year or,**
  - **The time between compounding approaches "0".**
  - **We will see that a limiting value will be approached for a given value of "r"**

4.8 Derivation of Continuous Compounding

- **We can state, in general terms for the EAIR:**

$$i = (1 + \frac{r}{m})^m - 1$$

Now, examine the impact of letting "m" approach infinity.

4.8 Derivation of Continuous Compounding

- **We re-define the EAIR general form as:**

$$(1+\frac{r}{m})^m - 1 = \left[\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^r - 1$$

Note – the term in brackets has the exponent changed but all is still the same….

4.8 Derivation of Continuous Compounding

- **There is a reason for the re-definition.**
- **From the calculus of limits there is an important limit that is quite useful.**
- **Specifically:**

$$\lim_{h \to \infty}\left(1+\frac{1}{h}\right)^{h} = e = 2.71828$$

4.8 Derivation of Continuous Compounding

- **Substituting we can see:**

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{\frac{m}{r}} = e,$$

4.8 Derivation of Continuous Compounding

- **So that:**

$$i = \lim_{m\to\infty}\left[\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^{r} - 1 = e^{r} - 1.$$

Summarizing…….

4.8 Derivation of Continuous Compounding

- **The EAIR when interest is compounded continuously is then:**

$$\text{EAIR} = e^{r} - 1$$

Where "r" is the nominal rate of interest compounded continuously.

This is the max. interest rate for any value of "r" compounded continuously.

4.8 Derivation of Continuous Compounding

- **Example:**
- **What is the true, effective annual interest rate if the nominal rate is given as:**
  - **r = 18%, compounded continuously**
  - **Or, r = 18% c.c.**

Solve $e^{0.18} - 1 = 1.1972 - 1 =$ 19.72%/year

The 19.72% represents the MAXIMUM EAIR for 18% compounded anyway you choose!

4.8 Finding “r” from the EAIR/cont. compounding

- **To find the equivalent nominal rate given the EAIR when interest is compounded continuously, apply:**

$$r = \ln(1 + i)$$

4.8 Example

- **Given r = 18% per year, cc, find:**
  - **A. the effective monthly rate**
  - **B. the effective annual rate**

a. r/month = 0.18/12 = 1.5%/month

Effective monthly rate is $e^{0.015} - 1$ = 1.511%

b. The effective annual interest rate is $e^{0.18} - 1$ = 19.72% per year.

4.8 Example

- **An investor requires an effective return of at least 15% per year.**
- **What is the minimum annual nominal rate that is acceptable if interest on his investment is compounded continuously?**

To start: $e^r - 1 = 0.15$

Solve for "r" ....

4.8 Example

- $e^r - 1 = 0.15$
- $e^r = 1.15$
- $\ln(e^r) = \ln(1.15)$
- $r = \ln(1.15) = 0.1398 =$ 13.98%

A rate of 13.98% per year, cc. generates the same as 15% true effective annual rate.

4.8 Final Thoughts

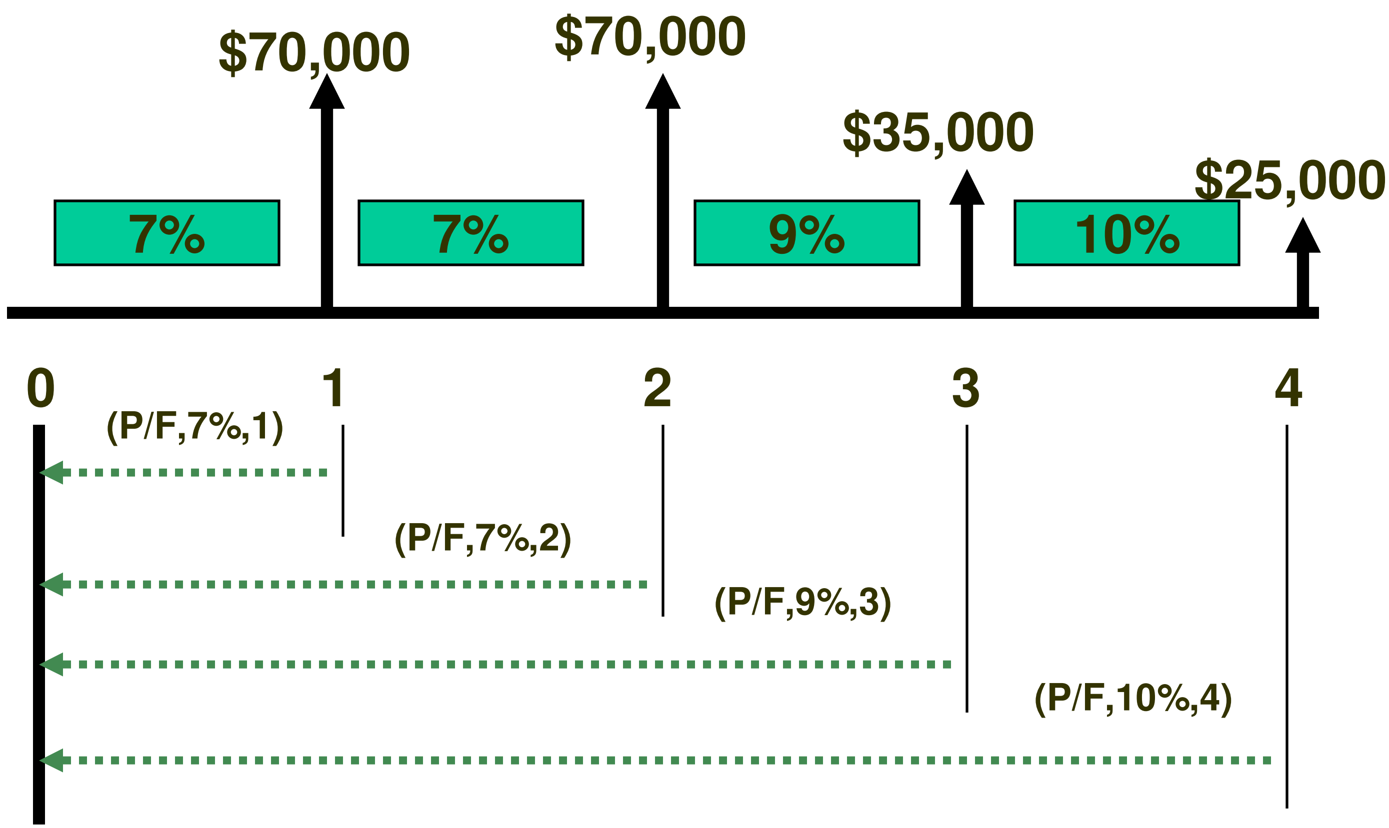
- **When comparing different statements of interest rate one must always compute to true, effective annual rate (EAIR) for each quotation.**
- **Only EAIR’s can be compared!**
- **Various nominal rates cannot be compared unless each nominal rate is converted to its respective EAIR!**

Section 4.9
Interest Rates that vary over time

- **In practice – interest rates do not stay the same over time unless by contractual obligation.**
- **There can exist "variation" of interest rates over time – quite normal!**
- **If required, how do you handle that situation?**

4.9 Interest Rates that vary over time

- **Best illustrated by an example.**
- **Assume the following future profits:**



4.9 Varying Rates: Present Worth

- **To find the Present Worth:**
  - **Bring each cash flow amount back to the appropriate point in time at the interest rate according to:**
- $\mathbf{P = F_1(P/F.i_1,1) + F_2(P/F,i_1)(P/F,i_2) + \ldots}$
- $\mathbf{+ F_n(P/F,i_1)(P/F,i_2)(P/F,i_3)\ldots(P/F,i_n,1)}$

This Process can get computationally involved!

4.9 Period-by-Period Analysis

- **P_0 =:**

1. \$7000(P/F,7%,1)
2. \$7000(P/F,7%,1)(P/F,7%,1)
3. \$35000(P/F,9%,1)(P/F,7%,1)^2
4. \$25000(P/F,10%,1)(P/F,9%,1)(P/F,7%,1)^2

Equals: <u>\$172,816</u> at t = 0…

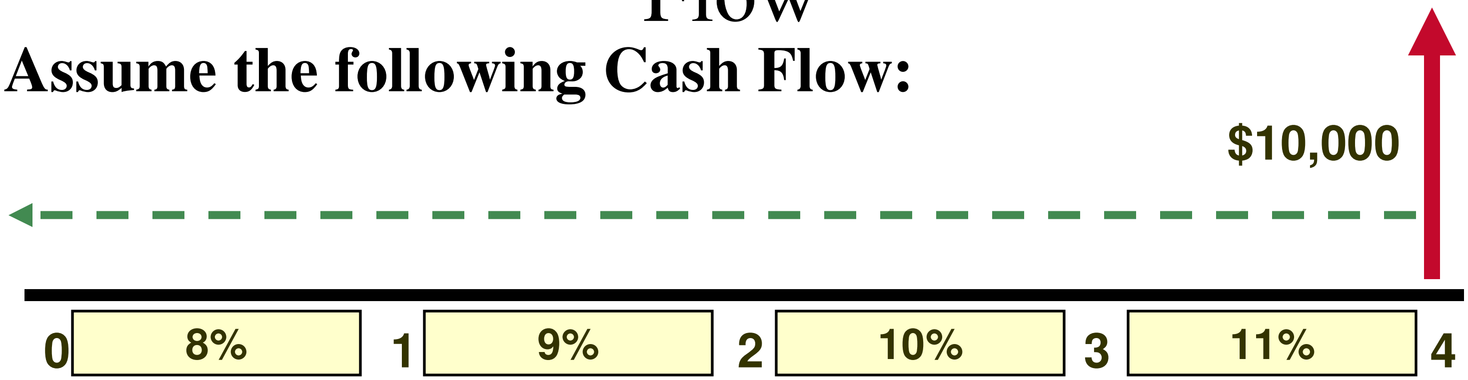
Work backwards one period at a time until you get to "0".

4.9 Varying Rates: Approximation

- **An alternative approach that approximates the present value:**
- **Average the interest rates over the appropriate number of time periods.**
- **Example:**
  - **{7% + 7% + 9% + 10%}/4 = 8.25%;**
  - **Work the problem with an 8.25% rate;**
  - **Merely an approximation.**

4.9 Varying Rates: Single, Future Cash Flow

- **Assume the following Cash Flow:**



Objective: Find P_0 at the varying rates

P_0 = \$10,000(P/F,8%,1)(P/F,9%,1)(P/F,10%,1)(P/F,11%,1)

= \$10,000(0.9259)(0.9174)(0.9091)(0.9009)

= \$10,000(0.6957) = <u>\$6,957</u>

4.9 Varying Rates: Observations

- **We seldom evaluate problem models with varying interest rates except in special cases.**
- **If required, best to build a spreadsheet model**
- **A cumbersome task to perform.**

Chapter 4 Summary

- **Many applications use and apply nominal and effective compounding**
- **Given a nominal rate – must get the interest rate to match the frequency of the payments.**
- **Apply the effective interest rate per payment period.**
  – PP >= CP (***adjusting the interest period to match the payment period)***
  – PP < CP (Consider inter-period compounding or not?)
- **When comparing varying interest rates, must calculate the EAIR in order to compare.**

Chapter Summary – cont.

- **All time value of money interest factors require use of an effective (true) periodic interest rate.**
- **The interest rate, *i*, and the payment or cash flow periods must have the same time unit.**
- **One may encounter varying interest rates and an exact answer requires a sequence of interest rates – cumbersome!**

Assignments and Announcements

- Homework 3 due a week from today
- Assignments due at the beginning of next class:
  - Answer the three online quizzes on chapter 4 on the text’s website.
  - Read chapter 5.1, 5.2 and 5.3