

Chapter 4 – Nominal and Effective Interest Rates

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- Nominal and Effective Interest Rates
- Comparing Payment Period and Compounding Period Lengths (PP vs. CP)
- Interest Periods Equal to Payment Periods
- Interest Periods Smaller than Payment Periods
- Interest Periods Larger than Payment Periods
- Interest Rates That Vary over Time

Nominal and Effective Interest Rates

- Generally interest can be quoted in more than one way:
 1. Quotation using a Nominal Interest Rate (or **APR** - the Annual Percentage Rate)
 2. Quoting an Effective Interest Rate (or **APY** - the Annual Percentage Yield)

Nominal and Effective Interest Rates

- Interest may be computed (compounded):
 - Annually – One time a year (at the end)
 - Every 6 months – 2 times a year (semi-annual)
 - Every quarter – 4 times a year (quarterly)
 - Every Month – 12 times a year (monthly)
 - Every Day – 365 times a year (daily)
 - ...
 - Continuous – infinite number of compounding periods in a year.

Terminology

Payment Period, T_p - Length of time during which cash flows are not recognized except as end of period cash flows.

Compounding Period (CP or interest period), T_c - Length of time between compounding operations.

Interest Rate Period, T - Interest rates are stated as % per time period. T is the time period.

Compounding frequency, m - the number of times that compounding occurs within the time period T .

Nominal Interest Rate

Nominal Interest Rate (r) - rate at which money grows, $\%/T$, *without considering compounding*. It can be stated for any time period: 1 year, 6 months, quarter, month, week, day, etc.

To find equivalent nominal rate r for any other time period:

$$r = \text{interest rate per period} * \text{number of periods}$$

Nominal Interest Rate

For example, if the **nominal** interest rate, r , were:
 $r = 1.5\%$ per month,

the **nominal** rate would also be:
 $r = 0.05\%$ per day, if 30 days/mo.
 $r = 4.5\%$ per quarter
 $r = 9\%$ semiannually
 $r = 18\%$ per year

Effective Interest Rate

Effective Interest Rate (i) - rate at which money grows, $\%/T$, *considering compounding*.

- It is usually expressed on an annual basis (i_a), but any time basis can be used.
- **Effective Interest Rate, i , equals the nominal interest rate, r , if the interest rate period T and the compounding period CP are equal.**

- All the interest formulas, factors, tabulated values, and spreadsheet relations must have **the effective interest rate** to properly account for the time value of money (effect of compounding).
- Effective rate has the *compounding frequency* attached to the nominal rate statement.

- **Nominal Rates:**
 - Format: " $r\%$ per time period T "
 - *Ex: 5% per 6-months ($r = 5\%$, $T = 6$ months)*
- **Effective Interest Rates:**
 - Format: " $r\%$ per time period T , compounded ' m ' times in time period T ."
 - ' m ' denotes the number of times per T that interest is compounded.
 - *Ex: 18% per year, compounded monthly ($r = 18\%$, $T = 1$ year, $CP = 1$ month, $m = 12$)*

$$\text{Effective rate per CP} = \frac{r\% \text{ per } T}{m \text{ per } T}$$

Example C4.1:

What is the CP , m , and effective rate per CP if

- a) $r = 9\%$ per year, compounded quarterly
- b) $r = 15\%$ per month, compounded daily

Solution

	<u>CP</u>	<u>m</u>	<u>Effective rate per CP</u>
a) quarter		4	$9\%/4 = 2.25\%$
b) day		30	$15\%/30 = 0.5\%$

Example C4.2:

Given $r = 9\%$ per year compounded monthly, find the Effective Monthly Rate (effective rate per CP).

Solution:

$m = 12$ compounding periods within a year (T).

Effective Monthly Rate (effective rate per CP):

$$0.09/12 = 0.0075 = 0.75\%/\text{month}$$

Example C4.3:

Statement: 4.5% per 6 months – compounded weekly.
Find the effective weekly rate (effective rate per CP).

Solution:

Nominal Rate $r = 4.5\%$.

$T = 6$ months.

$m = 26$ weeks per 6 months T

effective weekly rate is:

$$0.045/26 = 0.00173 = 0.173\% \text{ per week}$$

Varying Statements of Interest Rates

- Effective Rate is directly stated

“Effective rate is 8.243% per year,
compounded quarterly;

No nominal rate given;

Compounding frequency $m = 4$;

No need to calculate the true effective rate!

It is already given: 8.243% per year!

Varying Statements of Interest Rates

- Only the interest rate is stated

“ 8% per year”.

*No information on the frequency of compounding;
Must assume it is for one year!*

Assume that “ 8% per year” is a true, effective
annual rate!

Effective Interest Rate, i , equals the nominal interest rate, r , if the interest rate period T and the compounding period CP are equal.

$$\text{Effective rate per payment period} = (1 + r/m)^m - 1$$

r = nominal interest rate per payment period

m = number of compounding periods per payment period

Example C4.4:

What is the future worth of \$100 after 1 year if a bank pays 12% interest per year compounded semiannually?

Solution: First find effective interest rate per year
 $r = 12\%$ per year (nominal); $T = 1$ year; $CP = 6$ months;
 $m = 2$; payment period $PP = 1$ year

Effective rate per payment period $= (1 + r/m)^m - 1$
 i per year $= (1 + 0.12/2)^2 - 1 = 0.1236 = 12.36\%$

$$F = \$100 (1 + 0.1236) = \$112.36$$

Example C4.5:

A bank pays a 12% per year compounded semiannually. What is the effective interest rate for

a) 6 mo.? b) 12 mo? c) 24 mo.?

Solution:

a) Compounding period CP is 6 mo.

The nominal interest is 12% per year, so the nominal interest rate per 6 mo is 6%. The effective interest rate per 6 mo. is also 6% since the compounding period equals to the interest rate period (6 mo.).

Formally:

$r = 12\% / \text{year}$

$CP = 6 \text{ months}$

$i/6 \text{ months} = r/6 \text{ months} = 12\% / 2 = 6\%$

b) What is the effective interest rate for 12 mo.?

$r/12 \text{ months} = 0.12 \text{ (given)}; m = 2$

$i/12 \text{ months} = (1 + .12/2)^2 - 1 = 12.4\%$

c) What is the effective interest for 24 mo.?

$i = (1 + .12/4)^4 - 1 = 12.68\%$

Example C4.6:

The nominal rate is 1% per month and compounding occurs monthly, what is the effective rate for 12 months.

Solution: r must be in %/PP for PP=12 months.

$r = 12\%$

$m = \#CPs \text{ in } 1PP$

→ $m = 1 \text{ year} / 1 \text{ month} = 12$

$$i = (1 + \frac{.12}{12})^{12} - 1 = 1.1268 - 1 = 12.68\%$$

Terminology

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Comparing payment period and compounding period lengths (PP vs. CP)

Often the frequency of cash flows doesn't equal the frequency of interest compounding. *To correctly perform any equivalence computation, it is essential that the payment period and compounding period be placed on the same time basis, and that the interest rate and the number of periods be adjusted accordingly.*

Case 1. PP = CP

In this case all the results obtained in Chapter 2 are correct.

Example C4.9: An engineer plans to borrow \$3,000 from his company credit union, to be paid in 24 equal monthly installments. The credit union *charges interest at the rate of 1% per month* on the unpaid balance. How much money must the engineer repay each month?

Solution:

Here both the payment and compound periods are equal to one month.

Therefore, $A = P(A/P, 1\%, 24) = \$3,000 (0.04707) = \141.21 .

Case 2. PP > CP

Method 1: Determine the effective interest rate over the given compounding period, and treat each payment separately. i.e. change the time unit of the cash flow diagram to CP

Method 2: Calculate an effective interest rate for the payment period and then proceed as when the interest period and payment period coincide. i.e. retain the time unit as imposed by the cash flows for the cash flow diagram

$$i \text{ per PP} = ?$$

Example C4.10:

An engineer deposits \$1,000 in a savings account at the end of each year. If the bank pays interest at the rate of 6% per year, compounded *quarterly*, how much money will be accumulated in the account after 5 years?

Example C4.10, Solution

Method 1:

$m = 4$; $n = 5$; $mn = 20$; $r = 0.06$; PP = 1 year;
CP = 1 quarter; i per CP is $i = r/m = 0.06/4 = 0.015$

$$F = 1,000(F/P, 1.5\%, 16) + 1,000(F/P, 1.5\%, 12) + 1,000(F/P, 1.5\%, 8) + 1,000(F/P, 1.5\%, 4) + 1,000 = \$5,652.50.$$

Method 2:

$$i = (1 + r/m)^m - 1 = (1 + 0.06/4)^4 - 1 = 0.06136 \text{ (effective interest rate per year).}$$

$$F = \$1,000(F/A, 6.136\%, 5).$$

Note that there are no tables for $i = 6.136\%$. Using the formula, we get

$$F = A \left[\frac{(1+i)^n - 1}{i} \right] = \$1,000 \frac{(1+0.06136)^5 - 1}{0.06136} = \$5,652.40$$

Example C4.11:

A deposit of \$3000 is made after 3 years and \$5000 after 5 years, what is the value of the account after 10 years if the interest rate is 12% per year compounded semi-annually?

Solution: PP = 1 year > CP = 1/2 year

Method 1

$$i \text{ per CP} = .12/2 = .06$$

$$F = 3000 (F/P, 6\%, 14) + 5000 (F/P, 6\%, 10) = 3000 (2.2609) + 5000 (1.7908) = 15,736.70$$

Method 2

$$i = (1 + (.12/2))^2 - 1 = 12.36\% \text{ per year}$$

$$F = 3000 (F/P, 12.36\%, 7) + 5000 (F/P, 12.36\%, 5) = 15,736.70$$

Example C4.12:

\$100 is deposited each quarter for 5 years.
What is the future value of the account if interest is 12% per year compounded monthly.

Solution:

PP = 1 quarter; CP = 1 month
Nominal rate per month is $.12/12 = .01$ which is also the effective interest rate

The effective interest per quarter is:

$$i = (1 + .03/3)^3 - 1 = .0303$$

$$\rightarrow F = \$100 (F/A, .0303, 20) = \$2695.76$$

Interest Periods Smaller than Payment Periods

Table 3-3 Examples of n and i values where $PP = CP$ or $PP > CP$.

(1) Cash-flow Sequence	(2) Interest Rate	(3) What to Find; What is Given	(4) Standard Notation
\$500 semiannually for 5 years	16% per year compounded semiannually	Find P ; given A	$P = 500(F/A, 8\%, 10)$
\$75 monthly for 3 years	24% per year compounded monthly	Find P ; given A	$P = 75(F/A, 2\%, 36)$
\$180 quarterly for 15 years	5% per quarter	Find P ; given A	$P = 180(F/A, 5\%, 60)$
\$25 per month increase for 4 years	1% per month	Find P ; given G	$P = 25(P/G, 1\%, 48)$
\$5000 per quarter for 6 years	1% per month	Find A ; given P	$A = 5000(A/P, 3.03\%, 24)$

CP > PP

Case 1

Interest is earned only by those payments that have been deposited or invested for the entire interest period (CP).

This is the usual mode of operation of banks and lending institutions.

The following rules are applied:

1. All deposits made during an interest period are "moved" to the end of the interest period.
2. All withdrawals made during an interest period are "moved" to the beginning of the interest period.
3. Proceed as in the case where interest periods and payment periods coincide.

CP > PP

Case 2

Any amount of money that is deposited between compounding times earns compound interest.
Number of CPs per PP is now less than 1.
Use the effective rate formula to find effective rate. Use that rate assuming compounding occurs at that rate every PP.

Example: PP=1 week; CP=1 quarter; $r=3\%$ per quarter. Effective weekly $i\% = (1.03)^{1/13} - 1$.

Effective Interest Rate for Continuous Compounding

$$i_a = \left(1 + \frac{r}{m}\right)^m - 1 \quad m \rightarrow \infty$$

The base of natural logarithm $e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x$

The effective cont. int. rate:

$$\begin{aligned}
 i_a &= \lim_{m \rightarrow \infty} \left(1 + \frac{r}{m}\right)^m - 1 \\
 &= \lim_{m \rightarrow \infty} \left[\left(1 + \frac{1}{m/r}\right)^{m/r} \right]^r - 1 \\
 &= \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m/r}\right)^{m/r} \right]^r - 1 \\
 &= e^r - 1
 \end{aligned}$$

$i_a = e^r - 1$

Interest Rates That Vary over Time

Example C4.13:

How much money would be accumulated in 5 years with an initial deposit of \$10,000, if the account earned interest at 12% per year for the first 3 years and at 15% per year for the last 2 years?

Solution:

$$\begin{aligned} F &= \$10,000(F/P, 12\%, 3)(F/P, 15\%, 2) \\ &= \$10,000(1.4049)(1.3225) = \$18,579.80 \end{aligned}$$

Example 4.16:

Year	1	2	3	4
Net Profit	70k	70k	35k	25k
Annual Rate	7%	7%	9%	10%

Given above cash flow and corresponding interest, find its present worth

Solution:

$$\begin{aligned} P &= 70k(P/A, 7\%, 2) + 35k(P/F, 9\%, 1)(P/F, 7\%, 2) \\ &\quad + 25k(P/F, 10\%, 1)(P/F, 9\%, 1)(P/F, 7\%, 2) \\ &= \$172,816 \end{aligned}$$