01204211 Discrete Mathematics
Lecture 11: Counting 3

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Quick recap

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  - We count the number of ways one can choose a subset.
  - We provide a bijection between subsets and binary strings.
  - We prove the fact by induction.

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- The number of subsets of a set with n elements is 2^n. In fact, we know 3 proofs of this fact:
    - We count the number of ways one can choose a subset.
    - We provide a bijection between subsets and binary strings.
    - We prove the fact by induction.
- For a set with n elements, the number of its permutations is $n!$.

This lecture's goals[1]

- Consider set $\{1, 2, 3, 4, 5\}$. How many subsets with 3 elements does this set have?

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- There are 10 subsets with 3 elements: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.
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Abbreviations: We shall call a set with n elements as an n-**set**. We shall call a subset with k elements as a k-**subset**.

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- We will also discuss the inclusion-exclusion priciples.

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However, sometimes, it is useful to treat sets as ordered.
For example, for set $\{1, 2, 3\}$, there are 6 ordered subsets with 2 elements: $\{1, 2\}$, $\{1, 3\}$, $\{2, 1\}$, $\{2, 3\}$, $\{3, 1\}$, $\{3, 2\}$.

Example: runners

Question: There are 10 runners for a given competition. There are 3 awards: 1st price, 2nd price and 3rd price. In how many possible ways these 3 awards can be given? (No runner can get more than one award.)

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- For any 1st price winner, there are 9 choices to choose the 2nd price winner.

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- For any 1st price winner, there are 9 choices to choose the 2nd price winner.
- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.

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- First, we pick the 1st price winner: there are 10 choices.
- For any 1st price winner, there are 9 choices to choose the 2nd price winner.
- For any 1st and 2nd price winners, there are 8 choices for the 3rd winner.
- Therefore, we conclude that the number of ways is $10 \cdot 9 \cdot 8$.

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  - The number of running results is the number of permutation of the other 7 non-winning runners; thus, there are $7!$ of them.
- We can think of a process of choosing a permutation as having two big steps: (1) pick 3 top winners, then (2) pick the rest of runners. This provide a different way to count the number of permutations.

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- Let X be the set of ordered subsets with 3 elements of an 10-set. We then have $|X| \times 7! = 10!$, because they count the same objects. Solving this yields

$$|X| = \frac{10!}{7!} = 10 \cdot 9 \cdot 8.$$

General answers: numbers of ordered subsets

Using the same arguments (either one), we have this theorem.

Theorem 1

The number of ordered subsets with k elements of an n-set is

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

How big is $100!$?

- With computers, we may be able to answer the exact long number. But mathematicians usually enjoy a "quick" estimate just to have a rough idea on how things are.[2]
- How can we start?

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- Let's think about $n!$.
  - The first lower bound that comes to mind for $n!$ is $1^n = 1$.
  - Can we get a better lower bound? (Here, better lower bounds should be closer to the actual value.) How about 2^n? Is it a lower bound? How about 3^n or 5^n? Are they lower bounds of $n!$?

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Bounds for $n!$

Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since all its factor, except the first one is at least 2, we have that

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Are they any good?

n	2^{n-1}	$n!$	n^{n-1}
1	1	1	1
2	2	2	2
3	4	6	9
4	8	24	64
10	512	$3,628,800$	$1,000,000,000$

A better bound?

Let's consider $n!$ again, but for simplicity, let's consider only the case when n is an even number:

$$1 \cdot 2 \cdot 3 \cdots (n/2 - 1) \cdot (n/2) \cdot (n/2 + 1) \cdots n$$

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To get a better lower bound, we may move our cutting point from 2 to, say, $n/2$. Note that at least $n/2$ factors are at least $n/2$. Thus,

$$\begin{aligned} n! &= 1 \cdot 2 \cdots n \\ &\geq \underbrace{1 \cdot 1 \cdots 1}_{n/2} \times \underbrace{(n/2) \cdots (n/2)}_{n/2} \\ &= (n/2)^{n/2} = \sqrt{(n/2)^n}. \end{aligned}$$

Better?

n	2^{n-1}	$\sqrt{(n/2)^n}$	$n!$	n^{n-1}
1	1	-	1	1
2	2	1	2	2
3	4	-	6	9
4	8	4	24	64
6	32	27	720	7,776
10	512	3,125	3,628,800	1,000,000,000
12	2,048	46,656	479,001,600	743,008,370,688

OK. A bit better.

Stirling's formula

An even better estimate for $n!$ exists.

Theorem 2 (Stirling's formula)

$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$

When we write $a(n) \sim b(n)$, we mean that $\frac{a(n)}{b(n)} \to 1$ as $n \to \infty$.

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$$(100/e)^{100} \cdot \sqrt{200\pi}$$

Thus, the number of digits is its logarithm, in base 10, i.e.,

$$\log\left((100/e)^{100} \cdot \sqrt{200\pi}\right) = 100\log(100/e) + \log(200\pi) \approx 157.9696.$$

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Note that the correct answer is 158 digits.

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- While know that it is $n(n+1)/2$, we can get a very easy upper bound by noting that each term in the sum is at most n; thus,

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- This upper bound of n^2 is very good as the gaps between the upper bounds and the actual values will not be larger than 2, as $\frac{n^2}{n(n+1)/2} < 2$.

The number of subsets

Theorem: The number of k-subsets of an n-set is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

Proof.

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$$B \cdot k! = n \cdot (n-1) \cdots (n-k+1).$$

Therefore, the number of k-subsets is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!},$$

as required. $\square$

Binomial coefficients

The number of k-subsets of an n-set is very useful. Hence, there is a notation for it, i.e.,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

(which reads "n choose k"). These numbers are called **binomial coefficients**.

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Note that

- $\binom{n}{n} = 1$ (why?),
- $\binom{n}{0} = 1$ (why?), and,
- when $k > n$, $\binom{n}{k} = 0$.

Properties (1)

Theorem:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Properties (2)

Theorem: When $n, k > 0$, then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Properties (3)

Theorem: When $n, k > 0$, then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Quick questions (1)

There are 40 students in the classroom. There are 35 students who like Naruto, 10 students who like Bleach, and 7 students who like both of them. How many students in this classroom who do not like either Bleach or Naruto?

Quick questions (2)

There are 35 students in the classroom. There are 25 students who like Naruto, 15 students who like Bleach, 12 students who like One Piece. There are 10 students who like both Naruto and Bleach, 7 students who like both Bleach and One Piece, and 9 students who like both Naruto and One Piece. There are 5 students who like all of them.
How many students in this classroom who do not like any of Bleach, Naruto, or One Piece?

Is this correct?

The answer from the previous quick question is

$$35 - (25 + 15 + 12 - 10 - 7 - 9 + 5) = 4.$$

Is this correct? Why?

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Is this correct? Why?

Let's try to argue that this answer is, in fact, correct and try to find general answers to this kind of counting questions.

Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O									

Let's look at an individual student (1)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O	*	*		*			*		
Bobby	B									

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Bobby	B	*		*						
Cathy	B,O									

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Alfred	N,O	*	*		*			*		
Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O									

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Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-									

Let's look at an individual student (1)

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Bobby	B	*		*						
Cathy	B,O	*		*	*		*			
Dave	N,B,O	*	*	*	*	*	*	*	*	
Eddy	-	*								
⋮	⋮									

Let's look at an individual student (2)

			N	B	O	NB	BO	NO	NBO	
		35	-25	-15	-12	$+10$	$+7$	$+9$	-5	4
Alfred	N,O	1	-1		-1			$+1$		0
Bobby	B	1		-1						0
Cathy	B,O	1		-1	-1		$+1$			0
Dave	N,B,O	1	-1	-1	-1	$+1$	$+1$	$+1$	-1	0
Eddy	-	1								1
⋮	⋮									

Let's see how each one is counted

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Bobby (B):

$$1 - \binom{1}{1} =$$

Let's see how each one is counted

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Bobby (B):

$$1 - \binom{1}{1} = 1 - 1 = 0$$

Dave (N,B,O):

Let's see how each one is counted

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$$1 - \binom{2}{1} + \binom{2}{2} = 1 - 2 + 1 = 0$$

Bobby (B):

$$1 - \binom{1}{1} = 1 - 1 = 0$$

Dave (N,B,O):

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} =$$

Let's see how each one is counted

Alfred (N,O):

$$1 - \binom{2}{1} + \binom{2}{2} = 1 - 2 + 1 = 0$$

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$$1 - \binom{1}{1} = 1 - 1 = 0$$

Dave (N,B,O):

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0$$

Let's see how each one is counted

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Do you see any patterns here?

Let's see how each one is counted

Alfred (N,O):

$$1 - \binom{2}{1} + \binom{2}{2} = 1 - 2 + 1 = 0$$

Bobby (B):

$$1 - \binom{1}{1} = 1 - 1 = 0$$

Dave (N,B,O):

$$1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 1 - 3 + 3 - 1 = 0$$

Do you see any patterns here? How about

$$1 - \binom{5}{1} + \binom{5}{2} - \binom{5}{3} + \binom{5}{4} - \binom{5}{5} \quad ?$$

Underlying structures

Let's write 1 as $\binom{5}{0}$. Also, let's separate plus terms and minus terms:

$$\binom{5}{0} + \binom{5}{2} + \binom{5}{4} \quad \heartsuit \quad \binom{5}{1} + \binom{5}{3} + \binom{5}{5}$$

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Note that the left terms are the number of even subsets and the right terms are the number of odd subsets. Do you recall one of the homework?

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Note that the left terms are the number of even subsets and the right terms are the number of odd subsets. Do you recall one of the homework? We have proved this:

> **Theorem:** The number of even subsets is equal to the number of odd subsets.

This theorem also shows that our calculation technique is correct. This technique is usually called the **Inclusion-Exclusion principle**.