01204211 Discrete Mathematics
Lecture 15: Binomial Coefficients (2)

Jittat Fakcharoenphol

September 12, 2018

The binomial coefficients[1]

There is a reason why the term $\binom{n}{k}$ is called the binomial coefficients. In this lecture, we will discuss

- advanced counting with binomial coefficients.

[1] This lecture mostly follows Chapter 3 of [LPV].

More on counting

We shall see more techniques for counting when we consider the following problems.

- How many anagrams does the word "KASETSARTUNIVERSITY" have? (They do not have to be real English words.)
- How can you give out n different presents to k students when student i has to get n_i pieces of presents?
- How many ways can you distribute n baht coins to k children?

Easy anagrams

- An anagram of a particular word is a word that uses the same set of alphabets. For example, the anagrams of ADD are ADD, DAD, and DDA.

Easy anagrams

- An anagram of a particular word is a word that uses the same set of alphabets. For example, the anagrams of ADD are ADD, DAD, and DDA.
- How many anagrams does "$ABCD$" have?

Easy anagrams

- An anagram of a particular word is a word that uses the same set of alphabets. For example, the anagrams of ADD are ADD, DAD, and DDA.
- How many anagrams does "$ABCD$" have?
  - $4!$, because every permutation of A B C or D is a different anagram.

Easy anagrams

- An anagram of a particular word is a word that uses the same set of alphabets. For example, the anagrams of ADD are ADD, DAD, and DDA.
- How many anagrams does "$ABCD$" have?
  - $4!$, because every permutation of A B C or D is a different anagram.

Harder anagrams

- How many anagrams does "$ABCC$" have? Is it $4!$?

Harder anagrams

- How many anagrams does "$ABCC$" have? Is it $4!$?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams, i.e., it "distinguishes" the two C's.

Harder anagrams

- How many anagrams does "$ABCC$" have? Is it $4!$?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams, i.e., it "distinguishes" the two C's.
  - Let's try to be concrete. How many times does "$CABC$" get counted in $4!$?

Harder anagrams

- How many anagrams does "$ABCC$" have? Is it $4!$?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams, i.e., it "distinguishes" the two C's.
  - Let's try to be concrete. How many times does "$CABC$" get counted in $4!$?
  - If we treat two C's differently as C_1 and C_2, we can see that $CABC$ is counted twice as C_1ABC_2 and C_2ABC_1. This is true for any anagram of $ABCC$.

Harder anagrams

- How many anagrams does "$ABCC$" have? Is it $4!$?
  - This time we have to be careful because the answer of $4!$ is too large as it over counts many anagrams, i.e., it "distinguishes" the two C's.
  - Let's try to be concrete. How many times does "$CABC$" get counted in $4!$?
  - If we treat two C's differently as C_1 and C_2, we can see that $CABC$ is counted twice as C_1ABC_2 and C_2ABC_1. This is true for any anagram of $ABCC$.
  - Since each anagram is counted in $4!$ twice, the number of anagrams is $4!/2 = 4 \cdot 3 = 12$.

General anagrams

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 L's, 2 O's, H, E, W, R, and D.)

General anagrams

Let's try to use the same approach to count the anagram of $HELLOWORLD$. (It has 3 L's, 2 O's, H, E, W, R, and D.)

The number of permutation of alphabets in $HELLOWORLD$, treating each character differently is $10!$. However, each anagram is counted for $3!2!$ times because of the 3 copies of L and the 2 copies of O. Therefore, the number of anagrams is

$$\frac{10!}{3!2!}.$$

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents.

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents.

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents. If we distinguish the order which each child chooses the presents, then there are $9!$ ways.

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents. If we distinguish the order which each child chooses the presents, then there are $9!$ ways. However, in this case, we consider the distribution of presents, i.e., we consider the set of presents each child gets.

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents. If we distinguish the order which each child chooses the presents, then there are $9!$ ways. However, in this case, we consider the distribution of presents, i.e., we consider the set of presents each child gets.
- To see how many times each distribution is counted in the $9!$ ways, we can let children form a line and let each child permute his or her presents. Each child has $3!$ choices. Thus, one distribution appears $3!3!3!$ times.

Distributing presents

I have 9 different presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Let's think about the process of distributing the presents. We can first let A choose 3 presents, then B chooses the next 3 presents, and C chooses the last 3 presents. If we distinguish the order which each child chooses the presents, then there are $9!$ ways. However, in this case, we consider the distribution of presents, i.e., we consider the set of presents each child gets.
- To see how many times each distribution is counted in the $9!$ ways, we can let children form a line and let each child permute his or her presents. Each child has $3!$ choices. Thus, one distribution appears $3!3!3!$ times.
- Thus, the number of ways we can distribute presents is

$$\frac{9!}{3!3!3!}$$

Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.

Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.
- Another way to look at this distribution is to fix the order of the presents and see who gets each of the presents. Thus, the previous distribution is represented in the following table:

Presents	1	2	3	4	5	6	7	8	9
Children	A	B	A	B	C	B	C	A	C

Another way to look at the present distribution

- Let's look closely at a particular present distribution in the previous question. Let $\{1, 2, \ldots, 9\}$ be the set of presents.
- Consider the case where A gets $\{1, 3, 8\}$, B gets $\{2, 4, 6\}$, and C gets $\{5, 7, 9\}$.
- Another way to look at this distribution is to fix the order of the presents and see who gets each of the presents. Thus, the previous distribution is represented in the following table:

Presents	1	2	3	4	5	6	7	8	9
Children	A	B	A	B	C	B	C	A	C

- This is essentially an anagram problem. You can think of one particular way of present distribution as anagram of AAABBBCCC. Thus, we reach the same solution of

$$\frac{9!}{3!3!3!}.$$

Distributing identical presents

Now suppose that I have 9 identical presents. I want to give them to 3 students: A, B, and C. I want to give each student 3 presents. In how many ways can I do it?

- Note that when we state that the presents are identical, we mean that we do not distinguish them, i.e., the first present and the second present are indistinguishable.

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins.

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins. We place all 9 coins in a line. We let the first student picks some coin, then the second student, then the last one.

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins. We place all 9 coins in a line. We let the first student picks some coin, then the second student, then the last one.
- Since each coin is identical, we can let the first student picks the coin from the beginning of the line. Then the second one pick the next set of coins, and so on.

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins. We place all 9 coins in a line. We let the first student picks some coin, then the second student, then the last one.
- Since each coin is identical, we can let the first student picks the coin from the beginning of the line. Then the second one pick the next set of coins, and so on.
- One possible distribution is

$$\underbrace{oo}_{1}\underbrace{oooo}_{2}\underbrace{ooo}_{3}$$

Distributing coins (1)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it so that each student gets at least one coin?

- Let's first try to organize the distribution of coins. We place all 9 coins in a line. We let the first student picks some coin, then the second student, then the last one.
- Since each coin is identical, we can let the first student picks the coin from the beginning of the line. Then the second one pick the next set of coins, and so on.
- One possible distribution is

$$\underbrace{oo}_{1}\ \underbrace{oooo}_{2}\ \underbrace{ooo}_{3}$$

- In how many ways can we do that?

Distributing coins (2)

The example below provides us with a hint on how to count.

$$\underbrace{oo}_{1}\ \underbrace{oooo}_{2}\underbrace{ooo}_{3}$$

Distributing coins (2)

The example below provides us with a hint on how to count.

$$\underbrace{oo}_{1}\ \underbrace{oooo}_{2}\ \underbrace{ooo}_{3}$$

Since all coins are identical, what matters are where the first student and the second student stop picking the coins.

Distributing coins (2)

The example below provides us with a hint on how to count.

$$\underbrace{oo}_{1}\ \underbrace{oooo}_{2}\ \underbrace{ooo}_{3}$$

Since all coins are identical, what matters are where the first student and the second student stop picking the coins. I.e, the previous example can be depicted as

$$oo|oooo|ooo$$

Thus, in how many ways can we do that?

Distributing coins (2)

The example below provides us with a hint on how to count.

$$\underbrace{oo}_{1}\underbrace{oooo}_{2}\underbrace{ooo}_{3}$$

Since all coins are identical, what matters are where the first student and the second student stop picking the coins. I.e, the previous example can be depicted as

$$oo|oooo|ooo$$

Thus, in how many ways can we do that?
Since there are 8 places we can mark starting points, and there are 2 starting points we have to place, then there are $\binom{8}{2}$ ways to do so.

Distributing coins (2)

The example below provides us with a hint on how to count.

$$\underbrace{oo}_{1}\underbrace{oooo}_{2}\underbrace{ooo}_{3}$$

Since all coins are identical, what matters are where the first student and the second student stop picking the coins. I.e, the previous example can be depicted as

$$oo|oooo|ooo$$

Thus, in how many ways can we do that?
Since there are 8 places we can mark starting points, and there are 2 starting points we have to place, then there are $\binom{8}{2}$ ways to do so.
This is a fairly surprising use of binomial coefficients.

Distributing coins (3)

Let's consider a general problem where we have n identical coins to give out to k students so that each student gets at least one coin. In how many ways can we do that?

Distributing coins (3)

Let's consider a general problem where we have n identical coins to give out to k students so that each student gets at least one coin. In how many ways can we do that?
Since there are $n-1$ places between n coins and we need to place $k-1$ starting points, there are $\binom{n-1}{k-1}$ ways to do so.

Distributing coins (3)

Let's consider a general problem where we have n identical coins to give out to k students so that each student gets at least one coin. In how many ways can we do that?

Since there are $n-1$ places between n coins and we need to place $k-1$ starting points, there are $\binom{n-1}{k-1}$ ways to do so.

There are $\binom{n-1}{k-1}$ ways to distribute n identical coins to k children so that each child get at least one coin.

Distributing coins (4)

I have 9 indentical coins. I want to give them to 3 students: A, B, and C. In how many ways can I do it, given that some student may not get any coins?

Identities in the Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Odd and even subsets

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to prove this identity with the Pascal's triangle

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

A more formal proof

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n \binom{n}{n} = 0.$$

The next experiment

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$1^2 = 1$$

The next experiment

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Let's try to compute the sum of squares of numbers in each row.

$$
\begin{aligned}
1^2 &= 1 \\
1^2 + 1^2 &= 2
\end{aligned}
$$

The next experiment

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$\begin{aligned} 1^2 &= 1 \\ 1^2 + 1^2 &= 2 \\ 1^2 + 2^2 + 1^2 &= 6 \end{aligned}$$

The next experiment

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$\begin{aligned} 1^2 &= 1 \\ 1^2+1^2 &= 2 \\ 1^2+2^2+1^2 &= 6 \\ 1^2+3^2+3^2+1^2 &= 20 \end{aligned}$$

The next experiment

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Let's try to compute the sum of squares of numbers in each row.

$$\begin{aligned} 1^2 &= 1 \\ 1^2 + 1^2 &= 2 \\ 1^2 + 2^2 + 1^2 &= 6 \\ 1^2 + 3^2 + 3^2 + 1^2 &= 20 \\ 1^2 + 4^2 + 6^2 + 4^2 + 1^2 &= 70 \end{aligned}$$

Theorem:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Another identity

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Another identity

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

This suggests

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$

Theorem:

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}.$$