# 01204211 Discrete Mathematics
# Lecture 10: Counting 2

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# Listing all subsets[1]

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    In this case, we order by their cardinalities, then use dictionary ordering for subsets with the same numbers of elements.

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- Also, this representation can be considered backwards, i.e., if we start with an integer $6$, we can write down its binary representation: $110_2$ and turns it into a subset $\{a, b\}$.

# A correspondence

Let's see a full list of correspondence between $\{0, 1, 2, \ldots, 7\}$ and subsets of $\{a, b, c\}$.

- $0 \leftrightarrow 000_2 \leftrightarrow \{\}$
- $1 \leftrightarrow 001_2 \leftrightarrow \{c\}$
- $2 \leftrightarrow 010_2 \leftrightarrow \{b\}$
- $3 \leftrightarrow 011_2 \leftrightarrow \{c, b\}$
- $4 \leftrightarrow 100_2 \leftrightarrow \{a\}$
- $5 \leftrightarrow 101_2 \leftrightarrow \{a, c\}$
- $6 \leftrightarrow 110_2 \leftrightarrow \{a, b\}$
- $7 \leftrightarrow 111_2 \leftrightarrow \{a, b, c\}$

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Do you notice anything interesting?

# A general case

Similarly, we can describe a representation for each subset of a set $A$ with $n$ elements. As we consider each element $a$ of $A$, we put $1$ if $a \in A$ and put $0$ if $a \notin A$.

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How many bit strings of length $n$ are there?

There are $2^n$ bit strings; hence, the number of subsets is also $2^n$.
This is another proof of the following theorem:

> **Theorem:** The number of subsets of a set with $n$ elements is $2^n$.

# Two proofs

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Really, it does not make a statement stronger, truer, “more” correct. But each proof usually reveals additional facts related to the statement.

- The first proof considers a procedure for constructing subsets.
- The second proof introduces a nice technique for counting. I.e., instead of counting subsets directly, we show that we have a “special” correspondence between subsets and binary numbers, and then just count the numbers.

# A bijection

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What is so special about this correspondence?

- For each number, there is exactly **one** subset that corresponds to it.
- For each subset, there is exactly **one** number that it corresponds to.

With these two properties, we can conclude that both sets have the same cardinality.

> This type of correspondence is called a **one-to-one correspondence** or **bijection**.

# Sequences of choices

Previously, when we want to count the number of bit strings of length $n$, we use this argument:

> Suppose that to select an object, you have to make $k$ decisions. The first decision has $n_1$ choices, the second decision has $n_2$ choices, and so on. More precisely, for $1 \leq i \leq k$, the $i$-th decision has $n_i$ choices. Then the number of ways you can select an object is $n_1 \cdot n_2 \cdots n_{k-1} \cdot n_k$.

# Example 1

A car license number consists of two English letters and one number from 1 to 9999. How many possible license numbers are there?

# Example 2

10 students stand in a line. You want to give them ice cream. There are 4 flavours, but you don't want to give the same flavour to any consecutive students. In how many ways can you give out the ice cream to these students?

# Permutations

# Counting permutations: an example

We want to count the number of permutations. Let's try with a small example: permutations of set $\{a, b, c\}$.

# Counting permutations

# Number of permutations

We have proved this theorem.

**Theorem:** The number of permutations of a set with $n$ elements is $n!$.