By combining (7) and (8) we get

$$Q(g,a)Q(g,b) \geqq g(0)\left(g(1) - \int_0^1 x^{a+b}\,dg(x)\right) = g(0)Q(g,a+b),$$

and the proof is complete.

REFERENCES

1. K. B. Stolarsky, From Wythoff's Nim to Chebyshev's inequality, *Amer. Math. Monthly* 98 (1991), 889–900.
2. D. H. Luecking, Without Commensurability, *Amer. Math. Monthly* 99 (1992), 668.
3. D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., 1993.

Faculty of Textile Technology
Pierottijeva 6
41000 Zagreb, Croatia
pecaric@mahazu.hazu.hr

A Note on Some Irrational Decimal Fractions

A. McD. Mercer

Let $1 \leq a_1 < a_2 < \dots$ be a strictly increasing sequence of positive integers and write $\mathrm{Dec}\{a_k\}$ to mean the decimal fraction $0.(a_1)(a_2)\dots$. Recently the following result was proved in [**1**]:

Theorem A. *If*

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty$$

then $\mathrm{Dec}\{a_k\}$ *is irrational.*

Taking a_k to be the kth prime this theorem provides a new proof that the decimal $0.2357111317\dots$ is irrational (see also [**2**] and [**3**]). On the other hand, it does not throw any light on the nature of, say, the number $\mathrm{Dec}\{k^2\}$. Our purpose in this note is to prove the following generalization.

Theorem 1. *If there is an integer* $r \geq 0$ *such that the series*

$$\sum_{k=1}^{\infty} \frac{k^r}{a_k} = \infty,$$

then $\mathrm{Dec}\{a_k\}$ *is irrational.*

This theorem assures us, for example, that all of the numbers $\mathrm{Dec}\{k^m\}$ $m = 1, 2, \dots$ are irrational.

It should also be noted that the proof of Theorem 1, specialized to the case $r = 0$, differs from that of Theorem A and so provides a new proof of that result.

Proof of Theorem 1: Let us suppose that $x = 0.(a_1)(a_2)\ldots$ is a rational number which, written in the usual decimal form, reads $x = 0.b_1b_2\ldots$ $(0 \le b_k \le 9)$. Since x is rational its decimal expansion will be periodic (at least after a first block of digits). Let the period be p. To fix ideas we shall assume that the periodic behaviour commences immediately after the decimal point. (If this were not the case we could simply move the decimal point sufficiently to the right and treat the fractional part of the new number.)

Let N_k be the number of digits in a_k. Clearly $N_k \le N_{k+1}$ for each k and since the left-most digit of a_k is at least 1 then

$$a_k \ge 10^{N_k - 1} \quad (= c_k \text{ say}) \tag{1}$$

Let the values taken by the sequence $\{N_k\}_1^\infty$ be $V_1 < V_2 < \cdots$ and let the value V_k be taken exactly m_k times. We note that:

(i) If $N_k = N_{k+1}$ then $c_{k+1} = c_k$ and if $N_k < N_{k+1}$ then $c_{k+1} \ge 10c_k$ (2)

(ii) At most p consecutive N_k's can have the same value. To see this we observe that $N_\nu = N_{\nu+1} = \cdots = N_{\nu+p}$ $(= N \text{ say})$ would imply $a_\nu = a_{\nu+p}$ because pN is a period of the decimal representation of x. Hence

$$m_k \le p \tag{3}$$

(iii) If the jump $N_k < N_{k+1}$ occurs at the k-values $k = k_1, k_2, k_3, \ldots$ then $k_j = m_1 + m_2 + \cdots + m_j$. (4)

Let $r \ge 0$ be an integer. According to (1) and (4) the partial sums of the series $\sum_{k=1}^\infty (k^r/a_k)$ do not exceed

$$\frac{m_1(m_1)^r}{c_1} + \frac{m_2(m_1 + m_2)^r}{c_2} + \frac{m_3(m_1 + m_2 + m_3)^r}{c_3} + \cdots$$

and this series converges since by (2) and (3) it is dominated by the series

$$\frac{p^{r+1}}{c_1} \sum_{k=1}^\infty \frac{k^r}{(10)^{k-1}}.$$

Hence $\sum_{k=1}^\infty (k^r/a_k)$ is convergent. This completes the proof of the theorem.

REFERENCES

1. Norbert Hegyvari, On Some Irrational Decimal Fractions. *Amer. Math. Monthly*, 100 (1993), 779–780.
2. Hardy–Wright, *An Introduction to the Theory of Numbers*, 5th edition, Oxford, Clarendon Press, 1979.
3. Polya–Szego, *Problems and Theorems in Analysis II*, Springer-Verlag, 1976, (exercise 257.)

Department of Mathematics and Statistics.
University of Guelph.
Ontario. N1G 1J4.
Canada.
amercer@msnet.mathstat.uoguelph.ca.