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Some New Irrational Decimal Fractions

Patrick Martinez

Let $1 \leq a_1 < a_2 < \cdots < a_n < \cdots$ be a strictly increasing sequence of positive integers and denote by $Dec\{a_k\}$ the decimal fraction $0.(a_1)(a_2)\cdots$. We ask whether $Dec\{a_k\}$ is a rational number or not. Our main result is the following:

Theorem 1. *Assume that the decimal fraction $Dec\{a_k\}$ is rational. Then there exists a real number $x > 1$ and a positive constant C such that*

$$a_k \geq Cx^k \qquad \textit{for all} \quad k \geq 1. \tag{1}$$

In other words, if $Dec\{a_k\}$ is rational, then the sequence of blocks a_k must grow at least exponentially. It yields at once the

Corollary 2. *Assume that*

$$\sum_{k=1}^{\infty} \frac{y^k}{a_k} = \infty \qquad \textit{for all} \quad y > 1. \tag{2}$$

Then the decimal fraction $Dec\{a_k\}$ is irrational.

Indeed, if $Dec\{a_k\}$ is rational, then choosing C and x as in (1), we have

$$\sum_{k=1}^{\infty} \frac{(\sqrt{x})^k}{a_k} \leq C^{-1} \sum_{k=1}^{\infty} \frac{1}{(\sqrt{x})^k} < \infty,$$

so that (2) is not satisfied for $y = \sqrt{x}$.

Hegyvári proved in [**2**] that $Dec\{a_k\}$ is irrational if

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty; \tag{3}$$

this also provided a new proof of the irrationality of the number $Dec\{p_k\} = 0.23571113171923\ldots$ (where the sequence of digits is formed by the primes in ascending order) because $\sum 1/p_k = \infty$. See e.g. [**1**, Theorem 138] or [**4**, Exercise 257] for earlier proofs of the irrationality result.

Subsequently, Mercer [**3**] generalized Hegyvári's theorem by proving that $Dec\{a_k\}$ is irrational if

$$\sum_{k=1}^{\infty} \frac{k^r}{a_k} = \infty \quad \text{for some } r \geq 0. \tag{4}$$

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Our corollary is stronger than the theorems of Hegyvári and Mercer. Indeed, on the one hand, both conditions (3) and (4) imply (2) because for any given $r \geq 0$ and $y > 1$, we have $1 \leq k^r \leq y^k$ for all k large enough.

On the other hand, our corollary shows that $Dec\{a_k\}$ is irrational if a_k grows like $e^{\sqrt{k}}$, or more generally like e^{k^s} for some $0 < s < 1$, although in these cases neither of the conditions (3) and (4) is satisfied.

Theorem 1 is optimal in the following sense:

Theorem 3. *For each real $z > 1$, there exists a rational decimal fraction $Dec\{a_k\}$ and a positive constant C such that*

$$a_k \leq Cz^k \quad \textit{for all} \quad k \geq 1. \tag{5}$$

Remark. Our proof exhibits an explicit sequence $(a_k)_k$ satisfying (5). Note also that it is not true that a given rational number r and a real number $z > 1$, there exists a block decomposition (a_k) and $C > 0$ such that $r = Dec\{a_k\}$ and (5) is satisfied. For example, consider the number $r = 1/9 = 0.111\ldots$: here every block decomposition (a_k) satisfies $a_{k+1} \geq 10a_k$ for all k, so that $a_k \geq 10^{k-1}$.

Proof of Theorem 1. First we follow [**3**]. Assume that $Dec\{a_k\}$ is a rational number. Then its usual decimal form is periodic with some period $b_1 \ldots b_p$, $0 \leq b_i \leq 9$, at least after a first block of $m - 1$ digits. Thus (a_m) belongs already to the periodic part of $Dec\{a_k\}$.

Mercer observed that for $k \geq m$, a_{k+p} necessarily has at least one more digit than a_k. Indeed, if a_k and a_{k+p} have N digits, then $a_k, a_{k+1}, \ldots, a_{k+p}$ all have N digits. Then the block $(a_k)\ldots(a_{k+p-1})$ has exactly Np digits. Since it belongs to the p-periodic part of $Dec\{a_k\}$, it follows that $a_k = a_{k+p}$, which is impossible.

Now the key point is to notice that for $k \geq m$, a_{k+2p} has at least one more digit than a_{k+p}, and hence has at least two more digits than a_k; thus $a_{k+2p} \geq 10a_k$. We easily prove by induction that $a_{k+2np} \geq 10^n a_k$. Hence for every $\ell \geq m$, and denoting by n the integer part of $(\ell - m)/2p$, we have

$$a_\ell \geq a_{m+2np} \geq 10^n a_m \geq 10^{(\ell-m-2p)/2p} a_m.$$

Putting $C' = 10^{-(m+2p)/2p}$ and $x = 10^{1/2p}$ it follows that $a_l \geq C'x^\ell$ for all $\ell \geq m$. Changing C' to a smaller $C > 0$, these inequalities hold for all $k \geq 1$, so that the estimate (1) is satisfied. ∎

Proof of Theorem 3. Choose any $z > 1$ and any $q \in \mathbb{N}$, $q \geq 2$. Define $n := q!$ and consider the rational number r whose decimal expansion is n-periodic with a period $1 \cdots 1\,2$ of $n - 1$ consecutive 1 digits follows by one 2 digit: $r = 0.1 \cdots 1\,2\,1 \cdots 1\,2 \cdots$. We claim that there exists a block decomposition $(a_k)_k$ of r and a constant C that satisfy (3) if q is large enough.

First consider the $n(n + 1)$ first digits, and cut them into n blocks of $n + 1$ digits; denote by $a_1^{n+1} := 1 \cdots 1\,2\,1$ the first block built that way, a_2^{n+1} the second block, a_p^{n+1} the p^{th} block, and $a_n^{n+1} := 2\,1 \cdots 1\,2$ the last block. It is easy to verify that for all $p \in \{1, \ldots, n - 1\}$, $a_p^{n+1} < a_{p+1}^{n+1}$.

Next consider the $n(n + 2)/2 = n((n/2) + 1)$ following digits, and cut them into $n/2$ blocks of $n + 2$ digits; denote by $a_1^{n+2} := 1 \cdots 1\,2\,11$ the first block built that

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way, and by $a_{n/2}^{n+2} := 12\,1\cdots 1\,2$ the last block. Once again we easily verify that for all $p \in \{1, \ldots, (n/2)-1\}$, $a_p^{n+2} < a_{p+1}^{n+2}$ (and clearly $a_n^{n+1} < a_1^{n+2}$).

Then we repeat this process: fix $d|n$ and assume that for all $d'|n$, $d' < d$, we have already built the n/d' blocks $a_p^{n+d'}$ of $n+d'$ digits. Then consider the following $n(n+d)/d = n((n/d)+1)$ digits and cut them into n/d blocks of $n+d$ digits; denote by $a_1^{n+d} := 1\cdots 1\,2\,1\cdots 1$ the first block built that way (a_1^{n+d} is ended by d consecutive 1 digits), and $a_{n/d}^{n+d} := 1\cdots 1\,1\cdots 1\,2$. It is easy to see that for all $p \in \{1, \ldots, (n/d)-1\}$, $a_p^{n+d} < a_{p+1}^{n+d}$.

This allows us to construct n/d blocks a_p^{n+d} of $n+d$ digits for all the divisors d of n, and it is clear than if $d' < d$, then $a_{p'}^{n+d'} < a_p^{n+d}$ for all $p' \in \{1, \ldots n/d'\}$ and $p \in \{1 \ldots n/d\}$.

When $d = n$, there is only one block of $2n$ digits: $a_1^{n+n} := 1\cdots 1\,2\,1\cdots 1\,2$, constituted by two consecutive periods. The block a_1^{n+n} is the $\ell(n)^{th}$ block of our decomposition, where

$$\ell(n) = \sum_{d|n} \frac{n}{d}.$$

Then we can repeat *identically* the construction: consider the $n(2n+1)$ digits following the block a_1^{n+n}, and cut them into n blocks of $2n+1$ digits, denoted by $a_1^{2n+1}, \ldots, a_n^{2n+1}$. For all $d|n$, construct n/d blocks of $2n+d$ digits. Similarly, for each $k \geq 1$ and when d divides n, we can repeat the construction and build n/d blocks of $kn+d$ digits. This defines a block decomposition of r.

Now we verify (3): a_1^{n+n} has $2n$ digits, a_1^{2n+n} has $3n$ digits,..., a_1^{kn+n} has $(k+1)n$ digits, thus

$$a_1^{kn+n} \leq 10^{n(k+1)}.$$

On the other hand, a_1^{n+n} is the $\ell(n)^{th}$ block of our decomposition, a_1^{2n+n} is the $2\ell(n)^{th}$ block of our decomposition, and a_1^{kn+n} is the $k\ell(n)^{th}$ block of our decomposition. Hence, if we denote now $A_1 := a_1^{n+1}$, $A_2 := a_2^{n+1}, \ldots, A_{\ell(n)} := a_1^{n+n}, \ldots$, we obtain

$$A_{k\,\ell(n)} \leq 10^{n(k+1)},$$

which implies that

$$A_k \leq 10^{2n}\left(10^{n/\ell(n)}\right)^k \qquad \text{for all } k \geq 0.$$

Now let us estimate $n/\ell(n)$:

$$\frac{n}{\ell(n)} = \frac{1}{\sum_{d|n}\frac{1}{d}} \leq \frac{1}{\sum_{d=1}^{q}\frac{1}{d}} \leq \frac{1}{\ln q}.$$

Thus it is sufficient to choose q large enough so that $10^{n/\ell(n)} < z$, and we obtain (5). ∎

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Monic Polynomials in Z[x] with Roots in the Unit Disc

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A THEOREM OF KRONECKER. This note is motivated by an old result of Kronecker on monic polynomials with integer coefficients having all their roots in the unit disc. We call such polynomials *Kronecker polynomials* for short. Let $k(n)$ denote the number of Kronecker polynomials of degree n. We describe a canonical form for such polynomials and use it to determine the sequence $k(n)$, for small values of n. The first step is to show that the number of Kronecker polynomials of degree n is finite. This fact is included in the following theorem due to Kronecker [**6**]; see also [**5**] for a more accessible proof. The theorem also says that the non-zero roots of such polynomials are on the boundary of the unit disc; we use this fact to show that these polynomials are essentially products of cyclotomic polynomials.

Theorem 1. *For each* $n = 1, 2, \ldots$, *there are finitely many monic polynomials of degree* n *with integer coefficients and all zeros in the unit disc* $\{z \in \mathbb{C} \mid |z| \leq 1\}$. *All the zeros of such polynomials have modulus zero or one.*

Proof. Write

$$f(z) = z^n + a_{n-1}a^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0 = \prod_{j=1}^{n}(z - z_j),$$

where all $a_j \in \mathbf{Z}$ and the z_j are the zeros of f. Since all $|z_j| \leq 1$ we have

$$|a_{n-1}| = |z_1 + z_2 + \cdots + z_n| \leq n = \binom{n}{1},$$

$$|a_{n-2}| = \left|\sum_{j,k} z_j z_k\right| \leq \binom{n}{2},$$

$$|a_{n-3}| = \left|\sum_{j,k,l} z_j z_k z_l\right| \leq \binom{n}{3},$$

$$\vdots$$

$$|a_0| = |z_1 z_2 \cdots z_n| \leq 1 = \binom{n}{n}.$$

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