UNIT III

3. Matrices & determinants

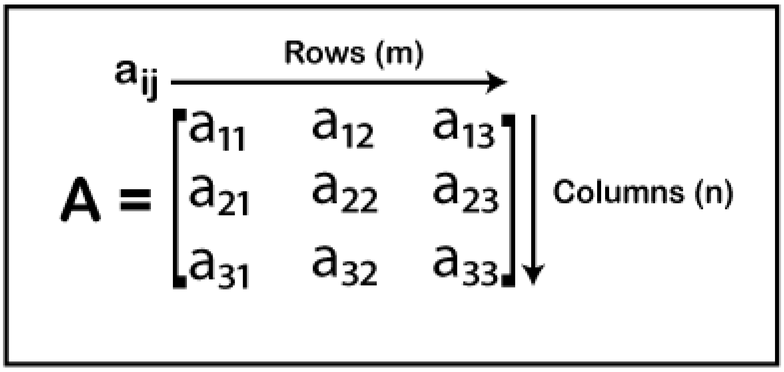
3.1 Matric & types
3.2 Algebra & Matrices
3.3 Definition of determinants
3.4 Adjoint of matrix
3.5 Inverse of matrix

What is the matrix?

The **matrix** is a set of numbers that are arranged in horizontal and vertical lines of entries. The horizontal entries called **rows**, and the vertical entries called **columns**. The numbers are called the **elements** or **entries** of the matrix. It is written inside a pair of **square brackets []**. In other words, it is an **array** of numbers. It is a rectangular representation of numbers in the form of an array.

Matrix Notation

Matrix is usually denoted by a **capital** letter and its elements denoted by the **small** letters along with **subscript** of the **row** and **column** number. The row and column are denoted by the lower-case letter **m** and **n**, respectively. The size of the matrix is defined by the number of rows and columns that it contains. A matrix with **m** rows and **n** columns is called **m** × **n** matrix. It contains a total of **m** × **n** elements. For example:



In the above matrix, a_ij (i represents row number, and j represents column number) are the elements of the matrix. There are three rows and three columns, so there is a total of **nine** elements in the matrix.

The matrix may contain any number of rows and columns. For example:

$[a_{11}]$ Represents the 1×1 matrix.

$[a_{11}\ a_{12}\ a_{13}]$ Represents the 1×3 matrix.

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Represents the 2×2 matrix.

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ Represents the 2×3 matrix.

$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$ Represents the 2×4 matrix.

Types of Matrices

There are the following types of matrices:

Empty Matrix: A Matrix with **no rows** and **no columns** is called an **empty matrix.** For example:

[ ]

Row Matrix: A matrix that has only **a** row is called a **row matrix**. It is also known as the **row vector**. For example:

[4 3 6]

Column Matrix: A matrix that has only **a** column is called a **column matrix**. It is also known as the **column vector**. For example:

$$\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}$$

Zero Matrix: A matrix whose all elements are **zero** is called a **zero matrix**. It is also known as the **null matrix**. For example:

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Square Matrix: A matrix in which row and column dimensions are **equal** (m=n) is called the **square matrix**. For example:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Diagonal Matrix: A square matrix in which all the non-diagonal elements are zero and contain at least one no-zero element in its principal diagonal is called the **diagonal matrix.** For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Scalar Matrix: A diagonal matrix whose all diagonal elements are equal is called the **scalar matrix**. The scalar matrix cannot be a unit matrix, while a unit matrix can be a scalar matrix. For example:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Unit Matrix: A scalar matrix whose principal diagonal elements are one and all non-diagonal elements are zero is called the **unit matrix**. It is also called an **identity matrix**. It is denoted by the letter **I**. It is also a scalar matrix.

$$\text{n} \times \text{n square matrix with } a_{ij} = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$$

For example:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular Matrix: It is a special kind of square matrix that forms a **triangle** either upper or lower to its principal diagonal. There are two types of the triangular matrix:

- **Upper Triangular Matrix:** A square matrix in which all the elements below the leading diagonal are zero. In other words, a square matrix **A=[a_{ij}]** is upper triangular if it satisfies the following condition:

a_{ij}=0 for i<j

For example:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 9 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Lower Triangular Matrix:** A square matrix in which all the elements above the principal diagonal are zero. In other words, a square matrix **A=[a_{ij}]** is a lower triangular if it satisfies the following condition:

a_{ij}=0 for<ij

For example:

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 11 & 0 \\ 6 & 9 & 12 \end{bmatrix}$$

Submatrix: A submatrix of a matrix is determined by deleting any rows or columns or both. For example, consider the following matrix:

$$A = \begin{bmatrix} 2 & 14 & 0 \\ 1 & 11 & 23 \\ 6 & 9 & 12 \end{bmatrix}$$

From the above matrix A, we can generate a submatrix. We are deleting the **2^{nd} row** and **3^{rd} column**. After deleting, we get the following submatrix:

$$\begin{bmatrix} 2 & 14 \\ 6 & 9 \end{bmatrix}$$

Operations on Matrix

We can perform the following operations on the matrix:

- Addition
- Subtraction
- Multiplication
- Division
- Scalar Multiplication
- Inverse
- Transpose
- Negative of a Matrix

Addition of Matrices

The sum om of two matrices can be done by adding the elements matching with the positions. Remember that both matrices must be of the same size. The resultant matrix is also of the same size.

$(A+B)_{ij} = A_{ij} + B_{ij}$

Suppose, there are two matrices A and B, each of size 3×3.

$$A = \begin{bmatrix} a & \mathrm{b} & \mathrm{c} \\ d & \mathrm{e} & \mathrm{f} \\ g & \mathrm{h} & \mathrm{i} \end{bmatrix} \, and \; B = \begin{bmatrix} j & \mathrm{k} & \mathrm{l} \\ \mathrm{m} & \mathrm{n} & \mathrm{o} \\ p & \mathrm{q} & \mathrm{r} \end{bmatrix}$$

The sum of A + B will be:

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} \boldsymbol{a} & \mathbf{b} & \mathbf{c} \\ \boldsymbol{d} & \mathbf{e} & \mathbf{f} \\ \boldsymbol{g} & \mathbf{h} & \mathbf{i} \end{bmatrix} + \begin{bmatrix} \boldsymbol{j} & \mathbf{k} & \mathbf{l} \\ \mathbf{m} & \mathbf{n} & \mathbf{o} \\ \boldsymbol{p} & \mathbf{q} & \mathbf{r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}+\boldsymbol{j} & \mathbf{b}+\mathbf{k} & \mathbf{c}+\mathbf{l} \\ \mathbf{d}+\mathbf{m} & \mathbf{e}+\mathbf{n} & \mathbf{f}+\mathbf{o} \\ \boldsymbol{g}+\boldsymbol{p} & \mathbf{h}+\mathbf{q} & \mathbf{i}+\mathbf{r} \end{bmatrix}$$

Properties of Addition

- **Commutative Law:** A + B = B + A
- **Associative Law:** A + (B + C) = (A + B) + C
- **Additive Identity:** A + 0 = 0 + A = A
- **Additive Inverse:** A + (-A) = (-A) + A = 0

Example: Add the following matrices A and B.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{6} \\ \mathbf{0} & \mathbf{9} \end{bmatrix} \, and \; \boldsymbol{B} = \begin{bmatrix} \mathbf{8} & \mathbf{14} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 8 & 14 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3+8 & 6+14 \\ 0+1 & 9+5 \end{bmatrix}$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} \mathbf{11} & \mathbf{20} \\ \mathbf{1} & \mathbf{14} \end{bmatrix}$$

Subtraction of Matrices

The subtraction of two matrices can be done by subtracting the elements matching with the positions. In other words, it is an addition of a negative matrix. Remember that both matrices must be of the same size. The resultant matrix is also of the same size.

$(A-B)_{ij} = A_{ij} - B_{ij}$

Suppose there are two matrices A and B, each of size 3×3.

$$A = \begin{bmatrix} a & \mathrm{b} & \mathrm{c} \\ d & \mathrm{e} & \mathrm{f} \\ g & \mathrm{h} & \mathrm{i} \end{bmatrix} and\ B = \begin{bmatrix} j & \mathrm{k} & \mathrm{l} \\ \mathrm{m} & \mathrm{n} & \mathrm{o} \\ p & \mathrm{q} & \mathrm{r} \end{bmatrix}$$

The subtraction of A - B will be:

$$\boldsymbol{A - B} = \begin{bmatrix} a & \mathbf{b} & \mathbf{c} \\ d & \mathbf{e} & \mathbf{f} \\ g & \mathbf{h} & \mathbf{i} \end{bmatrix} - \begin{bmatrix} j & \mathbf{k} & \mathbf{l} \\ \mathbf{m} & \mathbf{n} & \mathbf{o} \\ p & \mathbf{q} & \mathbf{r} \end{bmatrix} = \begin{bmatrix} a - j & \mathbf{b - k} & \mathbf{c - l} \\ \mathbf{d - m} & \mathbf{e - n} & \mathbf{f - o} \\ g - p & \mathbf{h - q} & \mathbf{i - r} \end{bmatrix}$$

Example: Subtract the following matrices A and B.

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{16} \\ \mathbf{0} & \mathbf{9} \end{bmatrix} and\ \boldsymbol{B} = \begin{bmatrix} \mathbf{8} & \mathbf{14} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}$$

Solution:

$$A - B = \begin{bmatrix} 3 & 16 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 14 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 - 8 & 16 - 14 \\ 0 - 1 & 9 - 5 \end{bmatrix}$$

$$\boldsymbol{A - B} = \begin{bmatrix} \mathbf{-5} & \mathbf{2} \\ \mathbf{-1} & \mathbf{4} \end{bmatrix}$$

Multiplication of Matrices

Matrix multiplication is the **dot product** of the rows and columns. The dot product is the sum of the products of the matching entries of the two sequences of the numbers.

First matrix's number of columns = Second matrix's number of rows

Properties of Multiplication

- **Non-commutative:** AB ≠ BA
- **Associative:** A(BC) = (AB)C
- **Left Distributive:** A(B + C) = AB + AC
- **Right Distributive:** (A + B)C = AC + BC
- **Scalar:** k(AB)=(kA)B (where k is scalar)
- **Identity:** IA=AI=A
- **Transpose:** $(AB)^T=A^TB^T$

Example : Multiply the following matrices.

$$\boldsymbol{A} = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix} \; \boldsymbol{and} \; \boldsymbol{B} = \begin{bmatrix} 9 & -3 \\ 6 & 1 \end{bmatrix}$$

Solution:

$$A \times B = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & -3 \\ 6 & 1 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} -1(9) + 4(6) & -1(-3) + 4(1) \\ 2(9) + 3(6) & 2(-3) + 3(1) \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 15 & 7 \\ 36 & -3 \end{bmatrix}$$

The multiplication of the matrices A and B is $\begin{bmatrix} \mathbf{15} & \mathbf{7} \\ \mathbf{36} & \mathbf{-3} \end{bmatrix}$.

Division of Matrices

The division of the matrices is a tricky process. To divide the two matrices, we perform the following steps:

- Find the **inverse** of the **divisor**
- Multiply the dividend matrix by the inverse matrix.

Suppose A and B are two matrices, then:

$$\frac{\boldsymbol{A}}{\boldsymbol{B}} = \boldsymbol{A} \times \frac{\mathbf{1}}{\boldsymbol{B}} = \boldsymbol{A} \times \boldsymbol{B}^{-1}$$

Where B^{-1} represents the inverse of B.

Example: Divide the following matrices A and B.

$$\boldsymbol{A} = \begin{bmatrix} 6 & 7 \\ 1 & 5 \end{bmatrix} \; and \; \boldsymbol{B} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution:

A is the numerator, and B is the denominator.

First, we will find the inverse of B.

$$B^{-1} = \frac{1}{(3)(3) - (2)(4)} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{1}{9 - 8} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$B^{-1} = -1 \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}$$

Now multiply the dividend matrix by the inverse.

$$A \times B^{-1} = \begin{bmatrix} 6 & 7 \\ 1 & 5 \end{bmatrix} \times \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}$$

$$A \times B^{-1} = \begin{bmatrix} 6(-3) + & 7(2) & 6(4) + & 7(-3) \\ 1(-3) + & 5(2) & 1(4) + & 5(-3) \end{bmatrix}$$

$$A \times B^{-1} = \begin{bmatrix} -18 + 14 & 24 - 21 \\ -3 + 10 & 4 - 15 \end{bmatrix}$$

$$\boldsymbol{A \times B^{-1} = \begin{bmatrix} -4 & 3 \\ 7 & -11 \end{bmatrix}}$$

Scalar Multiplication

When a matrix is multiplied by a **scalar** (constant) is called **scalar multiplication**. In the scalar multiplication, we multiply each element of the matrix by the scalar.

Suppose a matrix A of size 3×3 is given.

$$A = \begin{bmatrix} a & \mathrm{b} & \mathrm{c} \\ d & \mathrm{e} & \mathrm{f} \\ g & \mathrm{h} & \mathrm{i} \end{bmatrix}$$

It is multiplied by a constant **k** then the scalar multiplication **k × A** will be:

$$\boldsymbol{k \times A = k \times \begin{bmatrix} a & \mathrm{b} & \mathrm{c} \\ d & \mathrm{e} & \mathrm{f} \\ g & \mathrm{h} & \mathrm{i} \end{bmatrix} = \begin{bmatrix} k \times a & k \times \mathrm{b} & k \times \mathrm{c} \\ k \times \mathrm{d} & k \times \mathrm{e} & k \times \mathrm{f} \\ k \times g & k \times \mathrm{h} & k \times \mathrm{i} \end{bmatrix}}$$

Properties of Scalar Multiplication

Let, A and B two matrices of size m × n, and a and b are two scalars. Then:

- **Associative Property:** a (b A) = (a b) A
- **Commutative Property:** aA = Aa
- **Distributive Property:** (a + b) A = aA + b A and a (A + B) = aA + a B
- **Identity Property:** 1 A = A
- **Multiplicative Property:** O A = O (where O is a zero matrix)

Example: If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$ finds the value of 2A.

Solution:

$$2A = 2 \times \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2 \times 1 & 2 \times 4 & 2 \times 7 \\ 2 \times 2 & 2 \times 5 & 2 \times 8 \\ 2 \times 3 & 2 \times 6 & 2 \times 9 \end{bmatrix}$$

$$\mathbf{2A} = \begin{bmatrix} \mathbf{2} & \mathbf{8} & \mathbf{14} \\ \mathbf{4} & \mathbf{10} & \mathbf{16} \\ \mathbf{6} & \mathbf{12} & \mathbf{18} \end{bmatrix}$$

The inverse of a Matrix

Suppose that we have a square matrix A, whose determinant is not equal to zero, then there exists an m×n matrix $\mathbf{A^{-1}}$ that is called the inverse of A such that: $\mathbf{AA^{-1} = A^{-1}A = I}$, where **I** is the identity matrix.

It is easy to find the inverse of a 2×2 matrix in comparison to 3×3 or 4×4 matrix. Follow the steps to find the inverse of a 2×2 matrix.

- **Swap** the positions of the elements **a** and **d**.
- Put a **negative** sign in front of the **b** and **c**
- Divide each element of the matrix by the **determinant**.

For example, A is a 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Its determinant is **(ad-bc)** that is not equal to zero, then the inverse of the matrix will be:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

There are three methods to find the inverse of the large matrix.

- Gauss-Jordan Method
- Using Adjugate
- Use a matrix calculator

Properties of Inverse Matrix

- $A \times A^{-1} = I$
- $A^{-1} \times A = I$
- $(A^{-1})^{-1} = A$
- $(A^{-1})^T = (A^T)^{-1}$

Transpose Matrix

When we convert the rows into columns and columns into rows and generates a new matrix with this conversion is called the **transpose** matrix. It is denoted by $\mathbf{A^T}$ or $\mathbf{A'}$, or $\mathbf{A^{tr}}$, or $\mathbf{A^t}$. For example, consider the following matrix:

$$A = \begin{bmatrix} a & \mathrm{d} & \mathrm{g} \\ \mathrm{b} & \mathrm{e} & \mathrm{h} \\ c & \mathrm{f} & \mathrm{i} \end{bmatrix}$$

The transpose of the above matrix is:

$$\boldsymbol{A^T} = \begin{bmatrix} \boldsymbol{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \boldsymbol{g} & \mathbf{h} & \mathbf{i} \end{bmatrix}$$

Properties of Transpose Matrix

Let, A and B are two matrices and k is a real number, then:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(kA)^T = kA^T$

Example: Find the transpose of the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{7} \\ \mathbf{2} & \mathbf{5} & \mathbf{8} \\ \mathbf{3} & \mathbf{6} & \mathbf{9} \end{bmatrix}$.

Solution:

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Negative of a Matrix

Let, $A=a_{ij}$ be a m×n matrix. The negative of the matrix A is the m×n matrix $B=b_{ij}$ such that $b_{ij}=-a_{ij}$ for all i, j. The negative of the matrix A is written as **-A**.

Suppose, $A = \begin{bmatrix} a & \mathrm{b} & \mathrm{c} \\ d & \mathrm{e} & \mathrm{f} \\ g & \mathrm{h} & \mathrm{i} \end{bmatrix}$ then $-A = \begin{bmatrix} -a & -\mathrm{b} & -\mathrm{c} \\ -d & -\mathrm{e} & -\mathrm{f} \\ -g & -\mathrm{h} & -\mathrm{i} \end{bmatrix}$.

Example: Find the negative of the matrix $A = \begin{bmatrix} \mathbf{1} & \mathbf{-4} & \mathbf{7} \\ \mathbf{-2} & \mathbf{5} & \mathbf{8} \\ \mathbf{3} & \mathbf{6} & \mathbf{-9} \end{bmatrix}$.

Solution:

The negative matrix of A is $-A = \begin{bmatrix} -1 & 4 & -7 \\ 2 & -5 & -8 \\ -3 & -6 & 9 \end{bmatrix}$.

Determinant of a Matrix

The determinant is a **special number** that can be calculated from a matrix. The matrix has to be square (same number of rows and columns) like this one:

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$$

A Matrix

(This one has 2 Rows and 2 Columns)

Let us calculate the determinant of that matrix:

$$\begin{aligned} &= 3\times6 - 8\times4 \\ &= 18 - 32 \\ &= \mathbf{-14} \end{aligned}$$

Example 2:

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The **symbol** for determinant is two vertical lines either side like this:

$$\begin{aligned} |\mathbf{B}| &= 1\times4 - 2\times3 \\ &= 4 - 6 \\ &= \mathbf{-2} \end{aligned}$$

What is it for?

The determinant helps us find the inverse of a matrix, tells us things about the matrix that are useful in systems of linear equations, calculus and more.

Calculating the Determinant

First of all the matrix must be **square** (i.e. have the same number of rows as columns). Then it is just arithmetic.

For a 2×2 Matrix

For a 2×2 matrix (2 rows and 2 columns):

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

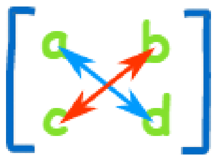
The determinant is:

$$|A| = ad - bc$$

"The determinant of A equals a times d minus b times c"

It is easy to remember when you think of a cross:

- Blue is positive (+ad),
- Red is negative (−bc).



Example: find the determinant of

$$C = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

Answer:

$$\begin{aligned} |C| &= 4\times8 - 6\times3 \\ &= 32 - 18 \\ &= 14 \end{aligned}$$

For a 3×3 Matrix

For a 3×3 matrix (3 rows and 3 columns):

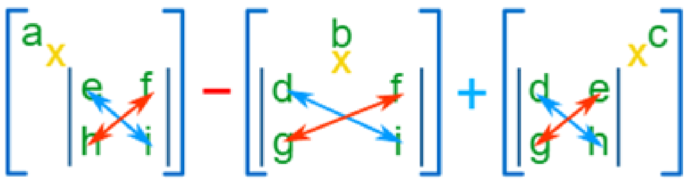
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

The determinant is:

$$\mathbf{|A| = a(ei - fh) - b(di - fg) + c(dh - eg)}$$

"The determinant of A equals ... etc"

It may look complicated, but **there is a pattern**:



3 x 3 Matrix determinant

To work out the determinant of a **3×3** matrix:

- Multiply **a** by the **determinant of the 2×2 matrix** that is **not in a**'s row or column.
- Likewise for **b**, and for **c**
- Sum them up, but remember the minus in front of the **b**

As a formula *(remember the vertical bars || mean "determinant of")*:

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"The determinant of A equals a times the determinant of ... etc"

For Example:

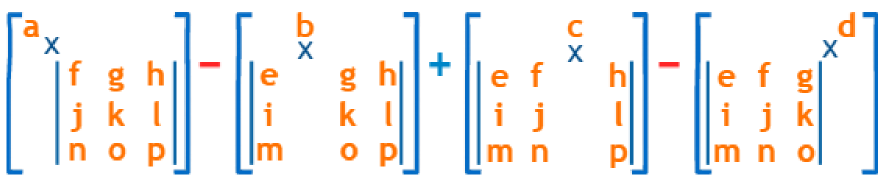
$$D = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$\begin{aligned} |D| &= 6\times(-2\times7 - 5\times8) - 1\times(4\times7 - 5\times2) + 1\times(4\times8 - (-2\times2)) \\ &= 6\times(-54) - 1\times(18) + 1\times(36) \\ &= -306 \end{aligned}$$

For 4×4 Matrices and Higher

The pattern continues for 4×4 matrices:

- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,



As a formula:

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Notice the +−+− pattern (+a... −b... +c... −d...). This is important to remember.

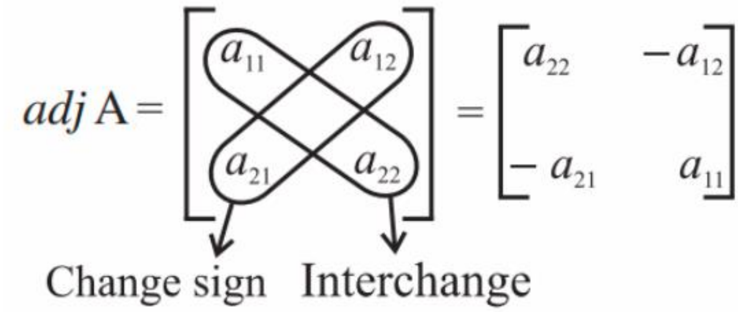
3.4 Adjoin of matrix

Adjointment of 2*2 matrix:

following two steps used to find adjointment of 2*2 matrix

Step1 :the adj A can also be calculated by interchanging al1 and a22. all will replace a2 and similarly a22 will replace al1.

Step2 : changing signs of a12 and a21. if there is + then make - and vice versa This can be shown as in following figure



Example 1: Find the adjoin of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$.

Solution:

Given

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Step1: interchange al1 with a2 and a2 with all.

Step2: changing signs of a12 and a21. if there is + then make - and vice versa which is shown in the following figure

$$\text{Therefore, adj } A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}.$$

Adjoint of a 3×3 Matrix

Adjoint of a Matrix

To find the Adjoint of a Matrix, first, we have to find the Cofactor of each element, and then find 2 more steps. see below the steps,

- **Step 1:** Find the Cofactor of each element present in the matrix.
- **Step 2:** Create another matrix with the cofactors and expand the cofactors, then we get a matrix
- **Step 3:** Now find the transpose of the matrix which comes from after Step 2.

Examples1:

Find the Adjoint of the given matrix **?**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 4 & 5 \\ 6 & 8 & 9 \end{bmatrix}$$

Solution:

Step 1: To find the cofactor of each element

To find the cofactor of each element, we have to delete the row and column of each element one by one and take the present elements after deleting.

Cofactor of element at A[0,0] = 1 $+\begin{bmatrix} 4 & 5 \\ 8 & 9 \end{bmatrix}$: = + (4×9 – 8×5) = -4

Cofactor of elements at A[0,1] =2: = $-\begin{bmatrix} 7 & 5 \\ 6 & 9 \end{bmatrix}$ -= (7×9 – 6×5) = -33

Cofactor of elements at A[0,2] = 3: $+\begin{bmatrix} 7 & 4 \\ 6 & 8 \end{bmatrix}$: = +(7×8 – 6×4) = 32

Cofactor of elements at A[2,0] = 7: = $-\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$ (2×9 – 8×3) = 6

Cofactor of elements at A[2,1] = 4 $+\begin{bmatrix} 1 & 3 \\ 6 & 9 \end{bmatrix}$: = +(1×9 – 6×3) = -9

Cofactor of elements at A[2,2] = 5 $-\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}$: = -(1×8 – 6×2) = 4

Cofactor of elements at A[3,0] = 6 $+\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}$: = +(2×5 – 4×3) = -2

Cofactor of elements at A[3,1] = 8 $-\begin{bmatrix}1 & 3\\7 & 5\end{bmatrix}$: = -(1×5 – 7×3) = 16

Cofactor of elements at A[3,2] = 9 $+\begin{bmatrix}1 & 2\\7 & 4\end{bmatrix}$: = +(1×4 – 7×2) = -10

The matrix looks like with the cofactors:

$$A = \begin{bmatrix} +\begin{bmatrix}4 & 5\\8 & 9\end{bmatrix} & -\begin{bmatrix}7 & 5\\6 & 9\end{bmatrix} & +\begin{bmatrix}7 & 4\\6 & 8\end{bmatrix} \\ -\begin{bmatrix}2 & 3\\8 & 9\end{bmatrix} & +\begin{bmatrix}1 & 3\\6 & 9\end{bmatrix} & -\begin{bmatrix}1 & 2\\6 & 8\end{bmatrix} \\ +\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} & -\begin{bmatrix}1 & 3\\7 & 5\end{bmatrix} & +\begin{bmatrix}1 & 2\\7 & 4\end{bmatrix} \end{bmatrix}$$

The final cofactor matrix:

$$A = \begin{bmatrix} -4 & -33 & 32 \\ 6 & -9 & 4 \\ -2 & 16 & -10 \end{bmatrix}$$

Step 2: Find the transpose of the matrix obtained in step 1

$$adj(A) = \begin{bmatrix} -4 & 6 & -2 \\ -33 & -9 & 16 \\ 32 & 4 & -10 \end{bmatrix}$$

This is the Adjoint of the matrix

Example 2: Find the Adjoint of the given matrix `

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Answer:

$$adj(A) = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

*This is the **Adjoint of the matrix.***

Example 3: Find the Adjoint of the given matrix `

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer:

$$= \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -4 \\ -3 & -2 & 7 \end{bmatrix}$$

*This is the **Adjoint of the matrix.***

Example 4: Find the Adjoint of the given matrix `

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

Answer:

$$= \begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

*This is the **Adjoint of the matrix***

Singular Matrix

A matrix is said to be a singular matrix if the determinant of that matrix is ZERO. This singularity is achieved with only square matrices because only square matrices have determinant. Also, the inversion of singular matrices is not possible because to find the inverse of a matrix we need to divide the adjoint of a matrix with the determinant of the matrix, but for a singular matrix, the value of the determinant is ZERO. So the division is not possible here. To find a matrix is singular or not there is some rule, see below:

- Rule 1: First check if the matrix square or not.
- Rule 2: If square, then calculate its determinant and check if the value is ZERO or not. If ZERO then it is a singular matrix.

Examples

Example 1: Check if the given matrix is singular or not ?

$$A = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

Solution:

Here this matrix is a square matrix, so let's find the determinant of this matrix,

det A = (4×2 – 8×1) = 0

Here determinant of A is ZERO,

Also $\frac{Adj(A)}{det(A)} = \frac{Adj(A)}{0} = \infty$ *= inversion not possible*

So we can say that matrix A is a singular matrix.