The speed of time in an interacting particle system

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March 11, 2021

1 Comparison of the speed of two models

Let (Λ, E) be a transitive graph. For each $i \in \Lambda$ we let $\mathcal{N}_i$ denote the set of vertices adjacent to i. We will be interested in an interating particle system on the lattice (Λ, E) whose local state space is $S = \{-1,+1\}^2$. We denote an element of S^Λ as (x, y) where $x = (x(i))_{i\in\Lambda} \in \{-1,+1\}^\Lambda$ and $y = (y(i))_{i\in\Lambda} \in \{-1,+1\}^\Lambda$. For $x \in \{0,1\}^\Lambda$ and $i \in \Lambda$, we let

$$F_i(x) := \frac{\#\big\{j \in \mathcal{N}_i : x(j) = x(i)\big\}}{\#\mathcal{N}_i}.$$

denote the fraction of neighbors of i where x has the same value as in i. We also write

$$1_{\{x(i)=y(i)\}} := \begin{cases} 1 & \text{if } x(i) = x(j), \\ 0 & \text{otherwise,} \end{cases}$$

and we define $1_{\{x(i)\neq y(i)\}}$ similarly. There are two parameters $\alpha, \beta \geq 0$. Each vertex $i \in \Lambda$ becomes active at the times of independent Poisson point set with rate one. Once a vertex i is active, we either flip $x(i)$, that is, we change its value from whatever it was to the other possible value, or we flip $y(i)$, or we do nothing. The probabilities are as follows:

- $x(i)$ flips with probability $\frac{1}{2}e^{-\beta F_i(x) - \alpha 1_{\{x(i)=y(i)\}}}$,
- $y(i)$ flips with probability $\frac{1}{2}e^{-\beta F_i(y) - \alpha 1_{\{x(i)\neq y(i)\}}}$.

In the note `aper.pdf` a model is considered such that for each $i \in \Lambda$:

- $x(i)$ flips with rate $e^{-x(i)[\beta' M_i(x) + \alpha' y(i)]}$,
- $y(i)$ flips with rate $e^{-y(i)[\beta' M_i(y) - \alpha' x(i)]}$,

where for $x \in \{0,1\}^\Lambda$ and $i \in \Lambda$, we call

$$M_i(x) := \frac{1}{\#\mathcal{N}_i} \sum_{j\in\mathcal{N}_i} x(j)$$

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the *local magnetisation* of x around i. We observe that

$$M_i(x) = x(i)\big[2F_i(x) - 1\big] \quad \text{and} \quad x(i)y(i) = 21_{\{x(i)\,=\,y(i)\}} - 1 = 1 - 21_{\{x(i)\,\neq\,y(i)\}}.$$

Using the fact that $x(i)^2 = 1$ always, it follows that

$$\begin{aligned}
e^{-x(i)[\beta' M_i(x) + \alpha' y(i)]} &= e^{-\beta' x(i) M_i(x) - \alpha' x(i) y(i)]} \\
&= e^{-\beta'[2F_i(x) - 1] - \alpha'[21_{\{x(i)=y(i)\}} - 1]} = e^{\beta' + \alpha'} e^{-2\beta' F_i(x) - 2\alpha' 1_{\{x(i)=y(i)\}}}.
\end{aligned}$$

and similarly

$$\begin{aligned}
e^{-y(i)[\beta' M_i(y) - \alpha' x(i)]} &= e^{-\beta' y(i) M_i(y) + \alpha' x(i) y(i)]} \\
&= e^{-\beta'[2F_i(y) - 1] + \alpha'[1 - 21_{\{x(i)\neq y(i)\}}]} = e^{\beta' + \alpha'} e^{-2\beta' F_i(y) - 2\alpha' 1_{\{x(i)\neq y(i)\}}}.
\end{aligned}$$

Thus, if we set $\alpha' := \frac{1}{2}\alpha$ and $\beta' := \frac{1}{2}\beta$, then this is the same model as before, except that all rates are a factor

$$2e^{(\beta + \alpha)/2}$$

larger. So for example, the model with $\beta = 3$ and $\alpha = 1$ corresponds to the model in `aper.pdf` with $\beta' = 3/2$ and $\alpha' = 1/2$, except for a factor

$$2e^{(\beta + \alpha)/2} = 2e^2 \approx 14.778$$

in the speed of time.

2 Faster simulation

In numerical simulations, the transitive graph (Λ, E) is finite. In each step, we choose a point (i, σ) uniformly from $\Lambda \times \{1, 2\}$. Then we apply the following rules:

- If $\sigma = 1$, then we flip $x(i)$ with probability $e^{-\beta F_i(x) - \alpha 1_{\{x(i)=y(i)\}}}$.
- If $\sigma = 2$, then we flip $y(i)$ with probability $e^{-\beta F_i(y) - \alpha 1_{\{x(i)\neq y(i)\}}}$.

We now measure time in steps of size

$$\frac{1}{\#(\Lambda \times \{1, 2\})} = \frac{1}{2\#\Lambda}.$$

When α and β are large, these probabilities are quite small for a lot of the sites, which means that most of the time our program does nothing. We can speed the program up as follows. We define:

$$R(i, \sigma) := \begin{cases} e^{-\beta F_i(x) - \alpha 1_{\{x(i)=y(i)\}}} & \text{if } \sigma = 1, \\ e^{-\beta F_i(y) - \alpha 1_{\{x(i)\neq y(i)\}}} & \text{if } \sigma = 2. \end{cases}$$

and set

$$R := \sum_{\sigma=1}^{2} \sum_{i \in \Lambda} R(i, \sigma).$$

Instead of choosing (i, σ) uniformly from $\Lambda \times \{1, 2\}$, we now choose (i, σ) with the probability

$$\frac{R(i, \sigma)}{R},$$

and after we have chosen (i, σ), we *always* flip, i.e., if $\sigma = 1$ we flip $x(i)$ and if $\sigma = 2$ we flip $y(i)$. This yields roughly the same as the previous model, provided we measure time in steps of size

$$\frac{1}{R}.$$

Note that R can change during our simulation, so not all time steps have the same size!