An Introduction to Interacting Particle Systems

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Contents

Chapter 1

Markov chains

1.1 Basic definitions

Let S be a finite set. By definition, a *probability law* on S is a function $\mu : S \to \mathbb{R}$ such that

$$\mu(x) \geq 0 \quad \forall x \in S \quad \text{and} \quad \sum_{x \in S} \mu(x) = 1.$$

By definition, a *probability kernel* on a finite set S is a collection of real numbers $P = \big(P(x,y)\big)_{x,y \in S}$ that satisfy

$$P(x,y) \geq 0 \qquad \forall x, y \in S \quad \text{and} \quad \sum_{y \in S} P(x,y) = 1 \qquad \forall x \in S.$$

In other words, this says that for each $x \in S$, the function $y \mapsto P(x,y)$ is a probability law on S. In particular, if $S = \{1, \ldots, N\}$, then we can write P in the form of a matrix

$$P = \begin{pmatrix} P(1,1) & \cdots & P(1,N) \\ \vdots & & \vdots \\ P(N,1) & \cdots & P(N,N) \end{pmatrix}.$$

We can multiply two probability kernels in the same way as we would multiply matrices, i.e.,

$$PQ(x,z) := \sum_{y \in S} P(x,y)Q(y,z) \qquad (x, z \in S).$$

It is easy to see that the product of two probability kernels is again a probability kernel. We let

$$P^k := \underbrace{P \cdots P}_{k \text{ times}}$$

denote the k-th moment of P. By definition, $P^0 = I$, the identity matrix, defined as $I(x, y) := 1$ if $x = y$ and $:= 0$ otherwise.

Let S be a finite set and let Δ be a finite or countably infinite set. Then we let S^Δ denote the set of all functions $k \mapsto x_k$ from Δ to S. In other words, elements of S^Δ are collections of the form $\vec{x} = (x_k)_{k\in\Delta}$, where $x_k \in S$ for all $k \in \Delta$. In particular, $S^{\mathbb{N}}$ is the set of all functions $k \mapsto x_k$ from $\mathbb{N}$ to S. A *stochastic process* with *state space* S is a random function $X = (X_k)_{k\in\mathbb{N}}$ from $\mathbb{N}$ to S. For each $n = 0, 1, 2, \ldots$, we let

$$\rho_n(\vec{x}) := \mathbb{P}[X_0 = x_0, \ldots, X_n = x_n] \qquad \big(\vec{x} \in S^{\{0,\ldots,n\}}\big). \tag{1.1}$$

denote the probability that $X_0 = x_0$ and X visits the states $x_1, \ldots, x_n$ in its first n steps. Then ρ_n is a probability law on $S^{\{0,\ldots,n\}}$ that we call the *law of the first n steps* of X. In probability theory, there is a mathematically precise way to give a meaning to the right-hand side of (1.1), but for this one needs measure theory which we do not require at this point, so for us the right-hand side of (1.1) will only be informal notation. We observe that from our interpretation, it follows that the probability laws $(\rho_n)_{n\geq 0}$ must satisfy the consistency relation

$$\rho_n(x_0, \ldots, x_n) = \sum_{x_{n+1}\in S} \rho_n(x_0, \ldots, x_n, x_{n+1}) \tag{1.2}$$

$\big(n \geq 0,\ (x_0, \ldots, x_n) \in S^{\{0,\ldots,n\}}\big)$. Kolomogorov's extension theorem says that each collection of probability laws $(\rho_n)_{n\geq 0}$ that satisfies (1.2) defines a stochastic process.

A *Markov chain* with *initial law* μ and *transition kernel* P is stochastic process X such that for each $n \geq 0$, the law of the first n steps of X is given by

$$\rho_n(\vec{x}) := \mu(x_0)P(x_0, x_1) \cdots P(x_{n-1}, x_n) \qquad \big(\vec{x} \in S^{\{0,\ldots,n\}}\big).$$

It is easy to check that this formula defines probability laws that are consistent in the sense of (1.2). We observe that

$$\begin{aligned}
\mathbb{P}[X_0 = x] &= \mu(x) && (x \in S),\\
\mathbb{P}[X_1 = y] &= \sum_{x\in S} \mathbb{P}[X_0 = x,\ X_1 = y] = \sum_{x\in S} \mu(x)P(x, y) && (y \in S),\\
\mathbb{P}[X_2 = y] &= \sum_{x\in S}\sum_{y\in S} \mathbb{P}[X_0 = x,\ X_1 = y,\ X_2 = z]\\
&= \sum_{x\in S}\sum_{y\in S} \mu(x)P(x, y)P(y, z) = \sum_{x\in S} \mu(x)P^2(x, z) && (z \in S).
\end{aligned}$$

More generally,

$$\mathbb{P}[X_n = z] = \sum_{y\in S} \mu(y)P^n(y, z) \qquad (z \in S). \tag{1.3}$$

For each $x \in S$, we let δ_x denote the probability law on S defined as $\delta_x(y) := 1$ if $x = y$ and $:= 0$ otherwise. If the initial law of X is δ_x, then we call X the Markov chain with *initial state* x. It is a custom to write $\mathbb{P}^x$ instead of $\mathbb{P}$ to indicate that we are considering the Markov chain with initial state x. Then

$$\mathbb{P}^x[X_n = z] = P^n(x, z) \qquad (x, z \in S).$$

1.2 Random mapping representations

Let $\mathcal{K}(S)$ denote the space of all probability kernels on S. Then $\mathcal{K}(S)$ is a convex set: if $P_1, \ldots, P_m \in \mathcal{K}(S)$ and $p_1, \ldots, p_m \in \mathbb{R}$ satisfy $p_i \geq 0$ for all $1 \leq i \leq m$ and $\sum_{i=1}^m p_i = 1$, then setting

$$P(x, y) := \sum_{i=1}^m p_i P_i(x, y) \qquad (x, y \in S)$$

defines a new probability kernel, that is a *convex combination* of the probability kernels $P_1, \ldots, P_m$. By definition, an *extremal element* of a convex set is an element that cannot be written as a nontrivial convex combination of other elements of the set. We claim that the extremal elements of $\mathcal{K}(S)$ are precisely the probability kernels of the form

$$P_m(x, y) := \begin{cases} 1 & \text{if } y = m(x), \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

where $m : S \to S$ is a map from S to S. Let $\mathcal{F}(S)$ denote the space of all maps (i.e., functions) from S to S. Each element of $\mathcal{K}(S)$ can be written as a convex combination of extremal elements of $\mathcal{K}(S)$. Thus, for each probability kernel on S, there exists a probability law ρ on $\mathcal{F}(S)$ such that

$$P(x, y) = \sum_{m \in \mathcal{F}(S)} \rho(m) P_m(x, y) \qquad (x, y \in S). \tag{1.5}$$

This way of writing P is usually not unique. Let us for example look at the case when $S = \{0, 1\}$. Then

$$\mathcal{F}(S) = \big\{\mathtt{id}, \mathtt{swap}, \mathtt{up}, \mathtt{down}\big\},$$

where $\mathtt{id}, \mathtt{swap}, \mathtt{up}, \mathtt{down}$ are the maps defined as

$$\begin{aligned} \mathtt{id}(x) &:= x, \\ \mathtt{swap}(x) &:= 1 - x, \\ \mathtt{up}(x) &:= 1, \\ \mathtt{down}(x) &:= 0. \end{aligned}$$

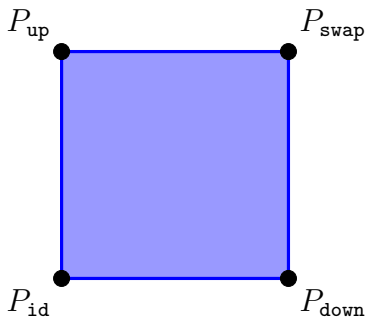

Figure 1.1: The space $\mathcal{K}(\{0,1\})$ of all probability kernels on $\{0,1\}$.

The space $\mathcal{K}(\{0,1\})$ has the shape of a square, see Figure 1.1, with the kernels $P_{\texttt{id}}, P_{\texttt{swap}}, P_{\texttt{up}}, P_{\texttt{down}}$ at the corners. The kernel

$$P := \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \tag{1.6}$$

can alternatively be written as

$$P = \tfrac{1}{2}P_{\texttt{id}} + \tfrac{1}{2}P_{\texttt{swap}} = \tfrac{1}{2}P_{\texttt{up}} + \tfrac{1}{2}P_{\texttt{down}}. \tag{1.7}$$

The expression (1.5) is called a *random mapping representation* of the probability kernel P. To explain this terminology, let M be a *random* map from S to S whose law is ρ. Then (1.5) says that

$$P(x,y) = \mathbb{P}\big[M(x) = y\big] \qquad (x, y \in S). \tag{1.8}$$

Let $(M_k)_{k\geq 1}$ be independent random maps with law ρ and let X_0 be an independent S-valued random variable with law μ. Then we can inductively define a stochastic process $X = (X_k)_{k\geq 0}$ by

$$X_k := M_k(X_{k-1}) \qquad (k \geq 1) \tag{1.9}$$

We claim that $X = (X_k)_{k\geq 0}$ is a Markov chain with initial law μ and transition kernel P given by (1.8). Indeed, the law of the first n steps of X is given by

$$\begin{aligned} &\mathbb{P}[X_0 = x_0, \ldots, X_n = x_n] \\ &\quad = \mathbb{P}[X_0 = x_0]\mathbb{P}\big[M_1(x_0) = x_1\big] \cdots \mathbb{P}\big[M_n(x_{n-1}) = x_n\big], \end{aligned}$$

where we have used the independence of $(M_k)_{k\geq 1}$ and the fact that X_0 is independent of the sequence of random maps $(M_k)_{k\geq 1}$.

Random mapping representations are used for simulating a Markov chain on a computer. Here, one usually starts with a sequence $(Z_k)_{k\geq 1}$ of independent and identically distributed random variables with some known distribution, that take values in some space Ω, and chooses a function $(x,z)\mapsto f(x,z)$ from[1] $S\times\Omega$ to S such that

$$P(x,y)=\mathbb{P}\big[f(x,Z_1)=y\big]\qquad(x,y\in S). \tag{1.10}$$

This says that the random maps $(M_k)_{k\geq 1}$ defined by $M_k(x):=f(x,Z_k)$ satisfy (1.8). Therefore, if X_0 has law μ and is independent of $(Z_k)_{k\geq 1}$, then setting

$$X_k:=f(X_{k-1},Z_k)\qquad(k\geq 1)$$

defines a Markov chain with initial law μ and transition kernel P.

Let us, for example, look at a Markov chain with state space $S=\{0,1\}$, initial state 0, and transition kernel P as in (1.6). A simple program that simulates the first n steps of this Markov chain could look like this:

```
X(0)=0;
for k=1:n
  X(k)=mod(X(k-1)+round(rand),2);
end
```

Here `rand` is a random number generator, i.e., a function that each time when it is called generates a pseudo-random number that is uniformly distributed on the interval $[0,1]$. We then use the function

$$f(x,z)=\texttt{mod}(x+\texttt{round}(z),2)\qquad\big(x\in S,\ z\in[0,1]\big)$$

to round `rand` to the nearest integer and then add it to $X(k-1)$ modulo 2. In this example, the random maps defined by $M_k(x):=f(x,Z_k)$ have the law

$$\mathbb{P}[M_k=\texttt{id}]=\tfrac{1}{2},\quad\mathbb{P}[M_k=\texttt{swap}]=\tfrac{1}{2},$$

and as a result the Markov chain has transition kernel $P=\frac{1}{2}P_{\texttt{id}}+\frac{1}{2}P_{\texttt{swap}}$.

Another, even simpler program that generates the same Markov chain could look like this:

```
X(0)=0;
for k=1:n
  X(k)=round(rand);
end
```

[1] $S\times\Omega$ denotes the space of all pairs (x,z) with $x\in S$ and $z\in\Omega$.

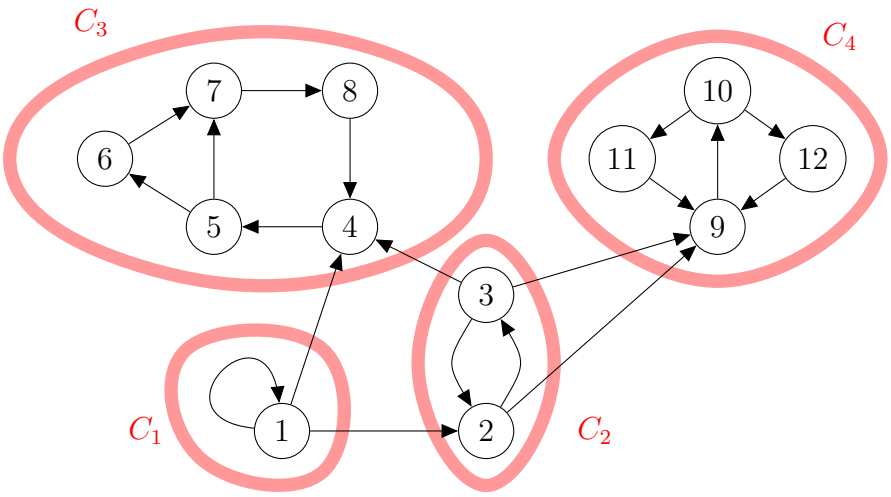

Figure 1.2: The state space of a Markov chain with four communicating classes $C_1, \ldots, C_4$. The classes C_1 and C_2 are open while the classes C_3 and C_4 are closed. The classes C_1 and C_3 are aperiodic. Class C_2 has period 2 and class C_4 has period 3. The classes C_3 and C_4 are closed.

In this example, the random maps defined by $M_k(x) := f(x, Z_k)$ have the law

$$\mathbb{P}[M_k = \texttt{up}] = \tfrac{1}{2}, \quad \mathbb{P}[M_k = \texttt{down}] = \tfrac{1}{2},$$

which in view of (1.7) yields the same transition kernel P.

1.3 Long-time behaviour

It is often illustrative to picture the state space of a Markov chain as an oriented graph, with an arrow pointing from state x to state y if and only if $P(x, y) > 0$, i.e., if it is possible to go from x to y in one step. We write $x \rightsquigarrow y$ if and only if $P^n(x, y) > 0$ for some $n \geq 0$, i.e., if it is possible to go from x to y in a finite number of steps. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y \rightsquigarrow x$. The relation $\leftrightsquigarrow$ is an equivalence relation, so we can divide S into subsets $C_1, \ldots, C_m$ called *communicating classes* such that $x \leftrightsquigarrow y$ if and only if x and y lie in the same communicating class. See Figure 1.2 for an example with four communicating classes.

We say that a communicating class C is *open* if it is possible to leave C, i.e., if there exists some $x \in C$ and $y \notin C$ such that $P(x, y) > 0$. Communicating classes that are not open are called *closed*. One can prove that each

Markov chain with probability one sooner or later enters a closed communicating class, i.e.,

$$\mathbb{P}\big[\,\text{there exists a } k \geq 0 \text{ and a closed communicating class } C \text{ such that } X_k \in C\big] = 1. \tag{1.11}$$

In particular, each state space must contain at least one closed communicating class. A closed communicating class that consists of a single element is called a *trap*.

If $f : S \to \mathbb{R}$ is a real function on S and P is a probability kernel on S, then we let Pf denote the real function on S defined by

$$Pf(x) := \sum_{y \in S} P(x,y) f(y) \qquad (x \in S),$$

i.e., we view f as a column vector upon which P acts as a matrix. Similarly, we let fP denote the real function on S defined by

$$fP(y) := \sum_{x \in S} f(x) P(x,y) \qquad (y \in S),$$

i.e., we view f as a row vector upon which we multiply with P from the left. In particular, for a Markov chain with initial law μ and transition kernel P, formula (1.3) says that μP^n is the law of the Markov chain at time n. By definition, a real function f on S such that $Pf = f$ is called a *harmonic function*. A probability law ν on S such that $\nu P = \nu$ is called an *invariant law*. Note that if we start a Markov chain in an invariant law, then the law at any time is equal to the initial law. The following two theorems describe all harmonic functions and invariant laws of a Markov chain. The first theorem moreover tells us how to calculate the probability that a Markov chain ends up in a given closed class.

Theorem 1.1 (Harmonic functions) *For a Markov chain X with state space S, let $C_1, \ldots, C_m$ be the closed communicating classes and let $T := \bigcup_{i=1}^m C_i$ be their union. Then for each $1 \leq i \leq m$, there exists a unique harmonic function f_i such that*

$$f_i(x) = 1 \quad \forall x \in C_i \quad \text{and} \quad f_i(x) = 0 \quad \forall x \in T \backslash C_i.$$

This function is given by the probability that the Markov chain ends up in the closed class C_i, as a function of the initial state, i.e.,

$$f_i(x) = \mathbb{P}^x\big[X_k \in C_i \text{ for some } k \geq 0\big] \qquad (x \in S).$$

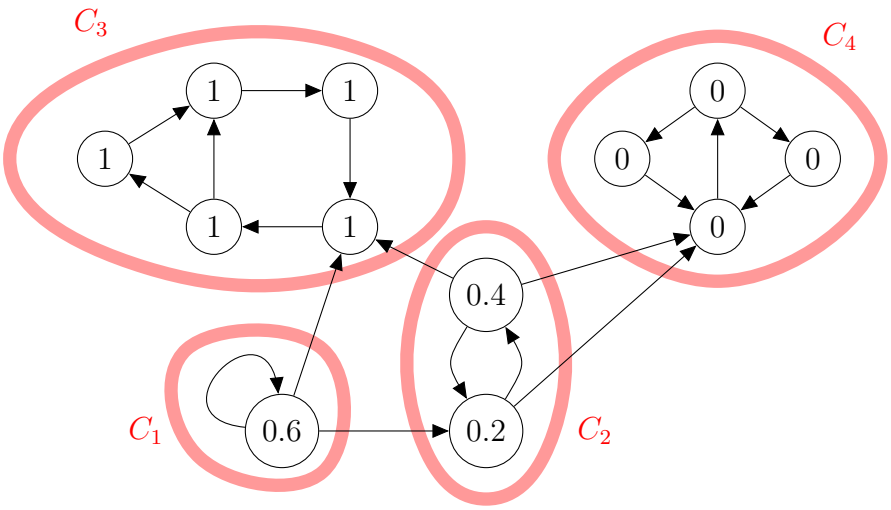

Figure 1.3: The harmonic function f_3 that describes the probability that the Markov chain ends up in the communicating class C_3 as a function of the initial state.

Every harmonic function f can uniquely be written as

$$f(x) = \sum_{i=1}^{m} a_i f_i(x) \qquad (x \in S),$$

where $a_1, \ldots, a_m$ are real constants.

Theorem 1.1 is illustrated in Figure 1.3. In this example, it is assumed that whenever there is more than one arrow pointing out of a given state, all outgoing arrows have equal probabilities.

Theorem 1.2 (Invariant laws) *For a Markov chain X with state space S, let $C_1, \ldots, C_m$ be the closed communicating classes. Then for each $1 \leq i \leq m$, there exists a unique invariant law ν_i such that*

$$\nu_i(x) = 0 \quad \forall x \in S \backslash C_i.$$

This invariant law satisfies $\nu_i(x) > 0$ for all $x \in C_i$. Every invariant law ν can uniquely be written as

$$\nu(x) = \sum_{i=1}^{m} p_i \nu_i(x) \qquad (x \in S),$$

where $p_1, \ldots, p_m$ are real constants such that $p_i \geq 0$ for all i and $\sum_{i=1}^{m} p_i = 1$.

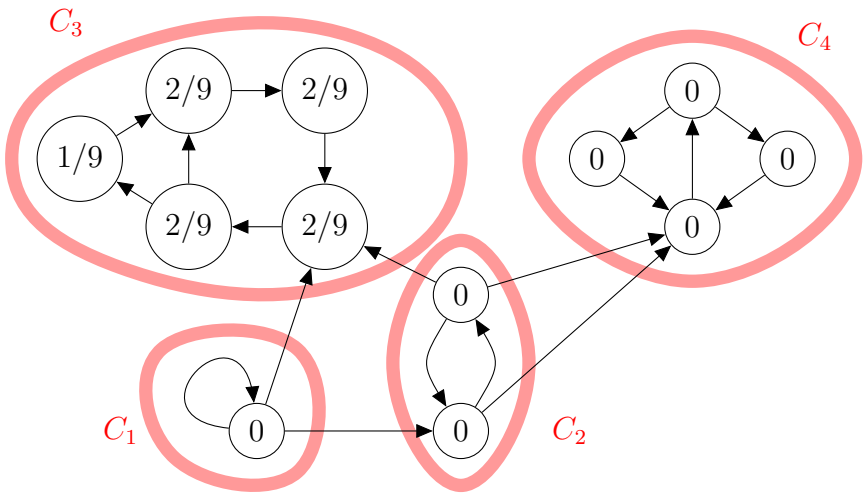

Figure 1.4: The invariant law ν_3 that is concentrated on the closed communicating class C_3.

Theorem 1.2 is illustrated in Figure 1.4.

By definition, the *period* of a state x is the greatest common divisor of the set $\{n \geq 1 : P^n(x,x) > 0\}$. All states in a communicating class C have the same period q, and we can partition C into subsets $D(0), \ldots, D(q-1)$ so that as long as the Markov chain does not leave C, if at some time it is in D_i, then in the next step it must be in $D(i+1 \bmod(q))$. For example, in Figure 1.2, the communicating class C_4 has period 3 and can be divided in $D(0) = \{9\}$, $D(1) = \{10\}$, and $D(2) = \{11, 12\}$.

A Markov chain that has only one communicating class is called *irreducible*. By Theorem 1.2, an irreducible Markov chain has a unique invariant law. If all states have period 1, then we say that the Markov chain is *aperiodic*. The following theorem says that as time tends to infinity, the law of an irreducible aperiodic Markov chain converges to its invariant law.

Theorem 1.3 (Ergodicity) *Let P be the transition kernel of an irreducible aperiodic Markov chain, and let ν be its invariant law. Then for each initial law μ,*

$$\lim_{n\to\infty} \mu P^n(x) = \nu(x) \qquad (x \in S).$$

By definition, a probability law ν such that

$$\nu(x)P(x,y) = \nu(y)P(y,x) \qquad (x,y \in S) \tag{1.12}$$

is called a *reversible law*. The equation (1.12) is called the *detailed balance* equation. A reversible law ν has the property that if we start the Markov

chain in the initial law ν, then the law of the first n steps is symmetric with respect to time reversal, i.e.,

$$\rho_n(x_0, \dots, x_n) = \rho_n(x_n, \dots, x_0),$$

which says that

$$\nu(x_0)P(x_0, x_1) \cdots P(x_{n-1}, x_n) = \nu(x_n)P(x_n, x_{n-1}) \cdots P(x_1, x_0).$$

Each reversible law is an invariant law, but not all invariant laws are reversible.

1.4 Poisson sets

By definition, the *Poisson distribution* with *mean* $r \geq 0$ is the probability law ρ_r on $\mathbb{N}$ defined as

$$\rho_r(k) = \frac{r^k e^{-r}}{k!} \qquad (k \geq 0).$$

The Poisson distribution naturally occurs in the study of rare events, as we now explain. For each $\varepsilon > 0$, let us write

$$\varepsilon\mathbb{Z} := \big\{\varepsilon k : k \in \mathbb{Z}\big\} = \big\{\dots, -2\varepsilon, -\varepsilon, 0, \varepsilon, 2\varepsilon, \dots\big\}.$$

Let $r \geq 0$ be a real number and assume that ε is small enough so that $\varepsilon r \leq 1$. Then we can select a random subset Π_ε of $\varepsilon\mathbb{Z}$ in such a way that independently for each element εk of $\varepsilon\mathbb{Z}$, we have $\mathbb{P}[\varepsilon k \in \Pi_\varepsilon] = \varepsilon r$ and $\mathbb{P}[\varepsilon k \notin \Pi_\varepsilon] = 1 - \varepsilon r$. It turns out the random sets Π_ε (or rather their laws) converge in an appropriate way as $\varepsilon \to 0$ to a limiting random set Π. This limiting set has the property that

$$\mathbb{P}\big[\#(\Pi \cap (a, b)) = k\big] = \rho_{r(b-a)}(k) \qquad (k \geq 0,\ a < b),$$

i.e., the number of points in an interval (a, b) has a Poisson distribution with rate $r(b - a)$. Moreover, if $(a_1, b_1), \dots, (a_1, b_n)$ are disjoint intervals, then the random variables $\#(\Pi \cap (a_1, b_1)), \dots, \#(\Pi \cap (a_n, b_n))$ are independent. It turns out that these properties determine the law of the random set Π uniquely. We call Π a *Poisson subset of the real line* with *intensity* r.

Poisson subsets of the plane can be defined in a similar way. In this case, one starts with subsets Π_ε of $\varepsilon\mathbb{Z} \times \varepsilon\mathbb{Z}$ where each point $(\varepsilon k, \varepsilon l)$ has a probability $\varepsilon^2 r$ to be an element of Π_ε. In the limit, the number of points in a subset of the plane has a Poisson distribution with rate r times the

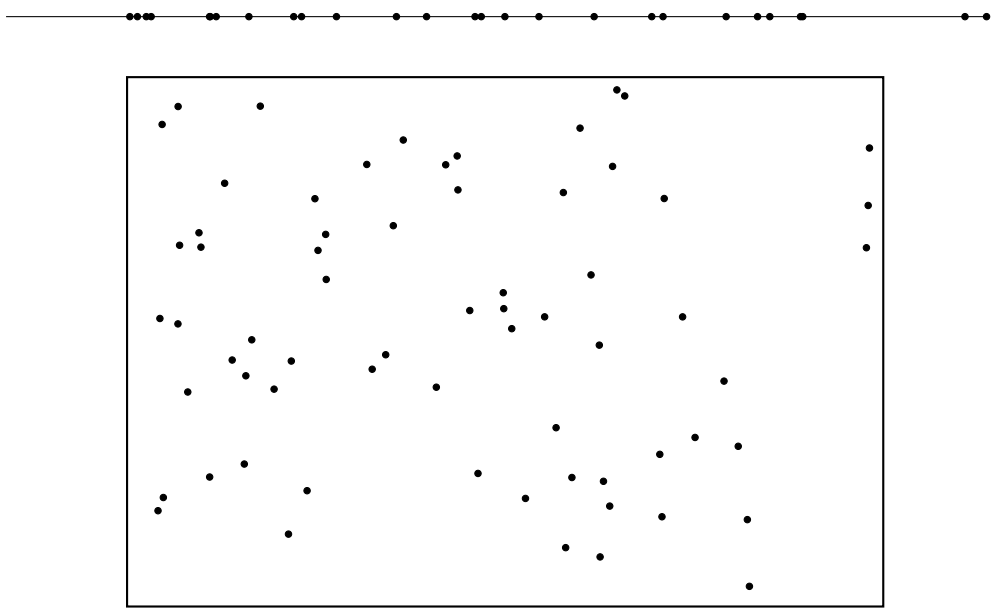

Figure 1.5: Poisson sets in one and two dimensions.

volume of the set. More generally, one can define Poisson subsets of $\mathbb{R}^d$ in any dimension $d \geq 1$. See Figure 1.5 for a picture of Poisson sets in one and two dimensions.

There is an alternative description of Poisson sets that works only for Poisson subsets of the real line (but not in higher dimensions). By definition, a random variable T has an *exponential distribution* with *mean* $\lambda > 0$ if

$$\mathbb{P}\big[T \geq t\big] = e^{-t/\lambda} \qquad (t \geq 0).$$

Such random variables can be simulated on a computer by setting:

```
T=-lambda*ln(rand);
```

Here `ln` is the natural logarithm, and we are using the fact that if Z is uniformly distributed on $[0, 1]$, then

$$\mathbb{P}\big[-\lambda \log(Z) \geq t\big] = \mathbb{P}\big[\log(Z) \leq -t/\lambda\big] = \mathbb{P}\big[Z \leq e^{-t/\lambda}\big] = e^{-t/\lambda}.$$

Let $(T_k)_{k\geq 1}$ and $(T'_k)_{k\geq 1}$ be independent exponentially distributed random variables with mean $1/r$ and let

$$U_n := \sum_{k=1}^{n} T_k \quad \text{and} \quad U'_n := \sum_{k=1}^{n} T'_k \quad (n \geq 1).$$

Then

$$\Pi := \{ \dots, U_2', U_1', U_1, U_2, \dots \}$$

is a Poisson set with intensity r. To understand this informally, we observe that the probability that moving at speed one to the right starting from the origin, we have to wait longer than time t to see the first point of Π is given by

$$\mathbb{P}\big[\Pi \cap (0,t) = \emptyset\big] = \lim_{\varepsilon \to 0} \mathbb{P}\big[\Pi_\varepsilon \cap (0,t) = \emptyset\big] = \lim_{\varepsilon \to 0} (1 - \varepsilon r)^{\lfloor t/\varepsilon \rfloor} = e^{-rt},$$

so the distance to the first point of Π on the right of the origin has an exponential distribution with mean $1/r$. Because of the independence properties of Poisson sets, after we have found the first point, the time we have to wait for the second point is independent and equally distributed with the first time, and so on.

1.5 Continuous-time Markov chains

Let Q be a probability kernel on a finite set S and let μ be a probability law on S. Fix $r > 0$. For $0 < \varepsilon \leq 1/r$, let $X^\varepsilon = (X_k^\varepsilon)_{k \geq 0}$ be the Markov chain with initial law μ and transition kernel

$$r\varepsilon Q + (1 - r\varepsilon) I,$$

where I denotes the identity matrix. In words, we can describe this Markov chain by saying that in each step, with probability $r\varepsilon$, the chain moves to a new position chosen according to the probability kernel Q, and with the remaining probability $1 - r\varepsilon$, nothing happens. Let $\varepsilon\mathbb{N} := \{0, \varepsilon, 2\varepsilon, \dots\}$ and let $Y^\varepsilon = (Y_{k\varepsilon}^\varepsilon)_{k\varepsilon \in \varepsilon\mathbb{N}}$ be defined by

$$Y_{k\varepsilon}^\varepsilon := X_k^\varepsilon \qquad (k \geq 0).$$

In other words, Y^ε is just the same as X^ε, except that we have sped up time by a factor ε^{-1}. It can be shown that the processes Y^ε (or rather their laws) converge in an appropriate way as $\varepsilon \to 0$ to a limiting process $Y = (Y_t)_{t \geq 0}$, where now the time variable t takes values continuously in the half-line $[0, \infty)$.

Let Π_ε denote the set of times in $\varepsilon\mathbb{N}$ when we apply the kernel Q. Then Π_ε converges to a Poisson subset of $[0, \infty)$ with intensity r. In view of this, we can construct the limit process $Y = (Y_t)_{t \geq 0}$ explicitly using exponentially distributed random variables. Let $(T_k)_{k \geq 1}$ be independent exponentially distributed random variables with mean $1/r$ and let

$$U_n := \sum_{k=1}^{n} T_k \qquad (n \geq 1)$$

with $U_0 := 0$. Let $X = (X_k)_{k\geq 1}$ be the Markov chain with initial law μ and transition kernel Q. Then we can construct $Y = (Y_t)_{t\geq 0}$ by setting

$$Y_t := X_k \quad \text{for} \quad t \in [U_k, U_{k+1}) \qquad (k \geq 0).$$

It is custom to define Y_t in such a way that at the time when it makes a jump, it already has the new value, i.e., we use intervals of the form $[U_k, U_{k+1})$ and not $(U_k, U_{k+1}]$. Note that it may happen that $X_{k-1} = X_k$ for some k and hence not at all the times U_k it will actually be true that Y_t jumps to a new position. The process $Y = (Y_t)_{t\geq 0}$ is called a *continuous-time Markov chain*.

Setting

$$P_t(x, y) := \mathbb{P}^x\big[Y_t = y\big] \qquad (t \geq 0,\ x, y \in S)$$

defines for each $t \geq 0$ a probability kernel P_t. These kernels satisfy

$$P_0 = I \quad \text{and} \quad P_s P_t = P_{s+t} \qquad (s, t \geq 0).$$

These relations mean that the operators $(P_t)_{t\geq 0}$ form a *semigroup*. The matrix

$$G(x, y) := r\big(Q(x, y) - I(x, y)\big) \qquad (x, y \in S) \tag{1.13}$$

is called the *generator* of the semigroup $(P_t)_{t\geq 0}$. For any matrix A, one defines a matrix e^A by the infinite sum

$$e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^n,$$

where $A^0 := I$. Then one has

$$P_t = e^{tG} \qquad (t \geq 0).$$

Thus,

$$P_t(x, y) = I(x, y) + tG(x, y) + \tfrac{1}{2}t^2 G^2(x, y) + \tfrac{1}{6}t^3 G^3(x, y) + \cdots \tag{1.14}$$

One can check that this formula is equivalent to

$$P_t(x, y) = \sum_{k=0}^{\infty} \rho_{rt}(k) Q^k(x, y) \qquad (t \geq 0,\ x, y \in S),$$

where ρ_{rt} is the Poisson distribution with mean rt.

The matrix G defined in (1.13) satisfies

$$G(x, y) \geq 0 \qquad (x, y \in S,\ x \neq y) \quad \text{and} \quad \sum_{y\in S} G(x, y) = 0 \qquad (x \in S).$$

Conversely, each matrix with these properties can be cast in the form (1.13) and hence is the generator of a continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$. For each $x \neq y$, the nonnegative number $G(x, y)$ is called the *rate* at which Y jumps from x to y. By (1.14), we have

$$\mathbb{P}^x\big[Y_t = y\big] = tG(x, y) + O(t^2) \qquad \text{as } t \to 0 \qquad (x \neq y),$$

so we can view the rate $G(x, y)$ as the "infinitesimal" probability to jump from x to Y in a very small time interval.

We write $x \rightsquigarrow y$ if and only if there exist $x = x_0, \ldots, x_n = y$ such that $G(x_{k-1}, x_k) > 0$ for all $1 \leq k \leq n$ and define communicating classes as before. Then the communicating classes of the continuous-time Markov chain with generator G are the same as the communicating classes of the Markov chain with transition kernel Q.

For a function $f : S \to \mathbb{R}$, the following statements are equivalent:

$$\text{(i) } Gf = 0, \quad \text{(ii) } P_t f = f \quad \forall t \geq 0.$$

Functions that satisfy these conditions are called *harmonic functions* for the continuous-time Markov chain $Y = (Y_t)_{t\geq 0}$. It follows immediately from (1.13) that $Gf = 0$ if and only if $Qf = f$, so a function is harmonic for the continuous-time Markov chain with generator G if and only if it is harmonic for the Markov chain with transition kernel Q. It follows from our construction of Y in terms of X that Y ends up in a given closed communicating class if and only if X does so. In view of this, the harmonic functions f_i from Theorem 1.1 also describe the probability that Y ends up in C_i, as a function of the initial state.

For a probability law ν on S, the following statements are equivalent:

$$\text{(i) } \nu G = 0, \quad \text{(ii) } \nu P_t = \nu \quad \forall t \geq 0.$$

Probability laws that satisfy these conditions are called *invariant laws* for the continuous-time Markov chain Y. It follows immediately from (1.13) that $\nu G = 0$ if and only if $\nu Q = \nu$, so the continuous-time Markov chain Y has the same invariant laws as the Markov chain X. In particular, if Y is irreducible (i.e., the whole states space is one communicating class), then Y has a unique invariant law. There is also a version of Theorem 1.3:

Theorem 1.4 (Ergodicity) *Let $(P_t)_{t\geq 0}$ be the semigroup of an irreducible continuous-time Markov chain, and let ν be its invariant law. Then for each initial law μ,*

$$\lim_{t\to\infty} \mu P_t(x) = \nu(x) \qquad (x \in S).$$

Note that contrary to Theorem 1.3, in the continuous-time setting, it suffices to assume irreducibility and there is no issue with aperiodicity.

Chapter 2

Interacting Particle Systems

2.1 Transitive graphs

The interacting particle systems that we will be interested are Markov chains $X = (X_n)_{n\geq 0}$ where at each time n, the state of the Markov chain is a function $X_n : \Lambda \to S$ that assigns to each element i of a finite set Λ a value $X_n(i)$, that is called the *local state* of X at the *site* i at time n. The set Λ is called the *lattice*. It will be covenient to assume that Λ has the structure of a transitive graph, as we now explain.

By definition, a *graph* is a pair (Λ, E) were:

- Λ is a set whose elements are called *vertices*,
- E is a set whose elements, called *edges*, are subsets $\{i, j\} \subset \Lambda$ that contain precisely two vertices.

When we draw a picture of a graph, we represent the vertices by points and we represent the edges by line segments connecting the concerned vertices.

By definition, an *automorphism* of a graph is a bijection $\phi : \Lambda \to \Lambda$ such that, for all i, j,

$$\{\phi(i), \phi(j)\} \in E \quad \text{if and only if} \quad \{i, j\} \in E.$$

In words, this says that ϕ maps edges to edges. In other words, automorphisms correspond to symmetries of the graph. A graph is *transitive*[1] if for each pair of vertices $i, j \in \Lambda$, there exists an automorphism $\phi : \Lambda \to \Lambda$ such

[1]More precisely, this concept is called *vertex-transitivity*. A related, but not equivalent concept is *edge-transitivity*, which means that for any two edges $\{i, j\}, \{i', j'\} \in E$, there exists an automorphism ϕ such that $\{\phi(i), \phi(j)\} = \{i', j'\}$. We will only consider vertex-transitive graphs and for simplicity simply call such graphs *transitive*.

that $\phi(i) = j$. In words, we can informally describe this by saying that the graph looks the same as seen from any vertex.

We will only be interested in graphs whose vertex set is either finite or at most countably infinite. We say that a graph (Λ, E) is finite if its vertex set Λ is finite. We call

$$\delta(i) := \#E_i \quad \text{with} \quad E_i := \big\{\{j,k\} \in E : j = i\big\} \qquad (i \in \Lambda)$$

the *degree* of the vertex i, i.e., $\delta(i)$ is the number of edges incident to the vertex i. We will always assume that (Λ, E) is *locally finite*, which means that $\delta(i) < \infty$ for all $i \in \Lambda$. If (Λ, E) is transitive, then there exists an integer δ such that $\delta(i) = \delta$ for all $i \in \Lambda$. We then simply call δ the *degree* of (Λ, E).

By definition, an *oriented graph* is a pair $(\Lambda, \mathcal{E})$ were as before, Λ is a set whose elements are called *vertices*, but contrary to what we had before now $\mathcal{E}$ is a set whose elements are *ordered* pairs (i, j) containing two distinct vertices. Elements (i, j) of $\mathcal{E}$ are called *oriented edges*. In pictures, we represent an oriented edge (i, j) by drawing an arrow that points from i to j.

An *automorphism* of an oriented graph is a bijection $\phi : \Lambda \to \Lambda$ such that, for all $i, j \in \Lambda$,

$$(\phi(i), \phi(j)) \in \mathcal{E} \quad \text{if and only if} \quad (i,j) \in \mathcal{E}.$$

An oriented graph is *transitive* if for each pair of vertices $i, j \in \Lambda$, there exists an automorphism $\phi : \Lambda \to \Lambda$ such that $\phi(i) = j$. For $i \in \Lambda$, we call

$$\mathcal{E}_{1,i} := \big\{(j,k) \in \mathcal{E} : j = i\big\} \quad \text{and} \quad \mathcal{E}_{2,i} := \big\{(j,k) \in \mathcal{E} : k = i\big\},$$

the sets of outgoing and incoming oriented edges at the vertex i, and we call

$$\delta_1(i) := \#\mathcal{E}_{1,i} \quad \text{and} \quad \delta_2(i) := \#\mathcal{E}_{2,i}$$

the *outdegree* and *indegree* of a vertex $i \in \Lambda$, i.e., this is the number of arrows starting at, respectively ending at i. If $(\Lambda, \mathcal{E})$ is transitive, then all vertices have the same indegree and the same outdegree, i.e., there exists integers δ_1 and δ_2 such that $\delta_1(i) = \delta_1$ and $\delta_2(i) = \delta_2$ for all $i \in \Lambda$. If $(\Lambda, \mathcal{E})$ is finite, then moreover $\delta_1 = \delta_2$, but for infinite transitive oriented graphs, it may happen that $\delta_1 \neq \delta_2$.

The *graph distance* $\rho(i, j)$ between two vertices i, j in an (unoriented) graph (Λ, E) is the length of the shortest path from i to j, i.e., $\rho(i, j)$ is the smallest integer r for which there exist vertices $i_0, \ldots, i_r$ with $i_0 = i$ and $i_r = j$ such that $\{i_{k-1}, i_k\} \in E$ for all $1 \leq k \leq r$. By definition $\rho(i, i) := 0$.

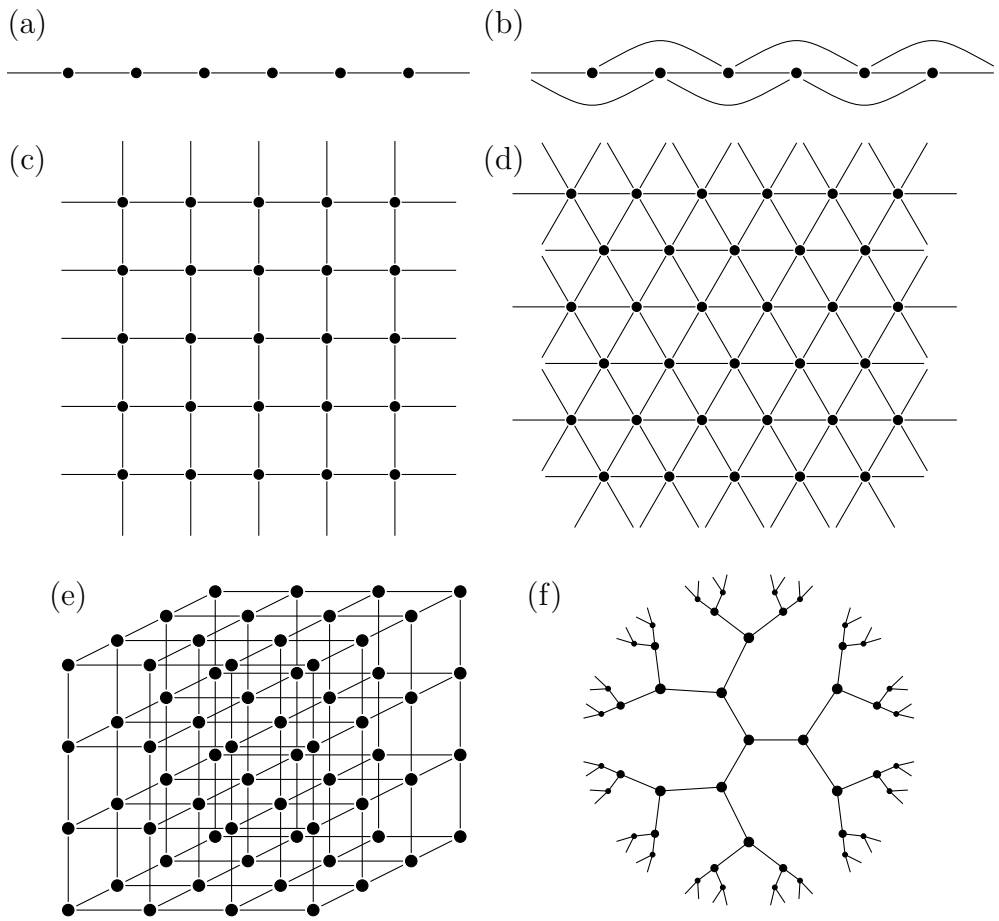

Figure 2.1: Some examples of transitive graphs: (a) the one-dimensional integer lattice, (b) the range 2 one-dimensional integer lattice, (c) the square lattice, (d) the triangular lattice, (e) the cubic lattice, (f) the 3-regular tree.

Each unoriented graph (Λ, E) naturally gives rise to an oriented graph $(\Lambda, \mathcal{E})$ defined by

$$\mathcal{E} := \big\{(i,j) \in \Lambda \times \Lambda : \{i,j\} \in E\big\}. \tag{2.1}$$

In words, this says that we replace each unoriented edge $\{i,j\}$ of (Λ, E) by two oriented edges (i,j) and (j,i). More generally, if (Λ, E) is an unoriented graph and $R \geq 1$ is an integer, then we can define an oriented graph $(\Lambda, \mathcal{E}_R)$ by setting

$$\mathcal{E}_R := \big\{(i,j) \in \Lambda \times \Lambda : 0 < \rho(i,j) \leq R\big\}.$$

In particular, for $R = 1$ this yields the oriented graph $(\Lambda, \mathcal{E})$ defined above. If the graph (Λ, E) is transitive, then so is the associated oriented graph

$(\Lambda, \mathcal{E}_R)$, for each $R \geq 1$.

In order to formulate the class of interacting particle systems that we will be interested in in this chapter, we need something a bit more general than an oriented graph. We will be interested in pairs $(\Lambda, \mathcal{E})$ where Λ is a set of vertices as before and $\mathcal{E}$ is a set whose elements are ordered sequences $(i_1, \ldots, i_h)$ of vertices, all different from each other, where $h \geq 2$ is some fixed integer. In particular, for $h = 2$ such an object is an oriented graph. For higher values of h, it seems there does not really exist an established name for such an object, so let us just call it an *h-tuple graph* and let us give the elements of $\mathcal{E}$ the name *h-edges*.[2]

By definition, an *automorphism* of an h-tuple graph $(\Lambda, \mathcal{E})$ is a bijection $\phi : \Lambda \to \Lambda$ such that, for all $i_1, \ldots, i_h \in \mathcal{E}$,

$$\big(\phi(i_1), \ldots, \phi(i_h)\big) \in \mathcal{E} \quad \text{if and only if} \quad (i_1, \ldots, i_h) \in \mathcal{E}.$$

An h-tuple graph is *transitive* if for each pair of vertices $i, j \in \Lambda$, there exists an automorphism $\phi : \Lambda \to \Lambda$ such that $\phi(i) = j$. Generalizing our earlier definition of the indegree and outdegree, for each vertex $i \in \Lambda$ we define

$$\delta_k(i) := \#\mathcal{E}_{k,i} \quad \text{with} \quad \mathcal{E}_{k,i} := \big\{(i_1, \ldots, i_h) \in \mathcal{E} : i_k = i\big\} \qquad (1 \leq k \leq h). \tag{2.2}$$

If $(\Lambda, \mathcal{E})$ is transitive, then $\delta_k(i) = \delta_k$ does not depend on $i \in \Lambda$. If Λ is finite, then moreover $\delta_1 = \cdots = \delta_h$. In this case we simply call $\delta := \delta_1 = \cdots = \delta_h$ the *degree* of $(\Lambda, \mathcal{E})$. For a finite h-tuple graph $(\Lambda, \mathcal{E})$, one has

$$\#\mathcal{E} = \delta \cdot \#\Lambda,$$

i.e., the number of h-edges is δ times the number of vertices.

For any (unoriented) graph (Λ, E) and vertex $i \in \Lambda$, we let

$$\mathcal{N}_i := \big\{j \in \Lambda : \{i, j\} \in E\big\}$$

denote the set of all vertices that are *adjacent* to i. If (Λ, E) is a transitive graph with degree δ, then we can define a transitive h-tuple graph $(\Lambda, \mathcal{E})$ with $h = \delta + 1$ by setting

$$\mathcal{E} = \big\{(i_1, \ldots, i_{\delta+1}) : \mathcal{N}_{i_1} = \{i_2, \ldots, i_{\delta+1}\}\big\}. \tag{2.3}$$

We will use this particular way of creating an h-tuple graph when we describe the Ising model. One can check that the degree of this h-tuple graph $(\Lambda, \mathcal{E})$ (according to our earlier definition) is δ!.

[2] *Hypergraphs* are a generalization of graphs, where the the edges $\{i, j\}$ are replaced by more general finite subsets of Λ that may contain more than two vertices and that are called *hyperedges*. It is tempting to call the objects that we need "oriented hypergraphs" but it seems that term is already in use for something else.

2.2 Interacting particle systems

In this section, we describe a general class of interacting particle systems, that will be our main objects of interest in this chapter. We will use the concept of an h-tuple graph that we have defined in the previous section. Recall that S^Λ denotes the Carthesian product space of Λ copies of S, i.e., elements x of S^Λ are of the form

$$x = \big(x(i)\big)_{i\in\Lambda} \quad \text{with} \quad x(i) \in S \;\forall\; i \in \Lambda.$$

Equivalently, S^Λ is nothing else than the set of all functions $x : \Lambda \to S$. In particular, we use the convention

$$S^m := S^{\{1,\ldots,m\}},$$

i.e., elements of S^m are sequences $(x(1), \ldots, x(m))$.

The interacting particle systems that we will be interested in in this chapter are built from three ingredients:

- a transitive h-tuple graph $(\Lambda, \mathcal{E})$ called the *lattice*,
- a finite set S called the *local state space*,
- a probability kernel K on S^h, called the *local transition kernel*.

Our main interest will be in the case that the lattice $(\Lambda, \mathcal{E})$ is finite, but large. We are then interested in the Markov chain $Y = (Y_n)_{n\geq 0}$ with state space S^Λ that evolves according to the following description:

1. In each time step, we first choose an h-edge $(i_1, \ldots, i_h)$ uniformly from the set $\mathcal{E}$ of all h-edges.
2. We then replace the old values $\big(Y_{n-1}(i_1), \ldots, Y_{n-1}(i_h)\big)$ of Y in this h-edge by new values $\big(Y_n(i_1), \ldots, Y_n(i_h)\big)$ that are chosen with probabilities depending on the old values according to the local transition kernel K.
3. All other values of Y stay the same, i.e., we set $Y_n(j) = Y_{n-1}(j)$ for all $j \notin \{i_1, \ldots, i_h\}$.

More formally, the transition kernel P of the Markov chain Y has the form

$$P(x, y) := \frac{1}{\#\mathcal{E}} \sum_{(i_1,\ldots,i_h)\in\mathcal{E}} K_{i_1,\ldots,i_h}(x, y) \qquad (x, y \in S^\Lambda),$$

where $\#\mathcal{E}$ denotes the number of elements of $\mathcal{E}$ and $K_{i_1,\ldots,i_h}$ denotes the probability kernel on S^Λ defined as:

$$K_{i_1,\ldots,i_h}(x,y) := \begin{cases} K\big((x(i_1),\ldots,x(i_h)),(y(i_1),\ldots,y(i_h))\big) \\ \quad \text{if } x(j)=y(j) \text{ for all } j \notin \{i_1,\ldots,i_h\}, \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $N := \#\Lambda$ denote the number of vertices. Recall the definition of $\mathcal{E}_{1,i}$ in (2.2). In each step, the probability that we choose an h-edge from $\mathcal{E}_{1,i}$ is $1/N$. In view of this, similar to what we did for continuous-time Markov chains, it is natural to rescale time by a factor N. We set

$$X_t^N := Y_{\lfloor Nt \rfloor} \qquad \big(t \in [0,\infty)\big), \tag{2.4}$$

where $r \mapsto \lfloor r \rfloor$ is the function that rounds off areal number r to the nearest integer below it. Then $(X_t^N)_{t\geq 0}$ is a process with continuous time parameter t that can jump only at times in $\{0, 1/N, 2/N, \ldots\}$ and that evolves as the Markov chain $(Y_n)_{n\geq 0}$ but with time sped up with a factor N.

Interacting particle systems can also be constructed in continuous time. Using the same ingredients as for the discrete-time system, we can define a continuous-time Markov chain $X = (X_t)_{t\geq 0}$ with state space S^Λ that evolves according to the following rules. Below, we use the convention that $X_{t-}(i)$ denotes the value of X at the site i just before the time t.

1. Each vertex $i \in \Lambda$ becomes active at the times of an independent Poisson point set with intensity one.

2. If a vertex $i \in \Lambda$ becomes active, then we choose an h-edge $(i_1,\ldots,i_h)$ uniformly from the set $\mathcal{E}_{1,i}$ of all h-edges with $i_1 = i$.

3. We then replace the old values $\big(X_{t-}(i_1),\ldots,X_{t-}(i_h)\big)$ of X in this h-edge by new values $\big(X_t(i_1),\ldots,X_t(i_h)\big)$ that are chosen with probabilities depending on the old values according to the local transition kernel K.

4. All other values of X stay the same, i.e., we set $X_t(j) = X_{t-}(j)$ for all $j \notin \{i_1,\ldots,i_h\}$.

An advantage of this sort of construction is that it can be shown that it leads to a well-defined process even if the lattice Λ is infinite. For finite but large lattices, we claim that the continuous-time process $X = (X_t)_{t\geq 0}$ is extremely similar to the sped-up discrete-time process $X^N = (X_t^N)_{t\geq 0}$. Indeed, for each fixed site $i \in \Lambda$, in the limit as N is large, the set

$$\big\{n/N : \text{ at time } n \text{ we choose an } h\text{-edge from } \mathcal{E}_{1,i}\big\}$$

approximates a Poisson set with intensity one. In theoretical considerations, it is often easier to work with the continuous-time process X, while in simulations, we use the sped-up discrete-time process X^N. When N is large, the difference between the two is indiscernable.

In the following sections, we will give examples of interacting particle systems and compare their behavior on finite and infinite lattices.

2.3 The voter model

For any finite set S, we define a *voter model map* $\mathtt{vot} : S^2 \to S^2$ by

$$\mathtt{vot}\big(x(1), x(2)\big) := \big(x(2), x(2)\big). \tag{2.5}$$

Let (Λ, E) be a transitive graph and let $(\Lambda, \mathcal{E})$ denote the associated oriented graph as defined in (2.1). Let S be a finite set, and let $P_{\mathtt{vot}}$ denote the probability kernel on S^2 associated with the map $\mathtt{vot}$, as defined in (1.4). The interacting particle system with these ingredients is known as the *voter model* on the graph (Λ, E) with local state space S. If Λ is finite and we work in discrete time, then the voter model has the following description:

1. In each step $n = 1, 2, \ldots$, we choose a vertex i uniformly from Λ.
2. We then choose another site j uniformly from the set $\mathcal{N}_i$ of sites adjacent to i.
3. We set $Y_n(i) := Y_{n-1}(j)$ and $Y_n(k) := Y_{n-1}(k)$ for all $k \neq i$.

In the continuous-time setting, which applies also to infinite Λ, the description is as follows:

1. For each vertex $i \in \Lambda$, there is an independent Poisson point subset of $[0, \infty)$ with rate one that determines when the vertex becomes active.
2. If a vertex i becomes active at some time t, then we choose another site j uniformly from the set $\mathcal{N}_i$ of sites adjacent to i.
3. We set $X_t(i) := X_{t-}(j)$ and $X_t(k) := X_{t-}(k)$ for all $k \neq i$.

In the context of the voter model, the local state $x(i)$ at a site i is often called the *type* at i. The voter model is often used to model biological populations, where organisms with different genetic types occupy sites in space. An alternative interpretation, that has given the voter model its name, is that sites represent people and types represent political opinions.

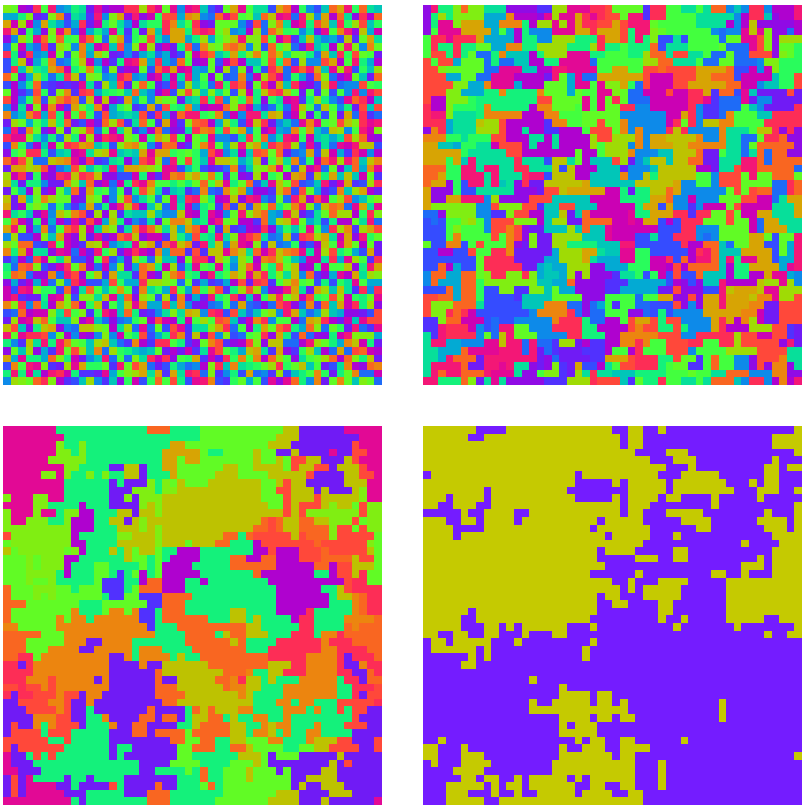

Figure 2.2: Four snapshots of a two-dimensional voter model with periodic boundary conditions. Initially, the types of sites are i.i.d. Time evolved in these pictures is 0, 1, 32, and 500.

With rate one, an individual becomes unsure what political party to vote for, asks a randomly chosen neighbor, and copies his/her opinion.

In Figure 2.2, we see the four snapshots of the time evolution of a two-dimensional nearest-neighbor voter model. The initial state is constructed by assigning i.i.d. types to the sites. Due to the sites copying the types of their neighbours, we see patches appear where locally all sites have the same type. As time proceeds, these patches, usually called *clusters*, grow in size, so that eventually, for any $R \geq 1$, the probability that all sites within distance R of the origin are of the same type tends to one.[3]

It turns out that this sort of behavior, called *clustering*, is dimension dependent. The voter model clusters in dimensions 1 and 2, but not in

[3]In spite of this, for the model on the infinite lattice, it is still true that the origin changes its type infinitely often.

dimensions 3 and more. In Figure 2.3, we see the four snapshots of the time evolution of a three-dimensional voter model. The model is simulated on a cube with periodic boundary conditions, and the types of the middle layer are shown in the pictures. In this case, we see that even after a long time, there are still many different types near the origin.

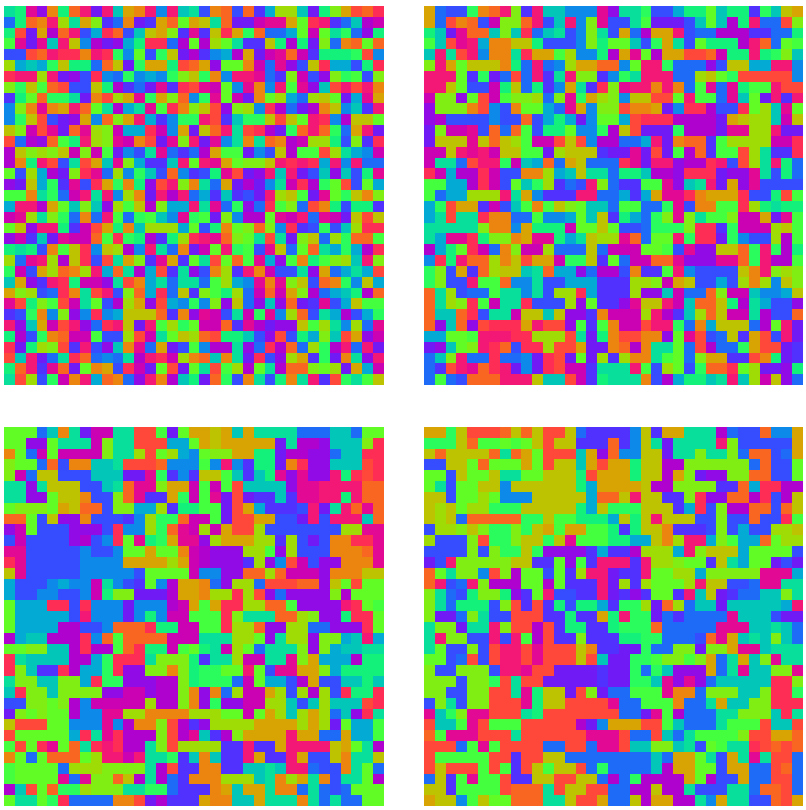

Figure 2.3: Four snapshots of the transsection of a three-dimensional voter model with periodic boundary conditions. Initially, the types of sites are i.i.d. Time evolved in these pictures is 0, 4, 32, and 250.

Indeed, it can be proved that on the cubic lattice $\mathbb{Z}^3$, the law of the contact process converges as $t \to \infty$ to a nontrivial law in which all types are present. On the other hand, on any finite connected graph, it follows from (1.11) that the voter model must eventually end up in one of the constant configurations. What this shows is that in dimensions 3 and more, the large volume limit $N \to \infty$ and the large time limit $t \to \infty$ cannot be interchanged. If we first send $t \to \infty$ and then $N \to \infty$, then we end up with a probability law that is concentrated on the constant configurations, but if we first send

$N \to \infty$ and then $t \to \infty$, then we end up with a probability law that is concentrated on configurations in which all types coexist.

If N is large, then even though the system eventually ends up in a constant configuration, the time we have to wait for this to happen becomes very large, so large, that we are not able to see this happen in our numerical simulations. To mathematically understand what we see in our numerical simulations, we use the infinite system, which is a good approximation of a large but finite system on the time scales that we are interested in.

2.4 The contact process

Let $S := \{0, 1\}$, let $0 \leq p \leq 1$ be a parameter, and let $K_p = K$ be the probability kernel on S^2 defined by

$$\begin{pmatrix} K(00,00) & K(00,01) & K(00,10) & K(00,11) \\ K(01,00) & K(01,01) & K(01,10) & K(01,11) \\ K(10,00) & K(10,01) & K(10,10) & K(10,11) \\ K(11,00) & K(11,01) & K(11,10) & K(11,11) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \end{pmatrix}.$$

Let (Λ, E) be a transitive graph and let $(\Lambda, \mathcal{E})$ denote the associated oriented graph as defined in (2.1). The interacting particle system defined by the lattice $(\Lambda, \mathcal{E})$, local state space $S := \{0, 1\}$, and local transition kernel K_p is called the *contact process*.

Using notation as in (1.4), we can write

$$K_p = pP_{\texttt{inf}} + (1-p)P_{\texttt{rec}},$$

where $\texttt{inf} : S^2 \to S^2$ and $\texttt{rec} : S^2 \to S^2$ are the maps defined as

$$\begin{aligned} \texttt{inf}\big(x(1), x(2)\big) &:= \big(x(1) \vee x(x), x(2)\big), \\ \texttt{rec}\big(x(1), x(2)\big) &:= \big(0, x(2)\big). \end{aligned} \tag{2.6}$$

The map `inf` has the effect that if $x(2) = 1$ and $x(1) = 0$, then $x(1)$ becomes 1 too. The map `rec` has the effect that $x(1)$ becomes 0, regardless of $x(2)$. We interpret sites i as organisms, e.g. trees along a road if $\Lambda = \mathbb{Z}$ or in an orchard if $\Lambda = \mathbb{Z}^2$, which can be in two states, where 0 means *healthy* and 1 means *infected*. If we apply the map `inf`, then the second site *infects* the first site, and if we apply the map `rec`, then the first site *recovers*. In discrete time, the contact process $Y = (Y_n)_{n \geq 0}$ has the following description:

1. In each step $n = 1, 2, \ldots$, we choose a vertex i uniformly from Λ.

2. We then choose another site j uniformly from the set $\mathcal{N}_i$ of sites adjacent to i.

3. With probability p, the site j infects the site i, and with the remaining probability $1-p$, the site i recovers.

In the continuous-time setting, the description is as follows:

1. For each vertex $i \in \Lambda$, there is an independent Poisson point set with rate one that determines at which times the vertex becomes active.

2. If a vertex i becomes active at some time t, then we choose another site j uniformly from the set $\mathcal{N}_i$ of sites adjacent to i.

3. With probability p, we apply the map `inf` to $\big(X_{t-}(i), X_{t-}(j)\big)$, and with the remaining probability $1-p$, we apply the map `rec`.

It is not hard to see that, for some fixed $(i,j) \in \mathcal{E}$, the set of times when we apply the map `inf` to $\big(X_{t-}(i), X_{t-}(j)\big)$ forms a Poisson point set with intensity p/δ, where δ is the degree of the graph (Λ, E). Similarly, for any fixed $i \in \Lambda$, the set of times when we apply the map `rec` to $\big(X_t(i), X_t(j)\big)$ for some $j \in \mathcal{N}_i$ forms a Poisson point set with intensity $1-p$. Therefore, we can alternatively describe the continuous-time process as follows:

1. For each oriented edge $(i,j) \in \mathcal{E}$, there is an independent Poisson point set $\Pi_{(i,j)}$ with rate p/δ that determines at which times the oriented edge becomes active.

2. For each vertex $i \in \Lambda$, there is an independent Poisson point set Π_i with rate $1-p$ that determines at which times the vertex becomes active.

3. At each time $t \in \Pi_{(i,j)}$, we set $\big(X_t(i), X_t(j)\big) := \texttt{inf}\big(X_{t-}(i), X_{t-}(j)\big)$.

4. At each time $t \in \Pi_i$, we set $X_t(i) := 0$.

This alternative description is more common in the literature. Moreover, it is common to rescale time by a factor $(1-p)^{-1}$, which has the effect that the rate of the Poisson point sets Π_i becomes 1. The rate of the Poisson point sets $\Pi_{(i,j)}$ then becomes

$$\lambda := \frac{p}{\delta(1-p)}.$$

Thus, if we define

$$X'_t := X_{t/(1-p)} \qquad (t \geq 0),$$

then the the process $(X'_t)_{t\geq 0}$ has the following informal description:

1. Each oriented edge $(i, j) \in \mathcal{E}$ becomes active at the times of an independent Poisson set with intensity λ, and at such times the site j infects i.

2. Each vertex $i \in \Lambda$ becomes active at the times of an independent Poisson set with intensity 1, and at such times the site i recovers.

The process $(X'_t)_{t\geq 0}$ is called the *contact process* on the graph (Λ, E) with *infection rate* λ.

Figure 2.4: Four snapshots of a two-dimensional contact process. Initially, only a single site is infected. The infection rate is 2, the death rate is 1, and time evolved in these pictures is 1, 5, 10, and 20.

In Figure 2.4, we see the four snapshots of the time evolution of a contact process on the square lattice $\mathbb{Z}^2$. Infected sites are black and healthy sites are white. Initially, only the origin is infected. The infection rate is 2 and the death rate is 1. In this example, the infection spreads through the whole population, eventually reaching a steady state. We note that even though it

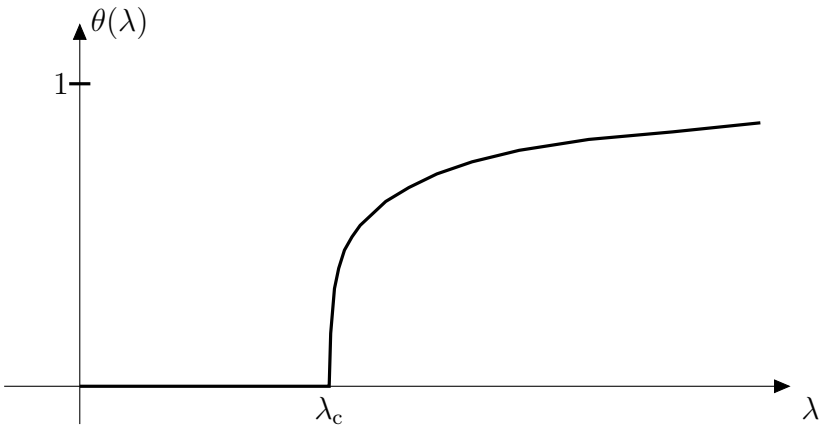

Figure 2.5: Survival probability of the one-dimensional contact process.

looks in the simulations as if we have reached an invariant law, for contact processes on finite lattices (for example a finite piece of $\mathbb{Z}^2$ with periodic boundary conditions), we know from (1.11) that the contact process must eventually end up in the all-zero configuration, which is the only trap of the Markov chain. The time we have to wait so see this happen, however, increases exponentially fast in the number of sites N and soon becomes so large that our computer is not powerful enough for us to actually see this happen. Similar to what we saw for the voter model, we use $t \to \infty$ limit of the infinite system as a mathematical idealisation for the behaviour of finite systems on the time scales that we are interested in.

Unlike the voter model, the behavior of the contact process is roughly similar in different dimensions. On the other hand, the infection rate λ is important for the behavior. Let $e_0 \in \{0,1\}^{\mathbb{Z}}$ be defined by $e_0(0) := 1$ and $e_0(i) := 0$ for all $i \neq 0$. In Figure 2.5, we have plotted the *survival probability*

$$\theta(\lambda) := \mathbb{P}^{e_0}[X_t \neq 0 \ \forall t \geq 0] \tag{2.7}$$

of the one-dimensional contact process, started in $X_0 = e_0$, i.e., with a single infected site at the origin, as a function of the infection rate λ. For reasons that we cannot explain here, this is in fact the same as the probability that the origin is infected in equilibrium.

It turns out that for the nearest-neighbor contact process on $\mathbb{Z}^d$, there exists a *critical value* $\lambda_c = \lambda_c(d)$ with $0 < \lambda_c < \infty$ such that $\theta(\lambda) = 0$ for $\lambda \leq \lambda_c$ and $\theta(\lambda) > 0$ for $\lambda > \lambda_c$. The function θ is continuous, strictly increasing and concave on $[\lambda_c, \infty)$ and satisfies $\lim_{\lambda\to\infty} \theta(\lambda) = 1$. One has

$$\lambda_c(1) = 1.6489 \pm 0.0002. \tag{2.8}$$

Proving these statements is not easy, however. For example, continuity of the function θ in the point λ_c was proved only in 1990 [BG90], seventeen years after the introduction of the model in [CS73, Har74]. The best[4] rigorous upper bound on the constant from (2.8) is $\lambda_c(1) \leq 1.942$ which is proved in [Lig95].

2.5 Ising and Potts models

In an *Ising model*, sites in the lattice $\mathbb{Z}^d$ are interpreted as atoms in a crystal, that can have two possible local states, usually denoted by -1 and $+1$. In the traditional interpretation, these states describe the direction of the magnetic field of the atom, and because of this, the local state $x(i)$ of a site i is usually called the *spin* at i. More generally, one can consider *Potts models* where each "spin" can have $q \geq 2$ possible values. In this case, the local state space is traditionally denoted as $S = \{1, \ldots, q\}$, the special case $q = 2$ corresponding to the Ising model (except for a small difference in notation between $S = \{-1, +1\}$ and $S = \{1, 2\}$).

Let $S := \{1, \ldots, q\}$. For each $z \in S$ and $(x(1), \ldots, x(\delta)) \in S^\delta$, we let

$$N\big(z \,|\, x(1), \ldots, x(\delta)\big) := \#\big\{i \in \{1, \ldots, \delta\} : x(i) = z\big\}$$

denote the number of elements of $\{x(1), \ldots, x(\delta)\}$ that are equal to z. For each $\beta \in \mathbb{R}$ and $(x(1), \ldots, x(\delta)) \in S^\delta$, we define a probability law $z \mapsto \mu\big(z \,|\, x(1), \ldots, x(\delta)\big)$ on S by

$$\mu_\beta\big(z \,|\, x(1), \ldots, x(\delta)\big) := \frac{1}{Z_{x(1),\ldots,x(\delta)}} e^{\beta N\left(z \,|\, x(1), \ldots, x(\delta)\right)}$$
$$\text{where} \quad Z_{x(1),\ldots,x(\delta)} := \sum_{z'=1}^{q} e^{\beta N\left(z' \,|\, x(1), \ldots, x(\delta)\right)}.$$

Note that this says that the probability of z is proportional to $e^{\beta N(z \,|\, x(1),\ldots,x(\delta))}$. The constant $Z_{x(1),\ldots,x(\delta)}$ is just a normalisation constant that guarantees that the total probability is one. If $\beta > 0$, then $\mu\big(z \,|\, x(1), \ldots, x(\delta)\big)$ is highest for those values of z that agree with the most common value among $x(1), \ldots, x(\delta)$. On the other hand, if $\beta < 0$, then $\mu\big(z \,|\, x(1), \ldots, x(\delta)\big)$ is highest for those values of z that occur the least frequently among $x(1), \ldots, x(\delta)$. For $\beta = 0$ we obtain the uniform distribution on S.

[4] There exists a sequence of rigorous upper bounds on the constant from (2.8) that is known to converge to the real value, but these bounds are so difficult to calculate that the best bound that has really been achieved by this method is much worse than the one in [Lig95].

Let (Λ, E) be a transitive graph with degree δ and let $(\Lambda, \mathcal{E})$ be the $(\delta+1)$-tuple graph defined in (2.3). By definition, the *q-state Potts model* with *Glauber dynamics* on the graph (Λ, E) is the interacting particle system on the $(\delta+1)$-tuple graph $(\Lambda, \mathcal{E})$ defined by the probability kernel K_β on $S^{\delta+1}$ given by

$$K_\beta\big((x(1),\ldots,x(\delta+1)),(y(1),\ldots,y(\delta+1))\big) := \begin{cases} \mu_\beta\big(y(1)\,|\,x(2),\ldots,x(\delta+1)\big) & \text{if } y(i)=x(i) \quad \forall 2\le i\le \delta+1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.9}$$

The description of the corresponding continuous-time process is as follows:

1. Each vertex $i \in \Lambda$ becomes active at the times of independent Poisson point set with rate one.
2. If the vertex i becomes active at time t, then we let $j_1,\ldots,j_d$ be the elements of $\mathcal{N}_i$, ordered in a random way.
3. We now give $X_t(i)$ a new value that is chosen according to the probability law $z \mapsto \mu_\beta\big(z \,\big|\, X_t(j_1),\ldots,X_t(j_d)\big)$.

In discrete time, the description is the same, except that in each time step we choose i uniformly from Λ. For $\beta > 0$, we speak of a *ferromagnetic* Potts model and for $\beta < 0$ of an *antiferromagnetic* Potts model.

In Figure 2.6 we see four snapshots of a two-dimensional nearest-neighbor ferromagnetic Potts model with $q = 4$ possible spin values. We have used periodic boundary conditions, and the value of the parameter β is 1.2. Superficially, the behavior is similar to that of a voter model, in the sense that the system forms clusters of growing size that in the end take over any finite neighborhood of the origin. Contrary to the voter model, however, even in the middle of large cluster that is predominantly of one color, sites can still flip to other values so in the simulations we see many small islands of different colors inside large clusters where one color dominates. Another difference is clustering actually happens only when the value of the parameter β is large enough. For small values of β, the behavior is roughly similar to the voter model in dimensions $d \ge 3$. There is a critical value $0 < \beta_c < \infty$ where the model changes from one type of behavior to the other type of behavior.

To make the critical value visible, let us start the Ising model with state space $S = \{-1,+1\}$ in an initial state such that $X_0(i) = +1$ for all $i \in \mathbb{Z}^2$. Then it can be shown that $X_t(i)$ has a limiting distribution as $t \to \infty$. We let

$$m_*(\beta) := \lim_{t\to\infty} \mathbb{E}[X_t(i)] = \mathbb{P}[X_t(i)=1] - \mathbb{P}[X_t(i)=-1] \tag{2.10}$$

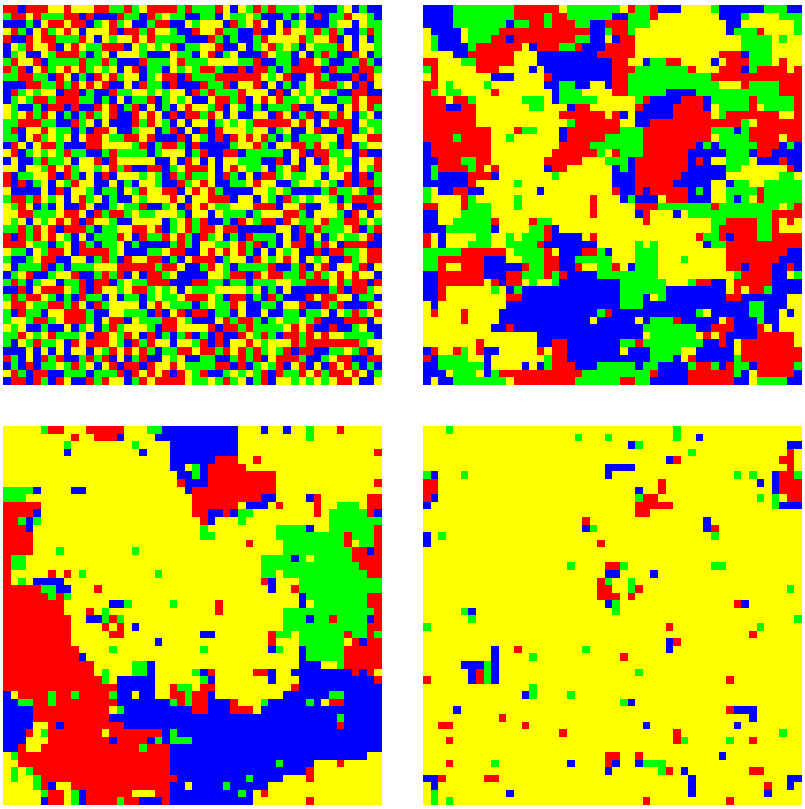

Figure 2.6: Four snapshots of a $q = 4$, $\beta = 1.2$ Potts model with Glauber dynamics and periodic boundary conditions. Initially, the types of sites are i.i.d. Time evolved in these pictures is 0, 4, 32, 500.

denote the average value of $X_t(i)$ as $t \to \infty$. This function is called the *spontaneous magnetization*. For the Ising model in two dimensions, the spontaneous magnetization can be explicitly calculated, as was first done my Onsager [Ons44]. The formula is

$$m_*(\beta) = \begin{cases} \left(1 - \sinh(\beta)^{-4}\right)^{1/8} & \text{for } \beta \geq \beta_c := \log(1+\sqrt{2}), \\ 0 & \text{for } \beta \leq \beta_c. \end{cases} \tag{2.11}$$

This function is plotted in Figure 2.7. In this case, the critical point β_c is known explicitly.

For ferromagnetic Ising models in dimensions $d \geq 3$, the graph of $m_*(\beta)$ looks roughly similar to Figure 2.7, with $\beta_c \approx 0.442$ [GPA01], but no explicit formulas are known. Ferromagnetic Potts models in dimensions $d \geq 2$ behave

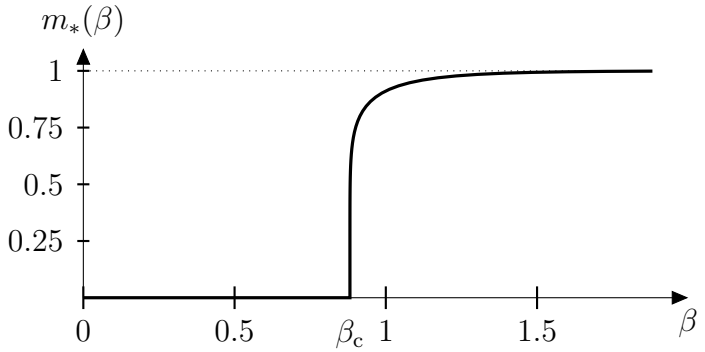

Figure 2.7: The spontaneous magnetization of the two-dimensional Ising model.

similarly. For the process started with $X_0(i) = 1$ for all $i \in \mathbb{Z}^d$, the limit

$$m'(\beta) := \lim_{t\to\infty} \mathbb{P}[X_t(i) = 1] \tag{2.12}$$

exists and there exists a $\beta_c(d, q)$ which depends on the dimension d and the number of colours q such that $m'(\beta) > 1/q$ for $\beta > \beta_c(d, q)$ and $m'(\beta) = 1/q$ for $\beta < \beta_c(d, q)$.

In dimension one, one has $m^*(\beta) = 0$ for *all* $\beta \geq 0$. More generally, one-dimensional Potts models do not show long range order, even if β is very large.[5] In Figure 2.8, we compare the time evolution of a one-dimensional Potts model (with a large value of β) with the time evolution of a one-dimensional voter model. In the voter model, the cluster size keeps growing, but in the Potts model, the typical cluster size converges to a finite limit.

The behaviour of antiferromagnetic Potts models is more complicated than that of the ferromagnetic models. In this case, sites like to have a colour different from (rather than the same as) their neighbors. The antiferromagnetic Ising model on $\mathbb{Z}^d$ can be transformed into a ferromagnetic Ising model by a simple trick. Let us call a vertex $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ *even* or *odd* depending on whether $\sum_{k=1}^d i_k$ is even or odd. Then, if $(X_t)_{t\geq 0}$ is an antiferromagnetic Ising model with Glauber dynamics and we set

$$X'_t(i) := \begin{cases} X_t(i) & \text{if } i \text{ is even,} \\ -X_t(i) & \text{if } i \text{ is odd,} \end{cases}$$

[5]This was first noticed by Ising [Isi25], who introduced the model but noticed that it was uninteresting, incorrectly assuming that what he had proved in dimension 1 would probably hold in any dimension. Peierls [Pei36] realized that dimension matters and proved that the Ising model in higher dimensions does show long range order.

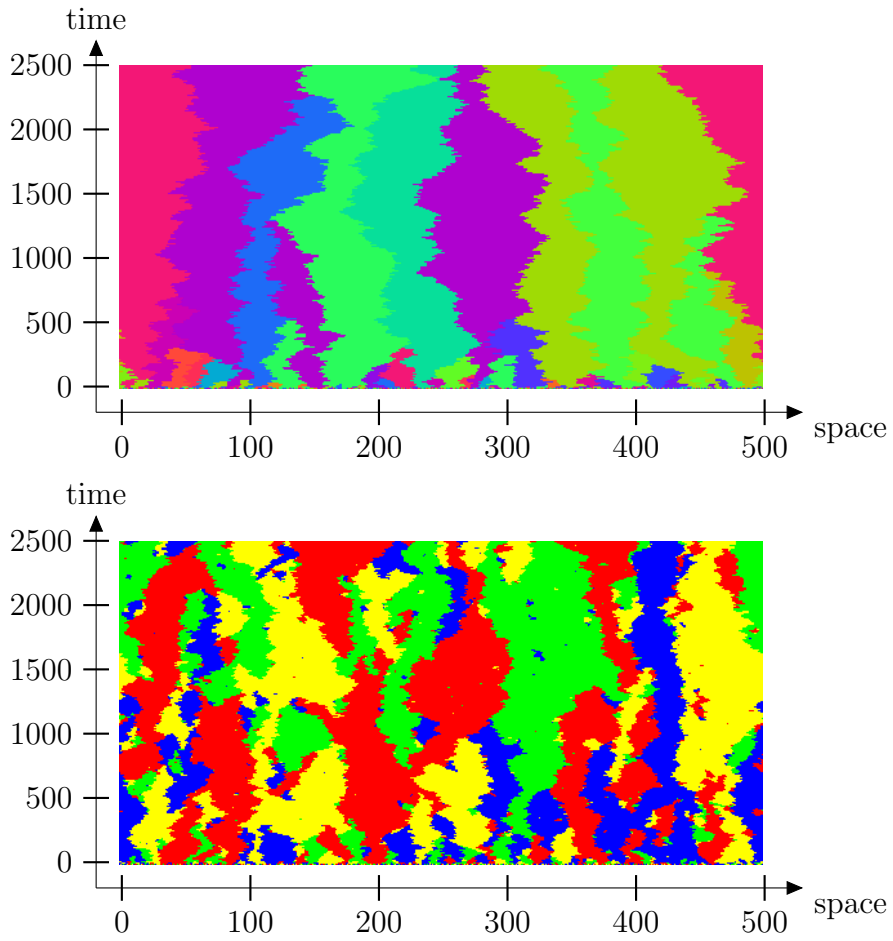

Figure 2.8: Time evolution of a one-dimensional voter model (above) and a one-dimensional Potts model (below) with a high value of β.

then $(X_t)_{t\geq 0}$ is a ferromagnetic Ising model with Glauber dynamics. This trick works only for the Ising model and only for *bipartite* lattices, that can nicely be divided into an even and odd sublattice, such as $\mathbb{Z}^d$, but not on the triangular lattice, for example.

Ising and Potts models are *reversible* (as defined in (1.12)). If the graph (Λ, E) is finite, then we can even give a formula for the invariant law. Let us write

$$H(x) := \#\big\{\{i,j\} \in E : x(i) = x(j)\big\} \qquad (x \in S^\Lambda).$$

Then setting

$$\mu_\beta(x) := \frac{1}{Z_\beta} e^{\beta H(x)} \quad \text{with} \quad Z_\beta := \sum_{y \in S^\Lambda} e^{\beta H(y)} \tag{2.13}$$

defines a probability law on S^Λ that satisfies the detailed balance equation (1.12)). Although in principle, this gives an explicit formula for the invariant law, in practice, if Λ is large, it is very difficult to calculate Z_β, which involves a sum over all possible configurations $y \in S^\Lambda$.

The probability law in (2.13) is called a *Gibbs measure* (for our particular choice of the energy function H). More generally, any interacting particle system that has has the probability laws in (2.13) as its reversible invariant law is called a *stochastic Potts model* (or *stochastic Ising model*, if $q = 2$). The particular choice for the dynamics we have described is called *Glauber dynamics*, but there are other choices possible. For the ferromagnetic Ising model, we can replace the probability kernel in (2.9) by setting

$$\begin{aligned}
&\tilde{K}_\beta\big((x(1), x(2), \ldots, x(\delta+1)), (-x(1), x(2), \ldots, x(\delta+1))\big) \\
&\qquad := e^{-\beta \#\{i \in \{2, \ldots, \delta+1\} : x(1) = x(i)\}}, \\
&\tilde{K}_\beta\big((x(1), x(2), \ldots, x(\delta+1)), (x(1), x(2), \ldots, x(\delta+1))\big) \\
&\qquad := 1 - e^{-\beta \#\{i \in \{2, \ldots, \delta+1\} : x(1) = x(i)\}},
\end{aligned} \tag{2.14}$$

and $\tilde{K}_\beta(x, y) := 0$ in all other cases. The description of the corresponding continuous-time process is as follows:

1. Each vertex $i \in \Lambda$ becomes active at the times of independent Poisson point set with rate one.

2. If the vertex i becomes active at time t, then we flip (i.e., change) the value of $X_t(i)$ with probability $e^{-\beta n}$, where $n := \#\{j \in \mathcal{N}_i : X_t(j) = X_t(i)\}$ is the number of neighbors of i that are in the same state as i.

This is called *Metropolis dynamics*. One can check that the probability law in (2.13) is a reversible law for this process. Nevertheless, as an interacting particle system, it is essentially different from the Ising model with Glauber dynamics. Even though the two processes behave roughly in a similar way and even have the same critical point, we cannot relate the two processes by a simple rescaling of time.

If the lattice (Λ, E) is finite, then the Potts model with Glauber dynamics is an irreducible Markov chain and hence (2.13) is its unique invariant law.

By symmetry, it is clear that under the law in (2.13), the law if each site is the uniform distribution on $S = \{1, \dots, q\}$. This means that on finite systems, the system started in $X_0(i) = 1$ for all $i \in \Lambda$ satisfies

$$\lim_{t\to\infty} \mathbb{P}[X_t(i) = 1] = 1/q \qquad (i \in \Lambda),$$

for all values of β. When $\beta > \beta_c$, however, the time we have to wait before we see this happens grows exponentially fast in $N := \#\Lambda$, so that even for systems with moderate number of sites like $N = 100$, we are no longer able to see this happen. Similar to what we had for the voter model and contact process, the limits $t \to \infty$ and $N \to \infty$ cannot be interchanged and we use the infinite system as a mathematical idealisation for the bevahiour of finite systems on the time scales we are interested in.

2.6 Phase transitions

Figures 2.5 and 2.7 are examples of a phenomenon that is often observed in interacting particle systems. As a parameter governing the dynamics is crosses a particular value, the system goes through an abrupt change in behavior. This is called a *phase transition* and the value of the parameter is called the *point of the phase transition* or, in the mathematical literature, *critical point*. As we will see in a moment, in the physics literature, the term critical point has a more restricted meaning. The term "phase transition" of course also describes the behavior that certain materials change from a gas, fluid, or solid phase into another phase at a particular value of the temperature, pressure etc., and from the theoretical physicist's point of view, this is indeed the same phenomenon.

In both Figure 2.5 and 2.7, the point of the phase transition in fact separates two regimes, one where the interacting particle systems (on the infinite lattice) has a unique invariant law (below λ_c and β_c) and another regime where there are more invariant laws (above λ_c and β_c). Indeed, for the contact process, the probability law that gives probability one to the empty configuration is always an invariant law, but above λ_c, a second, nontrivial invariant also appears. Potts models have q invariant laws (one corresponding to each color) above the critical point. Multiple invariant laws are a general phenomenon associated with phase transitions.

Phase transitions are classified into *first order* and *second order* phase transitions.[6] Second order phase transitions are also called *continuous* phase

[6]This terminology was introduced by Paul Ehrenfest. The idea is that in first order phase transitions, the first derivative of the free energy has a discontinuity, while in a

transitions. The phase transitions in Figures 2.5 and 2.7 are both second order, since the functions θ and m_* are continuous at the critical points λ_c and β_c, respectively. Also, second order phase transitions are characterized by the fact that at the critical point, there is only one invariant law. By contrast, if we would draw the function $m_*(\beta)$ of a Potts model for sufficiently large values of q (in dimension two, for $q > 4$), then the plot of m_* would make a jump at β_c and the system would have multiple invariant laws at this point, which means that this phase transition is first order.

It can be difficult to prove whether a given phase transition is first or second order. While for the two-dimensional Ising model, continuity of the magnetization follows from Onsager's solution [Ons44], the analogous statement for the three-dimensional Ising model was only proved recently [ADS15] (70 years after Onsager!).

For positive functions f and g, we write $f(z) \propto g(z)$ as $z \to z_0$ if there exists a constant $c > 0$ such that $\lim_{z\to z_0} f(z)/g(z) = c$. We write $z \downarrow z_0$ or $z \uparrow z_0$ if such formulas hold only when we approach z_0 from the right or left, respectively. For the Ising model, it is known (but only partially proved) that

$$m_*(\beta) \propto (\beta - \beta_c)^c \quad \text{as } \beta \downarrow \beta_c,$$

where c is a *critical exponent*, which is given by

$$c = 1/8 \text{ in dim } 2, \quad c \approx 0.326 \text{ in dim } 3, \quad \text{and} \quad c = 1/2 \text{ in dim} \geq 4.$$

For the contact process, one observes that

$$\theta(\lambda) \propto (\lambda - \lambda_c)^c \qquad \text{as } \lambda \downarrow \lambda_c,$$

with a critical exponent

$$c \approx 0.276 \text{ in dim } 1, \quad c \approx 0.583 \text{ in dim } 2,$$
$$c \approx 0.813 \text{ in dim } 3, \quad \text{and} \quad c = 1 \text{ in dim} \geq 4.$$

In theoretical physics, (nonrigorous) *renormalization group theory* is used to explain these critical exponents and calculate them. According to this theory, critical exponents are *universal*. For example, the nearest-neighbor model and the range R models with different values of R all have different values of the critical point, but the critical exponent c has the same value

second order phase transitions, the first derivative of the free energy is continuous and only the second derivative makes a jump.

for all these models.[7] Also, changing from the square lattice to, e.g., the triangular lattice has no effect on c.

Critical exponents are associated only with second order phase transitions. At the critical point of a second order phase transition, one observes *critical behavior*, which involves, e.g., power-law decay of correlations. For this reason, physicists use the term "critical point" only for second order phase transitions.

So far, there is no mathematical theory that can explain critical behavior, except in high dimensions (where one uses a technique called the *lace expansion*) and in a few two-dimensional models that can either be solved explicitly (like the two-dimensional Ising model) or that have a scaling limit that is conformally invariant and can be described with the help of the *Schramm-Loewner Equation* [Law05].

2.7 Moving particles

The voter model, the contact process, and the Ising model with Gauber dynamics are three classical interacting particle systems that have been studied in such detail that it would easily be possible to write a book solely devoted to any one of them. Apart from these classical models, lots of other interacting particle systems have been introduced ans studied in the literature to model a phlectora of phenomena. Some of these behave very similarly to the models we have already seen (and even appear to have the same critical exponents), while others are completely different. In this section, we discuss a couple of models whose common feature is that they involve particles that move around through the lattice while interacting with each other.

We define a *coalescing random walk map* $\mathtt{rw} : \{0,1\}^2 \to \{0,1\}^2$ by

$$\mathtt{rw}\big(x(1), x(2)\big) := \big(0, x(1) \vee x(2)\big),$$

and we let $P_{\mathtt{rw}}$ denote the associated probability kernel on $\{0,1\}$ as defined in (1.4). Let (Λ, E) be a transitive graph and let $(\Lambda, \mathcal{E})$ denote the associated oriented graph as defined in (2.1). The interacting particle system defined by the lattice $(\Lambda, \mathcal{E})$, local state space $S := \{0,1\}$, and local transition kernel $P_{\mathtt{rw}}$ has the following description:

1. If $X_t(i) = 0$ or $= 1$, then we say that the site i is empty, or occupied by a particle, respectively, at time t.

[7] Universality in the range R does not always hold. It has been proved that the $q = 3$ ferromagnetic Potts model in dimension two has a first order phase transition for large R [GB07], while the model with $R = 1$ is known to have a second order phase transition [DST17].

2. Each vertex $i \in \Lambda$ becomes active at the times of independent Poisson point set with rate one.

3. If an occupied site i becomes active, the particle on it jumps to neighboring site, chosen uniformly from $\mathcal{N}_i$.

4. If a particle lands on a site that is already occupied, then both particles *coalesce*, i.e., they merge into one particle.

Note that contrary to the particle systems we have seen before, it may happen that the local states of two sites change at the same time. (Indeed, this happens when a particle jumps to an empty site, vacating the site here it was before.) Interacting particle systems in which only one site changes its value at any time are called *spin-flip systems*.

We can combine the map `rw` with other maps we have already seen, for example the infection map `inf` and the recovery map `rec` defined in (2.6). For example, if we use the kernel

$$pP_{\texttt{inf}} + (1-p)P_{\texttt{rw}},$$

then we obtain a particle system where step 3 above is replaced by:

3'. If an occupied site i becomes active, then with probability p, the particle produces offspring on a neighboring site, chosen uniformly from $\mathcal{N}_i$, and with the remaining probability $1-p$, the particle jumps to a neighboring site, chosen uniformly from $\mathcal{N}_i$.

Two more maps of interest are the *annihilating random walk map* and the *exclusion map* defined as

$$\begin{aligned} \texttt{ann}\big(x(1),x(2)\big) &:= \big(0, x(1)+x(2) \bmod(2)\big),\\ \texttt{excl}\big(x(1),x(2)\big) &:= \big(x(2),x(1)\big). \end{aligned}$$

The map `ann` is similar to `rw` except that when two particles land on the same site, the *annihilate* each other, i.e., both particles disappear, see Figure 2.9. The interacting particle system defined by the kernel $P_{\texttt{excl}}$ is called the (symmetric) *exclusion process*. We can interpret it as follows:

3". If a site i becomes active, then we choose a neighboring site j uniformly from $\mathcal{N}_i$. If there is a particle at i, then it jumps to j, and if there is a particle at j, then it jumps to i.

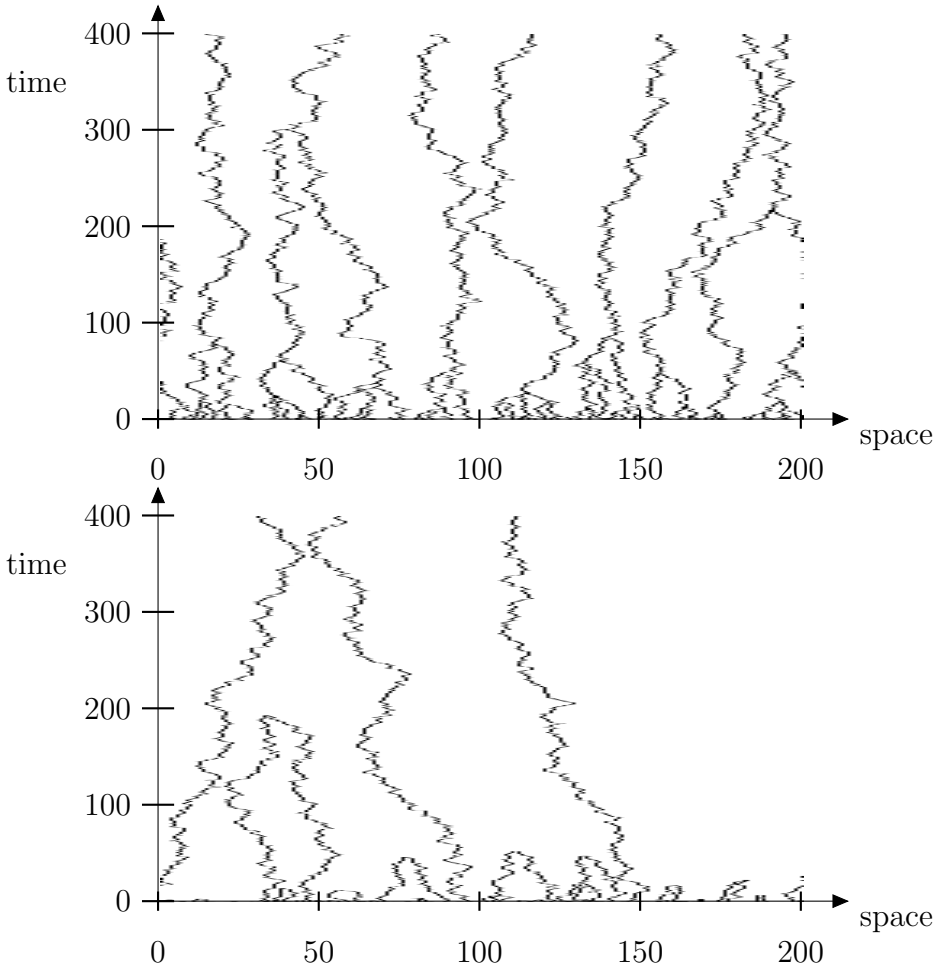

Figure 2.9: Systems of coalescing random walks (above) and annihilating random walks (below).

In particular, if i and j are both occupied, then the particles exchange positions. In this way, we never have to solve the situation when a particles lands on an already occupied position.

Another interesting map is the *cooperative branching map*

$$\texttt{coop}\big(x(1),x(2),x(3)\big) := \big(x(1) \vee (x(2) \wedge x(3)), x(2), x(3)\big),$$

which differs from the infection map defined in (2.6) in the sense that it takes two parents to produce offspring. Motivated by biological populations, several authors [Nob92, Dur92, Neu94, SS15a] have studied interacting particle systems with such a reproduction mechanism. Note that to construct an

interacting particle based on this map, we need a 3-tuple graph. Starting from an unoriented graph (Λ, E), we can define such a 3-tuple graph $(\Lambda, \mathcal{E})$, for example, by setting

$$\mathcal{E} := \big\{(i,j,k) \in \Lambda^3 : \{i,j\} \in E,\ \{j,k\} \in E,\ i \neq k\big\}.$$

See Figure 2.10 for a one-dimensional interacting particle system involving cooperative branching and coalescing random walks. In this example, when a site i becomes occupied, we choose a neighbor j and a next nearest neighbor k, uniformly from all possibilities. With probability p, we apply the map `coop`, which has the effect that if j and k are both occupied, then i becomes occupied too, and with the remaining probability $1-p$ we apply the map `rw`, which has the effect that the particle at i, if there is one, jumps to j and coalesces with the particle there, if there is one.

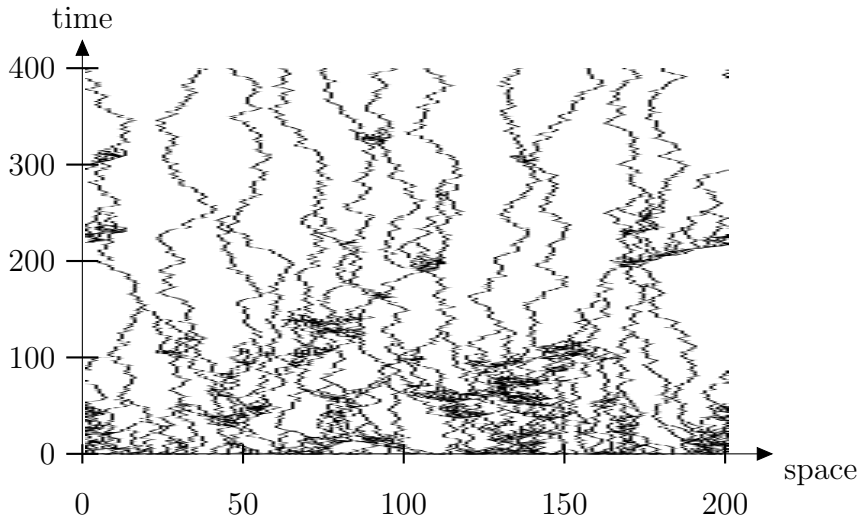

Figure 2.10: A one-dimensional interacting particle system with cooperative branching and coalescing random walk dynamics.

We define a *killing map* by

$$\begin{aligned} \texttt{kill}\big(1,1\big) &:= \big(1,0\big), \\ \texttt{kill}\big(x(1),x(2)\big) &:= \big(x(1),x(2)\big) \quad \text{in all other cases.} \end{aligned}$$

Sudbury [Sud97, Sud99] has studied a "biased annihilating branching process" that combines this map with the infection map `inf` from (2.6). When a site i becomes active, we choose a neighboring site j uniformly from j. With

probability p we apply the map `inf` and with probability $1-p$ we apply the map. In other words: if i is occupied while j is empty, then with probability p we place a new particle on j, and if i and j are both occupied, then with probability $1-p$ the particle at i kills the particle at j. Figure 2.11 shows a simulation of such a system when $p = 1/6$.

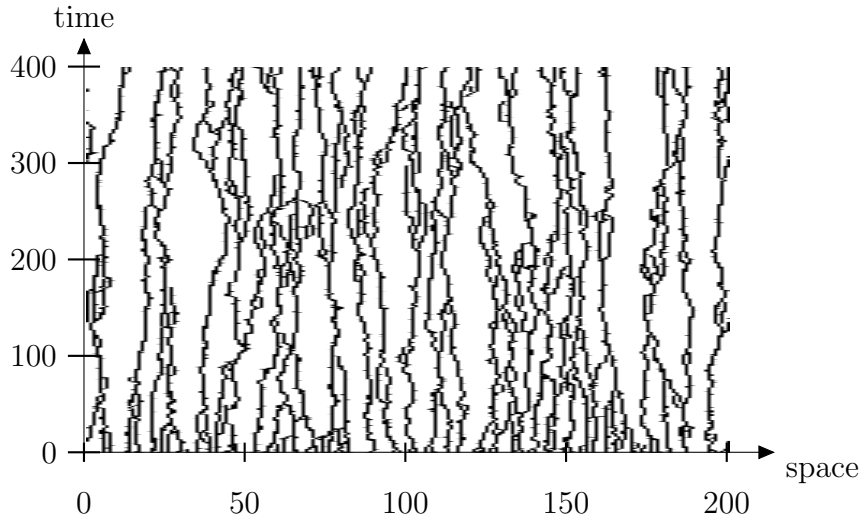

Figure 2.11: A system with branching and killing.

Although many interacting particle systems studied in the literature have only two possible local states (usually denoted by 0 and 1), this is not always so. For example, in [Kro99], a *two-stage contact process* is introduced. Here, the local state space is $\{0, 1, 2\}$ where 0 represents an empty site, 1 a young organism, and 2 an adult organism. The behavior of this model is similar to that of the contact process.

Chapter 3

Periodic Behaviour

3.1 Introduction

Usually, in the study of interacting particle systems, one is interested in invariant laws. There is often a big difference between the invariant laws of finite and infinite systems. A finite Ising model with Glauber dynamics is an irreducible Markov chain and hence has a unique invariant law. But the Ising model with Glauber dynamics on $\mathbb{Z}^d$ in dimensions $d \geq 2$ has at least two different invariant laws, which are the long-time limit laws started from the constant minus and plus initial configurations. Similarly, a finite voter model or a finite contact process eventually ends up in a trap and hence their invariant laws are trivial. But on $\mathbb{Z}^d$, voter models in dimensions $d \geq 3$ and contact processes with sufficiently large infection rates have additional, nontrivial invariant laws that are not present in the finite model.

A *periodic law* with *period* $r > 0$ of an interesting particle system is a probability law with the property that if the initial state X_0 is distributed according to this law, then so is X_r (and hence also $X_{2r}, X_{3r}, \ldots$), but at all times $0 < t < r$, the process X_t has a different distribution. Because of Theorem 1.4, in the continuous-time setting, interacting particle systems on finite lattices cannot have such periodic laws. Tom Mountford [Mou95] has shown that interacting particle systems on $\mathbb{Z}$ cannot have periodic laws. On the other hand, Jahnel and Külske [JK14] have constructed an interacting particle system on $\mathbb{Z}^3$ that has a periodic law. Strictly speaking, their construction does not fall into the class of interacting particle systems we are considering here, since the kernel they use for updating the state at a site uses information from infinitely many other sites. However, in a certain precise sense, the information they use from faraway sites is very small and decays exponentially fast as a function of the distance. For these reasons, their proof

makes it likely that there should also exist interacting particle systems of the type we are considering that have periodic laws. As we will see in the coming two sections, numerical simulations support this hypothesis.

The construction of Jahnel and Külske [JK14] works only in dimensions $d \geq 3$. It is not known if there exist interacting particle systems on $\mathbb{Z}^2$ that have periodic laws.

3.2 A mutually metastable Ising model

The model we are about to describe is loosely based on a model discussed in the paper [CFT16]. Let (Λ, E) be a transitive graph. We will be interested in an interating particle system on the lattice (Λ, E) whose local state space is $S = \{-1, +1\}^2$. We denote an element of S^Λ as (x, y) where $x = (x(i))_{i\in\Lambda} \in \{-1, +1\}^\Lambda$ and $y = (y(i))_{i\in\Lambda} \in \{-1, +1\}^\Lambda$. For $x \in \{0, 1\}^\Lambda$ and $i \in \Lambda$, we let

$$F_i(x) := \frac{\#\big\{j \in \mathcal{N}_i : x(j) = x(i)\big\}}{\#\mathcal{N}_i}.$$

denote the fraction of neighbors of i where x has the same value as in i. We also write

$$1_{\{x(i)=y(i)\}} := \begin{cases} 1 & \text{if } x(i) = x(j), \\ 0 & \text{otherwise,} \end{cases}$$

and we define $1_{\{x(i)\neq y(i)\}}$ similarly. The dynamics that we are about to define are similar to the Metropolis dynamics for the Ising model (compare formula (2.14)). There are two parameters $\alpha, \beta \geq 0$. The rules are as follows:

1. Each vertex $i \in \Lambda$ becomes active at the times of independent Poisson point set with rate one.

2. If the vertex i becomes active at time t, then we choose with equal probabilities whether we update $X_t(i)$ or $Y_t(i)$.

3. If we decide to update $X_t(i)$, then we flip its value with probability $e^{-\beta F_i(X_t) - \alpha 1_{\{X_t(i)=Y_t(i)\}}}$ and otherwise do nothing.

4. If we decide to update $Y_t(i)$, then we flip its value with probability $e^{-\beta F_i(Y_t) - \alpha 1_{\{X_t(i)\neq Y_t(i)\}}}$ and otherwise do nothing.

Rule 3 says that $X_t(i)$ is less likely to change its value if it has the same value as $X_t(j)$ for a large fraction of its neighbors $j \in \mathcal{N}_i$. The strength of this effect is governed by the parameter β. Rule 3 also says that $X_t(i)$ is less likely to change its value if it has the same value as $Y_t(j)$, and the strength

of this effect is governed by the parameter α. The rules for $Y_t(i)$ are almost the same, except that $Y_t(i)$ prefers to be *different* from $X_t(i)$.

Numerical simulations suggest that for suitable values of α and β, on the lattice $\mathbb{Z}^3$, this interacting particle system has a periodic law in which the majority of sites periodically cycle through the states

$$(+1,+1) \mapsto (+1,-1) \mapsto (-1,-1) \mapsto (-1,+1) \mapsto (+1,+1) \mapsto \cdots . \quad (3.1)$$

Heuristically, we can understand this as follows. If β is sufficiently small, then the Ising model with this parameter has long-range order. This means that if it were not for the interaction with Y (in particular, for $\alpha = 0$), then the system X has two invariant laws, one where the majority of sites has the value $+1$, and another where the majority of sites has the value -1. The same is true for Y. Now if we give α a small positive value, and both X and Y are distributed according to the invariant law where the majority of sites has the value $+1$, then X will be "perfectly satisfied", because it also likes to have the same value as Y, but Y will not be perfectly satisfied, since it likes to be different from X. As a result, Y will after some time change to the invariant law where the majority of sites has the value -1, but this makes X unhappy, and so on.

What is not so clear from this heuristic argument is what the role of the dimension is. So far, from the simulations, it seems that while on $\mathbb{Z}^3$ there is a periodic law, on $\mathbb{Z}^2$ there is none. On $\mathbb{Z}^2$, locally, it is stil true that most of the sites, for most of the time, cycle through the four possible states as in (3.1), but contary to what we see on $\mathbb{Z}^3$, they do not coordinate this behavior over large distances. This in spite of the fact that the Ising model on $\mathbb{Z}^2$ does exhibit long-range order when β is sufficiently large. It is also not clear for exactly which values of α and β there exists a periodic law on $\mathbb{Z}^3$. From the heuristic argument above, it would seem good to take β as large as possible and α as small as possible, but there are quite convincing arguments that indicate that actually, even on $\mathbb{Z}^3$, the sites cannot coordinate their behavior if β is too large and α is too small.

3.3 A dissipative Ising model

The interacting particle system we are about to describe was communicated to me by Paolo Dai Pra and Marco Formentin and is based on a model treated in the PhD thesis of Daniele Tovazzi [Tov19]. The system can be defined on any transitive graph (Λ, E). The local state space is again $S = \{-1,+1\}^2$ and we use the same notation for elements (x,y) of S^Λ as in the previous

section. For $x \in \{0,1\}^\Lambda$ and $i \in \Lambda$, we let

$$F_i^*(x) := \frac{\#\big\{j \in \mathcal{N}_i : x(j) = x(i),\ y(j) = +1\big\}}{\#\mathcal{N}_i}.$$

denote the fraction of neighbors of i where x has the same value as in i and moreover $y(j) = +1$. There are two parameters: $0 \le p \le 1$ and $\beta \ge 0$. The rules are as follows:

1. Each vertex $i \in \Lambda$ becomes active at the times of independent Poisson point set with rate one.

2. If the vertex i becomes active at time t, then with probability p we choose to "decay" and with the remaining probability $1 - p$ we choose to "update".

3. If we decide to decay, then we set $Y_t(i)$ to -1, regardless of its prior value.

4. If we decide to update, then with probability $e^{-\beta F_i^*(X_t)}$ we flip the value of $X_t(i)$ and set $Y_t(i)$ to $+1$. Otherwise, we do nothing.

We can view this as a model for a country ruled by two political parties, represented by the values ± 1. If $X_t(i) = -1$ or $+1$, then we interpret this as saying that the district i is ruled by a governor from the party -1 or $+1$, respectively. The value of $Y_t(i)$ indicates whether the local population is satisfied with the current governor, where $+1$ means satisfied and -1 means unsatisfied. When a site i becomes active, two things can happen: either people become unsatisfied (an irreversible process) or elections are held. In case of elections, people tend to vote for their old governor, even if they are unsatisfied, as long as a large proportion of the neighboring districts are satisfied with a governor from this party.

Again, simulations suggest that for a suitable choice of the parameters p and β, there are periodic laws on $\mathbb{Z}^3$ but not on $\mathbb{Z}^2$.

Bibliography

[ADS15] M. Aizenman, H. Duminil-Copin, and V. Sidoravicius. Random currents and continuity of Ising model's spontaneous magnetization. *Commun. Math. Phys.* 334 (2015), 719–742.

[AS04] S.R. Athreya and J.M. Swart. Branching-coalescing particle systems. *Prob. Theory Relat. Fields* 131(3) (2005), 376–414.

[AS09] S.R. Athreya and J.M. Swart. Branching-coalescing particle systems. Erratum. *Prob. Theory Relat. Fields.* 145(3-4) (2009), 639–640.

[AS12] S.R. Athreya and J.M. Swart. Systems of branching, annihilating, and coalescing particles. *Electron. J. Probab.* 17 (2012), paper No. 80, 1–32.

[BD01] W.A. Brock and S.N. Durlauf. Discrete Choice with Social Interactions. *The Review of Economic Studies* 68(2) (2001), 235–260.

[BG90] C. Bezuidenhout and G. Grimmett. The critical contact process dies out. *Ann. Probab.* 18(4) (1990), 1462–1482.

[CFT16] F. Collet, M. Formentin, and D. Tovazzi. Rhythmic behavior in a two-population mean-field Ising model. *Phys. Rev. E* 94 (2016), 042139.

[CP14] J.T. Cox and E.A. Perkins. A complete convergence theorem for voter model perturbations. *Ann. Appl. Probab.* 24(1) (2014), 150–197.

[CS73] P. Clifford and A. Sudbury. A model for spatial conflict. *Biometrika* 60(3) (1973), 581–588.

[DLZ14] R. Durrett, T. Liggett, and Y. Zhang. The contact process with fast voting. *Electron. J. Probab.* 19 (2014), Paper No. 28, 1–19.

[DN08] R.W.R. Darling and J.R. Norris. Differential equation approximations for Markov chains. *Probab. Surveys* 5 (2008), 37–79.

[DST17] H. Duminil-Copin, V. Sidoravicius, and V. Tassion. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. *Commun. Math. Phys.* 349(1) (2017), 47–107.

[Dur88] R. Durrett. *Lecture notes on particle systems and percolation.* Wadsworth & Brooks/Cole, Pacific Grove, 1988.

[Dur91] R. Durrett. A new method for proving the existence of phase transitions. *Spatial Stochastic Processes, Festschr. in Honor of Ted Harris 70th Birthday* (K.S. Alexander and J.C. Watkins eds.) 141–169. Prog. Probab. 19, Birkhäuser, Boston, 1991.

[Dur92] R. Durrett. Stochastic growth models: Bounds on critical values. *J. Appl. Probab.* 29(1) (1992), 11–20.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov Processes: Characterization and Convergence.* John Wiley & Sons, New York, 1986.

[FM01] J.A. Fill and M. Machida. Stochastic monotonicity and realizable monotonicity. *Ann. Probab.* 29(2), (2001), 938–978.

[GB07] T. Gobron and I. Merola. First-order phase transition in Potts models with finite-range interactions. *J. Stat. Phys.* 126(3) (2007), 507–583.

[GPA01] M. Ghaemi, G.A. Parsafar, and M. Ashrafizaadeh. Calculation of the critical temperature for 2- and 3-dimensional Ising models and for 2-dimensional Potts models using the transfer matrix method. *J. Phys. Chem. B* 105(42) (2001), 10355–10359.

[Har74] T.E. Harris. Contact interactions on a lattice. *Ann. Probab.* 2 (1974), 969–988.

[Har76] T.E. Harris. On a class of set-valued Markov processes. *Ann. Probab.* 4 (1976), 175–194.

[Isi25] E. Ising. Beitrag zur Theorie des Ferromagnetismus. *Zeitschrift für Physik* 31 (1925), 253–258.

[JK14] B. Jahnel and C. Külske. A class of nonergodic interacting particle systems with unique invariant measure. *Ann. Appl. Probab.* 24(6) (2014), 2595–2643.

[Kal97] O. Kallenberg. *Foundations of modern probability.* Springer, New York, 1997.

[KKO77] T. Kamae, U. Krengel, and G.L. O'Brien. Stochastic inequalities on partially ordered spaces. *Ann. Probab.* 5 (1977), 899–912.

[Kro99] S. Krone. The two-stage contact process. *Ann. Appl. Probab.* 9(2) (1999), 331–351.

[Law05] G.F. Lawler. *Conformally Invariant Processes in the Plane.* Mathematical Surveys and Monographs 114. American Mathematical Society, Providence, 2005.

[Lig85] T.M. Liggett. *Interacting Particle Systems.* Springer-Verlag, New York, 1985.

[Lig95] T.M. Liggett. Improved upper bounds for the contact process critical value. *Ann. Probab.* 23 (1995), 697–723.

[Lig99] T.M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes.* Springer-Verlag, Berlin, 1999.

[Lig10] T.M. Liggett. *Continuous time Markov processes. An introduction.* AMS, Providence, 2010.

[LSS97] T.M. Liggett, R.H. Schonmann and A.M. Stacey. Domination by product measures. *Ann. Probab.* 25(1) (1997), 71–95.

[Loe63] M. Loève. *Probability Theory, 3rd edition.* Van Nostrand, Princeton, 1963.

[Mou95] T.S. Mountford. A coupling of infinite particle systems. *J. Math. Kyoto Univ.* 35(1) (1995), 43–52.

[Neu94] C. Neuhauser. A long range sexual reproduction process. *Stochastic Processes Appl.* 53(2) (1994), 193–220.

[Nob92] C. Noble. Equilibrium behavior of the sexual reproduction process with rapid diffusion. *Ann. Probab.* 20(2) (1992), 724–745.

[NP99] C. Neuhauser and S.W. Pacala. An explicitly spatial version of the Lotka-Volterra model with interspecific competition. *Ann. Appl. Probab.* 9(4) (1999), 1226–1259.

[Ons44] L. Onsager. Crystal statistics, I. A two-dimensional model with an order-disorder transition. *Phys. Rev.* 65 (1944), 117–149.

[Pei36] R. Peierls. On Ising's model of ferromagnetism. *Proc. Camebridge Philos. Soc.* 36 (1936), 477-481.

[Pre74] C.J. Preston A generalization of the FKG inequalities. *Comm. Math. Phys.* 36(3) (1974), 233-241.

[SL95] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems. *Ann. Probab.* 23(4) (1995), 1816–1830.

[SL97] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. IV: Quasi-duality and thinnings of interacting particle systems *Ann. Probab.* 25(1) (1997), 96–114.

[SS08] A. Sturm and J.M. Swart. Voter models with heterozygosity selection. *Ann. Appl. Probab.* 18(1) (2008), 59–99.

[SS15a] A. Sturm and J.M. Swart. A particle system with cooperative branching and coalescence. *Ann. Appl. Probab.* 25(3) (2015), 1616–1649.

[SS15b] A. Sturm and J.M. Swart. Monotone and additive Markov process duality. Preprint, 2015, ArXiv:1510.06284.

[Sud97] A. Sudbury. The convergence of the biased annihilating branching process and the double-flipping process in $\mathbb{Z}^d$. *Stoch. Proc. Appl.* 68 (1997), 25–264.

[Sud99] A. Sudbury. Hunting submartingales in the jumping voter model and the biased annihilating branching process. *Adv. Appl. Probab.* 31(3) (1999), 839–854.

[Sud00] A. Sudbury. Dual families of interacting particle systems on graphs. *J. Theor. Probab.* 13(3) (2000), 695–716.

[Swa13a] J.M. Swart. Noninvadability implies noncoexistence for a class of cancellative systems. *Electron. Commun. Probab.* 18 (2013), paper No. 38, 1–12.

[Swa13b] J.M. Swart. Duality and Intertwining of Markov Chains. Lecture notes for a course at the ALEA in Europe School, October 21–25 (2013) at CIRM, Luminy (Marseille). Available at `http://staff.utia.cas.cz/swart/posters_index.html`.

[Tov19] D. Tovazzi. *Self-sustained periodic behaviors in interacting systems: macroscopic limits and fluctuations.* PhD Thesis University of Padua, 2019.