FINANCIAL ECONOMICS

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Homework 1

1. (a) Expand the model as

$$\log x_t = a_0 w_t + a_1 w_{t-1} + \sum_{j=2}^{\infty} a_j w_{t-j} \tag{1}$$

After imposing the restriction, this can be rewritten as

$$\log x_t = a_0 w_t + a_1 w_{t-1} + \varphi \sum_{j=2}^{\infty} a_{j-1} w_{t-j} = a_0 w_t + a_1 w_{t-1} + \varphi \sum_{j=1}^{\infty} a_j w_{t-1-j} \tag{2}$$

Note that

$$\log x_{t-1} = \sum_{j=0}^{\infty} a_j w_{t-1-j} = a_0 w_{t-1} + \sum_{j=1}^{\infty} a_j w_{t-1-j} \tag{3}$$

$$\implies \varphi(\log x_{t-1} - a_0 w_{t-1}) = \varphi \sum_{j=1}^{\infty} a_j w_{t-1-j} \tag{4}$$

Putting (4) into (2), we obtain

$$\log x_t = a_0 w_t + a_1 w_{t-1} + \varphi(\log x_{t-1} - a_0 w_{t-1}) \tag{5}$$

$$\implies \log x_t = \varphi \log x_{t-1} + a_0 w_t + (a_1 - \varphi a_0) w_{t-1} \tag{6}$$

(6) can be written in the desired formulation as

$$\log x_t = \varphi \log x_{t-1} + \sigma w_t + \theta \sigma w_{t-1} \tag{7}$$

For (7) to be equivalent to our original formulation, the relationships between a_0, a_1, φ, θ, and σ must be

$$\sigma = a_0 \tag{8}$$

$$\theta\sigma = (a_1 - \varphi a_0) \tag{9}$$

$$\implies \theta = \frac{a_1 - \varphi a_0}{a_0} = \frac{a_1}{a_0} - \varphi \tag{10}$$

(b)
- Rewrite (7) as

$$\log x_{t+1} = \varphi \log x_t + \sigma w_{t+1} + \theta\sigma w_t \tag{11}$$

Then, the conditional mean and variance are

$$E_t[\log x_{t+1}] = E_t[\varphi \log x_t + \sigma w_{t+1} + \theta\sigma w_t] = \varphi \log x_t + \theta\sigma w_t \tag{12}$$
$$V_t(\log x_{t+1}) = V_t(\varphi \log x_t + \sigma w_{t+1} + \theta\sigma w_t) = V_t(\sigma w_{t+1}) = \sigma^2 \tag{13}$$

- From our expression for $E_t[\log(x_{t+1})]$ in (12), we can find $E_{t-1}[\log(x_t)]$ and write

$$\varphi z_{t-1} + \lambda w_t = \varphi(\varphi \log x_{t-1} + \theta\sigma w_{t-1}) + (\varphi + \theta)\sigma w_t \tag{14}$$
$$= \varphi(\varphi \log x_{t-1} + \sigma w_t + \theta\sigma w_{t-1}) + \theta\sigma w_t = \varphi \log x_t + \theta\sigma w_t = z_t \tag{15}$$

Thus, we have verified that z_t follows an AR(1) process. Then, we can directly rewrite the ARMA(1,1) model in (11) as

$$\log x_{t+1} = z_t + \sigma w_{t+1} \tag{16}$$

- Note that

$$\log E[e^{s \log x_{t+1}}|z_t] = \log(E[e^{sz_t + s\sigma w_{t+1}}|z_t]) = \log(e^{sz_t} E[e^{s\sigma w_{t+1}}]) \tag{17}$$
$$= sz_t + \log E[e^{s\sigma w_{t+1}}] = sz_t + \frac{s^2\sigma^2}{2} \tag{18}$$

(18) is linear in z_t.

- Since $k_t(s; \log x_{t+2}) = \log E_t[E_{t+1}[e^{s \log x_{t+2}}]]$, let's first simplify $E_{t+1}[e^{s \log x_{t+2}}]$ as

$$E_{t+1}[e^{s \log x_{t+2}}] = e^{sz_{t+1}} E[e^{s\sigma w_{t+2}}] \tag{19}$$

Then, $k_t(s; \log x_{t+2})$ can be written as

$$\log E_t[e^{sz_{t+1}} E[e^{s\sigma w_{t+2}}]] = \log(E[e^{s\sigma w_{t+2}}] E_t[e^{sz_{t+1}}]) \tag{20}$$
$$= \log E[e^{s\sigma w_{t+2}}] + \log E_t[e^{sz_{t+1}}] \tag{21}$$
$$= \log E[e^{s\sigma w_{t+2}}] + \log E[e^{s(\varphi z_t + \lambda w_{t+1})}|z_t] \tag{22}$$
$$= \log E[e^{s\sigma w_{t+2}}] + s\varphi z_t + \log E_t[e^{s\lambda w_{t+1}}] \tag{23}$$
$$= \frac{s^2\sigma^2}{2} + s\varphi z_t + \frac{s^2\lambda^2}{2} = \varphi s z_t + \frac{s^2}{2}(\sigma^2 + \lambda^2) \tag{24}$$

In (24), we simply use the distributional properties of w_t to compute the expectation from its moment-generating-function.

- From equation (24), we can verify that $A_2 = \varphi A_1$ and $B_2 = \sigma^2 + \lambda^2$, where $A_1 = 1$ and $B_1 = \sigma^2$. Given the recursive relations $A_n = \varphi A_{n-1}$ and $B_n =$

$B_{n-1} + A_{n-1}^2\lambda^2$, let's write out the first few iterations by hand. These give

$$A_3 = \varphi^2,\ B_3 = \sigma^2 + \lambda^2 + \varphi^2\lambda^2 \tag{25}$$

$$A_4 = \varphi^3,\ B_4 = \sigma^2 + \lambda^2 + \varphi^2\lambda^2 + \varphi^4\lambda^2 \tag{26}$$

$$A_5 = \varphi^4,\ B_5 = \sigma^2 + \lambda^2 + \varphi^2\lambda^2 + \varphi^4\lambda^2 + \varphi^6\lambda^2 \tag{27}$$

Therefore, the general formulation can be captured through

$$A_n = \varphi^{n-1},\ B_n = \sigma^2 + \lambda^2 \sum_{i=0}^{n-2} (\varphi^2)^i = \sigma^2 + \lambda^2 \frac{1-(\varphi^2)^{n-1}}{1-\varphi^2} \tag{28}$$

(c)

- The ARMA(1,1) model can be exponentiated on both sides to result in

$$x_{t+1} = e^{\varphi \log x_t + \sigma w_{t+1} + \theta\sigma w_t} = e^{z_t} e^{\sigma w_{t+1}} \tag{29}$$

$$\implies \log E_t[x_{t+1}] = \log(e^{z_t} E_t[e^{\sigma w_{t+1}}]) \tag{30}$$

Notice that (30) is the same as (17) with $s = 1$. Moreover, as suggested by the hint, $\log E_t[x_{t+1}^\gamma] = \log E_t[e^{\gamma \log x_{t+1}}] = k_t(\gamma; \log x_{t+1})$. Combining these insights, we obtain

$$\pi_t = \log E_t[x_{t+1}] - \frac{1}{\gamma} \log E_t[x_{t+1}^\gamma] = k_t(1; \log x_{t+1}) - \frac{1}{\gamma} k_t(\gamma; \log x_{t+1}) \tag{31}$$

Based on the explicit solution in (18), this can be rewritten as

$$z_t + \frac{\sigma^2}{2} - \frac{1}{\gamma}(\gamma z_t + \frac{\gamma^2\sigma^2}{2}) = \frac{\sigma^2(1-\gamma)}{2} \tag{32}$$

This expression is independent of z_t.

- Note that

$$\pi_t^{(n)} = \log E_t[x_{t+n}] - \frac{1}{\gamma} \log E_t[x_{t+n}^\gamma] \tag{33}$$

$$= \log E_t[e^{\log x_{t+n}}] - \frac{1}{\gamma} \log E_t[e^{\gamma \log x_{t+n}}] = k_t(1; \log x_{t+n}) - \frac{1}{\gamma} k_t(\gamma; \log x_{t+n}) \tag{34}$$

Since $k_t(s; \log x_{t+n}) = A_n z_t s + B_n s^2/2$ and we derived expressions for A_n and B_n in (28), $\pi_t^{(n)}$ can be simplified as

$$\pi_t^{(n)} = A_n z_t + \frac{B_n}{2} - \frac{1}{\gamma}(A_n z_t \gamma + \frac{B_n \gamma^2}{2}) = \frac{B_n(1-\gamma)}{2} \tag{35}$$

$$= (\sigma^2 + \lambda^2 \frac{1-(\varphi^2)^{n-1}}{1-\varphi^2})(\frac{1-\gamma}{2}) \tag{36}$$

This expression is also independent of z_t. In addition, as $n \to \infty$,

$$(\sigma^2 + \lambda^2 \frac{1-(\varphi^2)^{n-1}}{1-\varphi^2})(\frac{1-\gamma}{2}) \to (\sigma^2 + \frac{\lambda^2}{1-\varphi^2})(\frac{1-\gamma}{2}) \tag{37}$$

2. In code.