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# NEAR-OPTIMAL CODED APERTURES FOR IMAGING VIA NAZAROV'S THEOREM

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## ABSTRACT

We characterize the fundamental limits of coded aperture imaging systems up to universal constants by drawing upon a theorem of Nazarov regarding Fourier transforms. Our work is performed under a simple propagation and sensor model model that accounts for thermal and shot noise, scene correlation, and exposure time. By focusing on linear mean square error as a measure of reconstruction quality, we show that appropriate application of a theorem of Nazarov leads to essentially optimal coded apertures, up to a constant multiplicative factor in exposure time. Our theoretical results are accompanied with a heuristically efficient algorithm to generate such patterns that explicitly takes into account scene correlations. Finally, for i.i.d scenes, we show improvements upon prior work by using spectrally flat sequences with bias. We focus on 1D apertures for conceptual clarity, but also discuss how all of our work here generalizes naturally to 2D.

***Index Terms—*** coded aperture cameras, computational photography, optical signal processing, Fourier analysis

## 1. INTRODUCTION

Certain modern imaging systems, especially those operating at high frequencies, use coded apertures. In these systems, a spatial mask that selectively blocks light from reaching the sensor is used as opposed to a traditional lens. The scene is then recovered by suitable post-processing. Perhaps the earliest and simplest instance of coded aperture imaging is the pinhole structure, see e.g [1] for a survey. The development of X-ray and gamma-ray astronomy gave rise to more sophisticated coded apertures [2, 3] to get around the lack of lenses and mirrors in such settings. Both proposed using random on-off occluders with a specified mean transmittance as a method to increase the aperture size as compared to the classical pinhole while retaining its resolution benefits.

More modern developments include the usage of uniformly redundant arrays (URA) to improve upon random on-off patterns [4], anti-pinhole imaging [5], as well as the combining of mask and lens in order to, e.g., facilitate depth estimation [6], deblur out-of-focus elements in an image [7], enable motion deblurring [8], and/or recover 4D lightfields [9]. Even more recent work seeks to forgo lenses altogether to decrease costs and meet physical constraints [10, 11]. Understanding coded apertures is also relevant in non-line-of-sight applications where masks naturally occur as scene occlusions [12, 13].

In light of the increased importance of coded apertures, prior work [14] set down a model to assist with understanding them. The proposed model used far-field geometric optics to model light propagation and a sensor model that includes thermal and shot noise components. Under this and mutual information (MI) as a performance metric, [14] compared the classical random on-off apertures [2, 3] of varying intensity to the "spectrally flat" patterns with transmissivity $1/2$ (same as the URA of [4]). Among other things, the analysis showed that when shot noise dominates thermal noise, randomly generated masks with lower transmissivity than $1/2$ offered greater performance compared to spectrally flat patterns.

This paper extends [14] in multiple avenues that may be broadly grouped into the following three main contributions.

First, we refine the model of [14] by incorporating exposure time. Here, we analyze linear minimum mean square error (LMMSE) as opposed to MI to draw tighter conclusions about an "operational" quantity, though we emphasize that our conclusions carry over to the setup of [14]; see Sec. 4.

Second, we remark upon the existence and construction of spectrally flat sequences with transmissivities $1/8, 1/4$ in addition to $1/2$. This extends the range of parameters where we have a sharp characterization of optimal coded apertures in our framework, and gives a tight answer to the problem of optimal coded apertures for i.i.d scenes; see Prop. 2, 3 for precise statements.

Third, we provide tight (up to a universal constant) coded apertures, both in 1D as well as in 2D, applicable for any prior on the spectrum of the scene at hand. The sense of tightness is given precisely in Prop. 4. This includes but is not limited to the naturally occuring power law [15]($f^{-\gamma}$-prior). Our aperture design naturally varies depending on the choice of prior, and we provide a (heuristically) efficient greedy algorithm for their generation. Mathematically, essentially all the content here comes from a beautiful theorem of Nazarov [16, pg 5] combined with classical waterfilling for power control. We note that [17, pg 9-11] has identified other applied problems for which Nazarov's theorem provides conceptual clarity and/or solutions.

## 2. MODEL

We shall first describe what we change and augment from the model of [14] and why.

We use a Poisson model, which is standard for photon counting. Photon counts naturally depend on exposure time, so we incorporate a parameter $t$ to describe it. For Poisson models analytical study of MI is difficult, and even with mean square error (MSE) it is often unclear how to achieve optimal MSE in practice. As such, the standard estimation process is linear; indeed, the work of [3, 2] used correlation decoders. In fact, [3] gives a beautiful analog realization of such a decoder. LMMSE addresses these issues. We note that if one used a Gaussian model instead, LMMSE is the same as MSE, and MSE is in turn essentially equivalent to MI in the low SNR limit [18, 19]. LMMSE depends purely on first and second moments, so in our mathematical study we do not emphasize the specific Poisson statistics.

We shall use a 1D model, as in [14], to keep the concepts and results simple to state and understand. We emphasize that all of the

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results of this paper generalize naturally to the analogous 2D model, and elaborate upon that in Sec. 4.

Let $\mathbf{f}$ denote the intensities of the unknown 1D scene of length $n$ of expected total power $J$. Let $\mathbb{E}[\mathbf{f}] = (J/n)\mathbf{1}, \mathbf{Cov}[\mathbf{f}] = \mathbf{Q}$. We assume $\mathbf{Q}$ is circulant and diagonalized as $\mathbf{Q} = \mathbf{F}_n^* \mathbf{D} \mathbf{F}_n$; $\mathbf{F}_n$ is the unitary DFT matrix and $\mathbf{D} = \mathrm{diag}(\mathbf{d})$. The measurements at the imaging plane are denoted $\mathbf{y}_j, j \in [n]$ and the $n \times n$ transfer matrix $\mathbf{A}$ model the aperture. We assume its entries all satisfy $0 \leq \mathbf{A}_{ji} \leq 1/n$ to model that the light can't be redirected, and $\sum_j \mathbf{A}_{ji} \leq 1$ to model local conservation of power. An ideal, perfectly focused, lens may be treated in this setup by $\mathbf{A} = \mathbf{I}$, as it redirects light perfectly.

We assume $\mathbf{A}_{ji} = (1/n)\mathbf{a}_{i-j \pmod n}$ for a $\mathbf{a} \geq 0$, i.e $\mathbf{A}$ is circulant. Let $\rho(\mathbf{a}) = (1/n)\sum_i \mathbf{a}_i$ be the *transmissivity* of the aperture. The noise component is denoted by $\mathbf{z}$ and its statistics are given by $\mathbb{E}[\mathbf{z}] = 0, \mathbf{Cov}[\mathbf{z}] = (t(W + J\rho)/n)\mathbf{I}$, where $W, J$ correspond to thermal and shot noise respectively, and $t$ is the *exposure time*. With these, our measurement model is then given by $\mathbf{y}_j = t\sum_i \mathbf{A}_{ji}\mathbf{f}_i + \mathbf{z}_j$, which leads to the following expression for the LMMSE of estimating $\mathbf{f}$ from $\mathbf{y}$:

$$m(n, t, W, J, \mathbf{d}, \mathbf{a}) = \sum_{i=0}^{n-1} \frac{1}{\frac{1}{\mathbf{d}_i} + \frac{t|\hat{\mathbf{a}}_i|^2}{n(W+J\rho(\mathbf{a}))}}. \quad (1)$$

Here, $\hat{\mathbf{a}}$ is the DFT of $\mathbf{a}$. In general, we assume $\mathbf{d}_i = (1/n)d(i/n)$ are $n$ equally spaced samples from a nonnegative, bounded, continuous function $d(x)$ on $[0, 1]$ with symmetry $d(x) = d(1 - x)$ and normalized so that $d(0) = \theta$. For example, i.i.d scenes correspond to $d(x) = \theta$. We note that our main result, Prop. 4, holds in greater generality. The above restriction on the form of $\mathbf{d}$ simply ensures correct physical scaling (invariant with respect to $n$) of the variance of total scene intensity coming from an arbitrary direction.

To get a better feel for (1), it is instructive to compare an ideal lens against a mask as a function of exposure time. An ideal lens satisfies $\mathbf{A} = \mathbf{I}$, (eqv.,$\mathbf{a} = (n, 0, \ldots, 0)$). Thus $\hat{\mathbf{a}} = (n, n, \ldots, n)$. Then from (1), it can be readily seen that for a $t$ growing with $n$ (say $t = \log(n)$), the LMMSE decays to 0 as $n \to \infty$. On the other hand, the entry-wise restriction $\mathbf{a} \in [0, 1]$ that holds for a mask results in a significant reduction in $\|\hat{\mathbf{a}}\|_2$. Due to this, in order to get an LMMSE that is bounded away from the trivial $\int d(x)dx$, one needs an exposure time that is $\Omega(n)$. Of course, this is not surprising; there are strong benefits to lenses when they are available. The need for long exposure times for coded apertures is also a known phenomenon, consistent with the emphasis of [3] on "hypothesis tests" between scenes as opposed to resolving full detail.

One way to interpret $t$ is that it reduces noise relative to the signal, changing the effective noise levels to $W/t, J/t$. All our main results established in the sequel ( eqs. (3) to (5)) show that one can construct apertures that are guaranteed to be tight within a constant factor of $t$. Under the above interpretation, what we establish rigorously is that our results are tight to within a universal constant number ($\approx$ 18.30) of dB, regardless of the scene correlation structure given by $\mathbf{d}$.

## 3. RESULTS

The goal of optimal aperture design (aka optimal $\mathbf{a}$) is to minimize the LMMSE formula subject to the scene model, denoted as follows:

$$m^*(n, t, W, J, \mathbf{d}) \triangleq \min_{\mathbf{a}} m(n, t, W, J, \mathbf{d}, \mathbf{a}).$$

Let us first understand why the minimization above is a challenging problem. Consider the even simpler problem in which the optimal transmissivity, say $\rho_0$, is given to us. Then, although $\mathbf{a} \in [0, 1], \rho(\mathbf{a}) = \rho_0$ is a convex constraint, the LMMSE (1) which we wish to minimize is neither convex nor quasiconvex in $\mathbf{a}$, since $1/(1 + cx^2)$ lacks any of these behaviors.

In order to solve this problem, our general approach is as follows. First, we use Parseval's identity that relates time and frequency space. Under a fixed power budget, it is easy to solve for the optimal power allocation $|\hat{\mathbf{a}}_i|^2$ by studying the well-behaved and convex $1/(1 + cx)$ that has a solution given by waterfilling (2). Next, we are faced with the "coefficient problem" of finding a $\mathbf{a} \in [0, 1]$ with given spectrum allocation. To address this, we appropriately apply a theorem of Nazarov [16, pg 5]. An exposition of Nazarov's work together with the context he draws from (e.g. the geometric ideas of [20], along with the analytic ideas of [21]) may be found in [22].

### 3.1. Lower bound

We first derive a lower bound for LMMSE (1) based on waterfilling (see e.g [23, Thm 19.7]). For notational ease, we let $\gamma = t/(n(W + J\rho))$ throughout.

**Proposition 1.** *Let* $\mathbf{a}$ *satisfy* $\rho(\mathbf{a}) = \rho$. *Then:*

$$m(n, t, W, J, \mathbf{d}, \mathbf{a}) \geq \frac{1}{\frac{n}{\theta} + \gamma n^2\rho^2} + \sum_{i=1}^{n-1} \frac{1}{\frac{1}{\mathbf{d}_i} + \gamma P_i}. \quad (2)$$

*Here* $P_i = (1/\gamma)(T - 1/d_i)^+$ *and total power* $P = \sum_{i=1}^{n-1} P_i = n(\lfloor n\rho\rfloor + (n\rho - \lfloor n\rho\rfloor)^2) - n^2\rho^2$. *Also note* $P \leq n^2\rho(1 - \rho)$. *We remark that* (2) *is sharp iff* $|\hat{\mathbf{a}}_i|^2 = P_i$ *for* $0 < i < n$.

*Proof.* We have $\hat{\mathbf{a}}_0 = n\rho$, giving the first term. For the nonzero frequencies, we use the fact that the maximum of $\sum_i x_i^2$ subjected to $x_i \in [0, 1]$ and $\sum x_i = r$ is $\lfloor r\rfloor + (r - \lfloor r\rfloor)^2$. This, together with Parseval's identity, yields an upper bound on the power of the nonzero frequencies. Waterfilling, modified to study $\frac{-1}{1+ax}$ as opposed to $\log(1 + ax)$, then gives the proposition. The floors are removed to get the $P$ upper bound by $x^2 \leq x$ for $0 \leq x \leq 1$. ☐

Note that minimizing the right hand side over $\rho$ gives a *lower bound* on $m^*(n, t, W, J, \mathbf{d})$. This task is trivial numerically, but in general difficult analytically. We denote this optimal $\rho$ by $\rho^*$ henceforth.

### 3.2. Upper bound

Our goal here has been set from (2). Conceptually, the design issue is finding a $\mathbf{a} \in [0, 1]$ with prescribed lower bounds $|\hat{\mathbf{a}}_i|^2 \geq P_i$. In general, this is impossible to do, and thus our lower bound (2) is not sharp in all settings. However, it should be noted that sharp cases do exist. Perhaps the conceptually simplest example is the analog of (2) for a lens, where our bound is sharp.

Our general approach is to simply step back by a factor $C$ and obtain a $\mathbf{a} \in [0, 1]$ with $|\hat{\mathbf{a}}_i|^2 \geq P_i/C$. What we do next is address how we can guarantee such a $C$. We shall move from simpler to more complex situations, and accordingly start off with i.i.d scenes where for infinitely many $n$ one does not need the full generality of Nazarov's solution.

#### *3.2.1. i.i.d scenes*

Recalling that $d(x) = \theta$ is constant, the waterfilling is trivial and asks for equal power allocation. As such, the goal is to have a 0, 1 sequence with a flat spectrum beyond the DC term. As already noted in [4, 14], one can certainly construct such spectrally flat sequences for infinitely many values of $n$, as long as they are unbiased with $\rho = 1/2 + o(1)$. This meets the lower bound (as $n \to \infty$) as long as the optimal $\rho^*$ is $1/2 + o(1)$ for the given $t, W, J$. A natural question is how good is using an "unbiased" spectrally flat sequence when $\rho^* \neq 1/2$? The answer is given in the following:

**Proposition 2.** *Let $\theta, W, J$ be fixed and let $\mathbf{d} = (\theta/n)\mathbf{1}$. Then for infinitely many $n$, there exists a $\mathbf{a} \in [0,1]$ such that:*

$$m(n, 2t, W, J, \mathbf{d}, \mathbf{a}) \leq m^*(n, t, W, J, \mathbf{d}). \tag{3}$$

In other words, "unbiased" spectrally flat sequences are always guaranteed to achieve optimal LMMSE at the expense of increasing the exposure time $t$ by at a factor of at most 2. In the sequel, we show how one can reduce this factor even further.

This is achieved by using spectrally flat sequences with $\rho = 1/8 + o(1)$ and $\rho = 1/4 + o(1)$, and allows us to refine 2 to $8/7$. The construction of these is number theoretic, and goes back quite far:$1/4$ corresponds to quartic residues [24], and $1/8$ corresponds to octic residues [25]. This allows us to refine the constant 2 to $8/7$. Unfortunately, one is not guaranteed the existence of infinitely many $n$ unlike the $1/2$ case, because we in fact don't know any single variable quadratic that takes on infinitely many primes [26].

Of perhaps greater importance is the fact that the octic residue constructions of [25] rely upon primes that come from a second order linear recurrence with rather large coefficients, arising as the solutions of Brahmagupta-Pell equations. There is thus a paucity of such constructions, indeed [25] gives only two such $n$ below $10^9$, namely $n = 73$ and $n = 26041$. On the other hand, the quartic residue constructions are reasonably numerous, with over 150 of them available below $10^7$. Even restricting ourselves to the quartic residues allows us to tighten from 2 to $4/3$. Summarizing all of the above, we have:

**Proposition 3.** *Let $\theta, W, J$ be fixed and let $\mathbf{d} = (\theta/n)\mathbf{1}$. Then for some values of $n$ that exist even beyond e.g $10^9$, there exists a $\mathbf{a} \in [0,1]$ such that:*

$$m(n, (8/7)t, W, J, \mathbf{d}, \mathbf{a}) \leq m^*(n, t, W, J, \mathbf{d}). \tag{4a}$$

*Moreover, for many ($> 150$ for $n < 10^7$) values of $n$ that exist even beyond e.g $10^9$, there exists a $\mathbf{a} \in [0,1]$ such that:*

$$m(n, (4/3)t, W, J, \mathbf{d}, \mathbf{a}) \leq m^*(n, t, W, J, \mathbf{d}). \tag{4b}$$

*Proof of Propn 2, 3.* The "difference sets" of [24, 25] are in our language spectrally flat sequences. The constant factor is given by the following single variable optimization. In view of (2), let $f_a(\rho) = (\rho(1-\rho))/(a+\rho)$ defined on $[0,1]$; $a$ corresponds to $W/J$. The numerator comes from the power bound, the denominator from the noise penalty. Then, $M(a,\rho) = (\sup_x f_a(x))/(f_a(\rho))$ is the multiplicative loss factor for a fixed $W/J$ and fixed $\rho \in \{0.125, 0.25, 0.5\}$. One may then optimize over $\rho, a$ to get the constant (4a). This proof, modified to $\rho \in \{0.25, 0.5\}$ and $\rho \in \{0.5\}$, also yields (4b) and (3) respectively. The fact that there are infinitely many $n$ for $\rho = 0.5 + o(1)$ follows from the quadratic residue construction together with the well known fact that there are infinitely many primes $p = 4k+1$ (see e.g [27, Chap 7]). □

### 3.2.2. Correlated scenes

We now turn to correlated scenes. Here the waterfilling is nontrivial, and asks for unequal power allocations. We therefore invoke Nazarov's solution to the coefficient problem [16, pg 5], and provide a statement here specialized to the DFT and $l_\infty$ that we use.

First, some notation. Let us define inner products with respect to the uniform probability distribution on $\{0, 1, \ldots, n-1\}$. Let $0 \leq i, j \leq n-1$, and let $\psi_j$ be a orthonormal basis for the DFT on real sequences. Explicitly, let $h = \lceil (n-1)/2 \rceil$. Let $\psi_0(i) = 1$, $\psi_j(i) = \sqrt{2}\cos(\omega j i)$ for $0 < j < h$, $\psi_j(i) = \sqrt{2}\sin(\omega j i)$ for $h < j < n$. If $n$ is even, let $\psi_h(i) = \cos(\omega h i)$, otherwise $\psi_h(i) = \sqrt{2}\cos(\omega h i)$. Finally, let $\beta(n) = \min_j |\psi_j|_1$.

**Theorem 1** (Nazarov)**.** *Let $M(n) = ((3\pi)/2)\beta(n)^{-2}$. Let $0 \leq p_0, p_1, \ldots, p_{n-1}$ be such that $\sum p_j = 1$. Then there exists a $\mathbf{b} \in [-M(n), M(n)]$ with $|(\mathbf{b}, \psi_j)|^2 \geq p_j$ for all $0 \leq j \leq n-1$.*

With 1 in hand, we are able to reach a far more general version of Prop. 2, 3 valid for any $n$ and any scene prior $\mathbf{d}$. Also, in Sec. 3.3 we show how to actually construct such tight sequences whose existence is guaranteed by 1.

**Proposition 4.** *For all $n, t, W, J, \mathbf{d}$, there exists a $\mathbf{a} \in [0,1]$ such that:*

$$m(n, 2M(n)^2 t, W, J, \mathbf{d}, \mathbf{a}) \leq m^*(n, t, W, J, \mathbf{d}). \tag{5}$$

*Furthermore, we have:*

$$M(n) \in [(3\pi^3)/16 + o(1), 3\pi + o(1)]. \tag{6}$$

The justification of the tightness of (5) lies in establishing (6), which we do first. The phenomenon is captured by the factorization of $n$, with the best, that is the largest, $\beta$ occurring for $n$ prime, and the worst occuring for $n$ divisible by 4. We have the following Lemma which establishes (6):

**Lemma 1.** *$\beta(n) \in \left[\frac{1}{\sqrt{2}} + o(1), \frac{2\sqrt{2}}{\pi} + o(1)\right]$ as $n \to \infty$. Moreover, if we restrict to $n$ being prime, $\beta(n) = \frac{2\sqrt{2}}{\pi} + o(1)$.*

*Proof sketch.* We give a full proof for the $n = p$ prime case. Then, for any $j \neq 0$, $ij$ sweeps over $\{0, 1, \ldots, p-1\}$, modulo $p$. Thus, really one is looking at a Riemann sum approximation to $\int_0^1 |\cos(2\pi x)| dx = 2/\pi$. The $l_2$ norm of $\cos(2\pi x)$ on $[0,1]$ is $1/\sqrt{2}$, completing the prime case. The composite case is more involved, as it needs to take into account the divisor structure of $n$, which prevents such symmetry of the cosine vectors. Once accounted for, the natural idea is to use Euler-Maclaurin summation, with standard modifications by e.g mollifiers to take into account the lack of smoothness of $|\cos(x)|$ at its zeros. However, the mechanics are perhaps simplest in our specific setting when one uses short quadratic splines around the zeros to get a $C^1$ approximation of any desired accuracy to $|\cos(x)|$ while not changing the uniform derivative bound. We omit a full proof due to space constraints, see e.g [28] for the mechanics of how this is done in general. □

We emphasize that by Lemma 1 $M(n) \leq C$ for some universal constant $C \approx 9.4248$, with even better values available at e.g prime $n > 100$. There, $C \approx 5.8146$ suffices.

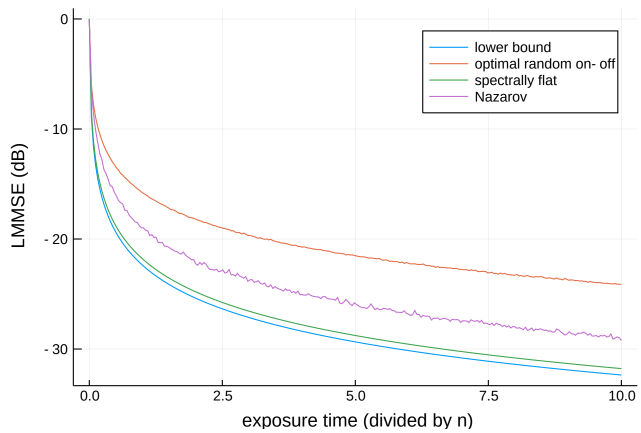
*Proof of Prop. 4.* Thm. 1, with $p_0 = 0$ and $p_j = P_j / \sum_j P_j$ for $0 < j < n$ yields a $\mathbf{b}$ with $|\mathbf{b}|_\infty \leq M(n)$ and $|(\mathbf{b}, \psi_j)|^2 \geq p_j$ for $0 < j < n$. Without loss, we may assume that $(\mathbf{b}, \psi_0) \leq 0$, else simply flip signs. Stitching the $\psi_j$ back to complex exponentials and recalling the upper bound $P \leq n^2\rho(1-\rho)$, this gives $|\hat{\mathbf{b}}_j|^2 \geq P_j/(\rho(1-\rho))$. Consider $\mathbf{a} = (\mathbf{b} + M(n))/2M(n)$. Then, $\mathbf{a} \in [0,1]$, $\rho(\mathbf{a}) \leq 0.5$, and $|\hat{\mathbf{a}}_j|^2 \geq P_j/(4M(n)^2\rho(1-\rho))$ for $0 < j < n$. We are now in a similar situation to that of Prop. 2, except with an extra $M(n)^2$ factor, and the fact that $\rho(\mathbf{a}) \leq 0.5$ instead of $\rho(\mathbf{a}) = 0.5 + o(1)$. The latter is no problem, as lower $\rho$ only helps us with the shot noise term, and the former simply multiplies the 2 of (3) by $M(n)^2$. □

### 3.3. Greedy algorithm

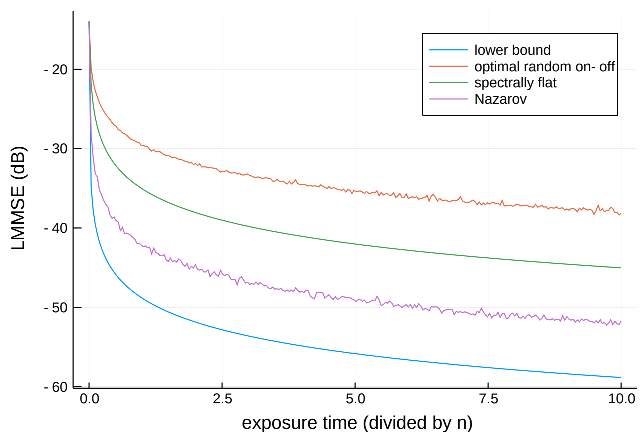
Here we propose a (heuristically) efficient algorithm to construct vectors $\mathbf{a}$ that satisfy the conditions of Prop. 4. This algorithm has its roots in Nazarov's original proof. At a high level, Nazarov's theoretical construction boils down to finding a "sign cortege" [16, pg 6] that is globally optimal for a certain real-valued Boolean function of $n$ signs, taking exponential time in the worst case. However, a closer examination of Nazarov's proof reveals that one simply needs a sign cortege that is locally optimal in the sense of Hamming geometry for the proof to work. Our observation suggests a natural greedy algorithm where one starts with a random cortege, and then flips one sign at a time if it improves the objective, repeating until no further improvement is possible. In our simulations [1] this runs very fast. For example, on our standard laptop, we can generate apertures for $n = 2000$ in 4 seconds. This resembles the situation of the simplex algorithm and the smoothed analysis of [29], or more recent work on max-cut [30]. Direct application of the methods of [30] to obtain theoretical guarantees runs into difficulties with the nonlinear change in objective with a single bit flip in our setting, unlike the linear change for max-cut. As such, we defer theoretical study of the greedy algorithm given here to future work.

### 3.4. Simulations

We give a simple illustration in Fig. 1 which confirms the following intuition based on our main results eqs. (3) to (5). With an i.i.d scene prior, one would prefer using the spectrally flat construction as opposed to the one coming from Nazarov's theorem due to the smaller constant. On the other hand, with a strong prior, e.g a bandlimited one, the waterfilling becomes highly skewed, and one would favor the one coming from Nazarov's theorem as it takes into account such strong skewing of the desired spectrum. For completeness, we also include the performance of a random on-off sequence with density $\rho$ [14], where $\rho$ is optimized over $[0, 1]$ for each $t$.




(a) i.i.d prior ($d(x) = \theta$ for $0 \leq x \leq 1/2$)




(b) bandlimited prior ($d(x) = \theta$ for $0 \leq x \leq s - r$, 0 for $x \geq s + r$, and $\theta(s + r - x)/(2r)$ otherwise for $0 \leq x \leq 1/2$)

**Fig. 1**. $n = 677, \theta = 1, W = J = 0.001, s = 0.02, r = 0.005$. We use the quartic residue construction for spectrally flat. Jaggedness of the Nazarov plot comes from the fact that in general the power control varies with $t$ and we randomly seed the sign cortege.

## 4. DISCUSSION AND FUTURE WORK

Our refined analysis of a model drawing heavily from [14] yields tight conclusions across all scene correlation patterns and noise regimes, with sharp conclusions available in some specific scenarios. Moreover, we give heuristically efficient algorithms for the generation of optimal coded apertures. We also note that similar conclusions to our main results eqs. (3) to (5) also hold for MI and Gaussian statistics of [14], simply because of the form of the expression for MI.

Furthermore, we note that our conclusions generalize naturally to 2D apertures, and in particular we have a tight characterization of optimal coded apertures in that setting. Concretely, one simply needs to take $\beta(n)^2$ as opposed to $\beta(n)$ due to the squaring of the $l_1$ lower bound for the 2D DFT. The rest of the analysis of Thm. 1 and Prop. 4 carries over naturally, with the orthogonal basis provided by products of $\psi_j$. We emphasize that this works regardless of the scene prior, even ones which are not separable. With an i.i.d prior, separable apertures are optimal up to constants as in 1D, and in fact taking a product of spectrally flat apertures yields natural analogs of Prop. 2, 3. However, with other priors, it seems like one needs the generality provided by Thm. 1. This work thus also answers the question of 2D apertures raised in [14].

As noted in [14], [9] raises the question of whether continuous-valued masks perform better than binary-valued ones. This work sheds some light on this: the solution of Nazarov which we have shown is tight does seem to use the flexibility of the $l_\infty$ norm in an essential way; see e.g [31, pg 12] for more on this. And more specifically, we have numerical evidence for finite $n$; to give a concrete example, for $n = 13$, the mask $[1, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0]$ has optimal LMMSE for an i.i.d scene over binary-valued masks for $\rho = 6/13, \theta = 0.01, W = 0.001, J = 0.001, t = 130$, but is improved upon by the continuous-valued mask whose first entry is equal to $\epsilon$ and whose $i^{\text{th}}$ entry is equal to $1 - \epsilon/6$ if $i - 1$ is a quadratic residue modulo 13, and 0 otherwise, for $0.26 \leq \epsilon \leq 0.34$.

Although Propn. 4 shows universal tightness across all priors, even "extreme" ones like bandlimited ones, the constant is worse than that for a spectrally flat construction for i.i.d scenes. The better

[1] Code available at: `https://github.com/gajjanag/apertures`

performance of spectrally flat constructions over the ones inspired by Nazarov's theorem seems to extend to other "natural" priors like the $f^{-\gamma}$ one, as the waterfilling still yields something that is nearly "flat". It might be interesting to quantify and understand the "flatness" of the waterfilling for "natural" priors.

One issue that we have not addressed here or in [14] is the equal scaling of $n$ at both sensor and scene. One natural way to address this is letting $\mathbf{A}$ be $m \times n$, or alternatively one could study a continuous model. Another issue is obtaining a good understanding of mask/lens combinations. This will require not only updates to the simple propagation model studied here and in [14], but also a refined understanding of the cost tradeoffs between lenses and apertures.

Stepping back from imaging problems, one may ask the question of where else Nazarov's theorem can be used in applied contexts, something also asked in [17]. For example, as Nazarov's theorem does not care about orthogonality, but merely a $l_2$ estimate like Parseval's theorem, one can use it for frames as well as bases, or for anything satisfying a restricted isometry property. Another example is the fact that we merely use the $l_\infty$ case of his theorem which works for all $l_p$ spaces.

## 5. REFERENCES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