M/M/1/B and M/G/1/B Queuing Systems

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Project Simulation project - Number 12

1 Objectives

In this project, we will simulate M/M/1/B and M/G/1/B queues using `Python` and will compare the blocking probability and delay observed with the analytical results. Later, we will extend this to cases where the service process follows a general distribution, and again compare it with analytical results. So our objectives in this project can be written as follows -

1. To simulate a M/M/1/B queuing system using Python.
2. To simulate a M/G/1/B queuing system using Python.
3. Derive blocking probability and delay for M/G/1/B queuing system.

In this report, we discuss the experiment design for our simulations and how we will go on calculating the important parameters of the queuing systems.

2 Derivations for M/M/1/B system

In this section, we discuss the derivation of the parameters of M/M/1/B queuing systems. We will use the Erlang-B formula to calculate delay and blocking probability for analyzing M/M/1/B queuing system. In our experiment, we have used Poisson arrivals and Poisson departures. The mean service rate is μ, mean arrival rate is μ and we also define $\rho = \frac{\lambda}{\mu}$.

2.1 Derivation of Blocking Probability

We have the steady state probabilties as -

$$\lambda p_{n-1} = \mu p_n \tag{1}$$

Using Equation (1), we can write the formulation of p_n in terms of p_0 which is as follows -

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0 \tag{2}$$

where $\rho = \frac{\lambda}{\mu}$. We sum up p_n to 1 and then get the formula for p_0.

$$\Sigma_{n=0}^{B} \rho^n p_0 = 1$$
$$p_0 = \frac{1-\rho}{1-\rho^{B+1}} \tag{3}$$

Thus blocking probability will be p_B and is given as -

$$p_B = \frac{\rho^B(1-\rho)}{1-\rho^{B+1}} \tag{4}$$

2.2 Derivation of Delay

We then calculate N and found that -

$$\begin{aligned} N &= \Sigma_{n=0}^{B} n p_n \\ &= p_0(\Sigma_{n=0}^{B} n\rho^n) \\ &= p_0\rho(\Sigma_{n=0}^{B} n\rho^{n-1}) \\ &= p_0\rho\frac{d}{d\rho}\left(\Sigma_{n=0}^{B}\rho^n\right) \\ &= \frac{p_0}{\rho}\left[\frac{1+(B-1)\rho^B - B\rho^{B-1}}{(1-\rho)^2}\right] \end{aligned} \tag{5}$$

Using Little's law, we have -

$$W = T - \frac{1}{\mu} \tag{6}$$

We can calculate T as -

$$\begin{aligned} N &= \lambda T \\ T &= \frac{1}{\lambda}N \\ T &= \frac{p_0}{\mu}\left[\frac{1+(B-1)\rho^B - B\rho^{B-1}}{(1-\rho)^2}\right] \end{aligned} \tag{7}$$

Therefore, W becomes -

$$W = \frac{p_0}{\mu}\left[\frac{1+(B-1)\rho^B - B\rho^{B-1}}{(1-\rho)^2}\right] - \frac{1}{\mu} \tag{8}$$

3 Derivations for M/G/1/B system

For M/G/1/B, we will using Pollaczek–Khinchine formula to derive the parameters. In our experiments, we will keep the distribution of departure to be Log-Normal distribution and Uniform distribution. Note that both of them are memoryless. We make the following assumptions as well -

1. If X_i is the time taken to serve i^{th} packet, then $(X_1, X_2, \cdots)$ is mutually independent and identically distributed. It is also independent of the inter-arrival time.
2. We also have $\bar{X} = E\{X_i\} = \frac{1}{\mu}$ which is basically the average servive time. μ is the capacity of the server or it serves at a rate of μ packets per second.
3. We know second moment of service time $\bar{X^2} = E\{X^2\}$.

3.1 Derivation of Blocking Probability

In the lecture, we read that for a M/G/1/B system also, we get the same blocking probability as of M/M/1/B. Erlang B formula applies to the M/G/1/B system. The reason is that it is dependent on the mean of the distribution of service times. So, the blocking probability will be given as -

$$P_B = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}} \tag{9}$$

3.2 Derivation of Delay

We start with some definitions. First is R_i which is the residual time of i^{th} entering packet. It is defined as the time left for j^{th} packet in the system to finish getting served. Second is N_i which is the number of packets in the queue when i^{th} packet arrives. W_i is the wait time of i^{th} packet. Now, W_i will be given as -

$$\begin{aligned} W_i &= (R_i + \sum_{j=1-N_i}^{i-1} X_j) \text{ if } N_i < B \\ &= 0 \text{ otherwise} \end{aligned} \tag{10}$$

So if we compare the two systems, we can write it as -

$$\frac{1}{S}\sum_{i=1}^{S} W_i <= \frac{1}{S}\sum_{i=1}^{S} W_i^{\phi} \tag{11}$$

where W^{ϕ} denotes the average wait time for a $M/G/1/\infty$ system, and S is the number of time stamps where the packet is entering the queue. This is because in M/G/1/B whenever $N_i \geq B$, that packet will not be able to enter the system. For those packets that enters when $N_i = B$, then it will get dropped. We can then define $W_i = \mathbb{I}(N_i < K)\ W_i^{\phi}$ where $\mathbb{I}$ is the indicator function. We can say the same argument for $\sum_{j=1-N_i}^{i-1} X_j$. So in total, we can get an upper bound as -

$$W <= W^{\phi} \tag{12}$$

Another way of seeing this is as follows. When a packet i enters a queue, the probability that it sees N_i is $(1-P_B)$, where P_B is the blocking probability. Thus, equation 10 can be written as -

$$W_i = (1 - P_B) \times (R_i + \sum_{j=1-N_i}^{i-1} X_j) \tag{13}$$

When i tends to ∞, we can write it as -

$$\begin{aligned} W &= (1 - P_B) \times (R + E\{X_j\}E\{N_i\}) \\ &= (1 - P_B) \times (R + \frac{1}{\mu} N_Q) \end{aligned}$$

where R is average residual time and W is average waiting time. Now using the derivation of R using graphical argument, we can write -

$$R = \frac{\lambda}{2} E\{X^2\} \tag{14}$$

Putting this and on simplification, we get -

$$W = \frac{1}{2}\left(\frac{\lambda E\{X\}^2 P_C}{1 - \rho P_C}\right) \tag{15}$$

where $P_C = 1 - P_B$.

4 Mean errors

The mean errors are reported in this report. This is for blocking probabilities.

Rho=0.2	Rho=0.4	Rho=0.6	Rho=0.8	Rho=1
0.95398172	.101425329	0.102053123	0.095893107	0.087861702

Table 1: Mean errors for different values of ρ. This is for Blocking probabilities. The errors are in %.

The mean errors are reported in this report. This is for waiting times.

Rho=0.2	Rho=0.4	Rho=0.6	Rho=0.8	Rho=1
1.00422884	9.17179384	11.682895	51.2527896	199.625527

Table 2: Mean errors for different values of ρ. This is for delay. The errors are in %.

5 Work distribution

The work distribution is given in table 3. Please note that the work is distributed equally among the team members.

Work distribution	
Aditya	Gajraj
Simulation for M/G/1/B	Simulation for M/M/1/B
Derive the parameters for M/G/1/B	

Table 3: Work distribution

6 References

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4. The art of computer systems performance analysis, R. Jain, 2013, Wiley India.
5. James Cohen's lectures.