M/M/1/B and M/G/1/B Queues

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Introduction

Problem Description

In this project, we will simulate an M/M/1/B queue using `Python` and will compare the blocking probability and delay observed with the analytical results. Later, we will extend this to cases where the service process follows a general distribution, and again compare it with analytical results. So our objectives in this project can be written as follows -

1. To simulate a M/M/1/B queuing system using Python.
2. To simulate a M/G/1/B queuing system using Python.
3. Derive blocking probability and delay for M/G/1/B queuing system.

In this report, we discuss the experiment design for our simulations and how we will go on calculating the important parameters of the queuing systems.

Parameters of M/M/1/B queue

The analytical results of blocking probability (P_B) and delay (T) for M/M/1/B queues are given as follows -

1. Blocking Probability -

$$P_B = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}} \tag{1}$$

2. Delay -

$$W = \frac{p_0}{\mu}\left[\frac{1+(B-1)\rho^B - B\rho^{B-1}}{(1-\rho)^2}\right] - \frac{1}{\mu} \tag{2}$$

We have used the Erlang-B formula to calculate delay and blocking probability for analyzing M/M/1/B queuing system. Derivation can be found at this Link.

Parameters of M/G/1/B queue

The analytical results of blocking probability (P_B) and delay (T) for M/G/1/B queues are given as follows -

1. Blocking Probability -

$$P_B = \frac{(1-\rho)\rho^B}{1-\rho^{B+1}} \tag{3}$$

2. Delay -

$$W = \frac{1}{2}\left(\frac{\lambda E\{X\}^2 P_C}{1-\rho P_C}\right) \leq \frac{p_0}{\mu}\left[\frac{1+(B-1)\rho^B - B\rho^{B-1}}{(1-\rho)^2}\right] - \frac{1}{\mu} \tag{4}$$

P_B is the same Erlang B formula as given in equation 1, and $P_C = 1 - P_B$. This is because the Erlang-B formula is dependent on the mean of the service distribution. And as long as the mean is kept same, the formula will remain same.

Derivation

We provide a sketch of the derivation of the blocking probability for M/G/1/B queues. Note that the complete derivation can be found at this Link.

1. Pollaczek-Khinchin formula gives the delay for a M/G/1 system and assumes the buffer size to be infinity. For a finite buffer size, We had to derive the Pollaczek-Khinchin formula again with the given constraints. This will give us W.
2. In the lecture, we read that for a M/G/1/B system also, we get the same blocking probability as of M/M/1/B. Erlang B formula applies to the M/G/1/B system. The reason is that it is dependent on the mean of the distribution of service times.

Results and Inferences

Experiment design

We simulate the queue by randomly choosing arrival times for the jobs. The experiment was repeated 10 times to minimize any random errors in analysis. We had the following parameters -

1. For both the queuing systems, we kept the calling population to be large i.e. 10^6 and queuing discipline as FCFS - First Come First Serve. Service times will be kept independent of arrival process in the code.
2. For M/M/1/B systems, the arrival and departures were Poisson distributed. For M/G/1/B, we made the departures follow a Normal distribution $\mathcal{N}(1/\mu, 5)$ and Uniform distribution $\mathit{Unif}(\frac{1}{2\mu}, \frac{3}{2\mu})$. Note that the mean of these distributions is $\frac{1}{\mu}$.

Results

B	Normal		Uniform		Poisson	
	Pb	W	Pb	W	Pb	W
1	0.20405	0	0.1962	0	0.2036	0
2	0.179	0.01416196	0.0307	0.00194313	0.045	0.00308356
5	0.1564	0.05200441	0.0006	0.00297634	0.00105	0.00487498
10	0.1017	0.09655041	0.0005	0.00282823	0.0005	0.00506428
15	0.04415	0.11353528	0.0005	0.00285083	0.0005	0.00531817
16	0.0241	0.0241	0.00290434	0.0005	0.0005	0.00531817
20	0.02695	0.02695	0.00280478	0.0005	0.0005	0.00497262

Table: Table showing Blocking probabilities, Waiting times and Buffer size of all three distributions. Number of experiments done were 10. ρ is kept 0.25.

Comparison between M/M/1/B and M/G/1/B

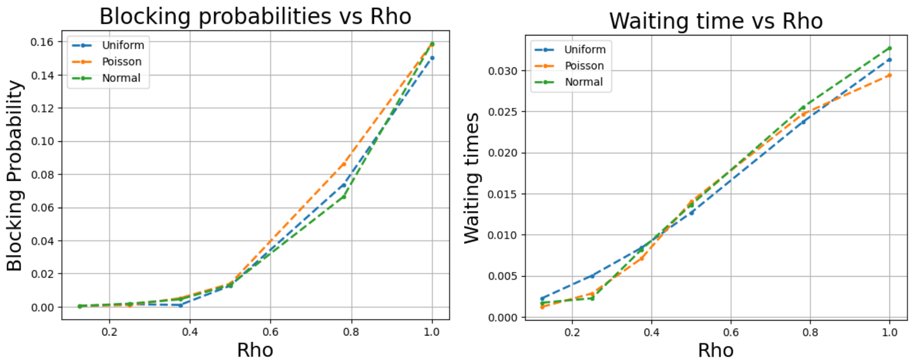

Figure: The left plot shows blocking probability vs ρ, and the right one shows waiting time vs ρ.

Inferences I

We made the following inferences.

1. As shown in Figure 1(a), the blocking probabilities of all the service distributions are coming out to be similar. In theory, it should be exactly same because all three distributions use Erlang-B formula and we have fixed the mean service time for all of them. Thus theory matches with the experimental results.
2. In figure 1(b), we can see that waiting time of the Possion distribution (Markovian process) is larger than the other general distributions for values of ρ up to a limit. After that, the mean waiting times becomes less. As per our derivation, it should be consistently less.
3. In Table 1, we have observed that as we increase the value of buffer, the queuing system was behaving more like the $M/M/1/\infty$ system. We can see the error between these systems to decrease as we increase B. Thus theoretical results matches the experimental results.

Inferences II

4. Further for M/M/1/B systems, our experimental values are matching with the theoretical values. We checked for 2-3 readings. For M/G/1/B systems, our blocking probability is matching, but our waiting time or delay is not matching. One reason we could think of is that the probability P_C which we have defined as (1-P_B) is wrong. It should be Probability that i^{th} packet sees a queue given that it is able to enter the queue of the system. Another approach could be using Pollaczak Khinchine's Tranform formula for deriving the queue length, which we were not able to understand completely.

Conclusion

Work Division

The work distribution is given in table below.

Work distribution	
Aditya	Gajraj
Simulation for M/G/1/B	Simulation for M/M/1/B
Derive the parameters for M/G/1/B	

Table: Work distribution

Please note that the work is distributed equally among the team members.

Prospect

There are multiple directions in which the project can extend -

1. We can extend this for a general arrival as well. It means the arrival distribution is not Markovian. For that system, we cannot get the exact value, but we will be able to derive the upper bound on the Delay and the Blocking Probability.
2. We can use the concept of Pollaczak-Khinchine Transform formula to derive the Semi-markovian systems. For that, we need to do an in-depth course on Queuing theory.

Fin..