Assignment 7
Sampling and Applications

1. Write a python program to estimate the value of π using Monte Carlo simulation (raindrop experiment). Generate animation of the simulation where estimates of π converges with an increase in the number of drops/samples. Plot the Monte Carlo estimate of π with 90% confidence interval where x-axis represent number of sample (1 to 3000) and y-axis represent estimate of π.

2. Evaluate the integral

$$\int_0^1 f(x)dx \quad \text{with} \quad f(x) = \frac{1}{27}\left(-65536x^8 + 262144x^7 - 409600x^6 + 311296x^5 - 114688x^4 + 16384x^3\right)$$

   using a Monte Carlo approach. Graphically show that the graph of $f(x)$ is fully contained in unit square $[0,1]^2$. Generate animation of the simulation where estimates of the integral converges with an increase in the number of drops/samples. Plot the Monte Carlo estimate of integral with 90% confidence interval where x-axis represent number of sample (1 to 2000) and y-axis represent estimate of $\int_0^1 f(x)dx$.

3. Sample points from $\mathcal{N}(0,1)$ distribution (target distribution) with destiny

$$f(x) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)$$

   using rejection sampling method by taking Cauchy distribution with density

$$g(x) = \frac{1}{\pi(1+x^2)}$$

   as a proposal distribution. Find the smallest value of M such that $f(x) \leq Mg(x)$. Generate animation of rejection sampling containing graphs of target distribution, proposal distribution, scale proposal distribution and histogram of selected random samples. The histogram of the selected random samples should keep on updating with number of iterations/ samples.

4. Let

$$p(x) = \frac{1}{2}e^{-|x|}$$

   is a density function of target distribution and $\mathcal{N}(0,4)$ is density function of proposal distribution. Write a python program to estimate $E[X]$, $E[X^2]$ and $E[X^5]$ where $X \sim p$ using importance sampling technique. Report the values of estimates for $N = 100, 500, \text{and} 1000$, also report the error in each estimates.

5. Let $f_X = \frac{1}{40} \times (2x+3),\ 0 < x < 5$ be a density function. Sample 1000 random draws from this density using the inverse transform sampling method. Plot the graph of given density function and histogram plot of the drawn samples in a single figure.

6. Simulate a 2D random walk inside given upper and lower bound of x and y-axis for given number of steps. Write a python code to generate animation of the simulated 2D random walk.

7. Generate 100000 random samples from bivariate normal distribution and draw scatter plot using following covariance matrices

   (a) $cov = c * I$, where $c \in \mathbb{R}^+$ and I is 2×2 identity matrix.

   (b) $cov = diag(a_1, a_2)$ where a_1 and $a_2 \in \mathbb{R}^+$.

   (c) $cov = A_{2\times 2}$ where A is 2×2 symmetric matrix with positive diagonal entries. Repeat the experiment for fixed value of diagonal entries and by taking different value for off diagonal entries like $\pm 0.2, \pm 0.5, \pm 0.8, \pm 1, \pm 2, \pm 3, \pm 5, \pm 10$ etc.

   Write the insights you get from all these scatter plot experiments. Is this insight is also applicable to other distributions.