Compressive/Compressed Sensing

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Compressive sensing

- A signal, $x \in \mathbb{R}^N$, is K-sparse when,

$$x = \Psi\theta, \tag{1}$$

  and $||\theta||_0 = K$.
  Ψ is a transform.

- A signal, $x \in \mathbb{R}^N$, is called approximately sparse when,

$$x = \Psi\theta, \tag{2}$$

  and most of the energy is concentrated in K transformed coefficients.

Encoding

Random linear measurements of a vector, x, is given as,

$$y = \Phi x \tag{3}$$

where $y \in \mathbb{R}^M$ is a measurement vector and $\Phi \in \mathbb{R}^{M \times N}$ is a sensing matrix.

Reconstruction

- x is sparse in the canonical form

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} ||x||_1 \text{ s.t. } y = \Phi x. \tag{4}$$

- x is approximately sparse

$$\hat{\theta} = \arg\min_{\theta \in \mathbb{R}^N} ||\theta||_1 \text{ s.t. } y = \Phi \Psi \theta. \tag{5}$$

Linear program

Conversion of Eq. 4 to linear program.
$y = \Phi x$ can be written as
$y = [\Phi, -\Phi]z$
where $z = [x^+; x^-] \in \mathbb{R}^{2N}$.
the equivalent optimization problem is,

$$\hat{z} = \arg \min_{z \in \mathbb{R}^{2N}} 1^T z \qquad \text{s.t. } [\Phi, -\Phi]z = y \tag{6}$$

Reconstructed $\hat{z}$ can be represented as,
$\hat{z} = [\hat{x}^+; \hat{x}^-]$.
The reconstructed signal is given as,
$\hat{x} = \hat{x}^+ - \hat{x}^-$.

Exercise 1

why l_1 not l_2: Empirical evidence
l1_l2_comparison.m

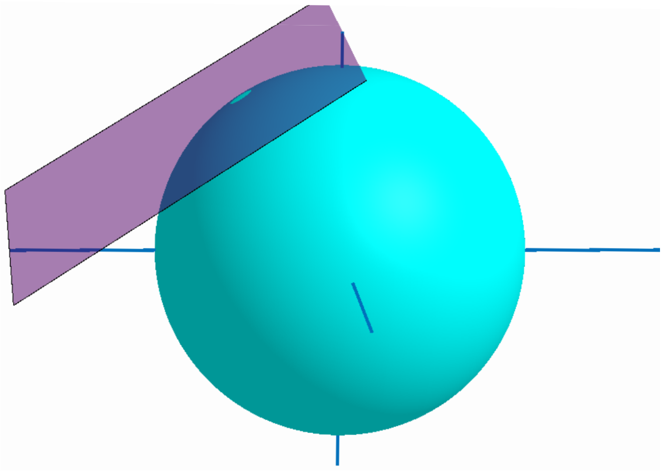


Figure: l_2 norm ball

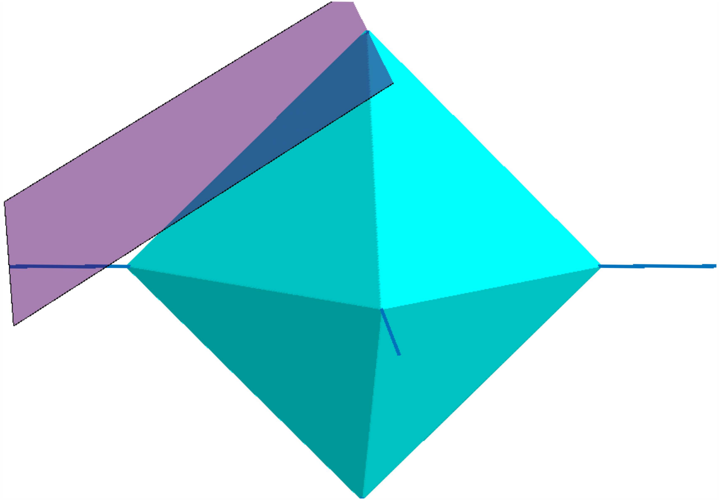


Figure: l_1 norm ball

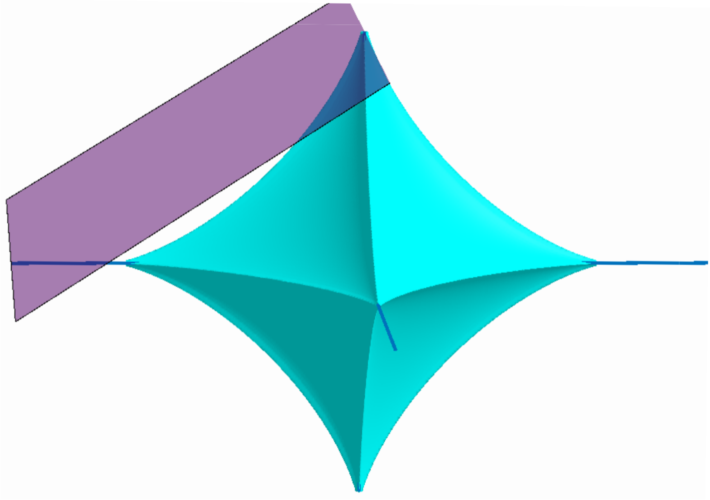


Figure: $l_{.7}$ norm ball

Exercise 2

exactlySparse.m
code demonstrates how a sparse signal in time domain is sampled and reconstructed using l_1 minimization.

Exercise 3

approximatelyECG.m
code demonstrates how a sparse signal in DCT domain is sampled and reconstructed using l_1 minimization

Exercise 4

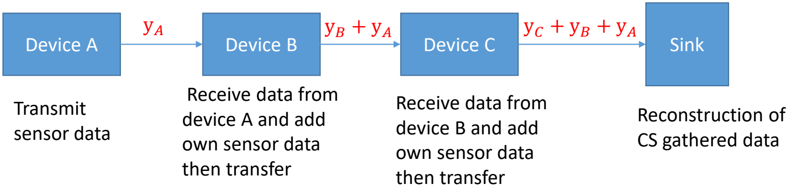
approximatelyAudio.m
code demonstrates how an audio signal is sampled and reconstructed using l_1 minimization.

Exercise 5

approximatelyImage.m
code demonstrates how an image is sampled and reconstructed using l_1 minimization.

Exercise 6

Data gathering



dataGatheringCS.m
code demonstrates how sequential data gathering is performed using CS.

Exercise 7

One-time sensing(OTS):
CS encoding on the i^{th} block of plaintext, x_i, we get the i^{th} measurement vector, y_i, as,

$$\mathrm{y}_i = \Phi_i \mathrm{x}_i, \tag{7}$$

exactlySparseConfidentiality.m
code demonstrates why CS is a good candidate for information secrecy

Homework

Extend the Exercise 5 for large image size
for example 256×256 or 512×512.

- Block encoding can be used. Image can be divided into blocks of size say 16×16 then CS encoding is performed on each block.
- Reconstruction using block reconstruction algorithms.

Code : https://my.ece.msstate.edu/faculty/fowler/BCSSPL/

L. Gan, "Block compressed sensing of natural images," in Proc. 15th Int. Conf. Digit. Signal Process., Cardiff, U.K, pp. 403–406, 2007.
S. Mun and J. E. Fowler, "Block compressed sensing of images using directional transforms," in Proc. 16th IEEE Int. Conf. Image Process. (ICIP), Cairo, Egypt, pp. 3021–3024, 2009.