Ordinary Differential Equations

Lab 4

Goal

The goal is to solve

$$y'(t) = \frac{dy}{dt} = f(t, y),\ a \le t \le b,\ y(a) = \alpha$$

Example 1

Solve

$$\begin{cases} y' = t^2 \\ y(1) = 2 \end{cases}$$

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$$y(1) = 2 \Longrightarrow C = \frac{5}{3}$$

$$y = \frac{t^3}{3} + \frac{5}{3}$$

Example 2

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$$\begin{cases} y' = 2y \\ y(0) = 5 \end{cases}$$

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$$\begin{cases} y' = 2y \\ y(0) = 5 \end{cases}$$

$$y' = 2y \Longrightarrow y = Ce^{2x}$$

$$y(0) = 5 \Longrightarrow C = 5$$

$$y = 5e^{2x}$$

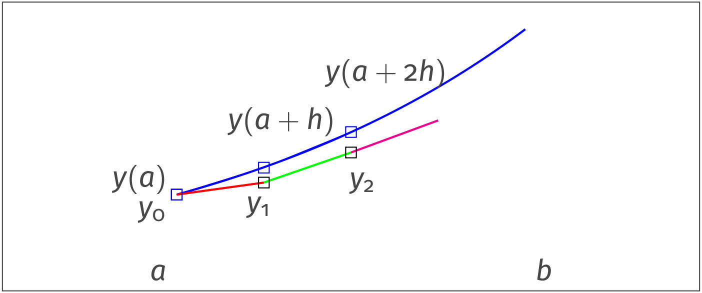
Example 3

Solve

$$\begin{cases} y' = 2y + t \\ y(0) = 5 \end{cases}$$

Idea

$$y'(t) = \frac{dy}{dt} = f(t, y),\ a \leq t \leq b,\ y(a) = \alpha$$



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Let us pick $t = t_i$ then

$$y'(t_i) \simeq \frac{y(t_{i+1}) - y_(t_i)}{h} \simeq \frac{y_{i+1} - y_i}{h}$$

Therefore,

$$\frac{yi + 1 - y_i}{h} \simeq y'(t_i) = f(t_i, y(t_i)) \simeq f(t_i, y_i)$$

we get,

$$y_{i+1} = y_i + hf(t_i, y_i)$$

General form

$$y'(t) = \frac{dy}{dt} = f(t, y),\ a \leq t \leq b,\ y(a) = \alpha$$

Euler's Explicit Method

We first discretive the domain $[a, b]$, into $n + 1$ points $t_i = a + ih$ where $h = \frac{b-a}{n}$, then

$$y_{i+1} = y_i + hf(t_i, y_i),\ i = 0 \ldots n$$

where y_i approximate $y(t_i)$

Remark

$y_0 = \alpha = y(a)$ is given and $y_n \simeq y(b)$.

Example 4

Compute the first couple of steps using Euler Explicite of $y(t)$, with $h = \frac{1}{2}$, such that

$$\begin{cases} y' = 2ty \\ y(1) = 3 \end{cases}$$

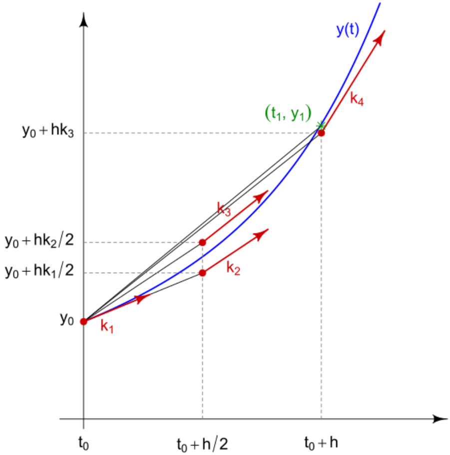
Test and Error

Pick y, $y(t) = 2t^2e^t$ then $f(t, y) = y'(t) = 4te^t + 2t^2e^t$ and solve for different h

- **Method 1:** $Error(h) = |y(b) - y_n|$
- **Method 2:** $Error(h) = \sqrt{\frac{1}{n}\sum_{i=0}^{n} |y(t_i) - y_i|}$

Make it better

$$y'(t) = \frac{dy}{dt} = f(t, y),\ a \leq t \leq b,\ y(a) = \alpha$$



Runge Kutta of Order 4

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We first discretive the domain $[a, b]$, into $n+1$ points $t_i = a + ih$ where $h = \frac{b-a}{n}$, then

$$k_1 = hf(t_i, y_i)$$
$$k_2 = hf(t_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$
$$k_3 = hf(t_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$
$$k_4 = hf(t_i + h, y_i + k_3)$$
$$y(t_{i+1}) = y(t_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Example 5

Compute the first couple of steps using RK4 of $y(t)$, with $h = \frac{1}{2}$, such that

$$\begin{cases} y' = 2ty \\ y(1) = 3 \end{cases}$$

Euler's Implicit Method

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We first discretive the domain $[a, b]$, into $n + 1$ points $t_i = a + ih$ where $h = \frac{b-a}{n}$, then

$$y(t_{i+1}) = y(t_i) + hf(t_{i+1}, y_{i+1})$$

System of ODE

For $a \le t \le b$

$$\begin{cases} y_1'(t) = f_1(t, y) \\ y_2'(t) = f_2(t, y) \end{cases}$$

where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ and $y(a) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

Example 5

Compute the first couple of steps using Euler of $y(t)$, with $h = \frac{1}{2}$, such that

$$\begin{cases} y_1'(t) = ty_1 + y_2 \\ y_2'(t) = y_1 t^2 \end{cases}$$

and $y(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

System of ODE

Write the function $f(t, y)$ and y_0 in python.
(see scipy.integrate.solve_ivp).

Code

```
def f(t,y):
        return np.array([t*y[0]+y[1], y[0]*t***2])

import scipy.integrate as spi

t = np.linspace(0, 200, 200)
res = spi.solve_ivp(f, (0, 200), [1,-1], t_eval=t)

plt.plot(res.t, res.y[0])
plt.plot(res.t, res.y[1])
```