

Minimization



Lab 6



Goal

The goal is to minimize

$$F(\mathbf{x}) = F(x_1, x_2, \dots, x_k), \text{ where } \mathbf{x} = [x_1, x_2, \dots, x_k]$$

This is equivalent to solving

$$\nabla F(x_1, x_2, \dots, x_n) = \begin{bmatrix} F_1(x_1, x_2, \dots, x_n) \\ F_2(x_1, x_2, \dots, x_n) \\ \vdots \\ F_n(x_1, x_2, \dots, x_n) \end{bmatrix} = \mathbf{0}$$

Definitions

Definitions

The gradient:

$$\nabla F(\mathbf{x}) = \nabla F(x_1, x_2, \dots, x_n) = \left[\frac{\partial F}{\partial x_1} \quad \frac{\partial F}{\partial x_2} \quad \cdots \quad \frac{\partial F}{\partial x_k} \right]$$

The Hessian matrix:

$$HF(\mathbf{x}) = HF(x_1, x_2, \dots, x_n) = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_k \partial x_1} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_2} & \cdots & \frac{\partial^2 F}{\partial x_k \partial x_2} \\ \vdots & \cdot & \cdots & \vdots \\ \frac{\partial^2 F}{\partial x_1 \partial x_k} & \frac{\partial^2 F}{\partial x_2 \partial x_k} & \cdots & \frac{\partial^2 F}{\partial x_k \partial x_k} \end{bmatrix}$$

Example

Example 1

Compute the gradient and Hessian matrix of

$$F(x_1, x_2) = x_1 \sin(x_1 x_2^2)$$

Newton's Method

Newton's Method

It comes from the Taylor's formula in higher dimension. We find a root of \mathbf{x} , close to your initial guess \mathbf{x}^0 , such that $\nabla F(\mathbf{x}) = 0$ which correspond to a minimum of $F(\mathbf{x})$. You start with the function ∇F , the initial guess \mathbf{x}^0 , a tolerance ϵ , and a max number of iteration *ITMAX*.

1. Calculate the next iteration (pick a small constant α)

$$\mathbf{x}^{n+1} = \mathbf{x}^n - \alpha [HF(\mathbf{x}^n)]^{-1} \nabla F(\mathbf{x}^n)$$

2. If convergence is satisfactory, i.e. $\|\nabla F(\mathbf{x}^{n+1})\| < \epsilon$, or $\|\mathbf{x}^{n+1} - \mathbf{x}^n\| < \epsilon$, or iteration is *ITMAX*, return \mathbf{x}^{n+1} and stop.

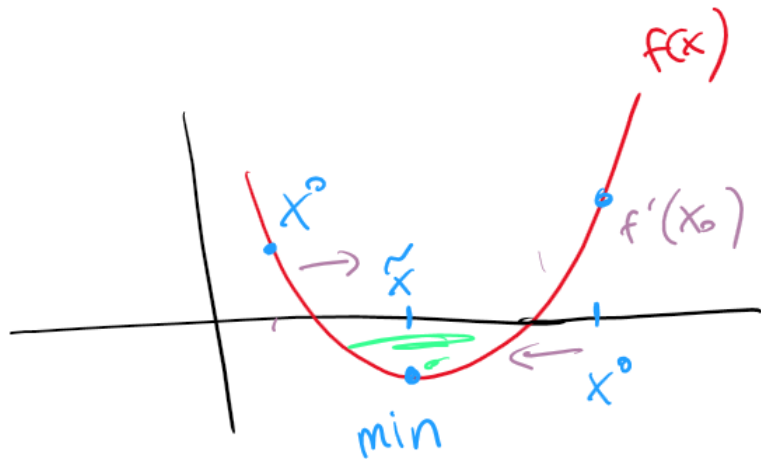
Example

Example 2

Minimize

$$F(x, y) = (x - 1)^2 + y^2 + 5$$

Idea for Gradient Descent



Gradient Descent

Gradient Descent

We find the minimum of $F(\mathbf{x})$. You start with the function ∇F , the initial guess \mathbf{x}^0 , a tolerance ε , and a max number of iteration $ITMAX$.

1. Calculate the next iteration (pick a small constant α)

$$\mathbf{x}^{n+1} = \mathbf{x}^n - \alpha \nabla F(\mathbf{x}^n)$$

2. If convergence is satisfactory, i.e. $\|\nabla F(\mathbf{x}^{n+1})\| < \varepsilon$, or $\|\mathbf{x}^{n+1} - \mathbf{x}^n\| < \varepsilon$, or iteration is $ITMAX$, return \mathbf{x}^{n+1} and stop.

Example

Example 3

Minimize

$$F(x_1, x_2) = (x_1 - 1)^2 + x_2^2 + 5$$