COMP9024 20T0

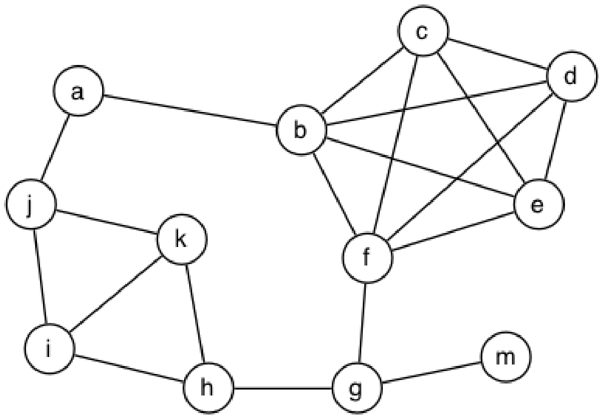
Week 2b Problem Set

Data Structures and Algorithms

Graph Data Structures and Graph Search

1. (Graph fundamentals)

   For the graph



   give examples of the smallest (but not of size/length 0) and largest of each of the following:

   a. path
   b. cycle
   c. spanning tree
   d. vertex degree
   e. clique

   Answer:

   a. path
      - smallest: any path with one edge (e.g. a-b or g-m)
      - largest: some path including all nodes (e.g. m-g-h-k-i-j-a-b-c-d-e-f)
   b. cycle
      - smallest: need at least 3 nodes (e.g. i-j-k-i or h-i-k-h)
      - largest: path including most nodes (e.g. g-h-k-i-j-a-b-c-d-e-f-g) (can't involve m)
   c. spanning tree
      - smallest: any spanning tree must include all nodes (the largest path above is an example)
      - largest: same
   d. vertex degree
      - smallest: there is a node that has degree 1 (vertex m)
      - largest: in this graph, 5 (b or f)
   e. clique
      - smallest: any vertex by itself is a clique of size 1
      - largest: this graph has a clique of size 5 (nodes b,c,d,e,f)

2. (Graph properties)

   a. Write pseudocode for computing

      - the minimum and maximum vertex degree
      - all 3-cliques (i.e. cliques of 3 nodes, "triangles")

      of a graph *g* with *n* vertices.

      Your methods should be representation-independent; the only function you should use is to check if two vertices *v,w*∈*{0,…n-1}* are adjacent in *g*.

   b. Determine the asymptotic complexity of your two algorithms. Assume that the adjacency check is performed in constant time, *O(1)*.

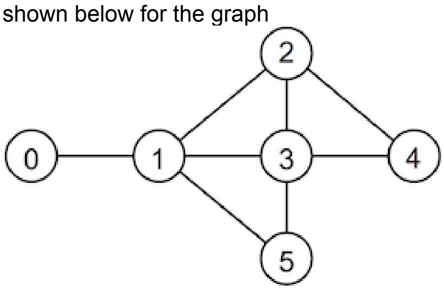
   c. Implement your algorithms in a program `graphAnalyser.c` that

      1. builds a graph from user input:
         - first, the user is prompted for the number of vertices
         - then, the user is repeatedly asked to input an egde by entering a "from" vertex followed by a "to" vertex
         - until any non-numeric character(s) are entered
      2. computes and outputs the minimum and maximum degree of vertices in the graph
      3. prints all vertices of minimum degree in ascending order, followed by all vertices of maximum degree in ascending order
      4. displays all 3-cliques of the graph in ascending order.

      Your program should use the Graph ADT from the lecture (`Graph.h` and `Graph.c`). These files should not be changed.

      Hint: You may assume that the graph has a maximum of 1000 nodes.

An example of the program executing is shown below for the graph




```
prompt$ ./graphAnalyser
Enter the number of vertices: 6
Enter an edge (from): 0
Enter an edge (to): 1
Enter an edge (from): 1
Enter an edge (to): 2
Enter an edge (from): 4
Enter an edge (to): 2
Enter an edge (from): 1
Enter an edge (to): 3
Enter an edge (from): 3
Enter an edge (to): 4
Enter an edge (from): 1
Enter an edge (to): 5
Enter an edge (from): 5
Enter an edge (to): 3
Enter an edge (from): 2
Enter an edge (to): 3
Enter an edge (from): done
Done.
Minimum degree: 1
Maximum degree: 4
Nodes of minimum degree:
0
Nodes of maximum degree:
1
3
Triangles:
1-2-3
1-3-5
2-3-4
```

Note that any non-numeric data can be used to 'finish' the interaction.

We have created a script that can automatically test your program. To run this test you can execute the `dryrun` *program that corresponds to this exercise. It expects to find a program named* `graphAnalyser.c` *in the current directory. You can use dryrun as follows:*

```
prompt$ 9024 dryrun graphAnalyser
```

Please ensure that your program output follows exactly the format shown in the sample interaction above. In particular, the vertices of minimum and maximum degree and the 3-cliques should be printed in ascending order.

Answer:

The following algorithm uses two nested loops to compute the degree of each vertex. Hence its asymptotic running time is $O(n^2)$.

```
MinMaxDegree(g):
   Input  graph g
   Output minimum and maximum vertex degree in g

   min=#vertices(g)-1, max=0
   for all vertices v∈g do
      deg[v]=0
      for all vertices w∈g, v≠w do
         if v,w adjacent in g then
            deg[v]=deg[v]+1
         end if
      end for
      if deg[v]<min then
         min=deg[v]
      end if
      if deg[v]>max then
         max=deg[v]
      end if
```

```
   end for
   return min,max
```

The following algorithm uses three nested loops to print all 3-cliques in order. Hence its asymptotic running time is *O(n^3)*.

```
show3Cliques(g):
   Input graph g of n vertices numbered 0..n-1

   for all i=0..n-3 do
      for all j=i+1..n-2 do
         if i,j adjacent in g then
            for all k=j+1..n-1 do
               if i,k adjacent in g and j,k adjacent in g then
                  print i"-"j"-"k
               end if
            end for
          end if
      end for
   end for
```

Sample `graphAnalyser.c`:

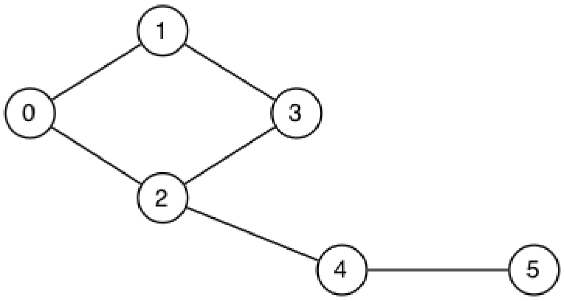
```
#include <stdio.h>
#include <stdlib.h>
#include <assert.h>
#include "Graph.h"

#define MAX_NODES 1000

// determine minimum and maximum degree of graph g with n vertices
// and output all nodes of those degrees
void MinMaxDegree(Graph g) {
   int degree[MAX_NODES];
   int nV = numOfVertices(g);
   int mindegree = nV-1;
   int maxdegree = 0;
   int v, w;
   for (v = 0; v < nV; v++) {
      degree[v] = 0;
      for (w = 0; w < nV; w++) {
         if (adjacent(g, v, w))
            degree[v]++;
      }
      if (degree[v] < mindegree)
         mindegree = degree[v];
      if (degree[v] > maxdegree)
         maxdegree = degree[v];
   }
   printf("Minimum degree: %d\n", mindegree);
   printf("Maximum degree: %d\n", maxdegree);

   printf("Nodes of minimum degree:\n");
   for (v = 0; v < nV; v++) {
      if (degree[v] == mindegree)
         printf("%d\n", v);
   }
   printf("Nodes of maximum degree:\n");
   for (v = 0; v < nV; v++) {
      if (degree[v] == maxdegree)
         printf("%d\n", v);
   }
}

// show all 3-cliques of graph g
void Show3Cliques(Graph g) {
   int i, j, k;
   int nV = numOfVertices(g);

   printf("Triangles:\n");
   for (i = 0; i < nV-2; i++)
      for (j = i+1; j < nV-1; j++)
         if (adjacent(g,i,j))
            for (k = j+1; k < nV; k++)
               if (adjacent(g,i,k) && adjacent(g,j,k))
                  printf("%d-%d-%d\n", i, j, k);
}

int main(void) {
   Edge e;
```

```
   int n;

   printf("Enter the number of vertices: ");
   scanf("%d", &n);
   Graph g = newGraph(n);

   printf("Enter an edge (from): ");
   while (scanf("%d", &e.v) == 1) {
      printf("Enter an edge (to): ");
      scanf("%d", &e.w);
      insertEdge(g, e);
      printf("Enter an edge (from): ");
   }
   printf("Done.\n");

   MinMaxDegree(g);
   Show3Cliques(g);
   freeGraph(g);

   return 0;
}
```

3. (Graph representations)

   Show how the following graph would be represented by

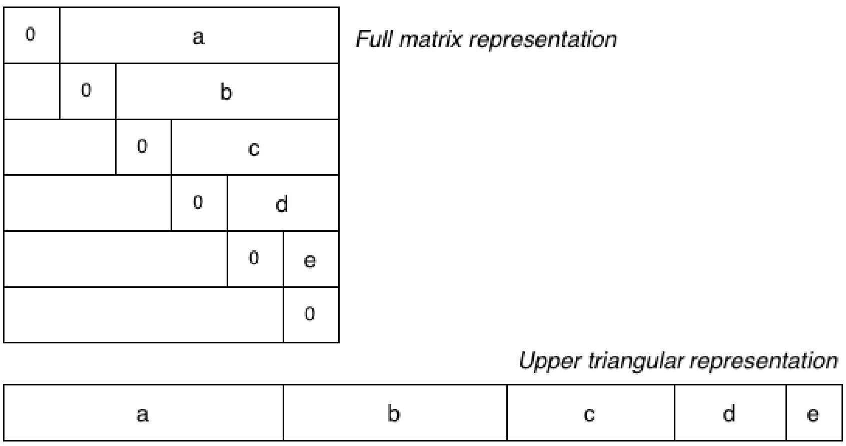
   i. an adjacency matrix representation (V×V matrix with each edge represented twice)
   ii. an adjacency list representation (where each edge appears in two lists, one for *v* and one for *w*)



   a. Consider the adjacency matrix and adjacency list representations for graphs. Analyse the storage costs for the two representations in more detail in terms of the number of vertices *V* and the number of edges *E*. Determine roughly the V:E ratio at which it is more storage efficient to use an adjacency matrix representation vs the adjacency list representation.

      For the purposes of the analysis, ignore the cost of storing the `GraphRep` structure. Assume that: each pointer is 8 bytes long, a `Vertex` value is 4 bytes, a linked-list *node* is 16 bytes long and that the adjacency matrix is a complete *V*×*V* matrix. Assume also that each adjacency matrix element is **1 byte** long. (*Hint:* Defining the matrix elements as 1-byte boolean values rather than 4-byte integers is a simple way to improve the space usage for the adjacency matrix representation.)

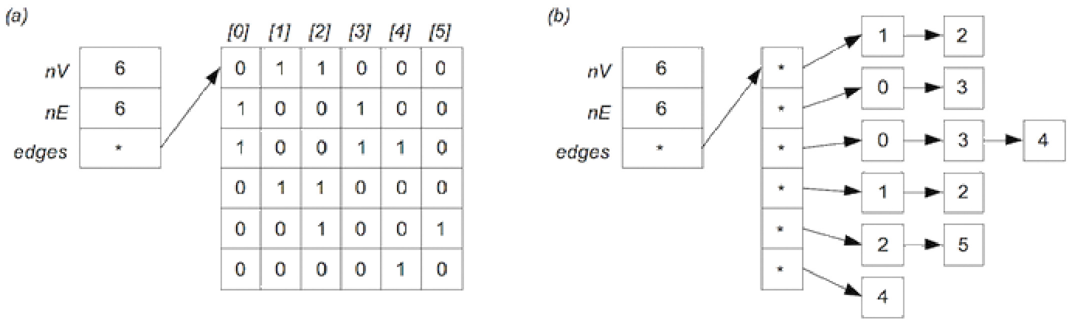
   b. The standard adjacency matrix representation for a graph uses a full *V*×*V* matrix and stores each edge twice (at [v,w] and [w,v]). This consumes a lot of space, and wastes a lot of space when the graph is sparse. One way to use less space is to store just the upper (or lower) triangular part of the matrix, as shown in the diagram below:



      The *V*×*V* matrix has been replaced by a single 1-dimensional array `g.edges[]` containing just the "useful" parts of the matrix.

      Accessing the elements is no longer as simple as `g.edges[v][w]`. Write pseudocode for a method to check whether two vertices `v` and `w` are adjacent under the upper-triangle matrix representation of a graph `g`.

Answer:

a. 

b. The adjacency matrix representation always requires a $V \times V$ matrix, regardless of the number of edges, where each element is 1 byte long. It also requires an array of V pointers. This gives a fixed size of $V \cdot 8 + V^2$ bytes.

The adjacency list representation requires an array of V pointers (the start of each list), with each being 8 bytes long, and then one list node for each edge in each list. The total number of edge nodes is $2E$ (each edge (v,w) is stored twice, once in the list for v and once in the list for w). Since each node requires 16 bytes (vertex+padding+pointer), this gives a size of $V \cdot 8 + 16 \cdot 2 \cdot E$. The total storage is thus $V \cdot 8 + 32 \cdot E$.

Since both representations involve V pointers, the difference is based on V^2 vs $32E$. So, if $32E < V^2$ (or, equivalently, $E < V^2/32$), then the adjacency list representation will be more storage-efficient. Conversely, if $E > V^2/32$, then the adjacency matrix representation will be more storage-efficient.

To pick a concrete example, if $V=25$ and if we have 19 or fewer edges ($25 \cdot 25/32 = 19.53$), then the adjacency list will be more storage-efficient, otherwise the adjacency matrix will be more storage-efficient.

c. The following solution uses a loop to compute the correct index in the 1-dimensional `edges[]` array:

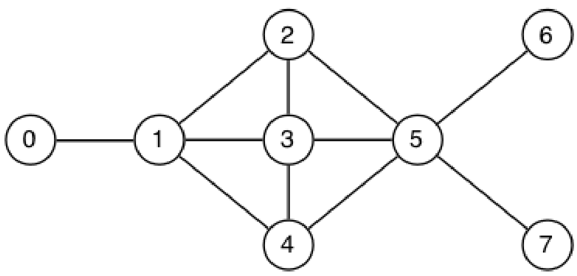
```
adjacent(g,v,w):
   Input  graph g in upper-triangle matrix representation
          v, w vertices such that v≠w
   Output true if v and w adjacent in g, false otherwise

   if v>w then
      swap v and w        // to ensure v<w
   end if
   chunksize=g.nV-1, offset=0
   for all i=0..v-1 do
      offset=offset+chunksize
      chunksize=chunksize-1
   end if
   offset=offset+w-v-1
   if g.edges[offset]=0 then return false
                        else return true
   end if
```

Alternatively, you can compute the overall offset directly via the formula $(nV-1)+(nV-2)+\ldots+(nV-v)+(w-v-1) = \frac{v}{2}(2 \cdot nV - v - 1) + (w-v-1)$ (assuming that $v < w$).

4. (Graph traversal: DFS and BFS)

Show the order in which the nodes of the graph depicted below are visited by

a. DFS starting at node 0
b. DFS starting at node 3
c. BFS starting at node 0
d. BFS starting at node 3



Assume the use of a stack for depth-first search (DFS) and a queue for breadth-first search (BFS), respectively. Show the state of the stack or queue explicitly in each step. When choosing which neighbour to visit next, always choose the smallest unvisited neighbour.

Answer:

a. DFS starting at 0:

```
Current     Stack (top at left)
-           0
0           1
1           2 3 4
2           3 5 3 4
3           4 5 5 3 4
4           5 5 5 3 4
5           6 7 5 5 3 4
6           7 5 5 3 4
7           5 5 3 4
-           -
```

b. DFS starting at 3:

```
Current     Stack (top at left)
-           3
3           1 2 4 5
1           0 2 4 2 4 5
0           2 4 2 4 5
2           5 4 2 4 5
5           4 6 7 4 2 4 5
4           6 7 4 2 4 5
6           7 4 2 4 5
7           4 2 4 5
-           -
```

c. BFS starting at 0:

```
Current     Queue (front at left)
-           0
0           1
1           2 3 4
2           3 4 5
3           4 5
4           5
5           6 7
6           7
7           -
```

d. BFS starting at 3:

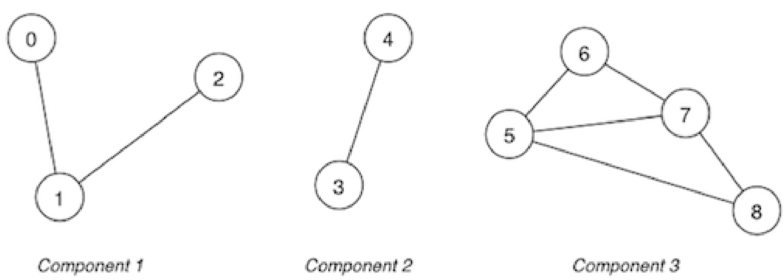
```
Current     Queue (front at left)
-           3
3           1 2 4 5
1           2 4 5 0
2           4 5 0
4           5 0
5           0 6 7
0           6 7
6           7
7           -
```

5. (Cycle check)

   a. Take the "buggy" cycle check from the lecture and design a correct algorithm, in pseudocode, to use depth-first search to determine if a graph has a cycle.

   b. Write a C program `cycleCheck.c` that implements your solution to check whether a graph has a cycle. The graph should be built from user input in the same way as in exercise 2. Your program should use the Graph ADT from the lecture (`Graph.h` and `Graph.c`). These files should not be changed.

   An example of the program executing is shown below for the following graph:




```
prompt$ ./cycleCheck
Enter the number of vertices: 9
Enter an edge (from): 0
Enter an edge (to): 1
Enter an edge (from): 1
Enter an edge (to): 2
```

```
Enter an edge (from): 4
Enter an edge (to): 3
Enter an edge (from): 6
Enter an edge (to): 5
Enter an edge (from): 6
Enter an edge (to): 7
Enter an edge (from): 5
Enter an edge (to): 7
Enter an edge (from): 5
Enter an edge (to): 8
Enter an edge (from): 7
Enter an edge (to): 8
Enter an edge (from): done
Done.
The graph has a cycle.
```

If the graph has no cycle, then the output should be:

```
prompt$ ./cycleCheck
Enter the number of vertices: 3
Enter an edge (from): 0
Enter an edge (to): 1
Enter an edge (from): #
Done.
The graph is acyclic.
```

You may assume that a graph has a maximum of 1000 nodes.

To test your program you can execute the `dryrun` *program that corresponds to this exercise. It expects to find a program named* `cycleCheck.c` *in the current directory. You can use dryrun as follows:*

```
prompt$ 9024 dryrun cycleCheck
```

Note: Please ensure that your output follows exactly the format shown above.

Answer:

a.

```
hasCycle(G):
|  Input  graph G
|  Output true if G has a cycle, false otherwise
|
|  mark all vertices as unvisited
|  for each vertex v∈G do          // make sure to check all connected components
|  |  if v has not been visited then
|  |     if dfsCycleCheck(G,v,v) then
|  |        return true
|  |     end if
|  |  end if
|  end for
|  return false

dfsCycleCheck(G,v,u):        // look for a cycle that does not go back directly to u
|  mark v as visited
|  for each (v,w)∈edges(G) do
|  |  if w has not been visited then
|  |  |  if dfsCycleCheck(G,w,v) then
|  |  |     return true
|  |  |  end if
|  |  else if w≠u then
|  |  |  return true
|  |  end if
|  end for
|  return false
```

b. The following two C functions implement this algorithm:

```
#define MAX_NODES 1000
int visited[MAX_NODES];

bool dfsCycleCheck(Graph g, Vertex v, Vertex u) {
   visited[v] = true;
   Vertex w;
   for (w = 0; w < numOfVertices(g); w++) {
      if (adjacent(g, v, w)) {
         if (!visited[w]) {
            if (dfsCycleCheck(g, w, v))
               return true;
         } else if (w != u) {
```

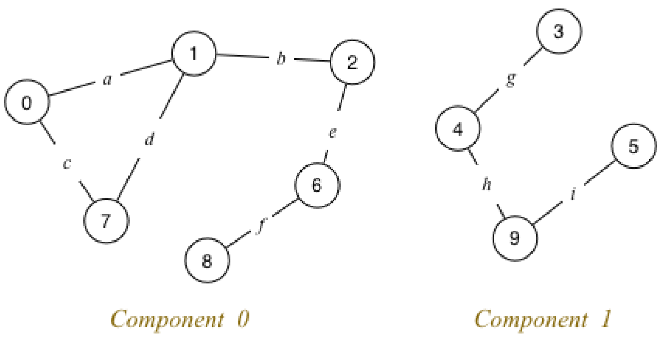
```
            return true;
         }
      }
   }
   return false;
}

bool hasCycle(Graph g) {
   int v, nV = numOfVertices(g);
   for (v = 0; v < nV; v++)
      visited[v] = false;
   for (v = 0; v < nV; v++)
      if (!visited[v])
         if (dfsCycleCheck(g, v, v))
            return true;
   return false;
}
```

6. (Connected components)

   a. Computing connected components can be avoided by maintaining a vertex-indexed connected components array as part of the `Graph` representation structure:

```
typedef struct GraphRep *Graph;

struct GraphRep {
   ...
   int nC;  // # connected components
   int *cc; /* which component each vertex is contained in
               i.e. array [0..nV-1] of 0..nC-1 */
   ...
}
```

   Consider the following graph with multiple components:



   Assume a vertex-indexed connected components array cc[0..nV-1] as introduced above:

```
nC   = 2
cc[] = {0,0,0,1,1,1,0,0,0,1}
```

   Show how the `cc[]` array would change if

   1. edge *d* was removed
   2. edge *b* was removed

   b. Consider an adjacency matrix graph representation augmented by the two fields

   - `nC` (number of connected components)
   - `cc[]` (connected components array)

   These fields are initialised as follows:

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV=V, g.nE=0, g.nC=V
|  allocate memory for g.edges[][]
|  for all i=0..V-1 do
|     g.cc[i]=i
|     for all j=0..V-1 do
|        g.edges[i][j]=0
|     end for
|  end for
|  return g
```

Modify the pseudocode for edge insertion and edge removal from the lecture to maintain the two new fields.

Answer:

a. After removing *d*, `cc[] = {0,0,0,1,1,1,0,0,0,1}` (i.e. unchanged)
After removing *b*, `cc[] = {0,0,2,1,1,1,2,0,2,1}` with nC=3

b. Inserting an edge may reduce the number of connected components:

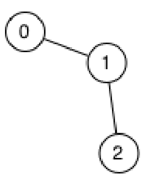
```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]=0 then                    // (v,w) not in graph
|  |  g.edges[v][w]=1, g.edges[w][v]=1        // set to true
|  |  g.nE=g.nE+1
|  |  if g.cc[v]≠g.cc[w] then                 // v,w in different components?
|  |  |  c=min{g.cc[v],g.cc[w]}               // ⇒ merge components c and d
|  |  |  d=max{g.cc[v],g.cc[w]}
|  |  |  for all vertices v∈g do
|  |  |     if g.cc[v]=d then
|  |  |        g.cc[v]=c                      // move node from component d to c
|  |  |     else if g.cc[v]=g.nC-1 then
|  |  |        g.cc[v]=d                      // replace largest component ID by d
|  |  |     end if
|  |  |  end for
|  |  |  g.nC=g.nC-1
|  |  end if
|  end if
```

Removing an edge may increase the number of connected components:

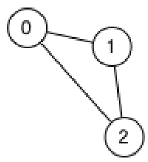
```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]≠0 then                    // (v,w) in graph
|  |  g.edges[v][w]=0, g.edges[w][v]=0        // set to false
|  |  if not hasPath(g,v,w) then              // v,w no longer connected?
|  |     dfsNewComponent(g,v,g.nC)            // ⇒ put v + connected vertices into new component
|  |     g.nC=g.nC+1
|  |  end if
|  end if

dfsNewComponent(g,v,componentID):
|  Input graph g, vertex v, new componentID for v and connected vertices
|
|  g.cc[v]=componentID
|  for all vertices w adjacent to v do
|     if g.cc[w]≠componentID then
|        dfsNewComponent(g,w,componentID)
|     end if
|  end if
```

7. (Hamiltonian/Euler paths and circuits)

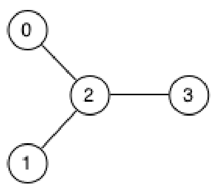
a. Identify any Hamiltonian/Euler paths/circuits in the following graphs:



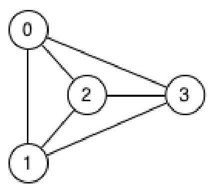
Graph 1



Graph 2

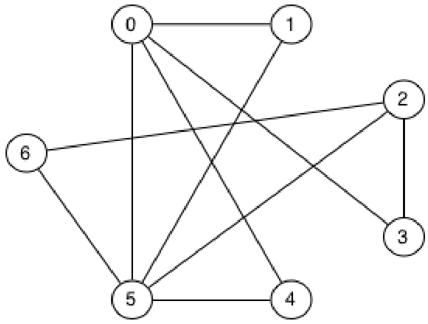


Graph 3



Graph 4

b. Find an Euler path and an Euler circuit (if they exist) in the following graph:



Answer:

a. Graph 1: has both Euler and Hamiltonian paths (e.g. 0-1-2), but cannot have circuits as there are no cycles.

   Graph 2: has both Euler paths (e.g. 0-1-2-0) and Hamiltonian paths (e.g. 0-1-2); also has both Euler and Hamiltonian circuits (e.g. 0-1-2-0).

   Graph 3: has neither Euler nor Hamiltonian paths, nor Euler nor Hamiltonian circuits.

   Graph 4: has Hamiltonian paths (e.g. 0-1-2-3) and Hamiltonian circuits (e.g. 0-1-2-3-0); it has neither an Euler path nor an Euler circuit.

b. An Euler path: 2-6-5-2-3-0-1-5-0-4-5

   No Euler circuit since two vertices (2 and 5) have odd degree.