Game theory

(14.1 - 14.3)

Game theory studies mathematical models of decision-making when two or more decision makers are involved, and their decisions affect each other's outcome.

In this class we will study the classic two-player zero-sum game model.

Penalty kicks: A right-footed soccer player must choose to kick left or right, and the goalie must choose to lean left or right. If the goalie guesses right, the kick is blocked. If they guess wrong, the kick will go in 80% of the time if the kicker kicked left, and 40% if they kicked right.

Both players are trying to maximize the difference between their score and their opponent score.

Payoff matrix

Kicker \ Goalie	Lean left	Lean right
Kick left	0, 0	0.8, -0.8
Kick right	0.4, -0.4	0, 0

first number = kicker's score − goalie's score

second number = goalie's score − kicker's score

<u>Rock-paper-scissors</u>: Two players each choose rock (R), paper (P), or scissors (S). P beats R, S beats P, R beats S. The winning player wins 1 point, and 0 points are awarded in a tie. Each player is trying to maximize the difference between their points and their opponent's points.

P1 \ P2	R	P	S
R	0, 0	-1, 1	1, -1
P	1, -1	0, 0	-1, 1
S	-1, 1	1, -1	0, 0

These are examples of two-player zero-sum games.

1) Two players, which we'll call "P1" and "P2" or "row player" and "column player".

2) P1 chooses one of m strategies, P2 chooses one of n strategies (choices can be different for each player).

3) If P1 chooses strategy i and P2 chooses strategy j, then P1 earns c_{ij} and P2 earns $-c_{ij}$, where c_{ij} is any real number. The matrix $(c_{ij}, -c_{ij})$ is the payoff matrix.

Rival networks: During the 8-9 PM timeslot, two networks are competing for an audience of 100 million viewers. Each network can air either a comedy, drama, or reality show. The number of viewers network 1 will have for each pair of choices is shown. Assume all other viewers will watch network 2.

N1 \ N2	C	D	R
C	35, 65	15, 85	60, 40
D	45, 55	58, 42	50, 50
R	45, 55	58, 42	70, 30

Payoff matrix for N1

Payoff matrix for N2

This is an example of a constant-sum game. The payoff for N1 plus the payoff for N2 is always 100 million, no matter the choices. A zero-sum game is a constant-sum game where the payoff sum is 0.

Any constant-sum game can be converted to a zero-sum game by subtracting $\underbrace{c}_{\text{sum of payoffs}}/2$ from all payoffs. All our analysis of zero-sum games will carry over to constant-sum games.

Analyzing rock-paper-scissors

Suppose two players are playing RPS repeatedly. What is the "best" strategy for player 1?

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Payoff matrix for P1.

Strategy 1 : Rock only

If P2 knows this strategy, P2 would choose P every time.

The payoff for P1 would be -1 every time.

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Strategy 2: $\frac{1}{2}$ rock, $\frac{1}{2}$ paper

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

If P2 knows this strategy:

If P2 plays R, P1's expected payoff is

$$\frac{1}{2}(0)+\frac{1}{2}(1)=\frac{1}{2}$$

If P2 plays P, P1's expected payoff is

$$\frac{1}{2}(-1)+\frac{1}{2}(0)=-\frac{1}{2}$$

If P2 plays S, P1's expected payoff is

$$\frac{1}{2}(1)+\frac{1}{2}(-1)=0$$

Of these, the lowest expected payoff for P1 is if P2 plays P. So P2 would play P, and P1's expected payoff is $-\frac{1}{2}$.

<u>Strategy 3</u>: $\frac{1}{3}R$, $\frac{1}{3}P$, $\frac{1}{3}S$

If P2 knows this, P1's expected payoff is 0 no matter what P2 does.

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

General strategy: Let's consider a strategy where we pick R with probability x_R, P with probability x_P, and S with probability x_S. The tuple (x_R, x_P, x_S) is called a mixed strategy. What is the best mixed strategy for RPS?

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Suppose P2 plays R

Then P1's expected payoff is

$$x_R(0) + x_P(1) + x_S(-1)$$

$$= x_P - x_S$$

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

P2 plays P

P1's expected payoff is

$$x_R(-1) + x_P(0) + x_S(1)$$

$$= -x_R + x_S$$

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

P2 plays S

P1's expected payoff is

$$x_R(1) + x_P(-1) + x_S(0)$$

$$= x_R - x_P$$

So P1's expected payoff is one of

$x_P - x_S$, $-x_R + x_S$, $x_R - x_P$ depending on what P2 does.

Let's assume the worst case scenario, that our opponent knows our mixed strategy (x_R, x_P, x_S). Since the opponent will always choose the strategy which minimizes our expected payoff (equivalently, maximizes their expected payoff since this is a zero-sum game), our expected payoff will be

$$\min(x_P - x_S, -x_R + x_S, x_R - x_P)$$

Therefore, our objective is to maximize

$$\min(x_P - x_S, -x_R + x_S, x_R - x_P)$$

over the possible mixed strategies (x_R, x_P, x_S).

Let

$$z = \min(x_P - x_S, -x_R + x_S, x_R - x_P)$$

$$\text{Max } z$$

$$\text{s.t. } z \le x_P - x_S \qquad x_R + x_P + x_S = 1$$

$$z \le -x_R + x_S \qquad 0 \le x_R, x_P, x_S \le 1$$

$$z \le x_R - x_P$$

If we solve this LP, we get an optimal solution of

$$z = 0 \qquad x_R = 1/3 \qquad x_P = 1/3 \qquad x_S = 1/3$$

Thus, under the assumption that the opponent knows our mixed strategy, the best we can do is a mixed strategy of $(1/3, 1/3, 1/3)$ with payoff 0.