Linear programs in two variables (3.2 and 3.3)

Recall that a linear program is an optimization problem where the objective function (what we are trying to maximize or minimize) is a linear function, and the constraints are linear equations and inequalities.

Any vector which satisfies all the constraints is a feasible solution. Any feasible solution which optimizes the objective function is an optimal solution, and the value of the objective function at one of these points is the optimal value. There can be multiple optimal solutions, but only one optimal value.

Max $x+y$

s.t. $2x+y \le 6$

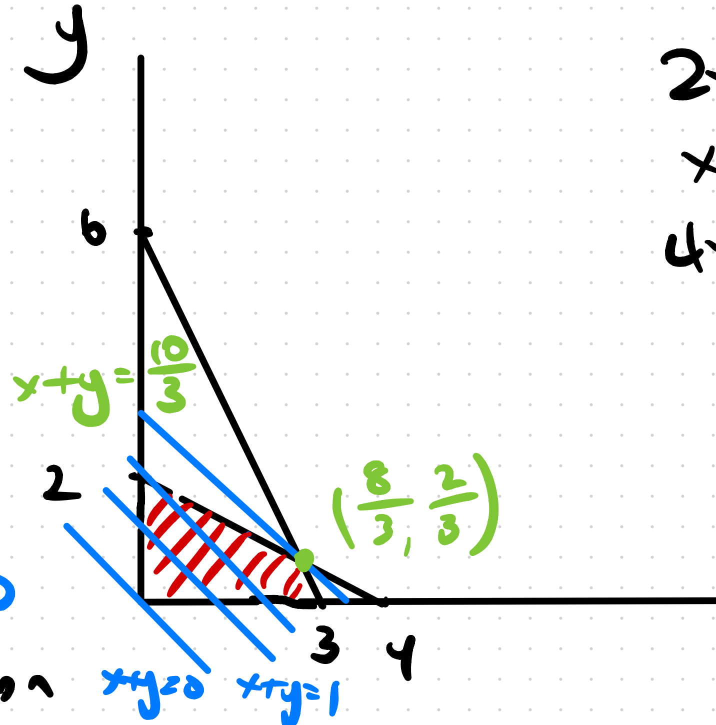
$x+2y \le 4$

$x, y \ge 0$

Level set method:

Construct the level set of the objective function $x+y$. These are the lines $x+y=c$ for some constant c.

Change c to see how the level set move.



$2x+y=6$

$x+2y=4$

$4x+2y=12$

$3x=8$

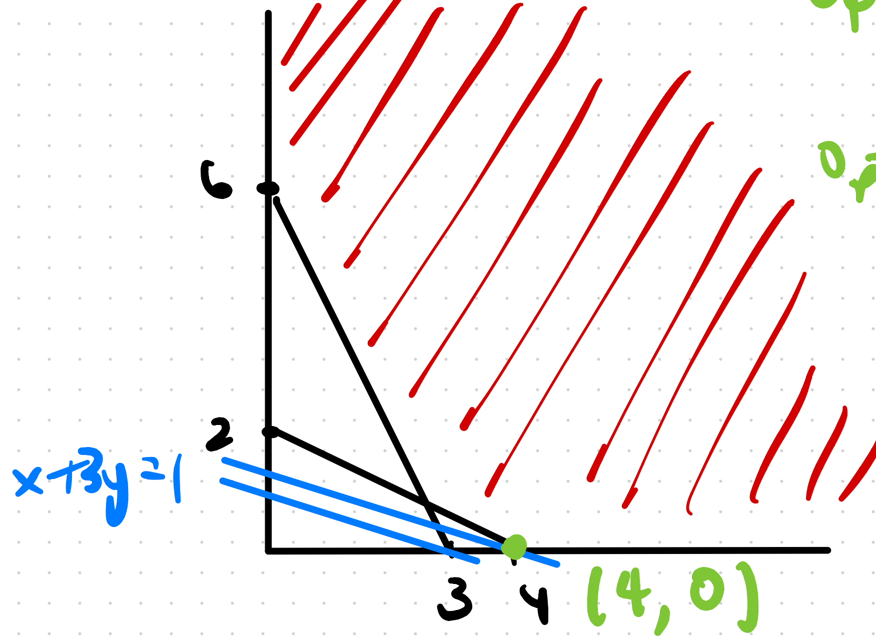
$x=\frac{8}{3}$

$y=\frac{2}{3}$

Optimal solution: $\left(\frac{8}{3}, \frac{2}{3}\right)$

Optimal value: $\frac{10}{3}$

$$\text{Min } x+3y$$
$$\text{s.t. } 2x+y \geq 6$$
$$x+2y \geq 4$$
$$x,y \geq 0$$



Optimal solution:
(4,0)

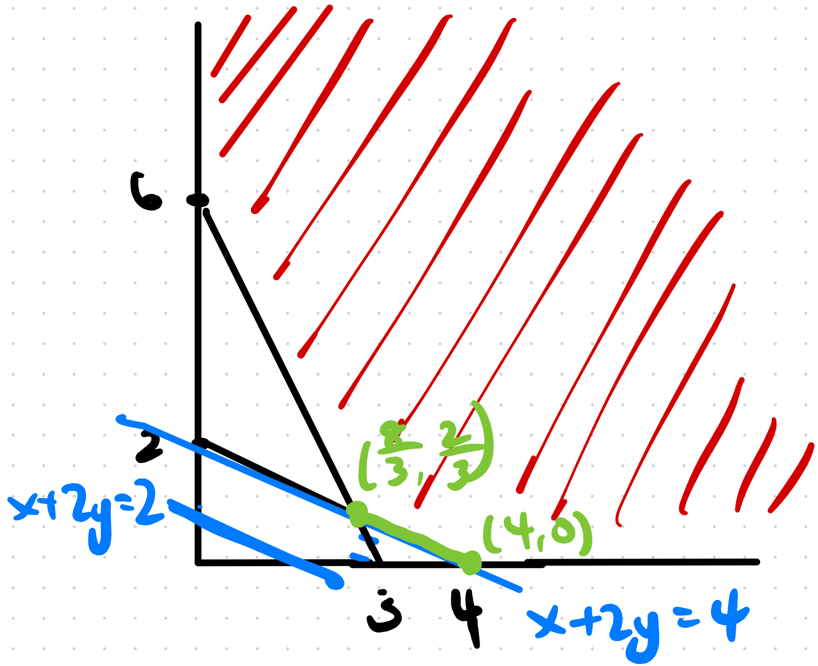
Optimal value: 4

Min $x + 2y$

s.t. $2x + y \geq 6$

$x + 2y \geq 4$

$x, y \geq 0$



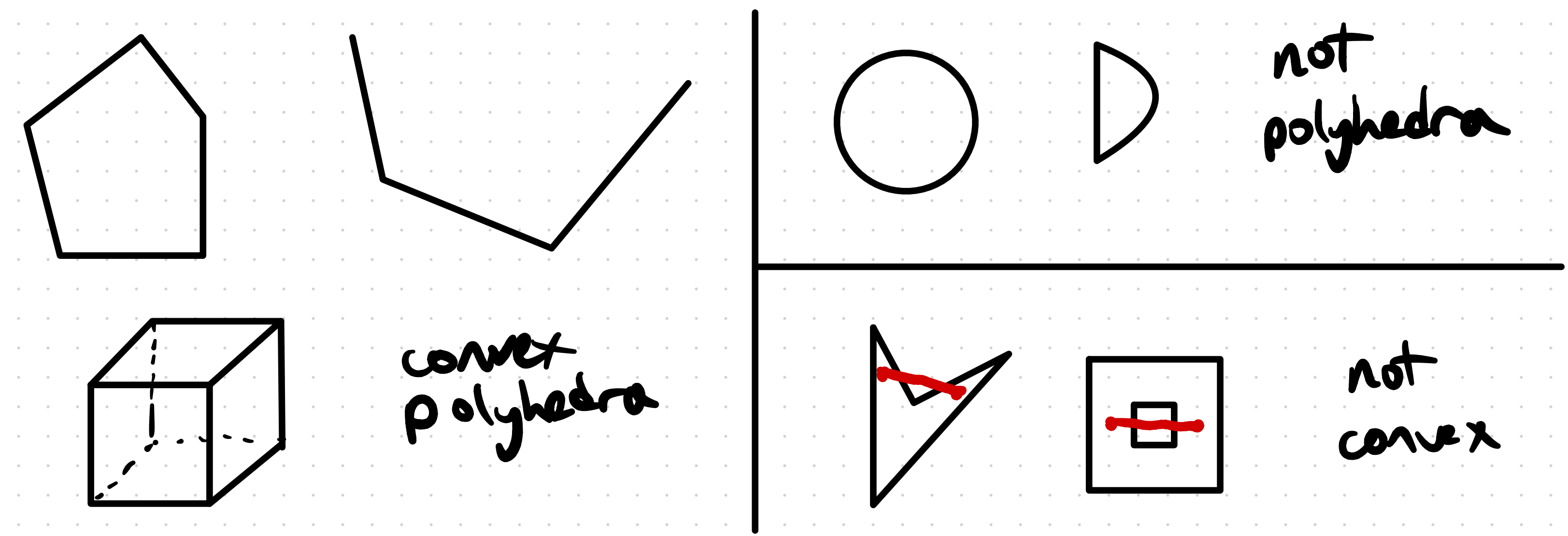
Every point on the segment $\left[\left(\frac{8}{3}, \frac{2}{3}\right), (4,0)\right]$ is an optimal solution.

Optimal value is 4

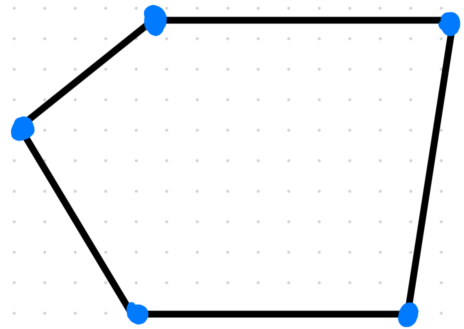
The feasible region of a linear program is a **convex polyhedron**.

Convex: If two points are feasible, then any point on the line segment between them is feasible.

Polyhedron: A shape with "flat sides".



A "corner" of a polyhedron is called a vertex.



<u>Thm</u>: If a LP is bounded and the feasible region has vertices then it has an optimal solution at a vertex.

As a result, to solve an LP we can check the objective function at the vertices of the feasible region.

A farmer must decide how many acres of corn and wheat to plant this year.

- An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week.
- An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week.
- Wheat is sold at \$4 / bushel, corn at \$3 / bushel.
- 7 acres of land and 40 hours/week of labor are available.
- Government requires at least 30 bushels of corn to be produced.

What should the farmer do to maximize profit?

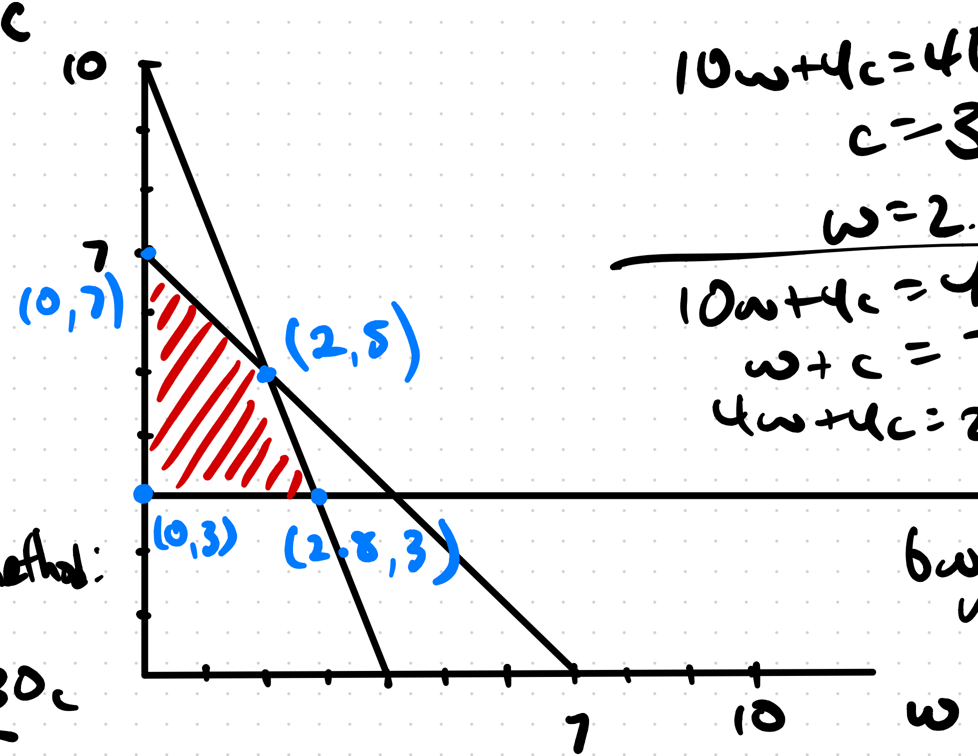
Max $100w + 30c$

s.t. $w + c \le 7$

$10w + 4c \le 40$

$10c \ge 30$

$w, c \ge 0$



$10w + 4c = 40$
$c = 3$
$w = 2.8$

$10w + 4c = 40$
$w + c = 7$
$4w + 4c = 28$

$6w = 12$
$w = 2$
$c = 5$

"Check the vertices" method:

Vertices	$100w + 30c$
(0, 3)	90
(0, 7)	210
(2.8, 3)	370
(2, 5)	350

Optimal solution: (2.8, 3)
Optimal value: 370

If there are more than two variables, then the feasible region lives in higher-dimensional space. In theory, both the level set method and check-vertices method will work.

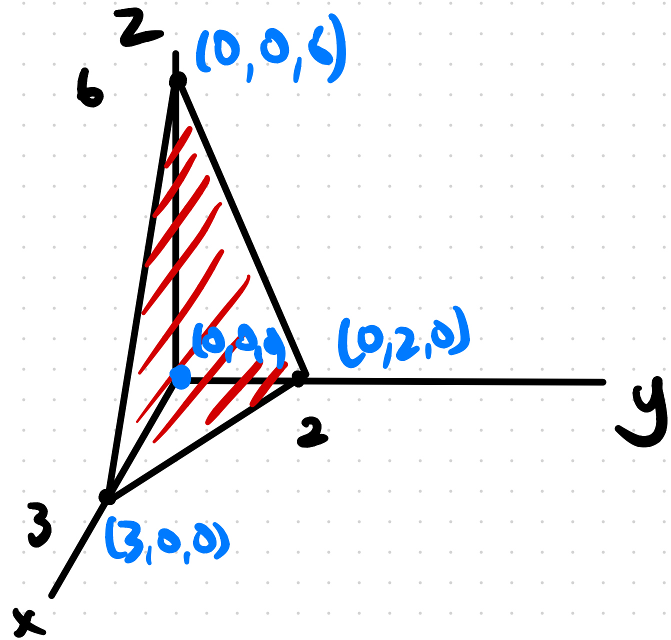
Max $x+y+z$

s.t. $2x+3y+z \leq 6$

$x, y, z \geq 0$

Optimal solution: $(0,0,6)$

Optimal value: 6



In practice, it's not reasonable to solve LP's with more than two variables by hand. (Typical LP's in applications have thousands of variables.) We instead use computer algorithms.