MATH 381

Discrete Mathematical Modeling

Instructor: Gaku Liu, he/him, "Gaku", "Professor Liu", gakuliu@uw.edu

TA: Nathan Cheung, ncheung@uw.edu

Office hours: TBA

A skeleton of each lecture will be provided before that lecture. It will then be updated afterwards.

Textbook: Operations Research, 4^{th} ed by Wayne Winston. See syllabus for link.

Weekly assignments, two in-class exams.
No exam during finals week.

First homework due Friday, another homework due next Thursday.

Programming: Some assignments will require you to write code. Some of this will be through Google Colab notebooks. See Homework 0.

You may be asked to read and write Python code on the exam.

Two independent units:

(1) Linear programming, integer programming, graphs, game theory

(2) Stochastic processes, Monte Carlo simulation, Markov chains.

Exams will be independent of each other.

Overview of mathematical modeling (Chapter 1)

A **mathematical model** is a mathematical description of some aspect of reality.

- Laws describing the physical world: Newton's laws, relativity, quantum mechanics.
- Economic models of markets and consumer behavior.
- A map is a flat representation of a portion of the Earth's surface.

The purpose of a model is to better understand a system, or suggest a course of action.

A model is almost never a completely accurate description of a system. In fact, the main reason models are used is to make a problem simple enough to be tractable.

"All models are wrong, but some are useful."

— George Box

"Truth is much too complicated to allow anything but approximations."

— John von Neumann

The goal of this class is to learn some useful models, recognize when to use them, and understand their limitations.

Linear programs
(Section 3.1)

A farmer must decide how many acres of corn and wheat to plant this year.

- An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week.
- An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week.
- Wheat is sold at \$4 / bushel, corn at \$3 / bushel.
- 7 acres of land and 40 hours/week of labor are available.
- Government requires at least 30 bushels of corn to be produced.

What should the farmer do to maximize profit?

This is an example of an optimization problem. You've encountered these in calculus. The basic principles for modeling them are as follows:

1) State the variables that are relevant (often the hardest part!) what we are allowed to change

2) State the objective function you're trying to optimize.

3) State the constraints on the variables.

The problem I gave is already simplified from a real situation. In practice, you would need to simplify yourself.

A farmer must decide how many acres of corn and wheat to plant this year.

- An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week.
- An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week.
- Wheat is sold at \$4 / bushel, corn at \$3 / bushel.
- 7 acres of land and 40 hours/week of labor are available.
- Government requires at least 30 bushels of corn to be produced.

What should the farmer do to maximize profit?

1) State variables:

c = # of acres of corn the farmer plants

w = # of acres of wheat " "

2) Objective function:

Want to maximize profit

$10c$ bushels of corn produced

$25w$ bushels of wheat produced

$4 \cdot 25w + 3 \cdot 10c$ profit

Maximize $100w + 30c$

3) Constraints:

$10c \geq 30$ government constraint

$c+w \leq 7$ land constraint

$10w + 4c \leq 40$ labor constraint

$c, w \geq 0$ nonnegativity constraints

We'll assume fractional amount of acres are possible.

Maximize $100w + 30c$

s.t.

such that

$$10c \geq 30$$

$$c + w \leq 7$$

$$10w + 4c \leq 40$$

$$c, w \geq 0$$

Let $x_1, x_2, \ldots, x_n$ be real-valued variables. A linear function is a function $f(x_1, \ldots, x_n)$ of the form

$$f(x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

where $c_1, \ldots, c_n$ are real constants.

Linear equation:

$$f(x_1, \ldots, x_n) = b$$

a real constant

Linear inequality (non-strict):

$$f(x_1, \ldots, x_n) \leq b \quad \text{or} \quad f(x_1, \ldots, x_n) \geq b$$

A **linear program (LP)** is an optimization problem where the objective function is linear, there are finitely many constraints, and all constraints are linear equations and non-strict linear inequalities.

Examples:

$$\begin{aligned} \text{Max } & a+b \\ \text{s.t. } & a-c \leq 1 \\ & 3a+2b+c=1 \\ & -10 \leq a,b,c \leq 10 \end{aligned}$$

$$a \geq -10 \quad b \geq -10 \quad c \geq -10$$
$$a \leq 10 \quad b \leq 10 \quad c \leq 10$$

$$\begin{aligned} \text{Min } & 3x-2y+z+0w \\ \text{s.t. } & x+y \geq 1 \\ & w+y+z \leq 2 \end{aligned}$$

$$\begin{aligned} \text{Max } & 0 \\ \text{s.t. } & x+y \geq 3 \\ & z = \pi \end{aligned}$$

To "solve" an optimization problem, we want to find the optimal value of the objective function, as well as assignment of variables that achieve that optimum.

Non-examples

$$\begin{aligned} &\text{Max} \quad x^2+y^2 \\ &\text{s.t.} \quad 0 \le x, y \le 1 \end{aligned}$$

← nonlinear objective function

$$\begin{aligned} &\text{Min} \quad x+y \\ &\text{s.t.} \quad xy \le 1 \\ &\qquad\; x, y \ge 0 \end{aligned}$$

← nonlinear constraint

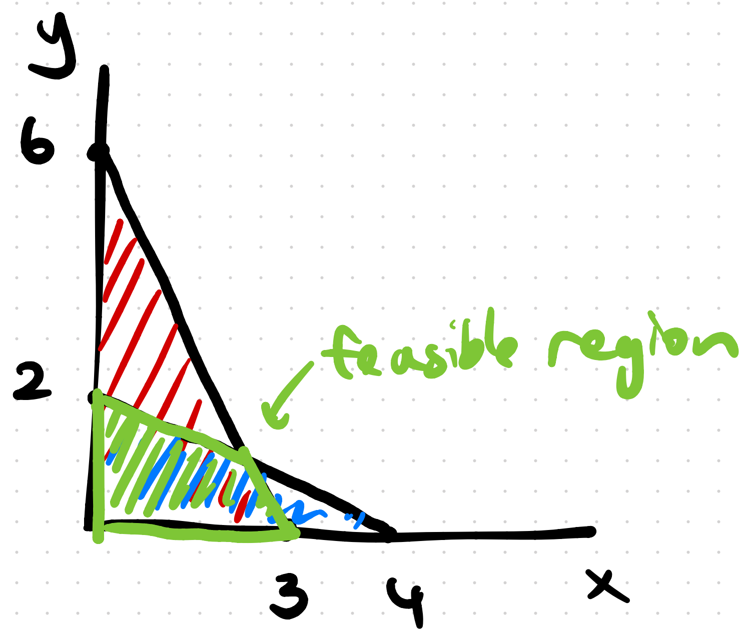
$$\begin{aligned} &\text{Max} \quad x+y \\ &\text{s.t.} \quad 0 \le x, y < 1 \end{aligned}$$

↑ strict inequality

} This does not have an optimum because $x+y$ can be arbitrarily close to 2, but not equal to 2.

The feasible region of a linear program in variables $x_1, \dots, x_n$ is the set of all $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ which satisfy the constraints.

$$\begin{aligned} \text{Max} \quad & x+y \\ \text{s.t.} \quad & 2x+y \le 6 \\ & x+2y \le 4 \\ & x, y \ge 0 \end{aligned}$$



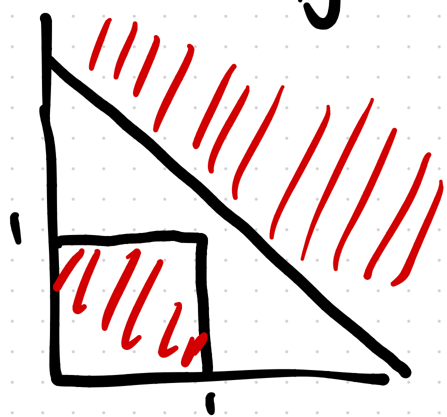
A linear program is ...

infeasible if the feasible region is empty

Max 0

s.t. $x+y \geq 3$

$0 \leq x, y \leq 1$



unbounded if the optimal value of the objective function is ∞ or $-\infty$

Max $x+y$

s.t. $2x+y \geq 5$

$x, y \geq 0$