Game theory cont.

Analyzing rock-paper-scissors

Suppose two players are playing RPS repeatedly. What is the "best" strategy for player 1?

P1 \ P2	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Player 1's payoff matrix

A **mixed strategy** (x_R, x_P, x_S) is a strategy where we play R with probability x_R, P with probability x_P, and S with probability x_S.

Last time, we showed through linear programming that if the opponent knows our mixed strategy, then the optimal mixed strategy is $(1/3, 1/3, 1/3)$ with expected payoff 0.

What if the opponent doesn't know our mixed strategy?

If we decide to use the mixed strategy (1/3, 1/3, 1/3), the worst case scenario is an expected 0 payoff. This means that in any scenario, this strategy guarantees at least 0 expected payoff.

If the opponent uses the mixed strategy (1/3, 1/3, 1/3), this guarantees the opponent at least 0 expected payoff, and therefore guarantees at most 0 expected payoff for us.

<u>Penalty kicks</u>: A right-footed soccer player must choose to kick left or right, and the goalie must choose to lean left or right. If the goalie guesses right, the kick is blocked. If they guess wrong, the kick is more likely to go in if the kicker kicked left (80% vs 40%).

K \ G	Lean left	Lean right
Kick left	0	0.8
Kick right	0.4	0

Assume the kicker uses mixed strategy (x_L, x_R).

If the goalie knows (x_L, x_R) . . .

K \ G	L	R
L	0	0.8
R	0.4	0

If the goalie chooses lean left, the kicker's expected payoff will be

$$x_L(0) + x_R(0.4)$$

$$0.4x_R$$

If goalie chooses lean right, kicker's expected payoff is

$$x_L(0.8) + x_R(0)$$

$$0.8x_L$$

The goalie will choose the smaller of these two numbers.

Therefore, in the worst case scenario, the kicker's expected payoff is

$$\min(0.4x_R,\ 0.8x_L) = z$$

Kicker wants to choose x_L, x_R so that this function is maximized.

$$\begin{aligned} \text{Max } & z \\ \text{s.t. } & z \leq 0.4x_R \qquad 0 \leq x_L, x_R \leq 1 \\ & z \leq 0.8x_L \qquad x_L + x_R = 1 \end{aligned}$$

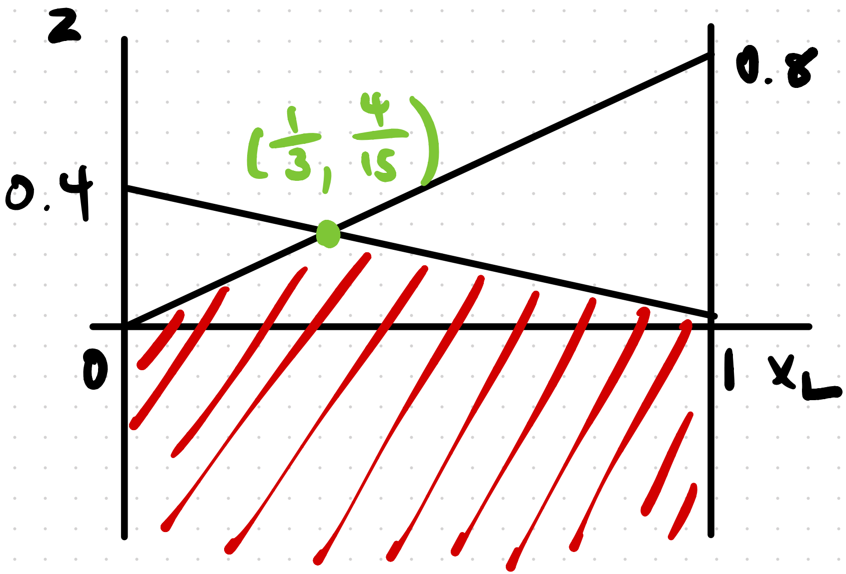
$x_R = 1 - x_L$

Substitution

Max z

s.t. $0 \le x_L \le 1$ $\quad z \le 0.4(1 - x_L)$

$z \le 0.8 x_L$



$x_L = \frac{1}{3}$ $\quad z = \frac{4}{15}$ $\quad x_e = 1 - x_L = \frac{2}{3}$

$$0.4(1 - x_L) = 0.8 x_L$$

$$0.4 = 1.2 x_L$$

$$\frac{1}{3} = x_L$$

$$0.8 \cdot \frac{1}{3} = \frac{4}{5} \cdot \frac{1}{3} = \frac{4}{15}$$

Optimal solution is

$$z = \frac{4}{15} \qquad x_L = \frac{1}{3} \qquad x_R = \frac{2}{3}$$

Therefore, if the kicker kicks left with probability 1/3 and right with probability 2/3, then they have an expected payoff of 4/15 in the worst case scenario. Thus, this strategy guarantees at least 4/15 expected payoff in any scenario.

Let's do the same for the goalie.
Assume the goalie uses mixed strategy (y_L, y_R).

K \ G	L	R
L	0	0.8
R	0.4	0

Still payoff matrix for Kicker

Assume kicker knows y_L, y_R

If kicker chooses to kick L, then the kicker's expected payoff is

$$y_L(0) + y_R(0.8)$$

$$0.8 y_R$$

If kicker chooses to kick R then the kicker's expected payoff

$$y_L(0.4) + y_R(0)$$

$$0.4 y_L$$

The kicker will choose the larger of these two numbers.

Therefore, in the best case scenario (for the kicker), the kicker's expected payoff is

$$\max(0.8y_R, 0.4y_L) = z$$

Goalie wants to choose y_L, y_R so that this number is as small as possible.

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Min z

s.t. $z \geq 0.8y_R$ $\qquad 0 \leq y_L, y_R \leq 1$

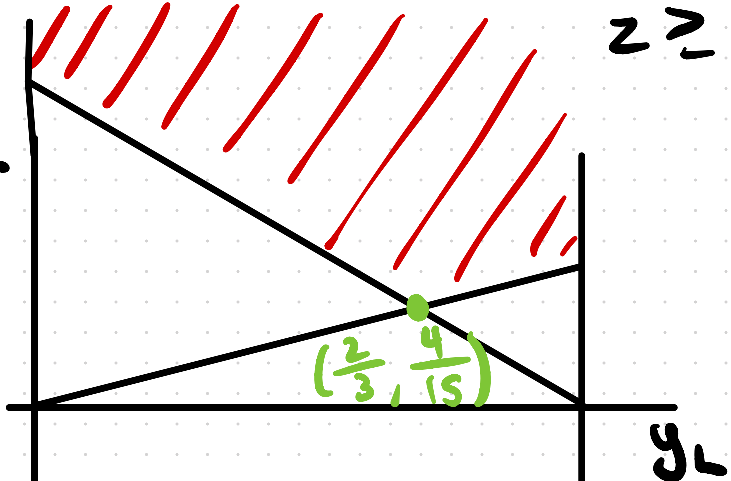
$z \geq 0.4y_L$ $\qquad y_L + y_R = 1$ $\rightarrow$ $y_R = 1 - y_L$

substitution

$$\text{Min } z$$

$$\text{s.t. } 0 \le y_L \le 1 \qquad z \ge 0.8(1-y_L)$$

$$z \ge 0.4 y_L$$



$$0.8(1-y_L) = 0.4 y_L$$

$$0.8 = 1.2 y_L$$

$$\frac{2}{3} = y_2$$

$$z = 0.4 \cdot \frac{2}{3} = \frac{2}{5} \cdot \frac{2}{3} = \frac{4}{15}$$

$$x_L = \frac{2}{3}$$

$$z = \frac{4}{15}$$

$$x_R = 1 - \frac{2}{3} = \frac{1}{3}$$

Optimal solution is

$$z = \frac{4}{15} \qquad y_L = \frac{2}{3} \qquad y_R = \frac{1}{3}$$

Therefore, if the goalies uses the mixed strategy $(2/3, 1/3)$, the kicker has expected payoff $4/15$ in their best case scenario. Thus, by using this strategy, the goalie guarantees at most $4/15$ expected payoff for the kicker.

Theorem: For any two-player zero-sum (or constant-sum) game, there exist a number z such that

(1) P1 has a mixed strategy that guarantees at least z expected payoff for P1.

(2) P2 has a mixed strategy that guarantees at most z expected payoff for P1.

The number z is called the value of the game. The strategies from (1) and (2) are called optimal strategies for their respective players.

The game theory model predicts that two rational players who assume the other player is rational will play their respective optimal strategies.

The optimal strategies form an equilibrium: If both players are playing optimal strategies (or believe their opponent is doing so), then there is no incentive for either to use a different strategy.

How close is this model to reality?

Chiappori, Levitt, Groseclose (2002): Studied behavior of professional soccer players. Kickers and goalies do randomize, and kicker's kick to their better side 45% of the time, while goalies play to that side 57% of the time.

How to find optimal strategy for P1:

1) Assume P1 uses mixed strategy $(x_1, \dots, x_m)$

2) For each of P2's strategies $j = 1, \dots, n$, determine P1's expected payoff if P2 uses j. This will be

$$f_j(x_1, \dots, x_m) = \sum_{i=1}^{m} c_{ij} x_i$$ sum down column j

3) Solve Max min$(f_1, \dots, f_n)$. This can be done through the LP

Max z → value of game (expected payoff for P1)

s.t. $z \leq f_j(x_1, \dots, x_m)$ for all columns j

$x_1 + \dots + x_m = 1$ $(x_1, \dots, x_m)$ optimal strategy for P1

$x_1, \dots, x_m \geq 0$

How to find optimal strategy for P2:

1) Assume P2 uses mixed strategy $(y_1, \ldots, y_n)$

2) For each of P1's strategies $i = 1, \ldots, m$, determine P1's expected payoff if P1 uses i. This will be

$$g_i(y_1, \ldots, y_n) = \sum_{j=1}^{n} c_{ij} y_j \quad \text{sum across row } i$$

3) Solve $\text{Min} \max(g_1, \ldots, g_m)$. This can be done through the LP

$$\text{Min } z \rightarrow \text{value of the game (expected payoff for P1)}$$

$$\begin{aligned} \text{s.t. } & z \geq g_i(y_1, \ldots, y_n) \quad \text{for all rows } i \\ & y_1 + \cdots + y_n = 1 \qquad (y_1, \ldots, y_n) \text{ optimal strategy for P2} \\ & y_1, \ldots, y_n \geq 0 \end{aligned}$$