Integer programming (9.1, 9.2)

An integer program (IP) is a linear program with the additional constraint that the variables must be integers.

A mixed integer program (MIP) is a linear program where some of the variables are constrained to integers, some are not.

A 0-1 integer program is an integer program where the variables are all 0 or 1.

Max $2x+y$

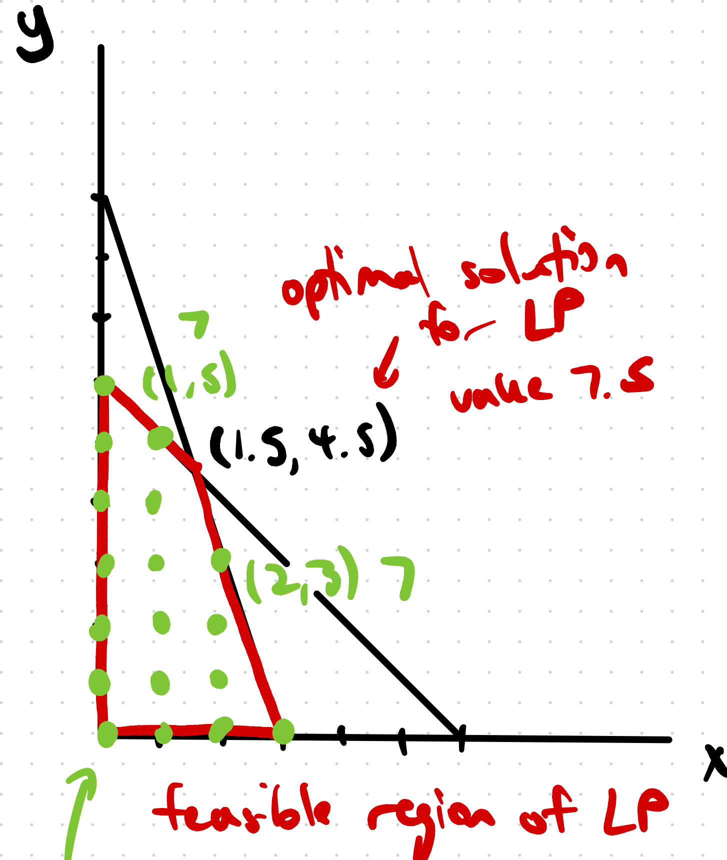
s.t. $3x+y \leq 9$

$x+y \leq 6$

$x,y \geq 0$

$x,y \in \mathbb{Z}$

x and y are integers



$3x+y=9$

$x+y=6$

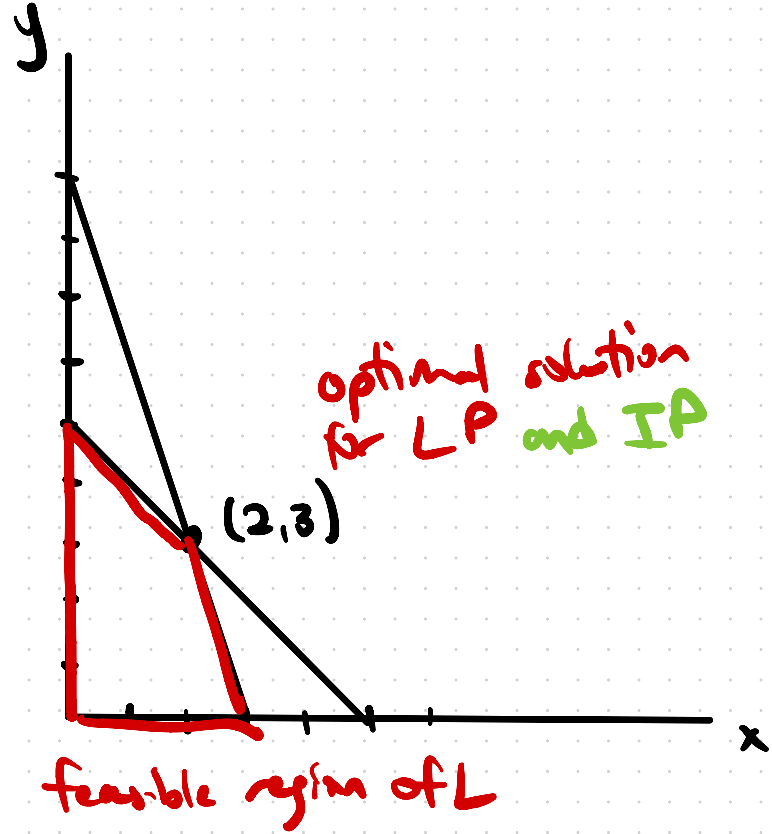
$2x=3$

$x=1.5$

$y=4.5$

(1,5) and (2,3) are optimal solutions optimal value is 7

$$\text{Max } 2x+y$$
$$\text{s.t. } 3x+y \le 9$$
$$x+y \le 5$$
$$x,y \ge 0$$
$$x,y \in \mathbb{Z}$$



Integer constraints shrink the feasible region of a program. The optimal solution might no longer be a vertex of the original polyhedron.

As we'll discuss next lecture, integer constraints can greatly increase the difficulty of the problem. It is up to you to decide whether they are needed.

E.g.: When working with large variables like money, integer constraints are probably not needed. When dealing with small variables like 0-1 variables, they are probably necessary.

A camper is going on a hike. There are four items they are considering taking. The weight of each item and "benefit" of each are given below. The camper can carry 14 lb. How can the camper maximize benefit?

	weight (lb)	benefit
1	5	16
2	7.5	22
3	4	12
4	3	8.5

Model as an IP

$$x_i = \begin{cases} 1 & \text{if we bring item } i \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Max } 16x_1 + 22x_2 + 12x_3 + 8.5x_4$$

$$\text{s.t. } 5x_1 + 7.5x_2 + 4x_3 + 3x_4 \le 14$$

$$0 \le x_i \le 1 \text{ for all } i$$

$x_i \in \mathbb{Z}$ for all i

$x_i = 0$ or 1 for all i

"Knapsack problem"

What if at most two items can be brought?

$$x_1 + x_2 + x_3 + x_4 \leq 2$$

What if item 2 cannot be brought without item 1?

$$x_2 \leq x_1$$

What if bringing item 2 means item 1 can't be brought?

$$x_1 + x_2 \leq 1$$

A county wants to build fire stations in its six cities. They require that every city is within 15 minutes of at least one fire station. The time it takes to drive between cities is in the below table. Find the minimum number of stations that can be built and where to build them.

	1	2	3	4	5	6
1	0	10	20	30	30	20
2	10	0	25	35	20	10
3	20	25	0	15	30	20
4	30	35	15	0	15	25
5	30	20	30	15	0	14
6	20	10	20	25	14	0

$$x_i = \begin{cases} 1 & \text{if we build in city } i \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Minimize} \sum_{i=1}^{6} x_i$$

s.t. $x_1 + x_2 \geq 1$ city 1 must be within 15 min of a station

$x_1 + x_2 + x_6 \geq 1$ city 2

$x_3 + x_4 \geq 1$ ⋮

$x_3 + x_4 + x_5 \geq 1$ ⋮

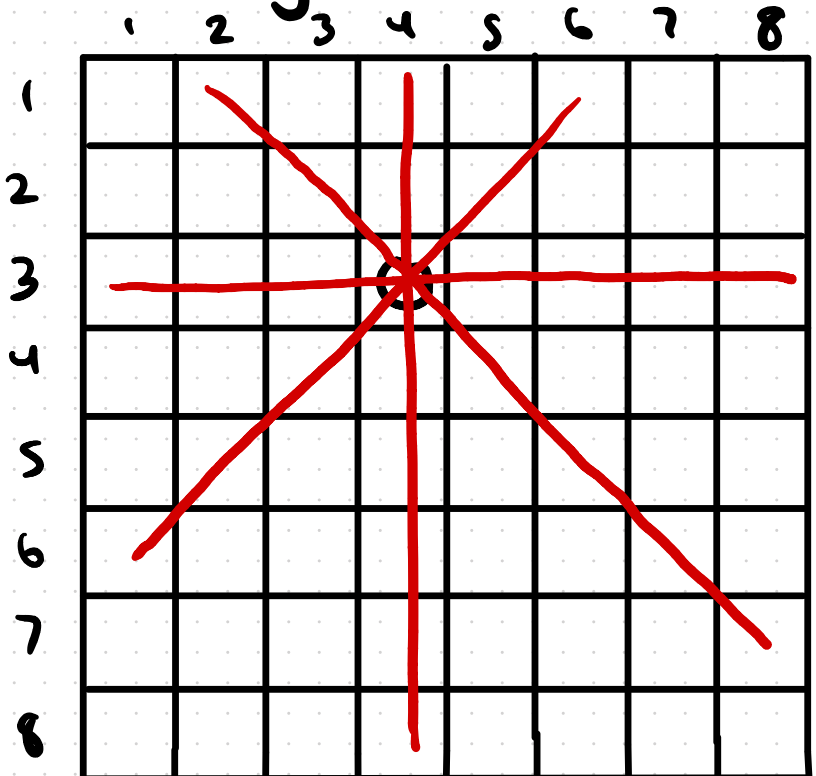
$x_4 + x_5 + x_6 \geq 1$ ⋮

$x_2 + x_5 + x_6 \geq 1$ ⋮

$x_i = 0$ or 1 for all i

$\left(\begin{matrix} 0 \leq x_i \leq 1 \\ x_i \in \mathbb{Z} \end{matrix} \text{ for all } i \right)$

How many queens can be placed on an 8×8 chessboard so that no two are attacking each other?



$$x_{ij} = \begin{cases} 1 & \text{if square } (i,j) \text{ has a queen} \\ 0 & \text{otherwise.} \end{cases}$$

Maximize $\sum_{i=1}^{8} \sum_{j=1}^{8} x_{ij}$

s.t. $x_{ij} = 0$ or 1 for all i, j

$$\underbrace{\text{For all } i, \quad \sum_{j=1}^{8} x_{ij} \leq 1}_{\text{list of 8 constraints}}$$

each row has at most one queen.

For all j, $\sum_{i=1}^{8} x_{ij} \leq 1$

For every diagonal,

$$\sum_{\substack{(i,j) \text{ in that} \\ \text{diagonal}}} x_{ij} \leq 1$$