Graphs

(8.1, 8.2)

Suppose we want to color a map of the world so that

(a) Each country is colored with one color.

(b) If two countries share a border, they have different colors.

How can we produce such a coloring that uses only five colors?

This problem can be modeled with a graph.

A **graph** consists of the following data:

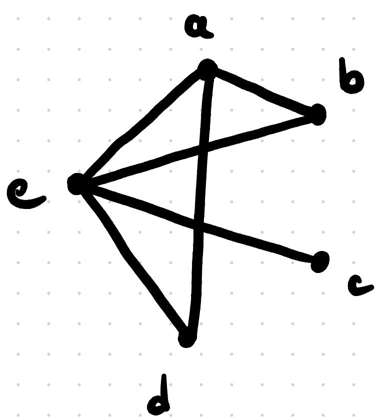
(1) A set of **vertices**. 

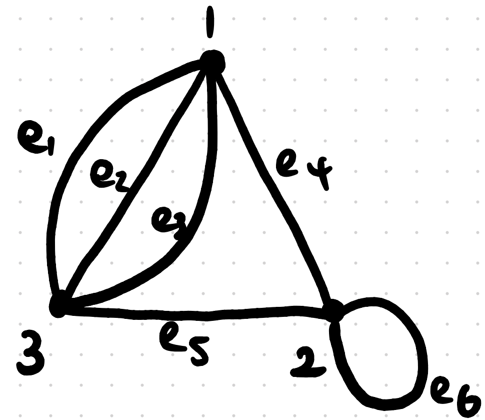
(2) A set of **edges**. Each edge connects two vertices, called the **endpoints**. (The endpoints can be the same, in which the edge is called a **loop**.) 

(3) (optional) A direction for each edge, pointing from one endpoint to the other. A graph with directed edges is called a **directed graph** or **digraph**.

NOTE: We can communicate graphs through pictures, but picture itself is not a graph.

Examples of undirected graphs:





Vertices: a, b, c, d, e

Edges: ab, ad, ae,
be, ce, de

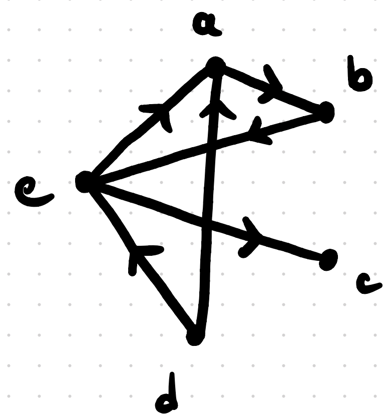
(order of vertices in an edge doesn't matter, ab is same as ba)

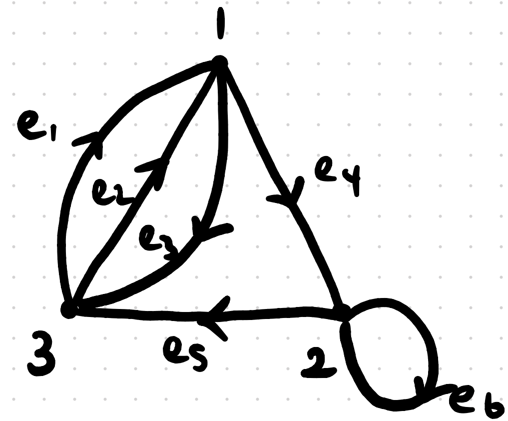
Vertices: 1, 2, 3

Edges: $e_1, e_2, e_3, e_4, e_5, e_6$

e_1, e_2, e_3 are parallel, they have the same endpoints.
e_6 is a loop.

Examples of directed graphs:





Vertices: a, b, c, d, e

Edges: ea, da, ab, be, ec, de

Order in each edge matters

Vertices: 1, 2, 3

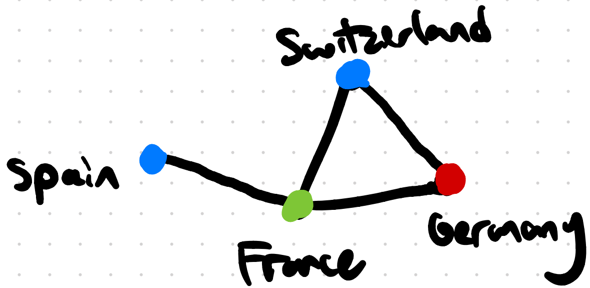
Edges: $e_1, e_2, e_3, e_4, e_5, e_6$

e_1 and e_2 are parallel, and not parallel to e_3.

"Orientation" of e_6 in drawing doesn't matter.

Modeling the map problem as a graph:

Create a graph where the vertices are countries, with an edge between two vertices if they share a border.



We want to color the vertices with 5 colors so that no two vertices of the same color share an edge. This is called a graph coloring.

Two vertices are adjacent if they are connected by an edge.

We will solve this through integer programming.

For each vertex i, we will define 5 variables

$x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}$

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is colored } j \\ 0 & \text{otherwise} \end{cases}$$

Constraints:

for every vertex i,

$$x_{i1} + x_{i2} + x_{i3} + x_{i4} + x_{i5} = 1$$

Every vertex has one color

For every edge uv,

$$x_{u1} + x_{v1} \le 1$$

$$x_{u2} + x_{v2} \le 1$$

$$x_{u3} + x_{v3} \le 1$$

$$x_{u4} + x_{v4} \le 1$$

$$x_{u5} + x_{v5} \le 1$$

$x_{ij} = 0$ or 1 for all i, j

Max 0 (just want a feasible solution)