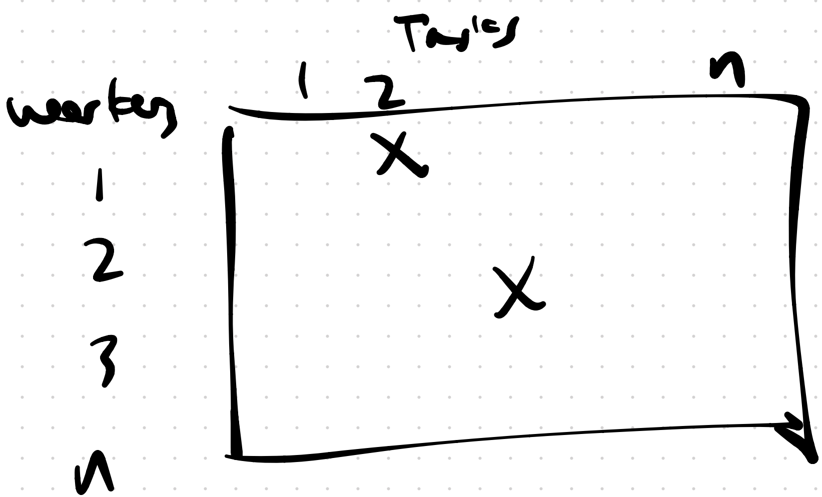
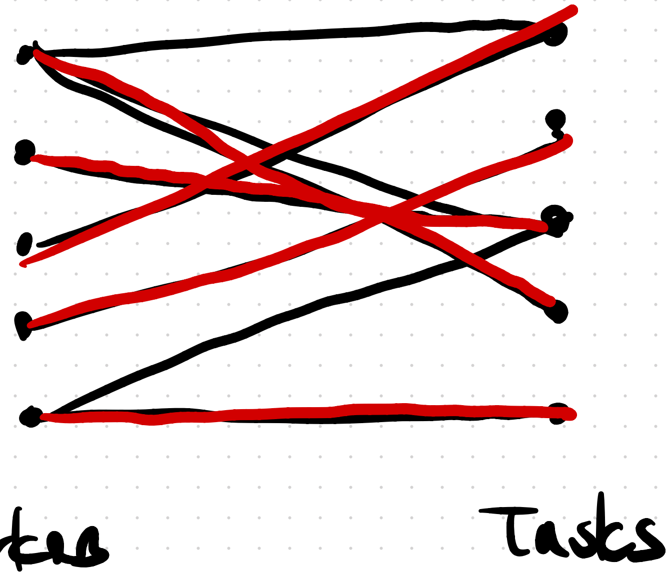
Graphs cont.

(8.3)

We have n workers and n tasks that need completing. Each worker has a different set of jobs they can perform. We want to assign tasks so that each worker does exactly one task, and every task has a worker assigned.





We construct a **bipartite graph** where one part is workers, the other part is tasks, with an edge from i to j if worker i can do task j.

Want to choose some set of edges so that every worker is incident to one edge, and every task is incident to one edge.

This is called a perfect matching.

IP solution:

For every edge ij in the graph,

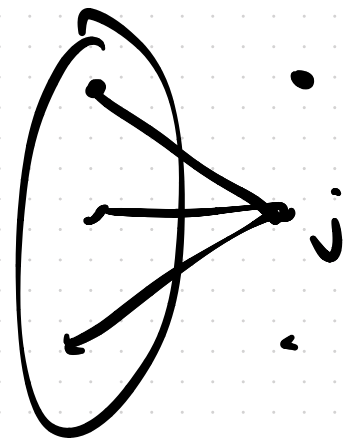
$$x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ is chosen to do task } j \\ 0 & \text{otherwise} \end{cases}$$

Max 0

s.t. $x_{ij} = 0$ or 1 for all i, j

For every task j,

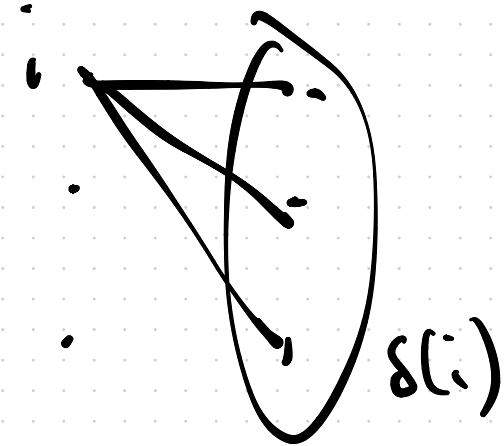
$$\sum_{i \in \underbrace{\delta(j)}_{\text{neighborhood, all vertices adjacent to } j}} x_{ij} = 1$$

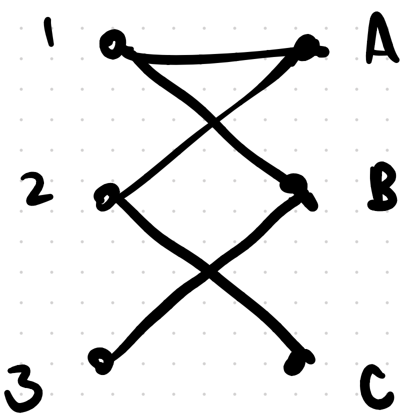


$\delta(j)$

For every worker i,

$$\sum_{j \in \delta(i)} x_{ij} = 1$$





Max 0

s.t.

$x_{1A} + x_{1B} = 1 \qquad x_{1A} + x_{2A} = 1$

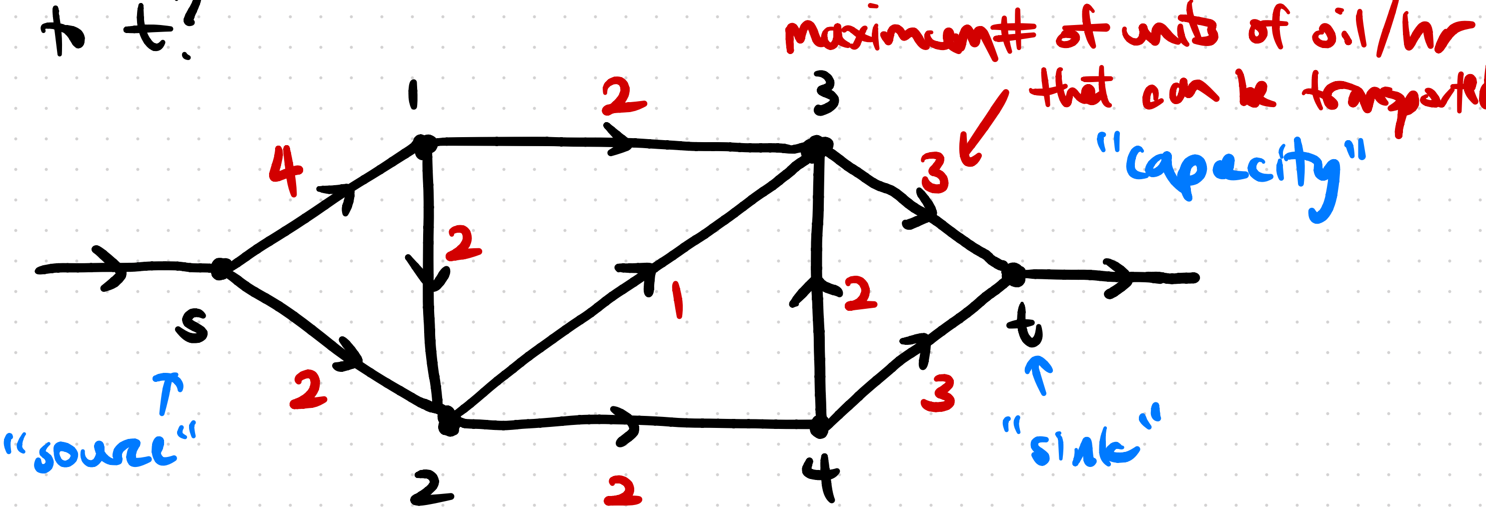
$x_{2A} + x_{2C} = 1 \qquad x_{1B} + x_{3B} = 1$

$x_{3B} = 1 \qquad x_{2C} = 1$

What if some workers can do some tasks more efficiently than others?

Exercise.

Oil is being shipped from point s to point t through the following network of pipes. The pipes have different diameters, and the maximum amount of oil per hour each pipe can transport is shown. How much oil per hour can be transported from s to t?



Max-flow problem

LP solution:

For every directed edge uv,

x_{uv} = amount of oil sent through pipe uv per hour.

Max . . . (come back to later)

s.t. For every vertex v other than s and t,

$$\sum_{u \in \delta^-(v)} x_{uv} = \sum_{u \in \delta^+(v)} x_{vu}$$

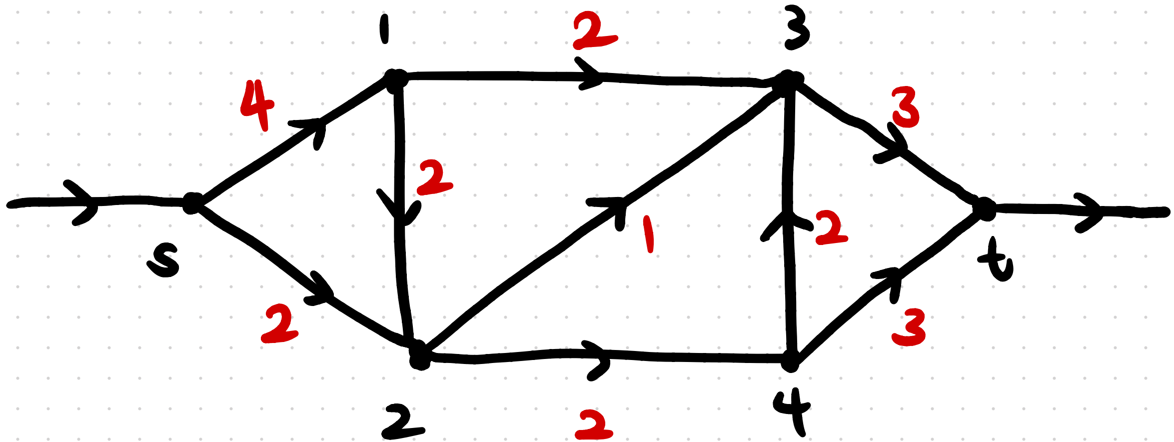
Let c_{uv} be the capacity of edge uv (given to us)

$$0 \le x_{uv} \le c_{uv} \quad \text{for all } uv$$

$$\text{Max} \sum_{u \in \delta^-(t)} x_{ut}$$

$$\left(\text{or Max} \sum_{u \in \delta^+(s)} x_{su}\right)$$

Oil is being shipped from point s to point t through the following network of pipes. The pipes have different diameters, and the maximum amount of oil per hour each pipe can transport is shown. How much oil per hour can be transported from s to t?



$$\text{Max} \quad x_{3t} + x_{4t}$$

s.t.

$$x_{s1} = x_{12} + x_{13} \qquad x_{13} + x_{23} + x_{43} = x_{3t}$$

$$x_{s2} + x_{12} = x_{23} + x_{24} \qquad x_{24} = x_{43} + x_{4t}$$

$$0 \le x_{s1} \le 4 \qquad \ldots$$

$$0 \le x_{s2} \le 2$$

How are graphs implemented in computer programs?

1) Adjacency matrix

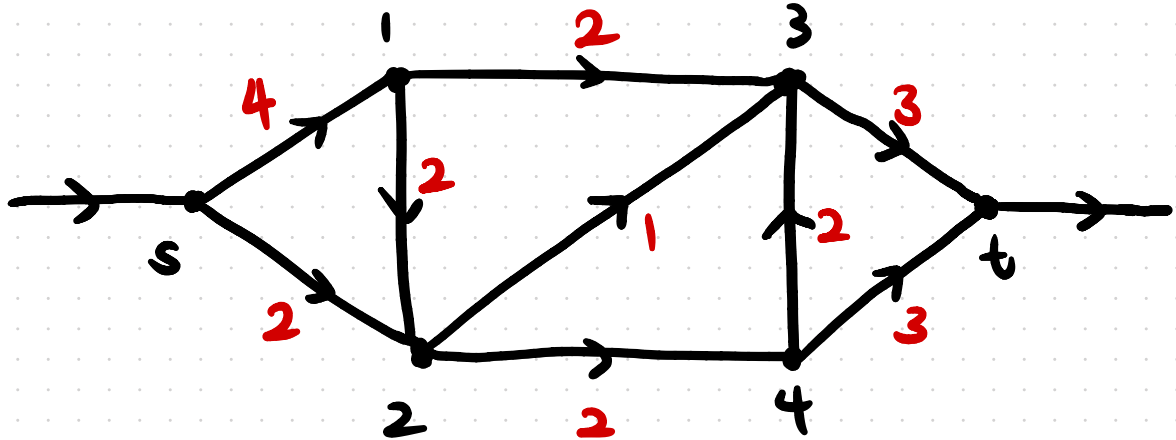
Matrix whose rows and columns are indexed by vertices, and

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

for undirected graphs

$$a_{ij} = \begin{cases} 1 & \text{if } ij \text{ is an edge} \\ -1 & \text{if } ji \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

for directed graphs

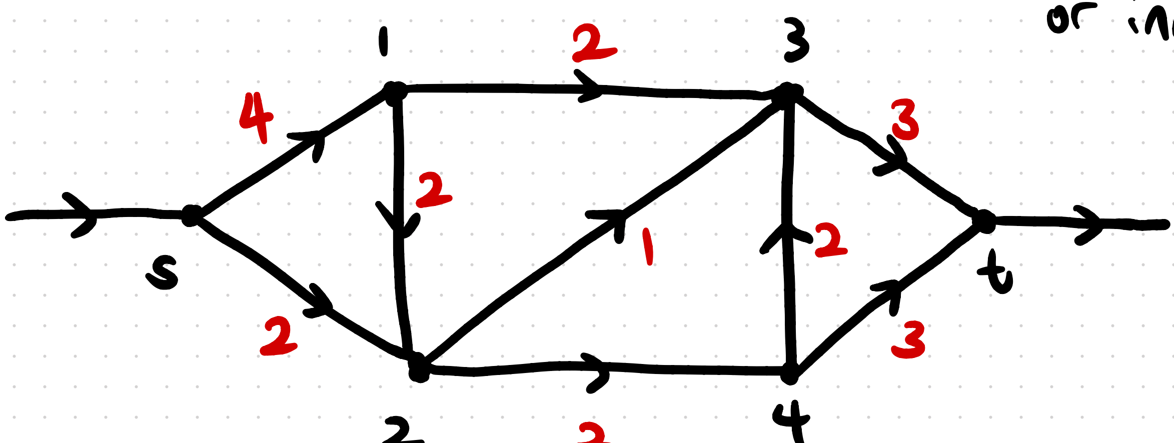


$$
\begin{array}{c} \\ s \\ 1 \\ 2 \\ 3 \\ 4 \\ t \end{array}
\begin{array}{c}
\begin{array}{cccccc} s & 1 & 2 & 3 & 4 & t \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 \\
0 & -1 & -1 & 0 & -1 & 1 \\
0 & 0 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & -1 & -1 & 0
\end{pmatrix}
\end{array}
$$

2) Adjacency list

Dictionary (hash table) whose keys are vertices, and the value of each key is its set of neighbors.

(or outneighbors or inneighbors)



Key	Value (outneighbors)
s	$\{1,2\}$
1	$\{2,3\}$
2	$\{3,4\}$
3	$\{t\}$
4	$\{3,t\}$
t	$\emptyset$

Adjacency matrices are good for dense graphs (graphs with many more edges than vertices).

Adjacency lists are more common, and good for sparse graphs (# of edges is comparable to # of vertices.)