Math 562

Geometric combinatorics

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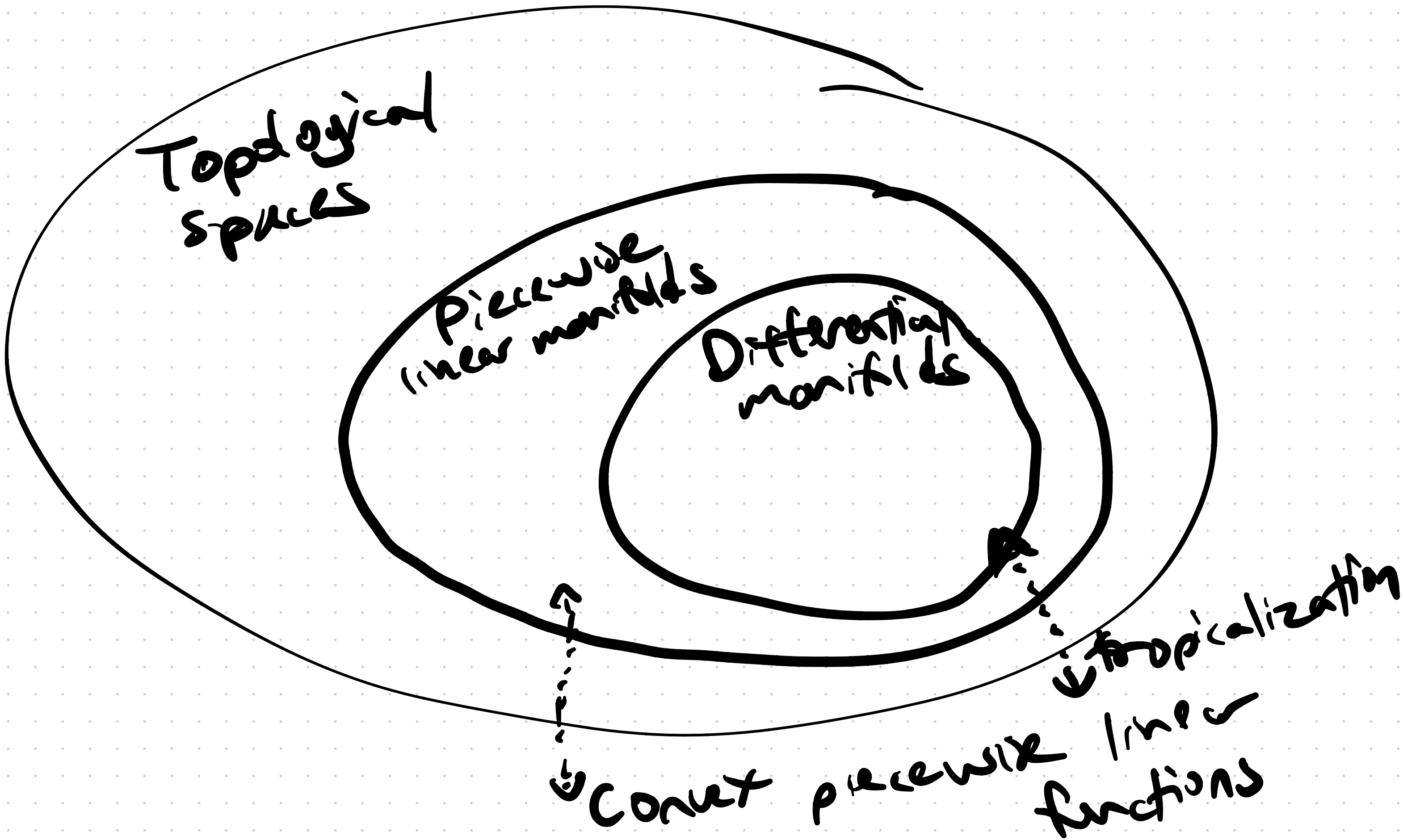
Lecture slides and homework will be posted on the class Github repository. Link on Canvas.

Grading is 100% homework, due every two weeks.

Later lectures will loosely follow the notes "Polyhedral Combinatorics" on Github. I recommend you at least read the first chapter soon.


Topological spaces
Piecewise linear manifolds
Differential manifolds
tropicalization
Convex piecewise linear functions

Simplicial complexes

A simplicial complex Δ is a finite set of finite sets such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. In this class we'll usually assume $\Delta \subset 2^{[n]}$ for some $[n] = \{1, \ldots, n\}$.

$$\Delta = \{ \{1,2,3\}, \{2,3,4\},$$
$$\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\},$$
$$\{1\}, \{2\}, \{3\}, \{4\},$$
$$\emptyset \}$$

Visually, we often view the elements $1, \dots, n$ as points, and sets $\sigma \subset [n]$ as simplices of dimension $|\sigma| - 1$ with its element as vertices.

$$\Delta = \{ \{1,2,3\}, \{2,3,4\},$$
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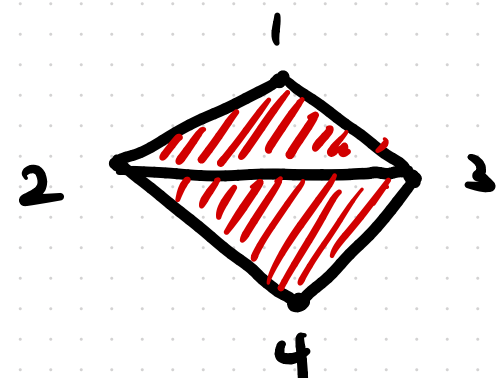
$$\{1\}, \{2\}, \{3\}, \{4\},$$
$$\phi \}$$



The elements of a simplicial complex are called **faces** of the complex. The dimension of a face is its cardinality minus one. Faces of dimension 0 are **vertices**. Maximal by inclusion elements of a simplicial complex are called **facets**.

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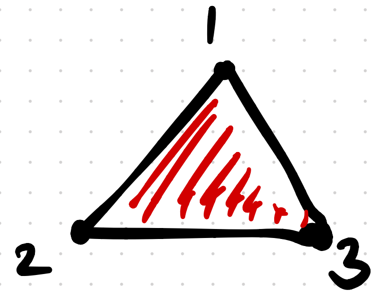
By convention, the empty set has dimension -1.

Example: $\Delta = 2^{[3]} = \{\text{all subset of } [3]\}$

Facet: $\{1,2,3\}$

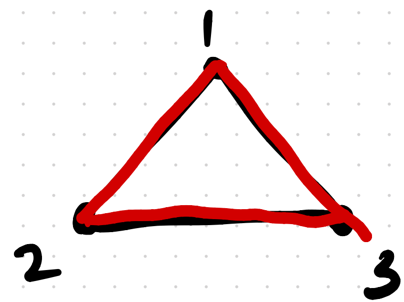


Example: $\Delta = 2^{[3]} / [3]$

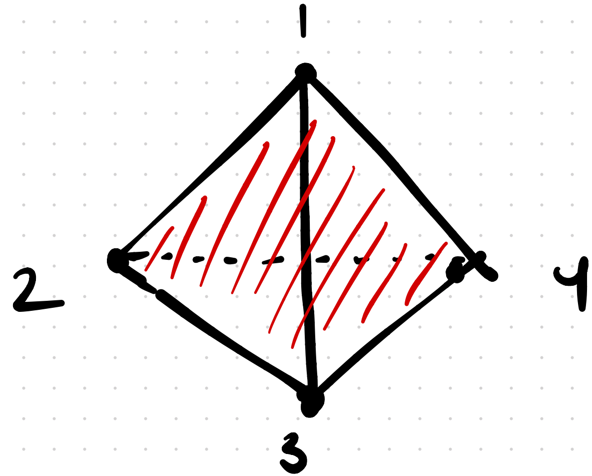
Facet: $\{1,2\}$

$\{2,3\}$

$\{1,3\}$

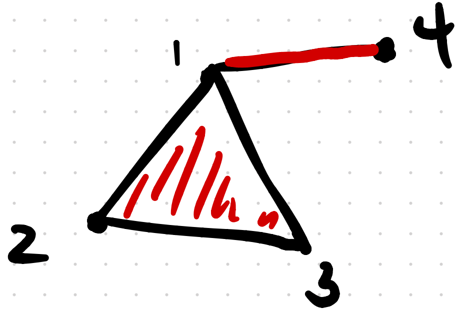


$\Delta = 2^{[4]}$



Facet: $[4]$

$\Delta = 2^{[3]} \cup \{\{1,4\}\}$



Facets: $\{1,2,3\}$

$\{1,4\}$

A simplex of dimension d is a set in $\mathbb{R}^n$ which is the convex hull of d+1 affinely independent points.

The geometric realization of a simplicial complex Δ is a topological space formed by associating a simplex of matching dimension to each facet of Δ, and then gluing these facets along the faces they share.

The geometric realization of Δ is unique up to homeomorphism, and is denoted $|\Delta|$.

One of the goals of geometric combinatorics is how the combinatorial properties of Δ relate to the topological properties of $|\Delta|$.

Given a simplicial complex Δ of dimension d, let $f_i(\Delta)$ (or just f_i) be the number of faces of Δ of dimension i.

The tuple $(f_{-1}, f_0, f_1, \ldots, f_d)$ is the **f-vector** of Δ, denoted $f(\Delta)$.

$$\Delta = 2^{[n]}$$

$$f_{-1} = 1$$

$$f_0 = n$$

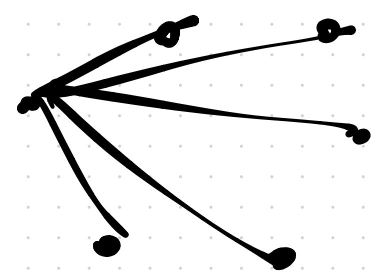
$$f_1 = \binom{n}{2}$$

$$f_i = \binom{n}{i+1}$$

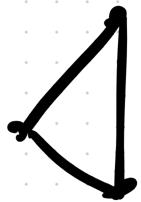
Question: What are the possible f-vectors over all simplicial complexes?

What vectors of positive integers (f_{-1}, f_0, f_1) can be f-vectors of simplicial complexes?

A vector (a, b, c) is an f-vector iff $a=1$ and $c \leq \binom{b}{2}$.



$(1, 4, 3,$



Let N be a positive integer. For every nonnegative integer i there is a unique way to write N as

$$N = \binom{a}{i+1} + \binom{b}{i} + \cdots + \binom{c}{j}$$

where $a > b > \cdots > c \geq j \geq 1$. Let

$$N^{(i)} = \binom{a}{i} + \binom{b}{i-1} + \cdots + \binom{c}{j-1}$$

<u>Theorem</u> (Kruskal-Katona): A tuple of positive integers $(f_{-1}, f_0, \ldots, f_d)$ is the f-vector of some simplicial complex if and only if $f_i^{(i)} \leq f_{i-1}$ for all i.

Sufficiency: We need to prove that if $f_i^{(i)} \leq f_{i-1}$ for all i, then $(f_{-1}, f_0, \ldots, f_d)$ is an f-vector.

Define the colexicographic order (colex) on $\underbrace{\binom{[n]}{i}}_{}$ as follows:
i-element subsets of $[n]$

If $A = \{a_1 < a_2 < \cdots < a_i\}$
$B = \{b_1 < b_2 < \cdots < b_i\}$
then $A < B$ if there exists j such that $a_j < b_j$ and $a_k = b_k$ for all $k > j$.

Alternatively, $A < B$ if $\sum_{a \in A} 2^a < \sum_{b \in B} 2^b$

<u>Claim</u>: Let Δ consist of the first f_i elements of $\binom{[n]}{i+1}$ in colex order, for all i. Then Δ is a simplicial complex if $f_i^{(i)} \leq f_{i-1}$ for all i.