

Homework 1

Due Wednesday, January 21

1. A *geometric simplicial complex* in \mathbb{R}^n is a simplicial complex Δ where every vertex is a point in \mathbb{R}^n , and if $\sigma, \sigma' \in \Delta$, then $(\text{conv } \sigma) \cap (\text{conv } \sigma') = \text{conv}(\sigma \cap \sigma')$. (Here, $\text{conv } \sigma$ is the convex hull of the points in the set σ .) Prove that every simplicial complex of dimension $\leq d$ can be represented as a geometric simplicial complex in \mathbb{R}^{2d+1} .
2. Given a simplicial complex Δ , the *barycentric subdivision* Δ' of Δ is the simplicial complex of all flags of Δ . In other words, the vertices of Δ' are the nonempty faces of Δ , and

$$\Delta' = \{\{\sigma_1, \dots, \sigma_k\} \subset 2^\Delta : \emptyset \subsetneq \sigma_1 \subsetneq \dots \subsetneq \sigma_k\}.$$

Prove that

$$f_i(\Delta') = (i+1)! \sum_{k \geq 0} S(k, i+1) f_{k-1}(\Delta)$$

where $S(k, i)$ are the Stirling numbers of the second kind.

3. For each $n \geq 1$, construct a simplicial complex homeomorphic to $\mathbb{R}P^n$.
4. Let Δ and Δ' be simplicial complexes with vertex sets V and V' , respectively. A *simplicial map* is a function $f : V \rightarrow V'$ such that $f(\sigma) \in \Delta'$ for all $\sigma \in \Delta$. Given a simplicial map $f : V \rightarrow V'$, for each k define a linear map $f_k : C_k \rightarrow C_k$ on k -chains by

$$f_k(\sigma) = \begin{cases} f(\sigma) & \text{if } \dim f(\sigma) = k \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f_k induces a map $H_k(\Delta) \rightarrow H_k(\Delta')$ on homology.