Simplicial Homology

Let Δ be a simplicial complex with vertex set $[n]$.

Let $\Delta^{(k)}$ denote the set of all simplices of Δ of of dimension k.

We will represent elements of $\Delta^{(k)}$ by tuples $(v_0, \ldots, v_k)$ where $1 \leq v_0 < \cdots < v_k \leq n$.

Throughout, we will fix a field $\mathbb{F}$. The properties of $\mathbb{F}$ will usually not be important, but we will sometimes want $\mathbb{F}$ to be infinite.

A **k-chain** is a formal sum

$$\sum_{\sigma \in \Delta^{(k)}} c_\sigma \sigma$$

where $c_\sigma \in \mathbb{F}$ for all σ. Let C_k denote the set of of all k-chains of Δ. C_k is a vector space over $\mathbb{F}$ with basis $\Delta^{(k)}$.

(If $\Delta^{(k)} = \phi$, then $C_k = \{0\}$.)

The boundary operator $\partial_k : C_k \to C_{k-1}$ is the linear map such that for all $(v_0, \dots, v_k) \in \Delta^{(k)}$,

$$\partial_k(v_0, \dots, v_k) = \sum_{i=0}^{k} (-1)^i \underbrace{(v_0, \dots, \hat{v}_i, \dots, v_k)}_{\text{$(v_0, \dots, v_k)$ with the i coordinate deleted}}$$

$$\partial_2(v_0, v_1, v_2) = (v_1, v_2) - (v_0, v_2) + (v_0, v_1)$$

<u>Proposition</u>: For all k, $\partial_{k-1} \circ \partial_k = 0$.

Proof:

$$\partial_{k-1} \circ \partial_k (v_0, \ldots, v_k) =$$

$$\sum_{\substack{0 \le i, j \le k \\ i \neq j}} T^{(v_0, \ldots, \overset{\hat{v}_i}{\hat{v}_j}, \ldots, v_k)}$$

If $i < j$, $(-1)^{i+j-1}$

If $i > j$, $(-1)^{i+j}$

$$= 0$$

It follows that

$$\cdots \to C_k \xrightarrow{\partial_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to C_{-1} \to 0$$

is a **chain complex**.

$$\cdots \to C_k \xrightarrow{\partial_k} C_{k-1} \to \cdots \to C_1 \to C_0 \to 0$$

is also a chain complex.

Given any chain complex $\cdots \xrightarrow{d_{k+1}} A_k \xrightarrow{d_k} A_{k-1} \to \cdots$, the **$k^{th}$ homology group** of the complex is

$$H_k = \frac{\ker d_k}{\operatorname{Im} d_{k+1}} \quad \begin{matrix} \leftarrow \text{cycles} \\ \leftarrow \text{boundaries} \end{matrix}$$

For the chain complex

$$\to C_k \xrightarrow{\partial_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0,$$

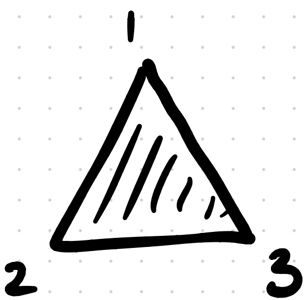
the homology groups are called the **simplicial homology groups** of Δ and denoted $H_k(\Delta; \mathbb{F})$ (or $H_k(\Delta)$).

For the chain complex

$$\to C_k \xrightarrow{\partial_k} C_{k-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to C_{-1} \to 0$$

the homology groups are called the reduced simplicial homology groups of Δ and denoted $\tilde{H}_k(\Delta; \mathbb{F})$.

Example: $\Delta = 2^{[3]}$



$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

$$H_2(\Delta) = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \ker \partial_2 = 0$$

$$C_2 = \operatorname{span}\,(1,2,3)$$

$$\partial_2(1,2,3) = (2,3) - (1,3) + (1,2)$$

$$H_1(\Delta) = \frac{\ker \partial_1}{\operatorname{im} \partial_2}$$

$$\partial_1 = \begin{array}{c c} & \begin{array}{ccc} 12 & 13 & 23 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$\downarrow$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\dim \ker \partial_1 = 1$ $\quad$ $\dim \operatorname{im} \partial_2 = 1$

$H_1(\Delta) = 0$

$$H_0(\Delta) = \frac{\ker 0}{\operatorname{im}\partial_1} = \frac{C_0}{\operatorname{im}\partial_1}$$

$$\dim H_0(\Delta) = 1$$

$$H_0(\Delta) \cong \mathbb{F}$$

$$\dim \operatorname{im}\partial_1 = 2$$

$$\dim C_0 = 3$$

Quotient map $C_0 \to \frac{C_0}{\operatorname{im}\partial_1} \cong$

$$c_1(1) + c_2(2) + c_3(3) \mapsto c_1 + c_2 + c_3$$

$$\to C_1 \to C_0 \xrightarrow{\partial_0} C_{-1} \to 0$$

$$\tilde{H}_0(\Delta) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = 0$$

∂_0 is the sum of coordinates

$\operatorname{im}\partial_1$ is the subspace of C_0 with sum of coordintes 0

$$\tilde{H}_{-1}(\Delta) = 0$$

Example: $\Delta = 2^{[n]}$.

__Prop__: $\tilde{H}_k(\Delta) = 0$ for all k.

Proof: we need to show $\ker \partial_k = \operatorname{im} \partial_{k+1}$ for all k. Suppose $x \in \ker \partial_k$

$$x = \sum_{\sigma \in \Delta^{(k)}} c_\sigma \sigma$$

For any $(v_0, \ldots, v_k)$ where $v_i \in [n]$, we identify it with the following element in C_k

$(v_0, \dots, v_k) = \text{sign}(\pi)(\pi v_0, \dots, \pi v_k)$

where $\pi v_0 < \cdots < \pi v_k$, if $v_0, \dots, v_k$ are distinct

$(v_0, \dots, v_k) = 0$ otherwise.

Given $x = \sum_{\sigma \in \Delta^{(k)}} c_\sigma \sigma$

If $\sigma = (v_0, \dots, v_k)$

$(1, \sigma) = (1, v_0, \dots, v_k)$

Define $y = \sum_{\sigma \in \Delta^{(k)}} c_\sigma (1, \sigma) \in C_{k+1}$

Claim: $\partial_{k+1} y = x$