Polyhedral Combinatorics

By Gaku Liu

Updated January 5, 2026

Contents

Chapter 1

Preliminaries

1.1 Vector and affine spaces

Vector notation

Throughout the text, let $V = \mathbb{R}^n$ denote a real vector space of finite dimension n. Let V^* denote the dual vector space of V, that is, the vector space of all linear functionals $V \to \mathbb{R}$. Elements of V will commonly be named x, while elements of V^* will commonly be named y. The symbol "0" will be used to denote both the scalar 0 and vector 0.

For most of this text we will work without coordinates. In case we wish to use coordinates, we fix a *standard basis* $e_1, \ldots, e_n$ of V. This determines a *standard dual basis* $e^1, \ldots, e^n$ of V^* defined by $e^i(e_j) = 0$ if $i \neq j$ and $e^i(e_i) = 1$ for all i.

It is common practice to always fix an inner product $\langle \cdot, \cdot \rangle$ and identify V and V^* using $x \mapsto \langle x, \cdot \rangle$. In this text I have decided to keep V and V^* formally distinct.

We denote the linear span of a set $S \subset V$ by $\operatorname{span} S$. Given two sets $S \subset V$, $T \subset W$ in vector spaces V, W, we say that S and T are *linearly equivalent* if there is a linear isomorphism $f : \operatorname{span} S \to \operatorname{span} T$ such that $f(S) = T$.

Given a vector subspace $W \subset V$, we define the *orthogonal complement* or *annihilator* of W to be the following vector subspace of V^*:

$$W^\perp = \{y \in V^* : y(x) = 0 \text{ for all } x \in W\}.$$

Affine spaces

An *affine subspace* of V is a set of the form $x + W$, where W is a linear subspace of V and $x + W$ denotes the translation of W by a vector $x \in V$. The *dimension* of $x + W$ is the dimension of W. In this text, the term *affine space* will refer to an affine subspace, although we may not explicitly name the ambient vector space.

An *affine combination* of points $x_1, \ldots, x_m \in V$ is any point of the form $\sum_{i=1}^m \lambda_i x_i$ where $\lambda_1, \ldots, \lambda_m$ are real numbers such that $\sum_{i=1}^m \lambda_i = 1$. If A is an affine subspace of V, then any affine combination of elements of A is also in A. The *affine span* of a nonempty set $S \subset V$ is the smallest affine subspace of V containing S. Equivalently, it is the set of all affine combinations of elements of S. We denote the affine span of S by $\operatorname{aff} S$. We define the *dimension* of S to be the dimension of its affine span. We say a set is *full-dimensional* if its affine span is V.

The points $x_1, \ldots, x_m \in V$ are *affinely independent* if for any $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\sum_{i=1}^m \lambda_i x_i = 0$ and $\sum_{i=1}^m \lambda_i = 0$, we have $\lambda_1 = \cdots = \lambda_m = 0$. For points $x_1, \ldots, x_m \in V$, the following are equivalent.

(i) $x_1, \ldots, x_m$ are affinely independent.

(ii) $x_1 - x_m, x_2 - x_m, \ldots, x_{m-1} - x_m$ are linearly independent.

(iii) $(x_1, 1), \ldots, (x_m, 1)$ are linearly independent in $V \times \mathbb{R}$.

It follows that the maximum size of an affinely independent set in V is $\dim V + 1$.

Let A, B be affine spaces (with possibly different ambient vector spaces). An *affine map* is a function $f : A \to B$ such that for any $x_1, \ldots, x_m \in A$ and affine combination $\sum_{i=1}^m \lambda_i x_i$, we have $f(\sum_{i=1}^m \lambda_i x_i) = \sum_{i=1}^m \lambda_i f(x_i)$. An *affine isomorphism* is an affine map which is a bijection. Given two subsets S, T of (possibly different) vector spaces, we say that S and T are *affinely equivalent* if there is an affine isomorphism $f : \operatorname{aff} S \to \operatorname{aff} T$ such that $f(S) = T$.

Given any set $S \subset V$, we define $S^\perp \subset V^*$ to be the orthogonal complement of the linear subspace parallel to $\operatorname{aff} S$. In other words, $S^\perp$ is the set of all linear functionals in V^* which are constant on S.

1.2 Polyhedra

A *convex polyhedron* in V is a set of the form

$$\{x \in V : y_i(x) \le b_i \text{ for all } i \in I\} \tag{1.2.1}$$

where I is a finite set and $y_i \in V^*$, $b_i \in \mathbb{R}$ for all $i \in I$. We call a representation of a convex polyhedron in the form (1.2.1) a *constraint presentation* of the polyhedron. If there is no risk of confusion, we may abbreviate the above presentation as

$$\{y_i(x) \le b_i,\ i \in I\}.$$

The inequalities $y_i(x) \le b_i$ are called *linear constraints*, or *constraints* on the polyhedron. When writing a presentation, we may write some constraints as $y_i(x) \ge b_i$ (equivalent to $(-y_i)(x) \le -b_i$) or $y_i(x) = b_i$ (equivalent to the combined constraints $y_i(x) \le b_i$ and $y_i(x) \ge b_i$).

Every polyhedron has many presentations. For example, if $y_1(x) \le b_1$ and $y_2(x) \le b_2$ are constraints in a presentation, we may add to the presentation any nonnegative linear combination of these constraints (i.e., $(c_1y_1 + c_2y_2)(x) \le c_1b_1 + c_2b_2$ for c_1, $c_2 \ge 0$) without changing the polyhedron. However, we will see in Theorem 3.2.5 that there is an essentially unique minimal way to present a polyhedron.

One can equivalently define convex polyhedra in terms of affine spaces and affine functionals (Exercise 1.1). In particular, this makes it clear that the property of being a polyhedron is preserved under affine isomorphism.

For the rest of this text, we will refer to convex polyhedra as simply *polyhedra*. In other contexts, the term "polyhedron" refers more generally to a finite union of convex polyhedra.

It is clear from the definition that all polyhedra are closed sets, and the intersection of a finite number of polyhedra is a polyhedron. A polyhedron which is bounded is called a *polytope*.

A *cone* is a subset $C \subset V$ satisfying the following conditions: $0 \in C$, and if $x \in C$, then $\lambda x \in C$ for all real $\lambda > 0$. A *polyhedral cone* is a polyhedron which is a cone. We note the following fact (Exercise 1.2).

Lemma 1.2.2. If C is a cone and there is some $y \in V^*$ and $b \in \mathbb{R}$ such that $y(x) \le b$ for all $x \in C$, then $b \ge 0$ and $y(x) \le 0$ for all $x \in C$.

It follows from the lemma that a subset of V is a polyhedral cone if and only if it is of the form

$$\{y_i(x) \le 0,\ i \in I\}$$

for some finite set I and $y_i \in V^*$ for all $i \in I$.

Example 1.2.3 (Simplicial cones)**.** Recall that e^1,...,e^n is the standard dual basis of V^*. The polyhedral cone

$$\mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n : e^i(x) \geq 0,\ i = 1, \dots, n\}$$

is the *nonnegative orthant* of $\mathbb{R}^n$. Any cone linearly equivalent to this cone is called a *simplicial cone*.

Example 1.2.4 (Half-spaces)**.** Given any $y \in V^*$ and $b \in \mathbb{R}$, the set

$$\{y(x) \leq b\}$$

defined by a single constraint is a polyhedron. If $y \neq 0$, this polyhedron is a *(closed) half-space* of V. If $y = 0$, then the polyhedron is empty if $b < 0$ and the whole space V if $b \geq 0$.

Example 1.2.5 (Affine subspaces)**.** Every affine subspace of V can be written as

$$\{y_i(x) = b_i,\ i \in I\}$$

for some finite set I and $y_i \in V^*$, $b_i \in \mathbb{R}$. Hence every affine subspace is a polyhedron. An affine subspace of the form $\{y(x) = b\}$ is an *affine hyperplane*, or just *hyperplane*. A *linear hyperplane* is an affine hyperplane which contains the origin.

Example 1.2.6 (Cubes)**.** The polyhedron

$$Q = \{x \in \mathbb{R}^n : -1 \leq e^i(x) \leq 1,\ i = 1, \dots, n\}$$

is an n-dimensional polytope called a *cube* or *n-cube*.

Example 1.2.7 (Cross-polytopes)**.** The polyhedron

$$R = \{x \in \mathbb{R}^n : \pm e^1(x) \pm \dots \pm e^n(x) \leq 1\},$$

where the constraints go through all 2^n combinations of $+$ or $-$, is an n-dimensional polytope called a *cross polytope*.

Example 1.2.8 (Simplices)**.** The polyhedron

$$\Delta = \{x \in \mathbb{R}^{n+1} : e^1(x) + \cdots + e^{n+1}(x) = 1,\ e^i(x) \geq 0,\ i = 1, \ldots, n\}$$

is an n-dimensional polytope called a *simplex* or n-*simplex*. Note that we have defined this polytope in $\mathbb{R}^{n+1}$, but it has an n-dimensional affine span, specifically $\{x \in \mathbb{R}^{n+1} : e^1(x) + \cdots + e^{n+1}(x) = 1\}$. In general, an n-*simplex* is defined to be any polytope affinely equivalent to Δ.

Another example of an n-simplex is

$$\{x \in \mathbb{R}^n : e^1(x) + \cdots + e^{n+1}(x) \leq 1,\ e^i(x) \geq 0,\ i = 1, \ldots, n\}$$

which is full-dimensional in $\mathbb{R}^n$.

You're probably also familiar with the regular *dodecahedron* and *icosahedron*, which, along with the regular 3-simplex, 3-cube, and 3-cross polytope, form the Platonic solids in $\mathbb{R}^3$.

Exercises

1.1. Let A be an affine space. An *affine functional* on A is an affine map $f : A \to \mathbb{R}$. A subset of A is a convex polyhedron if and only if it is of the form

$$\{x \in A : y_i(x) \leq 0 \text{ for all } i \in I\}$$

where I is a finite set and each y_i is an affine functional on A.

1.2. If C is a cone and there is some $y \in V^*$ and $b \in \mathbb{R}$ such that $y(x) \leq b$ for all $x \in C$, then $b \geq 0$ and $y(x) \leq 0$ for all $x \in C$.

Chapter 2

Convex sets

2.1 Convex sets and cones

A set $K \subset V$ is *convex* if for any $x_1,\ x_2 \in K$, the line segment with endpoints x_1, x_2 is contained in K; in other words, if $(1-\lambda)x_1 + \lambda x_2 \in K$ for all $0 < \lambda < 1$.

It is easy to check that all polyhedra are convex (hence the name "convex polyhedron"). In fact, any set of the form

$$\{x \in V : y_i(x) \leq b_i,\ i \in I\}, \tag{2.1.1}$$

where I is a (possibly infinite) set and $y_i \in V^*$, $b_i \in \mathbb{R}$ for all $i \in I$, is a closed convex set. We call a representation of the form (2.1.1) a *constraint presentation* of a convex set. We will show in Corollary 2.2.5 that all closed convex sets have a constraint presentation. Polyhedra are precisely the sets which have a finite constraint presentation.

Convex hull

A *convex combination* of points $x_1,\ \ldots,\ x_m \in V$ is any point of the form $\sum_{i=1}^m \lambda_i x_i$ where $\lambda_1,\ \ldots,\ \lambda_m$ are nonnegative real numbers satisfying $\sum_{i=1}^m \lambda_i = 1$. In other words, a convex combination is an affine combination in which the weights are restricted to be in $[0, 1]$. If K is convex, then any convex combination of elements of K is also in K.

Let $S \subset V$. The *convex hull* of S, denoted $\operatorname{conv} S$, is the smallest convex subset of V which contains S. This is well-defined because the intersection of all convex sets which contain S is again a convex set. Equivalently, the convex hull of S is the set of all convex combinations of elements of S.

Example 2.1.2. The cube in Example 1.2.6 is the convex hull of the 2^n points $\pm e_1 \pm \cdots \pm e_n$. The cross-polytope in Example 1.2.7 is the convex hull of the $2n$ points $\pm e_1, \ldots, \pm e_n$.

Example 2.1.3. The simplex Δ in Example 1.2.8 is the convex hull of the $n+1$ points $e_1, \ldots, e_{n+1}$. In general, a set is an n-simplex if and only if it is the convex hull of $n+1$ affinely independent points.

Convex cones

A *conical combination* of vectors $x_1, \ldots, x_m \in V$ is any vector of the form $\sum_{i=1}^m \lambda_i x_i$ where $\lambda_1, \ldots, \lambda_m \geq 0$. A conical combination of an empty set of vectors is defined to be 0. If C is a convex cone, then every conical combination of elements of C is also in C.

Let $S \subset V$. The *conical hull* of S, denoted $\operatorname{cone} S$, is the smallest convex cone which contains S. The conical hull of S equals the set of all conical combinations (including 0) of elements of S.

Example 2.1.4. The nonnegative orthant in Example 1.2.3 is the conical hull of the n vectors $e_1, \ldots, e_n$. In general, a set is an n-dimensional simplicial cone if and only if it is the conical hull of n linearly independent vectors.

Convex cones are often more convenient to work with than general convex sets. We can use the following trick to move between the two settings. Given a vector space V, we can embed it as the affine subspace $V \times \{1\}$ in the larger vector space $V \times \mathbb{R}$. Then affine (resp. convex) combinations in V are precisely linear (resp. conical) combinations of elements of $V \times \{1\}$ which are contained in $V \times \{1\}$. For example, for any set $S \subset V$, we have

$$\operatorname{conv} S = \{x \in V : (x, 1) \in \operatorname{cone}(S \times \{1\})\}. \tag{2.1.5}$$

Carathéodory's Theorem

As previously discussed, every point in the convex hull (resp. conical hull) of a set S can be written as a convex combination (resp. conical combination) of finitely many points in S. Carathéodory's Theorem says that in fact only a bounded number of points are needed.

Theorem 2.1.6 (Carathéodory's Theorem)**.** Let S be a subset of $V = \mathbb{R}^n$.

(a) If $x \in \operatorname{conv} S$, then x is a convex combination of $n+1$ or fewer elements of S.

(b) If $x \in \operatorname{cone} S$, then x is a conical combination of n or fewer elements of S.

Proof. Part (a) can be proved by applying part (b) to $S \times \{1\}$ and using (2.1.5). Thus, it suffices to prove (b).

Suppose that $x \in \operatorname{cone} S$. Then we can write $x = \sum_{i=1}^m \lambda_i x_i$ for some $x_1, \ldots, x_m \in S$ and nonnegative $\lambda_1, \ldots, \lambda_m$. If $m \le n$, we are done. Otherwise, suppose $m > n$. Since V has dimension n, the vectors $x_1, \ldots, x_m$ are linearly dependent. Let $\mu_1, \ldots, \mu_m$ be real numbers such that $\sum_{i=1}^m \mu_i x_i = 0$ and the μ_i are not all 0. Then for any real number α, we have

$$x = \sum_{i=1}^m \lambda_i x_i - \alpha \sum_{i=1}^m \mu_i x_i = \sum_{i=1}^m (\lambda_i - \alpha\mu_i) x_i.$$

By replacing every μ_i with $-\mu_i$ if necessary, we may assume at least one of the μ_i is positive. Start with $\alpha = 0$ and move α towards ∞. At some point, at least one of the $\lambda_i - \alpha\mu_i$ will equal 0, and the other $\lambda_i - \alpha\mu_i$ will be nonnegative. Hence, we can write x as a conical combination of $m - 1$ elements of S. Repeat this procedure until x is written as a conical combination of n elements of S. $\square$

We will use this theorem to prove the following.

Proposition 2.1.7. The convex hull or conical hull of a finite set is closed.

Proof. We will prove the result for conical hulls; the result for convex hulls follows using (2.1.5). We induct on the dimension of V. Let $S \subset V = \mathbb{R}^n$ be finite. By Carathéodory's Theorem, $\operatorname{cone} S$ is a union of cones of the form $\operatorname{cone} T$, where T ranges over all subsets of S with size at most n. Let T be a subset of S with size at most n. If T is contained in a proper linear subspace of V, then $\operatorname{cone} T$ is closed by the inductive hypothesis. Otherwise, T is a set of n linearly independent vectors. Then $\operatorname{cone} T$ is a simplicial cone, and hence is a polyhedron and therefore closed. Since S is finite, there are finitely many such T, and so $\operatorname{cone} S$ is closed. $\square$

We will later prove that the convex or conical hull of a finite set is in fact a polyhedron (Theorem 2.5.3).

Interior and boundary

Recall that the *interior* of a set $S \subset V$ is the set of all $x \in S$ such that there exists an open set U of V with $x \in U \subset S$. We denote the interior of S by $\operatorname{int} S$. The *boundary* of S, denoted $\operatorname{bd} S$, is defined as $\operatorname{cl} S \smallsetminus \operatorname{int} S$, where $\operatorname{cl} S$ is the closure of S in V.

The definitions of interior and boundary depend on the ambient space V. If S is not full-dimensional, then we always have $\operatorname{int} S = \varnothing$ and $\operatorname{bd} S = \operatorname{cl} S$. In many cases it is more useful to use a concept of interior and boundary which is intrinsic to S. We define the *relative interior* and *relative boundary* of S to be its interior and boundary with respect to the subspace topology on $\operatorname{aff} S$. We denote them $\operatorname{rint} S$ and $\operatorname{rbd} S$, respectively. The relative interior of any nonempty convex set is nonempty (Exercise 2.5).

Example 2.1.8. Let $\Delta \subset \mathbb{R}^{n+1}$ be the simplex from Example 1.2.8. We have

$$\begin{aligned} \operatorname{int} \Delta &= \varnothing \qquad & \operatorname{rint} \Delta &= \Delta \cap \{e^i(x) > 0,\ i = 1, \ldots, n+1\} \\ \operatorname{bd} \Delta &= \Delta \qquad & \operatorname{rbd} \Delta &= \Delta \cap \bigcup_{i=1}^{n+1} \{e^i(x) = 0\} \end{aligned}$$

The relative interior of Δ can also be characterized as the set of all *positive convex combinations* of $e_1, \ldots, e_{n+1}$; that is, all convex combinations $\sum_{i=1}^{n+1} \lambda_i e_i$ where $\lambda_i > 0$ for all i.

Lineality space and recession cone

For a convex set $K \subset V$, we define the *lineality space* and *recession cone* of K to be the following sets, respectively:

$$\operatorname{lins} K := \begin{cases} \{v \in V : x + \lambda v \in K \text{ for all } x \in K,\ \lambda \in \mathbb{R}\} & \text{if } K \neq \varnothing \\ \{0\} & \text{if } K = \varnothing \end{cases}$$

$$\operatorname{recc} K := \begin{cases} \{v \in V : x + \lambda v \in K \text{ for all } x \in K,\ \lambda > 0\} & \text{if } K \neq \varnothing \\ \{0\} & \text{if } K = \varnothing \end{cases}.$$

We have that $\operatorname{lins} K$ is a vector subspace and $\operatorname{recc} K$ is a convex cone (Exercise 2.7). If K is bounded, then $\operatorname{lins} K = \operatorname{recc} K = \{0\}$. Lineality spaces and recession cones describe the behavior of unbounded convex sets at infinity. Exercises 2.8–2.10 cover some basic properties of these sets. We highlight the following in particular (Exercise 2.10).

Proposition 2.1.9. Let $K = \{y_i(x) \leq b_i,\, i \in I\}$ where I is any set and $y_i \in V^*$, $b_i \in \mathbb{R}$ for all i. Assume K is nonempty. Then

$$\begin{aligned} \operatorname{lins} K &= \{y_i(x) = 0,\, i \in I\} \\ \operatorname{recc} K &= \{y_i(x) \leq 0,\, i \in I\}. \end{aligned}$$

Let $K \subset V$ be a convex set and let $L = \operatorname{lins} K$. Let $q : V \to V/L$ be the quotient map with kernel L. Then $q(K) = K/L$ is a convex set in V/L with lineality space $\{0\}$. This is a way to reduce questions about general convex sets to convex sets with trivial lineality space. Alternatively, choose any linear subspace W of V such that $V = W \oplus L$. Then we can write $K = K' \oplus L$, where $K' := K \cap W$ is a convex set with $\operatorname{lins} K' = \{0\}$.

Minkowski summation

Given two sets $S, T \subset V$, the *Minkowski sum* of S and T is the set

$$S + T := \{s + t : s \in S, t \in T\}.$$

Minkwoski summation is commutative and associative. If either S or T are empty then $S + T = \varnothing$. It is easy to show that if S and T are convex then $S + T$ is convex.

Example 2.1.10. The cube from Example 1.2.6 is the Minkowski sum of the line segments $[-e_i, e_i]$ over all $i = 1, \ldots, n$.

The Minkowski sum of two closed convex sets is not necessarily closed (Exercise 2.12). However, we will show that the Minkowski sum of two polyhedra is a polyhedron (Theorem 2.5.7).

2.2 Support and separation

In this section we prove the *supporting hyperplane theorem* and *separating hyperplane theorem*, two fundamental theorems in geometry. Our proof is an inductive argument. We also give a standard metric argument in Exercise 2.13.

The following is the key lemma.

Theorem 2.2.1. *Let $C \subset V$ be a convex cone with $C \neq V$. Then there exists $y \in V^* \smallsetminus \{0\}$ such that $y(x) \leq 0$ for all $x \in C$.*

Given a set $S \subset V$, a point $x \in S$, and an affine hyperplane H of V, we say that H *supports* S at x if $x \in H$ and S lies in one of the closed half-spaces bounded by H. Theorem 2.2.1 says that any convex cone which is not the entire vector space is supported by a hyperplane at 0.

Proof of Theorem 2.2.1. We induct on $\dim V$. We leave the case $\dim V \leq 2$ as an exercise (Exercise 2.14). Assume $\dim V \geq 3$. Let H be a linear hyperplane of V which contains a vector not in C. Then $C' := C \cap H$ is a convex cone in H with $C' \neq H$. By the inductive hypothesis, there is a hyperplane H' of H that supports C' at 0. Consider the quotient map $q : V \to V/H'$. Then $q(C)$ is a convex cone in V/H', and $\dim(V/H') = 2$. Since $C \cap (x + H') = \varnothing$ for any x in H on the opposite side of H' as C', we have $q(C) \neq V/H'$. Thus, by the inductive hypothesis, there is a hyperplane H'' of V/H' that supports $q(C)$ at 0. Then $q^{-1}(H'')$ is a hyperplane of V that supports C at 0, as desired. □

Recall that for a set $S \subset V$, the set $S^\perp \subset V^*$ is the set of all linear functionals which are constant on S. We have the following slight generalization of Theorem 2.2.1.

Corollary 2.2.2. *Let $C \subset V$ be a convex cone which is not a linear subspace. Then there exists $y \in V^* \smallsetminus C^\perp$ such that $y(x) \leq 0$ for all $x \in C$.*

Proof. This follows from applying Theorem 2.2.1 to C when viewed as a cone in the vector space $\operatorname{span} C$. Note that $(\operatorname{span} C)^* = V^*/C^\perp$. □

We can now easily prove the supporting and separating hyperplane theorems.

Theorem 2.2.3 (Supporting hyperplane theorem)**.** *Let K be a convex subset of V and let $u \in \operatorname{rbd} K$. Then there exist $y \in V^* \smallsetminus K^\perp$ and $b \in \mathbb{R}$ such that $y(u) = b$ and $y(x) \leq b$ for all $x \in K$.*

Proof. For any $x \in K$, define the *direction cone* of K at x to be the set

$$D(K, x) := \operatorname{cone}(K - x).$$

(This is also called the *cone of feasible directions.*) If u is on the relative boundary of K, then $D(K, u)$ is not a linear subspace. Hence, by Corollary 2.2.2, there is a hyperplane supporting $D(K, u)$ at 0 which does not contain $D(K, u)$. Translating this hyperplane by u gives a hyperplane supporting K at u which does not contain K, as desired. □

Theorem 2.2.4 (Separating hyperplane theorem)**.** Let K be a convex subset of V and let $u \in V \smallsetminus \operatorname{cl} K$. Then there exist $y \in V^* \smallsetminus \{0\}$ and $b \in \mathbb{R}$ such that $y(u) > b$ and $y(x) < b$ for all $x \in K$.

Proof. If $u \notin \operatorname{aff} K$, then we can find a hyperplane containing K and not u, and the result follows by shifting this hyperplane. Thus, assume $u \in \operatorname{aff} K$. By restricting to aff K, we may assume K is full-dimensional.

Since K is full-dimensional, it has an interior point w. Since $u \notin \operatorname{cl} K$, the line segment $[u, w]$ contains a boundary point v of K, and $v \neq u, w$. By the supporting hyperplane theorem applied to v, there exist $y \in V^* \smallsetminus \{0\}$ and $b \in \mathbb{R}$ such that $y(v) = b$ and $y(x) \leq b$ on K. Since u and w are on opposite sides of v in $[u, w]$, they must be on opposite sides of the hyperplane $\{y(x) = b\}$ through v. Hence $y(u) > b$. Replacing b with $b + \varepsilon$ for small enough $\varepsilon > 0$ gives the desired y and b. □

Corollary 2.2.5. If K is a nonempty closed convex set in V, then

$$K = \{x \in V : y(x) \leq \sup_K y \text{ for all } y \in V^*\}$$

Proof. It is clear that K is contained in the right hand side. If $u \notin K$, then by the separating hyperplane theorem there is some $y \in V^*$ such that $y(u) > \sup_K y$. Hence u is not in the right hand side, as desired. □

In particular, this shows that every closed convex set has a constraint presentation.

When applied to cones, the separating hyperplane theorem is a useful corollary known as Farkas' lemma.

Corollary 2.2.6 (Farkas' lemma)**.** Let C be a convex cone in V and let $u \in V \smallsetminus \operatorname{cl} C$. Then there exists $y \in V^* \smallsetminus \{0\}$ such that $y(u) > 0$ and $y(x) \leq 0$ for all $x \in C$.

Proof. Applying the separating hyperplane theorem to C and u, there exist $y \in V^* \smallsetminus \{0\}$ and $b \in \mathbb{R}$ such that $y(u) > b$ and $y(x) < b$ for all $x \in C$. By Lemma 1.2.2, we have $b \geq 0$ and $y(x) \leq 0$ for all $x \in C$. $\square$

2.3 Polar cones

Let $C \subset V$. The *polar cone* of C is the set $C^\circ \subset V^*$ defined by

$$C^\circ = \{y \in V^* : y(x) \leq 0 \text{ for all } x \in C\}.$$

Theorem 2.3.1. Let $C \subset V$.
(a) C° is closed convex cone.
(b) $C \subset C^{\circ\circ}$.
(c) We have $C = C^{\circ\circ}$ if and only if C is a closed convex cone.

Proof. Proof of (a) and (b). Exercise 2.16.

Proof of (c). The forward direction follows by (a). For the other direction, suppose C is a closed convex cone. By (b), we only need to prove that $C^{\circ\circ} \subset C$. Suppose that $u \in V \smallsetminus C$. By Farkas' lemma, there exists $y \in V^*$ such that $y(u) > 0$ and $y(x) \leq 0$ for all $x \in C$. The latter inequality implies $y \in C^\circ$. Then since $y(u) > 0$, we have $u \notin C^{\circ\circ}$, as desired. $\square$

We have discussed two ways to present a convex cone: through a set of constraints, or as the conical hull of a set of vectors. A presentation of C in one way gives a presentation of C° in the other way, as described next.

Proposition 2.3.2. Let C be a subset of V.
(a) If $C = \operatorname{cone} S$ for some $S \subset V$, then

$$C^\circ = \{y \in V^* : y(x) \leq 0 \text{ for all } x \in S\}.$$

(b) Suppose

$$C = \{x \in V : y(x) \leq 0 \text{ for all } y \in T\}$$

where $T \subset V^*$ and $\operatorname{cone} T$ is closed. Then $C^\circ = \operatorname{cone} T$.

Proof. Proof of (a). Clearly $C^\circ \subset S^\circ$. Moreover, if y is an element of V^* such that $y(x) \leq 0$ for all $x \in S$, then by taking nonnegative linear combinations of these inequalities we have $y(x) \leq 0$ for all $x \in \operatorname{cone} S = C$. Hence $S^\circ \subset C^\circ$.

Proof of (b). We have $C = T^\circ = (\operatorname{cone} T)^\circ$, by part (a). Since $\operatorname{cone} T$ is closed, by Theorem 2.3.1 we have $C^\circ = (\operatorname{cone} T)^{\circ\circ} = \operatorname{cone} T$. $\square$

Proposition 2.3.3. Let $C \subset V$. Then $\operatorname{span} C$ and $\operatorname{lins}(C^\circ)$ are orthogonal complements. If C is a closed convex cone, then $\operatorname{lins} C$ and $\operatorname{span}(C^\circ)$ are orthogonal complements.

Proof. Exercise 2.17. $\square$

Balanced support

Closed convex cones satisfy a stronger support property than Theorem 2.2.1. This property will be important in the next chapter.

Given a convex cone $C \subset V$, we say that C has *balanced support* if there exists $y \in V^*$ such that $y(x) \leq 0$ for all $x \in C$ (that is, $y \in C^\circ$), and

$$\{x \in C : y(x) = 0\} = \operatorname{lins} C.$$

For example, $C = V$ has balanced support with $y = 0$. If $C \neq V$, then C has balanced support if and only if it is supported by a hyperplane H at 0 and $C \cap H = \operatorname{lins} C$.

Theorem 2.3.4. Let $C \subset V$ be a closed convex cone and let $y \in C^\circ$. Then $\{x \in C : y(x) = 0\} = \operatorname{lins} C$ if and only if $y \in \operatorname{rint} C^\circ$. In particular, every closed convex cone has balanced support.

Proof. For the "if" direction, suppose $y \in \operatorname{rint} C^\circ$. By Proposition 2.3.3, $\operatorname{lins} C$ and $\operatorname{span} C^\circ$ are orthogonal complements. In particular $\operatorname{lins} C \subset \{x \in C : y(x) = 0\}$. For the other inclusion, suppose $x \in C$ and $y(x) = 0$. Since $x \in C = C^{\circ\circ}$, we have $y'(x) \leq 0$ for all $y' \in C^\circ$. Thus, by Exercise 2.6, we have $y'(x) = 0$ for all $y' \in C^\circ$. Again since $\operatorname{lins} C$ and $\operatorname{span} C^\circ$ are orthogonal complements, this means $x \in \operatorname{lins} C$, as desired.

For the "only if" direction, suppose $y \in C^\circ \setminus \operatorname{rint} C^\circ$. By the supporting hyperplane theorem applied to y and C°, there exists $x \in V \setminus (C^\circ)^\perp$ such that $y(x) = 0$ and $y'(x) \leq 0$ for all $y' \in C^\circ$. The latter statement implies $x \in C^{\circ\circ} = C$. However, $(C^\circ)^\perp = \operatorname{lins} C$, so $x \in C \setminus \operatorname{lins} C$. Hence there is some $x \in C \setminus \operatorname{lins} C$ such that $y(x) = 0$, as desired. $\square$

2.4 Extreme rays and points

Faces and extreme rays

Our next goal is to show that closed convex cones with trivial lineality space can be written as a conical hull in a unique minimal way.

Given a convex set K, a *face* of K is a nonempty convex set $F \subset K$ such that if $x \in F$ and x is a positive convex combination of $x_1, \dots, x_m \in K$, then $x_1, \dots, x_m \in F$. For cones, we can replace "convex set" with "convex cone" and "convex combination" with "conical combination" for an equivalent definition.

Example 2.4.1. If K is a convex set and H is a supporting hyperplane of K, then $H \cap K$ is a face of K. If a face F of K arises in this way or $F = K$, then F is called a *support set* of K.

It follows from the definition that if F is a face of K and $F' \subset F$, then F' is a face of F if and only if F' is a face of K.

In this text, a *ray* will mean a cone of dimension 1 (so the endpoint is at 0). Any nonzero vector in a ray is a *generator* for the ray. Given a convex set, the *extreme points* and *extreme rays* are the faces of the convex set which are points and rays, respectively. Extreme points of polyhedra are called *vertices*.

Theorem 2.4.2. Let $C \subset V$ be a closed convex cone with $\operatorname{lins} C = \{0\}$. Let $S \subset C$. Then $C = \operatorname{cone} S$ if and only if S contains a generator for every extreme ray of C.

Proof. Suppose $C = \operatorname{cone} S$. If S does not contain a generator for some extreme ray R, then the definition of a face implies $R \not\subset \operatorname{cone} S$, a contradiction. Thus S must contain a generator for every extreme ray.

For the other direction, it suffices to show that C equals the conical hull of the union of its extreme rays. We induct on the dimension of V. If $\dim V \leq 1$, then C is either $\{0\}$ or a single ray, and the statement holds.

Assume $\dim V \geq 2$. Let $x \in C$. Since C is closed and $\operatorname{lins} C = \{0\}$, by Theorem 2.3.4, C is supported by a hyperplane H with $C \cap H = \{0\}$. Hence, there exists $u \in V$ such that $u, -u \notin C$. Since C is closed, we have that $x + \lambda u$ and $x - \lambda u$ are not in C for large enough $\lambda > 0$ (Exercise 2.9). Thus there exist $\lambda_1, \lambda_2 \geq 0$ such that $x_1 := x + \lambda_1 u$ and $x_2 := x - \lambda_2 u$ are in the boundary of C. Let H be a supporting hyperplane of C at x_1. Then

$C \cap H$ is a closed convex cone in H with lineality space $\{0\}$. Applying the inductive hypothesis, x_1 is in the conical hull of the extreme rays of $C \cap H$. Moreover, $C \cap H$ is a face of C, so any extreme ray of $C \cap H$ is an extreme ray of C. Hence x_1 is in the conical hull of the extreme rays of C. The same argument holds for x_2. Since x is a conical combination of x_1 and x_2, it follows that x is in the conical hull of the extreme rays of C. $\square$

Minkowski's theorem and homogenization

We now use Theorem 2.4.2 to prove the following decomposition theorem for convex sets.

Theorem 2.4.3 (Minkowski)**.** Let K be a closed convex set with $\operatorname{lins} K = \{0\}$. Let $S \subset K$. Then $K = \operatorname{conv} S + \operatorname{recc} K$ if and only if S contains every extreme point of K.

Corollary 2.4.4. Let K be a closed, bounded convex set in V, and let $S \subset K$. Then $K = \operatorname{conv} S$ if and only if S contains every extreme point of K.

Theorem 2.4.3 can be proved in a similar way to Theorem 2.4.2. However, we will take this opportunity to introduce a useful construction called homogenization. Similar to (2.1.5), this gives a way to change questions about general convex sets to questions about convex cones. The difference is that homogenization also takes into account the behavior at infinity.

Given a convex set $K \subset V$, define its *homogenization* $\operatorname{homog} K$ to be the set in $V \times \mathbb{R}$ given by

$$\operatorname{homog} K := \operatorname{cone}(K \times \{1\}) \cup (\operatorname{recc} K \times \{0\}).$$

Equivalently, $\operatorname{homog} K$ is $\operatorname{cone}(K \times \{1\})$ along with all limit points of $\operatorname{cone}(K \times \{1\})$ in $V \times \{0\}$ (Exercise 2.21). The set $\operatorname{homog} K$ is a convex cone which is closed if K is closed. This and other properties can be found in Exercises 2.22–2.23. We use these to sketch a proof of Theorem 2.4.3.

Proof of Theorem 2.4.3. By Exercise 2.23, we have $K = \operatorname{conv} S + \operatorname{recc} K$ if and only if

$$\operatorname{homog} K = \operatorname{cone}((S \times \{1\}) \cup (\operatorname{recc} K \times \{0\}))$$

By Theorem 2.4.2 and Exercise 2.24, this occurs if and only if $S \times \{1\}$ contains $(x, 1)$ for every extreme point x of K. $\square$

2.5 Hull descriptions of polyhedra

Polyhedra were defined to be the solution sets to finite systems of linear inequalities. We now show that polyhedra can also be characterized as sets with finite convex/conical hull presentations. We first prove some intermediate results.

Proposition 2.5.1. If P is a polyhedron, then $\operatorname{recc} P$ and $\operatorname{homog} P$ are polyhedra.

Proof. See Exercises 2.10 and 2.25. □

Proposition 2.5.2. (a) A polyhedral cone has finitely many extreme rays.

(b) A polyhedron has finitely many vertices.

Proof. Proof of (a). Let C be a polyhedral cone. If $\operatorname{lins} C \neq \{0\}$ then C has no extreme rays. (See Exercise 2.20.) So assume $\operatorname{lins} C = \{0\}$. We induct on the dimension of V. It is obvious for $\dim V \leq 1$. Assume $\dim V \geq 2$. Write $C = \{y_i(x) \leq 0,\ i \in I\}$ for some finite set I. We may assume $y_i \neq 0$ for all i. Let $H_i = \{y_i(x) = 0\}$. If an element of C is not in any of the H_i, then it is in the set $\{y_i(x) < 0,\ i \in I\}$, which is the interior of C. Since $\dim V \geq 2$, a ray containing an interior element of C cannot be an extreme ray. Thus every extreme ray of C is contained in some H_i. An extreme ray of C contained in H_i is also an extreme ray of $C \cap H_i$. Each $C \cap H_i$ is a polyhedral cone in H_i with trivial lineality space, so by the inductive hypothesis it has finitely many extreme rays. There are finitely many H_i, so C has finitely many extreme rays.

Proof of (b). This follows from part (a) after homogenization, using Exercise 2.24. □

We now state our main result.

Theorem 2.5.3. (a) A subset of V is a polyhedral cone if and only if it is the conical hull of a finite set of vectors.

(b) A subset of $P \subset V$ is a polyhedron if and only if it can be written as $\operatorname{conv} S + \operatorname{cone} T$ for finite sets S, $T \subset V$. If this is the case and P is nonempty, then $\operatorname{recc} P = \operatorname{cone} T$.

Proof. Proof of (a). For the "only if" direction, let C be a polyhedral cone. Write $C = C' \oplus L$ where $L = \operatorname{lins} C$ and C' is a polyhedral cone with $\operatorname{lins} C' = \{0\}$. Since L is a vector space, it is the conical hull of a finite set. By Theorem 2.4.2 and Proposition 2.5.2, C' is the conical hull of a finite set. Hence C is the conical hull of a finite set.

For the "if" direction, let C be the conical hull of a finite set. By Proposition 2.3.2(a), C° is a polyhedral cone in V^*. Then by what we just proved, C° is the conical hull of a finite set. By Proposition 2.3.2(a) again, $(C^\circ)^\circ$ is a polyhedral cone. Since C is closed by Proposition 2.1.7, we have $C = (C^\circ)^\circ$, proving the result.

Proof of (b). This first part follows from (a) and homogenization, using Exercise 2.26. The second part follows from Exercise 2.26. □

Corollary 2.5.4. A subset of V is a polytope if and only if it is the convex hull of a finite set of points.

Corollary 2.5.5. The polar cone of a polyhedral cone is polyhedral.

We conclude by using Theorem 2.5.3 to prove the following statements.

Theorem 2.5.6. If $f : V \to W$ is an affine map and $P \subset V$ is a polyhedron, then $f(P)$ is a polyhedron.

Proof. An affine map $f : V \to W$ can be written as $f(x) = f_0(x) + b$, where $f_0 : V \to W$ is a linear map and $b \in \mathbb{R}$. It is straightforward to check that

$$f(\operatorname{conv} S + \operatorname{cone} T) = \operatorname{conv} f(S) + \operatorname{cone} f_0(T).$$ □

Theorem 2.5.7. The Minkowski sum of two polyhedra is a polyhedron.

Proof. Let $P_1 = \operatorname{conv} S_1 + \operatorname{cone} T_1$ and $P_2 = \operatorname{conv} S_2 + \operatorname{cone} T_2$ be polyhedra, where S_i, T_i are finite for $i = 1, 2$. Then by Exercise 2.27,

$$P_1 + P_2 = \operatorname{conv}(S_1 + S_2) + \operatorname{cone}(T_1 \cup T_2)$$

so $P_1 + P_2$ is a polyhedron. □

Theorem 2.5.8. If P, Q are polyhedra, then $\operatorname{conv}(P \cup Q)$ is a polyhedron.

Proof. Exercise 2.28. □

Theorem 2.5.9. Let P be a nonempty polyhedron and $y \in V^*$ a linear functional.
(a) If $y \in (\operatorname{recc} P)^\circ$, then y achieves a maximum on P.
(b) If $y \notin (\operatorname{recc} P)^\circ$, then y is unbounded from above on P.

Proof. By Theorem 2.5.3, we have $P = \operatorname{conv} S + \operatorname{recc} P$ for some nonempty finite set S. If $y \in (\operatorname{recc} P)^\circ$, then y achieves a maximum of 0 on $\operatorname{recc} P$, so y achieves a maximum of $\max_S y$ on P. If $y \notin (\operatorname{recc} P)^\circ$, then $y(u) > 0$ for some $u \in \operatorname{recc} P$, and thus $y(x + \lambda u) \to \infty$ for $x \in P$ and $\lambda \to \infty$. $\square$

Exercises

2.1. The convex hull of a compact set is compact.
2.2. Proposition 2.1.7 is not true if we replace "finite" with "compact".
2.3. If S is a compact and $0 \notin \operatorname{conv} S$, then $\operatorname{cone} S$ is closed.
2.4. The closure, interior, and relative interior of a convex set are convex.
2.5. The relative interior of any nonempty convex set is nonempty.
2.6. Let K be a convex set and $x_0 \in \operatorname{rint} K$. Suppose $y \in V^*$ and $b \in \mathbb{R}$ such that $y(x) \leq b$ for all $x \in K$. Then $y(x) = b$ for all $x \in K$ if and only if $y(x_0) = b$.
2.7. Let K be a convex subset of V. Then $\operatorname{lins} K$ is a vector subspace of V, and $\operatorname{recc} K$ is a convex cone.
2.8. (a) Let K be a convex set. Then

$$\operatorname{lins} K = \operatorname{lins} \operatorname{recc} K = (\operatorname{recc} K) \cap (-\operatorname{recc} K).$$

(b) If K is a convex cone, then $\operatorname{recc} K = K$.
2.9. Let K be a closed convex subset of V. For a nonzero vector $v \in V$, the following two statements are equivalent:
(i) $v \in \operatorname{lins} K$.
(ii) For some $x_0 \in K$, we have $x_0 + \lambda v \in K$ for all $\lambda \in \mathbb{R}$.
In addition, the following two statements are equivalent:
(iii) $v \in \operatorname{recc} K$.
(iv) For some $x_0 \in K$, we have $x_0 + \lambda v \in K$ for all $\lambda \geq 0$.
2.10. Let $K = \{y_i(x) \leq b_i,\ i \in I\}$ where I is any set and $y_i \in V^*$, $b_i \in \mathbb{R}$ for all i. Assume K is nonempty. Then

$$\begin{aligned} \operatorname{lins} K &= \{y_i(x) = 0,\ i \in I\} \\ \operatorname{recc} K &= \{y_i(x) \leq 0,\ i \in I\}. \end{aligned}$$

2.11. Let $K \subset V$ be a nonempty closed convex set. The following are true.
(a) $\operatorname{recc} K$ is the largest cone $C \subset V$ such that $K = S + C$ for some set $S \subset V$.
(b) $\operatorname{lins} K$ is the largest cone $C \subset V$ such that $K = S + C + (-C)$ for some set $S \subset V$.

2.12. The Minkowski sum of two closed convex sets is not necessarily closed. *Hint:* Consider the sets $\{x \in \mathbb{R}^2 : e^1(x)e^2(x) \geq 1\}$ and $\{x : \mathbb{R}^2 : e^1(x) \geq 0\}$.

2.13. We give a metric proof of Theorem 2.2.4. Let K be a nonempty closed convex subset of V and let $u \in V \setminus \operatorname{cl} K$. Fix an inner product $\langle \cdot, \cdot \rangle$ and norm $||x|| = \langle x, x \rangle$.
(a) There exists $v \in K$ such that $||u-v||$ is the minimum distance between u and any point in K.
(b) Let $y = \langle u-v, \cdot \rangle$ and $b = y(v)$. Then $y(u) > b$ and $y(x) \leq b$ for $x \in K$.

2.14. Let $\dim V \leq 2$, and let $C \subset V$ be a convex cone with $C \neq V$. Then there exists $y \in V^* \setminus \{0\}$ such that $y(x) \leq 0$ for all $x \in C$.

2.15. Let K and L be two disjoint nonempty closed convex sets in V such that $\operatorname{recc} K \cap \operatorname{recc} L = \{0\}$. Then there exist $y \in V^* \setminus \{0\}$ and $b \in \mathbb{R}$ such that $y(x) > b$ for all $x \in K$ and $y(x) < b$ for all $x \in L$.
Hint: Write $K = \{y_i(x) \leq b_i : i \in I\}$ and $L = \{z_j(x) \leq c_j : j \in J\}$. Apply Theorem 2.2.1 to the conical hull of $\{(y_i, -b_i)\} \cup \{(z_j, -b_j)\}$ in $V^* \times \mathbb{R}$.

2.16. Let C be a subset of V.
(a) C° is a closed convex cone.
(b) $C \subset (C^\circ)^\circ$.

2.17. Let $C \subset V$. Then $\operatorname{span} C$ and $\operatorname{lins}(C^\circ)$ are orthogonal complements. If C is a closed convex cone, then $\operatorname{lins} C$ and $\operatorname{span}(C^\circ)$ are orthogonal complements.

2.18. Let $C \subset V$.
(a) If $C = (\operatorname{cone} S) + L$ where $S \subset V$ and L is a vector subspace, then

$$C^\circ = \{y \in V^* : y(x) \leq 0 \text{ for all } x \in S\} \cap L^\perp$$

(b) Suppose

$$C = \{x \in V : y(x) \leq 0 \text{ for all } y \in T\} \cap L$$

where $T \subset V^*$, $\operatorname{cone} T$ is closed, and L is a vector subspace. Then $C^\circ = (\operatorname{cone} T) + L^\perp$.

2.19. Let K be a convex set. A nonempty convex set $F \subset K$ is a face of K if and only for every $x \in K$, if $x = \frac{1}{2}(x_1 + x_2)$ where $x_1, x_2 \in K$, then $x_1, x_2 \in F$.

2.20. Let K be a convex set. Every face of K contains a translation of $\operatorname{lins} K$.
2.21. Let $K \subset V$ be a convex set. Then $\operatorname{homog} K$ is the union of $\operatorname{cone}(K \times \{1\})$ with the set of limits points of $\operatorname{cone}(K \times \{1\})$ in $V \times \{0\}$.
2.22. Let K be a convex set. The following are true.
(a) $\operatorname{homog} K$ is a convex cone. If K is closed, then so is $\operatorname{homog} K$.
(b) $\operatorname{lins}(\operatorname{homog} K) = \operatorname{lins} K \times \{0\}$.
2.23. Let K be a closed convex set in V. Let $S \subset V$. Then $K = \operatorname{conv} S + \operatorname{recc} K$ if and only if

$$\operatorname{homog} K = \operatorname{cone}((S \times \{1\}) \cup (\operatorname{recc} K \times \{0\})).$$

2.24. The extreme rays of $\operatorname{homog} K$ are exactly the rays $\operatorname{cone}\{(x, 1)\}$ over all extreme points x of K, along with the extreme rays of $\operatorname{recc} K \times \{0\}$.
2.25. Let $K = \{y_i(x) \leq b_i,\ i \in I\}$ where I is any set. Assume K is nonempty. Then

$$\operatorname{homog} K = \{(x, t) \in V \times \mathbb{R} : t \geq 0,\ y_i(x) - b_i t \leq 0 \text{ for all } i \in I\}.$$

2.26. Let B be a nonempty bounded convex set and C a closed cone. Then

$$\begin{aligned} \operatorname{recc}(B + C) &= C \\ \operatorname{homog}(B + C) &= \operatorname{cone}((B \times \{1\}) \cup (C \times \{0\})). \end{aligned}$$

2.27. For any two sets S_1, $S_2 \subset V$,

$$\begin{aligned} \operatorname{cone}(S_1 \cup S_2) &= \operatorname{cone}(S_1 \cup S_2) \\ \operatorname{conv}(S_1) + \operatorname{conv}(S_2) &= \operatorname{conv}(S_1 + S_2). \end{aligned}$$

2.28. If P, Q are polyhedra, then $\operatorname{conv}(P \cup Q)$ is a polyhedron.

Chapter 3

Faces

3.1 Faces of convex sets and polyhedra

Faces of convex sets

Let $K \subset V$ be a convex set. Recall that a *face* of K is a nonempty convex set $F \subset K$ such that if $x \in F$ and x is a positive convex combination of $x_1, \ldots, x_m \in K$, then $x_1, \ldots, x_m \in F$. Equivalently, a nonempty convex set $F \subset K$ is a face of K if and only for every $x \in F$, if $x = \frac{1}{2}(x_1 + x_2)$ for some $x_1, x_2 \in K$, then $x_1, x_2 \in F$ (Exercise 2.19). If K is nonempty, then K is a face of itself. A *proper face* of K is a face which is not K. If $K = \varnothing$, then K has no faces.

We begin by proving a general characterization of faces. Recall that for a point $x \in K$, the *direction cone* of K at x is

$$D(K, x) = \operatorname{cone}(K - x).$$

In other words, $v \in D(K, x)$ if and only if $x + \lambda v \in K$ for some $\lambda > 0$. By Exercise 2.8, we have

$$\operatorname{lins} D(K, x) = D(K, x) \cap (-D(K, x)).$$

In other words, $v \in \operatorname{lins} D(K, x)$ if and only if $x + \lambda v \in K$ for some $\lambda > 0$ and some $\lambda < 0$.

Proposition 3.1.1. Let K be a convex set, and let $x \in K$. The following are true.

(a) There is exactly one face F of K such that $x \in \operatorname{rint} F$, and for any face G of K containing x, we have $G \supset F$.

(b) The face F from part (a) is given by

$$F = K \cap (x + \operatorname{lins} D(K, x)).$$

In addition, $\operatorname{aff} F = x + \operatorname{lins} D(K, x)$.

Proof. We first prove that if x is in the relative interior of a face F, then any other face containing x must contain F. Suppose F, G are faces such that $x \in \operatorname{rint} F$ and $x \in G$. Let $x_1 \in F$. Since $x \in \operatorname{rint} F$ and F is convex, there is $x_2 \in F$ such that x is in the relative interior of the segment $[x_1, x_2]$. Since G is face, we must therefore have $x_1 \in G$. Hence $G \supset F$, proving our claim that any face containing x must contain F. This also implies that x is in the relative interior of at most one face.

Now let $L = \operatorname{lins} D(K, x)$ and let $F = K \cap (x + L)$. To complete the proof of the theorem, it suffices to show that F is a face of K, that $x \in \operatorname{rint} F$, and $\operatorname{aff} F = x + L$. We show the latter two statements first. By definition F is contained in the affine space $x + L$. Also, by definition of L, we have that K contains a neighborhood of x in the relative topology of $x + L$, and hence F does as well. Hence $\operatorname{aff} F = x + L$, and $x \in \operatorname{rint} F$.

We now prove F is a face. Since F is the intersection of two convex sets, it is convex. Let $x' \in F$ and suppose $x' = \frac{1}{2}(x_1 + x_2)$ for some x_1, $x_2 \in K$. Since $x' \in F$ and $x \in \operatorname{rint} F$, there is $x_3 \in F$ such that x is in the relative interior of the segment $[x', x_3]$. Then x is in the relative interior of $T := \operatorname{conv}(x_1, x_2, x_3)$. Since $T \subset K$ by convexity, we have $T - x \subset L$ by definition of L. Thus $T \subset x + L$, so $T \subset F$. In particular x_1, $x_2 \in F$, so F is a face of K. $\square$

Corollary 3.1.2. Every convex set is the disjoint union of the relative interiors of its faces.

If x, $x' \in \operatorname{rint} F$ for some face F of K, then $D(K, x) = D(K, x')$ (Exercise 3.4). We can therefore define $D(K, F)$ to be $D(K, x)$ for any $x \in \operatorname{rint} F$. By Proposition 3.1.1, $\operatorname{lins} D(K, F)$ equals the linear subspace parallel to $\operatorname{aff} F$.

Support sets

Let K be a convex set. Recall from Example 2.4.1 that a *support set* of K is a nonempty set F such that either $F = K$ or $F = H \cap K$ for some supporting hyperplane H of K. Equivalently, a set F is a support set of K if and only if it is nonempty and there exists a linear functional $y \in V^*$ such that $F = \operatorname{argmax}_K y$. (The case $F = K$ is when $y = 0$.) Support sets are also called *exposed faces*.

All support sets of a convex set are faces, but not every face is a support set (Exercise 3.9). The precise condition for a face to be a support set is as follows.

Proposition 3.1.3. Let K be a convex set and F a face of K. Then F is a support set of K if and only if $D(K, F)$ has balanced support.

This is part (b) of the following proposition (Exercise 3.10).

Proposition 3.1.4. Let K be a convex set and F a face of K. Let $y \in V^*$. The following are true.

(a) $F \subset \operatorname{argmax}_K y$ if and only if $y(x) \leq 0$ for all $x \in D(K, F)$.

(b) $F = \operatorname{argmax}_K y$ if and only if $y(x) \leq 0$ for all $x \in D(K, F)$ and $D(K, F) \cap \{y(x) = 0\} = \operatorname{lins} D(K, F)$.

Faces of polyhedra

Let P be a polyhedron. By Proposition 3.1.1, every face of P is the intersection of P with an affine space, and is therefore a polyhedron. We start by showing the following.

Proposition 3.1.5. Let $P = \{y_i(x) \leq b_i,\ i \in I\}$ be a polyhedron, where I is finite. Let F be a face of P, and let $J \subset I$ be the set of all j such that $y_j(x) = b_j$ for all $x \in F$. Then

$$D(P, F) = \{y_j(x) \leq 0,\ j \in J\}.$$

Proof. Let $x_0 \in \operatorname{rint} F$. By Exercise 2.6, we have $j \in J$ if and only if $y_j(x_0) = b_j$. It is easy to check that $D(P, x_0) \subset \{y_j(x) \leq 0,\ j \in J\}$. For the reverse inclusion, let $v \in \{y_j(x) \leq 0,\ j \in J\}$. By the definition of J, we have $y_i(x_0) < b_i$ for all $i \in I \setminus J$. Since $I \setminus J$ is finite, we can choose $\lambda > 0$ small enough so that $y_i(x_0 + \lambda v) < b_i$ for all $i \in I \setminus J$. Moreover, for all $j \in J$ and $\lambda > 0$, we have $y_j(x_0 + \lambda v) \leq y_j(x_0) = b_j$. Hence for small enough $\lambda > 0$, we have $x_0 + \lambda v \in P$. Thus $v \in D(P, x_0)$, as desired. □

Corollary 3.1.6. Every face of a polyhedron is a support set of the polyhedron.

Proof. By Proposition 3.1.5, for any face F we have that $D(P, F)$ is polyhedral and therefore closed. Thus $D(P, F)$ has balanced support by Theorem 2.3.4. □

Corollary 3.1.7. Let P, F, and J be as in Proposition 3.1.5. Then

$$F = P \cap \{y_j(x) = b_j,\ j \in J\}.$$

Proof. This follows from Propositions 3.1.1 and 2.1.9. □

Corollary 3.1.8. A polyhedron has finitely many faces.

Proof. In Corollary 3.1.7, there are finitely many possible J. □

3.2 Face posets

For a convex set K, let $\mathscr{F}(K)$ denote the set of all faces of K. We view $\mathscr{F}(K)$ as a poset ordered by inclusion. We call $\mathscr{F}(K)$ the *face poset* of K.

Proposition 3.2.1. Let K be a convex set. The following are true.

(a) Let F be a face of K and let G be any subset of F. Then G is a face of F if and only if G is a face of K.

(b) If $\{F_i : i \in I\}$ is a collection of faces of K and $\bigcap_{i\in I} F_i \neq \varnothing$, then $\bigcap_{i\in I} F_i$ is a face of K.

(c) If $K = K' \oplus L$ where L is a vector space, then the map $F \mapsto F \oplus L$ is a poset isomorphism $\mathscr{F}(K') \to \mathscr{F}(K)$.

(d) If K is nonempty, then K is the unique maximal element of $\mathscr{F}(K)$.

(e) If K is nonempty and closed, the minimal elements of $\mathscr{F}(K)$ are translates of $\operatorname{lins} K$.

(f) If K is a cone, then $\operatorname{lins} K$ is the unique minimal element of $\mathscr{F}(K)$.

Proof. Parts (a)–(d) are straightforward. For part (e), write $K = K' \oplus L$ where L is the lineality space of K and K' is a closed convex set with $\operatorname{lins} K' = \{0\}$. By Theorem 2.4.3, every face of K' has an extreme point, and by (a), these are also extreme points of K'. Hence the minimal elements of $\mathscr{F}(K')$ are the extreme points of K'. By (c), the minimal elements of $\mathscr{F}(K)$ are thus translates of L. Part (f) is Exercise 3.5. □

Proposition 3.2.2. Let K, K' be convex sets and $f : K \to K'$ an affine isomorphism. Then the map $F \mapsto f(F)$ is a poset isomorphism $\mathscr{F}(K) \to \mathscr{F}(K')$.

Proof. Easy from the definition of a face. $\square$

Proposition 3.2.3. Let K be a convex set and F a face of K.

(a) The poset $\mathscr{F}(K)_{\leq F} := \{G \in \mathscr{F}(K) : G \subset F\}$ is the face poset of F.

(b) The poset $\mathscr{F}(K)_{\geq F} := \{G \in \mathscr{F}(K) : G \supset F\}$ is isomorphic to the face poset of $D(K, F)$.

Proof. Part (a) is Proposition 3.2.1(a). For part (b), consider the map $\mathscr{F}(K)_{\geq F} \to \mathscr{F}(D(K, F))$ given by $G \mapsto D(G, F)$. We leave it as an exercise to show that this map is well-defined and an isomorphism (Exercise 3.7). $\square$

Grading

Let $(\mathscr{P}, \leq)$ be a poset. For x, $y \in \mathscr{P}$, we say that x *covers* y if $x > y$ and there is no z such that $x > z > y$. A *grading* on $\mathscr{P}$ is a function $\rho : \mathscr{P} \to \mathbb{Z}$ such that $\rho(x) > \rho(y)$ if $x > y$, and $\rho(x) = \rho(y) + 1$ if x covers y.

Proposition 3.2.4. For any polyhedron P, the dimension function is a grading on $\mathscr{F}(P)$.

Proof. Suppose F, $G \in \mathscr{F}(P)$ and G covers F. Then the interval $[F, G]$ in $\mathscr{F}(P)$ has only two elements. By Proposition 3.2.3, $[F, G]$ is isomorphic to $\mathscr{F}(D(G, F))$. Write $D(G, F) = C \oplus \operatorname{lins} D(G, F)$, where C is a polyhedral cone with $\operatorname{lins} C = \{0\}$. By Proposition 3.2.1(c), we have $\mathscr{F}(C) \cong \mathscr{F}(D(G, F)) \cong [F, G]$. So $\mathscr{F}(C)$ has exactly two elements $\{0\}$ and C. Since C is polyhedral, by Theorem 2.4.2 it has at least one extreme ray, and therefore C itself is a ray. Hence $\dim C = 1$. Thus, $\dim D(G, F) - \dim \operatorname{lins} D(G, F) = 1$. Since $\dim D(G, F) = \dim G$ and $\dim \operatorname{lins} D(G, F) = \dim F$, we have $\dim G - \dim F = 1$, as desired. $\square$

Facets

Given a polyhedron P of dimension d, the $(d-1)$-dimensional faces of P are called the *facets* of P. Given a facet F of P, an inequality $y(x) \leq b$ such that $\max_P y = b$ and $\operatorname{argmax}_P y = F$ is called a *facet-defining inequality* of P for F. The facet-defining inequalities satisfy a minimal presentation theorem which is the polar dual to Theorem 2.4.2.

Theorem 3.2.5. Let $P \subset V$ be a polyhedron with $\dim P = \dim V$. Let I be a finite set and for each $i \in I$, let $y_i(x) \leq b_i$ be a linear inequality which is true on P. Then

$$P = \{y_i(x) \leq b_i,\ i \in I\}$$

if and only if for every facet F of P, there is some $i \in I$ such that $y_i(x) \leq b_i$ is a facet-defining inequality for F.

We leave the proof as an exercise. One proof sketch is the following: If P is a polyhedral cone with $\dim P = \dim V$, we show that an inequality $y(x) \leq 0$ is facet-defining if and only if y is an extreme ray of P°. (This follows, for example, from Proposition 3.4.7 later in this chapter.) We can then deduce Theorem 3.2.5 for cones from Theorem 2.4.2. The general result follows from homogenization.

Corollary 3.2.6. Every proper face of a polyhedron is the intersection of the facets that contain it.

Proof. The statement does not change if we change the ambient space to the affine span of the polyhedron. Thus we may assume the polyhedron is full-dimensional. The result then follows from Theorem 3.2.5 and Corollary 3.1.7. □

3.3 Examples

Example 3.3.1 (Simplicial cones)**.** Let

$$\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : e^i(x) \geq 0 \text{ for all } i\}$$

be the nonnegative orthant of $\mathbb{R}^n$ from Example 1.2.3. The extreme rays are generated by the vectors e_i, $i \in [n]$. The facet-defining inequalities are $e^i(x) \geq 0$. The faces are all sets of the form

$$\mathbb{R}^S_{\geq 0} := \operatorname{cone}\{e_i : i \in S\} = \mathbb{R}^n_{\geq 0} \cap \{e^i(x) = 0,\ i \notin S\}$$

where S is any subset of $[n]$. The map $S \mapsto \mathbb{R}^S_{\geq 0}$ is a poset isomorphism from the *Boolean lattice* $\mathscr{B}_n$ of subsets of $[n]$ to $\mathscr{F}(\mathbb{R}^n_{\geq 0})$.

Example 3.3.2 (Simplices)**.** Let

$$\Delta = \{x \in \mathbb{R}^{n+1} : e^1(x) + \cdots + e^{n+1}(x) = 1,\ e^i(x) \geq 0,\ i = 1, \ldots, n\}$$

be the n-dimensional simplex from Example 1.2.8. The vertices are the points e_i, and the facet-defining inequalities are $e^i(x) \geq 0$. The faces are the sets of the form

$$\Delta_S := \operatorname{conv}\{e_i : i \in S\} = \Delta \cap \{e^i(x) = 0,\, i \notin S\}$$

where S is any nonempty subset of $[n+1]$. The map $S \mapsto \Delta_S$ is a poset isomorphism from $\mathscr{B}_{n+1} \setminus \{\varnothing\}$ to $\mathscr{F}(\Delta)$.

Example 3.3.3 (Cube)**.** Let

$$Q = \{x \in \mathbb{R}^n : -1 \leq e^i(x) \leq 1,\, i = 1, \dots, n\}$$

be the n-dimensional cube from Example 1.2.6. The vertices of Q are all points of the form $\pm e_1 \pm \cdots \pm e_n$. The facet-defining inequalities are $e^i(x) \geq -1$ and $e^i(x) \leq 1$ over all $i \in [n]$. The faces of Q are

$$Q_{S,T} := Q \cap \{e^i(x) = -1,\, i \in S\} \cap \{e^i(x) = 1,\, i \in T\}$$

where S, T are any disjoint subsets of $[n]$. There are 3^n faces in total.

Example 3.3.4 (Cross-polytope)**.** Let

$$R = \{x \in \mathbb{R}^n : \pm e^1(x) \pm \cdots \pm e^n(x) \leq 1\},$$

be the d-dimensional cross-polytope from Example 1.2.7. The vertices of R are the points $\pm e_1, \dots, \pm e_n$, and the facet-defining inequalities are the inequalities in the above presentation. The faces of R are

$$R_{S,T} := \operatorname{conv}(\{e_i : i \in S\} \cup \{-e_i : i \in T\})$$

where S, T are any disjoint subsets of $[n]$ with $S \cup T \neq \varnothing$, as well as R itself. There are 3^n faces in total.

3.4 Normal fans

Normal cones

Let $K \subset V$ be a convex set and $x \in K$. The *(outward) normal cone* of K at x is defined to be

$$N(K, x) := D(K, x)^\circ.$$

Recall that if F is a face of K and x, $x' \in \operatorname{rint} F$, then $D(K,x) = D(K,x')$. Thus we can define $N(K,F)$ to be $N(K,x)$ for any $x \in \operatorname{rint} F$. Elements of $N(K,x)$ or $N(K,F)$ are called *normal vectors* to K at x or F.

The normal cone of K at x is the set of all linear functionals which achieve a maximum on K at x. We state this below.

Proposition 3.4.1. Let K be a convex set, F a face of K, and $x \in \operatorname{rint} F$. The following are true.

(a) We have $y \in N(K,F)$ if and only if $F \subset \operatorname{argmax}_K y$, if and only if $x \in \operatorname{argmax}_K y$.

(b) If $D(K,F)$ is closed, then $y \in \operatorname{rint} N(K,F)$ if and only if $F = \operatorname{argmax}_K y$.

Proof. Proof of (a). The first part is Proposition 3.1.4(a), and then the second part follows from Exercise 2.6.

Proof of (b). This follows from Proposition 3.1.4(b) and Proposition 2.3.4. □

Corollary 3.4.2. If F, G are faces of K and $F \subset G$, then $N(K,F) \supset N(K,G)$.

Proposition 3.4.3. If $D(K,F)$ is closed, then

$$\operatorname{span} N(K,F) = F^{\perp}.$$

Proof. This follows from Proposition 2.3.3. □

Example 3.4.4. Let $P = \{y_i(x) \leq b_i,\ i \in I\}$ be a polyhedron, where I is finite. By Proposition 3.1.5, for any face F of P we have $D(P,F) = \{y_j(x) \leq 0,\ j \in J\}$ where J is the set of all j such that $y_j(x) = b_j$ for all $x \in F$. Then by Proposition 2.3.2,

$$N(P,F) = \operatorname{cone}(\{y_j : j \in J\}).$$

Suppose instead that P is presented as $\{y_i(x) \leq b_i,\ i \in I\} \cap A$, where I is finite and A is an affine space. Then for any face F of P, we have $D(P,F) = \{y_j(x) \leq 0,\ j \in J\} \cap A$ where J is as before, and by Exercise 2.18,

$$N(P,F) = \operatorname{cone}(\{y_j : j \in J\}) + A^{\perp}.$$

Normal fans

For a convex set K, we define the *normal fan* $\mathscr{N}(K)$ of K to be the set of all normal cones of K, that is

$$\mathscr{N}(K) = \{N(K,x) : x \in K\} = \{N(K,F) : F \in \mathscr{F}(K)\}.$$

We make $\mathscr{N}(K)$ a poset by partially ordering by inclusion. We will focus on the case where K is a polyhedron.

Theorem 3.4.5. Let $P \subset V$ be a polyhedron. The following are true.

(a) The map $F \mapsto N(P,F)$ is a poset anti-isomorphism $\mathscr{F}(P) \to \mathscr{N}(P)$ with $\dim F + \dim N(P,F) = \dim V$.

(b) If P is nonempty, then $\mathscr{N}(P)$ has unique minimal element $N(P,P) = P^{\perp}$. In particular, the minimal element is $\{0\}$ if P is full-dimensional.

(c) For any face F of P, the faces of $N(P,F)$ are precisely the sets $N(P,G)$ where G is a face of P containing F.

(d) For nonempty P, we have $\bigcup_{F\in\mathscr{F}(P)} N(P,F) = (\operatorname{recc} P)^{\circ}$.

Proof. Proof of (a). The map is order-reversing by 3.4.2, and it is surjective by definition. Since P is a polyhedron, $D(P,F)$ is closed for all F. Thus the injectivity follows from Proposition 3.4.1(b). The equality $\dim F + \dim N(P,F) = \dim V$ follows from Proposition 3.4.3.

Proof of (b). From (a), $\mathscr{N}(P)$ has unique minimal element $N(P,P) = D(P,P)^{\circ}$. Since $D(P,P)$ is the linear subspace parallel to $\operatorname{aff} P$, we have $D(P,P)^{\circ} = D(P,P)^{\perp} = P^{\perp}$.

Proof of (c). Exercise 3.12.

Proof of (d). By Proposition 3.4.1, a linear functional is in $\bigcup_{F\in\mathscr{F}(P)} N(P,F)$ if and only if it achieves a maximum on P. The result thus follows from Theorem 2.5.9. □

Example 3.4.6 (Polyhedral cones)**.** If C is a polyhedral cone, then it has a unique minimal face $\operatorname{lins} C$. So by Theorem 3.4.5(a), $\mathscr{N}(C)$ has a unique maximal element $N(C,\{0\}) = D(C,\{0\})^{\circ} = C^{\circ}$. By Theorem 3.4.5(c), the other normal cones $N(C,F)$ are exactly the faces of C°. We thus obtain the following.

Proposition 3.4.7. If C is a polyhedral cone, then $\mathscr{N}(C) = \mathscr{F}(C^{\circ})$. The map $F \mapsto N(C,F)$ is a poset anti-isomorphism $\mathscr{F}(C) \to \mathscr{F}(C^{\circ})$.

Example 3.4.8 (Simplex)**.** Let Δ be the n-dimensional simplex from Example 1.2.8. The minimal element L of $\mathscr{N}(\Delta)$ is

$$\Delta^\perp = \{e^1(x) + \cdots + e^{n+1}(x) = 0\}^\perp = \operatorname{span}\{e^1 + \cdots + e^{n+1}\}.$$

The facet-defining inequalities are $-e^i(x) \le 0$. Therefore, by Example 3.3.2, the next-to-minimal elements of $\mathscr{N}(\Delta)$ are $\operatorname{cone}\{-e_i\} + L$. From Example 3.3.2, the normal cone of the face Δ_S is $\operatorname{cone}\{-e_i : i \notin S\} + L$.

Exercises

3.1. Every face of a closed convex set is closed.
3.2. Let K, L be convex sets. The faces of $K \times L = \{(k, l) : k \in K, l \in L\}$ are precisely the sets $F \times G$ where F is a face of K and G is a face of L.
3.3. Let K, L be convex sets in the same space. The faces of $K \cap L$ are precisely the nonempty sets of the form $F \cap G$ where F is a face of K and G is a face of L.
3.4. Let K be a convex set. If x, $x' \in \operatorname{rint} F$ for some face F of K, then $D(K, x) = D(K, x')$.
3.5. If C is a cone, then $\operatorname{lins} C$ is the unique minimal element of $\mathscr{F}(C)$.
3.6. (a) Let C be a closed convex set with $\operatorname{lins} C = \{0\}$ and F a face of C. Then F is the conical hull of all extreme rays of C contained in F.
(b) Let K be a closed bounded convex set and F a face of K. Then F is the convex hull of all extreme points of K contained in F.
3.7. Let K be a convex set and F a face of K. The map $G \mapsto D(G, F)$ is a poset isomorphism $\mathscr{F}(K)_{\ge F} \to \mathscr{F}(D(K, F))$.
3.8. Let K be a convex set and $\operatorname{homog} K$ its homogenization. Then the map $F \mapsto \operatorname{homog} F$ is an injective map of posets $\mathscr{F}(K) \to \mathscr{F}(\operatorname{homog} K)$. The faces of $\operatorname{homog} K$ not in the image of this map are $F \times \{0\}$, where F is a face of $\operatorname{recc} K$.
3.9. Let K be the following convex subset of $\mathbb{R}^2$:

$$K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} \cup \{0 \le x \le 1,\ 0 \le y \le 1\}.$$

Then $(0, 1)$ and $(1, 0)$ are faces of K but not support sets of K.
3.10. Let K be a convex set and F a face of K. Let $y \in V^*$. The following are true.
(a) The support set of K with respect to y contains F if and only if the support set of $D(K, F)$ with respect to y contains $D^*(K, F)$.

(b) The support set of K with respect to y equals F if and only if the support set of $D(K, F)$ with respect to y equals $D^*(K, F)$.

3.11. Let K be d-dimensional convex set and F a face of dimension $d - 1$. Then F is a support set of K.

3.12. For any face F of P, the faces of $N(P, F)$ are precisely the sets $N(P, G)$ where $G \supset F$ is a face of P.