Kuwait University

Faculty of Science

Computer Science Department

CS 301: Algorithms Design and Analysis

Summer 2021/2022

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Assignment #2

Date: 7th of July, 2022

Question 1:

a) $Let\ c\ , n_0\ \geq 0,\ then\ there\ exists\ a\ function\ g(n),\ f(n)\ Such\ that$:

$$\boldsymbol{O}(g(n)) = \{ f(n) \mid 0 \leq f(n) \leq c \cdot g(n),\ \forall n \geq n_0\}$$

b) $Let\ c_1\ , c_2,\ n_0 \geq 0,\ then\ there\ exists\ a\ function\ g(n),\ f(n)\ Such\ that$:

$$\boldsymbol{\Theta}(g(n)) = \{ f(n) \mid 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n),\ \forall n \geq n_0\}$$

c) $Let\ c\ , n_0\ \geq 0,\ then\ there\ exists\ a\ function\ g(n),\ f(n)\ Such\ that$:

$$\boldsymbol{\Omega}(f(n)) = \{ f(n) \mid 0 \leq c \cdot g(n) \leq f(n),\ \forall n \geq n_0\}$$

d) Given a certain algorithm, the best case scenario (in terms of time complexity) is a set of inputs that provide the least amount of time.

e) Given a certain algorithm, the worst case scenario (in terms of time complexity) is a set of inputs that provide the most amount of time

f) Given a certain algorithm, the average case (in terms of time complexity) is a means of calculating the average time needed using inputs with their respective probabilities.

Question 2:

Expression	Dominating term	Big O
$5 + 0.001n^3 0.025n$	$0.001n^3$	$O(n^3)$
$n! + n^n$	n^n	$O(n^n)$
$2^{3^2} + 3^{2^n}$	9^n	$O(9^n)$
$n^2 log(n) + n(log(n))^2$	$n^2 log(n)$	$O(n^2 log(n))$
$nlog(n) + 9^{999999999}$	$nlog(n)$	$O(nlog(n))$

Big O's sorted from smallest to largest:

$O(nlog(n)),\ O(n^2 log(n)),\ O(n^3),\ O(9^n),\ O(n^n)$

Question 3:

a) $T(n) = 2T(\frac{n}{4}) + 1$

$\Rightarrow 2 > 4^0 \Rightarrow 2 > 1$

$\therefore$ Since $a > b^d$, then Case 3 applies

$\therefore T(n) = O(n^{\log_b a}) = O(\sqrt{n})$

b) $T(n) = 2T(\frac{n}{4}) + n^{1/2}$

$\Rightarrow 2 = 4^{1/2} \Rightarrow 2 = 2$

$\therefore$ Since $\cancel{2}\ a = b^d$, then case 1 applies

$\therefore T(n) = O(\sqrt{n}\log(n))$

c) $T(n) = 2T(\frac{n}{4}) + n$

$\Rightarrow 2 < 4^1$

$\therefore$ Since $a < b^d$, then case 2 applies

$\therefore T(n) = O(n)$

d) $T(n) = 2T(\frac{n}{4}) + n^2 \Rightarrow 2 < 4^2 \Rightarrow 2 < 16$

$\therefore$ Since $a < b^d$, then Case 2 applies

$\therefore T(n) = O(n^2)$

Question 4:

a) $T(n) = T(n-1) + 1, \quad T(1) = 1$

$T(2) = T(2-1) + 1 = T(1) + 1 = 1 + 1 = 2$

$T(3) = T(3-1) + 1 = T(2) + 1 = 2 + 1 = 3$

$T(4) = T(4-1) + 1 = T(3) + 1 = 3 + 1 = 4$

$\therefore$ from the following results, The assumption is: $T(n) = n$

Proof:

Base Step: $(n=1)$

$T(1) = 1$ —— ①

Inductive Step:

Assume that $T(n) = n$ is true $\forall n \geq 1$, then $T(n+1) = n+1$ is also true $\forall n \geq 1$

$\therefore T(n+1) = T(n) + 1 \Rightarrow T(n+1) = n+1$ —— ②

$\therefore$ from ①, ②, $T(n) = n, \ \forall n \geq 1$ is true ∎

b) $T(n) = T(n-1) + n, \quad T(1) = 10$

$T(2) = T(2-1) + 2 = T(1) + 2 = 12$

$T(3) = T(3-1) + 3 = T(2) + 3 = 15$

$T(4) = T(3) + 4 = 19$

By going backwards from $T(4)$, we get:

$T(4) = T(3) + 4 = T(2) + 3 + 4 = T(1) + 2 + 3 + 4 = 10 + 2 + 3 + 4$

We can assume that $T(n) = 10 + \left(\sum_{i=1}^{n} i\right) - 1$

upon further simplification, we get:

$$T(n) = \frac{n(n+1) + 18}{2}$$

Proof:

Base step: $(n = 1)$

$T(1) = \frac{(1)(1+1) + 18}{2} = \frac{2 + 18}{2} = \frac{20}{2} = 10$ — ①

Inductive step:

Assume that $T(n) = \frac{n(n+1) + 18}{2}$ is true $\forall n \geq 1$, then

$T(n+1) = \frac{(n+1)(n+2) + 18}{2}$ is also true $\forall n \geq 1$

$\therefore T(n+1) = T(n) + n+1 = \frac{n(n+1) + 18}{2} + n+1 = \frac{n(n+1) + 18 + 2(n+1)}{2}$

$= \frac{n(n+1) + 2n+2 + 18}{2} = \frac{n^2 + n + 2n + 2 + 18}{2} = \frac{n^2 + 3n + 2 + 18}{2}$

$= \frac{(n+1)(n+2) + 18}{2}$ — ②

$\therefore$ from ①, ②, $T(n) = \frac{n(n+1) + 18}{2}$ is true $\forall n \geq 1$

c) $T(n) = T(n-1) + 2n^2$, $T(1) = 10$

Theorem

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$T(2) = T(1) + 2(2)^2 = 10 + 8 = 18$

$T(3) = T(2) + 2(3)^2 = 18 + 18 = 36$

$T(4) = T(3) + 2(4)^2 = 36 + 32 = 68$

$\therefore$ By going backwards from $T(4)$, we get:

$T(4) = T(3) + 2(4)^2 = T(2) + 2(3)^2 + 2(4)^2 = T(1) + 2(2)^2 + 2(3)^2 + 2(4)^2$

$= 10 + 2(2^2 + 3^2 + 4^2) \Rightarrow T(4) = 8 + 2(1^2 + 2^2 + 3^2 + 4^2)$

$\therefore$ We can assume that $T(n) = 8 + \frac{n(n+1)(2n+1)}{3}$

Proof:

Base step: $(n=1)$

$T(1) = 8 + \frac{1(1+1)(2+1)}{3} = 8 + \frac{6}{3} = 8 + 2 = 10$ — ①

Inductive step.

Assume that $T(n) = \frac{n(n+1)(2n+1)}{3} + 8$ is true $\forall n \geq 1$, then $T(n+1) = \frac{(n+1)(n+2)(2n+3)}{3} + 8$ is also true $\forall n \geq 1$

$\therefore T(n+1) = T(n) + 2(n+1)^2 = 8 + \frac{n(n+1)(2n+1)}{3} + 2(n^2 + 2n + 1)$

$= 8 + \frac{2n^3 + 3n^2 + n + 6n^2 + 12n + 6}{3} = \frac{2n^3 + 9n^2 + 13n + 6}{3} + 8 = 8 + \frac{(n+1)(n+2)(2n+3)}{3}$ ②

$\therefore$ from ①, ②, $T(n) = 8 + \frac{n(n+1)(2n+1)}{3}$ $\forall n \geq 1$ ∎

d) $T(n) = 2T(n-1)$, $T(1) = 1$

$T(2) = 2\,T(1) = 2(1) = 2$

$T(3) = 2\,T(2) = 2(2) = 2^2$

$T(4) = 2T(3) = 2(2^2) = 2^3$

∴ from the previous results, we can assume that $T(n) = 2^{n-1}$

Proof:

Base step: $(n=1)$

$T(1) = 2^{1-1} = 2^0 = 1$ — ①

Inductive step:

Assume that $T(n) = 2^{n-1}$ is true $\forall n \geq 1$, then $T(n+1) = 2^n$ must also be true $\forall n \geq 1$

∴ $T(n+1) = 2T(n) = 2(2^{n-1}) = 2^{n-1+1} = 2^n$ — ②

∴ from ①, ② $T(n) = 2^{n-1}$ is true $\forall n \geq 1$ ∎

e) $T(3^n) = T(3^n/3) + 1, \quad T(1) = 1$

let $X = 3^n$, then $T(X) = T(\frac{X}{3}) + 1$

By using The tree method:

$$
\begin{array}{ccc}
T(X) & & \text{Cost:} \\
| & & \\
T(\frac{X}{3}) & & 1 \\
| & & \\
T(\frac{X}{3^2}) & & 1 \\
| & & \\
T(\frac{X}{3^3}) & & 1 \\
\vdots & & \vdots \\
T(\frac{X}{3^i}) & & 1
\end{array}
$$

Stopping Case : $\frac{X}{3^i} = 1 \Rightarrow 3^i = X \Rightarrow \log_3 3^i = \log_3 X \Rightarrow \underline{i = \log_3 X}$

$\therefore \sum_{i=0}^{\log_3 X} 1 = \log_3 X + 1$. Since $3^n = X$, then the closed form Will be $T(n) = n + 1 \quad \forall n \geq 0$

Proof:

Base Step: $(n=0)$

$T(0) = 0+1 = 1$ — ①

Inductive Step:

Assume that $T(n) = n+1$ is true $\forall n \geq 0$, then $T(n+1) = n+2$ is also true $\forall n \geq 0$

$\therefore$ ~~T(n+1) =~~ $T(3^{n+1}) = T(3^n) + 1$

$= T(n) + 1 = (n+1) + 1 = n+2$ — ②

$\therefore$ from ①, ②, $T(n) = n+1$ is true $\forall n \geq 0$

Question 5:

a) $f(n) = O(g(n))$ $because\ as\ n\ increases,\ f(n)\ will\ keep\ decreasing\ as\ opposed\ to\ g(n)$
$which\ will\ keep\ increasing.$

b) $f(n) = O(g(n))$ $since\ O(n \cdot log(n))\ is\ always\ greater\ than\ O(log(n))$
$when\ n\ gets\ very\ large$

c) $f(n) = O(g(n))$ $since\ 2^n\ is\ always\ greater\ than\ n(log(n))^2\ \forall n \geq 1$

d) $f(n) = O(g(n))$ and $g(n) = O(f(n))$ $since\ they\ both\ have\ the\ same\ dominating\ term$
$and\ both\ increase\ roughly\ as\ much$

e) $f(n) = O(g(n))$ and $g(n) = O(f(n))$ $since\ they\ are\ both\ linear\ and\ roughlt\ have$
$are\ always\ withing\ the\ same\ range$

f) $g(n) = O(f(n))$ $since\ n^n\ is\ always\ much\ greater\ n!$

g) $g(n) = O(f(n))$ $since\ 2^n\ is\ always\ greater\ than\ n^2\ when\ n\ is\ very\ large$