Question1.1:

T(n) = T(n/5) + 5, T(1) = 31

T(5) = T(5/5) + 5 = T(1) + 5 = 31 + 5 = 36

T(25) = T(25/5) + 5 = T(5) + 5 = 36 + 5 = 41

T(125) = T(125/5) + 5 = T(25) + 5 = 41 + 46

From these three examples, the general expression is: T(n) = 31 + 5$log_{5}(n)$

Let n = 5^{x}, then T(5^{x}) = 31 + 5x

Proof:

Base Case (x = 0):

T(5^{0}) = 31 + 5(0) = 31 + 0 = 31

Inductive step:

Assume that T(5^{x}) is true, then T(5^{x+1}) is true.

The expected answer should be: T(5^{x+1}) = 31 + 5(x + 1)

now for (x + 1): T(5^{x+1}) = T(5^{x}) + 5 = 31 + 5x + 5 = 31 + 5(x + 1)

∴ T(5^{x}) = T(5^{x-1}) + 5 is equivalent to T(5^{x}) = 31 + 5x

Using mathematical induction. ▌

Question 1.2:

$T(n) = 2T(n/2) + n$, $T(1) = 1$

$T(2) = 2T(2/2) + 2 = 2T(1) + 2 = 2(1) + 2 = 2 + 2 = 4$

$T(4) = 2T(4/2) + 4 = 2T(2) + 4 = 2(4) + 4 = 8 + 4 = 12$

$T(8) = 2T(8/2) + 8 = 2T(4) + 8 = 2(12) + 8 = 24 + 8 = 32$

From these three examples, the general expression is: $T(n) = n(log_2(n) + 1)$

Let $n = 2^x$, then: $T(2^x) = 2^x(x + 1)$

Proof:

Base step ($x = 0$):

$$T(2^0) = 2^0(0 + 1) = 1(1) = 1$$

Inductive step:

Assume that $T(2^x)$ is true, then $T(2^{x+1})$ is true

The expected answer should be: $T(2^{x+1}) = 2^{x+1}(x + 2)$

Now for $(x + 1)$:

$$T(2^{x+1}) = 2T(2^x) + 2^{x+1} = 2(2^x(x + 1)) + 2^{x+1} = 2^{x+1}(x + 1) + 2^{x+1}$$

$$= 2^{x+1}(x + 2)$$

$\therefore$ $T(2^x) = 2T(2^{x-1}) + 2^x$ is equivalent to $T(2^x) = 2^x(x + 1)$

Using mathematical induction. ▮

Question 1.3:

Theorem: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

T(n) = T(n − 1) + n^2 , T(1) = 23

T(2) = T(1) + 4 = 23 + 4 = 27

T(3) = T(2) + 9 = 27 + 9 = 36

T(4) = T(3) + 16 = 36 + 16 = 52

From these three examples, the general expression is: T(n) = 22 + $\sum_{i=1}^{n} i^2$

= 22 + $\frac{n(n+1)(2n+1)}{6}$

Proof:

Base step (n = 1):

$$T(1) = 22 + \frac{1(1+1)(2(1)+1)}{6} = 22 + 1 = 23$$

Inductive step:

Assume that T(n) is true, then T(n + 1) is true.

The expected answer should be: T(n + 1) = 22 + $\frac{(n+1)(n+2)(2n+3)}{6}$

Now for (n + 1):

$$T(n+1) = T(n) + (n + 1)^2 = 22 + \frac{(n+1)(n+2)(2n+3)}{6} + (n + 1)^2$$

$$= 22 + \frac{(n+1)(n+2)(2n+3) + 6(n+1)^2}{6} = 22 + \frac{(n+1)(n+2)(2n+3)}{6}$$

∴ T(n) = T(n - 1) + n^2 is equivalent to T(n) = 22 + $\frac{n(n+1)(2n+1)}{6}$

Using mathematical induction. ▮