Price Formulation of Constant Function Market Makers

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1 Price Functions

1.1 Definition of Price Functions

The purpose of this research note is to explore a way of defining Constant Function Market Makers (CFMMs) by their price functions $p(x, y)$, where x is the quantity of one asset, y the quantity of the second asset, and $p(x, y)$ the spot price of x with numeraire y.

Given intervals $I_1 \subset (0, \infty)$ and $I_2 \subset (0, \infty)$, we define $p : I_1 \times I_2 \to (0, \infty)$ to be a **price function** provided that $p(x, y)$

1. is non-increasing in x,
2. is non-decreasing in y,
3. is non-negative,
4. is continuous,
5. is Lipschitz continuous in y on any closed interval $I \subset I_2$.

We next prove existence of a CFMM with spot price equal to the any given price function.

1.2 Existence of CFMM for $p(x, y)$

We would like to solve the following ODE:

$$u'(x) = -p(x, u(x))$$
$$u(x_0) = y_0$$

It follows from the last condition that the Picard-Lindelof theorem gives us a unique soluction $u(x)$ to the ODE on any closed interval $I \subset I_2$. Since the solution is unique, it can be extended to all of I_2.

Observe then that $K(x, y) = \frac{y}{u(x)} = 1$ is a constant function from which a CFMM can be built.

Furthermore,

$$\begin{aligned}
\frac{\partial K}{\partial x} &= -\frac{y}{u^2(x)}u'(x) = \frac{y}{u^2(x)}p(x,y) \\
\frac{\partial K}{\partial y} &= \frac{1}{u(x)}
\end{aligned}$$

By the chain rule, the spot price is

$$\frac{dy}{dx} = -\frac{\frac{\partial K}{\partial x}}{\frac{\partial K}{\partial y}} = -\frac{y}{u(x)}p(x,y)$$

Since $y = u(x)$, the spot price of the CFMM using $K(x,y)$ as the constant function will be $p(x,y)$.

Note that the $K(x,y)$ constructed here is a "trading function" by the definition in [1]. That work is more general than this, and more exotic trading functions may not have price functions satisfying the criteria listed above (in particularly, Lipschitz continuity).

1.3 Properties

We define the weights intuitively as the percentages of the pool made up of each asset:

$$\begin{aligned}
W_x &= \frac{xp(x,y)}{xp(x,y)+y} \\
W_y &= \frac{y}{xp(x,y)+y}
\end{aligned}$$

Observe that this implies that $p(x,y) = \frac{W_x}{W_y}\frac{y}{x}$, the familiar constant product CFMM price formula.

2 Example: Reweighting CFMM

2.1 Reweighting CFMM Definition

We consider as an example price functions of the general form

$$p(x,y) = C\left(\frac{y+\alpha}{x+\beta}\right)^{a+1},$$

where $C, \alpha, \beta > 0$, and $a \geq -1$. Clearly $a = 0$ is the constant product AMM and $a = -1$ is the constant sum AMM.

It turns out that $a > 0$ gives us a family of reweighting AMMs. Our ODE turns into

$$\begin{aligned}
u'(x) &= -C\left(\frac{u+\alpha}{x+\beta}\right)^{a+1} \\
u(x_0) &= y_0
\end{aligned}$$

This is separable, so we must simply solve

$$\int (u+\alpha)^{-a-1} du = -C \int (x+\beta)^{-a-1} dx$$

Doing this, we find the following swap invariant function:

$$K(x,y) = \left((y+\alpha)^{-a} + C(x+\beta)^{-a}\right)^{-\frac{1}{a}}$$

Note that

$$W_x = \frac{xp(x,y)}{xp(x,y)+y} = \frac{xC\,(y+\alpha)^{a+1}}{xC\,(y+\alpha)^{a+1} + y(x+\beta)^{a+1}}$$

When $\alpha = \beta = 0$, we see

$$W_x = \frac{Cy^a}{Cy^a + x^a}$$

It's clear from this equation that at $a = 0$ the weight is constant (since it's just a constant product AMM), but with $a > 0$, the AMM *reweights* towards the token being purchased (that is, if x decreases and y increases, W_x increases). The curvature of the reweighting CFMM is higher than that of the constant product CFMM, resulting in this reweighting. In the language of AMMs, higher a produces lower impermanent loss at the expense of subjecting traders to increased slippage.

2.2 Asymptotes of the Reweighting CFMM

We introduced the Reweighting AMM with α and β not just for the sake of generalization, but because the choice of $\alpha = \beta = 0$ is problematic. We would like $\lim_{x\to\infty} y = 0$ and $\lim_{y\to\infty} x = 0$, but we see that this requires

$$\alpha = K(x,y)$$
$$\beta = C^{\frac{1}{a}} K(x,y)$$

We therefore adjust to

$$K^{-a}(x,y) = (y + K(x,y))^{-a} + (C^{-\frac{1}{a}}x + K(x,y))^{-a}$$

recalling that $K(x,y)$ is constant during a swap.

Note that this transition has actually adjusted our price function to

$$p(x,y) = C\left(\frac{y + K(x,y)}{x + C^{\frac{1}{a}}K(x,y)}\right)^{a+1},$$

3 Composite CFMMs

Throughout this section we use p_A^B to indicate the spot price of asset A denominated in B.

3.1 CFMMs sharing a token

If a particular asset x is in two CFMMs given by price functions $p_1(x, y)$ and $p_2(x, z)$, observe that $p(y, z) = \frac{p_2(x,z)}{p_1(x,y)}$ is also a price function. This is a simple way to compose two CFMMs.

This can also be used to construct a composite CFMM with two different pools for token x, by considering the pools to be different tokens x_1 and x_2 related by $p(x_1, x_2) = 1$.

3.2 Liquidity Provider Tokens

The Liquidity Provider (LP) token of a CFMM entitles the holder to some portion of the assets in the CFMM reserves.

Suppose we have a CFMM defined by price function $p_1(x, y)$. Let L represent the LP token of this CFMM, and denote by L_T the quantity of LP tokens for this pool in existence. Since the LP tokens altogether entitle their holders to the assets in the CFMM, we see that the spot price

$$p_L^y = \frac{xp_1(x, y) + y}{L_T}$$

Note that this is a price function in L_T and y.

We can then consider what happens if the LP tokens are provided to a CFMM defined by price function $p_2(L, z)$. For prices to be arbitrage free, we must have

$$p_z^y = \frac{p_L^y}{p_L^z} = \frac{xp_1(x, y) + y}{L_T p_2(L, z)}$$

Note that this is a price function. We now have an arbitrage-free three token pool.

References

[1] Guillermo Angeris, Alex Evans, and Tarun Chitra. Replicating market makers, 2021.