Price Formulation of Constant Function Market Makers

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1 Price Functions

1.1 Definition of Price Functions

The purpose of this research note is to explore a way of defining Constant Function Market Makers (CFMMs) by their price functions $p(x, y)$, where x is the quantity of one asset, y the quantity of the second asset, and $p(x, y)$ the spot price of x with numeraire y.

Given intervals $I_1 \subset (0, \infty)$ and $I_2 \subset (0, \infty)$, we define $p : I_1 \times I_2 \to [0, \infty)$ to be a **price function** provided that $p(x, y)$

1. is non-increasing in x,
2. is non-decreasing in y,
3. is non-negative,
4. is continuous,
5. is Lipschitz continuous in y on any closed interval $I \subset I_2$.

We next prove existence of a CFMM with spot price equal to the any given price function.

1.2 Existence of CFMM for $p(x, y)$

We would like to solve the following ODE:

$$u'(x) = -p(x, u(x))$$
$$u(x_0) = y_0$$

It follows from the last condition that the Picard-Lindelof theorem gives us a unique solcution $u(x)$ to the ODE on any closed interval $I \subset I_2$. Since the solution is unique, it can be extended to all of I_2.

Observe then that $K(x, y) = \frac{y}{u(x)} = 1$ is a constant function from which a CFMM can be built.

Furthermore,

$$
\begin{aligned}
\frac{\partial K}{\partial x} &= -\frac{y}{u^2(x)}u'(x) = \frac{y}{u^2(x)}p(x,y) \\
\frac{\partial K}{\partial y} &= \frac{1}{u(x)}
\end{aligned}
$$

By the chain rule, the spot price is

$$
\frac{dy}{dx} = -\frac{\frac{\partial K}{\partial x}}{\frac{\partial K}{\partial y}} = -\frac{y}{u(x)}p(x,y)
$$

Since $y = u(x)$, the spot price of the CFMM using $K(x,y)$ as the constant function will be $p(x,y)$.

1.3 Properties

We define the weights intuitively as the percentages of the pool made up of each asset:

$$
\begin{aligned}
W_x &= \frac{xp(x,y)}{xp(x,y)+y} \\
W_y &= \frac{y}{xp(x,y)+y}
\end{aligned}
$$

Observe that this implies that $p(x,y) = \frac{W_x}{W_y}\frac{y}{x}$, the familiar constant product CFMM price formula.

2 Example: Reweighting CFMM

2.1 Reweighting CFMM Definition

We consider as an example price functions of the general form

$$
p(x,y) = C\left(\frac{y+\alpha}{x+\beta}\right)^{a+1},
$$

where $C, \alpha, \beta > 0$, and $a \geq -1$. Clearly $a = 0$ is the constant product AMM and $a = -1$ is the constant sum AMM.

It turns out that $a > 0$ gives us a family of reweighting AMMs. Our ODE turns into

$$
\begin{aligned}
u'(x) &= -C\left(\frac{u+\alpha}{x+\beta}\right)^{a+1} \\
u(x_0) &= y_0
\end{aligned}
$$

This is separable, so we must simply solve

$$\int (u+\alpha)^{-a-1} du = -C \int (x+\beta)^{-a-1} dx$$

Doing this, we find the following swap invariant function:

$$K(x,y) = \left((y+\alpha)^{-a} + C(x+\beta)^{-a}\right)^{-\frac{1}{a}}$$

Note that

$$W_x = \frac{xp(x,y)}{xp(x,y)+y} = \frac{xC\,(y+\alpha)^{a+1}}{xC\,(y+\alpha)^{a+1} + y(x+\beta)^{a+1}}$$

When $\alpha = \beta = 0$, we see

$$W_x = \frac{Cy^a}{Cy^a + x^a}$$

It's clear from this equation that at $a = 0$ the weight is constant (since it's just a constant product AMM), but with $a > 0$, the AMM *reweights* towards the token being purchased (that is, if x decreases and y increases, W_x increases). The curvature of the reweighting CFMM is higher than that of the constant product CFMM, resulting in this reweighting. In the language of AMMs, higher a produces lower impermanent loss at the expense of subjecting traders to increased slippage.

2.2 Asymptotes of the Reweighting CFMM

We introduced the Reweighting AMM with α and β not just for the sake of generalization, but because the choice of $\alpha = \beta = 0$ is problematic. We would like $\lim_{x\to\infty} y = 0$ and $\lim_{y\to\infty} x = 0$, but we see that this requires

$$\begin{aligned}\alpha &= K(x,y)\\ \beta &= C^{\frac{1}{a}} K(x,y)\end{aligned}$$

We therefore adjust to

$$K^{-a}(x,y) = (y + K(x,y))^{-a} + (C^{-\frac{1}{a}}x + K(x,y))^{-a}$$

recalling that $K(x,y)$ is constant during a swap.

Note that this transition has actually adjusted our price function to

$$p(x,y) = C\left(\frac{y+K(x,y)}{x + C^{\frac{1}{a}}K(x,y)}\right)^{a+1},$$