Price Formulation of Constant Function Market Makers

Colin Grove

HydraDX

1 Price Functions

1.1 Definition of Price Functions

The purpose of this research note is to explore a way of defining Constant Function Market Makers (CFMMs) by their price functions $p(x, y)$, where x is the quantity of one asset, y the quantity of the second asset, and $p(x, y)$ the spot price of x with numeraire y.

Given intervals $I_1 \subset [0, \infty)$ and $I_2 \subset [0, \infty)$, we define $p : I_1 \times I_2 \to [0, \infty)$ to be a **price function** provided that $p(x, y)$

1. is non-increasing in x,
2. is non-decreasing in y,
3. is non-negative,
4. is continuous,
5. is Lipschitz continuous in y on any closed interval $I \subset I_2$.

We next prove existence of a CFMM with spot price equal to the any given price function.

1.2 Existence of CFMM for $p(x, y)$

We would like to solve the following ODE:

$$
\begin{aligned}
u'(x) &= -p(x, u(x)) \\
u(x_0) &= y_0
\end{aligned}
$$

It follows from the last condition that the Picard-Lindelof theorem gives us a unique soluction $u(x)$ to the ODE on any closed interval $I \subset I_2$. Since the solution is unique, it can be extended to all of I_2.

Observe then that $K(x, y) = \frac{y}{u(x)} = 1$ is a constant function from which a CFMM can be built.

Furthermore,

$$\begin{aligned}\frac{\partial K}{\partial x} &= -\frac{y}{u^2(x)}u'(x) = \frac{y}{u^2(x)}p(x,y)\\ \frac{\partial K}{\partial y} &= \frac{1}{u(x)}\end{aligned}$$

Thus the spot price of the CFMM using $K(x,y)$ as the constant function will be $p(x,y)$.

1.3 Properties

We define the weights intuitively as the percentages of the pool made up of each asset:

$$\begin{aligned}W_x &= \frac{xp(x,y)}{xp(x,y)+y}\\ W_y &= \frac{y}{xp(x,y)+y}\end{aligned}$$

Observe that this implies that $p(x,y) = \frac{W_x}{W_y}\frac{y}{x}$, the familiar constant product CFMM price formula.

2 Scale-Invariant Price Functions

We say a price function is **scale-invariant** if $p(kx,ky) = p(x,y)$.

3 Composing Price Functions