

Yield Curve Construction: A Mathematical Guide

Quant Insider

September 8, 2025

Notation

- $P(t, T)$: price at time t of a ZCB paying 1 at T ; $D(t, T) \equiv P(t, T)$ (“discount factor”).
- $r(t)$: risk-free short rate; $y(t, T)$: (continuously compounded) spot yield for maturity T .
- $f(t, T)$: instantaneous forward rate; $f(t, T) = -\partial_T \ln P(t, T)$ and $P(t, T) = e^{-\int_t^T f(t, u) du}$.
- α_i : accrual fraction of period $[T_{i-1}, T_i]$; c : coupon rate; S : par swap rate.

1 Bootstrapping Methods

1.1 Par Curve Bootstrapping (Coupons/Bonds)

Given a set of coupon bonds with maturities $T_1 < \dots < T_n$ and price B_k for maturity T_k ,

$$B_k = \sum_{i=1}^{k-1} c \alpha_i D(t, T_i) + (1 + c \alpha_k) D(t, T_k). \quad (1)$$

Solve recursively for $D(t, T_k)$:

$$D(t, T_k) = \frac{B_k - \sum_{i=1}^{k-1} c \alpha_i D(t, T_i)}{1 + c \alpha_k}. \quad (2)$$

When issuing prices are quoted in YTM, convert to clean prices or equivalently to par coupon quotes.

1.2 On-the-Run Treasuries Bootstrapping

Select most liquid benchmarks; apply the same recursion. If only a sparse set exists, combine with bills (short end) and extrapolate smoothly between benchmarks (see §2).

1.3 Swap Curve Bootstrapping (Deposits, Futures/FRAs, Swaps)

Step 1: Short end (Deposits). For an overnight/term deposit over $[t, T_1]$ with simple rate L ,

$$D(t, T_1) = \frac{1}{1 + L \alpha_1}. \quad (3)$$

Step 2: Forwards via FRAs/futures. If forward simple rates $L(T_{i-1}, T_i)$ (from FRAs or futures, with convexity adjustment if needed) are available, then

$$\frac{D(t, T_i)}{D(t, T_{i-1})} = \frac{1}{1 + L(T_{i-1}, T_i) \alpha_i}, \quad i = 2, \dots, m. \quad (4)$$

Step 3: Swaps (par conditions). For a fixed-for-floating swap maturing at T_n (floating leg on tenor α_i), the par swap rate S_n satisfies

$$S_n \sum_{i=1}^n \alpha_i D(t, T_i) = 1 - D(t, T_n). \quad (5)$$

Given $D(t, T_1), \dots, D(t, T_{n-1})$, solve (5) for $D(t, T_n)$ and iterate along maturities. In the modern multi-curve setting, discounting is done on an OIS curve D_d while forwards come from a tenor-specific forwarding curve D_f ; then the par condition becomes

$$S_n \sum_{i=1}^n \alpha_i D_d(t, T_i) = 1 - D_d(t, T_n), \quad (6)$$

with forward cashflows projected from *forwarding* rates consistent with D_f .

2 Interpolation and Smoothing

Let $\{T_k\}$ be bootstrapped nodes with discount factors $D_k = D(t, T_k)$. Interpolate either D , $\ln D$, y , or f .

2.1 Linear Interpolation

Piecewise linear on D . For $T \in [T_k, T_{k+1}]$,

$$D(T) = D_k + \frac{D_{k+1} - D_k}{T_{k+1} - T_k} (T - T_k). \quad (7)$$

Piecewise constant/linear on forwards f is also common: choose f constant on each bucket, then D integrates exactly.

2.2 Cubic Splines (Smoothing Splines)

Estimate a smooth function $g(T)$ (yield or log-discount) minimizing

$$\sum_j w_j (g(T_j) - \hat{g}_j)^2 + \lambda \int (g''(T))^2 dT, \quad (8)$$

with a possibly time-varying roughness penalty $\lambda = \lambda(T)$ (VRP). Afterwards derive D or f from g .

2.3 B-Splines / Penalized Splines

Represent $g(T) = \sum_\ell \beta_\ell B_\ell(T)$ and penalize differences of coefficients (or integrated curvature). Ensures local control and stable fitting.

2.4 Monotone Convex (Hagan–West)

Interpolate the *forward* curve with a piecewise cubic that enforces monotonicity and shape-preserving convexity between nodes, avoiding spurious oscillations. Given node times T_k and forwards f_k , construct cubic segments with slope choices guaranteeing

$$f'(T_k) \text{ consistent with data,} \quad D(T) = \exp\left(-\int_t^T f(u) du\right).$$

This yields arbitrage-free (non-negative) instantaneous forward rates under mild conditions and excellent locality.

2.5 Hermite (PCHIP)

Use piecewise cubic Hermite polynomials with data-dependent slopes to preserve monotonicity. Applied to D , $\ln D$, or f , PCHIP avoids overshoot while ensuring C^1 continuity.

3 Parametric Curve Fitting

3.1 Nelson–Siegel (NS)

Forward rate (time to maturity $m = T - t$):

$$f(m) = \beta_0 + \beta_1 \frac{1 - e^{-m/\tau_1}}{m/\tau_1} + \beta_2 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} - e^{-m/\tau_1} \right). \quad (9)$$

Spot yield follows by integrating f :

$$y(m) = \frac{1}{m} \int_0^m f(u) du = \beta_0 + \beta_1 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} \right) + \beta_2 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} - e^{-m/\tau_1} \right). \quad (10)$$

$\beta_0, \beta_1, \beta_2$ map to level, slope, curvature; τ_1 controls decay.

3.2 Svensson (Extended NS)

Add a second curvature term:

$$f(m) = \beta_0 + \beta_1 \frac{1 - e^{-m/\tau_1}}{m/\tau_1} + \beta_2 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} - e^{-m/\tau_1} \right) + \beta_3 \left(\frac{1 - e^{-m/\tau_2}}{m/\tau_2} - e^{-m/\tau_2} \right). \quad (11)$$

Gives more flexibility for medium/long maturities.

3.3 Smith–Wilson (Solvency II Extrapolation)

Given market instruments (cashflows \mathbf{c} at maturities \mathbf{u}) and an ultimate forward rate (UFR) with long-run convergence parameter α , the discount function is

$$P(m) = e^{-\text{UFR} \cdot m} + \mathbf{W}(m)^\top \boldsymbol{\zeta}, \quad \boldsymbol{\zeta} = (\mathbf{W}\mathbf{C})^{-1}(\mathbf{p} - \mathbf{e}), \quad (12)$$

where $m = T - t$, $\mathbf{W}(m, \mathbf{u})$ is the Wilson kernel

$$\mathbf{W}(m, \mathbf{u}) = e^{-\text{UFR}(m+\mathbf{u})} \left(\alpha \min(m, \mathbf{u}) - e^{-\alpha \max(m, \mathbf{u})} \sinh(\alpha \min(m, \mathbf{u})) \right), \quad (13)$$

\mathbf{C} maps instrument prices to cashflows, \mathbf{p} are observed prices, and \mathbf{e} are the discounted cashflows under UFR alone. This enforces smooth convergence of forward rates to UFR beyond the last liquid point (LLP).

3.4 Exponential Polynomials

Fit $y(m) = \sum_{k=0}^K a_k m^k + \sum_{j=1}^J b_j e^{-\lambda_j m}$ (or analogous forward specification). Choose (K, J) to balance fit and parsimony; impose positivity/arbitrage constraints if working on f .

4 Stochastic / Model-Based Approaches

4.1 Short-Rate Models

Vasicek. $dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$. Zero-coupon bond:

$$P(t, T) = \exp\left(A(T - t) - B(T - t)r_t\right), \quad B(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}, \quad A(\tau) = \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)(B(\tau) - \tau) - \frac{\sigma^2}{4\kappa}B(\tau)^2. \quad (14)$$

Hull–White (extended Vasicek). $dr_t = (\theta(t) - \kappa r_t)dt + \sigma dW_t$; retains exponential-affine $P(t, T) = \exp\{A(t, T) - B(t, T)r_t\}$ with $B(t, T) = (1 - e^{-\kappa(T-t)})/\kappa$ and time-dependent A calibrated to today's curve.

CIR / Black–Karasinski. Mean-reverting specifications with positivity (CIR) or lognormal rates (BK) implying closed/semi-closed bond prices; parameters are calibrated to the initial $P(0, T)$.

4.2 HJM (Forward-Rate) Framework

Model $f(t, T)$ directly under risk-neutral measure:

$$df(t, T) = \mu(t, T) dt + \sum_{k=1}^d \sigma_k(t, T) dW_k(t), \quad \mu(t, T) = \sum_{k=1}^d \sigma_k(t, T) \int_t^T \sigma_k(t, u) du \quad (15)$$

(no-arbitrage drift restriction). Choosing σ_k specifies the model; the initial curve $P(0, T)$ is matched exactly.

4.3 LIBOR Market Model (BGM/LMM)

For tenor dates $\{T_j\}$, forward LIBOR $L_j(t)$ for $[T_j, T_{j+1}]$ under $Q_{T_{j+1}}$:

$$dL_j(t) = \sigma_j(t) L_j(t) dW^{Q_{T_{j+1}}}(t), \quad (16)$$

(lognormal under its own forward measure). Under a common numéraire Q_{T_p} , drifts adjust via Girsanov:

$$dL_j(t) = L_j(t) \sigma_j(t) dW^{Q_{T_p}}(t) \pm L_j(t) \sum_k \frac{\delta L_k(t)}{1 + \delta L_k(t)} \sigma_j(t) \sigma_k(t) \rho_{jk} dt, \quad (17)$$

linking the entire family of forwards; caplets/swaptions recover Black-style prices.

5 Market-Specific Curves

5.1 Government Bond Curves

Estimate ZC curves from sovereign bonds/bills; common choices are Svensson/NS fits or smoothing splines with VRP. Illiquid points are filtered; callable/short residual maturities often excluded.

5.2 Swap Curves: OIS Discounting vs LIBOR Forwarding

Post-2007, collateralized trades are discounted on the overnight indexed swap (OIS) curve. For pricing a fixed–float IRS:

$$\text{NPV} = S \sum_i \alpha_i D_{\text{OIS}}(t, T_i) - \sum_i \alpha_i D_{\text{OIS}}(t, T_i) \mathbb{E}^{Q_{T_i}}[L^{(\text{tenor})}(T_{i-1}, T_i)], \quad (18)$$

with forwards generated from a tenor-specific forwarding curve (3M, 6M, ...). Multi-curve bootstrapping builds D_{OIS} and forwarding curves jointly using deposits, FRAs/futures, basis swaps, and IRS quotes.

5.3 Corporate Bond Curves

From government/swap curve plus spread: price $B = \sum c_i \alpha_i D(t, T_i) e^{-s(T_i)T_i}$. If a constant Z -spread is assumed, discount by $D(t, T_i) e^{-ZT_i}$; otherwise fit term-dependent $s(T)$ (e.g., spline on OAS).

5.4 Inflation-Linked Curves

Construct nominal P^N and real P^R curves from nominals and linkers (TIPS/ILBs). The breakeven inflation curve follows from

$$1 + \pi_{BE}(t, T) \approx \frac{(P^N(t, T))^{-1/T}}{(P^R(t, T))^{-1/T}}. \quad (19)$$

Interpolate P^R (or real forwards) with methods as above; price cashflows using real discounting and indexation mechanics.

Implementation Notes (Practical)

- **Choice of state space:** Interpolate $\ln D$ or f to preserve positivity of D and avoid arbitrage.
- **Locality vs smoothness:** Hagan–West/PCHIP give strong locality with shape control; smoothing splines give global smoothness with penalty tuning.
- **Multi-curve:** Calibrate OIS discounting first; then each forwarding curve using matching-tenor instruments (FRAs/futures, IRS, basis swaps).
- **Extrapolation:** Use Smith–Wilson (with LLP, UFR, α) when regulatory convergence to a long-run rate is required.