

Quantum substitutions of Pisot type, their quantum topological entropy and their use for optimal spacing

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Quantum substitutions of Pisot type and their topological entropy are introduced.
Their utility as algorithms for optimal spacing is analyzed.

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II. INTRODUCTION

The mathematical Theory of Formal Systems, the Theoretical Physics' Theory of Dynamical Systems and the Theoretical Computer Science's Theory of Computation may be seen as three different perspectives from which to investigate a same object:

actually, as it has been strongly remarked by John. L. Casti in the 9th chapter "The Mystique of Mechanics: Computation, Complexity and the Limits to Reason" of [1], one can pass from one perspective to an other through the following translation's diagram:

MATHEMATICS	THEORETICAL PHYSICS	THEORETICAL COMPUTER SCIENCE
formal system	dynamical system	computational device
axioms	initial conditions	input
logical inference	dynamical evolution	computation
theorems	final state	output

that is particularly fruitful as soon as one looks at the phenomenon of Incompleteness.

Gödel's Theorem [2] stating the existence of propositions undecidable (id est that cannot neither be proved nor disproved) in suitable formal systems is seen with indifference, if not with suspicion, by Mathematicians not working on foundational issues, the involved undecidable propositions being seen as self-referential intellectual masturbations (such as Gödel's sentence being a numerical translation, realized through Gödel numbering, of a statement asserting its own non provability).

Even more, the same occurs as to Theoretical Physicists whose formalisms so often prefer aesthetical elegance to the mathematical rigor, the translation of these formalisms in consistent mathematical terms being seen as a not particularly interesting bureaucratic task left to mathematical physicists.

Contrary, the phenomenon of incompleteness pervades different fields of Mathematics and might have astonishing consequences also in Theoretical Physics:

- as it has been suggested by Geroch and Hartle [3], Markov's Theorem stating the undecidability of the Homeomorphism Problem for four-manifolds (see for instance [4]) might compromise our possibility of quantizing gravity;
- as it has been suggested by Agnes and Rasetti [5], the theorems by Boone and Novikov (and their consequences such as the theorem stating the recursive undecidability of Dehn's isomorphism problem) stating the unsolvability of the Word Problem of suitable finitely presented groups might affect the predictability in suitable classical dynamical systems;
- Berger's Theorem (see [6]) stating the undecidability of the Tiling Problem might have consequences in the theory of quasicrystals [7];
- Matyasevich's proof of the undecidability of Hilbert Tenth Problem [8], [9] is a Damocles' sword above the head as to all the diophantine equations occurring in Physics.

From the other side Theoretical Computer Science has affected Mathematics, allowing to look at the phenomenon of Undecidability from a different perspective:

from the viewpoint of Gregory Chaitin's two Information-Theoretic Incompleteness Theorems [10], [11] the existence of limitations in the predictive power of formal systems appears quite reasonable and its role in Physics appears natural [12], in particular as far as Chaos Theory is concerned, as we will show in a specifically dedicated section of this paper.

It would seem rather natural to think that all this classical stuff could be generalized to the quantum case.

Unfortunately, the problem of extending Algorithmic Information Theory to the quantum domain is a not yet settled issue [13], [14], [15], [16], [17], [18], [19].

In particular the problem, first proposed by Karl Svozil, that almost a decade ago raised our interest in the subject during the Phd-studies [20], id est that of formulating and proving quantum analogues of Chaitin's Information-theoretic Incompleteness Theorems, is still open.

Since there exists also a combinatorial approach to Information Theory coarser than the algorithmic approach but incomparably easier, it appears natural, as a first step in such a direction, to introduce and study the quantum combinatorial information of the quantum analogues of the simplest formal systems, id est those whose inference rules are simple substitution rules on the involved symbols [21].

In this way we have been naturally led to the first result of this paper, namely the definition of the quantum topological entropy for quantum substitutions.

Among all the substitutions, furthermore, there exists a particular class, the substitutions of Pisot type, whose very useful properties leading to their role in the Theory of quasicrystals and of nonperiodic tilings [22], [23], [24], [25] are naturally derived from the mathematical properties of a remarkable set of numbers, the Pisot-Vijayaraghavan numbers [26], underlying much of the idolatrous mystique of the *golden number* τ (for this reason we have emphasized how the same properties are shared with the poor relative of τ , the so called *plastic number* ρ .)

This has led us to the second result of this paper, id est the formulation of a quantum algorithm for optimal spacing.

III. TOPOLOGICAL ENTROPY OF SUBSTITUTIONS

Given a finite alphabet $A = \{a_1, \dots, a_{|A|}\}$ (id est a set A such that the cardinality $|A|$ of A is such that $2 \leq |A| < \aleph_0$) let us denote, following [11], [21], [27] and [28], by $A^+ := \cup_{n \in \mathbb{N}_+} A^n$ the free semi-group generated by A , id est the set of all the finite words over A .

Given $\vec{x}, \vec{y} \in A^+$ let us denote by $\vec{x} \cdot \vec{y}$ the concatenation of \vec{x} and \vec{y} , id est the string $(x_1, \dots, x_{|\vec{x}|}, y_1, \dots, y_{|\vec{y}|})$ where $|\vec{x}|$ denotes the length of the strings \vec{x} .

Introduced the set $A^{\mathbb{N}_+}$ of the *sequences over A* let us endow A with the discrete topology, and let us endow A^+ and $A^{\mathbb{N}_+}$ with the induced product topology.

Such a topology over $A^{\mathbb{N}_+}$ is the metric topology induced by the following distance:

$$d(\vec{x}, \vec{y}) := \begin{cases} 0, & \text{if } \vec{x} = \vec{y}; \\ \frac{1}{2^{\min\{n \in \mathbb{N}_+ : x_n \neq y_n\}}}, & \text{otherwise.} \end{cases} \quad (3.1)$$

In the following we will denote sequences through a bar, id est we will adopt the following notation:

$$\bar{x} := \{x_n\}_{n \in \mathbb{N}_+} \in A^{\mathbb{N}_+} \quad (3.2)$$

Given a word $\vec{w} = (w_1, \dots, w_r) \in A^+$ and a sequence $\bar{x} \in A^{\mathbb{N}_+}$:

Definition III.1

\vec{w} occurs in \bar{x} at the position $n \in \mathbb{N}_+$:

$$x_n = w_1 \wedge \dots \wedge x_{n+r-1} = w_r \quad (3.3)$$

Definition III.2

\vec{w} is a factor of \bar{x} :

$$\vec{w} \leq_{occ} \bar{x} := \exists n \in \mathbb{N}_+ : \vec{w} \text{ occurs in } \bar{x} \text{ at the position } n \in \mathbb{N}_+ \quad (3.4)$$

In an analogous way, given two words $\vec{x}, \vec{y} \in A^+$:

Definition III.3

\vec{x} is a factor of \vec{y} :

$$\vec{x} \leq_{occ} \vec{y} := \exists n \in \mathbb{N}_+ : y_n = x_1 \wedge \dots \wedge y_{n+|\vec{x}|} = x_{|\vec{x}|} \quad (3.5)$$

Given two words $\vec{x}, \vec{y} \in A^+$ let us denote by $|\vec{x}|_{\vec{y}}$ the number of occurrences of \vec{y} in \vec{x} .

In a similar manner, given a sequence $\bar{x} \in A^{\mathbb{N}_+}$ and a word $\vec{y} \in A^+$, let us denote by $|\bar{x}|_{\vec{y}}$ the number (eventually infinite) of occurrences of \vec{y} in \bar{x} .

Definition III.4

language of length n of \bar{x} :

$$\mathcal{L}_n(\bar{x}) := \{\vec{y} \in A^n : \vec{y} \text{ is a factor of } \bar{x}\} \quad (3.6)$$

Definition III.5

language of \bar{x} :

$$\mathcal{L}(\bar{x}) := \cup_{n \in \mathbb{N}_+} \mathcal{L}_n(\bar{x}) \quad (3.7)$$

Given a factor $\vec{y} \in A^+$ of the sequence $\bar{x} \in A^{\mathbb{N}_+}$:

Definition III.6

frequency of \vec{y} in \bar{x} :

$$f_{\vec{y}}(\bar{x}) := \lim_{n \rightarrow +\infty} \frac{|\vec{x}(n)|_{\vec{y}}}{n} \quad (3.8)$$

Let us introduce some remarkable classes of sequences:

Definition III.7

set of the periodic sequences:

$$PERIODIC(A^{\mathbb{N}_+}) := \{\bar{x} \in A^{\mathbb{N}_+} : (\exists T \in \mathbb{N}_+ : (x_{n+T} = x_n \ \forall n \in \mathbb{N}_+))\} \quad (3.9)$$

Definition III.8

set of the ultimately periodic sequences:

$$PERIODIC_{ult}(A^{\mathbb{N}_+}) := \{\bar{x} \in A^{\mathbb{N}_+} : (\exists T \in \mathbb{N}_+, \exists N \in \mathbb{N}_+ : (x_{n+T} = x_n \ \forall n \in \mathbb{N}_+ : n \geq N))\} \quad (3.10)$$

Definition III.9

set of the Borel-normal sequences over A :

$$NORMAL(A^{\mathbb{N}_+}) := \{\bar{x} \in A^{\mathbb{N}_+} : f_{\vec{y}}(\bar{x}) = \frac{1}{|A|^{|\vec{y}|}} \ \forall \vec{y} \in A^+\} \quad (3.11)$$

Let us now introduce the following:

Definition III.10

combinatorial information function of \bar{x} :

$$p_n(\bar{x}) = |\mathcal{L}_n(\bar{x})| \quad (3.12)$$

Obviously:

$$1 \leq p_n(\bar{x}) \leq |A|^n \ \forall \bar{x} \in A^{\mathbb{N}_+}, \forall n \in \mathbb{N}_+ \quad (3.13)$$

Proposition III.1

Properties of the periodic and ultimately periodic sequences:

1.

$$PERIODIC(A^{\mathbb{N}_+}) \subset PERIODIC_{ult}(A^{\mathbb{N}_+}) \quad (3.14)$$

2.

$$\exists n \in \mathbb{N}_+ : p_n(\bar{x}) \leq n \Rightarrow \bar{x} \in PERIODIC_{ult}(A^{\mathbb{N}_+}) \quad (3.15)$$

Definition III.11

topological entropy of \bar{x} :

$$H_{top}(\bar{x}) := \lim_{n \rightarrow +\infty} \frac{\log_{|A|}(p_n(\bar{x}))}{n} \quad (3.16)$$

where the existence of the limit follows from the subadditivity of the function $n \mapsto \log_{|A|}(p_n(\bar{x}))$.

Remark III.1

Clearly the topological entropy of a sequence is different from zero only if $p_n(\bar{x})$ grows exponentially with n .

For instance introduced the following:

Definition III.12

Sturmian sequences:

$$STURMIAN(A^{\mathbb{N}^+}) := \{\bar{x} \in A^{\mathbb{N}^+} : p_n(\bar{x}) = n + 1 \ \forall n \in \mathbb{N}_+\} \quad (3.17)$$

it may be easily proved that:

Proposition III.2

$$H_{top}(\bar{x}) = 0 \ \forall \bar{x} \in STURMIAN(A^{\mathbb{N}^+}) \quad (3.18)$$

PROOF:

Applying the definition III.11 and the definition III.12 it follows that:

$$H_{top}(\bar{x}) = \lim_{n \rightarrow +\infty} \frac{\log_{|A|}(n + 1)}{n} = 0 \ \forall \bar{x} \in STURMIAN(A^{\mathbb{N}^+}) \quad (3.19)$$

■

Remark III.2

Let us recall how Andrei Nikolaevic Kolmogorov, in his seminal paper on Algorithmic Information Theory [29], showed with his usual intellectual clearness that there exist three possible approaches to Information Theory:

- the *combinatorial approach* in which the information of an object is defined relatively to a context of objects distributed uniformly
- the *probabilistic approach* in which the information of an object is defined relatively to a context whose objects are weighted according to a non necessarily uniform probability distribution
- the *algorithmic approach* furnishing an absolute measure of the information of an object, id est not relative to any context of different objects of which it is thought to be a member

As we will now show quantitatively the topological entropy furnishes a measure of the information contained in a sequence coarser than the one given by Algorithmic Information Theory.

From the other side it has the positive feature of being purely of combinatorial nature, id est not to require the introduction of concepts from Computability Theory.

Let us then introduce the following useful partial ordering over A^+ :

Definition III.13

\vec{x} is a prefix of \vec{y} :

$$\vec{x} <_p \vec{y} := \exists \vec{z} \in A^+ : \vec{y} = \vec{x} \cdot \vec{z} \quad (3.20)$$

Given a set $S \subset A^+$:

Definition III.14

S is prefix-free:

$$\vec{x} \not<_p \vec{y} \ \forall \vec{x}, \vec{y} \in S \quad (3.21)$$

Let us now briefly recall some necessary rudiments of Computability Theory [30], [31], [32], [33].

Given a partial function $C : A^+ \rightharpoonup A^+$ we can look at it as a total function adding a "non-halting" symbol" (that following [11] we will assume to be the infinity symbol ∞) and posing $C(\vec{x}) := \infty$ whether C doesn't halt on \vec{x} .

We can then introduce the following:

Definition III.15

halting set of C :

$$HALT_C := \{\vec{x} \in A^+ : C(\vec{x}) \neq \infty\} \quad (3.22)$$

We will say that:

Definition III.16

C is a partial computable function:

$$\exists f : \mathbb{N} \overset{\circ}{\mapsto} \mathbb{N} \text{ partial recursive} : C(\vec{x}) = string(f(string^{-1}(\vec{x}))) \quad (3.23)$$

where $string : \mathbb{N} \mapsto A^+$ is the map associating to an integer n the n^{th} word of A^+ in lexicographic order and where we demand to the mentioned literature as to the definition of a partial recursive function on natural numbers.

Given a set $X \subset A^+$:

Definition III.17

X is computable:

the characteristic function χ_X of X is computable

A notion weaker than computability is the following:

Definition III.18

X is computably enumerable:

$$\exists C : A^+ \overset{\circ}{\mapsto} A^+ \text{ partial computable function} : X = HALT_C \quad (3.24)$$

Definition III.19

Chaitin computer:

a partial recursive function $C : A^+ \overset{\circ}{\mapsto} A^+$ such that $HALT_C$ is prefix free.

Definition III.20

Universal Chaitin computer:

a Chaitin computer U such that for every Chaitin computer C there exist a constant $c_{U,C} \in \mathbb{R}_+$ such that:

$$\forall \vec{x} \in HALT_C, \exists \vec{y} \in A^+ : U(\vec{y}) = C(\vec{x}) \wedge |\vec{y}| \leq |\vec{x}| + c_{U,C} \quad (3.25)$$

Given a universal Chaitin computer U and a word $\vec{x} \in A^+$:

Definition III.21

algorithmic information of \vec{x} with respect to U :

$$I_U(\vec{x}) := \begin{cases} \min\{|\vec{y}| : \vec{y} \in A^+, U(\vec{y}) = \vec{x}\}, & \text{if } \{\vec{y} \in A^+ : U(\vec{y}) = \vec{x}\} \neq \emptyset; \\ +\infty, & \text{otherwise.} \end{cases} \quad (3.26)$$

Given a universal Chaitin computer U and $\vec{x} \in A^+$:

Definition III.22

algorithmic probability of \vec{x} with respect to U :

$$P_U(\vec{x}) := \sum_{\vec{y} \in U^{-1}(\vec{x})} \frac{1}{|A|^{|\vec{y}|}} \quad (3.27)$$

Definition III.23

Chaitin's halting probability with respect to U :

$$\Omega_U := \sum_{\vec{x} \in A^+} P_U(\vec{x}) \quad (3.28)$$

In order to show the conceptual importance of Ω_U let us recall first of all the following Alan Turing's celebrated result [34]:

Theorem III.1

Undecidability of the Halting Problem:

$$HALT_U \text{ is not computable}$$

Chaitin's halting probability codifies the undecidability stated by the theorem III.1 in a very useful form owing to the following:

Theorem III.2

Chaitin's Theorem on the Halting Problem:

Given the first n cbits of $r_{\{0,1\}}(\Omega_U)$ one can decide whether $U(\vec{x}) < +\infty$ for every $\vec{x} \in \cup_{k=1}^n \{0,1\}^k$

where $r_{\{0,1\}}$ is the nonterminating symbolic representation with respect to the alphabet $\{0,1\}$ of the definition A.5.

Let us now introduce the concept of *algorithmic randomness* (introduced in independent but equivalent ways by Martin-Löf, by Solovay and by Chaitin) by defining the algorithmically random sequences as those sequences with maximal algorithmic information (and that hence cannot be algorithmically compressed):

Definition III.24

algorithmically random sequences on A :

$$RANDOM(A^{\mathbb{N}+}) := \{ \bar{x} \in A^{\mathbb{N}+} : \forall U \text{ universal Chaitin computer } \exists c_U \in (0, +\infty) : I(\bar{x}(n)) \geq n - c_U \ \forall n \in \mathbb{N}_+ \} \quad (3.29)$$

where $\bar{x}(n)$ is the prefix of length n of the sequence \bar{x} .

Such a notion doesn't depend upon the chosen alphabet in the following sense:

Theorem III.3

Independence of algorithmic randomness from the adopted alphabet:

$$RANDOM(A_2^{\mathbb{N}+}) = T_{A_1, A_2}[RANDOM(A_1^{\mathbb{N}+})] \ \forall A_1, A_2 : \max\{|A_1|, |A_2|\} \in \mathbb{N} \quad (3.30)$$

where T_{A_1, A_2} is the alphabet's transition map of the definition A.6.

Definition III.25

algorithmically random elements of the interval $(0,1)$:

$$RANDOM[(0,1)] := v_A(RANDOM(A^{\mathbb{N}+})) , \ 2 \leq |A| < \aleph_0 \quad (3.31)$$

where v_A is the numerical value map of the definition A.4.

Remark III.3

Let us remark that, owing to the theorem III.3, the definition III.25 doesn't depend on the adopted alphabet A .

The paradigmatic example of a random real is given by Chaitin's halting probabilities:

Proposition III.3

Halting probabilities are random:

$$\Omega_U \in RANDOM[(0,1)] \quad \forall U \text{ universal Chaitin computer} \quad (3.32)$$

Let us now observe that clearly:

Proposition III.4

Periodic and ultimately periodic sequences are not random:

$$PERIODIC_{ult}(\mathbb{A}^{\mathbb{N}_+}) \cap RANDOM(\mathbb{A}^{\mathbb{N}_+}) = \emptyset \quad (3.33)$$

PROOF:

Let \bar{x} be a sequence periodic of period $T \in \mathbb{N}_+$ after a prefix of $N \in \mathbb{N}$ digits.

By the subadditivity of algorithmic information:

$$I(\bar{x}(nT)) \leq I(\bar{x}(N)) + I(\{x_{N+1}, \dots, x_T\}) + O(1) \quad \forall n \in \mathbb{N}_+ \quad (3.34)$$

and is hence bounded.

■

Let us now introduce the following:

Definition III.26

recurrent sequences on A :

$$RECURRENT(A^{\mathbb{N}_+}) = \{\bar{x} \in \mathbb{A}^{\mathbb{N}_+} : |\bar{x}|_{\vec{y}} = +\infty \quad \forall \vec{y} \in A^+\} \quad (3.35)$$

Definition III.27

uniformly recurrent sequences on A :

$$RECURRENT_{un}(A^{\mathbb{N}_+}) := \{\bar{x} \in RECURRENT(A^{\mathbb{N}_+}) : \forall \vec{y} \in A^+, \exists r \in \mathbb{N}_+ \vec{y} \leq_{occ} \{x_n, \dots, x_{n+r}\} \quad \forall n \in \mathbb{N}_+\} \quad (3.36)$$

Demanding to [11] for the proofs we can now state some remarkable property of the algorithmically random sequences:

Theorem III.4

Calude-Chitescu Theorem:

$$RANDOM(A^{\mathbb{N}_+}) \subset RECURRENT(A^{\mathbb{N}_+}) \quad (3.37)$$

Remark III.4

The Calude-Chitescu Theorem shows how theorem III.3 has not be equivocated in the following sense.

Let us consider two finite alphabets A_1 and A_2 such that $A_1 \subset A_2$; then clearly:

$$|\bar{x}|_a = 0 \quad \forall \bar{x} \in RANDOM(A_1^{\mathbb{N}_+}), \forall a \in A_2 - A_1 \quad (3.38)$$

and hence, according to the theorem III.4:

$$\bar{x} \notin RANDOM(A_2^{\mathbb{N}_+}) \quad \forall \bar{x} \in RANDOM(A_1^{\mathbb{N}_+}) \quad (3.39)$$

as it can be appreciated considering the alphabet $A_1 := \{0, 2\}$ and $A_2 := \{0, 1, 2\}$ involved in the construction of Cantor's middle third set extensively analyzed in the section C.

Remark III.5

Can the Calude-Chitescu Theorem (id est the theorem III.4) be strengthened by substituting the set $RECURRENT(A^{\mathbb{N}+})$ with the set $RECURRENT_{un}(A^{\mathbb{N}+})$?

Up to our knowledge such a question is still open.

Furthermore, since uniformly recurrent sequences are deeply linked with the notion of almost periodicity, the issue is deeply linked with the problem of estimating the algorithmic information's content of an almost periodic function, a problem discussed in the section C.

Theorem III.5

$$NORMAL(A^{\mathbb{N}+}) \subset RANDOM(A^{\mathbb{N}+}) \quad (3.40)$$

Remark III.6

The strict inclusion of theorem III.5 implies that though an algorithmically random sequence is always Borel normal, there exist Borel normal sequences that are not algorithmically random.

The classical counterexample is the following:

Definition III.28

Champerknowne sequence:

$$\bar{x}_{Champerknowne} := \cdot_{n \in \mathbb{N}_+} string(n) \quad (3.41)$$

where $string(n)$ is the n^{th} string in lexicographical ordering.

By construction:

$$\bar{x}_{Champerknowne} \in NORMAL(A^{\mathbb{N}+}) \quad (3.42)$$

Anyway the Champerknowne sequence is obviously highly algorithmically compressible, being defined through a very short algorithm:

$$\bar{x}_{Champerknowne} \notin RANDOM(A^{\mathbb{N}+}) \quad (3.43)$$

Let us remark that:

Proposition III.5

Link between topological entropy and algorithmic randomness:

1.

$$H_{top}(\bar{x}) = 1 \quad \forall \bar{x} \in RANDOM(A^{\mathbb{N}+}) \quad (3.44)$$

2.

$$H_{top}(\bar{x}) = 1 \not\Rightarrow \bar{x} \in RANDOM(A^{\mathbb{N}+}) \quad (3.45)$$

PROOF:

1. The theorem III.4 implies that:

$$p_n(\bar{x}) = |A|^n \quad \forall \bar{x} \in RANDOM(A^{\mathbb{N}+}), \forall n \in \mathbb{N}_+ \quad (3.46)$$

and hence:

$$H_{top}(\bar{x}) = \lim_{n \rightarrow \infty} \frac{n}{n} = 1 \quad \forall \bar{x} \in RANDOM(A^{\mathbb{N}+}) \quad (3.47)$$

2. the fact that the Champernowne sequence of the definition III.28 is Borel normal implies that:

$$p_n(\bar{x}_{Champernowne}) = |A|^n \quad \forall n \in \mathbb{N}_+ \quad (3.48)$$

and hence:

$$H_{top}(\bar{x}_{Champernowne}) = \lim_{n \rightarrow \infty} \frac{n}{n} = 1 \quad (3.49)$$

though, as we have already remarked, \bar{x} is not algorithmically random.

■

Let us now suppose to have a probability distribution f over the alphabet A , id est a map $f : A \mapsto [0, 1]$ such that:

$$\sum_{a \in A} f(a) = 1 \quad (3.50)$$

and let us introduce the following:

Definition III.29

probabilistic information of f :

$$S_{prob}(f) := \langle -\log_{|A|} f \rangle = - \sum_{a \in A} f(a) \log_{|A|} f(a) \quad (3.51)$$

As it is natural for different approaches devoted to formalize from different perspectives the same concept, the algorithmic information and the probabilistic information are deeply linked.

Let us consider, at this purpose, a stochastic process in which, at the n^{th} temporal step, a letter x_n from the alphabet A is chosen at random according to the distribution f .

Supposed to have chosen n letters, the random variables x_1, \dots, x_n are independent and identical distributed so that the random vector (x_1, \dots, x_n) has probability distribution:

$$f^{(n)}(x_1, \dots, x_n) := \prod_{i=1}^n f(x_i) \quad (3.52)$$

Then:

Theorem III.6

Link between the probabilistic information and the algorithmic information:

$$\lim_{n \rightarrow +\infty} \frac{\langle I(\vec{x}(n)) \rangle}{n} = S(f) \quad (3.53)$$

where of course:

$$\langle I(\vec{x}(n)) \rangle = \sum_{\vec{x} \in A^n} f^{(n)}(\vec{x}) I(\vec{x}) \quad (3.54)$$

Remark III.7

A first generalization of the theorem III.6 may be obtained removing the hypothesis that different letters x_1, \dots, x_n are identically distributed and independent, id est considering the case in which they form an arbitrary stochastic process (an arbitrary shift in the language of Abstract Dynamical System Theory [35], [36]).

It can be further generalized to arbitrary abstract dynamical systems through a suitable symbolic codification of its trajectories.

The resulting Brudno Theorem [37], [38], [39], [40] states that the dynamical entropy of an abstract dynamical system, defined as the asymptotic rate of production of probabilistic-information produced by its dynamics, is equal to the asymptotic rate of algorithmic information of almost every its orbit.

Let us now introduce the following basic:

Definition III.30

substitution over A:

$$\text{a map } \sigma : A \mapsto A^+$$

A substitution σ over A may be extended to a morphism of A^+ by concatenation, id est posing:

$$\sigma[(x_1, \dots, x_n)] := \cdot_{i=1}^n \sigma(x_i) \quad (3.55)$$

In the same way it may be naturally extended to a map over $A^{\mathbb{N}+}$ by posing:

$$\sigma(\{x_n\}_{n \in \mathbb{N}}) := \cdot_{n \in \mathbb{N}} \sigma(x_n) \quad (3.56)$$

Remark III.8

Substitutions are very efficient ways for producing sequences owing to the following:

Proposition III.6

Fixed point of a substitution beginning with a given letter:

HP:

$$a \in A$$

$$\sigma \text{ substitution such that } \sigma(a) \text{ begins with } a \text{ and } |\sigma(a)| \geq 2$$

TH:

$$\exists! \bar{\sigma}(a) \in A^{\mathbb{N}+} : \sigma(\bar{\sigma}(a)) = \bar{\sigma}(a)$$

Given a substitution σ it is natural to define its topological entropy as the sum of the topological entropy of its fixed point starting with different letters:

Definition III.31

topological entropy of σ :

$$H_{top}(\sigma) := \sum_{a \in A: |\sigma(a)| \geq 2} H_{top}(\bar{\sigma}(a)) \quad (3.57)$$

Let us now briefly recall some basic notions concerning Matrices' Theory.

Given $n \in \mathbb{N}_+ : n \geq 2$ and a square matrix $A \in M_n(\mathbb{Z})$:

Definition III.32

A is primitive:

$$(A_{ij} \in \mathbb{N} \forall i, j \in \{1, \dots, n\}) \wedge (\exists n \in \mathbb{N}_+ : (A^n)_{ij} \in \mathbb{N}_+ \forall i, j \in \{1, \dots, n\}) \quad (3.58)$$

Definition III.33

A is irreducible:

$$\nexists S \text{ linear subspace of } \mathbb{R}^n : (A\vec{v} \in S \forall \vec{v} \in S) \quad (3.59)$$

Let us recall the following basic:

Theorem III.7

Perron-Frobenius' Theorem:

HP:

A irreducible primitive

TH:

A admits a strictly positive eigenvalue α (called the *leading eigenvalue of A*) whose absolute value is greater than the absolute value of all the other eigenvalues, α is a simple eigenvalue and there exists an eigenvector with positive entries associated to α

Given a substitution σ let us introduce the following:

Definition III.34

σ is primitive:

$$\exists n \in \mathbb{N}_+ : a \text{ occurs in } \sigma^n(b) \quad \forall a, b \in A \quad (3.60)$$

Definition III.35

incidence matrix of σ :

$$M_\sigma \in \mathbb{M}_{|A|}(\mathbb{N}) := (M_\sigma)_{ij} := |\sigma(a_j)|_{a_i} \quad (3.61)$$

Then:

Proposition III.7

$$\sigma \text{ is primitive} \Leftrightarrow M_\sigma \text{ is primitive}$$

Demanding to the appendix B for the involved mathematical notions let us introduce the following:

Definition III.36

σ is of Pisot type:

M_σ has a leading eigenvalue λ such that for every other eigenvalue α one gets $\lambda > 1 > |\alpha|$

It may be proved that:

Proposition III.8

Basic properties of substitutions of Pisot type:

HP:

σ substitution of Pisot type

TH:

1. σ is primitive
2. the characteristic polynomial $P_\sigma := \det(\lambda \mathbb{I} - M_\sigma) \in \mathbb{Z}[\lambda]$ is irreducible over \mathbb{Q} [41] and hence the leading eigenvalue α of M_σ is a Pisot-Vijayaraghavan number.
3. in the fixed point $\bar{\sigma}(a)$ of σ starting with a letter a such that $|\sigma(a)| \geq 2$ the frequencies of the letters are given by the coordinates of the positive eigenvector associated with the leading eigenvalue normalized in such a way that the sum of its coordinates equal 1.

{0,1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,0,\
 1,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,0,\
 1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,\
 1,0,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,1,0,0,1,0,0,1,0,\
 1,0,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,1,0,0}

...

Let us remark that:

$$\sigma^n(0) = \sigma^{n-1}(0) \cdot \sigma^{n-2}(0) \quad \forall n \in \mathbb{N} : n \geq 2 \quad (3.64)$$

and hence:

$$|\sigma^n(0)| = F_{n+2} \quad \forall n \in \mathbb{N} \quad (3.65)$$

where F_n is the n^{th} Fibonacci number (equal to the number of adult couples of rabbits at the n^{th} month) defined recursively by

$$F_{n+2} := F_{n+1} + F_n \quad (3.66)$$

$$F_0 := 0 \quad (3.67)$$

$$F_1 := 1 \quad (3.68)$$

Furthermore:

$$|\sigma^n(0)|_0 = F_{n+1} \quad \forall n \in \mathbb{N}_+ \quad (3.69)$$

$$|\sigma^n(0)|_1 = F_n \quad \forall n \in \mathbb{N}_+ \quad (3.70)$$

The incidence matrix of σ is::

$$M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.71)$$

whose leading eigenvalue is the golden number $\tau := \frac{1+\sqrt{5}}{2}$, id est the least Pisot-Vijayaraghavan number of order two (see the example B.1).

Since:

$$M_\sigma^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad \forall n \in \mathbb{N} \quad (3.72)$$

it follows that σ is also primitive and hence is of Pisot type.

The eigenvector of M_σ normalized so that the sum of its components equals to one is $(\frac{\tau}{1+\tau}, \frac{1}{1+\tau})$; hence:

$$f_0(\bar{\sigma}(0)) = \frac{\tau}{1+\tau} \quad (3.73)$$

$$f_1(\bar{\sigma}(1)) = \frac{1}{1+\tau} \quad (3.74)$$

and hence in particular:

$$\frac{f_0(\bar{\sigma}(0))}{f_1(\bar{\sigma}(1))} = \tau \quad (3.75)$$

that is consistent with the equation 3.69 and the equation 3.70 since:

$$\lim_{n \rightarrow +\infty} \frac{F_{n+1}}{F_n} = \tau \quad (3.76)$$

Let us remark that since:

$$\frac{f_0(\bar{\sigma}(0))}{f_1(\bar{\sigma}(0))} = \tau \neq 1 \quad (3.77)$$

it follows that the Fibonacci sequence is not Borel-normal and hence, in particular, is not algorithmically-random, as we could have inferred by the fact that it is Sturmian and hence:

$$H_{top}(\sigma) = H_{top}(\bar{\sigma}(1)) = 0 \quad (3.78)$$

The fact that $\frac{f_0(\bar{\sigma}(1))}{f_1(\bar{\sigma}(0))} = \tau \notin \mathbb{Q}$ implies that the Fibonacci sequence is not ultimately periodic.

Example III.2

Let us consider the following substitution over the alphabet of three letters $\{0, 1, 2\}$:

$$\sigma(0) := \{1, 2\} \quad (3.79)$$

$$\sigma(1) := \{2\} \quad (3.80)$$

$$\sigma(2) := \{0\} \quad (3.81)$$

from which one derives the *Padovan sequence* $\bar{\sigma}(0)$:

$\{0\}$

$\{1, 2\}$

$\{2, 0\}$

$\{0, 1, 2\}$

$\{1, 2, 2, 0\}$

$\{2, 0, 0, 1, 2\}$

$\{0, 1, 2, 1, 2, 2, 0\}$

$\{1, 2, 2, 0, 2, 0, 0, 1, 2\}$

$\{2, 0, 0, 1, 2, 0, 1, 2, 1, 2, 2, 0\}$

$\{0, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 1, 2\}$

$\{1, 2, 2, 0, 2, 0, 0, 1, 2, 2, 0, 0, 1, 2, 0, 1, 2, 1, 2, 2, 0\}$

$\{2, 0, 0, 1, 2, 0, 1, 2, 1, 2, 2, 0, 0, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 1, 2\}$

$\{0, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 1, 2, 1, 2, 2, 0, 2, 0, 0, 1, 2, 2, 0, 0, 1, 2, 0, 1, 2, 1, 2, 2, 0\}$

...

It may be easily proved that:

$$|\sigma^n(0)| = P_{n+2} \quad \forall n \in \mathbb{N}_+ \quad (3.82)$$

where P_n is the n^{th} *Padovan number* defined by:

$$P(0) := P(1) := P(2) := 1 \quad (3.83)$$

$$P_n := P_{n-2} + P_{n-3} \quad (3.84)$$

Furthermore:

$$\lim_{n \rightarrow +\infty} \frac{|\sigma^n(0)|}{|\sigma^{n-1}(0)|} = \lim_{n \rightarrow +\infty} \frac{P_{n+2}}{P_{n+1}} = \lim_{n \rightarrow +\infty} \frac{P_n}{P_{n-1}} = \rho \quad (3.85)$$

where $\rho := \frac{(9-\sqrt{69})^{\frac{1}{3}} + (9+\sqrt{69})^{\frac{1}{3}}}{2^{\frac{1}{3}} 3^{\frac{2}{3}}}$ is the *plastic number*, id est the least Pisot-Vijayaraghavan number (see the example B.2).

The incidence matrix of σ is:

$$M_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad (3.86)$$

whose leading eigenvalue is ρ .

Since:

$$M_\sigma^n = \begin{pmatrix} P_{n-2} & P_{n-4} & P_{n-3} \\ P_{n-3} & P_{n-5} & P_{n-4} \\ P_{n-1} & P_{n-3} & P_{n-2} \end{pmatrix} \quad \forall n \in \mathbb{N} : n \geq 5 \quad (3.87)$$

σ is also primitive and hence is of Pisot type.

The eigenvector of M_σ associated to the eigenvalue ρ and normalized so that the sum of its components equals to one is $\{\rho - 1, \frac{1+\rho-\rho^2}{1+\rho}, \frac{1}{1+\rho}\}$ and hence:

$$f_0(\bar{x}(0)) = \rho - 1 \quad (3.88)$$

$$f_1(\bar{x}(0)) = \frac{1+\rho-\rho^2}{1+\rho} \quad (3.89)$$

$$f_2(\bar{x}(0)) = \frac{1}{1+\rho} \quad (3.90)$$

Since:

$$\frac{f_1(\bar{x}(0))}{f_0(\bar{x}(0))} = \frac{1+\rho-\rho^2}{-1+\rho^2} \neq 1 \quad (3.91)$$

$$\frac{f_2(\bar{x}(0))}{f_0(\bar{x}(0))} = \frac{1}{-1+\rho^2} \neq 1 \quad (3.92)$$

it follows that the Padovan sequence is not Borel-normal and hence, in particular, it is not-algorithmically random.

Since $\frac{f_1(\bar{x}(0))}{f_0(\bar{x}(0))} \notin \mathbb{Q}$ and $\frac{f_2(\bar{x}(0))}{f_0(\bar{x}(0))} = \frac{1}{-1+\rho^2} \notin \mathbb{Q}$ it follows that the Padovan sequence is not ultimately periodic.

Example III.3

Let us consider the Pell substitution, id est the substitution σ over the binary alphabet $\{0, 1\}$:

$$\sigma(0) := \{0, 1\} \tag{3.93}$$

$$\sigma(1) := \{0, 0, 1\} \quad (3.94)$$

from which one derives the Pell sequence:

 $\{0\}$ $\{0, 1\}$ $\{0, 1, 0, 0, 1\}$
$$\{0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1\}$$
$$\{0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,0,1\}$$
$$\{0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,0,1,0,1,0,1,\backslash$$

$$0,0,1,0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1\}$$
$$\{0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,\backslash$$

$$0,0,1,0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,\backslash$$

$$0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,\backslash$$

$$0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,\backslash$$

$$0,1,0,1,0,0,1,0,1,0,1,0,1,0,0,1\}$$

```
{0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,\
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0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1}
```

[illegible]

1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,\
0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,0,1,\
0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,\
0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,\
0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,\
0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,1,0,1,0,1,0,0,1,0,1,0,0,1,0,1,0,\
1,0,0,1,0,1,0,1,0,0,1\}

...

It may be easily proved that:

$$|\sigma^n(0)| = a_{n+1} \quad \forall n \in \mathbb{N}_+ \quad (3.95)$$

where $\{a_n\} \in \mathbb{N}^{\mathbb{N}}$ are the Pell numbers defined as:

$$a_0 := 0 \quad (3.96)$$

$$a_1 := 1 \quad (3.97)$$

$$a_n := 2a_{n-1} + a_{n-2} \quad \forall n \in \mathbb{N} : n \geq 2 \quad (3.98)$$

Furthermore:

$$|\sigma^n(0)|_0 = a_n + a_{n-1} \quad \forall n \in \mathbb{N}_+ \quad (3.99)$$

$$|\sigma^n(0)|_1 = a_n \quad \forall n \in \mathbb{N}_+ \quad (3.100)$$

The incidence matrix of σ is:

$$M_\sigma = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad (3.101)$$

Since:

$$M_\sigma^n = \begin{pmatrix} a_n + a_{n-1} & 2a_n \\ a_n & a_n + a_{n-1} \end{pmatrix} \quad \forall n \in \mathbb{N}_+ \quad (3.102)$$

it follows that M_σ is a primitive matrix and hence σ is a primitive substitution.

Since furthermore the leading eigenvalue of M_σ is $1 + \sqrt{2}$ that is a Pisot-Vijayaraghavan number (see the example B.3) it follows that the Pell substitution is of Pisot type.

The eigenvector of M_σ associated to the leading eigenvalue $1 + \sqrt{2}$ and normalized so that the sum of its components equals to one is $\{\frac{\sqrt{2}}{1+\sqrt{2}}, \frac{1}{1+\sqrt{2}}\}$. Therefore:

$$f_0(\bar{\sigma}(0)) = \frac{\sqrt{2}}{1 + \sqrt{2}} \quad (3.103)$$

$$f_1(\bar{\sigma}(0)) = \frac{1}{1 + \sqrt{2}} \quad (3.104)$$

consistently with the equation 3.99 and the equation 3.100 since:

$$\lim_{n \rightarrow +\infty} \frac{a_n}{a_{n-1}} = 1 + \sqrt{2} \quad (3.105)$$

Since:

$$\frac{f_0(\bar{\sigma}(0))}{f_1(\bar{\sigma}(0))} = \sqrt{2} \notin \mathbb{Q} \quad (3.106)$$

it follows that the Pell sequence is neither Borel normal nor ultimately periodic.

IV. SUBSTITUTIONS OF PISOT TYPE AS ALGORITHMS FOR OPTIMAL SPACING

Let us suppose to have to allocate $n \in \mathbb{N}_+ : n \geq 2$ petals of a flower so that:

1. the average distance between neighbors is maximal in order to maximize the exposition of each petal to sun and rain.
2. the distribution of the distances among neighbour petals is the more uniform one.

Supposed all the petals have unit length so that their vertices belong to the unit circumference of the complex plane let us assume that their positions are $e^{i\theta_1}, \dots, e^{i\theta_n}$ where $0 \leq \theta_1 < \theta_2 < \dots < \theta_k < \theta_{k+1} < \dots < \theta_n < 2\pi$ and let us assume:

$$\alpha_k := d_{\text{geodesic}}(\theta_{k+1} - \theta_k) \quad k = 1, \dots, n \quad (4.1)$$

where:

$$\theta_{n+1} := \theta_1 \quad (4.2)$$

and where the distance $d_{\text{geodesic}} : [0, 2\pi)^2 \mapsto \mathbb{R}$:

$$d_{\text{geodesic}}(\theta_{k+1}, \theta_k) = \begin{cases} \theta_{k+1} - \theta_k, & \text{if } \theta_{k+1} - \theta_k \leq \pi; \\ 2\pi - (\theta_{k+1} - \theta_k), & \text{otherwise.} \end{cases} \quad (4.3)$$

is the geodesic distance between $e^{i\theta_k}$ and $e^{i\theta_{k+1}}$.

Our optimization problem consists in maximizing the mean of α :

$$\bar{\alpha} := \frac{\sum_{i=1}^n \alpha_i}{n} \quad (4.4)$$

while minimizing the variance of α :

$$\sigma^2(\alpha) := \frac{\sum_{i=1}^n (\alpha_i - \bar{\alpha})^2}{n} \quad (4.5)$$

The solution of such an optimization problem consists in locating the n^{th} petal in the n^{th} root of unit:

$$e^{i\theta_k} = r_{n,k} := e^{i\frac{2\pi}{n}k} \quad k \in \{1, \dots, n\} \quad (4.6)$$

so that:

$$\alpha_k = \bar{\alpha} = \frac{2\pi}{n} \quad k = 1, \dots, n \quad (4.7)$$

$$\sigma^2(\alpha) = 0 \quad (4.8)$$

To prove it let us observe that, by the equation 4.3, any configuration such that $\max\{\theta_k - \theta_{k-1}\} < \pi$ and $\sum_k \alpha_k = 2\pi$ furnishes the maximum value of the mean of α :

$$\bar{\alpha} = \frac{2\pi}{n} \quad (4.9)$$

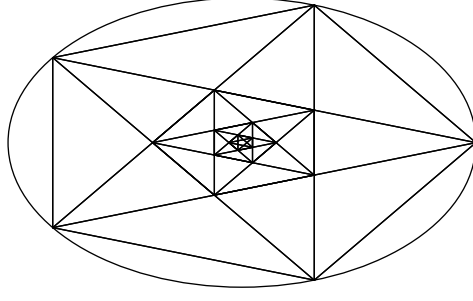
Among these configurations the one with minimal variance is, of course, that in which $\alpha_k = \theta_k - \theta_{k-1} = \frac{2\pi}{n}$ for every k so that the variance of α is equal to zero.

The roots of the identity have some beautiful properties; first of all:

Proposition IV.1

Cyclotomic identity:

$$\sum_{k=1}^n r_{n,k} = 0 \quad \forall n \in \mathbb{N}_+ : n \geq 2 \quad (4.10)$$

FIG. 1: The 5th roots of unity generate self-similarity.

PROOF:

It is sufficient to observe that:

$$\sum_{k=1}^n r_{n,k} = \sum_{k=1}^n r_{1,k}^k = \frac{r_{1,k}(1 - r_{1,k}^n)}{1 - r_{1,k}} = 0 \quad (4.11)$$

where we have used the fact that:

$$\sum_{k=1}^n z^k = \frac{z(1 - z^n)}{1 - z} \quad \forall z \in \mathbb{C} - \{1\}, \forall n \in \mathbb{N}_+ : n \geq 2 \quad (4.12)$$

■

Let us now denote by $polygon_{n,1}$ the regular polygon having the n^{th} roots of unity as vertices.

Given $i, j \in \{1, \dots, n\}$ such that $i \neq j, i \neq j \pm 1$ let us furthermore denote by $diagonal(i, j)$ the segment connecting $r_{n,i}$ and $r_{n,j}$.

Let us then consider the set of the intersections of distinct diagonals, id est the set of points $\{diagonal(i_1, j_1) \cap diagonal(i_2, j_2), i_1, i_2, j_1, j_2 \in \{1, \dots, n\}, i_1 \neq j_1, i_2 \neq j_2, i_1 \neq i_2, j_1 \neq j_2\}$ and let us call $polygon_{n,2}$ the polygon having these points as vertices.

We will say that:

Definition IV.1

the n^{th} roots of unity generate self-similarity:

$polygon_{n,2}$ is regular

Then name of the definition IV.1 is owed to the fact that if $polygon_{n,2}$ is regular then there exists a scaling factor $r_{scaling} \in (0, 1)$ and a rotation angle $\phi_{rotation} \in [0, 2\pi)$ such that:

$$polygon_{n,2} = r_{scaling} e^{i\phi_{rotation}} polygon_{n,1} \quad (4.13)$$

Clearly in this case the procedure may be iterated generating a sequence $\{polygon_{n,k}, k \in \mathbb{N}_+\}$ defined recursively by:

$$polygon_{n,k} = r_{scaling} e^{i\phi_{rotation}} polygon_{n,k-1} \quad (4.14)$$

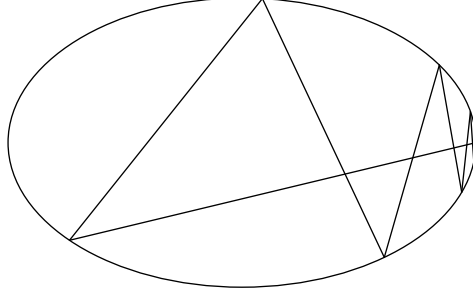
Example IV.1

The 5th roots of unity generate self similarity, with $r_{scaling} := \frac{\tau}{1+2\tau}$ and $\phi_{rotation} := \pi$ as it can be easily computed and verified looking at the figure 1.

Let us now suppose to have a countable infinity of petals, so that the uniform distribution given by the equation 4.6 become meaningless.

Let us then follow another strategy and let us pose the petals in the points:

$$p_{\lambda,k} := \exp(i\theta_{\lambda,k}) \quad k \in \mathbb{N}_+ \quad (4.15)$$

FIG. 2: The first 10 cusps of C_τ .

where:

$$\theta_{\lambda,k} := \text{Mod}_{2\pi}[(2\pi\lambda)^k] \quad \forall k \in \mathbb{N}_+ \quad (4.16)$$

Let us furthermore call $side_{\lambda,k}$ the segment connecting $p_{\lambda,k}$ and $p_{\lambda,k+1}$ and let us then introduce the following:

Definition IV.2

piece-wise linear curve associated to λ :

$$C_\lambda := \cup_{k \in \mathbb{N}_+} side_k \quad (4.17)$$

Definition IV.3

∞^{th} cusp of C_λ :

$$p_{\lambda,\infty} := \lim_{k \rightarrow \infty} p_{\lambda,k} \pmod{2\pi} \quad (4.18)$$

Then:

Theorem IV.1

Geometrical beauty of Pisot-Vijayaraghavan numbers:

HP:

$$\lambda \in PV(\mathbb{A}) \quad (4.19)$$

TH:

$$p_{\lambda,\infty} = p_{\lambda,1} \quad (4.20)$$

PROOF:

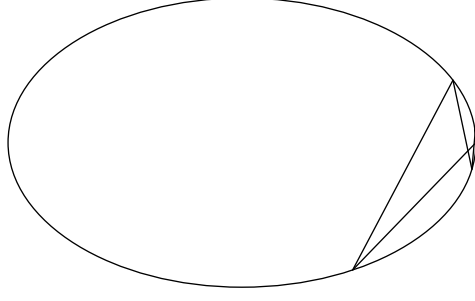
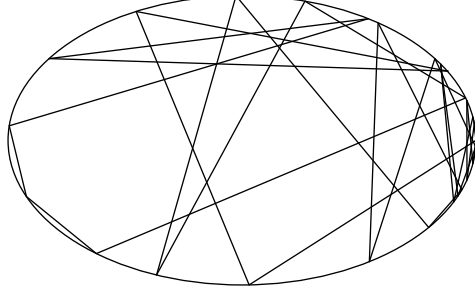
The proposition B.2 implies that if λ is chosen to be a Pisot-Vijayaraghavan number, then:

$$\lim_{k \rightarrow +\infty} \theta_{\lambda,k} = 0 \pmod{2\pi} \quad (4.21)$$

from which the thesis follows. ■

The meaning of the theorem IV.1 is shown in the figure 2, in the figure 3 and in the figure 4.

Remark IV.1

FIG. 3: The first 5 cusps of $C_{1+\sqrt{2}}$ FIG. 4: The first 30 cusps of C_ρ .

As for the n^{th} roots of unity one can introduce the segment $\text{diagonal}(i,j)$ connecting $p_{\lambda,i}$ and $p_{\lambda,j}$ for every $i, j \in \mathbb{N} : i \neq j, i \neq j \pm 1$ and then considerate the polygon having as vertices the intersections of the diagonals.

As shown in in the figure 5 and in the figure 6 no self-similarity is generated in this way.

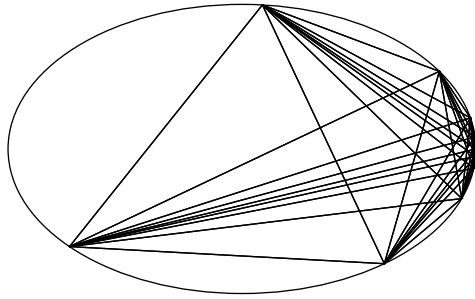
As a corollary of the theorem IV.1 it follows that the cusps of C_λ are a solution of our optimization problem for a countable infinity of petals if and only if λ is a Pisot-Vijayaraghavan number.

Our model is a simplified version of the optimization problem that has led Nature to fix the role of $\lambda = \tau$ in the phyllotaxis of many flowers (see for instance the 11th chapter "The golden section and phyllotaxis" of [42] and [43]).

Let us now suppose to have a substitution σ of Pisot type over the binary alphabet $\{0, 1\}$ such that $|\sigma(0)| \geq 2$ and let us call again λ the leading eigenvalue of its incidence matrix.

Let suppose to choose two fixed angles $\beta_{(0)}, \beta_{(1)} \in [0, 2\pi)$ such that:

$$\text{Mod}_{2\pi}\left[\frac{\beta_{(0)}}{\beta_{(1)}}\right] = \lambda \quad (4.22)$$

FIG. 5: C_τ doesn't give rise to self-similarity.

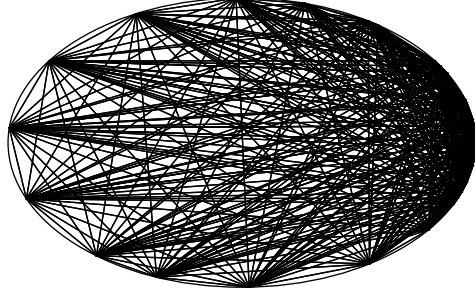


FIG. 6: C_ρ doesn't give rise to self-similarity.

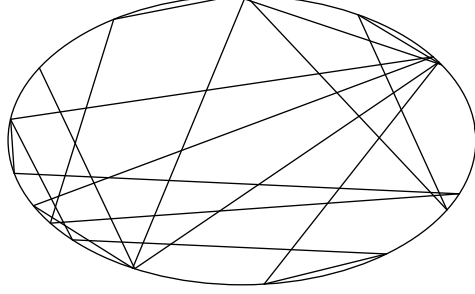


FIG. 7: The spacing given by the first $\sum_{k=1}^{20} F_k + 1 = 17711$ digits of the Fibonacci sequence, associating spacing τ to the letter 0 and spacing 1 to the letter 1.

If we suppose to construct our flower by following the algorithmic procedure:

$$\theta_k := \begin{cases} \text{Mod}_{2\pi}(\theta_{k-1} + \beta_{(0)}), & \text{if } (\bar{\sigma}(0))_k = 0 \\ \text{Mod}_{2\pi}(\theta_{k-1} + \beta_{(1)}), & \text{if } (\bar{\sigma}(0))_k = 1 \end{cases} \quad (4.23)$$

(where $(\bar{\sigma}(0))_k$ is the k^{th} digit of the sequence fixed point $\bar{\sigma}(0)$ starting with 0) the optimal solution is reached asymptotically.

Examples obtained using the Fibonacci substitution and the Pell substitution are shown, respectively, in the figure 7 and in the figure 8.

Remark IV.2

Since to build flowers may be a problem for Nature but is not precisely a concrete problem appearing in common life, the optimization problems discussed in this paper might appear as mathematical curiosities with no practical application.

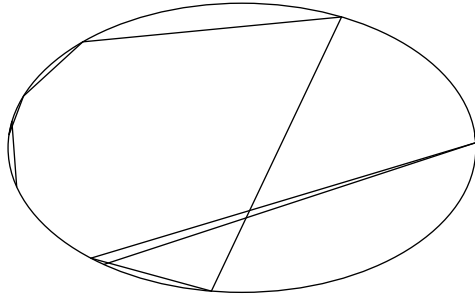


FIG. 8: The spacing given by the first $a_{11} = 5741$ digits of the Pell sequence, associating spacing $1 + \sqrt{2}$ to the letter 0 and spacing 1 to the letters 1.

The whole matter appears from a different perspective as soon as one realizes their deep link with certain searching problems, such as the problem of localizing with the minimum possible uncertainty the minimum of an unknown physical observable f whose functional dependence from an other physical observable x is unknown, the only information available being the fact that it has a unique minimum on an interval (a, b) , through a finite number n of compounds measurement $\{x_k, f(x_k)\}_{k=1}^n$, each measurement comporting a cost so that the number of measurements n has to be minimized too (see the 9th chapter "Optimal spacing and search algorithms" of [44], the 6th chapter "Search Techniques and nonlinear programming" of [45], the 1th chapter "Basic Concepts" of [46], the 6th chapter "Searching" of [47], the 10th chapter "Minimization and Maximization of Functions" of [48] and the 10th chapter "Search and games" of [49]).

V. CHAOS AS A PHENOMENON OF UNDECIDABILITY

Let us start by giving some more information concerning Chaitin's almost mystic Ω number.

With this regard we have, preliminarily, to introduce some basic notion of Computability Theory over the Reals on which, fortunately, agreement has been reached [50], [51].

Given a sequence of rational numbers $\{r_n\} \in \mathbb{Q}^{\mathbb{N}}$:

Definition V.1

$\{r_n\}$ is computable:

$$\exists a, b, s : \mathbb{N} \mapsto \mathbb{N} \text{ recursive} : (b(n) \neq 0 \wedge r_n = (-1)^{s(n)} \frac{a(n)}{b(n)}) \quad \forall n \in \mathbb{N} \quad (5.1)$$

Given $x \in \mathbb{R}$:

Definition V.2

$\{r_n\}$ converges computably to x :

$$\text{comp} - \lim_{n \rightarrow +\infty} r_n = x := \exists e : \mathbb{N} \mapsto \mathbb{N} \text{ recursive} : (|r_k - x| \leq 2^{-n} \quad \forall k \geq e(n)) \quad \forall n \in \mathbb{N} \quad (5.2)$$

Definition V.3

x is computable:

$$\exists \{r_n\} \in \mathbb{Q}^{\mathbb{N}} \text{ computable} : \text{comp} - \lim_{n \rightarrow +\infty} r_n = x \quad (5.3)$$

Let us introduce also the weaker notion:

Definition V.4

x is computably enumerable:

$$\exists \{r_n\} \in \mathbb{Q}^{\mathbb{N}} \text{ increasing, computable} : \lim_{n \rightarrow +\infty} r_n = x \quad (5.4)$$

We will denote the set of computable enumerable reals by $c.e.(\mathbb{R})$.

Then [11]:

Proposition V.1

Characterization of Chaitin's halting probabilities:

$$\{\Omega_U \mid U \text{ universal Chaitin computer}\} = \text{RANDOM}[(0, 1)] \cap c.e.(\mathbb{R}) \quad (5.5)$$

Let us now consider a formal system whose rules of inference form a computably enumerable set of order pairs $F := \{(a_n, T_n)\}_{n \in \mathbb{N}}$, the ordered pair (a_n, T_n) indicating that the theorem T_n is deducible from the axiom a_n .

We will adopt the Mathematical Logic's usual notation:

Definition V.5

T is deducible in the formal system F from the axiom a :

$$a \vdash_F T := (a, T) \in F \quad (5.6)$$

We can then state the following fundamental (demanding once more to the basic reference [11] for three different proofs):

Theorem V.1

Chaitin's 1st Information-theoretic Incompleteness Theorem:

HP:

F formal system with one axiom a

$$a \vdash_F "I(x) > n" \Rightarrow I(x) > n \quad (5.7)$$

TH:

$$\exists c_F \in \mathbb{R}_+ : a \vdash_F "I(x) > n" \Rightarrow n < I(a) + c_F \quad (5.8)$$

Example V.1

Let us look at a generic substitution σ over the finite alphabet A as to a formal system F_σ having as axioms the letters of A and as inference rules the substitution rule:

$$a \vdash_{F_\sigma} \sigma(a) \quad \forall a \in A \quad (5.9)$$

$$\vec{x} \vdash_{F_\sigma} \sigma(\vec{x}) \quad \forall \vec{x} \in A^+ \quad (5.10)$$

Let us suppose that the substitution σ is Sturmian (for instance it might be the Fibonacci substitution of the example III.1) so that the algorithmic information of all the deducible theorems is rather low.

It follows that F_σ may be used only to estimate the information content of objects of rather low information.

Let us then state also the following:

Theorem V.2

Chaitin's 2th Information-theoretic Incompleteness Theorem:

HP:

A finite alphabet

F formal system such that the set T_F of theorems deducible in F is computably enumerable and such that any statement of the form "the n^{th} cbit of $r_{\{0,1\}}(\Omega_U)$ is a 0" , "the n^{th} cbit of $r_{\{0,1\}}(\Omega_U)$ is a 1" can be represented in T_F and such a statement is a theorem of T_F only if it is true

TH:

T_f can enable us to determine the positions and values of at most finitely many scattered cbits of $r_{\{0,1\}}(\Omega_U)$

Let us now analyze how Chaitin Information-theoretic Incompleteness Theorems are at the heart of Chaos Theory. Let us, at this purpose, briefly recall some basic notions.

Given a finite alphabet A : $2 \leq |A| < \aleph_0$

Definition V.6

Brudno algorithmic entropy of $\bar{x} \in A^{\mathbb{N}+}$:

$$B(\bar{x}) := \lim_{n \rightarrow \infty} \frac{I(\vec{x}(n))}{n} \quad (5.11)$$

As it has been proved by Brudno himself [37]:

Proposition V.2

$$\{\bar{x} \in A^{\mathbb{N}^+} : B(\bar{x}) > 0\} \supset RANDOM(A^{\mathbb{N}^+}) \quad (5.12)$$

given a classical probability space (X, μ) :

Definition V.7

endomorphism of (X, μ) :

$T : HALT_\mu \rightarrow HALT_\mu$ surjective :

$$\mu(A) = \mu(T^{-1}A) \quad \forall A \in HALT_\mu \quad (5.13)$$

where $HALT_\mu$ is the halting-set of the measure μ , namely the σ -algebra of subsets of X on which μ is defined.

Definition V.8

classical dynamical system:

a triple (X, μ, T) such that:

- (X, μ) is a classical probability space
- $T : HALT_\mu \rightarrow HALT_\mu$ is an endomorphism of (X, μ)

Given a classical dynamical system (X, μ, T) :

Definition V.9

(X, μ, T) is *ergodic*:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^k(B)) = \mu(A) \mu(B) \quad \forall A, B \in HALT_\mu \quad (5.14)$$

Given a classical probability space (X, μ) :

Definition V.10

finite measurable partition of (X, μ) :

$$\begin{aligned} P &= \{P_0, \dots, P_{n-1}\} \quad n \in \mathbb{N} : \\ P_i &\in HALT(\mu) \quad i = 0, \dots, n-1 \\ P_i \cap P_j &= \emptyset \quad \forall i \neq j \\ \mu(X - \cup_{i=0}^{n-1} P_i) &= 0 \end{aligned} \quad (5.15)$$

We will denote the set of all the finite measurable partitions of (X, μ) by $\mathcal{P}(X, \mu)$.

Given two partitions $P = \{P_i\}_{i=0}^{n-1}$, $Q = \{Q_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)$:

Definition V.11

P is a *coarse-graining* of Q ($P \preceq Q$):

every atom of P is the union of atoms of Q

Definition V.12

coarsest refinement of $P = \{P_i\}_{i=0}^{n-1}$ and $Q = \{Q_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)$:

$$\begin{aligned} P \vee Q &\in \mathcal{P}(X, \mu) \\ P \vee Q &:= \{P_i \cap Q_j \mid i = 0, \dots, n-1, j = 0, \dots, m-1\} \end{aligned} \quad (5.16)$$

Clearly $\mathcal{P}(X, \mu)$ is closed both under coarsest refinements and under endomorphisms of (X, μ) .

Definition V.13

Shannon probabilistic entropy of $P = \{P_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

$$H(P) := - \sum_{i=0}^{n-1} \mu(P_i) \log_n [\mu(P_i)] \quad (5.17)$$

Given a classical dynamical system $CDS := (X, \mu, T)$ [52], [53], [54], [35], [36]:

Definition V.14

Kolmogorov-Sinai entropy of CDS:

$$h_{CDS} := \sup_{P \in \mathcal{P}(X, \mu)} \lim_{n \rightarrow \infty} \frac{1}{n} H(\vee_{k=0}^{n-1} T^{-k} P) \quad (5.18)$$

Let us now briefly recall how the orbits of a classical dynamical system may be symbolically codified.

Let $A_n := \{a_0, \dots, a_{n-1}\}$ be an alphabet of n letters.

Considered a partition $P = \{P_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

Definition V.15

symbolic translator of CDS with respect to P :

$\psi_P : X \rightarrow A_n$:

$$\psi_P(x) := \text{string}(i) : x \in P_i \quad (5.19)$$

where $\text{string}(i)$ is the i^{th} letter of A_n in lexicographic order.

Definition V.16

k -point symbolic translator of CDS with respect to P :

$\psi_P^{(k)} : X \rightarrow A_n^k$:

$$\psi_P^{(k)}(x) := \cdot_{j=0}^{k-1} \psi_P(T^j x) \quad (5.20)$$

Definition V.17

orbit symbolic translator of CDS with respect to P :

$\psi_P^{(\infty)} : X \rightarrow A_n^\infty$:

$$\psi_P^{(\infty)}(x) := \cdot_{j=0}^\infty \psi_P(T^j x) \quad (5.21)$$

Let us finally introduce the following basic:

Definition V.18

CDS is chaotic:

$$\exists P \in \mathcal{P}(X, \mu) : \psi_P^{(\infty)}(x) \in \text{RANDOM}(A_{|P|}^{\mathbb{N}_+}) \quad \forall -\mu - \text{almost } x \in X \quad (5.22)$$

Then:

Theorem V.3

Brudno Theorem:

$$CDS \text{ is chaotic} \Rightarrow h_{CDS} > 0 \quad (5.23)$$

Remark V.1

Let us remark that, owing essentially to the proposition V.2, the fact of having strictly positive Kolmogorov-Sinai entropy, though being (according to the theorem V.3) a necessary condition for chaoticity, is not a sufficient condition for chaoticity.

Let us now look at such a dynamical system from a mathematical logic point of view, id est looking at it as a formal system.

Let us suppose that CDS is chaotic and let $P \in \mathcal{P}(X, \mu)$ be a partition such that:

$$\psi_P^{(\infty)}(x) \in RANDOM(A_{|P|}^{\mathbb{N}_+}) \quad \forall -\mu - almost \ x \in X \quad (5.24)$$

Given the initial condition $x \in X$, the letter $\psi_P(x) \in A_{|P|}$ is the axiom of such a formal system,
Let us now observe that the equation 5.24 implies that:

$$I(\psi_P^{(n)}(x)) = O_{n \rightarrow +\infty}(n) \quad \forall -\mu - almost \ x \in X \quad (5.25)$$

Chaitin's 1th Information-theoretic Incompleteness Theorem (id est the theorem V.1) implies that the eventuality that the divergence of $I(\psi_P^{(n)}(x))$ is faster cannot be decided:

Corollary V.1

Undecidability underlying chaotic dynamical systems:

$$\psi_P(x) \not\vdash_{CDS} "I(\psi_P^{(n)}(x)) = o_{n \rightarrow +\infty}(n)" \quad \forall -\mu - almost \ x \in X \quad (5.26)$$

Remark V.2

Let us remark that such an explanation of the mathematical logic undecidability underlying the phenomenon of chaos, holding for discrete-time dynamical systems, is completely independent from Pesin Theorem that, for continuous-time dynamical systems, states that the Kolmogorov-Sinai entropy is equal to the sum of the positive Lyapunov exponents [55].

Also the 2th Chaitin Information-theoretic Incompleteness Theorem (id est the theorem V.2) affects the Theory of Dynamical Systems.

To appreciate this fact it is enough to observe that a universal Chaitin computer U may be seen, from the viewpoint of the Theory of Abstract Dynamical Systems, as a particular kind of shift (see [56], [57] and the references therein).

The Undecidability of the Halting Problem (id est the theorem III.2) appears then as a property of such a dynamical system ruled by Ω_U owing to the theorem III.2.

VI. QUANTUM TOPOLOGICAL ENTROPY VERSUS QUANTUM ALGORITHMIC RANDOMNESS

Given the finite alphabet $A := \{a_1, \dots, a_{|A|}\}$

Definition VI.1

Quantum computational Hilbert space with respect to A :

$$\mathcal{H}_A := \mathbb{C}^{|A|} \quad (6.1)$$

$$\hat{\mathbb{I}}_{\mathcal{H}_A} = \sum_{a \in A} |a\rangle\langle a| \quad (6.2)$$

$$\langle a|b\rangle = \delta_{a,b} \quad \forall a, b \in A \quad (6.3)$$

Given $n \in \mathbb{N}_+$:

Definition VI.2

n qu- $|A|$ -its' Hilbert space:

$$\mathcal{H}_A^{\otimes n} := \otimes_{i=1}^n \mathcal{H}_A = \mathbb{C}^{|A|^n} \quad (6.4)$$

$$\hat{\mathbb{I}}_{\mathcal{H}_A^{\otimes n}} := \sum_{\vec{x} \in A^n} |\vec{x}\rangle\langle \vec{x}| \quad (6.5)$$

$$\langle \vec{x}|\vec{y}\rangle = \delta_{\vec{x},\vec{y}} \quad \forall \vec{x}, \vec{y} \in A^n \quad (6.6)$$

The quantum analogue of the set A^+ of all strings over A is the following:

Definition VI.3

Hilbert space of quantum strings with respect to A :

$$\mathcal{H}_A^{\otimes +} := \oplus_{n \in \mathbb{N}_+} \mathcal{H}_A^{\otimes n} := \oplus_{n \in \mathbb{N}_+} \mathbb{C}^{|A|^n} \quad (6.7)$$

$$\hat{\mathbb{I}}_{\mathcal{H}_A^{\otimes +}} := \sum_{\vec{x} \in A^+} |\vec{x}\rangle\langle \vec{x}| \quad (6.8)$$

$$\langle \vec{x}|\vec{y}\rangle = \delta_{\vec{x},\vec{y}} \quad \forall \vec{x}, \vec{y} \in A^+ \quad (6.9)$$

The quantum analogue of the set $A^{\mathbb{N}_+}$ of all sequences over A is the following:

Definition VI.4

Hilbert space of quantum sequences with respect to A :

$$\mathcal{H}_A^{\otimes \mathbb{N}_+} := \otimes_{n \in \mathbb{N}_+} \mathcal{H}_A \quad (6.10)$$

$$\hat{\mathbb{I}}_{\mathcal{H}_A^{\otimes \mathbb{N}_+}} := \int_{A^{\mathbb{N}_+}} d\bar{x} |\bar{x}\rangle\langle \bar{x}| \quad (6.11)$$

$$\langle \bar{x}|\bar{y}\rangle = \delta(\bar{x} - \bar{y}) \quad \forall \bar{x}, \bar{y} \in A^{\mathbb{N}_+} \quad (6.12)$$

where $d\bar{x}$ is the Lebesgue measure over $A^{\mathbb{N}_+}$ and where the δ in the right hand side of 6.12 is the Dirac delta tempered distribution (defined in the appendix A).

Given a quantum sequence $|\psi\rangle \in \mathcal{H}_A^{\otimes \mathbb{N}_+}$ the natural quantum analogue of the definition III.10 is the following:

Definition VI.5

combinatorial quantum information function of $|\psi\rangle$:

$$p_n(|\psi\rangle) := \int_{A^{\mathbb{N}_+}} d\bar{x} |\langle \bar{x} | \psi \rangle|^2 p_n(\bar{x}) \quad (6.13)$$

Remark VI.1

The definition VI.5 is nothing but the expectation value of the classical combinatorial information function when a measurement of the sequence operator \hat{x}^1 is performed.

The natural quantum analogue of the definition III.11 is then the following:

Definition VI.6

topological quantum entropy of $|\psi\rangle$:

$$H_{top}(|\psi\rangle) := \lim_{n \rightarrow +\infty} \frac{\log_{|A|}(p_n(|\psi\rangle))}{n} \quad (6.15)$$

Clearly:

Proposition VI.1

$$H_{top}(|\bar{x}\rangle) = H_{top}(\bar{x}) \quad \forall \bar{x} \in A^{\mathbb{N}_+} \quad (6.16)$$

PROOF:

It is sufficient to observe that:

$$p_n(|\bar{x}\rangle) = \int_{A^{\mathbb{N}_+}} d\bar{y} |\langle \bar{y} | \bar{x} \rangle|^2 p_n(|\bar{y}\rangle) = \int_{A^{\mathbb{N}_+}} d\bar{y} \delta^2(\bar{x} - \bar{y}) p_n(|\bar{y}\rangle) = p_n(\bar{x}) \quad (6.17)$$

■

Example VI.1

Given the binary alphabet $\{0,1\}$ let us consider the quantum sequence:

$$|\psi\rangle := \frac{1}{\sqrt{2}}(|0000000000\dots\rangle + |1111111111\dots\rangle) \quad (6.18)$$

Then:

$$p_n(|\psi\rangle) = \frac{p_n(|0000000000\dots\rangle) + p_n(|1111111111\dots\rangle)}{2} \quad (6.19)$$

Since obviously:

$$\mathcal{L}_n(0000000000\dots) = \{\cdot_{i=1}^n 0\} \quad (6.20)$$

$$p_n(0000000000\dots) = 1 \quad (6.21)$$

$$\mathcal{L}_n(1111111111\dots) = \{\cdot_{i=1}^n 1\} \quad (6.22)$$

¹ where obviously the sequence operator is defined by:

$$\hat{x}|\bar{x}\rangle := \bar{x}|\bar{x}\rangle \quad (6.14)$$

PROOF:

1. Clearly:

$$p_n(|\psi\rangle) = \int_{RANDOM(A^{\mathbb{N}+})} d\bar{x} |\langle \bar{x} | \psi \rangle|^2 p_n(|\bar{x}\rangle) = \int_{RANDOM(A^{\mathbb{N}+})} d\bar{x} |\langle \bar{x} | \psi \rangle|^2 |A|^n > 0 \quad \forall |\psi\rangle \in RANDOM(\mathcal{H}_A^{\otimes \mathbb{N}+}) \quad (6.33)$$

2. the quantum Champerknowne sequence is not algorithmically random:

$$|\bar{x}_{Champerknowne}\rangle \notin RANDOM(\mathcal{H}_A^{\otimes \mathbb{N}+}) \quad (6.34)$$

Anyway:

$$H_{top}(|\bar{x}_{Champerknowne}\rangle) = H_{top}(\bar{x}_{Champerknowne}) = 1 \quad (6.35)$$

■

Remark VI.2

The proposition VI.2 shows that the considerations of the remark III.2 may be thoroughly extended to the quantum case:

the quantum topological entropy furnishes a measure of the quantum information contained in a quantum sequence coarser than the one given by Quantum Algorithmic Information Theory [13], [14], [15], [16], [17], [18], [19].

Since no general agreement has been reached in the scientific community as to the right quantum analogue of the definition III.21 (see [40] for a discussion of the involved issues), the combinatorial approach has the advantages not to involve Quantum Computability Theory with all its still not yet settled issues.

Example VI.3

Given $\Omega_1, \Omega_2 \in RANDOM[(0, 1)] \cap c.e.(\mathbb{R})$ and a finite alphabet A let us consider the quantum sequence of qu- $|A|$ -its:

$$|\psi\rangle := \frac{1}{\sqrt{2}}(|r_A(\Omega_1)\rangle + |r_A(\Omega_2)\rangle) \quad (6.36)$$

where r_A is the nonterminating symbolic representation with respect to A of the definition A.5.

Obviously:

$$|\psi\rangle \in RANDOM(\mathcal{H}_A^{\mathbb{N}+}) \quad (6.37)$$

VII. QUANTUM ALGORITHMS OF SUBSTITUTION

Given two finite alphabet A and B and a map $f : A^+ \rightarrow B^+$ there exists a canonical way of constructing a quantum algorithm associated to f as the linear operator $\hat{f} : \mathcal{H}_A^{\otimes+} \mapsto \mathcal{H}_B^{\otimes+}$ acting as f on the computational basis (about some remarkable considerations concerning the implementation of such an algorithm see the section 4.9.13 "Quantum implementations" of [58]).

Applied in particular to a substitution σ such a strategy leads to the following:

Definition VII.1

quantum algorithm of the first kind associated to σ :

the linear operator over $\mathcal{H}_A^{\otimes+} \cup \mathcal{H}_A^{\otimes\mathbb{N}+}$ defined by the following action on the computational bases:

$$\hat{\sigma}|\vec{x}\rangle := |\sigma(\vec{x})\rangle \quad \vec{x} \in A^+ \quad (7.1)$$

$$\hat{\sigma}|\bar{x}\rangle := |\sigma(\bar{x})\rangle \quad \bar{x} \in A^{\mathbb{N}+} \quad (7.2)$$

We will refer to $\hat{\sigma}$ as to a *quantum algorithm of substitution* or, more concisely, as to a *quantum substitution*. Clearly:

Proposition VII.1

1.

$$\hat{\sigma}|\vec{x}\rangle = \bigotimes_{i=1}^{|\vec{x}|} |\sigma(x_i)\rangle \quad \forall \vec{x} \in A^+ \quad (7.3)$$

2.

$$\hat{\sigma}|\bar{x}\rangle = \bigotimes_{n \in \mathbb{N}+} |\sigma(x_n)\rangle \quad \forall \bar{x} \in A^{\mathbb{N}+} \quad (7.4)$$

3.

$\hat{\sigma}$ is not self-adjoint

4.

$$\hat{\sigma}|_{\mathcal{H}_A^{\otimes+}} = \sum_{\vec{x} \in A^+} |\sigma(\vec{x})\rangle \langle \vec{x}| \quad (7.5)$$

PROOF:

1. It is sufficient to observe that:

$$\hat{\sigma}|\vec{x}\rangle = |\sigma(\vec{x})\rangle = |\sigma(\cdot_{i=1}^{|\vec{x}|} x_i)\rangle = |\cdot_{i=1}^{|\vec{x}|} \sigma(x_i)\rangle = \bigotimes_{i=1}^{|\vec{x}|} |\sigma(x_i)\rangle \quad \forall \vec{x} \in A^+ \quad (7.6)$$

2. In an analogous way:

$$\hat{\sigma}|\bar{x}\rangle = |\sigma(\bar{x})\rangle = |\sigma(\cdot_{n \in \mathbb{N}+} x_n)\rangle = |\cdot_{n \in \mathbb{N}+} \sigma(x_n)\rangle = \bigotimes_{n \in \mathbb{N}+} |\sigma(x_n)\rangle \quad \forall \bar{x} \in A^{\mathbb{N}+} \quad (7.7)$$

3. It is sufficient to observe that:

$$\langle \vec{x} | \hat{\sigma} | \bar{y} \rangle = \delta_{\vec{x}, \sigma(\bar{y})} \neq \langle \bar{y} | \hat{\sigma} | \vec{x} \rangle = \delta_{\bar{y}, \sigma(\vec{x})} \quad (7.8)$$

4. Clearly:

$$\begin{aligned} \hat{\sigma}|_{\mathcal{H}_A^{\otimes+}} &= \sum_{\vec{x} \in A^+} |\vec{x}\rangle \langle \vec{x}| \hat{\sigma} \sum_{\bar{y} \in A^+} |\bar{y}\rangle \langle \bar{y}| = \\ &= \sum_{\vec{x} \in A^+} \sum_{\bar{y} \in A^+} \langle \vec{x} | \hat{\sigma} | \bar{y} \rangle |\vec{x}\rangle \langle \bar{y}| = \sum_{\vec{x} \in A^+} \sum_{\bar{y} \in A^+} \delta_{\vec{x}, \sigma(\bar{y})} |\vec{x}\rangle \langle \bar{y}| \end{aligned} \quad (7.9)$$

from which the thesis follows.

■

Let us now suppose that $a \in A$ is such that $|\sigma(a)| \geq 2$. Denoted as usual with $\bar{\sigma}(a)$ the fixed point of σ starting with a let us observe that:

Proposition VII.2

$$\hat{\sigma}|\bar{\sigma}(a) \rangle = |\bar{\sigma}(a) \rangle \quad (7.10)$$

PROOF:

It is sufficient to observe that:

$$\hat{\sigma}|\bar{\sigma}(a) \rangle = |\sigma[\bar{\sigma}(a)] \rangle = |\bar{\sigma}(a) \rangle \quad (7.11)$$

■

Let us introduce also the following:

Definition VII.2

symmetric state of n qu- $|A|$ -its:

$$|S, n \rangle := \frac{1}{\sqrt{|A|^n}} \sum_{\vec{x} \in A^n} |\vec{x} \rangle \quad (7.12)$$

The Mathematica 5 notebook of the section E may be used to compute the action of quantum substitutions on symmetric states.

Example VII.1

Let σ denote the Fibonacci substitution of the example III.1. Then:

$$\hat{\sigma}|S, 1 \rangle = \hat{\sigma}\left[\frac{1}{\sqrt{2}}(|0 \rangle + |1 \rangle)\right] = \frac{1}{\sqrt{2}}(|01 \rangle + |0 \rangle) \quad (7.13)$$

$$\hat{\sigma}^2|S, 1 \rangle = \frac{1}{\sqrt{2}}(|010 \rangle + |01 \rangle) \quad (7.14)$$

$$\hat{\sigma}^3|S, 1 \rangle = \frac{1}{\sqrt{2}}(|01001 \rangle + |010 \rangle) \quad (7.15)$$

and in general:

$$\hat{\sigma}^n|S, 1 \rangle = \frac{1}{\sqrt{2}}(|\sigma^n(0) \rangle + |\sigma^{n-1}(0) \rangle) \quad \forall n \in \mathbb{N}_+ \quad (7.16)$$

and hence:

$$\lim_{n \rightarrow +\infty} \hat{\sigma}^n|S, 1 \rangle = \frac{1}{\sqrt{2}}(|\bar{\sigma}(0) \rangle + |\bar{\sigma}(0) \rangle) = \sqrt{2}|\bar{\sigma}(0) \rangle \quad (7.17)$$

Furthermore:

$$\hat{\sigma}|S, 2 \rangle = \hat{\sigma}\left[\frac{1}{2}(|00 \rangle + |01 \rangle + |10 \rangle + |11 \rangle)\right] = \frac{1}{2}(|0101 \rangle + |010 \rangle + |001 \rangle + |00 \rangle) \quad (7.18)$$

$$\hat{\sigma}^2|S, 2 \rangle = \frac{1}{2}(|010010 \rangle + |01001 \rangle + |01010 \rangle + |0101 \rangle) \quad (7.19)$$

$$\hat{\sigma}^3|S, 2 \rangle = \frac{1}{2}(|0100101001 \rangle + |01001010 \rangle + |01001001 \rangle + |010010 \rangle) \quad (7.20)$$

Remark VII.1

The quantum algorithms of substitution of the first kind have the great disadvantage of loosing the connection with the powerful and beautiful Mathematics of Pisot-Vijayaraghavan numbers underlying the incidence matrix and are of doubtful practical utility.

This suggests the introduction of quantum algorithms of substitution of the second kind.

Given a substitution σ :

Definition VII.3

quantum algorithm of the second kind associated to σ :

the linear operator $\hat{\sigma}$ represented by M_σ in the computational basis of \mathcal{H}_A .

We will refer again to $\hat{\sigma}$ as to a quantum algorithm of substitution or, more concisely, as to a quantum substitution. Then:

Proposition VII.3

HP:

σ Pisot substitution with leading eigenvalue λ

$$a \in A : |\sigma(a)| \geq 2 \quad (7.21)$$

$\hat{\sigma}$ quantum algorithm of the second kind associated to σ

TH:

$$\lim_{n \rightarrow +\infty} \hat{\sigma}^n = |e_\lambda \rangle \langle e_\lambda| \quad (7.22)$$

where:

$$\hat{\sigma}|e_\lambda \rangle = \lambda|e_\lambda \rangle \quad (7.23)$$

$$\langle e_\lambda | e_\lambda \rangle = 1 \quad (7.24)$$

PROOF:

Since:

$$\hat{\sigma} = \lambda|e_\lambda \rangle \langle e_\lambda| + \sum_{y \in \text{Con}(\lambda)} y|e_y \rangle \langle e_y| \quad (7.25)$$

it follows that:

$$\hat{\sigma}^n = \lambda^n|e_\lambda \rangle \langle e_\lambda| + \sum_{y \in \text{Con}(\lambda)} y^n|e_y \rangle \langle e_y| \quad (7.26)$$

Since $\lambda \in PV(\mathbb{A})$:

$$\lim_{n \rightarrow +\infty} \sum_{y \in \text{Con}(\lambda)} y^n|e_y \rangle \langle e_y| = 0 \quad (7.27)$$

from which the thesis follows. ■

Example VII.2

The quantum algorithm of the second kind associated to the Fibonacci substitution is clearly the operator $\hat{\sigma} : \mathcal{H}_{\{0,1\}} \mapsto \mathcal{H}_{\{0,1\}}$ such that:

$$\hat{\sigma}|0\rangle = |0\rangle + |1\rangle \quad (7.28)$$

$$\hat{\sigma}|1\rangle = |0\rangle \quad (7.29)$$

From the example III.1 we may immediately infer that:

$$\hat{\sigma} = \tau|e_\tau\rangle\langle e_\tau| + \left(-\frac{1}{\tau}\right)|e_{-\frac{1}{\tau}}\rangle\langle e_{-\frac{1}{\tau}}| \quad (7.30)$$

$$|e_\tau\rangle = \frac{\tau}{\sqrt{\tau+2}}|0\rangle + \frac{1}{\sqrt{\tau+2}}|1\rangle \quad (7.31)$$

$$|e_{-\frac{1}{\tau}}\rangle = -\frac{1}{\tau}\sqrt{\frac{\tau+1}{\tau+2}}|0\rangle + \sqrt{\frac{\tau+1}{\tau+2}}|1\rangle \quad (7.32)$$

(where we have used the fact that $\tau^2 = \tau + 1$) so that the probability that a measurement of the qubit operator in the state $\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle = |e_\tau\rangle$ gives as result zero is:

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle}(0) = \frac{\tau^2}{\tau+2} \quad (7.33)$$

while the probability that such a measurement gives as result one is:

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle}(1) = \frac{1}{\tau+2} \quad (7.34)$$

Example VII.3

The quantum algorithm of second kind associated to the Padovan substitution σ is clearly the linear operator $\hat{\sigma} : \mathcal{H}_{\{0,1,2\}} \mapsto \mathcal{H}_{\{0,1,2\}}$ such that:

$$\hat{\sigma}|0\rangle := |1\rangle + |2\rangle \quad (7.35)$$

$$\hat{\sigma}|1\rangle := |2\rangle \quad (7.36)$$

$$\hat{\sigma}|2\rangle := |0\rangle \quad (7.37)$$

From the example III.1 we may immediately infer that:

$$\hat{\sigma}|e_\rho\rangle = \rho|e_\rho\rangle \quad (7.38)$$

$$|e_\rho\rangle = \frac{\rho^2 - 1}{\sqrt{(\rho^2 - 1)^2 + (1 + \rho - \rho^2)^2 + 1}}|0\rangle + \frac{1 + \rho - \rho^2}{\sqrt{(\rho^2 - 1)^2 + (1 + \rho - \rho^2)^2 + 1}}|1\rangle + \frac{1}{\sqrt{(\rho^2 - 1)^2 + (1 + \rho - \rho^2)^2 + 1}}|2\rangle \quad (7.39)$$

(where we have used the fact that $\rho^3 = \rho + 1$) so that the probability distribution of the outcome of a measurement of the qutrit operator in the state $\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle = |e_\rho\rangle$ is:

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle}(0) = \frac{(\rho^2 - 1)^2}{(\rho^2 - 1)^2 + (1 + \rho - \rho^2)^2 + 1} \quad (7.40)$$

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle}(1) = \frac{(1 + \rho - \rho^2)^2}{(\rho^2 - 1)^2 + (1 + \rho - \rho^2)^2 + 1} \quad (7.41)$$

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n|0\rangle}(2) = \frac{1}{(\rho^2 - 1)^2 + (1 + \rho - \rho^2)^2 + 1} \quad (7.42)$$

Example VII.4

The quantum algorithm of the second kind associated to the Pell substitution is clearly the operator $\hat{\sigma} : \mathcal{H}_{\{0,1\}} \mapsto \mathcal{H}_{\{0,1\}}$ such that:

$$\hat{\sigma}|0\rangle := |0\rangle + |1\rangle \quad (7.43)$$

$$\hat{\sigma}|1\rangle := 2|0\rangle + |1\rangle \quad (7.44)$$

From the example III.3 we immediately infer that:

$$\hat{\sigma} = (1 + \sqrt{2})|e_{1+\sqrt{2}}\rangle\langle e_{1+\sqrt{2}}| + (1 - \sqrt{2})|e_{1-\sqrt{2}}\rangle\langle e_{1-\sqrt{2}}| \quad (7.45)$$

$$|e_{1+\sqrt{2}}\rangle = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \quad (7.46)$$

$$|e_{1-\sqrt{2}}\rangle = -\sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle \quad (7.47)$$

so that the probability distribution of the outcome of a measurement of the qubit operator in the state $\lim_{n \rightarrow +\infty} \hat{\sigma}^n |0\rangle = |e_{1+\sqrt{2}}\rangle$ is:

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n |0\rangle}(0) = \frac{2}{3} \quad (7.48)$$

$$Pr_{\lim_{n \rightarrow +\infty} \hat{\sigma}^n |0\rangle}(1) = \frac{1}{3} \quad (7.49)$$

Remark VII.2

Quantum algorithms mimicking the recursive structure of Fibonacci numbers have been introduced in [59]. In our modest opinion it doesn't seem, anyway, to be any connection between those remarkable algorithms and the quantum substitutions of Pisot type discussed in this paper.

Remark VII.3

The idea of taking into account suitable (specifically almost periodic Schrödinger) operators associated to substitutions has been already considered in the Mathematical-Physics' literature concerning quasicrystals (see for instance [60], [61] and references therein).

In order to allow the reader to appreciate the possible conceptual connections a brief review of the basic notions is presented in the section D.

From the mathematical side the great achievement obtained by Barry Simon and coworkers, namely the discovery that almost periodic Schrödinger operators has commonly continuous spectral measures supported on Cantor sets [62] has never been looked from an information theoretic perspective.

As we have already observed in the remark III.5, this would be a necessary step in order to generalize the Calude-Chitescu Theorem (id est the theorem III.4) clarifying the algorithmic information theoretic status of uniformly recurrent sequences.

The attempt to make some step forward in this direction is presented in the section C.

Remembering, furthermore, that an almost periodic function $f : \mathbb{R}^D \mapsto \mathbb{C}$ can be seen as an ergodic stationary random function over the probability space $(\Gamma, Hull(\Gamma), \mu_{Haar}(\Gamma))$, $\Gamma := \overline{\mathcal{T}_{\vec{x}}f}$ being the closure of all the shifts of f (where of course $\mathcal{T}_{\vec{x}}f(\vec{y}) := f(\vec{x} + \vec{y})$) (see the section 7.2 "Periodic and Almost Periodic Potentials" of [63]) such a task could be part of a more ambitious ompletely unexplored research project investigating algorithmically-random Schrödinger operators that, anyway, is far beyond the purposes of this paper.

The natural analogue of the definition III.31 for the quantum algorithm of first kind associated to a substitution is the following:

Definition VII.4

quantum topological entropy of $\hat{\sigma}$:

$$H_{top}(\hat{\sigma}) := \sum_{a \in A : |\sigma(a)| \geq 2} \max\{H_{top}(|\psi\rangle), |\psi\rangle \in \mathcal{H}_A^{\otimes \mathbb{N}^+} : \langle \bar{\sigma}(a) | \psi \rangle \neq 0\} \quad (7.50)$$

Remark VII.4

Quantum analogues of the theorem III.6 expressing the link between the quantum probabilistic information (id est Von Neumann entropy) and the quantum algorithmic information have been established in the same papers in which this latter notion has been introduced [14], [15], [16], [17], [18], [19].

After having defined chaotic a quantum dynamical system whose orbits, simbolically codified in a suitable way, are algorithmically-random, as we have observed in the remark III.7 the next conceptual step would then consists in its generalization to arbitrary quantum stochastic processes and then, through a suitable symbolic codification, to arbitrary quantum dynamical systems resulting in a quantum analogue of the Brudno Theorem (id est the theorem V.3) stating that the vanishing of the quantum dynamical entropy of a quantum dynamical system (defined as the asymptotic rate of production of probabilistic information of such a dynamical system) is a necessary condition for chaoticity also at the quantum level.

The first problem, with this regard, consists in the fact that, according to whether or not one assumes that measurements are performed on the quantum dynamical system during its dynamical evolution, one results in different notions of quantum dynamical entropy: respectively the *Connes-Narnhofer-Thirring entropy* [64], [65], [66], [67] and the *Alicki-Lindblad-Fannes entropy* [68].

A remarkable result going in the right direction has been obtained in [69] though, in our modest opinion, it is not yet a Quantum Brudno Theorem, since it in no way takes into account individual trajectories and their eventual quantum algorithmic randomness that, according to the analysis presented in the section V, is the key point as to the underlying undecidability phenomenon.

The definition VII.4, involving only the combinatorial approach to quantum information, might be a tool useful in order to proceed in the indicated direction.

VIII. A QUANTUM ALGORITHM FOR OPTIMAL SPACING

In the section IV we have seen that the optimization problem consisting in allocating $n \in \mathbb{N}_+ : n \geq 2$ petals of a flower so that:

1. the average distance between neighbors is maximal in order to maximize the exposition of each petal to sun and rain.
2. the distribution of the distances among neighbour petals is the more uniform one.

has as solutions the n^{th} -roots of unity:

$$r_{n,k} := e^{i\frac{2\pi}{n}k} \quad k = 1, \dots, n \quad (8.1)$$

Given a generic finite alphabet A and $n \in \mathbb{N}_+$ [70]:

Definition VIII.1

quantum Fourier transform on n qu- $|A|$ -its:

the linear map \hat{F} over $\mathcal{H}_A^{\otimes n}$ defined by its matrix's elements in the computational basis:

$$\langle \vec{y} | \hat{F} | \vec{x} \rangle := \frac{1}{\sqrt{|A|^n}} r_{|A|^n, l_{loc}(\vec{x}) l_{loc}(\vec{y})} \quad \forall \vec{x}, \vec{y} \in A^n, \forall n \in \mathbb{N}_+ \quad (8.2)$$

where l_{loc} denotes the *local lexicographic number* on A^n defined as:

$$l_{loc}(\cdot_{k=1}^n a_1) = 0 \quad (8.3)$$

\vdots

\vdots

$$l_{loc}(\cdot_{k=1}^n a_{|A|}) = |A|^n - 1 \quad (8.4)$$

Let us face, anyway, the more interesting optimization problem in which the number of petals is countable.

Let us now suppose to have a substitution σ of Pisot type over the binary alphabet $\{0, 1\}$ such that $|\sigma(0)| \geq 2$ and let us call again λ the leading eigenvalue of its incidence matrix.

Let suppose to choose two fixed angles $\beta_{(0)}, \beta_{(1)} \in [0, 2\pi)$ such that:

$$Mod_{2\pi} \left[\frac{\beta_{(0)}}{\beta_{(1)}} \right] = \lambda \quad (8.5)$$

Let us then construct our flower according to the following algorithmic procedure:

1. set $n=0$
2. label[start]
3. prepare the state $|0\rangle$ on the 1^{th} register
4. apply n times the operator $\hat{\sigma}$ on the 1^{th} register
5. copy the status of the 1^{th} register on the 2^{th} register
6. perform on the 2^{th} register a measurement of the qubit operator
7. pose the n^{th} petal at the angle:

$$\theta_n = \begin{cases} Mod_{2\pi}(\theta_{n-1} + \beta_{(0)}), & \text{if the result of the measurement was 0} \\ Mod_{2\pi}(\theta_{n-1} + \beta_{(1)}), & \text{if the result of the measurement was 1} \end{cases} \quad (8.6)$$

8. increase n of one unit

9. goto the label[start]

It is important to stress that the fifth step of such an algorithm doesn't violate the No Cloning Theorem as one can infer taking into account the precise formulation of such a theorem [70]:

Theorem VIII.1

No Cloning Theorem:

HP:

\mathcal{H} Hilbert space

TH:

$$\nexists \hat{U} : \mathcal{H} \mapsto \mathcal{H} \otimes \mathcal{H} \text{ unitary} : \hat{U}|\psi_1\rangle = |\psi_1\rangle \otimes |\psi_1\rangle \wedge \hat{U}|\psi_2\rangle = |\psi_2\rangle \otimes |\psi_2\rangle \wedge |\psi_1\rangle \neq |\psi_2\rangle \wedge \langle \psi_1 | \psi_2 \rangle \neq 0 \quad (8.7)$$

PROOF:

Let us suppose ad absurdum that such a unitary operator exists. Then:

$$\langle \psi_1 | \hat{U}^\dagger \hat{U} | \psi_2 \rangle = (\langle \psi_1 | \otimes \langle \psi_1 |) (|\psi_2\rangle \otimes |\psi_2\rangle) = (\langle \psi_1 | \psi_2 \rangle)^2 = \langle \psi_1 | \psi_2 \rangle \quad (8.8)$$

and hence:

$$\langle \psi_1 | \psi_2 \rangle = 0 \quad (8.9)$$

contradicting the hypothesis that the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are non-orthogonal. ■

Theorem VIII.1 forbids the simultaneous cloning of non-orthogonal states, but it doesn't forbid the simultaneous cloning of distinct orthogonal states.

Given a finite alphabet A and considered the cloning function $f_{clone} : A^+ \mapsto A^+$ one can immediately introduce the quantum algorithm $\hat{f}_{clone} : \mathcal{H}_A^{\otimes+} \mapsto \mathcal{H}_A^{\otimes+}$ acting as f_{clone} on the computational basis:

$$\hat{f}_{clone}|\vec{x}\rangle := |\vec{x} \cdot \vec{x}\rangle \quad \vec{x} \in A^+ \quad (8.10)$$

perfectly compatible with the theorem VIII.1.

The presented algorithm reaches asymptotically the optimal solution.

APPENDIX A: GENERALIZED FUNCTIONS ON THE SPACE OF ALL SEQUENCES OVER A FINITE ALPHABET

Given a topological space (X, \mathcal{T}) [71] and a subset of its $S \subset X$:

Definition A.1

set of the limit points of S :

$$LP(S) := \{x \in X : (O \in \mathcal{T} \wedge x \in O) \Rightarrow O \cap S \neq \emptyset\} \quad (\text{A1})$$

Definition A.2

S is a Cantor set:

$$(\bar{S} = S) \wedge (S^\circ = \emptyset) \wedge (LP(S) = S)$$

where \bar{S} is the closure of S while S° is its interior.

Given a finite alphabet $A = \{a_1, \dots, a_{|A|}\}$ (id est a set A such that the cardinality of A $|A| \in \mathbb{N}_+$) we will denote, following the notation of [11], [21], [27] and [28], by $A^+ := \cup_{n \in \mathbb{N}_+} A^n$ the free semi-group generated by A , id est the set of all the finite words over A .

Given $\vec{x}, \vec{y} \in A^+$ let us denote by $\vec{x} \cdot \vec{y}$ the concatenation of \vec{x} and \vec{y} , id est the string $(x_1, \dots, x_{|\vec{x}|}, y_1, \dots, y_{|\vec{y}|})$ where $|\vec{x}|$ denotes the length of the strings \vec{x} .

Introduced the set $A^{\mathbb{N}_+}$ of the *sequences over A* let us endow A with the discrete topology, and let us endow A^+ and $A^{\mathbb{N}_+}$ with the product topology.

Such a topology over $A^{\mathbb{N}_+}$ is the metric topology induced by the following distance:

$$d(\vec{x}, \vec{y}) = \begin{cases} 0, & \text{if } \vec{x} = \vec{y}; \\ \frac{1}{2^{\min\{n \in \mathbb{N}_+ : x_n \neq y_n\}}}, & \text{otherwise.} \end{cases} \quad (\text{A2})$$

The Borel- σ algebra $\mathcal{B}(A^{\mathbb{N}_+})$ associated to such a topology is the σ -algebra generated by the cylinder sets:

$$C_{\vec{x}} := \{\vec{y} \in A^{\mathbb{N}_+} : \vec{y}(|\vec{x}|) = \vec{x} \mid \vec{x} \in A^+\} \quad (\text{A3})$$

where $\vec{y}(n)$ denotes the prefix of length n of the sequence \vec{y} .

We can then introduce the following:

Definition A.3

Lebesgue measure associated to A :

the probability measure $\mu_{Lebesgue_A}$ over the measurable space $(A^{\mathbb{N}_+}, \mathcal{B}(A^{\mathbb{N}_+}))$ such that:

$$\mu_{Lebesgue, A}(C_{\vec{x}}) := \frac{1}{|A|^{|\vec{x}|}} \quad (\text{A4})$$

Given $f \in L^1(A^{\mathbb{N}_+}, d\mu_{Lebesgue_A})$ let us pose $d\vec{x} := d\mu_{Lebesgue, A}$ and hence:

$$\int_{A^{\mathbb{N}_+}} d\vec{x} f(\vec{x}) := \int_{A^{\mathbb{N}_+}} d\mu_{Lebesgue_A}(\vec{x}) f(\vec{x}) \quad (\text{A5})$$

The applicability of Probability Theory to the real world lies, from a foundational perspective, on the following:

Theorem A.1

On the Foundation of Probability Theory:

$$\mu_{Lebesgue, A}[RANDOM(A^{\mathbb{N}_+})] = 1 \quad \forall A : 2 \leq |A| < \aleph_0 \quad (\text{A6})$$

Given a sequence $\vec{x} \in A^{\mathbb{N}_+}$:

Definition A.4

numerical value of \bar{x} :

$$v_A(\bar{x}) := \sum_{n=1}^{\infty} \frac{lex(x_n)}{|A|^n} \quad (\text{A7})$$

Remark A.1

Let us remark that the map $v : A^{\mathbb{N}^+} \mapsto [0, 1]$ is surjective but it is not injective since:

Proposition A.1

$$v_A(\vec{x} \cdot a_i \cdot \prod_{k=|\vec{x}|+2}^{\infty} a_{|A|}) = v_A(\vec{x} \cdot a_{i+1} \cdot \prod_{k=|\vec{x}|+2}^{\infty} a_1) \quad \forall \vec{x} \in A^+, \forall i \in \{1, \dots, |A| - 1\} \quad (\text{A8})$$

Example A.1

Let us suppose that $A := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then we are taught from childhood that:

$$0.\bar{9} = 1 \quad (\text{A9})$$

$$0.1\bar{9} = 0.2 \quad (\text{A10})$$

$$0.2\bar{9} = 0.3 \quad (\text{A11})$$

and so on.

Let us then introduce the following:

Definition A.5

nonterminating symbolic representation with respect to A:

the map $r_A : [0, 1] \mapsto A^{\mathbb{N}^+}$:

$$r_A\left(\sum_{n=1}^{\infty} \frac{lex(x_n)}{|A|^n}\right) := \cdot_{n \in \mathbb{N}^+} x_n \quad (\text{A12})$$

with the nonterminating choice in the cases of the proposition A.1.

Given two finite alphabets A_1 and A_2 we can then introduce the following:

Definition A.6

alphabet's transition map from A_1 to A_2 :

the map $T_{A_1, A_2} : A_1^{\mathbb{N}^+} \mapsto A_2^{\mathbb{N}^+}$:

$$T_{A_1, A_2}(\bar{x}) = r_{A_2}[v_{A_1}(\bar{x})] \quad (\text{A13})$$

The structure of topological space that we have given to $A^{\mathbb{N}^+}$ is sufficient, as it is well known, to define limits and derivatives.

Given a smooth (id est infinitely differentiable) map $f : A^{\mathbb{N}^+} \mapsto \mathbb{C}$ and $n, m \in \mathbb{N}$:

Definition A.7

$$\|f\|_{n, m} := \sup_{\bar{x} \in A^{\mathbb{N}^+}} |(v_A(\bar{x}))^n \frac{d^m}{dx^m} f(v_A(\bar{x}))| \quad (\text{A14})$$

Definition A.8

Schwartz space of rapid decrease functions over $A^{\mathbb{N}^+}$:

$$\mathcal{S}(A^{\mathbb{N}^+}) := \{f : A^{\mathbb{N}^+} \mapsto \mathbb{C} \text{ smooth} : \|f\|_{n, m} < +\infty \forall n, m \in \mathbb{N}\} \quad (\text{A15})$$

The family of seminorms $\|f\|_{n, m}$ induces a natural topology over $\mathcal{S}(A^{\mathbb{N}^+})$ that can be used to introduce the following:

Definition A.9

space of tempered distributions over $A^{\mathbb{N}_+}$:

$$\mathcal{S}'(A^{\mathbb{N}_+}) := \text{the topological dual of } \mathcal{S}(A^{\mathbb{N}_+}).$$

Given $\bar{x} \in A^{\mathbb{N}_+}$:

Definition A.10

Dirac delta in \bar{x} :

$$\delta_{\bar{x}} := \text{the linear functional } \delta_{\bar{x}}(f) := f(\bar{x})$$

It may be easily proved that:

Proposition A.2

$$\delta_{\bar{x}} \in \mathcal{S}'(A^{\mathbb{N}_+}) \quad \forall \bar{x} \in A^{\mathbb{N}_+} \tag{A16}$$

In this paper we will adopt the usual notation:

$$\int_{A^{\mathbb{N}_+}} d\bar{x} f(\bar{x}) \delta(\bar{x} - \bar{y}) := \delta_{\bar{y}}(f) \tag{A17}$$

APPENDIX B: PISOT-VIJAYARAGHAVAN NUMBERS

Given $x \in \mathbb{C}$:

Definition B.1

x is algebraic:

$$\exists P \in \mathcal{P}[\mathbb{Q}] : P(x) = 0 \quad (\text{B1})$$

where we have denoted with $\mathcal{P}[\mathbb{K}]$ the linear space of all the monic polynomials with coefficients on the generic ring \mathbb{K} .

We will denote the set of all the algebraic numbers by \mathbb{A} .

Definition B.2

x is algebraic integer:

$$\exists P \in \mathcal{P}[\mathbb{Z}] : P(x) = 0 \quad (\text{B2})$$

We will denote the set of all the algebraic numbers by $\mathbb{A}_{\mathbb{Z}}$.

Given $n \in \mathbb{N}_+$ let us denote with $\mathcal{P}_n[\mathbb{K}]$ the set of all the monic polynomials of degree n with coefficients belonging to the generic ring \mathbb{K} .

Given $x \in \mathbb{A}$:

Definition B.3

degree of x :

$$\deg(x) := \min\{n \in \mathbb{N}_+ : P(x) = 0 \ P \in \mathcal{P}_n[\mathbb{Q}]\} \quad (\text{B3})$$

Definition B.4

minimal polynomial of x :

$$P_{x-\text{minimal}} \in \mathcal{P}_{\deg(x)}[\mathbb{Q}] : P_{x-\text{minimal}}(x) = 0 \quad (\text{B4})$$

Definition B.5

set of the conjugates of x :

$$\text{Con}(x) := \{y \in \mathbb{C} : y \neq x \wedge P_{x-\text{minimal}}(y) = 0\} \quad (\text{B5})$$

We can then state the following [72], [73] :

Proposition B.1

Fundamental property of algebraic integers:

HP:

$$x \in \mathbb{A}_{\mathbb{Z}}$$

TH:

$$x^n + \sum_{y \in \text{Con}(x)} y^n \in \mathbb{Z} \quad \forall n \in \mathbb{N}_+ : n \geq \deg(x)$$

Let us now finally introduce the main actor of this paper:

Definition B.6

set of the Pisot-Vijayaraghavan numbers:

$$PV(\mathbb{A}) := \{x \in \mathbb{A}_{\mathbb{Z}} : x > 1 \wedge (|y| < 1 \forall y \in \text{Con}(x))\} \quad (\text{B6})$$

Let us recall that given $a, b \in \mathbb{R}$:

Definition B.7

floor of a :

$$\lfloor a \rfloor := \max\{n \in \mathbb{N} : n \leq a\} \quad (\text{B7})$$

Definition B.8

$$\text{Mod}_b(a) := a - \lfloor \frac{a}{b} \rfloor b \quad (\text{B8})$$

Given $a \in \mathbb{R}_+$ let us endow the interval $[0, a)$ with the topology of the circle, id est by the topology induced by the following metric:

Definition B.9

S^1 -geodesic distance over $[0, a)$:

$$d_{\text{geodesic}}(x, y) := \begin{cases} |x - y|, & \text{if } |x - y| \geq \frac{a}{2}; \\ a - |x - y|, & \text{otherwise.} \end{cases} \quad (\text{B9})$$

Given a sequence $\{x_n\}_{n \in \mathbb{N}_+} \in [0, a)^{\mathbb{N}_+}$ and $y \in [0, a)$:

Definition B.10

convergence modulo a :

$$\lim_{n \rightarrow +\infty} x_n = y \pmod{a} := \forall \epsilon > 0, \exists N \in \mathbb{N} : (d_{\text{geodesic}}(x_n, y) < \epsilon \forall n \in \mathbb{N} : n > N) \quad (\text{B10})$$

Then [26], [74]:

Proposition B.2

Fundamental property of the Pisot-Vijayaraghavan numbers:

HP:

$$x \in PV(\mathbb{A}) \quad (\text{B11})$$

TH:

$$\lim_{n \rightarrow +\infty} \text{Mod}_1(x^n) = 0 \quad (\text{B12})$$

PROOF:

By the proposition B.1:

$$x^n + \sum_{y \in \text{Con}(x)} y^n \in \mathbb{Z} \quad \forall n \in \mathbb{N}_+ : n \geq \deg(x) \quad (\text{B13})$$

$$\text{Mod}_1(x^n + \sum_{y \in \text{Con}(x)} y^n) = 0 \quad \forall n \in \mathbb{N}_+ : n \geq \deg(x) \quad (\text{B14})$$

let us observe that since $x \in PV(\mathbb{A})$:

$$\sum_{y \in \text{Con}(x)} y^n = O_{n \rightarrow \infty}(\frac{1}{n}) \quad (\text{B15})$$

so that:

$$\text{Mod}_1[x^n + O_{n \rightarrow \infty}(\frac{1}{n})] = 0 \quad \forall n \in \mathbb{N}_+ : n \geq \deg(x) \quad (\text{B16})$$

from which the thesis follows ■

Another important property of Pisot–Vijayaraghavan numbers concerns their link with suitable linear recurrence numeric sequences, according to the following:

Definition B.11

linear recurrence numeric sequence:

$$\{f_n\}_{n \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} : (\exists d(f) \in \mathbb{N}_+, \exists c_1, \dots, c_{d(f)} \in \mathbb{Z}^{d(f)} : f_n = \sum_{k=1}^{d(f)} c_k f_{n-k} \quad \forall n \in \mathbb{N} : n \geq d(f)) \quad (\text{B17})$$

We will call the integer $d(f)$ the *degree* of the linear recurrence numeric sequence f .

Remark B.1

It is important to remark that a linear recurrence numeric sequence is completely defined by primitive recursion once the values of $f_0, \dots, f_{d(f)-1}$ have been assigned.

Then (see the 13th chapter "Pisot sequences, Boyd sequences and linear recurrence" of [26]):

Proposition B.3

Linear recurrence numeric sequence associated to a Pisot–Vijayaraghavan number:

HP:

$$\lambda \in PV(\mathbb{A}) \quad (\text{B18})$$

TH:

$$\exists \{f_n\}_{n \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}} \text{ linear recurrence numeric sequence} : \lim_{n \rightarrow +\infty} \frac{f_n}{f_{n-1}} = \lambda \quad (\text{B19})$$

Remark B.2

Let us remark that, in in general, there might exist many primitive recursive numeric sequences associated to a Pisot–Vijayaraghavan number.

Let us observe furthermore that:

Proposition B.4

HP:

$$\lambda \in PV(\mathbb{A}) : ord(\lambda) = 2$$

$\{f_n\}_{n \in \mathbb{N}} \in \mathbb{Z}^{\mathbb{N}}$ linear recurrence numeric sequence associated to λ

TH:

$$\exists \{c_n\} \in \mathbb{R}^{\mathbb{N}} : f_n = c_n \lambda^n + O_{n \rightarrow +\infty}(\frac{1}{n}) \wedge \lim_{n \rightarrow +\infty} \frac{c_n}{c_{n-1}} = 1 \quad (\text{B20})$$

PROOF:

Let $y \in Con(\lambda) : y \neq \lambda$ be the conjugate of λ .

Posed:

$$c_n := \frac{f_n}{(\lambda^n - y^n)} \quad (\text{B21})$$

one obtains that:

$$\frac{f_n}{f_{n-1}} = \frac{c_n \lambda^n + O_{n \rightarrow +\infty}(\frac{1}{n})}{c_{n-1} \lambda^{n-1} + O_{n \rightarrow +\infty}(\frac{1}{n})} \quad (\text{B22})$$

Using the fact that $\{f_n\}$ is associated to λ :

$$\lambda + O_{n \rightarrow +\infty}(\frac{1}{n}) = \frac{c_n}{c_{n-1}} \lambda \quad (\text{B23})$$

from which the thesis follows ■

Remark B.3

If the linear recurrence numeric sequence is of the form $f_n = m f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$ Binet has proved that the sequence c_n of the proposition B.4 is constant not only asymptotically:

$$c_n = \frac{1}{\sqrt{m^2 + 4}} \quad \forall n \in \mathbb{N} \quad (\text{B24})$$

Example B.1

Given the *golden number* $\tau := \frac{1+\sqrt{5}}{2}$:

$$P_{\tau-\text{minimal}}(x) = x^2 - x - 1 \quad (\text{B25})$$

and hence $P_{\tau-\text{minimal}} \in \mathcal{P}[\mathbb{Z}]$.

Furthermore $\tau > 1$ while:

$$Con(\tau) = \{-\frac{1}{\tau}\} \quad (B26)$$

where $|\frac{1}{\tau}| < 1$.

Hence $\tau \in PV(\mathbb{A})$.

The *golden number* is the least Pisot-Vijayaraghavan number of order two.

Furthermore it is the least limit point of Pisot-Vijayaraghavan numbers:

$$\tau = \min LP[PV(\mathbb{A})] \quad (B27)$$

A linear recurrence numeric sequence (of degree 2) associated to τ is the sequence of Fibonacci numbers:

$$F_{n+2} := F_{n+1} + F_n \quad (B28)$$

$$F_0 := 0 \quad (B29)$$

$$F_1 := 1 \quad (B30)$$

Example B.2

Given the *plastic number* $\rho := \frac{(9-\sqrt{69})^{\frac{1}{3}} + (9+\sqrt{69})^{\frac{1}{3}}}{2^{\frac{1}{3}}3^{\frac{2}{3}}}$:

$$P_{\rho-minimal}(x) = x^3 - x - 1 \quad (B31)$$

and hence $P_{\rho-minimal} \in \mathcal{P}[\mathbb{Z}]$.

Furthermore $\rho > 1$ while:

$$Con(\rho) = \left\{ \frac{i(i+\sqrt{3})(9-\sqrt{69})^{\frac{1}{3}} + (-1-i\sqrt{3})(9+\sqrt{69})^{\frac{1}{3}}}{2 \cdot 2^{\frac{1}{3}}3^{\frac{2}{3}}}, \frac{(-1-i\sqrt{3})(9-\sqrt{69})^{\frac{1}{3}} + i(i+\sqrt{3})(9+\sqrt{69})^{\frac{1}{3}}}{2 \cdot 2^{\frac{1}{3}}3^{\frac{2}{3}}} \right\} \quad (B32)$$

with $|\frac{i(i+\sqrt{3})(9-\sqrt{69})^{\frac{1}{3}} + (-1-i\sqrt{3})(9+\sqrt{69})^{\frac{1}{3}}}{2 \cdot 2^{\frac{1}{3}}3^{\frac{2}{3}}}| < 1$ and $|\frac{(-1-i\sqrt{3})(9-\sqrt{69})^{\frac{1}{3}} + i(i+\sqrt{3})(9+\sqrt{69})^{\frac{1}{3}}}{2 \cdot 2^{\frac{1}{3}}3^{\frac{2}{3}}}| < 1$.

Hence $\rho \in PV(\mathbb{A})$.

The *plastic number* is the least Pisot-Vijayaraghavan number.

Furthermore, introduced the notion of the *height* $h(x)$ of an algebraic number $x \in \mathbb{A}$ (a sort of measure of the algebraic complication needed to describe x for whose definition and properties we demand to the 3th chapter "Pisot and Salem Numbers" of [75], to the section 3.5 "Smyth's theorem" of [26] and to the 4th chapter "Small points" of [76]), the plastic number appears in the following:

Theorem B.1

Smith Theorem:

HP:

$$x \in \mathbb{A}_{\mathbb{Z}} : x \neq 0 \wedge \frac{1}{x} \notin Con(x) \quad (B33)$$

TH:

$$h(x) \geq \frac{\log \rho}{\deg(x)} \quad (\text{B34})$$

A linear recurrence numeric sequence (of degree 3) associated to ρ is the sequence of the Padovan numbers:

$$P(0) := P(1) := P(2) := 1 \quad (\text{B35})$$

$$P_n := P_{n-2} + P_{n-3} \quad (\text{B36})$$

Example B.3

The number $1 + \sqrt{2} > 1$ has as minimal polynomial:

$$P_{1+\sqrt{2}-\text{mimimal}}(x) = x^2 - 2x - 1 \quad (\text{B37})$$

and is hence an algebraic integer.

Furthermore:

$$\text{Con}(1 + \sqrt{2}) = \{1 - \sqrt{2}\} \quad (\text{B38})$$

where obviously:

$$|1 - \sqrt{2}| < 1 \quad (\text{B39})$$

Hence $1 + \sqrt{2} \in PV(\mathbb{A})$.

A linear recurrence numeric sequence (of degree 2) associated to $1 + \sqrt{2}$ is the sequence of Pell numbers:

$$a_0 := 0 \quad (\text{B40})$$

$$a_1 := 1 \quad (\text{B41})$$

$$a_n := 2a_{n-1} + a_{n-2} \quad \forall n \in \mathbb{N} : n \geq 2 \quad (\text{B42})$$

We want to conclude this Number Theoretic section with a celebrated well known result, the recursive undecidability of Hilbert 10th problem showing its deep link with Theoretical Computer Science [8], [9].

Given a set $S \subseteq \mathbb{N}$:

Definition B.12

S is Diophantine:

$$\exists P \in \mathcal{P}[\mathbb{Z}] : S = \{x \in \mathbb{N} : P(x) = 0\} \quad (\text{B43})$$

Theorem B.2

Matiyasevich Theorem:

HP:

$$S \subseteq \mathbb{N} \quad (\text{B44})$$

TH:

S is Diophantine if and only if it is recursively enumerable

Theorem B.2 implies that it doesn't exist an algorithm that, receiving as input a polynomial $P \in \mathcal{P}(\mathbb{Z})$ outputs "yes" if it has non-negative integer solutions while it outputs "no" if it doesn't.

APPENDIX C: A BRIEF INFORMATION THEORETIC ANALYSIS OF SINGULAR LEBESGUE-STIELTJES MEASURES SUPPORTED ON CANTOR SETS AND ALMOST PERIODICITY

Demanding to [77], [78], [79] for any further information let us recall that given a continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ non decreasing, id est such that:

$$f(x) \geq f(y) \quad \forall x, y \in \mathbb{R} : x > y \quad (\text{C1})$$

we can introduce the following:

Definition C.1

Lebesgue-Stieltjes measure associated to the f :
the measure $\mu_f : \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty)$ such that:

$$\mu_f[(a, b)] := \lim_{\epsilon \rightarrow 0^-} f(b - \epsilon) - \lim_{\epsilon \rightarrow 0^+} f(a + \epsilon) \quad \forall a, b \in \mathbb{R} : a < b \quad (\text{C2})$$

where $\mathcal{B}(\mathbb{R})$ is the Borel- σ -algebra on \mathbb{R} , id est the σ -algebra induced by the topology induced by the usual metric $d(x, y) := |x - y|$.

For instance, choosing the identity function $Id : \mathbb{R} \rightarrow \mathbb{R}$:

$$Id(x) := x \quad (\text{C3})$$

the Lebesgue-Stieltjes measure μ_{Id} is nothing but the Lebesgue measure on numbers $v_A(\mu_{Lebesgue})$ (where $\mu_{Lebesgue}$ is the Lebesgue measure on sequences of the definition A.3).

Given a measure $\mu : \mathcal{B} \rightarrow [0, +\infty)$ let us recall that:

Definition C.2

set of pure points of μ :

$$PP(\mu) := \{x \in \mathbb{R} : \mu(\{x\}) \neq 0\} \quad (\text{C4})$$

Definition C.3

μ is pure point:

$$\mu(X) = \sum_{x \in X} \mu(\{x\}) \quad \forall X \in \mathcal{B}(\mathbb{R}) \quad (\text{C5})$$

Definition C.4

μ is absolutely continuous (with respect to the Lebesgue measure):

$$\mu \prec \mu_{Id} := \mu(B) = 0 \Rightarrow \mu_{Id}(B) = 0 \quad (\text{C6})$$

Let us recall the basic:

Theorem C.1

Radon-Nikodym Theorem:
HP:

$$\mu \prec \mu_{Id}$$

TH:

There exists a μ_{Id} -measurable function, called the Radon-Nikodym derivative of μ with respect to μ_{Id} and consequentially denoted as $\frac{d\mu}{d\mu_{Id}}$, such that $\int_{\mathbb{R}} f(x) d\mu(x) = \int_{\mathbb{R}} f(x) \frac{d\mu}{d\mu_{Id}}(x) d\mu_{Id}(x)$ for every μ -measurable function f

Definition C.5

μ is singular (with respect to the Lebesgue measure):

$$\mu \perp \mu_{Id} := \exists S : \mu(S) = 0 \wedge \mu_{Id}(\mathbb{R} - S) = 0 \quad (C7)$$

Let us then recall the following:

Theorem C.2

Lebesgue Decomposition Theorem:

HP:

$$\mu : \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty) \text{ measure}$$

TH:

$$\exists!(\mu_{p.p}, \mu_{a.c.}, \mu_{sing}) \text{ triple of measures} : \mu_{p.p} \text{ is pure point} \wedge \mu \prec \mu_{a.c.} \wedge \mu \perp \mu_{sing}$$

Given a map $f : [a, b] \rightarrow \mathbb{R}$:

Definition C.6

f is absolutely continuous:

$$\forall \epsilon > 0, \exists \delta > 0 : \sum_{i=1}^n |f(b_i) - f(a_i)| \leq \epsilon$$

$$\forall \{(a_i, b_i)\}_{i=1}^n : (a_i, b_i) \subset (a, b) \wedge (a_i, b_i) \cap (a_j, b_j) = \emptyset \wedge \sum_{i=1}^n \mu_{Id}[(a_i, b_i)] < \delta \forall n \in \mathbb{N}_+ \quad (C8)$$

A remarkable feature of Lebesgue-Stieltjes measures is the following:

Theorem C.3

On the singularity of Lebesgue-Stieltjes measures:

HP:

$$\mu_f \text{ Lebesgue-Stieltjes measure}$$

TH:

$$\mu_f \text{ is an absolutely continuous measure if and only if } f \text{ is an absolutely continuous function}$$

Let us now show how Algorithmic Information Theory enters in the game.

Given a trinary alphabet $A := \{a_1, a_2, a_3\}$ let us consider the set of all sequences over A not containing the letter a_2 :

$$NO_{A, a_2} := \{\bar{x} \in A^{\mathbb{N}^+} : |\bar{x}|_{a_2} = 0\} \quad (C9)$$

As we have already observed in the remark III.4, the theorem III.4 implies that:

$$NO_{A, a_2} \cap RANDOM(A^{\mathbb{N}^+}) = \emptyset \quad (C10)$$

Let us now consider the binary alphabet $B := \{a_1, a_3\}$.

Clearly:

$$NO_{A,a_2} = B^{\mathbb{N}_+} \supset RANDOM(B^{\mathbb{N}_+}) \quad (C11)$$

At a first sight the equation C10 and the equation C11 might appear incompatible.

A deeper analysis allows, anyway, to understand that everything is absolutely consistent:

simply we have to remember that the correct way of passing from the alphabet A to the alphabet B is through the transition map $T_{A,B}$ of the definition A.6.

Actually the theorem III.3 guarantees that:

$$RANDOM(B^{\mathbb{N}_+}) = T_{A,B}[RANDOM(A^{\mathbb{N}_+})] \quad (C12)$$

Remark C.1

In a very interesting paper [80] that has greatly inspired this section, Cristian Calude, Ludwig Staiger and Karl Svozil have introduced a generalization of the notion of algorithmic randomness, id est of the definition III.24, to sequences $\{x_n\}$ each letter of which belongs to a different alphabet A_n . The analysis of the probably deep link existing between such a generalization and what we are going to discuss here is something that, at present, we are not able to formalize and is left as an interesting challenge for the reader.

Let us now introduce the set:

Definition C.7

Cantor's middle third set:

$$\mathcal{C}_{A,a_2} := v_A(NO_{A,a_2}) \quad (C13)$$

It may be easily proved that:

Proposition C.1

1.

$$\mu_{Id}(\mathcal{C}_{A,a_2}) = 0 \quad (C14)$$

2.

$$|\mathcal{C}_{A,a_2}| > \aleph_0 \quad (C15)$$

3.

$$\dim_{Hausdorff}(\mathcal{C}_{A,a_2}) = \frac{\log |B|}{\log |A|} \quad (C16)$$

Given $n \in \mathbb{N}_+$ let us introduce the following:

Definition C.8

Cantor function:

the map $f_{A,a} : B^+ \mapsto \mathbb{R}$:

$$f_{A,a}(\vec{x}) := v_B(\vec{x}) + \frac{|B|}{|B|^{\|\vec{x}\|+1}} \forall \vec{x} \in B^+ : v_B(\vec{x}) \in (v_A(\vec{x}) + \frac{1}{|A|^{\|\vec{x}\|+1}}, v_A(\vec{x}) + \frac{|B|}{|A|^{\|\vec{x}\|+1}}) \quad (C17)$$

It may be easily proved that the map $\tilde{f} : [0, 1] \mapsto \mathbb{R}$:

$$\tilde{f}(x) := f(r_B(x)) \quad (C18)$$

is a continuous non-decreasing function so that it induces a Lebesgue-Stieltjes measure $\mu_{\tilde{f}}$.

Since \tilde{f} is not absolutely continuous it follows that $\mu_{\tilde{f}}$ is singular with respect to the Lebesgue measure; actually:

$$\mu_{\tilde{f}}[\mathcal{C}_{A,a_2}] = 1 \quad (\text{C19})$$

while we know by the proposition C.1 that \mathcal{C}_{A,a_2} has vanishing Lebesgue measure.

Let us remark that \mathcal{C}_{A,a_2} is a Cantor set in the sense of the definition A.2.

Now the measure $\mu_{\tilde{f}} : \mathcal{B}[[0, 1]] \mapsto [0, +\infty)$ induces naturally the measure $\mu_f : \mathcal{B}[A^{\mathbb{N}^+}] \mapsto [0, +\infty)$ defined by:

$$\mu_f(X) := \mu_{\tilde{f}}(v_A(X)) \quad (\text{C20})$$

Then:

$$\mu_{\text{Lebesgue}, B}[NO_{A,a_2}] = \mu_{\text{Lebesgue}, B}[B^{\mathbb{N}^+}] = 1 \quad (\text{C21})$$

$$\mu_{\text{Lebesgue}, B}[NO_{A,a_2} \cap \text{RANDOM}(B^{\mathbb{N}^+})] = \mu_{\text{Lebesgue}, B}[\text{RANDOM}(B^{\mathbb{N}^+})] = 1 \quad (\text{C22})$$

$$\mu_{\text{Lebesgue}, A}[NO_{A,a_2}] = 0 \quad (\text{C23})$$

$$\mu_f[NO_{A,a_2}] = 1 \quad (\text{C24})$$

where we have used the theorem A.1.

The whole story may be easily generalized in the following way:

given a finite alphabet $A : 2 \leq |A| < \aleph_0$ and a letter $a \in A$ let us introduce the set of all the sequences on A not containing the digit a :

$$NO_{A,a} := \{\bar{x} \in A^{\mathbb{N}^+} : |\bar{x}|_a = 0\} \quad (\text{C25})$$

and the new alphabet B obtained from A excluding the letter a :

$$B := A - \{a\} \quad (\text{C26})$$

Then one can introduce the set:

Definition C.9

generalized Cantor set associated to (A, a) :

$$\mathcal{C}_{A,a} := v_A(NO_{A,a}) \quad (\text{C27})$$

Given $n \in \mathbb{N}_+$ let us introduce the following:

Definition C.10

generalized Cantor function associated to (A, a) :

the map $f_{A,a} : B^+ \mapsto \mathbb{R}$:

$$f_{A,a}(\vec{x}) := v_B(\vec{x}) + \frac{|B|}{|B|^{|\vec{x}|+1}} \forall \vec{x} \in B^+ : v_B(\vec{x}) \in (v_A(\vec{x}) + \frac{1}{|A|^{|\vec{x}|+1}}, v_A(\vec{x}) + \frac{|B|}{|A|^{|\vec{x}|+1}}) \quad (\text{C28})$$

and the associated map $\tilde{f} : [0, 1] \mapsto \mathbb{R}$:

$$\tilde{f}_{A,a}(x) := f(r_B(x)) \quad (\text{C29})$$

Then:

Proposition C.2

Properties of the generalized Cantor functions:

HP:

$$0 < lex(a) < |A| - 1 \quad (C30)$$

TH:

1. $\tilde{f}_{A,a}$ is a continuous non decreasing function
2. $\tilde{f}_{A,a}$ is not absolutely continuous

PROOF:

It is sufficient to follow step by step the analogous proof holding in the particular case of the Cantor middle-third set and generalize in the natural way ■

The proposition C.2 implies that one can introduce the Lebesgue-Stieltjes measure $\mu_{\tilde{f}}$. Then:

Corollary C.1

$\mu_{\tilde{f}_{A,a}}$ is singular

PROOF:

The thesis follows by the proposition C.3 and the proposition C.2. ■

Furthermore:

$$\mu_{Lebesgue,B}[NO_{A,a}] = \mu_{Lebesgue,B}[B^{\mathbb{N}_+}] = 1 \quad (C31)$$

$$\mu_{Lebesgue,B}[NO_{A,a} \cap RANDOM(B^{\mathbb{N}_+})] = \mu_{Lebesgue,B}[RANDOM(B^{\mathbb{N}_+})] = 1 \quad (C32)$$

$$\mu_{Lebesgue,A}[NO_{A,a}] = 0 \quad (C33)$$

$$\mu_f[NO_{A,a}] = 1 \quad (C34)$$

We will now briefly outline the deep link existing between the singular Lebesgue-Stieltjes measures associated to generalized Cantor functions and the theory of almost periodic functions, sequences and measures [81], [82], [83].

Given a map $f : \mathbb{R} \mapsto \mathbb{C}$:

Definition C.11

f is periodic:

$$\exists T \in (0, +\infty) : f(t+T) = f(t) \quad \forall t \in \mathbb{R} \quad (C35)$$

Given a periodic function f:

Definition C.12

fundamental period of f:

$$Period_{fund}[f] := \min\{T \in (0, +\infty) : f(t+T) = f(t) \quad \forall t \in \mathbb{R}\} \quad (C36)$$

We will denote by $PER_T(\mathbb{R}, \mathbb{C})$ the set of all the periodic functions.

Remark C.2

In the section III we saw that the *combinatorial information* and the *algorithmic information* of a periodic sequence (according to the definition III.7) are very low since a periodic sequence is completely specified assigning the values it takes over a single period.

The same thing can clearly be said of periodic functions.

Clearly:

Proposition C.3

$$PER_T(\mathbb{R}, \mathbb{C}) \text{ is a complex linear space } \forall T \in (0, +\infty) \quad (C37)$$

Given $n \in \mathbb{N}_+$, $\omega_1, \dots, \omega_n \in (0, +\infty)$ and a map $f : \mathbb{R} \mapsto \mathbb{C}$:

Definition C.13

f is quasiperiodic with pulsations $\omega_1, \dots, \omega_n$:

$$(\exists \phi : \mathbb{R}^n \mapsto \mathbb{C} : \phi(x_1, \dots, x_i + 2\pi, \dots, x_n) = \phi(x_1, \dots, x_i, \dots, x_n) \forall i \in \{1, \dots, n\}) \wedge (f(t) = \phi(\omega_1 t, \dots, \omega_n t) \forall t \in \mathbb{R}) \quad (C38)$$

Definition C.14

$\omega_1, \dots, \omega_n$ are rationally independent:

$$\left(\sum_{i=1}^n k_i \omega_i = 0 \Rightarrow k_1 = \dots = k_n = 0 \right) \forall k_1, \dots, k_n \in \mathbb{Q} \quad (C39)$$

Then:

Proposition C.4

On the relation between periodicity and quasiperiodicity:

1. f is periodic $\Rightarrow f$ is quasiperiodic
2. f is periodic $\Leftrightarrow f$ is quasiperiodic with rationally dependent pulsations

Definition C.15

f is a complex trigonometric polynomial:

$$\exists n \in \mathbb{N}_+, \exists A_1, \dots, A_n \in \mathbb{C}, \exists \omega_1, \dots, \omega_n \in (0, +\infty) : f(t) = \sum_{k=1}^n A_k \exp(i\omega_k t) \quad \forall t \in \mathbb{R} \quad (C40)$$

Proposition C.5

$$f \text{ complex trigonometric polynomial } \Rightarrow f \text{ is quasi-periodic} \quad (C41)$$

PROOF:

Given the complex trigonometric polynomial:

$$f(t) := \sum_{k=1}^n A_k \exp(i\omega_k t) \quad (C42)$$

let us introduce the following map $\phi : \mathbb{R}^n \mapsto \mathbb{C}$:

$$\phi(x_1, \dots, x_n) := \sum_{k=1}^n A_k \exp(ix_k) \quad (C43)$$

Obviously:

$$\phi(x_1 + 2\pi, \dots, x_n) = \dots = \phi(x_1, \dots, x_n + 2\pi) \quad \forall x_1, \dots, x_n \in \mathbb{R} \quad (\text{C44})$$

and:

$$f(t) = \phi(\omega_1 t, \dots, \omega_n t) \quad \forall t \in \mathbb{R} \quad (\text{C45})$$

■

Let us now introduce Harald Bohr's notion of almost periodic function.

Definition C.16

almost periodic function:

a map $f : \mathbb{R} \mapsto \mathbb{C}$:

$$\forall \epsilon > 0 \exists T_\epsilon \text{ complex trigonometric polynomial} : |f(x) - T_\epsilon(x)| < \epsilon \quad \forall x \in \mathbb{R} \quad (\text{C46})$$

We will denote by $A - PER(\mathbb{R}, \mathbb{C})$ the set of all the almost periodic functions.

To appreciate the meaning of the definition C.16 let us recall that [77] given a metric space (M, d) and a set $S \subset M$:

Definition C.17

S is dense in (M, d) :

$$\forall y \in S \exists \{x_n\}_{n \in \mathbb{N}} \in M^{\mathbb{N}} : \lim_{n \rightarrow +\infty} d(x_n, y) = 0 \quad (\text{C47})$$

Then definition C.16 implies that:

Proposition C.6

{ complex trigonometric polynomials } is dense in the metric space $(A - PER(\mathbb{R}, \mathbb{C}), d_1)$

where:

$$d_1(f, g) := \sup_{t \in \mathbb{R}} |f(t) - g(t)| \quad (\text{C48})$$

Furthermore:

Proposition C.7

On the relation between quasiperiodicity and almost periodicity:

1. f is quasiperiodic $\Rightarrow f$ is almost periodic
2. f is almost periodic $\nRightarrow f$ is quasiperiodic

The basic properties of almost periodic functions are encoded in the following:

Proposition C.8

Basic properties of almost periodic functions:

- 1.

$$f \text{ is almost periodic} \Leftrightarrow \forall \epsilon \in (0, +\infty) \exists l(\epsilon) \in (0, +\infty) : \\ (\forall a \in \mathbb{R}, \exists \xi \in \mathbb{R} : |f(x + \xi) - f(x)| < \epsilon \quad \forall x \in (a, a + l(\epsilon))) \quad (\text{C49})$$

2. f complex trigonometric polynomial $\Rightarrow f$ almost periodic
3. $A - PER(\mathbb{R}, \mathbb{C})$ is a complex linear space

4.

$$f_1 \cdot f_2 \text{ is almost periodic } \forall f_1, f_2 \text{ almost periodic} \quad (\text{C50})$$

5.

$$\frac{d}{da} \lim_{T \rightarrow +\infty} \int_a^{a+T} f(x) dx = 0 \quad (\text{C51})$$

Given an almost periodic function f :

Definition C.18

mean value of f :

$$M[f] := \lim_{T \rightarrow +\infty} \int_a^{a+T} f(x) dx \quad (\text{C52})$$

Then:

Proposition C.9

Basic property of the mean value of an almost periodic function:

$$\text{support}_\omega M[\exp(-i\omega x)f(x)] \leq \aleph_0 \quad (\text{C53})$$

Given a generic function $f : \mathbb{R} \mapsto \mathbb{C}$:

Definition C.19

Fourier transform of f :

$$\mathcal{F}[f](\omega) := \begin{cases} \lim_{T \rightarrow +\infty} \int_{-T}^{+T} d\omega \exp(-i\omega x) f(x), & \text{if the limit exists;} \\ +\infty, & \text{otherwise} \end{cases} \quad (\text{C54})$$

The proposition C.9 implies that:

Proposition C.10

Fourier spectrum of almost periodic functions:

HP:

f almost periodic function

TH:

1.

$$\exists \{A_n\} \in \mathbb{C}^{\mathbb{Z}}, \exists \{\omega_n\} \in (0, +\infty)^{\mathbb{Z}} : \mathcal{F}[f](\omega) = \sum_{n=-\infty}^{+\infty} A_n \delta(\omega - \omega_n) \quad (\text{C55})$$

2. if f is periodic of fundamental period $\frac{2\pi}{\omega}$ then $\omega_n = n\omega \forall n \in \mathbb{Z}$

PROOF:

1. The thesis immediately follows combining the definition C.19 and the proposition C.9.
2. The thesis is nothing but the basic theorem concerning Fourier series of periodic functions

■

As to the existence of the Fourier transform of almost periodic functions there exists a very powerful result [82]:

Proposition C.11

About the specification of an almost periodic function through its Fourier transform:

HP:

$$\{\omega_n\} \in (0, +\infty)^{\mathbb{Z}}$$

$$\{A_n\} \in \mathbb{C}^{\mathbb{Z}} : \sum_{n=-\infty}^{+\infty} |A_n| \in \mathbb{R} \quad (\text{C56})$$

TH:

$$\exists f \in A - PER(\mathbb{R}, \mathbb{C}) : \mathcal{F}[f](\omega) = \sum_{n=-\infty}^{+\infty} A_n \delta(\omega - \omega_n)$$

Let us now consider a generic measure $\mu : \mathcal{B}(\mathbb{R}) \mapsto [0, +\infty)$ on the real line and let us introduce the following:

Definition C.20

Fourier transform of μ :

$$\mathcal{F}[\mu](\omega) := \int_{\mathbb{R}} e^{-i\omega x} d\mu(x) \quad (\text{C57})$$

Obviously:

Proposition C.12

HP:

$$\mu \prec \mu_{Id}$$

TH:

$$\mathcal{F}[\mu] = \mathcal{F}\left[\frac{d\mu}{d\mu_{Id}}\right] \quad (\text{C58})$$

where the Fourier transform of the left hand side is intended in the sense of the definition C.20 while the Fourier transform of the right hand side is intended in the sense of the definition C.19.

PROOF:

It is sufficient to observe that:

$$\mathcal{F}[\mu] = \int_{\mathbb{R}} e^{-i\omega x} \frac{d\mu}{d\mu_{Id}}(x) d\mu_{Id}(x) = \int_{-\infty}^{+\infty} e^{-i\omega x} \frac{d\mu}{d\mu_{Id}}(x) dx = \mathcal{F}\left[\frac{d\mu}{d\mu_{Id}}\right] \quad (\text{C59})$$

■

The situation is, instead, more complicated in the case in which the measure μ is singular with respect to the Lebesgue measure $d\mu_{Id}$.

Let us consider in particular the case in which $\mu = \mu_{\tilde{f}_{A,a}}$ is the singular measure associated to a generalized Cantor function $f_{A,a}$.

Though we have not yet a proof, we think that Eberlein's Theorem (see the 11th chapter "Almost periodicity of the generalized Fourier transform" of [83]) strongly supports the following:

Conjecture C.1

About the Fourier transform of the Lebesgue-Stieltjes measure associated to a generalized Cantor function:

$$\mathcal{F}[\mu_{\tilde{f}_{A,a}}] \text{ is almost periodic} \quad (\text{C60})$$

APPENDIX D: QUASICRYSTALS

Given $n \in \mathbb{N}_+$ let us introduce the following:

Definition D.1

n-dimensional euclidean space:

the Hilbert space $\mathbb{E}^n := (\mathbb{R}^n, \cdot)$, where $\vec{x} \cdot \vec{y} := \sum_{i=1}^n x_i y_i$ is the usual euclidean inner product.

Given $\vec{x} \in \mathbb{E}^n$ and $r \in \mathbb{R}_+$:

Definition D.2

ball with center \vec{x} and radius r :

$$B_r(\vec{x}) := \{\vec{y} \in \mathbb{E}^n : |\vec{y} - \vec{x}| < r\} \quad (\text{D1})$$

Given $S \subset \mathbb{E}^n$ let us introduce the following [7]:

Definition D.3

S is a Delone set in \mathbb{E}^n :

1.

$$\exists r_0 \in \mathbb{R}_+ : |\vec{x} - \vec{y}| > 2r_0 \quad \forall \vec{x}, \vec{y} \in S \quad \vec{x} \neq \vec{y}$$

2.

$$\exists R_0 \in \mathbb{R}_+ : B_r(\vec{x}) \cap S \neq \emptyset \quad \forall r > R_0, \forall \vec{x} \in \mathbb{E}^n \quad (\text{D2})$$

Following the generalization of the mathematical definition of a crystal given in [7], [24], [25] on the score of the indications given in 1992 by the Commission on Nonperiodic Crystals established by the International Unit of Crystallography, we will characterize a crystal by the condition that its diffraction pattern exhibits a countable infinity of Bragg peaks.

Given a Delone set S:

Definition D.4

density distribution of S:

$$\rho_S(\vec{x}) := \sum_{\vec{s} \in S} \delta(\vec{x} - \vec{s})$$

Definition D.5

autocorrelation of S:

$$\gamma_S(\vec{x}) := \rho(\vec{x}) \star \overline{\rho(-\vec{x})} \quad (\text{D3})$$

where \star denotes the *convolution operator*:

$$(f \star g)(\vec{x}) := \int_{\mathbb{R}^n} d\vec{y} f(\vec{x}) g(\vec{x} - \vec{y}) \quad (\text{D4})$$

Definition D.6

spectrum of S:

the measure μ_S on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ identified by $\lim_{L \rightarrow +\infty} \frac{1}{(2L)^n} \mathcal{F}[\gamma_{S \cap [-L, L]^n}(\vec{x})]$

where $\mathcal{B}(\mathbb{R}^n)$ is the Borel- σ -algebra of \mathbb{R}^n and where \mathcal{F} denotes the Fourier transform [77], [84].

By the Lebesgue Decomposition Theorem (i.e. the straightforward multidimensional generalization of the theorem C.2) we know that μ_S may be decomposed uniquely as:

$$\mu_S = \mu_S^{(p.p.)} + \mu_S^{(a.c.)} + \mu_S^{(sing)} \quad (\text{D5})$$

where $\mu_S^{(p.p.)}$ is a *pure point measure*, $\mu_S^{(a.c.)}$ is *absolutely continuous* w.r.t. the Lebesgue measure $\mu_{Lebesgue}$ while $\mu_S^{(sing)}$ is *singular* w.r.t. $\mu_{Lebesgue}$.

The mathematical characterization of crystals involves only $\mu_S^{(p.p.)}$; introduced the following [77]:

Definition D.7

set of pure points of μ_S :

$$PP(\mu_S) := \{\vec{x} \in \mathbb{R}^n : \mu_S(\{\vec{x}\}) \neq 0\} = \{\vec{x} \in \mathbb{R}^n : \mu_S^{(p.p.)}(\{\vec{x}\}) \neq 0\} \quad (\text{D6})$$

we can at last define:

Definition D.8

S is a crystal:

$$|PP(\mu_S)| = \aleph_0 \quad (\text{D7})$$

Remark D.1

It is claimed in the book review [85] of [7] that the definition D.6 is not mathematically rigorous.

It should be noticed, with this regard, that the author of such a criticism hasn't proposed [86] an, according to him, better definition.

Demanding to [24] for further details let us observe that since for every Delone set S:

$$\frac{1}{(2L)^n} \mathcal{F}[\gamma_{S \cap [-L, L]^n}(\vec{x})] \in \mathcal{S}'(\mathbb{R}^n) \quad \forall L \in \mathbb{R}_+ \quad (\text{D8})$$

its limit for $L \rightarrow \infty$ is perfectly well defined with respect to the natural topology of the Schwartz space of tempered distributions [77].

As to definition D.8 it is important to remind the double nature of the Dirac delta as a tempered distribution and as a Lebesgue-Stieltjes measure.

Let us recall that given $k \in \mathbb{N}_+$ vectors $\vec{a}_1, \dots, \vec{a}_k \in \mathbb{R}^n$:

Definition D.9

\mathbb{Z} -module generated by $\vec{a}_1, \dots, \vec{a}_k$:

$$\Gamma(\vec{a}_1, \dots, \vec{a}_k) := \left\{ \sum_{i=1}^k m_i \vec{a}_i \mid m_i \in \mathbb{Z} \ i = 1, \dots, k \right\} \quad (\text{D9})$$

Definition D.10

lattice in \mathbb{E}^n :

a \mathbb{Z} -module generated by a basis of \mathbb{E}^n

Let us now consider the particular case in which S is a lattice $\Gamma(\vec{a}_1, \dots, \vec{a}_n)$ (where hence $\{\vec{a}_1, \dots, \vec{a}_n\}$ is a basis of \mathbb{E}^n) and let us introduce the following:

Definition D.11

dual lattice of $\Gamma(\vec{a}_1, \dots, \vec{a}_n)$:

$$\Gamma'(\vec{a}_1, \dots, \vec{a}_n) := \Gamma(\vec{a}'_1, \dots, \vec{a}'_n)$$

where $\{\vec{a}'_1, \dots, \vec{a}'_n\}$ is called the basis dual to the basis $\{\vec{a}_1, \dots, \vec{a}_n\}$ and is defined by the condition:

$$\vec{a}_i \cdot \vec{a}'_j := \delta_{i,j} \quad i, j = 1, \dots, n \quad (\text{D10})$$

A corner stone of Mathematical Crystallography (both classical and quasi) is the following [7]:

Theorem D.1

Poisson's summation formula:

$$\mathcal{F}(\rho_{\Gamma(\vec{a}_1, \dots, \vec{a}_n)}) = \rho_{\Gamma'(\vec{a}_1, \dots, \vec{a}_n)} \quad (\text{D11})$$

from which it follows that:

Corollary D.1

$\Gamma(\vec{a}_1, \dots, \vec{a}_n)$ is a crystal

The problem of characterizing which nonperiodic Delone sets are crystals is an extremely high and steep mountain that has been challenged by some of the best minds in the scientific community (see e.g. [22] and [23]).

Let us now introduce the following:

Definition D.12

isometry in \mathbb{E}^n :

a map $\phi : \mathbb{E}^n \mapsto \mathbb{E}^n$ such that:

$$|\phi(\vec{x} - \vec{y})| = |\vec{x} - \vec{y}| \quad \forall \vec{x}, \vec{y} \in \mathbb{E}^n$$

The isometries of \mathbb{E}^n form a group that we will denote by $Is(\mathbb{E}^n)$.

Given a Delone set S :

Definition D.13

symmetry group of S :

$$Sim(S) := \{\phi \in Is(\mathbb{E}^n) : \phi \circ \mu_S = \mu_S\} \quad (\text{D12})$$

Given a group G and an element $g \in G$:

Definition D.14

order of g :

$$ord(g) := \min\{n \in \mathbb{N}_+ : g^n = 1\}$$

Definition D.15

Hiller's function:

$$Hil : \mathbb{N}_+ \mapsto \mathbb{N} : Hil(k) := \min\{n \in \mathbb{N} : g \in GL(n, \mathbb{Z}) \wedge ord(g) = k\}$$

Introduced the well-known:

Definition D.16

Euler's ϕ function:

$$\phi : \mathbb{N}_+ \mapsto \mathbb{N}_+ : \phi(n) := |\{k \in \mathbb{N}_+ : k < n \wedge gcd(k, n) = 1\}|$$

then [87]:

Theorem D.2

Hiller's Theorem:

$$Hil(\prod_{n=1}^{\infty} p_n^{\alpha_n}) = \sum_{\{n \in \mathbb{N}_+ : p_n^{\alpha_n} \neq 2\}} \phi(p_n^{\alpha_n}) \quad \forall \{\alpha_n \in \mathbb{N}\}_{n \in \mathbb{N}} : \exists N \in \mathbb{N} : \alpha_n = 0 \quad \forall n > N$$

where p_n is the n^{th} prime number.

The *Hiller function* is algorithmically implemented in the section E.

It may be easily verified that:

n	$Hil(n)$
1	0
2	0
3	2
4	2
5	4
6	2
7	6
8	4
9	6
10	4
11	10
12	4
13	12
14	6
15	6
16	8
17	16
18	6
19	18
20	6
21	8
22	10
23	22
24	6
25	20
26	12
27	18
28	8
29	28
30	6
31	30
32	16
33	12
34	16
35	10
36	8

Remark D.2

Let us remark that the constraint $p_n^{\alpha_n} \neq 2$ in theorem D.2 is essential as it is proved in [87] where, contrary to [7], such a theorem is reported correctly.

It can be easily verified that:

Corollary D.2

$$Hil(n) \leq 3 \Leftrightarrow n \in \{1, 2, 3, 4, 6\}$$

Given a Delone set S in \mathbb{E}^n :

Proposition D.1

Crystallographic restriction:

$$S \text{ is a lattice} \Rightarrow Hil(ord(g)) \leq n \quad \forall g \in Sim(S)$$

PROOF:

The thesis follows by the same definition D.15 of Hiller's function. ■

Definition D.17

S is a quasicrystal:

$$S \text{ is a crystal} \wedge \exists g \in Sim(S) : Hil(ord(g)) > n$$

Example D.1

By the corollary D.2 n -fold rotational symmetry axes, i.e. axes around which there is rotational symmetry for rotations by $\frac{2\pi}{n}$, can exist in two and three dimensions if and only if $n \in \{2, 3, 4, 6\}$. Hence a bidimensional or three-dimensional crystal exhibiting an n -fold rotational symmetry axis with $n \notin \{2, 3, 4, 6\}$ is a quasicrystal.

Example D.2

Passing from three-dimension to four-dimension the only symmetries that become allowed are those of order 5, 8, 10, 12. Hence a four-dimensional crystal exhibiting a symmetry whose order $\notin \{2, 3, 4, 5, 6, 8, 10, 12\}$ is a quasicrystal.

APPENDIX E: MATHEMATICA IMPLEMENTATION OF THIS PAPER

The notions introduced in this paper may be implemented algorithmically through the following Mathematica 5 [88], [89] notebook:

```

Off[General::spell1];

$MaxExtraPrecision=\[Infinity];

<<DiscreteMath`Combinatorica`;

<<NumberTheory`AlgebraicNumberFields`

(**words[alphabet,
    n] is the list of all the words of length n in lexicographic ordering**)

words[alphabet_,n_]:=Strings[alphabet,n]

(**wordsupto[alphabet,
    n] is the list of all the words of length less or equal to n in \
lexicographic ordering**)

wordsupto[alphabet_,n_]:=
  If[n\[Equal]1,words[alphabet,1],
    Join[wordsupto[alphabet,n-1],words[alphabet,n]]]

(** subwords[x] gives the list of all the sub-
    words of the word x in lexicographic ordering**)

subwords[x_]:=Flatten[Table[Take[x,{i,j}],{i,1,Length[x]},{j,i,Length[x]}],1]

(** generalizedselect[l,predicate,
    y] picks out all elements e of the list l for which the binary \
predicate predicate[e,y] is True **)

(** WARNING:
    since the instruction is implemented throwing away elements of pattern \
String the list l must not to contain strings **)

generalizedselect[l_,predicate_,y_]:=
  DeleteCases[
    Table[If[predicate[Part[l,i],y],Part[l,i],"throw away"],{i,1,Length[l]}],
    x_String]

havinggivenlengthQ[word_,n_]:=Equal[Length[word],n]

(** language[word,n] is the language of length n of the word x
**)

language[word_,n_]:=generalizedselect[subwords[word],havinggivenlengthQ,n]

(** topologicalentropy[alphabet,word,
    n] is the approximation at level n of the topological entropy of a \
sequence **)

topologicalentropy[alphabet_,word_,n_]:=
  Divide[Log[Length[alphabet],Length[language[word,n]]],n]

```

```

occurrences[x_,y_]:=If[AtomQ[x],Count[{x},y],Count[x,y]]

(*** substitution[rule,
    x] is the action of the substitution rule over the word x ***)

substitution[rule_,x_]:=Flatten[Map[rule,x]]

incidencematrix[alphabet_,rule_]:=
  Table[occurrences[rule[Part[alphabet,j]],Part[alphabet,i]],{i,1,
    Length[alphabet]},{j,1,Length[alphabet]}}

(*** sequence[rule,x,n] is the n-
    the iterate of the action of the substitution rule over the word x ***)

sequence[rule_,x_,n_]:=
  If[n\[Equal]1,substitution[rule,x],substitution[rule,sequence[rule,x,n-1]]]

fibonacci[x_]:=If[x\[Equal]0,{0,1},0]

padovan[x_]:=If[x\[Equal]0,{1,2},If[x\[Equal]1,2,0]]

padovannumber[n_]:=If[n\[LessEqual]2,1,padovannumber[n-2]+padovannumber[n-3]]

procedure[rule_,x_,n_]:=Do[Print[sequence[rule,x,k]],{k,1,n}]

algebraicorder[x_]:=
  If[x\[Element]Algebraics,Exponent[MinimalPolynomial[x,y],y],\[Infinity]]

differentQ[x_,y_]:=Unequal[FullSimplify[x-y],0]

conjugates[x_]:=
  If[x\[Element]Algebraics,
    generalizedselect[
      Table[Root[MinimalPolynomial[x],k],{k,1,algebraicorder[x]}],differentQ,
      x],\[Infinity]]

absolutevaluelessthanoneQ[x_]:=Abs[x]<1

pvnumberQ[x_]:=
  And[AlgebraicIntegerQ[x],x>1,
    Length[Select[conjugates[x],absolutevaluelessthanoneQ]]\[Equal]
    Length[conjugates[x]]]

polynomial[x_,listofcoefficients_]:=
  Sum[listofcoefficients[[i]]*x^i,{i,1,Length[listofcoefficients]}]

\[Tau]=GoldenRatio;

\[Rho]=Root[(#^3-#-1)&,1];

pointofpolygon[k_, i_] := {Cos[(2*\[Pi]*i)\k], Sin[(2*\[Pi]*i)/k]}

lineofpolygon[k_, i_, j_] := Line[{pointofpolygon[k,i],pointofpolygon[k,j]}]

polygonpentagram[k_]:=Flatten[Table[lineofpolygon[k,i,j],
  {i,1,k},{j,1,k}]]

```

```

polygonpicture[k_] := Show[Graphics[polygonpentagram[k]]]

pointofpentagram[n_, i_] :=
  If[n \[Equal] 1, {Cos[(2*\[Pi]*i)/5],
    Sin[(2*\[Pi]*i)/5]}, - (GoldenRatio/(1 + 2*GoldenRatio))
    pointofpentagram[n - 1, i]]

lineofpentagram[n_, i_, j_] := Line[{pointofpentagram[n, i], pointofpentagram[n, j]}]

pentagram[n_] := Flatten[Table[lineofpentagram[n, i, j],
  {i, 1, 5}, {j, 1, 5}]]

pentagrampicture[n_] := Show[Graphics[Table[pentagram[i], {i, 1, n}]]]

pvpoint[\[Lambda]_, n_] :=
  If[n \[Equal] 1, {1,
    0}, {Re[Exp[\[ImaginaryI]*Mod[(2*\[Pi])*(\[Lambda]^n), 2*\[Pi]]],
    Im[Exp[\[ImaginaryI]*Mod[(2*\[Pi])*(\[Lambda]^n), 2*\[Pi]]]}]

pvside[\[Lambda]_, n_] := Line[{pvpoint[\[Lambda], n], pvpoint[\[Lambda], n+1]}]

pvline[\[Lambda]_, i_, j_] := Line[{pvpoint[\[Lambda], i], pvpoint[\[Lambda], j]}]

pvcurve[\[Lambda]_, n_] := Table[pvside[\[Lambda], k], {k, 1, n-1}]

pvcurvepicture[\[Lambda]_, n_] :=
  Show[Graphics[{Circle[{0, 0}, 1], pvcurve[\[Lambda], n]}]]

vpentagram[\[Lambda]_, n_] :=
  Flatten[Table[pvline[\[Lambda], i, j], {i, 1, n}, {j, 1, n}]]

vpentagrampicture[\[Lambda]_, n_] :=
  Show[Graphics[{Circle[{0, 0}, 1], vpentagram[\[Lambda], n]}]]

elementaryrotation[alphabet_, spacings_, letter_, initialpoint_] :=
  Mod[Part[spacings, Extract[Position[alphabet, letter], {1, 1}]] + initialpoint,
    2*\[Pi]]

orbit[alphabet_, spacings_, word_, initialpoint_, n_] :=
  If[n > Length[word], "undefined",
    If[n \[Equal] 1,
      elementaryrotation[alphabet, spacings, Part[word, 1], initialpoint],
      elementaryrotation[alphabet, spacings, Part[word, n],
        orbit[alphabet, spacings, word, initialpoint, n-1]]]]

pointofpisotsequence[alphabet_, spacings_, rule_, x_,
  n_] := {Re[Exp[\[ImaginaryI]*
    orbit[alphabet, spacings, sequence[rule, x, n], 0,
      Length[sequence[rule, x, n]]]],
  Im[Exp[\[ImaginaryI]*
    orbit[alphabet, spacings, sequence[rule, x, n], 0,
      Length[sequence[rule, x, n]]]]}]

curveoffpisotsequence[alphabet_, spacings_, rule_, x_, n_] :=
  Flatten[Table[
    Line[{pointofpisotsequence[alphabet, spacings, rule, x, k],
      pointofpisotsequence[alphabet, spacings, rule, x, k+1]}], {k, 1, n-1}]]

```

```

pisotsequencepicture[alphabet_, spacings_, rule_, x_, n_] :=
  Show[Graphics[{Circle[{0,0},1],
    curveoffpisotsequence[alphabet, spacings, rule, x, n]}]]

Hiller[n_] :=
  If[n==1, 0,
    If[n==2, 0,
      If[Length[FactorInteger[n]]==1,
        EulerPhi[FactorInteger[n][[1]][[1]]^FactorInteger[n][[1]][[2]]],
        Sum[Hiller[
          FactorInteger[n][[i]][[1]]^FactorInteger[n][[i]][[2]], {i, 1,
            Length[FactorInteger[n]]}]]]]]]

```

-
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