Contents

1 The model

We will consider a domain $\Omega \subset \mathbb{R}^2$, bounded and with C^1 or polygonal boundary, separated by an internal planar interface Γ that represents the fault. The fault Γ partitions Ω into two disjoint subdomains Ω^+ and Ω^-, so that $\Omega = \mathrm{int}(\Omega^+ \cup \Omega^- \cup \Gamma)$. We set $\Omega_\Gamma = \Omega \setminus \Gamma = \Omega^+ \cup \Omega^-$. We indicate with $\boldsymbol{n}_\Gamma$ the normal to Γ, conventionally oriented from Ω^- to Ω^+, while on Γ $\boldsymbol{n}^+ = -\boldsymbol{n}^- = \boldsymbol{n}_\Gamma$ are the normal vectors oriented outwardly with respect to the respective side of the domain. While $\boldsymbol{\nu}_\Gamma$ is the vector tangent to Γ.

For a sufficiently regular scalar function $f : \Omega_\Gamma \to \mathbb{R}$ we define the average and jump across Γ as

$$\{f\} = \frac{f^+ + f^-}{2} \quad \text{and } [\![f]\!] = f^+ - f^-.$$

The definition extends naturally to vector valued functions vector valued functions $\mathbf{f} : \Omega_\Gamma \to \mathbb{R}^2$. We wish to note that

$$\{\mathbf{f} \cdot \boldsymbol{n}\} = \frac{\mathbf{f}^+ \cdot \boldsymbol{n}^+ + \mathbf{f}^- \cdot \boldsymbol{n}^-}{2} = \frac{[\![\mathbf{f} \cdot \boldsymbol{n}_\Gamma]\!]}{2},$$

and

$$[\![\mathbf{f} \cdot \boldsymbol{n}]\!] = \mathbf{f}^+ \cdot \boldsymbol{n}^+ - \mathbf{f}^- \cdot \boldsymbol{n}^- = 2\{\mathbf{f} \cdot \boldsymbol{n}_\Gamma\}.$$

Here, for $\mathbf{x} \in \Gamma$,

$$f^\pm(\mathbf{x}) = \lim_{\delta \to 0^+} f(\mathbf{x} + \delta \boldsymbol{n}_\Gamma^\pm)$$

and analogously for vector valued functions.

Remark 1. It may be convenient for stability to generalize the definition of the averaging operator as

$$\{f\}_\gamma = \gamma^+ f^+ + \gamma^- f^-, \tag{1}$$

for two positive constants γ^+ and γ^- such that $\gamma^+ + \gamma^- = 1$.

Modelling the fault as a geometrically one-codimensional planar manifold is indeed an approximation justified by the fact that the fault aperture, here indicated by l_Γ, is small compared to the linear dimensions of the domain. However, the fault region may have a permeability strongly different than that in the surrounding rock, therefore its effect on the flow will be modelled by using the model developed in [2] and exploited in [1]. In this model, flow pressure may be discontinuous across the fault surface. For instance, when the permeability of the rock at the two sides of the fault varies greatly pressure may experience a strong gradient across the fault zone, which is modelled as a discontinuity in the reduced model. We assume that it behaves as a (possibly) slipping interface. In particular, we assume that we are in a compressive regime, so the normal displacements may be assumed continuous across the fault. We define the following quantities. In the bulk region Ω_Γ, with p, $\mathbf{q}$ and $\mathbf{u}$ we indicate the fluid pressure, Darcy's velocity and rock displacement, respectively. In the fracture, p_Γ and $\mathbf{q}_\Gamma$ indicate the average pressure and Darcy's flux along the fracture plane, respectively.

Since we assume that the fluid viscosity is constant throughout the domain, we will indicate in the following with the term permeability what is in fact the permeability scaled with viscosity. In particular, with $\boldsymbol{K}$ we indicate the permeability in the bulk, which is in general a symmetric positive definite tensor with bounded components in Ω_Γ. For the sake of simplicity we assume that it takes the form $\boldsymbol{K} = K\mathbf{I}$, with $\mathbf{I}$ the identity tensor. We furthermore assume that for almost all $\mathbf{x} \in \Omega_\Gamma$ we have $0 < \underline{K} \leq K(\mathbf{x}) \leq \overline{K}$. As detailed in [], we assume that the permeability in the fracture can be decomposed into a normal, $\hat{K}_n$ and a tangential component $\hat{K}_\tau$, to take into account that indeed fracture resistance to flow may be different in the normal and tangential direction. The reduced model for the fracture makes use of effective permeability, to account for fracture aperture. In particular, we define the following quantities

$$K_\Gamma = l_\Gamma \hat{K}_\tau, \quad \eta = \frac{l_\Gamma}{\hat{K}_n}. \tag{2}$$

We assume that $0 < \underline{K_\Gamma} \leq K_\Gamma(\mathbf{x}) \leq \overline{K_\Gamma}$ and $0 < \underline{\eta} \leq \eta(\mathbf{x}) \leq \overline{\eta}$, for almost all $\mathbf{x} \in \Gamma$.

With this definitions, the differential model in the bulk and in the fracture reads:

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{1}{M}p + b\,\mathrm{div}\,\mathbf{u}\right) + \mathrm{div}\,\mathbf{q} = \mathbf{f} \\ \boldsymbol{K}^{-1}\mathbf{q} + \nabla p = \rho_f \mathbf{g} \\ -\,\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}, p) = -\rho \mathbf{g} \end{cases} \qquad \text{in } \Omega_\Gamma \quad t > 0, \tag{3}$$

$$\begin{cases} l_\Gamma \dfrac{\partial}{\partial t}\left(\dfrac{1}{M}p_\Gamma\right) + \mathrm{div}_{\boldsymbol{\tau}}\,\mathbf{q}_\Gamma - [\![\mathbf{q} \cdot \boldsymbol{n}_\Gamma]\!] = f_\Gamma \\ K_\Gamma^{-1}\mathbf{q}_\Gamma + \nabla_{\boldsymbol{\tau}}\, p_\Gamma = \rho_f \mathbf{g} \end{cases} \qquad \text{in } \Gamma \quad t > 0. \tag{4}$$

Here, M and b are the **check come si chiamano** the compressibility and Biot modules, $\mathbf{f}$ and $\mathbf{f}_\Gamma$ source terms representing possible injection/extraction of fluids, $\mathbf{g}$ the gravity acceleration, ρ_f is the (constant) fluid density, while ρ is the density of the rock, which is related to that of the fluid and that of the solid matrix, ρ_s, by $\rho = \phi\rho_f + (1-\phi)\rho_s$. Finally, $\mathrm{div}_{\boldsymbol{\tau}}$ and $\nabla_{\boldsymbol{\tau}}$ are the divergence and gradient operator on the fracture plane. The equations are supplemented by boundary conditions. In particular, we here assume that

$$\begin{cases} p = p^\partial, \quad \text{on } \partial\Omega_p \\ p_\Gamma = p_\Gamma^\partial, \quad \text{on } \partial\Gamma_{p_\Gamma} \\ \mathbf{q} \cdot \boldsymbol{n} = q^\partial, \quad \text{on } \partial\Omega_{\mathbf{q}} \\ \mathbf{q}_\Gamma \cdot \boldsymbol{\nu}_\Gamma = q_\Gamma^\partial, \quad \text{on } \partial\Gamma_{\mathbf{q}_\Gamma} \\ \mathbf{u} = \mathbf{u}^\partial \quad \text{on } \partial\Omega_{\mathbf{u}} \\ \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \boldsymbol{t}^\partial \quad \text{on } \partial\Omega_\sigma \end{cases} \tag{5}$$

Moreover, we should add initial condition for pressure and compatible values for the Darcy velocities at $t = 0$.

The total Cauchy stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}, p)$ takes into account of the combined action of the solid and fluid part that compose the rock. According to Biot's theory it is expressed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^e(\mathbf{u}) - bp\boldsymbol{I}, \tag{6}$$

where $\boldsymbol{I}$ is the identity tensor and $\boldsymbol{\sigma}^e$ the effective stress that accounts for the action of the solid part. In this work we assume small deformations and elastic behaviour, so we define the strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^T \mathbf{u})$$

and

$$\boldsymbol{\sigma}^e(\mathbf{u}) = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}). \tag{7}$$

Here λ and μ are the Lamè parameters, which we assume to be both positive and bounded in Ω_Γ. Indeed, the third equation in (4) can be written as

$$-\operatorname{\mathbf{div}} \boldsymbol{\sigma}^e(\mathbf{u}) + b\nabla p = -\rho \mathbf{g} \quad \text{in } \Omega_\Gamma. \tag{8}$$

Equations (3) and (4) must be closed by suitable interface conditions on Γ and boundary conditions. For the former, we adopt the model in [] for what concerns fluid flow, by which we have that on Γ **check con altri lavori perchè qui abbiamo il termine sorgente in frattura la II equazione cambia!**

$$\begin{cases} \eta\{\mathbf{q} \cdot \boldsymbol{n}_\Gamma\} = [\![p \cdot \boldsymbol{n}_\Gamma]\!], \\ \eta_0 [\![\mathbf{q} \cdot \boldsymbol{n}_\Gamma]\!] = \{p \cdot \boldsymbol{n}_\Gamma\} - p_\Gamma, \end{cases} \tag{9}$$

where $\eta_0 = \xi_0 \eta$ and $\xi_0 > 0$ is a closure parameter of the model, usually taken equal to **????**. For the mechanical part, we first define the normal component of the total and effective normal stress on Γ as

$$\sigma_{n_\Gamma} = (\boldsymbol{\sigma} \cdot \boldsymbol{n}_\Gamma) \cdot \boldsymbol{n}_\Gamma, \quad \sigma^e_{n_\Gamma} = (\boldsymbol{\sigma}^e \cdot \boldsymbol{n}_\Gamma) \cdot \boldsymbol{n}_\Gamma,$$

respectively, and the corresponding vectors

$$\boldsymbol{\sigma}_{n_\Gamma} = \sigma_{n_\Gamma} \cdot \boldsymbol{n}_\Gamma, \quad \boldsymbol{\sigma}^e_{n_\Gamma} = \sigma^e_{n_\Gamma} \cdot \boldsymbol{n}_\Gamma.$$

Note that they may be discontinuous on Γ.

The tangential stress on Γ is instead given by

$$\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \boldsymbol{n}_\Gamma - \boldsymbol{\sigma}_{n_\Gamma}, \quad \boldsymbol{\tau}^e = \boldsymbol{\sigma}^e \cdot \boldsymbol{n}_\Gamma - \boldsymbol{\sigma}^e_{n_\Gamma}, \tag{10}$$

and, because of the definition of effective stress, we have that $\boldsymbol{\tau} = \boldsymbol{\tau}^e$.

Analogously, we define tangential and normal component of the displacement

$$\mathbf{u}_{n_\Gamma} = u_{n_\Gamma} \cdot \boldsymbol{n}_\Gamma = (\mathbf{u} \cdot \boldsymbol{n}_\Gamma)\, \boldsymbol{n}_\Gamma, \quad \mathbf{u}_{t_\Gamma} = \mathbf{u} - \mathbf{u}_{n_\Gamma}. \tag{11}$$

To describe the sliding conditions, we define

$$\mathcal{G} = |\{\boldsymbol{\tau}\}| + \mu_f \overline{\sigma}^e_{n_\Gamma},$$

where, $|\cdot|$ indicates the Eucledean norm, $\overline{\sigma}^e_{n_\Gamma}$ is a reference value of the compressive part of the normal component of the effective stress on the fault, which may be taken as

$$\overline{\sigma}^e_{n_\Gamma} = \min(0, \{\sigma^e_{n_\Gamma}\}), \tag{12}$$

or, to be more conservative, as suggested in [**?**]

$$\overline{\sigma}^e_{n_\Gamma} = \min(0, \{\sigma_{n_\Gamma}\} - b\max(p^+, p^-)). \tag{13}$$

With this definition $\overline{\sigma}^e_{n_\Gamma} \leq 0$. To be consistent with what is found sometimes in the literature, we note that $\mathcal{G}$ may be also written as

$$\mathcal{G} = |\{\boldsymbol{\tau}\}| - \mu_f|\overline{\sigma}^e_{n_\Gamma}|.$$

We also define on Γ, the displacement velocity and its tangential component, as

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} \quad \text{and } \dot{\mathbf{u}}_{t_\Gamma} = \frac{\partial \mathbf{u}_{t_\Gamma}}{\partial t}. \tag{14}$$

We assume that the fault may only act as a potentially sliding surface. At each time t the interface conditions for the mechanical problem read as

$$\begin{cases} [\![u_{n_\Gamma}]\!] = 0, \\ [\![\boldsymbol{\sigma}_{n_\Gamma}]\!] = \mathbf{0}, \\ [\![\boldsymbol{\tau}]\!] = \mathbf{0}, \\ \mathcal{G} \leq 0, \\ \exists \beta \leq 0 \text{ s.t } [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] = \beta\{\boldsymbol{\tau}\} \\ \beta\mathcal{G} = 0. \end{cases} \quad \text{on } \Gamma. \tag{15}$$

Note that the balance of normal stresses at the fault interface involve the total stress $\boldsymbol{\sigma}$ and it implies that both σ_{n_Γ} and $\boldsymbol{\tau}$ are indeed continuous across Γ. So we can set, on Γ that $\sigma_{n_\Gamma} = \{\sigma_{n_\Gamma}\}$ and $\boldsymbol{\tau} = \{\boldsymbol{\tau}\}$. Moreover, since $\boldsymbol{\tau}^e = \boldsymbol{\tau}$ also the latter is continuous, while, since pressure may be discontinuous across the fracture, $\sigma^e_{n_\Gamma}$ may be discontinuous. However, also in view of the possible use of broken spaces in the numerical discretization we will keep indicating the jumps for the normal and tangential components of the total stress.

Thanks to (15) at each time t we are able to identify two (non necessarily connected and possibly empty) measurable portions of Γ, indicated by $\Gamma_D = \Gamma_D(t)$ and $\Gamma_D = \Gamma_D(t)$:

$$\begin{cases} \Gamma_D(t) = \overline{\{\mathbf{x} \in \Gamma : \, \mathcal{G}(\mathbf{x}) < 0\}}, \\ \Gamma_N(t) = \Gamma \setminus \Gamma_D(t), \end{cases} \tag{16}$$

as well as the convex set

$$\mathcal{K}_\mu = \{\mathbf{z} \in \mathbb{R}^d : |\mathbf{z}| - \mu_f\overline{\sigma}^e_{n_\Gamma} \leq 0\}.$$

Proposition 1.1. *The friction conditions in* (15)*:*

$$\begin{cases} \mathcal{G} \leq 0, \\ \exists \beta \leq 0 \ s.t \ [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] = \beta\{\boldsymbol{\tau}\} \\ \beta\mathcal{G} = 0. \end{cases}$$

are equivalent to

$$\begin{cases} \mathcal{G} \leq 0, \\ \{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - \mu_f \overline{\sigma}^e_{n_\Gamma} |[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| = 0 \end{cases} \tag{17}$$

Proof. Let the friction conditions in (15) previously recalled be true. Then

- if $\mathcal{G} < 0$ then $[\![\dot{\mathbf{u}}_{t_\Gamma}]\!] = 0$. Then, $\{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - \mu_f \overline{\sigma}^e_{n_\Gamma} |[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| = 0$.
- if $\mathcal{G} = 0$, then $\{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - \mu_f \overline{\sigma}^e_{n_\Gamma} |[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| = \{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - |\{\boldsymbol{\tau}\}||[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| = \beta|\{\boldsymbol{\tau}\}| - |\beta||\{\boldsymbol{\tau}\}| = 0$, since $\beta < 0$.

Conversely, let (17) be true, then

- if $\mathcal{G} < 0$ Let suppose $|[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| \neq 0$. We have, $|\{\boldsymbol{\tau}\}| < -\mu_f \overline{\sigma}^e_{n_\Gamma}$, thus $0 = \{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - \mu_f \overline{\sigma}^e_{n_\Gamma} |[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| > \{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] + |\{\boldsymbol{\tau}\}||[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]$, by which

  $$\{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] < -|\{\boldsymbol{\tau}\}||[\![\dot{\mathbf{u}}_{t_\Gamma}]\!].$$

  But this is not possible since, by Cauchy-Schwarz inequality, $\{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] \geq -|\{\boldsymbol{\tau}\}||[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]$. Thus, we must have $|[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| = 0$.
- If instead $\mathcal{G} = 0$, then $0 = \{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - \mu_f \overline{\sigma}^e_{n_\Gamma} |[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]| = \{\boldsymbol{\tau}\} \cdot [\![\dot{\mathbf{u}}_{t_\Gamma}]\!] - |\{\boldsymbol{\tau}\}||[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]|$. But this is possible only if $\{\boldsymbol{\tau}\} = \frac{1}{\beta}[\![\dot{\mathbf{u}}_{t_\Gamma}]\!]$, for a $\beta < 0$.

$\square$

1.1 Friction models

There are various friction model in the literature **qui bisogna arricchire la letteratura**, mostly **questa parte è ancora da scrivere...**.

2 Fixed Stress Split

The model represented by the systems of equations (3), (4), (8), with the interface conditions (**??**) and (15) form a closely coupled system of differential equations and inequalities whose solution may be simplified by adopting splitting strategies. A very successful approach presented in the literature [**?**] is the fixed-stress splitting, which may be interpreted as a particular preconditioner for the differential problem. It may be presented in different form, we here described it in differential form, postponing to a later stage its variational and fully discrete formulation. **Nota: nelle equazioni mancano ancora condizioni inizieli e al bordo, vanno aggiunte!**

The fixed-stress split may be seen as a particular time-advancing scheme where the equations for the fluid and the structural problem are solved sequentially within an iterative procedure. We assume that the time interval of interest $[0, T]$ has been partitioned into time steps Δt, for simplicity taken constant, however the scheme is directly extendable to variable time steps. With the suffix n we define quantities at time $t_n = n\Delta t$, e.g. $\mathbf{u}^n = \mathbf{u}(t^n)$. We assume to know all quantities time ^{n}. We set $p^{(0)}=p^n$, $p_\Gamma^{(0)}=p_\Gamma^n$, $\mathbf{q}^{(0)}=\mathbf{q}^n$, $\mathbf{u}^{(0)}=\mathbf{u}^n$. Let $\beta_{FS}>0$ be a coefficient, whose definition will be better specified later on. A fixed-stress splitting schemes for our problem can be described as follows

For $k=0,\ldots$*:*

1. Compute the modified flow problem:

$$\begin{cases} \left(\dfrac{1}{M}+\beta_{FS}\right)p^{(k+1)}+\Delta t\operatorname{div}\mathbf{q}^{(k+1)}=\beta_{FS}p^{(k)}-b\left(\operatorname{div}\mathbf{u}^{(k)}-\operatorname{div}\mathbf{u}^n\right)+\Delta t\mathbf{f}^{n+1} \\ \boldsymbol{K}^{-1}\mathbf{q}^{(k+1)}+\nabla p^{(k+1)}=\rho_f\mathbf{g} \end{cases} \quad \text{in } \Omega_\Gamma, \tag{18}$$

$$\begin{cases} \operatorname{div}_{\boldsymbol{\tau}}\mathbf{q}_\Gamma^{(k+1)}-[\![\mathbf{q}^{(k+1)}\cdot\boldsymbol{n}_\Gamma]\!]=f_\Gamma^{n+1} \\ K_\Gamma^{-1}\mathbf{q}_\Gamma^{(k+1)}+\nabla_{\boldsymbol{\tau}}p_\Gamma^{(k+1)}=\rho_f\mathbf{g} \end{cases} \quad \text{in } \Gamma, \tag{19}$$

$$\begin{cases} \eta\{\mathbf{q}^{(k+1)}\cdot\boldsymbol{n}_\Gamma\}-[\![p^{(k+1)}\cdot\boldsymbol{n}_\Gamma]\!]=0, \\ \eta_0[\![\mathbf{q}^{(k+1)}\cdot\boldsymbol{n}_\Gamma]\!]-\{p^{(k+1)}\cdot\boldsymbol{n}_\Gamma\}+p_\Gamma^{(k+1)}=0, \end{cases} \quad \text{on } \Gamma. \tag{20}$$

2. Solve the structural problem for the given pressure field

$$-\mathbf{div}\,\boldsymbol{\sigma}^e(\mathbf{u}^{(k+1)})=\rho\mathbf{g}-b\nabla p^{(k+1)} \quad \text{in } \Omega_\Gamma, \tag{21}$$

with interface conditions

$$\begin{cases} [\![u_{n_\Gamma}^{(k+1)}]\!]=0, \\ [\![\boldsymbol{\sigma}_{n_\Gamma}^{(k+1)}]\!]=\mathbf{0}, \\ [\![\boldsymbol{\tau}^{(k+1)}]\!]=\mathbf{0}, \\ \mathcal{G}^{(k+1)}\leq 0, \\ \exists\beta^{(k+1)}\leq 0 \quad \text{s.t.}[\![\mathbf{u}_{t_\Gamma}^{(k+1)}-\mathbf{u}_{t_\Gamma}^n]\!]=\beta^{(k+1)}\{\boldsymbol{\tau}^{(k+1)}\}, \\ \mathcal{G}^{(k+1)}\beta^{(k+1)}=0. \end{cases} \tag{22}$$

3. Stop the iteration when a suitable convergence condition is met (we postpone the discussion to Section **??**), and set $p^{n+1}=p^{(k+1)}$, $p_\Gamma^{n+1}=p_\Gamma^{(k+1)}$, $\mathbf{q}^{n+1}=\mathbf{q}^{(k+1)}$, $\mathbf{q}_\Gamma^{n+1}=\mathbf{q}_\Gamma^{(k+1)}$ and $\mathbf{u}^{n+1}=\mathbf{u}^{(k+1)}$.

3 Variational formulation and numerical scheme

3.1 Functional setting

We use the standard notation and norms for Lesbegue and Sobolev spaces for functions in $\Omega_\Gamma=\Omega\setminus\Gamma$, Ω^+ and Ω^-. We define

$$L^2(\Omega_\Gamma)=\{v:\Omega_\Gamma\to\mathbb{R}:\int_\Omega v^2<\infty)\},$$

and we will use the same notation for vector functions whose components are in $L^2(\Omega_\Gamma)$. Note that since Γ is a set of zero 2-measure, a function in $L^2(\Omega_\Gamma)$ are naturally identified with a function in $L^2(\Omega)$. So we will write that, for u and v in $L^2(\Omega_\Gamma)$ that

$$(u, v)_{L^2(\Omega_\Gamma)} = \int_{\Omega_\Gamma} uv = \int_\Omega uv,$$

and $\|u\|_{L^2(\Omega_\Gamma)} = \sqrt{(u, u)_{L^2(\Omega_\Gamma)}}$. We also define

$$H^1(\Omega_\Gamma) = \{v \in L^2(\Omega_\Gamma) : \nabla v \in L^2(\Omega_\Gamma)\}, \tag{23}$$

with

$$(u, v)_{H^1(\Omega_\Gamma)} = (u, v)_{L_2(\Omega)} + \int_\Omega \nabla u \cdot \nabla v, \tag{24}$$

and

$$\|u\|^2_{H^1(\Omega_\Gamma)} = \sqrt{(u, u)_{H^1(\Omega_\Gamma)}}. \tag{25}$$

Analogous definition for vector valued functions:

$$\mathbf{H}^1(\Omega_\Gamma) = \{\mathbf{v} : \Omega_\Gamma \to \mathbb{R}^d : \mathbf{v} \in \mathbf{L}^2(\Omega_\Gamma) : \nabla \mathbf{v} \in \mathbf{L}^2(\Omega_\Gamma)\},$$

with the usual definition of inner product and norm. We will often omit the bold character for vector valued functional spaces. It can be now useful to define the following subspaces

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega_\Gamma) : \mathbf{v}|_{\partial^D\Omega} = \mathbf{0}\}, \tag{26}$$

and

$$\mathbf{V}_\Gamma = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_\Gamma = \mathbf{0}\}. \tag{27}$$

The space $\mathbf{H}^1(\Omega_\Gamma)$ is a broken space, since its elements may be discontinuous across Γ. We thus define the following trace spaces

$$T = \{t : \Gamma \to \mathbb{R}, v \in H^{1/2}(\Gamma), \mathrm{supp}(v) \subset \Gamma\}, \tag{28}$$

and

$$T_\Gamma = \{t = (t^+, t^-) : \Gamma \to \mathbb{R} \times \mathbb{R}, \{t\} \in T, [\![t]\!] \in T\}, \tag{29}$$

$$\mathbf{T}_t = \{\mathbf{v} = (\mathbf{v}^+, \mathbf{v}^-) : v_n \in T_\Gamma, \mathbf{v}_t \in [T_\Gamma]^d\}, \tag{30}$$

$$\mathbb{T} = \{\mathbf{v} \in \mathbf{T}_t : v_n = 0\}, \tag{31}$$

where v_n and $\mathbf{v}_t$ denote the normal compoent and the tangential part of $\mathbf{v}$.

We claim that if $\mathbf{v} \in V$ then its trace on Γ is in $\mathbf{T}_t$. (**check!**)

Finally, we indicate with H_{div} the space of L^2 vector valued functions with (distributional) divergence in L^2, and

$$H_{\mathrm{div}}(\Omega_\Gamma) = \{\mathbf{v} \in L^2(\Omega_\Gamma) : \mathbf{v}|_{\Omega^\pm} \in H_{\mathrm{div}}(\Omega^\pm), \ \}, \tag{32}$$

with scalar product

$$(\mathbf{v}, \mathbf{u})_{H_{\mathrm{div}}(\Omega_\Gamma)} = \int_\Omega \mathbf{v} \cdot \mathbf{u} + \int_\Omega \mathrm{div}\, \mathbf{v}\, \mathrm{div}\, \mathbf{u} + \int_{\Omega^-} \mathrm{div}\, \mathbf{v}\, \mathrm{div}\, \mathbf{u},$$

and associated norm $\|\mathbf{v}\|_{H^1(\Omega_\Gamma)} = \sqrt{(\mathbf{u}, \mathbf{u})_{H^1(\Omega_\Gamma)}}$.

3.2 The mechanical step as variational inequality

We concentrate at the moment on the structural step (step 2) of the fixed stress splitting. And, for the sake of simplicity, we avoid here the index k and $k+1$. Therefore, in the context of this section, pressure p is a given data, as well as $[\![\mathbf{u}^n_{t_\Gamma}]\!]$. **Mancano ancora le condizioni al bordo esterne, qui le prenderei omogenee, si dovrà però precisare meglio gli spazi, per ora lascio nel vago per evitare di complicarmi la vita.**. We also define the following affine operator

$$\delta(\mathbf{v}) = [\![\mathbf{v}_{t_\Gamma}]\!] - [\![\mathbf{u}^n_{t_\Gamma}]\!], \tag{33}$$

where $[\![\mathbf{u}^n_{t_\Gamma}]\!]$ is given. The problem is then

Problem 3.1. *Find* $\mathbf{u}$ *that satisfies*

$$-\,\mathbf{div}\,\boldsymbol{\sigma}^e(\mathbf{u}) = \rho\mathbf{g} - b\nabla p \quad \textit{in } \Omega_\Gamma, \tag{34}$$

with interface conditions

$$\begin{cases} [\![u_{n_\Gamma}]\!] = 0, \\ [\![\boldsymbol{\sigma}\cdot\boldsymbol{n}_\Gamma]\!] = \mathbf{0}, \\ \mathcal{G} \le 0, \\ \exists\beta \le 0 \quad \textit{s.t. } \delta(\mathbf{u}_{t_\Gamma}) = \beta\{\boldsymbol{\tau}\}, \\ \mathcal{G}\beta = 0. \end{cases} \tag{35}$$

Note that we have written the condition on the jump of normal and tangential total stresses as a single relation. We can extend a previous result to this case

Proposition 3.1. *The friction conditions, i.e. the last two conditions in* (35), *are equivalent to*

$$\begin{cases} \mathcal{G} \le 0, \\ \{\boldsymbol{\tau}\}\cdot\delta(\mathbf{u}_{t_\Gamma}) - \mu_f\overline{\sigma}^e_{n_\Gamma}|\delta(\mathbf{u}_{t_\Gamma})| = 0. \end{cases} \tag{36}$$

Proof. Analogous to the proof of 1.1 □

We define $\Sigma = \partial\Omega_\Gamma \setminus \Gamma$ and $V = \{\mathbf{w} \in H^1(\Omega_\Gamma),\ \mathbf{w}|_\Sigma = \mathbf{0},\ [\![w_{n_\Gamma}]\!] = 0\}$ We state the following variational inequality, **bisognerebbe dare tutte le condizioni sui dati, per ora vado giù formalmente poi le scrivo meglio**

Problem 3.2. *Find* $\mathbf{u} \in V$ *such that*

$$a_s(\mathbf{u}, \mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) \ge F(\mathbf{v}-\mathbf{u}), \quad \forall \mathbf{v} \in V \tag{37}$$

where $a_s(\mathbf{u},\mathbf{v}) = \int_{\Omega_\Gamma} \boldsymbol{\sigma}^e(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})$,

$$j(\mathbf{u},\mathbf{v}) = \int_\Gamma \mu_f|\sigma^e_{n_\Gamma}(\mathbf{u})||\delta(\mathbf{v}_{t_\Gamma})|, \tag{38}$$

$$F(\mathbf{v}) = \int_{\Omega_\Gamma} \rho \mathbf{g}\cdot\mathbf{n} + b\int_{\Omega_\Gamma} p \operatorname{div}\mathbf{v} - b\int_\Gamma [\![p]\!]\{\mathbf{v}\cdot\boldsymbol{n}_\Gamma\} - b\int_\Gamma \{p\}[\![\mathbf{v}\cdot\boldsymbol{n}_\Gamma]\!].$$

in F ci sono anche gli altri termini di bordo, per ora omessi

Proposition 3.2. *A weak solution of Problem 3.1 is equivalent to the solution of the variational inequality in 3.2.* ***Da scrivere meglio***

Proof. Let p be regular enough and $\mathbf{u}$ be a regular solution of Problem 3.1. Let consider Eq. (37). By counterintegration, recalling that $[\![ab]\!] = [\![a]\!]\{v\} + \{a\}[\![b]\!]$, the convention on normal orientation, the symmetry of $\boldsymbol{\sigma}^e$, and the definition of $\boldsymbol{\sigma}$, we have

$$\begin{aligned} &a_s(\mathbf{u},\mathbf{v}-\mathbf{u}) - F(\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) = \\ &< -\operatorname{\mathbf{div}}\boldsymbol{\sigma}^e(\mathbf{u}) + b\nabla p - \rho\mathbf{g}, \mathbf{v}-\mathbf{u} > + \big([\![\boldsymbol{\sigma}(\mathbf{u})\cdot\boldsymbol{n}_\Gamma]\!], \{\mathbf{v}-\mathbf{u}\}\big)_\Gamma + \big(\{\boldsymbol{\sigma}(\mathbf{u})\cdot\boldsymbol{n}_\Gamma\}, [\![\mathbf{v}-\mathbf{u}]\!]\big)_\Gamma + \\ &\qquad + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}). \end{aligned} \tag{39}$$

where $(a,b)_\Gamma = \int_\Gamma ab$ denotes the $L^2(\Gamma)$ product and $< .,. >$ the $H^1(\Omega_\Gamma) - H^{-1}(\Omega_\Gamma)$ duality pairing. Since $-\operatorname{\mathbf{div}}\boldsymbol{\sigma}^e(\mathbf{u}) + b\nabla p - \rho\mathbf{g} = \mathbf{0}$ a.e. in Ω_Γ and $[\![\boldsymbol{\sigma}\cdot\boldsymbol{n}_\Gamma]\!] = \mathbf{0}$ we have

$$\begin{aligned} &a_s(\mathbf{u},\mathbf{v}-\mathbf{u}) - F(\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) = \\ &\qquad \big(\{\boldsymbol{\sigma}(\mathbf{u})\boldsymbol{n}_\Gamma\}, [\![\mathbf{v}-\mathbf{u}]\!]\big)_\Gamma + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) \end{aligned} \tag{40}$$

Now (we omit to indicate the dependence on $\mathbf{u}$),

$$\big(\{\boldsymbol{\sigma}\cdot\boldsymbol{n}_\Gamma\}, [\![\mathbf{v}-\mathbf{u}]\!]\big)_\Gamma = \big(\{\sigma_{n_\Gamma}\}, [\![v_{n_\Gamma} - u_{n_\Gamma}]\!]\big)_\Gamma + \big(\{\boldsymbol{\tau}\}, [\![\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!]\big)_\Gamma = \big(\{\boldsymbol{\tau}\}, [\![\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!]\big)_\Gamma. \tag{41}$$

since $(\mathbf{v}-\mathbf{u}) \in V$. Finally,

$$\begin{aligned} &a_s(\mathbf{u},\mathbf{v}-\mathbf{u}) - F(\mathbf{v}-\mathbf{u}) + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) = \\ &\qquad \big(\{\boldsymbol{\tau}\}, [\![\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!]\big)_\Gamma + j(\mathbf{u},\mathbf{v}) - j(\mathbf{u},\mathbf{u}) = \\ &\int_\Gamma \{\boldsymbol{\tau}\}\cdot[\![\mathbf{v}_{t_\Gamma}]\!] - \int_\Gamma \{\boldsymbol{\tau}\}\cdot[\![\mathbf{u}_{t_\Gamma}]\!] + \int_\Gamma \mu_f|\sigma^e_{n_\Gamma}||\delta(\mathbf{v}_{t_\Gamma})| - \int_\Gamma \mu_f|\sigma^e_{n_\Gamma}||\delta(\mathbf{u}_{t_\Gamma})| = \\ &\int_\Gamma \{\boldsymbol{\tau}\}\cdot\delta(\mathbf{v}_{t_\Gamma}) - \int_\Gamma \{\boldsymbol{\tau}\}\cdot\delta(\mathbf{u}_{t_\Gamma}) + \int_\Gamma \mu_f|\sigma^e_{n_\Gamma}||\delta(\mathbf{v}_{t_\Gamma})| - \int_\Gamma \mu_f|\sigma^e_{n_\Gamma}||\delta(\mathbf{u}_{t_\Gamma})|. \end{aligned} \tag{42}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} &\int_\Gamma \{\boldsymbol{\tau}\}\cdot\delta(\mathbf{v}_{t_\Gamma}) + \int_\Gamma \mu_f|\sigma^e_{n_\Gamma}||\delta(\mathbf{v}_{t_\Gamma})| \geq -\int_\Gamma \big[|\{\boldsymbol{\tau}\}| - \mu_f||\sigma^e_{n_\Gamma}||\delta(\mathbf{v}_{t_\Gamma})| = \\ &\qquad - \int_\Gamma \mathcal{G}|\delta(\mathbf{v}_{t_\Gamma})| \geq 0 \end{aligned} \tag{43}$$

Now, if $\mathcal{G} = 0$ then $|\{\boldsymbol{\tau}\}| = \mu_f |\sigma^e_{n\Gamma}|$ and $\delta(\mathbf{u}_{t_\Gamma}) = \beta\{\boldsymbol{\tau}\}$, with $\beta \leq 0$. Thus,

$$\int_\Gamma \{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \int_\Gamma \mu_f |\sigma^e_{n\Gamma}||\delta(\mathbf{u}_{t_\Gamma})| \;=\; \int_\Gamma \beta |\{\boldsymbol{\tau}\}|^2 + \int_\Gamma |\beta||\{\boldsymbol{\tau}\}|^2 \;=\; 0, \quad (44)$$

since $\beta \leq 0$. While, if $\mathcal{G} < 0$ then, $\delta(\mathbf{u}_{t_\Gamma}) = \mathbf{0}$ and

$$\int_\Gamma \{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \int_\Gamma \mu_f |\sigma^e_{n\Gamma}||\delta(\mathbf{u}_{t_\Gamma})| = 0. \quad (45)$$

In conclusion, we have,

$$a_s(\mathbf{u}, \mathbf{v} - \mathbf{u}) - F(\mathbf{v} - \mathbf{u}) + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq 0, \;\; \forall \mathbf{v} \in V.$$

We now look at the converse. If $\mathbf{u}$ is a sufficiently smooth solution of Problem 3.2 we have, by counter-integration

$$< -\,\mathbf{div}\, \boldsymbol{\sigma}^e(\mathbf{u}) + b\nabla p - \rho \mathbf{g}, \mathbf{v} - \mathbf{u} > + \big([\![\boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{n}_\Gamma]\!], \{\mathbf{v} - \mathbf{u}\}\big)_\Gamma + \\ \big(\{\boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{n}_\Gamma\}, [\![\mathbf{v} - \mathbf{u}]\!]\big)_\Gamma + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq 0, \quad (46)$$

for all $\mathbf{v} \in V$. We first take $\mathbf{v} \in V_u$ where V_u is the set $V_u = \{\mathbf{w} \in V : [\![\mathbf{w} - \mathbf{u}]\!] = \{\mathbf{w} - \mathbf{u}\} = \mathbf{0}\}$. Clearly we have that $(\mathbf{v} - \mathbf{u}) \in V_0 = \{\mathbf{w} \in V : [\![\mathbf{w}]\!] = \{\mathbf{w}\} = \mathbf{0}\}$, and (46) implies

$$< -\,\mathbf{div}\, \boldsymbol{\sigma}^e(\mathbf{u}) + b\nabla p - \rho \mathbf{g}, \mathbf{w} \geq 0, \, \forall \mathbf{w} \in V_0,$$

and since V_0 is a closed subspace of V, this means that

$$-\,\mathbf{div}\, \boldsymbol{\sigma}^e(\mathbf{u}) + b\nabla p - \rho \mathbf{g} = 0, \text{ a.e. in } \Omega_\Gamma,$$

and so it satisfies the differential problem (3.1). We are left with the other terms that we can now expand into the tangential and normal component (omitting again the dependence on $\mathbf{u}$ of the stresses). Remembering that functions in V have the jump of normal component equal to zero, we have

$$\big([\![\sigma_{n\Gamma}]\!], \{v_n - u_{n\Gamma}\}\big)_\Gamma + \big([\![\boldsymbol{\tau}]\!], \{\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}\}\big)_\Gamma + \\ \big(\{\boldsymbol{\tau}\}, [\![\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!]\big)_\Gamma + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq 0, \quad (47)$$

Let's now consider the various terms. We consider the set $V_n = \{\mathbf{w} \in V : [\![\mathbf{w}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!] = \{\mathbf{w}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}\} = \mathbf{0}\}$, clearly $\delta(\mathbf{v}) = \delta(\mathbf{u})$ for all $\mathbf{v} \in V_n$ and consequently $j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) = 0$. Moreover, functions of the form $\mathbf{v} - \mathbf{u}$ spans the subspace $V_{n,0} = \{\mathbf{w} \in V : [\![\mathbf{w}_{t_\Gamma}]\!] = \{\mathbf{w}_{t_\Gamma}\} = \mathbf{0}\}$, and consequently (47) implies

$$\big([\![\sigma_{n\Gamma}]\!], \{v_n\}\big)_\Gamma \geq 0 \quad \forall \mathbf{v} \in V_{n,0},$$

and thus, $[\![\sigma_{n\Gamma}]\!] = 0$ a.e. on Γ. We can now examine the second term by considering the set $V_t = \{\mathbf{w} \in V : [\![\mathbf{w}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!] = \mathbf{0},\, \{w_n - u_{n\Gamma}\} = 0\}$ to obtain by similar means that

$$\big([\![\boldsymbol{\tau}]\!], \{\mathbf{w}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}\}\big)_\Gamma = 0, \quad \forall \mathbf{w} \in V_t$$

and thus $[\![\boldsymbol{\tau}]\!] = 0$ a.e. in Γ. Thus, inequality (46) reduces to

$$\big(\{\boldsymbol{\tau}\}, [\![\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!]\big)_\Gamma + j(\mathbf{u}, \mathbf{v}) - j(\mathbf{u}, \mathbf{u}) \geq 0, \, \forall \mathbf{v} \in V. \tag{48}$$

We first note that $\big(\{\boldsymbol{\tau}\}, [\![\mathbf{v}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}]\!]\big)_\Gamma = \big(\{\boldsymbol{\tau}\}, \delta(\mathbf{v}_{t_\Gamma})\big)_\Gamma - \big(\{\boldsymbol{\tau}\}, \delta(\mathbf{u}_{t_\Gamma})\big)_\Gamma$ and, consequently, using the definition of j

$$\int_\Gamma \{\boldsymbol{\tau}\} \cdot \delta(\mathbf{v}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{v})| \, d\gamma - \int_\Gamma \{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{u}_{t_\Gamma})| \, d\gamma \geq 0 \tag{49}$$

Now, since $\mathbf{v}$ is any element of the space V, and thus we may choose a $\mathbf{v} = \mathcal{R}(\zeta\{\boldsymbol{\tau}\})$ with an arbitrarily $\zeta \in \mathbb{R}$, where $\mathcal{R}$ is an extension operator $H^{1/2}_{00}(\Gamma) \to V$. Then it is clear that the inequality can be satisfied only if

$$(\zeta - 1) \int_\Gamma \{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{u}_{t_\Gamma})| \, d\gamma \geq 0 \quad \forall \zeta \geq 0,$$

but this is possible only if

$$\int_\Gamma \{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{u}_{t_\Gamma})| \, d\gamma = 0. \tag{50}$$

We now exploit again inequality (49) with $\mathbf{v} = \mathcal{R}(\zeta\{\boldsymbol{\tau}\})$, where now ζ is a positive function in $C^\infty(\Gamma)$ with compact support in Γ. We have that (49), together with (50) is satisfied if and only if

$$\int_\Gamma \zeta(\{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{u}_{t_\Gamma})) \geq 0,$$

which means that $(\{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{u}_{t_\Gamma}))$ is a.e. positive in Γ, but then, because of (50) this means

$$\{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) + \mu_f |\overline{\sigma}^e_{n_\Gamma}| |\delta(\mathbf{u}_{t_\Gamma}) = 0 \quad \text{a.e. in } \Gamma,$$

which is nothing else that the second equation in (36), since $\overline{\sigma}^e_{n_\Gamma} \leq 0$. We are then left with

$$\mathbf{v} \in V.$$

Since $\{\boldsymbol{\tau}\} \cdot \delta(\mathbf{v}) \geq -|\{\boldsymbol{\tau}\}| |\delta(\mathbf{v})|$, and v is an arbitrary vector of V, the last inequality implies

$$|\{\boldsymbol{\tau}\}| - \mu_f |\overline{\sigma}^e_{n_\Gamma}| \leq 0,$$

i.e. $\mathcal{G} \leq 0$. And this completes the proof. □

La dimostrazione precedente va scritta formalmente meglio, ma dovrebbe essere moralmente a posto, il che mi rincuora

3.3 Variational formulation with Nitsche's penalization

mancano ancora tutte le condizioni sui dati.... In this work we propose to impose the coupling considions on Γ by the Nitsche's penalization approach, following the ideas of [**?**, **?**, **?**], extended and adapted to our case, in the context of a fixed-stress splitting. We present the method still in the continuous setting, even if its the penalization parameter will eventually be linked to the discretization. **Non sono sicuro sia una grande idea, magari poi cambiamo e usiamo direttamente gli spazi discreti...**. For the sake of simplicity we omit the index $(k+1)$ for quantities at the current iteration level.

Solution of the structural problem Since we are using fixed stress we consider the structural problem (21) omitting the index $(k+1)$ and assuming the pressure p^{k+1} as given. We proceed formally to construct a consistent penalization scheme. To this purpose we consider, for the sake of simplicity, homogeneous boundary conditions for the structural problem, the extension to more general condition can be made in the usual way. So $\mathbf{u} = \mathbf{0}$ on $\partial\Omega^D$ and $\boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega^N$, with $\partial\Omega^D \cup \partial\Omega^N = \partial\Omega_\Gamma$, and $|\partial\Omega^D| > 0$. We define $\mathbf{V} = \{\mathbf{w} \in \mathbf{H}^1(\Omega_\Gamma) : \mathbf{w} = \mathbf{0}, \text{ on } \partial\Omega^D\}$. Note that it is a broken space. We indicate with $< \cdot, \cdot >$ the duality pair between $\mathbf{V}$ and $\mathbf{V}'$.

We proceed formally, by considering a $\mathbf{v} \in V$. We use the following equalities

$$< -\mathbf{div}\, \boldsymbol{\sigma}^e(\mathbf{u}), \mathbf{v} >= \int_{\Omega_\Gamma} \boldsymbol{\sigma}^e(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_{\partial\Omega} ((\boldsymbol{\sigma}^e(\mathbf{u}) \cdot \boldsymbol{n}) \cdot \mathbf{v} - \int_\Gamma [\![(\boldsymbol{\sigma}^e(\mathbf{u}) \cdot \boldsymbol{n}_\Gamma) \cdot \mathbf{v}]\!]$$

The second integral in the rhs acts effectively only on $\partial\Omega^N$ and will be dealt with later. We have

$$- \int_\Gamma [\![(\boldsymbol{\sigma}^e(\mathbf{u}) \cdot \boldsymbol{n}_\Gamma) \cdot \mathbf{v}]\!] = - \int_\Gamma [\![\boldsymbol{\sigma}^e(\mathbf{u}) \cdot \boldsymbol{n}_\Gamma]\!] \cdot \{\mathbf{v}\} - \int_\Gamma \{\boldsymbol{\sigma}^e(\mathbf{u}) \cdot \boldsymbol{n}_\Gamma\} \cdot [\![\mathbf{v}]\!] =$$
$$- \int_\Gamma [\![\boldsymbol{\tau}(\mathbf{u})]\!] \cdot \{\mathbf{v}_t\} - \int_\Gamma [\![\sigma^e_{n_\Gamma}]\!]\{v_n\} - \int_\Gamma \{\boldsymbol{\tau}(\mathbf{u})\} \cdot [\![\mathbf{v}_t]\!] - \int_\Gamma \{\sigma^e_{n_\Gamma}\}[\![v_n]\!]. \quad (51)$$

We now write

$$< -b\nabla p, \mathbf{v} >= b \int_{\Omega_\Gamma} p \operatorname{div} \mathbf{v} - b \int_{\partial\Omega} p\mathbf{v} \cdot \boldsymbol{n} - b \int_\Gamma [\![p\mathbf{v} \cdot \boldsymbol{n}_\Gamma]\!] =$$
$$b \int_{\Omega_\Gamma} p \operatorname{div} \mathbf{v} - b \int_{\partial\Omega} p\mathbf{v} \cdot \boldsymbol{n} - b \int_\Gamma [\![p]\!]\{v_n\} - b \int_\Gamma \{p\}[\![v_n]\!]. \quad (52)$$

Not that

$$- \int_{\partial\Omega} ((\boldsymbol{\sigma}^e(\mathbf{u}) \cdot \boldsymbol{n}) \cdot \mathbf{v} + b \int_{\partial\Omega} p\mathbf{v} \cdot \boldsymbol{n} = - \int_{\partial\Omega} ((\boldsymbol{\sigma}(\mathbf{u}) \cdot \boldsymbol{n}) \cdot \mathbf{v},$$

while

$$- \int_\Gamma [\![\sigma^e_{n_\Gamma}]\!]\{v_n\} + b \int_\Gamma [\![p]\!]\{v_n\} = - \int_\Gamma [\![\sigma_{n_\Gamma}]\!]\{v_n\},$$

Using the previous relations, we now consider the boundary and interface conditions to write the following equality (still formal, since the problem is not well posed). Note that we are not imposing $[\![u_n]\!] = 0$ strongly.

$$\int_{\Omega_\Gamma} \boldsymbol{\sigma}^e(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_\Gamma \{\boldsymbol{\tau}(\mathbf{u})\} \cdot [\![\mathbf{v}_t]\!] - \int_\Gamma \{\sigma^e_{n_\Gamma}\}[\![v_n]\!] = \int_{\Omega_\Gamma} \rho \mathbf{g} \cdot \mathbf{v} + b \int_\Gamma \{p\}[\![v_n]\!] - b \int_{\Omega_\Gamma} p \operatorname{div} \mathbf{v},$$

for all $\mathbf{v} \in \mathbf{V}$.

Now we define two penalty coefficients $\alpha_n > 0$ and $\alpha_t > 0$ (maybe functions) whose expression will be clear later on, and

$$\begin{aligned} A(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\Gamma} \boldsymbol{\sigma}^e(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \int_\Gamma \{\boldsymbol{\tau}(\mathbf{u})\} \cdot [\![\mathbf{v}_t]\!] - \int_\Gamma \{\sigma^e_{n_\Gamma}(\mathbf{u})\}[\![v_n]\!] \\ + \int_\Gamma \alpha_n [\![u_{n_\Gamma}]\!][\![v_n]\!] + \int_\Gamma \alpha_t [\![\mathbf{u}_{t_\Gamma}]\!] \cdot [\![\mathbf{v}_{t_\Gamma}]\!] \\ - \int_\Gamma [\![\mathbf{u}_{t_\Gamma}]\!] \cdot \{\boldsymbol{\tau}(\mathbf{v})\} - \int_\Gamma [\![u_{n_\Gamma}]\!]\{\sigma^e_{n_\Gamma}(\mathbf{v})\} \end{aligned} \tag{53}$$

$$F(\mathbf{v}) = \int_{\Omega_\Gamma} \rho \mathbf{g} \cdot \mathbf{v} - b \int_\Gamma \{p\}[\![v_n]\!] + b \int_{\Omega_\Gamma} p \operatorname{div} \mathbf{v} + \int_\Gamma \alpha_t [\![\mathbf{u}^n]\!] \cdot [\![\mathbf{v}_{t_\Gamma}]\!] - \int_\Gamma [\![\mathbf{u}^n_{t_\Gamma}]\!] \cdot \{\boldsymbol{\tau}(\mathbf{v})\}, \tag{54}$$

$$h(\boldsymbol{\beta}, \mathbf{v}) = \int_\Gamma \alpha_t \boldsymbol{\beta} \cdot [\![\mathbf{v}_{t_\Gamma}]\!] - \int_\Gamma \{\boldsymbol{\tau}(\mathbf{v})\} \cdot \boldsymbol{\beta}. \tag{55}$$

We define

$$\mathcal{G}_p(\mathbf{u}) = |\{\boldsymbol{\tau}(\mathbf{u})\}| + \mu_f \sigma^e_{n_\Gamma}(\mathbf{u}, p) = |\{\boldsymbol{\tau}(\mathbf{u})\}| - \mu_f |\sigma^e_{n_\Gamma}(\mathbf{u}, p)|$$

and the projection operator $H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \subset \mathcal{K}_\mu$

$$\mathcal{P}(\mathbf{w}) = \mathbf{w} - \max(0, \mathcal{G}_p(\mathbf{w})) \frac{\mathbf{w}}{|\mathbf{w}|}, \tag{56}$$

and

$$D(\mathbf{w}) = \mathbf{w} - \mathcal{P}(\mathbf{w}) = \max(0, \mathcal{G}_p(\mathbf{w})) \frac{\mathbf{w}}{|\mathbf{w}|} \tag{57}$$

We have

$$D(\{\boldsymbol{\tau}(\mathbf{u})\}) = \max(0, 1 - \frac{\mu_f |\sigma^e_{n_\Gamma}(\mathbf{u}, p)|}{|\{\boldsymbol{\tau}(\mathbf{u})\}|})\{\boldsymbol{\tau}(\mathbf{u})\} \tag{58}$$

and we may note that it is a non-negative operator, in the sense that

$$\int_\Gamma D(\{\boldsymbol{\tau}(\mathbf{u})\}) \cdot \{\boldsymbol{\tau}(\mathbf{u})\} \geq 0. \tag{59}$$

4 The proposed procedure

We now investigate now an approach derived from the considerations in []. First of all we (re)define the following:

- Projection operator $\mathcal{P}_\rho : \mathbb{R}^d \to \mathbb{R}^d$, for a $\rho \in \mathbb{R}$:

$$\mathcal{P}_\rho(\mathbf{v}) = \begin{cases} \mathbf{0} & \text{if } \rho \leq 0 \\ \mathbf{v} & \text{if } |\mathbf{v}| \leq \rho \\ \rho \mathbf{v}|\mathbf{v}|^{-1} & \text{otherwise} \end{cases} = \mathbf{v} - \max(0, |\mathbf{v}| - \rho)\mathbf{v}|\mathbf{v}|^{-1}. \tag{60}$$

  Note that we may write the formula above more compactly as

$$\mathcal{P}_\rho(\mathbf{v}) = \mathbf{v} - \max\left(0, |\mathbf{v}| - \max(\rho, 0)\right)\mathbf{v}|\mathbf{v}|^{-1}. \tag{61}$$

- Augmented tangential stress $\mathbf{N} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, for $\gamma > 0$:

$$\mathbf{N} = \mathbf{N}(\mathbf{u}, \boldsymbol{\beta}) = \{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}. \tag{62}$$

  We will set $\mathbf{N}(\mathbf{u}) = \mathbf{N}(\mathbf{u}, \delta(\mathbf{u}_{t_\Gamma})) = \{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\delta(\mathbf{u}_{t_\Gamma})$, where we recall that $\delta(\mathbf{u}_{t_\Gamma}) = [\![\mathbf{u}_{t_\Gamma} - \mathbf{u}_{t_\Gamma}^{(n)}]\!]$.

- Complementary function $\mathcal{D}_\rho : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$:

$$\mathcal{D}_\rho(\mathbf{u}, \boldsymbol{\beta}) = \{\boldsymbol{\tau}(\mathbf{u})\} - \mathcal{P}_\rho(\mathbf{N}(\mathbf{u}, \boldsymbol{\beta})) = \{\boldsymbol{\tau}(\mathbf{u})\} - \mathcal{P}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}) \tag{63}$$

  and we set $\mathcal{D}_\rho(\mathbf{u}) = \mathcal{D}_\rho(\mathbf{u}, \delta(\mathbf{u}_{t_\Gamma})) = \{\boldsymbol{\tau}(\mathbf{u})\} - \mathcal{P}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\delta(\mathbf{u}_{t_\Gamma}))$

We define

$$\hat{\xi}_\rho = \hat{\xi}_\rho(\mathbf{u}, \boldsymbol{\beta}) = \max\left(0, 1 - \frac{\max(\rho, 0)}{|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}|}\right) = \max\left(0, 1 - \frac{\max(\rho, 0)}{|\mathbf{N}|}\right) \tag{64}$$

We have

$$\mathcal{D}_\rho(\mathbf{u}, \boldsymbol{\beta}) = (1 - \hat{\xi})\gamma\boldsymbol{\beta} + \hat{\xi}\{\boldsymbol{\tau}(\mathbf{u})\}, \tag{65}$$

where it is immediate to verify that, by construction, $0 \leq \hat{\xi} < 1$ for all admissible values of ρ and $\mathbf{u}$. So the distance vector $\mathcal{D}_\rho$ is a *convex combination of the tangential stress and the penalty term* $\gamma\delta(\mathbf{u}_{t_\Gamma})$.

Note that $\mathcal{P}_\rho$, $\hat{\xi}_\rho$, N_ρ, and $\mathcal{D}_\rho$ may also be written as functions of $\{\boldsymbol{\tau}\}$ and $\boldsymbol{\beta}$, we will use this fact later on, and we will use, with abuse of notation, the same symbol in the two cases.

We start with the following important result

Proposition 4.1. *The friction condition* (36) *or, equivalently, the last two conditions in* (35) *are equivalent to*

$$\begin{cases} \mathcal{G}_p(\mathbf{u}) \leq 0 & \textit{if } \delta(\mathbf{u}_{t_\Gamma}) = \mathbf{0}, \\ \{\boldsymbol{\tau}(\mathbf{u})\} = \mu_f \overline{\sigma}^e_{n_\Gamma} \dfrac{\delta(\mathbf{u}_{t_\Gamma})}{|\delta(\mathbf{u}_{t_\Gamma})|} & \textit{otherw.} \end{cases} \tag{66}$$

and also to

$$\mathcal{D}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{u}) = 0 \tag{67}$$

Proof. Let's start from the first statement and assume that $\mathbf{u}$ satisfies (36), let's indicate, as before, $\delta(\mathbf{u}_{t_\Gamma}) = \boldsymbol{\beta}$. Where $\boldsymbol{\beta} = \mathbf{0}$ we necessarily have $\mathcal{G}_p(\mathbf{u}) \le 0$, which is the first condition in (66). If $\boldsymbol{\beta} = \mathbf{0}$ then $\mathcal{G}_p(\mathbf{u}) = 0$, i.e.

$$|\{\boldsymbol{\tau}\}| = -\mu_f \overline{\sigma}^e_{n_\Gamma}, \tag{68}$$

by which $\overline{\sigma}^e_{n_\Gamma} \le 0$. Furthermore the second condition in (36) implies that $\boldsymbol{\beta} = \beta\{\boldsymbol{\tau}\}$ for a negative β, consequently $\{\boldsymbol{\tau}\} = \frac{1}{\beta}\boldsymbol{\beta}$ and therefore (68) implies $\{\boldsymbol{\tau}\} = \mu_f \overline{\sigma}^e_{n_\Gamma} \dfrac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}$, i.e. the second of (66).

Conversely, let's assume that (66) is satisfied. If $\mathbf{u}$ satisfies the first condition, i.e. $\boldsymbol{\beta} = \mathbf{0}$ and thus $\mathcal{G}_p(\mathbf{u}) \le 0$, then clearly (36) is satisfied. If $\boldsymbol{\beta} \neq \mathbf{0}$ then we have $\{\boldsymbol{\tau}\} = \mu_f \overline{\sigma}^e_{n_\Gamma} \dfrac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|}$ that is equivalent to the second condition in (36).

Lets now consider the second proposition and assume that (36) is satisfied. If $\boldsymbol{\beta} = \mathbf{0}$ then $|\{\boldsymbol{\tau}\} - \gamma\boldsymbol{\beta}| = |\{\boldsymbol{\tau}\}| \le -\mu_f \overline{\sigma}^e_{n_\Gamma}$. This implies $\mathcal{P}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{N}(\mathbf{u})) = \{\boldsymbol{\tau}(\mathbf{u})\}$, and thus $\mathcal{D}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{u}) = \mathbf{0}$. If instead $\boldsymbol{\beta} \neq \mathbf{0}$ then $\{\boldsymbol{\tau}\} \cdot \delta(\mathbf{u}_{t_\Gamma}) = \mu_f \overline{\sigma}^e_{n_\Gamma} |\delta(\mathbf{u}_{t_\Gamma})| < 0$ and $|\{\boldsymbol{\tau}\}| = -\mu_f \overline{\sigma}^e_{n_\Gamma} = \mu_f |\overline{\sigma}^e_{n_\Gamma}|$ (since necessarily $\overline{\sigma}^e_{n_\Gamma} \le 0$). Moreover, we have already seen that in this case (36) implies $\boldsymbol{\beta} = \beta\{\boldsymbol{\tau}(\mathbf{u})\}$ with $\beta < 0$, thus $\boldsymbol{N}(\mathbf{u}) = \{\boldsymbol{\tau}\} - \gamma\boldsymbol{\beta} = \{\boldsymbol{\tau}\} - \gamma\beta\{\boldsymbol{\tau}\} = (1+\omega)\{\boldsymbol{\tau}\}$, with $\omega = -\gamma\beta > 0$. So, $|\mathbf{N}(\mathbf{u})| = |\{\boldsymbol{\tau}\} - \gamma\boldsymbol{\beta}| = (1+\omega)|\{\boldsymbol{\tau}\}| = (1+\omega)\mu_f|\overline{\sigma}^e_{n_\Gamma}| > \mu_f|\overline{\sigma}^e_{n_\Gamma}|$. Consequently,

$$\begin{aligned}\mathcal{P}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{N}(\mathbf{u})) = -\mu_f \overline{\sigma}^e_{n_\Gamma} \frac{\mathbf{N}(\mathbf{u})}{|\mathbf{N}(\mathbf{u})|} = \mu_f |\overline{\sigma}^e_{n_\Gamma}| \frac{\mathbf{N}(\mathbf{u})}{(1+\omega)\mu_f|\overline{\sigma}^e_{n_\Gamma}|} &= \\ \frac{1}{1+\omega}\mathbf{N}(\mathbf{u}) = \frac{1+\omega}{1+\omega}\{\boldsymbol{\tau}(\mathbf{u})\} = \{\boldsymbol{\tau}(\mathbf{u})\}&,\end{aligned}$$

thus $\mathcal{D}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{u}) = \mathbf{0}$. So, (36) leads to the satisfaction of (67).

Now, the converse. Let's assume (67) be verified. We have two possibilities. The first is

$$|\mathbf{N}(\mathbf{u})| = |\{\boldsymbol{\tau}\} - \gamma\boldsymbol{\beta}| \le -\mu_f \overline{\sigma}^e_{n_\Gamma}, \tag{69}$$

(which implies $\overline{\sigma}^e_{n_\Gamma} \le 0$). Then, $\mathcal{D}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{u}) = \mathbf{0}$ implies $\{\boldsymbol{\tau}\} - \{\boldsymbol{\tau}\} + \gamma\boldsymbol{\beta} = \gamma\boldsymbol{\beta} = \mathbf{0}$, thus $\boldsymbol{\beta} = 0$ and (69) gives $|\{\boldsymbol{\tau}\}| \le -\mu_f \overline{\sigma}^e_{n_\Gamma}$, i.e. $\mathcal{G}_p(\mathbf{u}) \le 0$, which is in accord with (36) (and all other equivalent formulations).

We now assume that instead

$$|\mathbf{N}(\mathbf{u})| = |\{\boldsymbol{\tau}\} - \gamma\boldsymbol{\beta}| > -\mu_f \overline{\sigma}^e_{n_\Gamma}, \tag{70}$$

by which,

$$\mathcal{P}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{N}(\mathbf{u})) = -\mu_f \overline{\sigma}^e_{n_\Gamma} \frac{\mathbf{N}(\mathbf{u})}{|\mathbf{N}(\mathbf{u})|} = \frac{-\mu_f \overline{\sigma}^e_{n_\Gamma}}{|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}|}\left(\boldsymbol{\tau}(\mathbf{u}) - \gamma\boldsymbol{\beta}\right).$$

Consequently, $\mathcal{D}_{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{u}) = \mathbf{0}$ implies

$$\{\boldsymbol{\tau}(\mathbf{u})\} + \frac{\mu_f \overline{\sigma}^e_{n_\Gamma}}{|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}|}\left(\boldsymbol{\tau}(\mathbf{u}) - \gamma\boldsymbol{\beta}\right) = \mathbf{0},$$

and therefore,

$$\boldsymbol{\beta} = \frac{1}{\gamma}\left(\frac{|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}|}{\mu_f \overline{\sigma}^e_{n_\Gamma}} + 1\right)\{\boldsymbol{\tau}(\mathbf{u})\}.$$

With the assumption that $\overline{\sigma}^e_{n_\Gamma} < 0$, we can write

$$\begin{aligned} &\boldsymbol{\beta} = \beta\{\boldsymbol{\tau}(\mathbf{u})\}, \quad \text{with} \\ &\beta = \frac{1}{\gamma}\left(1 - \frac{|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}|}{\mu_f |\overline{\sigma}^e_{n_\Gamma}|}\right) < 0 \end{aligned} \tag{71}$$

since $|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}| > \mu_f|\overline{\sigma}^e_{n_\Gamma}|$. We then need only to check that $\mathcal{G}_p(\mathbf{u}) = 0$. Since we have already found that

$$\{\boldsymbol{\tau}(\mathbf{u})\} = \frac{\mu_f|\overline{\sigma}^e_{n_\Gamma}|}{|\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}|}\left(\{\boldsymbol{\tau}(\mathbf{u})\} - \gamma\boldsymbol{\beta}\right),$$

by taking the Euclidean norm on both sides we get

$$|\{\boldsymbol{\tau}(\mathbf{u})\}| = \mu_f|\overline{\sigma}^e_{n_\Gamma}|.$$

By which $\mathcal{G}_p(\mathbf{u}) = 0$, and this completes the proof. □

Remark 4.1. *Note that in all previous derivations we have considered only $\overline{\sigma}^e_{n_\Gamma} \leq 0$, indeed the results may be extended to consider also the case $\overline{\sigma}^e_{n_\Gamma} > 0$, for which we should have $\{\boldsymbol{\tau}\} = \mathbf{0}$, by setting that in this case $\mathcal{P}_{-\mu_f\overline{\sigma}^e_{n_\Gamma}}(\mathbf{u}) = \mathbf{0}$.*

4.1 Some remarks

This interpretation of the friction conditions opens up different possibility. The first, developed by Y. Renard, Chouly et al. is to develop a Nitsche's formulation that exploits this condition. We wil describe it in a following section. The other are

1. Use a control problem formulation where the functional to be minimized is equivalent to

$$\mathcal{D}_{-\mu_f\overline{\sigma}^e_{n_\Gamma}}(\mathbf{N}(\mathbf{u},\boldsymbol{\beta})) = \mathbf{0}.$$

For instance: Find $\boldsymbol{\beta}$ and $\mathbf{u}(\boldsymbol{\beta})$ such that

$$A\mathbf{u}(\boldsymbol{\beta}) = F + H\mathbf{u}(\boldsymbol{\beta})$$

and

$$\boldsymbol{\beta} = \text{argmin}_{\boldsymbol{\gamma}} \frac{1}{2}\int_\Gamma \left|\mathcal{D}_{-\mu_f\overline{\sigma}^e_{n_\Gamma}(\mathbf{u}(\boldsymbol{\gamma}))}(\mathbf{u},\boldsymbol{\gamma})\right|^2$$

2. Solve with an iterative method the problem

$$?\boldsymbol{\beta}: \ \mathcal{D}_{-\mu_f\overline{\sigma}^e_{n_\Gamma}(\mathbf{u}(\boldsymbol{\beta}))}(\mathbf{N}(\mathbf{u}(\boldsymbol{\beta}),\boldsymbol{\beta})) = \mathbf{0}$$

where

$$\mathbf{u}(\boldsymbol{\beta}) = A^{-1}(F + H\boldsymbol{\beta}). \tag{72}$$

4.2 Derivatives

To implement both strategies we need to have the expressions for some derivatives. For a $F : U \to V$ we indicate with $\partial_u F(w)$ the Frechèt derivative evaluated in $w \in U$, i.e. a linear operator $U \to V$, and in particular with $\partial_u F(w)h$ we indicate the element of V obtained by the application of $\partial_u F(w)$ to $h \in U$. We use bold symbols, as usual, to indicate vector quantities.

First of all we recall that $\{\boldsymbol{\tau}\}(\mathbf{u})$ is a linear function $H^1(\Omega_\Gamma) \to H^{-1/2}(\Gamma)$, and we have already identified it with the operator C, we may alternatively write $\partial_{\mathbf{u}}\{\boldsymbol{\tau}\}(\mathbf{u})\mathbf{w} = \{\boldsymbol{\tau}\}(\mathbf{w}) = \{\boldsymbol{\tau}(\mathbf{w})\}$. Moreover if we write our problem formally as in (72)

$$\partial_{\boldsymbol{\beta}}\mathbf{u}(\boldsymbol{\beta})\boldsymbol{\gamma} = A^{-1}H\boldsymbol{\gamma},$$

i.e. the solution of the mechanical problem with displacement jumps equal to $\boldsymbol{\gamma}$ (and zero normal displacements on Γ, and homogeneous b.c. everywhere else). In some cases it may be necessary to set $\boldsymbol{\beta} = \delta(\mathbf{u}_{t_\Gamma})$ (it depends on the way we want to tackle the problem). It is sufficient to note that

$$\partial_{\mathbf{u}}\delta(\mathbf{u}_{t_\Gamma})\mathbf{h} = [\![\mathbf{h}]\!], \tag{73}$$

being the jump a linear operator $\mathbf{V} \to \mathbf{H}^{1/2}_\Gamma$.

We have also

$$\partial_{\mathbf{u}}\mathbf{N}(\mathbf{u},\boldsymbol{\beta})\mathbf{h} = \{\boldsymbol{\tau}(\mathbf{h})\}$$

and

$$\partial_{\boldsymbol{\beta}}\mathbf{N}(\mathbf{u},\boldsymbol{\beta})\boldsymbol{\zeta} = \gamma\boldsymbol{\zeta}$$

Thus

$$\partial_{\boldsymbol{\beta}}\mathbf{N}(\mathbf{u}(\boldsymbol{\beta}),\boldsymbol{\beta})\boldsymbol{\zeta} = \{\boldsymbol{\tau}(A^{-1}H\boldsymbol{\zeta})\} + \gamma\boldsymbol{\zeta}.$$

Now, lets consider in general

$$\mathcal{D}_\rho(\mathbf{u},\boldsymbol{\beta}) = \{\boldsymbol{\tau}(\mathbf{u})\} - \mathcal{P}_\rho(\mathbf{N}(\mathbf{u},\boldsymbol{\beta})).$$

To find the derivative we proceed by considering each component in turn. First we consider $\mathcal{P}_\rho : \mathbf{W} \to \mathbf{W}$ where $\mathbf{W} = \{\mathbf{w} \in \mathbf{L}^2(\Gamma) : \ \mathbf{w}\cdot\boldsymbol{n}_\Gamma = 0\}$. In other words we interpret the projection as operating on *vector function tangent to* Γ. For simplicity we continue to consider $\boldsymbol{n}_\Gamma$ as a constant (i.e. Γ is a plane), even if in the thesis of Mlika [3] we have also the expression for the derivative w.r.t. the normal direction). As before we write the expression of the derivative applied to a vector, in this case a vector of $\mathbf{W}$.

$$\partial_{\mathbf{q}}\mathcal{P}_\rho(\mathbf{q})\mathbf{w} \in \begin{cases} \mathbf{0} & \text{if } \rho < 0 \\ \mathbf{w} & \text{if } |\mathbf{q}| < \rho,\ \rho > 0 \\ \dfrac{\rho}{|\mathbf{q}|}\left(\mathbf{w} - \dfrac{(\mathbf{q},\mathbf{w})}{|\mathbf{q}|^2}\mathbf{q}\right) & \text{if } |\mathbf{q}| > \rho,\ \rho > 0 \end{cases} \tag{74}$$

It's values for $\rho = 0$ or $|\mathbf{q}| = \rho$ are not defined, that's why it is better to think it as an inclusion. In practice, we will need to choose which values to take in those limit cases. Note that the last line in the previous relation may also be written as

$$\left(\frac{\rho}{|\mathbf{q}|}\mathbf{I} - \frac{\rho}{|\mathbf{q}|^3}\mathbf{q}\otimes\mathbf{q}\right)\mathbf{w} \tag{75}$$

where $\mathbf{I}$ is the $d\times d$ identity matrix and $\mathbf{q}\otimes\mathbf{q}$ denotes the symmetric matrix with components $[\mathbf{q}\otimes\mathbf{q}]_{ij} = q_i q_j$.

Rather interesting is also the derivative w.r.t. ρ applied to a variation of ρ.

$$\partial_\rho \mathcal{P}_\rho(\mathbf{q})\delta\rho \in \begin{cases} \mathbf{0} & \text{if } \rho < 0 \\ \mathbf{0} & \text{if } |\mathbf{q}| < \rho,\ \rho > 0 \\ \dfrac{\mathbf{q}}{|\mathbf{q}|}\delta\rho & \text{if } |\mathbf{q}| > \rho,\ \rho > 0 \end{cases} \tag{76}$$

Now, in our problem $\rho = -\mu_f \overline{\sigma}^e_{n_\Gamma}$, which is itself a function of $\mathbf{u}$. Therefore in computing $\partial_{\mathbf{u}}\mathbf{D}{-\mu_f \overline{\sigma}^e_{n_\Gamma}}(\mathbf{u},\boldsymbol{\beta})$ I should take into account the component illustrated in (76) as well as $\partial_{\mathbf{u}}\overline{\sigma}^e_{n_\Gamma}(\mathbf{u})$. This is overly complicated and introduces further strong linearities. Since we will solve the problem within an iterative procedure, we may try to replace the derivatives with approximations. We might consider the approximation

$$\partial_{\mathbf{u}}|_{\overline{\rho}}\mathbf{D}\rho(\mathbf{u},\boldsymbol{\beta}) = \partial_{\mathbf{u}}\mathbf{D}\overline{\rho}(\mathbf{u},\boldsymbol{\beta}), \tag{77}$$

i.e. the derivative assuming ρ fixed at the value $r\hat{h}o$.

It may be convenient also to consider, always taking for simplicity $\hat{\rho}$ fixed, the derivatives of $\mathcal{D}(\{\boldsymbol{\tau}\},\boldsymbol{\beta})$, where, for simplicity I indicate with $\partial_{\boldsymbol{\tau}}$ the derivative w.r.t. $\{\boldsymbol{\tau}\}$. Let's fist look at the directional derivative and use the definition in (**??**). After a few manipulations we have

$$\partial_{\boldsymbol{\tau}}\mathcal{D}_{\hat{\rho}}(\boldsymbol{\tau},\boldsymbol{\beta}) = \begin{cases} \boldsymbol{I} & \text{if } \hat{\rho} \le 0 \\ \hat{\xi}_{\hat{\rho}}\boldsymbol{I} + \dfrac{\hat{\rho}\mathbf{N}\otimes\mathbf{N}}{|\mathbf{N}|^3} & \text{if } 0 < \hat{\rho} < |\mathbf{N}| \\ \mathbf{0} & \text{if } \hat{\rho} > |\mathbf{N}| \end{cases} \tag{78}$$

while

$$\partial_{\boldsymbol{\beta}}\mathcal{D}_{\hat{\rho}}(\boldsymbol{\tau},\boldsymbol{\beta}) = \begin{cases} \mathbf{0} & \text{if } \hat{\rho} \le 0 \\ (1-\hat{\xi}_{\hat{\rho}})\gamma\boldsymbol{I} - \dfrac{\hat{\rho}\gamma\mathbf{N}\otimes\mathbf{N}}{|\mathbf{N}|^3} & \text{if } 0 < \hat{\rho} < |\mathbf{N}| \\ \gamma\mathbf{I} & \text{if } \hat{\rho} > |\mathbf{N}| \end{cases} \tag{79}$$

Or use the definition $\mathbf{D}_\rho$ in (**??**) assuming $\hat{\xi}$ fixed at a given value $\overline{\xi}$. Thus,

$$\partial_{\mathbf{u}}|_{\overline{\xi}}\mathbf{D}\rho(\mathbf{u},\boldsymbol{\beta}) = (\overline{\xi}\partial_{\mathbf{u}}\{\boldsymbol{\tau}(\mathbf{u})\}. \tag{80}$$

We have, after a few calculations,

$$\partial_{\mathbf{u}}|_{\overline{\xi}}\mathbf{D}\rho(\mathbf{u},\boldsymbol{\beta})\mathbf{w} = \overline{\xi}\{\boldsymbol{\tau}(\mathbf{w})\} \tag{81}$$

while

$$\partial_{\mathbf{u}}|_{\overline{\rho}}\mathbf{D}\rho(\mathbf{u},\boldsymbol{\beta})\mathbf{w} \in \begin{cases} \{\boldsymbol{\tau}(\mathbf{w})\} & \text{if } \rho < 0, \\ \mathbf{0} & \text{if } |\mathbf{N}| < \rho,\ \rho > 0, \\ \left((1 - \frac{\overline{\rho}}{|\mathbf{N}|})\mathbf{I} + \frac{\overline{\rho}\mathbf{N}\otimes\mathbf{N}}{|\mathbf{N}|^3}\right)\{\boldsymbol{\tau}(\mathbf{w})\} & \text{if } |\mathbf{N}| > \rho,\ \rho > 0, \end{cases} \tag{82}$$

where $\mathbf{N} = \mathbf{N}(\mathbf{u},\boldsymbol{\beta})$. We also note that in the previous two relations the term that multiplies $\{\boldsymbol{\tau}(\mathbf{w})\}$ is the approximate derivative of $\mathbf{D}_\rho$ with respect to $\{\boldsymbol{\tau}(\mathbf{w})\}$ (it is sufficient to apply the chain rule!). This is relevant for the latter.

The knowledge of those derivatives may help in setting up various techniques.

4.2.1 Properties of the derivatives

The derivative of $\mathbf{D}_\rho$ are not continuous at the point $|\mathbf{N}| = \rho > 0$, and at $\rho = 0$, but we might try to see how their norm behaves. We first consider the quantities as functions of $\{\boldsymbol{\tau}\}$, and $\boldsymbol{\beta}$ We have the following results We have the following

Proposition 4.2. *The derivatives of* $\mathbf{N}$ *as function of the tangential stress and slip satisfy*

$$\begin{cases} \|\partial_{\boldsymbol{\tau}}\mathbf{N}(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 = 1, \\ \|\partial_{\boldsymbol{\beta}}\mathbf{N}(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 = \gamma. \end{cases} \tag{83}$$

Proof. We have

$$\begin{cases} \partial_{\boldsymbol{\tau}}\mathbf{N}(\{\boldsymbol{\tau}\},\boldsymbol{\beta}) = \mathbf{I}, \\ \partial_{\boldsymbol{\beta}}\mathbf{N}(\{\boldsymbol{\tau}\},\boldsymbol{\beta}) = -\gamma\mathbf{I}, \end{cases}$$

which provides the result □

Proposition 4.3. *The derivative of the complementary function D are bounded uniformly with respect to ρ, $\{\boldsymbol{\tau}\}$ and $\boldsymbol{\beta}$. More precisely,*

$$\begin{cases} \|\partial_{\boldsymbol{\tau}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 < 2 \\ \|\partial_{\boldsymbol{\beta}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 < 2\gamma. \end{cases} \tag{84}$$

Proof. If $\hat{\rho} \leq 0$ then $\|\partial_{\boldsymbol{\tau}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 = 1$ and $\|\partial_{\boldsymbol{\beta}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 = 0$. If $0 \leq \hat{\rho} < |\mathbf{N}|$ then

$$\|\partial_{\boldsymbol{\tau}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 \leq 1 + \frac{\rho}{|\mathbf{N}|^3}\|\mathbf{N}\otimes\mathbf{N}^T\|_2 \leq 1 + \frac{\rho}{|\mathbf{N}|} < 2$$

while

$$\|\partial_{\boldsymbol{\beta}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 \leq \gamma + \frac{\gamma\rho}{|\mathbf{N}|^3}\|\mathbf{N}\otimes\mathbf{N}^T\|_2 \leq \gamma(1 + \frac{\rho}{|\mathbf{N}|}) < 2\gamma$$

If $\hat{\rho} > |\mathbf{N}|$ then $\|\partial_{\boldsymbol{\tau}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 = 0$, while $\|\partial_{\boldsymbol{\beta}}\mathbf{D}_\rho(\{\boldsymbol{\tau}\},\boldsymbol{\beta})\|_2 = \gamma$. Collecting the various inequalities we obtain the stated result. □

Proposition 4.4. *The complementary function is Lipschitz continuous with respect to both arguments, more precisely*

$$|\mathbf{D}_\rho(\{\boldsymbol{\tau}\}_1 - \{\boldsymbol{\tau}\}_2, \boldsymbol{\beta}_1 - \boldsymbol{\beta}_2)| \leq 2|\{\boldsymbol{\tau}\}_1 - \{\boldsymbol{\tau}\}_2| + 2\gamma|\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2| \tag{85}$$

Proof. It is an immediate consequence of the previous result. □

4.3 Techniques based on the solution of $\mathcal{D}_\rho(\mathbf{u}(\beta), \boldsymbol{\beta}) = 0$

4.3.1 Steklov Poincaré operator

First of all we note that the operator $\mathcal{S} : \mathbf{H}^{\frac{1}{2}}_{00}(\Gamma) \to \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ defined as

$$\mathcal{S}(\boldsymbol{\beta}) : \boldsymbol{\beta} \to \{\boldsymbol{\tau}(\mathbf{u})(\boldsymbol{\beta})\},$$

is a Steklov-Poincaré operator, which, using the operator notations of the previous sections, may be written as

$$\mathcal{S}(\boldsymbol{\beta}) = CA^{-1}(F + H\boldsymbol{\beta}),$$

or, equivalently, $\mathcal{S} = C \circ A^{-1} \circ H + CA^{-1}F$. It is an *affine* operator, and its linear counterpart is $\mathcal{S}_0 = C \circ A^{-1} \circ H$, i.e.

$$\mathcal{S}(\boldsymbol{\beta}) : \boldsymbol{\beta} \to \{\boldsymbol{\tau}(\mathbf{u})(\boldsymbol{\beta})\} - \{\boldsymbol{\tau}(\mathbf{u})(\mathbf{0})\}.$$

We have the following result

Proposition 4.5. *The Steklow-Poincarè operator is a bounded operator, in the sense that* ***check definition of norms and spaces...***

$$\|\{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta}))\} - \{\boldsymbol{\tau}(\mathbf{u}(\mathbf{0}))\}\|_{T'_\tau} \leq C\|\boldsymbol{\beta}\|_{T_\tau}, \tag{86}$$

for a constant C independent of $\boldsymbol{\beta}$.

Proof. It is a standard result of continuity of trace operators. □

An important note: Since we will be treating quasi-static mechanical problems, we will consider

$$\boldsymbol{\beta} = [\![\mathbf{u}_t]\!] - [\![\mathbf{u}_t^n]\!], \tag{87}$$

(the term Δt being in fact accounted for by the coefficient γ).

where $[\![\mathbf{u}_t^n]\!]$ is the slip already reached at the previous iteration. Thus the solution $\mathbf{u}(\boldsymbol{\beta})$ will satisfy the condition

$$[\![\mathbf{u}_t(\boldsymbol{\beta})]\!] = [\![\mathbf{u}_t^n]\!] + \boldsymbol{\beta}. \tag{88}$$

4.4 Derivatives

To implement first order methods we need to compute some derivatives w.r.t.$\boldsymbol{\beta}$.

4.5 Derivative of $\{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta}))\}$

Let $\mathbf{R} : \mathcal{T} \to \mathbf{V}$ a lifting operator, then $\mathbf{R}\boldsymbol{\beta} = \mathbf{R}_\Gamma\boldsymbol{\beta} + \mathbf{R}_1\boldsymbol{\beta}$ where $\mathbf{R}_\Gamma\boldsymbol{\beta} \in \mathbf{V}_\Gamma$ is the projection on $\mathbf{V}_\Gamma = \{\mathbf{v} \in \mathbf{V} : \ \mathbf{v}|_\Gamma = \mathbf{0}\}$ defined by

$$a(\mathbf{R}_\Gamma\boldsymbol{\beta}, \mathbf{v}) = a(\mathbf{R}\boldsymbol{\beta}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}_\Gamma.$$

Therefore, being $\mathbf{u}(\boldsymbol{\beta}) = \mathbf{u}_\Gamma(\boldsymbol{\beta}) + \mathbf{R}\boldsymbol{\beta}$, with $\mathbf{u}_\Gamma(\boldsymbol{\beta})$ the solution of the problem

$$a(\mathbf{u}_\Gamma(\boldsymbol{\beta}), \mathbf{v}) = F(\mathbf{v}) - a(\mathbf{R}\boldsymbol{\beta}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_\Gamma$$

by applying the Gauss theorem we have , for all $\boldsymbol{t} \in \mathcal{T}$,

$$\begin{aligned} < \{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta})\}, \boldsymbol{t} >= a(\mathbf{u}(\boldsymbol{\beta}), \mathbf{R}\boldsymbol{t}) - F(\mathbf{R}\boldsymbol{t}) = \\ a(\mathbf{u}_\Gamma(\boldsymbol{\beta}), \mathbf{R}\boldsymbol{t}) + a(\mathbf{R}\boldsymbol{\beta}, \mathbf{R}\boldsymbol{t}) - F(\mathbf{R}\boldsymbol{t}) = \\ a(\mathbf{u}_\Gamma(\boldsymbol{\beta}), \mathbf{R}_\Gamma\boldsymbol{t}) + a(\mathbf{u}_\Gamma(\boldsymbol{\beta}), \mathbf{R}_1\boldsymbol{t}) + a(\mathbf{R}\boldsymbol{\beta}, \mathbf{R}_\Gamma\boldsymbol{t}) + a(\mathbf{R}\boldsymbol{\beta}, \mathbf{R}_1\boldsymbol{t}) - F(\mathbf{R}_\Gamma\boldsymbol{t}) - F(\mathbf{R}_1\boldsymbol{t}) = \\ a(\mathbf{R}\boldsymbol{\beta}, \mathbf{R}_1\boldsymbol{t}) - F(\mathbf{R}_1\boldsymbol{t}) = a(\mathbf{R}_1\boldsymbol{\beta}, \mathbf{R}_1\boldsymbol{t}) - F(\mathbf{R}_1\boldsymbol{t}). \end{aligned} \tag{89}$$

, By which, $\partial_\beta\{\boldsymbol{\tau}\}$ may be represented as

$$< \partial_\beta\{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta})\}, \boldsymbol{t} >= a(\mathbf{R}_1\cdot, \mathbf{R}_1\boldsymbol{t})$$

This shows that $\partial_\beta\{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta})\}$ is a linear operator from $\mathcal{T} \subset \mathbf{H}^{1/2}_{00}(\Gamma)$ to $\mathbf{H}^{-1/2}(\Gamma)$.

4.5.1 A first fixed point iteration

A first scheme may simply be a fixed point iteration of the type: given $\boldsymbol{\beta}^{(0)}$ compute, for $k = 0, 1 \ldots$

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k+1)} - \rho^{(k)}\mathcal{D}_{-\mu_f\overline{\sigma}^e_{n_\Gamma}(\mathbf{u}(\boldsymbol{\beta}^{(k)}))}(\mathbf{u}(\beta^{(k)}), \boldsymbol{\beta}^{(k)}), \quad k = 0, 1, \cdots,$$

for a suitable $\rho^{(k)} > 0$. For the sake of simplicity, we will use the suffix $^{(k)}$ to indicate quantities at iteration k and $\boldsymbol{N}^{(k)} = \{\boldsymbol{\tau}\}^{(k)} - \gamma\boldsymbol{\beta}^{(k)}$ (the Nitsche corrected tangential stress). By simple computations, one has

$$\mathcal{D}^{(k)} = \begin{cases} \{\boldsymbol{\tau}\}^{(k)} \quad \text{if} \ \ \overline{\sigma}^e_{n_\Gamma}(\mathbf{u}(\boldsymbol{\beta}^{(k)})) \geq 0 \\ \text{otherwise:} \\ \begin{cases} \gamma\boldsymbol{\beta}^{(k)} & \text{if } |\boldsymbol{N}^{(k)}| \leq -\mu_f\overline{\sigma}^e_{n_\Gamma} \\ \{\boldsymbol{\tau}\}^{(k)} - \mu_f|\overline{\sigma}^e_{n_\Gamma}(\mathbf{u}(\boldsymbol{\beta}^{(k)}))|\dfrac{\boldsymbol{N}^{(k)}}{|\boldsymbol{N}^{(k)}|} & \text{otherw.} \end{cases} \end{cases}$$

which is *very similar to the Uzawa algorithm presented in Section* **??**.

Note that the fact that we can try to find an approximation of the local Liptschitz constant L using the results of the previous section. and set from $k = 1$

$$\rho^{(k)} = 1/L \simeq \frac{|\boldsymbol{\beta}^{(k)} - \boldsymbol{\beta}^{(k-1)}|}{2\left(|\{\boldsymbol{\tau}\}^{(k)} - \{\boldsymbol{\tau}\}^{(k-1)}| + |\boldsymbol{\beta}^{(k)} - \boldsymbol{\beta}^{(k-1)}|\right)}$$

implementing a L-scheme.

4.5.2 A second fixed point iteration

Relation (65) allows us to find a a different fixed point iteration.

Indeed, exploiting (65) the equality the form of a fixed point

$$\boldsymbol{\beta} = -\frac{\hat{\xi}}{\gamma(1-\hat{\xi})}\{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta}))\} = \boldsymbol{\Psi}(\boldsymbol{\beta}).$$

(remember that ξ depends on $\mathbf{u}(\boldsymbol{\beta})$ as well).

A standard fixed point iteration would read

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\Psi}(\boldsymbol{\beta}^{(k)}),$$

the convergence being assured if and only if $\boldsymbol{\Psi}$ is a contraction. Since this is not necessarily true (analysis needed!!) we propose the damped iteration

$$\boldsymbol{\beta}^{(k+1)} = \alpha_k \boldsymbol{\Psi}(\boldsymbol{\beta}^{(k)}) + (1-\alpha_k)\boldsymbol{\beta}^{(k)}. \tag{90}$$

The problem is to find the suitable damping. Alternatives are: Aitken or Anderson acceletation techniques, which may give the correct damping.

We write the algorithm for (90): Given $\boldsymbol{\beta}^{(0)}$, and $\mathbf{u}^{(0)}$, for $k = 0, \cdots$

1. compute $\hat{\xi}^{(k)}$ is using (64);
2. compute $\boldsymbol{\Psi}^{(k)} = -\dfrac{\hat{\xi}^{(k)}}{\gamma(1-\hat{\xi}^{(k)})}\{\boldsymbol{\tau}\}^{(k)}$;
3. Estimate (if possible) an optimal α_k;
4. Update: $\boldsymbol{\beta}^{(k+1)} = (1-\alpha_k)\boldsymbol{\beta}^{(k)} + \alpha_k \boldsymbol{\Psi}^{(k)}$.

4.6 Control problem

Here, we use a technique based on approximate derivatives, in particular we assume that ρ is given, se we ignore the variation with respect to ρ. We consider $\boldsymbol{\beta} \in U$ as our control variable. The space of control variable U will be taken as $\mathbb{T}$ defined in (**??**), while $\{\boldsymbol{\tau}\} \in O = \mathbb{T}'$ is our observed variable. More precisely we will observe $\mathbf{D}_\rho$ with $\rho = -\mu_f \overline{\sigma}^e_{n_\Gamma}$ taken constant within the optimization process. We assume that $\mathbf{D}_\rho(\{\boldsymbol{\tau}\}, \boldsymbol{\beta})$ belong to the space Z that will be made more precise afterwards. At the moment we proceed formally by assuming that D and U and Z are Hilbert spaces, thus equipped by an inner product.

We will consider the regolarised functional

$$J(\mathbf{u}, \boldsymbol{\beta}) = \frac{\omega}{2}(\mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\}, \boldsymbol{\beta}), \mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\}, \boldsymbol{\beta}))_Z + \frac{\zeta}{2}(\boldsymbol{\beta}, \boldsymbol{\beta})_U.$$

Clearly, we can consider this functional as a function only of the control variable by setting $J(\boldsymbol{\beta}) = J(\mathbf{u}(\beta), \boldsymbol{\beta})$, where we assume that for a given $\boldsymbol{\beta}$ there is a lifting function $R\boldsymbol{\beta} \in V$ with $[\![(R\boldsymbol{\beta})_t]\!] = \boldsymbol{\beta}$ and unique $\mathbf{u} = \mathbf{u}^0 + R\boldsymbol{\beta} \in V^0 + R\boldsymbol{\beta}$ so that $A(\mathbf{u}^0,\mathbf{v}) = F(\mathbf{v}) + h(\boldsymbol{\beta},\mathbf{v})$, for all $\mathbf{v} \in V^0 = \{\mathbf{v} \in V : \ [\![\mathbf{v}_t]\!] = 0\}$. Since we are in a quasi-static setting the friction condition relates the tangential stress to the **slip velocity**, that we assume have been discretized with a finite difference, we recall then relation (88): $[\![\mathbf{u}_t]\!] = [\![\mathbf{u}_t^n]\!] + \boldsymbol{\beta}$ on Γ. Furthermore, $F(\mathbf{v}) = (\boldsymbol{F},\mathbf{v})$

Here ω is a scaling factor needed to normalize the terms in the solution procedure. Its role will be clearer later on.

It is convenient to interpret J as a functional $T' \times T \to \mathbb{R}$, writing (with an abuse of notation)

$$J(\boldsymbol{\tau},\boldsymbol{\beta}) = \frac{\omega}{2}(\mathbf{D}_\rho(\boldsymbol{\tau},\boldsymbol{\beta}),\mathbf{D}_\rho(\boldsymbol{\tau},\boldsymbol{\beta}))_Z + \frac{\zeta}{2}(\boldsymbol{\beta},\boldsymbol{\beta})_U = J_D(\boldsymbol{\tau},\boldsymbol{\beta}) + \frac{\zeta}{2}\|\boldsymbol{\beta}\|_U^2,$$

where now $\boldsymbol{\tau}$ denotes here an element in T' and

$$J_D = \frac{\omega}{2}\|(\mathbf{D}_\rho(\boldsymbol{\tau},\boldsymbol{\beta})\|_Z^2. \tag{91}$$

Moreover, we have that

$$A(\mathbf{u}(\boldsymbol{\beta},\mathbf{v}) = F(\mathbf{v}) + < \{\boldsymbol{\tau}\}, [\![\mathbf{v}]\!] >, \quad \forall \mathbf{v} \in V, \tag{92}$$

which implies, that in this setting, $h(\boldsymbol{\beta},\mathbf{v}) = -A(R\boldsymbol{\beta},\mathbf{v})$.

4.6.1 The derivation of the control problem

We proceed formally by defining the following Lagrangian $L : V \times T \times V \times T' \to \mathbb{R}$

$$\begin{aligned}&L(\mathbf{w},\boldsymbol{\gamma};\mathbf{q},\boldsymbol{\alpha}) = \\ &J(\{\boldsymbol{\tau}(\mathbf{w})\},\boldsymbol{\gamma}) + < \boldsymbol{\alpha}, [\![\mathbf{w}_t]\!] - [\![\mathbf{u}_t^n]\!] - \boldsymbol{\gamma} > + A(\mathbf{w},\boldsymbol{q}) - F(\boldsymbol{q}) - < \{\boldsymbol{\tau}(\mathbf{w})\}, [\![\boldsymbol{q}]\!] >\end{aligned} \tag{93}$$

and we seek $\mathbf{u}$, $\boldsymbol{\beta}$, $\mathbf{p}$ and $\boldsymbol{\lambda}$ stationary point of L: i.e.

$$L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda}) = \inf_{(\mathbf{w},\boldsymbol{\gamma})} \sup_{(\mathbf{q},\boldsymbol{\alpha})} L(\mathbf{w},\boldsymbol{\gamma};\mathbf{q},\boldsymbol{\alpha})$$

The stationary point is found by setting to zero the derivatives. In the following we use the natation $\partial_{\mathbf{x}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\mathbf{v}]$ to denote the Gateaux derivative of L w.r.t $\mathbf{x}$ at point $(\mathbf{u},\boldsymbol{\beta},\mathbf{p},\boldsymbol{\lambda})$ applied to $\mathbf{v}$. We also indicate with $\Lambda : T' \to T$ the Ritz operator. We recall that $\boldsymbol{\tau}(\mathbf{u})$ is an affine function, so

$$\partial_{\mathbf{u}}\{\boldsymbol{\tau}(\mathbf{u})\}[\mathbf{v}] = \{\boldsymbol{\tau}(\mathbf{v})\}, \tag{94}$$

while

$$\partial_{\boldsymbol{\tau}} J_D(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta})[\mathbf{v}] = \omega < \Lambda\mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta}), \partial_{\boldsymbol{\tau}}\mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta})[\mathbf{v}] >, \tag{95}$$

and

$$\partial_{\boldsymbol{\beta}} J_D(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta})[\gamma] = \omega < \Lambda\mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta}), \partial_{\boldsymbol{\beta}}\mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta})[\gamma] > . \tag{96}$$

The primal problem We have

$$\begin{cases} \partial_{\mathbf{p}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\mathbf{v}] = A(\mathbf{u},\mathbf{v}) - F(\mathbf{v}) - < \{\boldsymbol{\tau}(\mathbf{u})\}, [\![\mathbf{v}_t]\!] >= 0 \\ \partial_{\boldsymbol{\lambda}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\boldsymbol{\phi}] = < \boldsymbol{\phi}, [\![\mathbf{u}_t]\!] - [\![\mathbf{u}_t]\!]^n - \boldsymbol{\beta} > \end{cases} \tag{97}$$

for all $\mathbf{v} \in V$ and for all $\boldsymbol{\phi} \in U'$. We can recognize the solution of problem (97) as the solution of our differential equation when we impose a slip equal to $[\![\mathbf{u}_t]\!]^n + \boldsymbol{\beta}$ on Γ. This is indeed our primal problem.

The dual problem Using (95) we can write

$$\begin{aligned} \partial_{\mathbf{u}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\mathbf{w}] = &< \partial_{\boldsymbol{\tau}} J_D(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta}), \{\boldsymbol{\tau}(\mathbf{w})\} > + \\ &\qquad < \boldsymbol{\lambda}, [\![\mathbf{w}_t]\!] > + A(\mathbf{w},\mathbf{p}) - < \{\boldsymbol{\tau}(\mathbf{w})\}, [\![\mathbf{p}_t]\!] > \end{aligned}$$

for all $W \in V$ while

$$\partial_{\boldsymbol{\beta}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\gamma] = < \partial_{\boldsymbol{\beta}} J_D(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta}), \boldsymbol{\gamma} > + \zeta(\boldsymbol{\beta},\boldsymbol{\gamma})_U - < \boldsymbol{\lambda}, \boldsymbol{\gamma} >, \tag{98}$$

for all $\boldsymbol{\gamma} \in U'$. In the following we will just write $\mathbf{D}_\rho$ and J_D for $\mathbf{D}_\rho(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta})$ and $J_D(\{\boldsymbol{\tau}(\mathbf{u})\},\boldsymbol{\beta})$, respectively.

With simple manipulations we can write,

$$\partial_{\mathbf{u}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\mathbf{w}] = A(\mathbf{p},\mathbf{w}) + < \left(\partial_{\boldsymbol{\tau}} J_D - [\![\mathbf{p}_t]\!]\right), \{\boldsymbol{\tau}(\mathbf{w})\} > + < \boldsymbol{\lambda}, [\![\mathbf{w}_t]\!] > .$$

while we have

$$\partial_{\boldsymbol{\beta}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\gamma] = < \partial_{\boldsymbol{\beta}} J_D, \boldsymbol{\gamma} > + \zeta(\boldsymbol{\beta},\boldsymbol{\gamma})_U - < \boldsymbol{\lambda}, \boldsymbol{\gamma} >$$

Luca: questo passaggio va giustificato meglio, usando stime di traccia... We now consider $\mathbf{p} \in V$ as solution of

$$A(\mathbf{p},\mathbf{w}) + < \left(\partial_{\boldsymbol{\tau}} J_D - [\![\mathbf{p}_t]\!]\right), \{\mathbf{w}\} >= 0, \forall \mathbf{w} \in V^0 \tag{99}$$

which is equivalent to solve the

$$\begin{cases} -\nabla \cdot (\boldsymbol{\sigma}(\mathbf{p})) = 0 & \text{in } \Omega_\Gamma \\ [\![\mathbf{p}_t]\!] = \partial_{\boldsymbol{\tau}} J_D & \text{on } \Gamma \\ \mathbf{p} = \mathbf{0} & \text{on } \partial\Omega^D \\ \boldsymbol{\sigma}(\mathbf{p}) \cdot \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega^N \\ [\![p_n]\!] = 0 & \text{on } \Gamma \\ [\![\boldsymbol{\sigma}(\mathbf{p}) \cdot \boldsymbol{n}_\Gamma]\!] = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{100}$$

Setting now to zero $\partial_{\mathbf{u}} L$ for all $\mathbf{w} \in V$ we get

$$A(\mathbf{p},\mathbf{w}) + < \left(\partial_{\boldsymbol{\tau}} J_D - [\![\mathbf{p}_t]\!]\right), \{\boldsymbol{\tau}(\mathbf{w})\} > + < \boldsymbol{\lambda}, [\![\mathbf{w}_t]\!] >= 0$$

by counterintegrating by parts (and setting to zero the terms that are zero) we have

$$
\begin{aligned}
< -\nabla\cdot(\boldsymbol{\sigma}(\mathbf{p})), \mathbf{w} > + < [\![\boldsymbol{\sigma}(\mathbf{p})\cdot \boldsymbol{n}_\Gamma]\!], \{\mathbf{w}\} > + < \{\boldsymbol{\sigma}(\mathbf{p})\cdot \boldsymbol{n}_\Gamma\}, [\![\mathbf{w}]\!] > + \\
< \left(\partial_{\boldsymbol{\tau}} J_D - [\![\mathbf{p}_t]\!]\right), \{\boldsymbol{\tau}(\mathbf{w})\} > + < \boldsymbol{\lambda}, [\![\mathbf{w}_t]\!] >= \\
< \{\boldsymbol{\tau}(\mathbf{p})\}, [\![\mathbf{w}_t]\!] > + < \boldsymbol{\lambda}, [\![\mathbf{w}_t]\!] >= 0.
\end{aligned}
$$

By which, using (100) and the definition of V,

$$
\boldsymbol{\lambda} = -\boldsymbol{\tau}(\mathbf{p}), \tag{101}
$$

in the sense of traces.

The previous relation allow us to derive the gradient of the Lagrangian with respect to the control as

$$
\partial_{\boldsymbol{\beta}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda})[\gamma] =< \partial_{\boldsymbol{\beta}} J_D, \boldsymbol{\gamma} > + \zeta(\boldsymbol{\beta},\boldsymbol{\gamma})_U + < \boldsymbol{\tau}(\mathbf{p}), \boldsymbol{\gamma} > . \tag{102}
$$

We also recall that in general

$$
\partial_{\boldsymbol{\beta}} L(\mathbf{u},\boldsymbol{\beta};\mathbf{p},\boldsymbol{\lambda}) = \partial_{\boldsymbol{\beta}} J(\mathbf{u}(\boldsymbol{\beta}),\boldsymbol{\beta}),
$$

so the quantity at the right hand side of (102) may drive a gradient based optimization scheme.

Some details We give some details of some of the terms. We recall that $\mathbf{N} = \{\boldsymbol{\tau}\} - \gamma\boldsymbol{\beta}$ and that $\hat{\xi}$ is defined in (64), and $\rho = -\mu_f \overline{\sigma}^e_{n_\Gamma}$.

By using the previous results we have

$$
\partial_{\boldsymbol{\tau}} J_D \in \begin{cases} \omega \mathbf{D}_\rho & \text{if } \rho \le 0 \\ \omega\left(\hat{\xi}\mathbf{D}_\rho + \dfrac{\rho \mathbf{N}\cdot\mathbf{D}_\rho}{|\mathbf{N}|^3}\mathbf{N}\right) & \text{if } 0 < \rho < |\mathbf{N}| \\ \mathbf{0} & \text{if } \rho > |\mathbf{N}| \end{cases} \tag{103}
$$

or, equivalently,

$$
\partial_{\boldsymbol{\tau}} J_D \in \begin{cases} \omega \mathbf{D}_\rho & \text{if } \rho \le 0, \\ \omega\left(\hat{\xi}\mathbf{D}_\rho + (1-\xi)\dfrac{\mathbf{N}\cdot\mathbf{D}_\rho}{|\mathbf{N}|^2}\mathbf{N}\right) & \text{if } 0 < \rho < |\mathbf{N}|, \\ \mathbf{0} & \text{if } \rho > |\mathbf{N}|. \end{cases}
$$

It is expressed as a differential inclusion since it is not defined for $\rho = |\mathbf{N}|$, where it is discontinuous unless $D = \mathbf{0}$. We also have (here we use the expression of the Gateaux derivative)

$$
< \partial_{\boldsymbol{\beta}} J_D, \boldsymbol{\gamma} >= \begin{cases} 0 & \text{if } \rho \le 0 \\ \gamma\omega(1-\hat{\xi}) < \mathbf{D}_\rho, \boldsymbol{\gamma} > - \dfrac{< \omega\gamma\rho(\mathbf{N}\cdot\mathbf{D}_\rho)\mathbf{N}, \boldsymbol{\gamma} >}{|\mathbf{N}|^3} & \text{if } 0 < \rho < |\mathbf{N}| \\ \omega < \gamma \mathbf{D}_\rho, \boldsymbol{\gamma} > & \text{if } \rho > |\mathbf{N}| \end{cases} \tag{104}
$$

which is equivalent to

$$< \partial_{\boldsymbol{\beta}} J_D, \boldsymbol{\gamma} >= \begin{cases} 0 & \text{if } \rho \leq 0 \\ \omega\gamma(1-\hat{\xi}) \left(< \mathbf{D}_\rho - (\frac{\mathbf{N}}{|\mathbf{N}|} \cdot \mathbf{D}_\rho) \frac{\mathbf{N}}{|\mathbf{N}|}, \boldsymbol{\gamma} > \right) & \text{if } 0 < \rho < |\mathbf{N}| \\ \omega < \gamma \mathbf{D}_\rho, \boldsymbol{\gamma} > & \text{if } \rho > |\mathbf{N}| \end{cases}$$

or, operator-wise,

$$\partial_{\boldsymbol{\beta}} J_D = \begin{cases} \mathbf{0} & \text{if } \rho \leq 0 \\ \gamma\omega(1-\hat{\xi}) \left(\mathbf{D}_\rho - \frac{\mathbf{N} \cdot \mathbf{D}_\rho}{|\mathbf{N}|^2} \mathbf{N} \right) & \text{if } 0 < \rho < |\mathbf{N}| \\ \gamma \mathbf{D}_\rho & \text{if } \rho > |\mathbf{N}| \end{cases} \tag{105}$$

One may note that if we set $\hat{\mathbf{N}} = \dfrac{\mathbf{N}}{|\mathbf{N}|}$ as the unitary vector in the direction of $\mathbf{N}$ we have that the term $\dfrac{\mathbf{N} \cdot \mathbf{D}_\rho}{|\mathbf{N}|^2} \mathbf{N}$ is equal to $|\mathbf{D}_\rho| \cos(\theta_{DN}) \hat{\mathbf{N}}$, where θ_{DN} is the angle between $\mathbf{D}_\rho$ and $\mathbf{N}$.

4.6.2 The algorithm

This algorithm is based on approximating the derivative of the distance function using the derivative at a fixed $\hat{\xi}$, as in (81). We indicate with V_h the discretization di V and T_h the space of tangential traces on Γ (they are vectors). We recall that in our case $\boldsymbol{\beta}$ is an approximation of the *slip velocity*: $\boldsymbol{\beta}^{(k)} = ([\![\mathbf{u}_t^{(k)}]\!] - [\![\mathbf{u}_t^n]\!])(\Delta t)^{-1}$ The Δt term might be hidden so I consider instead just $\boldsymbol{\beta}^{(k)} = ([\![\mathbf{u}_t^{(k)}]\!] - [\![\mathbf{u}_t^n]\!])$.

Starting from $\boldsymbol{\beta}^{(0)} = \mathbf{0} \in T_h$, for $k = 0, 1, \ldots$

1. Find $\mathbf{u}^{(k)} = \mathbf{u}(\boldsymbol{\beta}^{(k)}) \in V_h$ by solving the primal problem imposing $[\![\mathbf{u}_t^{(k)}]\!] = \boldsymbol{\beta}^{(k)} + [\![\mathbf{u}_t^n]\!]$.

2. Compute with a suitable projection method $\{\boldsymbol{\tau}\}^{(k)} \in T_h$ such that $\{\boldsymbol{\tau}\}^{(k)} \simeq \{\boldsymbol{\tau}(\mathbf{u}(\boldsymbol{\beta}^{(k)})\}$, be a convergent approximation. **LUCA: This has to be clarified**. Analogously, compute $(\overline{\sigma}^e_{n_\Gamma})^{(k)} \simeq \overline{\sigma}^e_{n_\Gamma}(\mathbf{u}^{(k)})$ and $\rho^{(k)} = -\mu_f(\overline{\sigma}^e_{n_\Gamma})^{(k)}$.

3. Construct $\mathbf{N}^{(k)} \in T_h$, $\hat{\xi}^k \in T_h$ and $\mathbf{D}^{(k)} \in T_h$ as

$$\begin{cases} \mathbf{N}^{(k)} = \{\boldsymbol{\tau}\}^{(k)} - \gamma \boldsymbol{\beta}^{(k)}, \\ \hat{\xi}^k = \hat{\xi}_{\rho^{(k)}}(\mathbf{u}^{(k)}, \boldsymbol{\beta}^{(k)}) = \max\left(0, 1 - \dfrac{\max(0, \rho^{(k)})}{|\mathbf{N}^{(k)}|}\right) \\ \mathbf{D}^{(k)} = (1-\hat{\xi}^k)\gamma\boldsymbol{\beta}^{(k)} + \hat{\xi}^k \{\boldsymbol{\tau}\}^{(k)}. \end{cases}$$

   You may also want to calibrate ζ so that it takes different values when $\hat{\xi}^k > 0$.

4. Find the adjoint solution $\mathbf{p}^{(k)}$ that satisfies the discretization of (100). In particular, we will set homogeneous Dirichlet boundary conditions on $\partial\Omega_\Gamma \setminus \Gamma$ and zero the source term, while

$$[\![\mathbf{p}_t^{(k)}]\!] = \begin{cases} \mathbf{D}^{(k)} & \text{if } \rho^{(k)} \leq 0 \\ \hat{\xi}^{(k)}\mathbf{D}^{(k)} + \dfrac{\rho\mathbf{N}^{(k)} \cdot \mathbf{D}^{(k)}}{|\mathbf{N}^{(k)}|^3}\mathbf{N}^{(k)} & \text{if } 0 < \rho^{(k)} < |\mathbf{N}| \\ \mathbf{0} & \text{if } \rho^{(k)} > |\mathbf{N}| \end{cases} \tag{106}$$

Important Note: in the 2D case, since both D and N are parallel we have that, indicating in non-bold the tangential components:

$$[\![p_t^{(k)}]\!] = \begin{cases} D^{(k)} & \text{if } \rho^{(k)} \leq 0 \\ D^{(k)} & \text{if } 0 < \rho^{(k)} < |N| \\ 0 & \text{if } \rho^{(k)} > |N| \end{cases}$$

5. Compute

$$d_\beta J_D^{(k)} = \begin{cases} \mathbf{0} & \text{if } \rho^{(k)} \leq 0 \\ \gamma\left(1 - \hat{\xi}^{(k)}\right)\mathbf{D}^{(k)} - \dfrac{\gamma\rho\mathbf{N}^{(k)} \cdot \mathbf{D}^{(k)}}{|\mathbf{N}^{(k)}|^3}\mathbf{N}^{(k)} & \text{if } 0 < \rho < |\mathbf{N}| \\ \gamma\mathbf{D}^{(k)} & \text{if } \rho > |\mathbf{N}| \end{cases} \tag{107}$$

Important Note: in the 2D case, since both D and N are parallel we have that, indicating in non-bold the tangential components:

$$d_\beta J_D^{(k)} = \begin{cases} 0 & \text{if } \rho^{(k)} \leq 0 \\ 0 & \text{if } 0 < \rho < |N| \\ \gamma D^{(k)} = \gamma^2\beta & \text{if } \rho > |N| \end{cases}$$

6. Compute the gradient $\mathbf{G}^{(k)}$ as solution of

$$\int_\Gamma \mathbf{G}^{(k)} \cdot [\![\mathbf{v}_{t_\Gamma}]\!] = \int_\Gamma \left(\omega\{\boldsymbol{\tau}(\mathbf{p}^{(k)})\} + \zeta\boldsymbol{\beta}^{(k)} + \omega d_\beta J_D^{(k)}\right) \cdot [\![\mathbf{v}_{t_\Gamma}]\!]\, d\gamma \tag{108}$$

for $\mathbf{v} \in \mathbf{V}$, (i.e $\forall [\![\mathbf{v}_{t_\Gamma}]\!] \in T_h$) or "pointwise" as

$$\mathbf{G}^{(k)} = \omega\{\boldsymbol{\tau}(\mathbf{p}^{(k)})\} + \zeta\boldsymbol{\beta}^{(k)} + \omega d_\beta J_D^{(k)}$$

7. Compute a suitable accelerator coefficient $\alpha^{(k)} > 0$ and correct $\boldsymbol{\beta}$.

$$\boldsymbol{\beta}^{(k+1)} = \boldsymbol{\beta}^{(k)} - \alpha^{(k)}\mathbf{G}^{(k)}, \tag{109}$$

for an $\alpha > 0$. Since I know that $\boldsymbol{\beta} = \beta\boldsymbol{\tau}$ for a $\beta \leq 0$ I can do

$$\delta\beta = \left(\boldsymbol{\beta}^{(k)} - \alpha\mathbf{G}^{(k)}\right) \cdot \{\boldsymbol{\tau}\}^{(k)}, \quad \boldsymbol{\beta}^{(k+1)} = \min(0, \delta\beta)\frac{\{\boldsymbol{\tau}\}^{(k)}}{|\{\boldsymbol{\tau}\}^{(k)}|^2}. \tag{110}$$

The iteration continues until $\|\mathbf{G}^{(k)}\| \leq tol$, being tol a given tolerance.

Some interesting result We gve a closer look at the terms at the rhs of (106) and (107) we have (we omit the affix k)

$$[\![\mathbf{p}]\!] \cdot \mathbf{D}_\rho \geq 0.$$

Proof. The first and last case are obvious. Let's look at the case when $0 < \rho < |\mathbf{N}|$. We have that

$$\rho \frac{\mathbf{N} \cdot \mathbf{D}_\rho}{|\mathbf{N}|^3} \mathbf{N} \cdot \mathbf{D}_\rho = \frac{\rho}{|\mathbf{N}|} \frac{(\mathbf{N} \cdot \mathbf{D}_\rho)^2}{|\mathbf{N}|^2} \geq 0,$$

and, since $\hat{\xi} \geq 0$ the proof for (106) is completed. For (107) we note that

$$(1 - \hat{\xi}) \mathbf{D}_\rho \cdot \mathbf{D}_\rho = \frac{\rho}{|\mathbf{N}|} |\mathbf{D}|^2.$$

So, for the given case,

$$d_\beta J_D \cdot \mathbf{D}_\rho = \frac{\rho}{|\mathbf{N}|} \left(|\mathbf{D}|^2 - \frac{(\mathbf{N} \cdot \mathbf{D}_\rho)^2}{|\mathbf{N}|^2} \right) \geq 0.$$

□

Since the Steklow-Poincarè operator is a positive-definite pseudo-differential operator, we may (probably) conclude that $-\mathbf{D}$ is a descent direction, which justifies the fixed point iterations presented in the previous sections.

4.6.3 The discrete algoritm

We now assume that we use finite elements to discretize the problem and Nitsche's technique to impose the slip. We have that Ω_Γ is discretized with the mesh $\mathcal{T}_h$, which is a mesh cut by Γ_h, approximation of Γ.

$$\mathbf{V}_h = \{\mathbf{v}_h \in V : \mathbf{v}_h|_K \in [\mathbb{P}^k(K)]^d, \forall K \in \mathcal{T}_h\}, \tag{111}$$

Note that it is a broken space since it allows for jumps across Γ_h. However the type of XFEM technique we use if of the class "cutfem", which means that the we have a geoetrical conformity between $\mathcal{T}_h$ and Γ_h. So we can identify a mesh $\mathcal{G}_h = \mathcal{T}_h \cap \Gamma_h$ and

$$G_h^v = \{v_h \in C^0(\Gamma_h) : v_h|_K = P^k(K), \forall K \in \mathcal{G}_h\},$$

and its vectorial counterpart

$$\mathbf{G}_h^v = \{\mathbf{v}_h \in [C^0(\Gamma_h)]^d : \mathbf{v}_h|_K = [P^k(K)]^d, \forall K \in \mathcal{G}_h\}.$$

We can finally, identify the *space of traces* of $\mathbf{V}_h$ as

$$\mathrm{trace}_{\Gamma_h}(\mathbf{V}_h) = \mathbf{T}_h^v = \mathbf{G}_h^v \times \mathbf{G}_h^v.$$

It may be convenient to use on Γ_h other space, for instance the space of piecewise constant functions

$$G_h^0 = \{v_h \in C^0(\Gamma_h) : v_h|_K = P^0(K),\, \forall K \in \mathcal{G}_h\}$$

and its analogouos vector counterparts $\mathbf{G}_h^0$ and $\mathbf{T}_0^v$. In that case we will indicate with Π_0^v a stable interpolator between G_h^0 to G_h^v with left-inverse Π_v^0.

We assume that our XFEM formulation, for a given *total tangential slip* $\boldsymbol{\gamma} \in \mathbf{G}_h^v$ produces a solution of our primal problem in the form

$$A\mathbf{u}_h = \mathbf{f} + H\boldsymbol{\gamma},$$

where A and H are linear operators and here we identify elements of the discrete space with the corresponding vectors of degree of freedom.

Thus, for a given *incremental slip* $\boldsymbol{\beta}_h = \boldsymbol{\gamma}_h - [\![\mathbf{u}_{h,t}^n]\!]$ we have

$$A\mathbf{u}_h(\boldsymbol{\beta}_h) = \mathbf{f} + H\left(\boldsymbol{\beta}_h + [\![\mathbf{u}_{h,t}^n]\!]\right). \tag{112}$$

For a given $\boldsymbol{\tau}_h \in \mathbf{G}_h^v$, the corresponding *discrete dual problem* reads

$$A\mathbf{p}_h(\boldsymbol{\tau}_h) = H\boldsymbol{\tau}_h, \tag{113}$$

where the forcing term is NOT present for the different boundary conditions. We also assume that we are able to compute an approximation of the stress-jump in $\mathbf{G}_h^v$ for a given $\mathbf{u}_h$

$$\{\boldsymbol{\tau}_h\} = C\mathbf{u}_h = C\left(A^{-1}\left(\mathbf{f} + H\left(\boldsymbol{\beta}_h + [\![\mathbf{u}_{h,t}^n]\!]\right)\right)\right) \tag{114}$$

and analogously for the dual problem.

At this point the discrete algoritm is just a replica of the continuos one.

4.7 Weak form

The weak form of the primal problem is standard. For the dual problem we have the difficulty of representing the action of the Ritz operator Λ_U. Maybe in practice it may be replaced by a suitable smoothing operator. We will come back later. As for the complementarity condition we note that if $e_\gamma \in V$ is any extension of a function $\gamma \in U$, we have

$$\left(\frac{d}{d\beta}J(u(\beta),\beta),\gamma\right)_\Gamma = a_p(p, e_\Gamma) + \alpha(\beta,\gamma)_\Gamma,$$

where a_p is the bilinear form associated to the dual problem, i.e. $a_p(p,w) = \int_\Omega \nabla p \nabla w$.

5 A Collection of results

Here I indicate with V a subspace of a Hilbert space and with W the space of the traces of W on Γ. We identify the sets (here and in the following the application of the trace operator is implicit)

$$K = \{v \in V : v \leq 0 \mathrm{on} \Gamma\} \tag{115}$$

$$\Lambda_N = \{\lambda \in W' :< \lambda, u >\geq 0, \forall u \in K\} \tag{116}$$

K is a closed and convex set. We have

The following statements are equivalent:

$$u \leq 0, \sigma_n \in \Lambda_N, \lambda u = 0 \qquad \text{on } \Gamma \tag{117}$$

$$u \in V, \sigma_n \in \Lambda_n \text{ and } < \mu, u >\geq< \sigma_n, u > \quad \forall \mu \in \Lambda_N \tag{118}$$

$$\sigma_n - \Pi_-(\sigma_n - \gamma u) = 0 \qquad \gamma > 0 \tag{119}$$

Here, $u\lambda = 0$ on Γ means that either $< lambda, u >= 0$.

Proof. Let's start with (117) $\rightleftarrows$ (118). Ancora da fare $\square$

We now define W_t as the space of tangential traces on Γ, and

$$K_\rho = \{\boldsymbol{\tau} \in W_t' :< \boldsymbol{\tau}, \mathbf{u}_t >\leq< \rho, |\mathbf{u}_t| >, \forall \mathbf{u}_t \in W_t\}$$

where ρ such that $< \rho, |\mathbf{u}_t| >\geq \xi \|\mathbf{u}_t\|$, with $\xi > 0$.

The following statements are equivalent:

$$|\boldsymbol{\tau}| \leq \rho,\, \exists \beta \geq 0 : \mathbf{u}_t = \beta \boldsymbol{\tau},\, (|\boldsymbol{\tau}| - \rho)\beta = 0; \tag{120}$$

$$\mathbf{u}_t \in W_t, \boldsymbol{\tau} \in K_\rho \; < \mathbf{u}_t, \boldsymbol{\lambda}_t >\geq< \mathbf{u}_t, \boldsymbol{\tau} >, \forall \boldsymbol{\lambda}_t \in W_t'; \tag{121}$$

$$\mathbf{u}_t \in W_t, \boldsymbol{\tau} \in W_t' : \boldsymbol{\tau} = \Pi_{K_\rho}(\boldsymbol{\tau} - \gamma \mathbf{u}_t),\, \gamma > 0. \tag{122}$$

6 Algebraic formulation

We reorder the degrees of freedom for the displacement appropriately and indicate with $\boldsymbol{\Sigma}_h$ the vector containing the degrees of freedom for the reconstructed discrete normal stresses $\boldsymbol{\sigma} \cdot \boldsymbol{n}_\Gamma$. More precisely, $\mathbf{u}_{\Omega^\pm}$ are the vectors of degrees of freedom for the displacement internal to $\Omega^\pm$, i.e. whese associated shape function has null trace on Γ, while $\mathbf{u}_{\Gamma^+}$ and $\mathbf{u}_{\Gamma^-}$ are the vectors of the dofs on Γ (i.e dofs whose corresponding shape function has a non-null trace on Γ), replicated on the two sides Γ^+ and Γ^- of Γ.

We can write a global algebraic system of the form (note that no conditions are yet applied on Γ!)

$$\begin{bmatrix} A_{\Omega^+\Omega^+} & \mathbf{0} & A_{\Omega^+\Gamma^+} & \mathbf{0} \\ \mathbf{0} & A_{\Omega^-\Omega^-} & \mathbf{0} & A_{\Omega^-\Gamma^-} \\ A_{\Gamma^+\Omega^+} & \mathbf{0} & A_{\Gamma^+\Gamma^+} & \mathbf{0} \\ \mathbf{0} & A_{\Gamma^-\Omega^-} & \mathbf{0} & A_{\Gamma^-\Gamma^-} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Omega^+} \\ \mathbf{u}_{\Omega^-} \\ \mathbf{u}_{\Gamma^+} \\ \mathbf{u}_{\Gamma^-} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\Omega^+} \\ \mathbf{f}_{\Omega^-} \\ \mathbf{f}_{\Gamma^+} + M\boldsymbol{\Sigma}_h \\ \mathbf{f}_{\Gamma^-} - M\boldsymbol{\Sigma}_h, \end{bmatrix}$$

where the various matrices indicated by A are the contributions to the standard finite element stiffness matrix, while, since the discretization is conforming, the elements of M are

$$M_{ij} = \int_{\Gamma^+} \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = \int_{\Gamma^-} \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j = \frac{1}{2}\left(\int_{\Gamma^+} \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j + \int_{\Gamma^-} \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j \right).$$

The unknowns on Γ may be expressed as

$$\mathbf{u}_{\Gamma^+} = \{\mathbf{u}_h\} + \frac{1}{2}[\![\mathbf{u}_h]\!], \quad \mathbf{u}_{\Gamma^-} = \{\mathbf{u}_h\} - \frac{1}{2}[\![\mathbf{u}_h]\!] \tag{123}$$

Thus, the previous system may be rewritten as

$$\begin{cases} A_{\Omega^+\Omega^+}\mathbf{u}_{\Omega^+} + A_{\Omega^+\Gamma^+}(\{\mathbf{u}_h\} + \frac{1}{2}[\![\mathbf{u}_h]\!]) = \mathbf{f}_{\Omega^+}, \\ A_{\Omega^-\Omega^-}\mathbf{u}_{\Omega^-} + A_{\Omega^-\Gamma^-}(\{\mathbf{u}_h\} - \frac{1}{2}[\![\mathbf{u}_h]\!]) = \mathbf{f}_{\Omega^-}, \\ A_{\Gamma^+\Omega^+}\mathbf{u}_{\Omega^+} + A_{\Gamma^+\Gamma^+}(\{\mathbf{u}_h\} + \frac{1}{2}[\![\mathbf{u}_h]\!]) = \mathbf{f}_{\Gamma^+} + M\boldsymbol{\Sigma}_h, \\ A_{\Gamma^-\Omega^-}\mathbf{u}_{\Omega^-} + A_{\Gamma^-\Gamma^-}(\{\mathbf{u}_h\} - \frac{1}{2}[\![\mathbf{u}_h]\!]) = \mathbf{f}_{\Gamma^+} - M\boldsymbol{\Sigma}_h, \end{cases} \tag{124}$$

that is,

$$\begin{bmatrix} A_{\Omega^+\Omega^+} & \mathbf{0} & A_{\Omega^+\Gamma^+} & \frac{1}{2}A_{\Omega^+\Gamma^+} \\ \mathbf{0} & A_{\Omega^-\Omega^-} & A_{\Omega^-\Gamma^-} & -\frac{1}{2}A_{\Omega^-\Gamma^-} \\ A_{\Gamma^+\Omega^+} & \mathbf{0} & A_{\Gamma^+\Gamma^+} & \frac{1}{2}A_{\Gamma^+\Gamma^+} \\ \mathbf{0} & A_{\Gamma^-\Omega^-} & A_{\Gamma^-\Gamma^-} & -\frac{1}{2}A_{\Gamma^-\Gamma^-} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Omega^+} \\ \mathbf{u}_{\Omega^-} \\ \{\mathbf{u}_h\} \\ [\![\mathbf{u}_h]\!] \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\Omega^+} \\ \mathbf{f}_{\Omega^-} \\ \mathbf{f}_{\Gamma^+} + M\boldsymbol{\Sigma}_h \\ \mathbf{f}_{\Gamma^-} - M\boldsymbol{\Sigma}_h \end{bmatrix}$$

We replace the last two block-rows with their sum and difference, respectively. We further impose $[\![\boldsymbol{\sigma} \cdot \boldsymbol{n}_\Gamma]\!] = \mathbf{0}$ by setting $[\![\boldsymbol{\Sigma}_h]\!] = \mathbf{0}$. After a further division by 2 of the last block-row, we obtain

$$\begin{bmatrix} A_{\Omega^+\Omega^+} & \mathbf{0} & A_{\Omega^+\Gamma^+} & \frac{1}{2}A_{\Omega^+\Gamma^+} \\ \mathbf{0} & A_{\Omega^-\Omega^-} & A_{\Omega^-\Gamma^-} & -\frac{1}{2}A_{\Omega^-\Gamma^-} \\ A_{\Gamma^+\Omega^+} & A_{\Gamma^-\Omega^-} & A_\Gamma & \mathbf{0} \\ \frac{1}{2}A_{\Gamma^+\Omega^+} & -\frac{1}{2}A_{\Gamma^-\Omega^-} & \mathbf{0} & \frac{1}{2}A_\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Omega^+} \\ \mathbf{u}_{\Omega^-} \\ \{\mathbf{u}_h\} \\ [\![\mathbf{u}_h]\!] \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\Omega^+} \\ \mathbf{f}_{\Omega^-} \\ \mathbf{f}_{\Gamma^+} + \mathbf{f}_{\Gamma^-} \\ \frac{1}{2}(\mathbf{f}_{\Gamma^+} - \mathbf{f}_{\Gamma^-}) + M\{\boldsymbol{\Sigma}_h\} \end{bmatrix}, \tag{125}$$

where we have exploited the fact that, being the mesh conforming, $A_{\Gamma^+\Gamma^+} = A_{\Gamma^-\Gamma^-}$ and we have set $A_\Gamma = \frac{1}{2}(A_{\Gamma^+\Gamma^+}A_{\Gamma^-\Gamma^-}) = A_{\Gamma^+\Gamma^+} = A_{\Gamma^-\Gamma^-}$.

The imposition of $[\![\mathbf{u}_h]\!] = \boldsymbol{\beta}_h$ allows to compute the actual unknowns $[\mathbf{u}_{\Omega^+}, \mathbf{u}_{\Omega^-}, \{\mathbf{u}_h\}]^T$ by solving the following reduced system

$$\begin{bmatrix} A_{\Omega^+\Omega^+} & \mathbf{0} & A_{\Omega^+\Gamma^+} \\ \mathbf{0} & A_{\Omega^-\Omega^-} & A_{\Omega^-\Gamma^-} \\ A_{\Gamma^+\Omega^+} & A_{\Gamma^-\Omega^-} & A_\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Omega^+} \\ \mathbf{u}_{\Omega^-} \\ \{\mathbf{u}_h\} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\Omega^+} - \frac{1}{2}A_{\Omega^+\Gamma^+}\boldsymbol{\beta}_h \\ \mathbf{f}_{\Omega^-} + \frac{1}{2}A_{\Omega^-\Gamma^-}\boldsymbol{\beta}_h \\ \mathbf{f}_{\Gamma^+} + \mathbf{f}_{\Gamma^-} \end{bmatrix} \tag{126}$$

and to reconstruct the degrees of freedom at the interface using (123). More precisely,

$$\mathbf{u}_{\Gamma^+} = \{\mathbf{u}_h\} + \frac{1}{2}\boldsymbol{\beta}_h, \quad \mathbf{u}_{\Gamma^-} = \{\mathbf{u}_h\} - \frac{1}{2}\boldsymbol{\beta}_h.$$

The average normal stress on Γ may then be computed solving

$$M\{\mathbf{\Sigma}_h\} = \frac{1}{2}(\mathbf{f}_{\Gamma^-} - \mathbf{f}_{\Gamma^+}) + \frac{1}{2}A_{\Gamma^+\Omega^+}\mathbf{u}_{\Omega^+} - \frac{1}{2}A_{\Gamma^-\Omega^-}\mathbf{u}_{\Omega^-} + \frac{1}{2}A_\Gamma \boldsymbol{\beta}_h. \qquad (127)$$

One can then easily compute the normal and tangential components at each node.

6.1 An alternative

If one does not want to use the reduced system(126), which corresponds to the lifting technique, one can solve the modified full system

$$\begin{bmatrix} A_{\Omega^+\Omega^+} & \mathbf{0} & A_{\Omega^+\Gamma^+} & \frac{1}{2}A_{\Omega^+\Gamma^+} \\ \mathbf{0} & A_{\Omega^-\Omega^-} & A_{\Omega^-\Gamma^-} & -\frac{1}{2}A_{\Omega^-\Gamma^-} \\ A_{\Gamma^+\Omega^+} & A_{\Gamma^-\Omega^-} & A_\Gamma & \mathbf{0} \\ \mathbf{0} & -\mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\Omega^+} \\ \mathbf{u}_{\Omega^-} \\ \{\mathbf{u}_h\} \\ [\![\mathbf{u}_h]\!] \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\Omega^+} \\ \mathbf{f}_{\Omega^-} \\ \mathbf{f}_{\Gamma^+} + \mathbf{f}_{\Gamma^-} \\ \boldsymbol{\beta}_h \end{bmatrix},$$

and then proceed as before for the computation of the stress. But be careful, you need to store the matices for the computation of the stress somewhere! And we loose symmetry (at least formally).

6.2 A note

A last note, if I collect all the unknowns into $\mathbf{U} = [\mathbf{u}_{\Omega^+}, \mathbf{u}_{\Omega^-}, \{\mathbf{u}_h\}]^T$ I can rewrite the global system (125)(I exploit the symmetry of the $A_{\Gamma\Omega}$) as

$$\begin{bmatrix} A_{UU} & A_{US} \\ A_{US}^T & A_{SS} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \boldsymbol{\beta}_h \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_U \\ \boldsymbol{F}_S + M\{\mathbf{\Sigma}_h\}, \end{bmatrix}.$$

It is then easy to show that

$$\{\mathbf{\Sigma}_h\} = M^{-1}(A_{SS} - A_{US}^T A_{UU}^{-1} A_{US})\boldsymbol{\beta}_h + \mathbf{g},$$

where $\mathbf{g}$ collects all terms not depending on $\boldsymbol{\beta}_h$. Since the global matrix

$$\begin{bmatrix} A_{UU} & A_{US} \\ A_{US}^T & A_{SS} \end{bmatrix}$$

is symmetric positive definite, then the Shur complement $(A_{SS} - A_{US}^T A_{UU}^{-1} A_{US})$ is symmetric positive definite and $M^{-1}(A_{SS} - A_{US}^T A_{UU}^{-1} A_{US})$ is symmetric positive definite with respect to the inner product $(\mathbf{a}, \mathbf{b})_M = \mathbf{a}^T M \mathbf{b}$.

This reflects that also at algebraic level the stresses on Γ are an affine and bounded function of $\boldsymbol{\beta}_h$. More precisely, there exists $\overline{C}_h$ and $\underline{C}_h$, with $0 < \underline{C}_h \leq \overline{C}_h$ such that

$$\underline{C}_h \|\boldsymbol{\beta}_h\| \leq (\mathbf{\Sigma}_h, \mathbf{\Sigma}_h)_M \leq \overline{C}_h \|\boldsymbol{\beta}_h\|.$$

6.3 A practical consideration

The matrices governing (126) and (127) can be computed via standard finite element assembly, by looping over the mesh elements and scattering the local stiffness matrix accordingly. Also matrix M may be computed during standard assembly.

References

[1] Paola Francesca Antonietti, Luca Formaggia, Anna Scotti, Marco Verani, and Nicola Verzotti. Mimetic finite difference approximation of flows in fractured porous media. *ESAIM: M2AN*, 50(3):809–832, 2016.

[2] Vincent Martin, Jérôme Jaffré, and Jean E. Roberts. Modeling fractures and barriers as interfaces for flow in porous media. *SIAM Journal on Scientific Computing*, 26(5):1667–1691, 2005.

[3] Rabii Mlika. *Nitsche method for frictional contact and self-contact : mathematical and numerical study.* PhD thesis, Universite de Lyon, France, 2018.