Numerical Approximation of Large Contrast Problems with the Unfitted Nitsche Method

Erik Burman and Paolo Zunino

Abstract These notes are concerned with the numerical treatment of the coupling between second order elliptic problems that feature large contrast between their characteristic coefficients. In particular, we study the application of Nitsche's method to set up a robust approximation of interface conditions in the framework of the finite element method. The notes are subdivided in three parts. Firstly, we review the weak enforcement of Dirichlet boundary conditions with particular attention to Nitsche's method and we discuss the extension of such technique to the coupling of Poisson equations. Secondly, we review the application of Nitsche's method to large contrast problems, discretised on computational meshes that capture the interface of discontinuity between coefficients. Finally, we extend the previous schemes to the case of unfitted meshes, which occurs when the computational mesh does not conform with the interface between subproblems.

1 A Review of Nitsche's Method

1.1 Weak Enforcement of Boundary Conditions for Poisson's Problem

The aim of this section is to review some well known techniques to enforce boundary conditions of Dirichlet type for second order problems. In particular, we will focus on the techniques that allow to enforce such boundary conditions within the

Erik Burman
Department of Mathematics, University of Sussex, Falmer, Brighton, BN1 9RF United Kingdom, e-mail: E.N.Burman@sussex.ac.uk

Paolo Zunino
MOX - Department of Mathematics, Politecnico di Milano, P.zza Leonardo da Vinci 32, 20133 Milano, Italy e-mail: paolo.zunino@polimi.it

definition of the bilinear form associated to the variational formulation of the problem at hand, rather than enforcing the constraints at the boundary in the search space for the solution. We refer to such schemes as those using weak enforcement of Dirichlet boundary conditions, in contrast to the case where Dirichlet boundary conditions appear in the definition of the trial space, often addressed as strong enforcement of boundary constraints. Concerning Neumann or mixed type boundary conditions we observe that they are naturally embedded in the set up of the problem bilinear form. Some alternatives for the treatment of natural boundary conditions have been recently addressed in [34].

We start from the simplest model problem, that is Poisson's problem with Dirichlet boundary conditions, which can be straightforwardly formulated as follows. Let Ω be a convex polygonal domain in $\mathbb{R}^d$. Given $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$, find $\hat{u} \in H^1(\Omega)$ a weak solution of

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The most straightforward way to enforce Dirichlet type constraints at the boundary is to embed the variational formulation into an Hilbert space whose functions satisfy the boundary constraints. Given $\mathscr{R}g \in H^1(\Omega)$, a lifting of g on the entire Ω, we aim to find $u \in H^1_0(\Omega)$ such that

$$\begin{aligned} &a(u,v) = F(v) - a(\mathscr{R}g, v) \quad \forall v \in H^1_0(\Omega), \\ &a(u,v) := (\nabla u, \nabla v)_\Omega\,, \\ &F(v) := (f, v)_\Omega\,, \end{aligned} \tag{2}$$

where $(\cdot,\cdot)_\Omega$ denotes the L^2 inner product on Ω. In the framework of the finite element method, the enforcement of Dirichlet boundary conditions in the trial space is also easily translated to the discrete level. Thus, for the approximation of classical second order problems there is no need to consider alternatives. However, the continuous expansion of computational analysis in several engineering disciplines often requires to consider non standard problem formulations. Among many other examples we mention problems that feature multiple domains, accounting for the contact between different materials or fluids, problems with moving boundaries, such as the ones arising from fluid-structure interaction analysis, problems set on domains with very complex dendritic shapes, which are often encountered in the application of computational analysis to life sciences. In these cases, the strong enforcement of Dirichlet boundary or interface conditions may turn out to be cumbersome when applied at the discrete level, while the weak treatment of Dirichlet constraints, which allows to relax their satisfaction, may lead to numerical schemes that are more efficient or easily implemented.

An effective technique for weak enforcement of Dirichlet boundary constraints is the application of Lagrange multipliers. The original idea, due to Babuška [3], is based on the fact that the weak formulation of the Poisson's problem is equivalent to the minimisation among all functions $v \in H^1_0(\Omega)$ of the energy functional

$$J(u) = \min_{v \in H^1_0(\Omega)} J(v) \tag{3}$$

$$J(v) := a(v,v) - 2F(v) \quad \forall v \in H^1(\Omega). \tag{4}$$

The problem of finding the minimum $u \in H^1_0(\Omega)$ can be seen as a constrained minimisation problem, because the solution is sought in a subspace $H^1_0(\Omega)$ of the natural space $H^1(\Omega)$ where the functional is well defined. This convex constrained minimisation problem can be translated into an unconstrained problem by resorting to the Lagrangian functional accounting for the constraint $u = 0$ on $\partial\Omega$. Let $H^{-\frac{1}{2}}(\partial\Omega)$ be the dual space of $H^{\frac{1}{2}}(\partial\Omega)$ with the duality pairing $\langle \cdot, \cdot \rangle_{\partial\Omega}$, then

$$L(v,\mu) := J(v) + \langle \mu, v \rangle, \quad \forall v \in H^1(\Omega),\ \mu \in H^{-\frac{1}{2}}(\partial\Omega)$$

is the Lagrangian functional and we look for a couple (u,λ), where the additional unknown λ is called Lagrange multiplier such that,

$$L(u,\lambda) = \inf_{v \in H^1(\Omega)} \sup_{\mu \in H^{-1/2}(\partial\Omega)} L(v,\mu).$$

This is an instance of a saddle point problem, involving minimisation with respect to one unknown and maximisation with respect to the other. Owing to fundamental results of convex analysis, this constrained minimisation problem admits the following equivalent formulation: setting $b(\lambda,v) := \langle \lambda, v \rangle_{\partial\Omega}$ and given $f \in L^2(\Omega)$, find $u \in H^1(\Omega)$, $\lambda \in H^{-\frac{1}{2}}(\partial\Omega)$ such that

$$\begin{cases} a(u,v) + b(\lambda,v) = F(v) \quad \forall v \in H^1(\Omega), \\ b(\mu,u) = b(\mu,g) \quad \forall \mu \in H^{-\frac{1}{2}}(\partial\Omega). \end{cases} \tag{5}$$

We notice that the new formulation with Lagrange multipliers involves an additional unknown that at the discrete level increases the computational cost of the problem. However, this is not only a drawback, because the unknown λ has a relevant physical meaning,

$$\lambda + \partial_n u = 0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega).$$

Anyway, the most relevant remark concerning the weak enforcement of Dirichlet boundary condition with Lagrange multipliers is the fact that the corresponding variational problem does not conform with the assumptions of Lax-Milgram's Lemma, which ensures well posedness of the usual weak formulation of Poisson's problem. The crucial point is that the introduction of Lagrange multipliers breaks the coercivity of the entire weak problem, whose well posedness holds true under the following set of conditions,

$$a(\cdot,\cdot) \text{ and } b(\cdot,\cdot) \text{ are bilinear and continuous,}$$

$$\text{coercivity: } \exists\, \alpha > 0 \text{ s.t. } a(v,v) \geq \alpha \|v\|^2_{1,\Omega}$$
$$\forall v \in Z := \{v \in H^1(\Omega) : b(\lambda, v) = 0,\ \forall \lambda \in H^{-\frac{1}{2}}(\partial\Omega)\},$$

$$\text{“inf-sup”: } \exists\, \beta > 0 \text{ s.t. } \forall \lambda \in H^{-\frac{1}{2}}(\partial\Omega),\ \sup_{v \in H^1(\Omega)\setminus\{0\}} \frac{b(\lambda, v)}{\|v\|_{1,\Omega}} \geq \beta.$$

It is immediately evident that the verification of such conditions is a more challenging task than the check of Lax-Milgram's assumptions. In the case of the variational formulation of problem (1) with weak enforcement of boundary conditions they are satisfied, see for instance [35].

However, a fundamental problem appears when we look at the discretisation by means of finite elements. With a conforming finite element discretisation, the classical Lax-Milgram's coercivity is automatically inherited at the discrete level, but this is not the case for the aforementioned “inf-sup”condition. According to the particular choices for the approximation spaces of $H^1(\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ such condition may not be verified at the discrete level. The correct formalisation of this difficulty and the constructive development of suitable couples of discrete spaces for the approximation of saddle point problems have been an important milestone of finite element analysis in the last decades, see [9, 39, 40] among many others.

More precisely, given the finite element spaces $V_h \subset H^1(\Omega)$, $\Lambda_h \subset H^{-\frac{1}{2}}(\partial\Omega)$, the application of Galerkin method to (5) consists in finding $u_h \in V_h$ and $\lambda_h \in \Lambda_h$ such that

$$\begin{cases} a(u_h, v_h) + b(\lambda_h, v_h) = F(v_h) \quad \forall v_h \in V_h, \\ b(\mu_h, u_h) = b(\mu_h, g) \quad \forall \mu_h \in \Lambda_h \end{cases} \tag{6}$$

and proceeding similarly to the infinite-dimensional case, it has been proved that, see [3], the discrete problem is well posed provided that

$$\exists\, \alpha_h > 0 \text{ s.t. } a(v_h, v_h) \geq \alpha_h \|v_h\|^2_{1,\Omega} \tag{7}$$
$$\forall v_h \in Z_h := \{v_h \in V_h\ s.t.\ b(v_h, \mu_h) = 0,\ \forall \mu_h \in \Lambda_h\},$$

$$\exists\, \beta_h > 0 \text{ uniformly independent of } h \text{ s.t.} \tag{8}$$
$$\forall\, \lambda_h \in \Lambda_h,\ \sup_{v_h \in V_h\setminus\{0\}} \frac{b(\lambda_h, v_h)}{\|v_h\|_{1,\Omega}} \geq \beta_h.$$

Note that, since the search space for the solution u_h has been extended, by removing the strong enforcement of the constraints at the boundary, coercivity of $a(\cdot,\cdot)$ is lost in $V_h \subset H^1(\Omega)$. For this reason, the Lax-Milgram's theory does not apply any more. Furthermore, the satisfaction of the discrete “inf-sup” condition is not straightforward, for instance intuitive choices of discrete spaces such as linear finite elements in Ω for u_h and linear finite elements on $\partial\Omega$ for λ_h lead to an unstable discrete problem. For standard H^1-conforming affine finite elements for the approximation of u, a piecewise constant approximation for λ is stable provided that the multiplier space is defined on a boundary mesh with size $3h$, with h being the characteristic element size. For Crouzeix-Raviart approximation of u, piecewise constant multipliers on the unrestricted boundary mesh are stable. Recalling the equation $\lambda + \partial_n u = 0$ one can expect that the regularity for the Lagrange multiplier space should be lower than the one for the primal unknown u_h. Such rule of thumb is also confirmed by observing that a generalisation of the previous stable couple of elements is given by k-order H^1-conforming finite elements on Ω combined with fully discontinuous $(k-1)$-order finite elements on $\partial\Omega$. We refer the interested reader to [39, 40, 41] for a detailed analysis.

The relaxation of the strong enforcement of Dirichlet boundary conditions by means of Lagrange multipliers leads to an accurate but expensive problem at the discrete level. For this reason, some alternatives have been developed, with the aim to perform the weak approximation of boundary conditions using a numerical method that can still be cast in the framework of Lax-Milgram's lemma.

Starting from the minimisation problem (3), the most straightforward strategy consists in the application of a penalty method. The idea is to enrich the energy functional $J(v)$ with an additional quadratic term that takes its minimum when the Dirichlet boundary conditions are exactly satisfied. The magnitude of the additional functional should be modulated by means of a constant factor that ensures that the minimum of the augmented functional accurately, but not exactly, satisfies the prescribed boundary conditions. Given $\varepsilon > 0$ the penalty method consists in finding $u_\varepsilon \in H^1(\Omega)$ such that

$$J_\varepsilon(u_\varepsilon) = \min_{v \in H^1(\Omega)} J_\varepsilon(v), \tag{9}$$

$$J_\varepsilon(v) := J(v) + \tfrac{1}{2}\varepsilon^{-1}\|v-g\|^2_{0,\partial\Omega}, \quad \forall v \in H^1(\Omega), \tag{10}$$

whose Euler equations require to find $u_\varepsilon \in H^1(\Omega)$ such that

$$a(u,v) + \varepsilon^{-1}(u-g,v)_{\partial\Omega} = F(v), \quad \forall v \in H^1(\Omega), \tag{11}$$

which seem to share all the good properties of (2) with the additional advantage that the natural search and test spaces are the entire $H^1(\Omega)$. The application of Galerkin method to (11) consists in finding $u_{h,\varepsilon} \in V_h \subset H^1(\Omega)$ such that

$$a(u_{h,\varepsilon}, v_h) + \varepsilon^{-1}\left(u_{h,\varepsilon} - g, v_h\right)_{\partial\Omega} = F(v_h), \quad \forall v_h \in V_h, \tag{12}$$

where V_h could be any H^1-conformal finite element space on Ω. However, to analyze the efficiency of the penalty method, we remind that (12) has been developed to approximate (2). In this respect, the first property to be considered is the consistency of such an approximation scheme. Starting from (12) and performing integration by parts on Ω we obtain a residual,

$$\mathscr{R}(u_{h,\varepsilon}) := \left(-\Delta u_{h,\varepsilon} - f, v_h\right)_\Omega + \left(\partial_n u_{h,\varepsilon}, v_h\right)_{\partial\Omega} + \varepsilon^{-1}\left(u_{h,\varepsilon} - g, v_h\right)_{\partial\Omega} \quad \forall v_h \in V_h.$$

Replacing $u_{h,\varepsilon}$ with $u \in H_0^1(\Omega)$ such that $-\Delta u - f = 0$ weakly in Ω and $u = g$ on $\partial\Omega$ we observe that the residual does not vanish, i.e.

$$\mathscr{R}(u) = (\partial_n u, v_h)_{\partial\Omega} \neq 0.$$

This proves that the penalty method is not strongly consistent with the original weak Poisson's problem. Then, the fundamental question is how to choose the penalty parameter ε with respect to the characteristic mesh size h and the finite element polynomial order k so that $u_{h,\varepsilon}$ converges to u with possibly optimal rate as h becomes infinitesimal. We refer to [4, 7] for a thorough discussion and error analysis of the penalty method, which will be briefly summarized later on. Anyway, the penalty method has received a considerable attention in literature, in particular for the approximation of problems where the computational mesh is not fitted to the boundary, because the penalty term can be easily implemented also in this setting.

Among several interpretations, Nitsche's method can be seen as a variant to override the major drawback of the penalty method, restoring the strong consistency of the discrete scheme with respect to (2). More precisely, we aim to find $u_{h,\varepsilon} \in V_h \subset H^1(\Omega)$ such that

$$a_\varepsilon(u_{h,\varepsilon}, v_h) = F_\varepsilon(v_h) \quad \forall v_h \in V_h, \tag{13}$$

with

$$\begin{aligned} a_\varepsilon(u_{h,\varepsilon}, v_h) &:= a(u_{h,\varepsilon}, v_h) - \left(\partial_n u_{h,\varepsilon}, v_h\right)_{\partial\Omega} - s\left(\partial_n v_h, u_{h,\varepsilon}\right)_{\partial\Omega} + \varepsilon^{-1}\left(u_{h,\varepsilon}, v_h\right)_{\partial\Omega}, \\ F_\varepsilon(v_h) &:= F(v_h) + \varepsilon^{-1}(g, v_h)_{\partial\Omega} - s(\partial_n v_h, g)_{\partial\Omega}, \end{aligned}$$

where ε plays the role of penalty parameter and $s\left(\partial_n v_h, u_{h,\varepsilon} - g\right)_{\partial\Omega}$ with $s \in \{-1, 0, 1\}$ is an additional term that if $s = 1$ restores the symmetry of $a_\varepsilon(u_{h,\varepsilon}, v_h)$, according to the fact that $a(u, v)$ is supposed to be a symmetric bilinear form. However, all choices $s = \pm 1$ and $s = 0$ are admissible and will be discussed later on. Another fundamental part of the scheme is the selection of the penalty parameter that will clearly emerge from the error analysis of the scheme.

Quoting R. Stenberg 1995, [45], *"In view of our analysis it seems that the Nitsche method is the most straightforward method to use. Unfortunately, this method seems to be quite unknown. We think, however, that it would be worthwhile to explore it in applications such as contact problems, for fictitious domain methods and for domain decomposition"*. Indeed, Nitsche's method has been recently applied to all of these cases with success and the scope of the present work is to review those studies, developing and discussing further extensions.

1.2 Analysis of Nitsche's method

Let $\mathscr{T}_h$ be a family of shape regular and quasi uniform triangulations of Ω. Let K be a generic element of $\mathscr{T}_h$ and let h_K be its diameter (the radius of the smallest ball containing this set) and the characteristic mesh size is $h := \max_{K\in\mathscr{T}_h} h_K$. Without loss of generality, we refer with our notation and choice of symbols to the case of two space dimensions. In particular, we apply the subscript E to denote element edges (or faces in three dimensions). Let $\mathscr{B}_h$ be the collection of mesh edges lying on the boundary $\partial\Omega$. On each mesh $\mathscr{T}_h$ we set up a Lagrangian finite element space of order k denoted as

$$V_h := \{v_h \in C^0(\Omega):\ v_h|_K \in \mathbb{P}^k(K)\ \forall K \in \mathscr{T}_h\}.$$

We endow the finite element space with the following norms that are adapted to the analysis of the scheme

$$\|v\|^2_{\pm\varepsilon,\partial\Omega} := \sum_{E\in\mathscr{B}_h} \varepsilon^{\mp 1}\|v\|^2_{0,E}, \quad \forall v \in L^2(\partial\Omega),$$
$$\|v\|^2_{1,\varepsilon,\Omega} := |v|^2_{1,\Omega} + \|v\|^2_{\varepsilon,\partial\Omega}, \quad \forall v \in H^1(\Omega).$$

For the forthcoming analysis we remind of the following basic inequalities, for which we refer to [10]. For simplicity of notation, we write $a \lesssim b$ if there exists a positive constant C independent of h such that $a \le Cb$. The standard L^2 Cauchy-Schwarz inequality can be straightforwardly extended to,

$$(v,w)_{\partial\Omega} \le \|v\|_{\pm\varepsilon,\partial\Omega}\|w\|_{\mp\varepsilon,\partial\Omega}, \quad \forall v,w \in L^2(\partial\Omega).$$

We will also make use of a generalised Poincaré inequality, also known as Poincaré-Friedrichs inequality, which holds in $H^1(\Omega)$ provided that an additional term is introduced to enrich the H^1-seminorm in order to account for constant functions,

$$\|v\|_{1,\Omega} \lesssim |v|^2_{1,\Omega} + \|v\|^2_{\varepsilon,\partial\Omega}, \quad \forall v \in H^1(\Omega).$$

Finally, the following discrete inequalities will be fundamental for the analysis of Nitsche's method,

$$h_E^{\frac{1}{2}}\|v_h\|_{0,E} \lesssim \|v_h\|_{0,K}, \quad h_K\|\nabla v_h\|_{0,K} \lesssim \|v_h\|_{0,K}, \quad \forall v_h \in V_h. \tag{14}$$

The first inequality implies that there exists a positive constant C_I such that

$$\sum_{E\in\mathscr{B}_h} h_E\|v_h\|^2_{0,E} \le C_I \sum_{K\in\mathscr{T}_h} \|v_h\|^2_{0,K}. \tag{15}$$

We notice that problem (13) consists of a standard Galerkin method using an H^1-conformal approximation space. Then, owing to Lax-Milgram's lemma its well posedness is ensured by consistency, stability and boundedness of $a_\varepsilon(\cdot,\cdot)$ together with linearity and boundedness of the right hand side.

Recalling that Nitsche's method can be seen as a correction of a simple penalty method in order to recover consistency, it is easy to verify that, given $u \in H^2(\Omega) \cap H^1_0(\Omega)$ the weak solution of $-\Delta u = f$ in Ω with $u = g$ on $\partial\Omega$, then problem (13) satisfies $a_\varepsilon(u-u_{h,\varepsilon},v_h)=0$ for any $v_h\in V_h$, which states that Nitsche's method is strongly consistent for any admissible value of ε and s.

In the framework of Lax-Milgram's lemma, stability is equivalent to coercivity of $a_\varepsilon(\cdot,\cdot)$ that holds true if there exists $\alpha>0$, uniformly independent of h, such that

$$a_\varepsilon(v_h,v_h)\geq \alpha\|v_h\|^2_{1,\varepsilon,\Omega},\ \forall v_h\in V_h.$$

To investigate the validity of such property in the particular case $s=1$, we proceed as follows

$$\begin{aligned}
a_\varepsilon(v_h,v_h) &= |v_h|^2_{1,\Omega}+\|v_h\|^2_{\varepsilon,\partial\Omega}-2\,(v_h,\partial_n v_h)\\
&\geq |v_h|^2_{1,\Omega}+\|v_h\|^2_{\varepsilon,\partial\Omega}-2\|v_h\|_{\varepsilon,\partial\Omega}\|\partial_n v_h\|_{-\varepsilon,\partial\Omega}\\
&\geq |v_h|^2_{1,\Omega}+\|v_h\|^2_{\varepsilon,\partial\Omega}-\delta^{-1}\|v_h\|^2_{\varepsilon,\partial\Omega}-\delta\|\partial_n v_h\|^2_{-\varepsilon,\partial\Omega}\\
&\leq |v_h|^2_{1,\Omega}+\|v_h\|^2_{\varepsilon,\partial\Omega}-\delta^{-1}\|v_h\|^2_{\varepsilon,\partial\Omega}-\delta\sum_{E\in\mathscr{B}_h}\varepsilon|v_h|^2_{1,E}.
\end{aligned}\tag{16}$$

In order to combine the first with the fourth term of previous inequality, it is convenient to select ε such that it is ε is directly proportional to h_E on $\mathscr{B}_h$. As a result of that, the norm $\|v\|_{\mp\varepsilon,\partial\Omega}$ is equivalent to

$$\|v_h\|^2_{\pm\frac{1}{2},h,\partial\Omega}:=\sum_{E\in\mathscr{B}_h}h_E^{\mp 1}\|v\|^2_{0,E},\qquad \forall v\in L^2(\partial\Omega)$$

and we denote $\|v\|^2_{1,h,\Omega}:=|v|^2_{1,\Omega}+\|v\|^2_{\frac{1}{2},h,\partial\Omega}$ accordingly. Owing to inverse inequality (14), we notice that it holds

$$\|v_h\|^2_{-\frac{1}{2},h,\partial\Omega}=\sum_{E\in\mathscr{B}_h}h_E\|v_h\|^2_E\lesssim\sum_{K\in\mathscr{T}_h}\|v_h\|^2_K\lesssim\|v_h\|^2_{0,\Omega},\quad\forall\ v_h\in V_h.$$

Given a positive constant γ we select $\varepsilon=h_E/\gamma$ for notational convenience. Then, the bilinear form $a_\varepsilon(\cdot,\cdot)$ and the right hand side $F_\varepsilon(\cdot)$ should be modified as follows,

$$\begin{aligned}
&a_h(u_h,v_h):=a(u_h,v_h)-(\partial_n u_h,v_h)_{\partial\Omega}-s\,(\partial_n v_h,u_h)_{\partial\Omega}+\gamma\sum_{E\in\mathscr{B}_h}h_E^{-1}\,(u_h,v_h)_E\,,\\
&F_h(v_h):=F(v_h)+\gamma\sum_{E\in\mathscr{B}_h}h_E^{-1}\,(g,v_h)_E-s\,(\partial_n v_h,g)_{\partial\Omega}\,,
\end{aligned}$$

and we aim to find $u_h\in V_h$ such that $a_h(u_h,v_h)=F_h(v_h)$ for any $v_h\in V_h$, which precisely define the Nitsche's method, except from the constant γ. To conclude the analysis of coercivity of $a_h(\cdot,\cdot)$, we mimic the reasoning of (16) and exploiting (15) we obtain

$$a_h(v_h,v_h)\gtrsim(1-\delta C_I)|v_h|^2_{1,\Omega}+(\gamma-\delta^{-1})\|v_h\|^2_{\frac{1}{2},h,\partial\Omega}$$

such that the coercivity of $a_h(\cdot,\cdot)$ holds true for any $C_I^{-1}>\delta>0$ provided that the penalty parameter γ is such that $\gamma>\delta^{-1}>C_I$.

Boundedness of Nitsche's method is equivalent to continuity of $a_h(\cdot,\cdot)$, that is, there exists $M > 0$ uniformly independent of h such that

$$a_h(u,v) \leq M\|u\|_{1,h,\Omega}\|v\|_{1,h,\Omega},\ \forall u,v \in H^1(\Omega)$$

The proof of such property follows from a combination of Cauchy-Schwarz inequalities,

$$\begin{aligned}
&a_h(u,v)\\
\leq&|u|_{1,\Omega}|v|_{1,\Omega} + \gamma\|u\|_{\frac{1}{2},h,\partial\Omega}\|v\|_{\frac{1}{2},h,\partial\Omega} + \|u\|_{\frac{1}{2},h,\partial\Omega}\|\partial_n v\|_{-\frac{1}{2},h,\partial\Omega}\\
&+\|v\|_{\frac{1}{2},h,\partial\Omega}\|\partial_n u\|_{-\frac{1}{2},h,\partial\Omega}\\
\lesssim&|u|_{1,\Omega}|v|_{1,\Omega} + \gamma\|u\|_{\frac{1}{2},h,\partial\Omega}\|v\|_{\frac{1}{2},h,\partial\Omega} + \|u\|_{\frac{1}{2},h,\partial\Omega}|v|_{1,\Omega} + \|v\|_{\frac{1}{2},h,\partial\Omega}|u|_{1,\Omega}\\
\lesssim&\|u\|_{1,h,\Omega}\|v\|_{1,h,\Omega}.
\end{aligned}$$

Combining consistency, stability and boundedness, we are able to perform the error analysis of Nitsche's method. We remind that the finite element space V_h satisfies a well known approximation property in the H^1 norm, which can be easily extended to the mesh dependent norm $\|\cdot\|_{1,h,\Omega}$ owning to inverse inequalities, in particular for any $v \in H^{k+1}(\Omega)$,

$$\inf_{v_h \in V_h}\|v - v_h\|_{1,h,\Omega} \lesssim h^k\|v\|_{k+1,\Omega}.$$

Then, Cea's lemma allows us to conclude that given $u \in H^{k+1}(\Omega)$ with $k \geq 1$ the weak solution of $-\Delta u = f$ in Ω with $u = g$ on $\partial\Omega$ and given u_h the solution of Nitsche's method with γ large enough, the following a-priori error estimate holds true,

$$\|u - u_h\|_{1,h,\Omega} \lesssim h^k\|u\|_{k+1,\Omega}, \tag{17}$$

and in case of self-adjoint problems and $s = 1$, exploiting Aubin-Nitsche's Lemma one obtains,

$$\|u - u_h\|_{0,\Omega} \lesssim h^{k+1}\|u\|_{k+1,\Omega}. \tag{18}$$

The optimality of approximation properties in the L^2-norm show the advantage of Nitsche's method with respect to the penalty technique, because the latter scheme turns out to be slightly suboptimal in this norm. Indeed, the analysis of [7] shows that, provided $u \in H^4(\Omega)$, for piece-wise linear elements on polygonal domains with perfectly fitted boundaries the optimal penalty choice is $\varepsilon \sim h^{\frac{5}{3}}$ and it leads to

$$\|u - u_h\|_{1,\Omega} \lesssim h\|u\|_{4,\Omega}, \quad \|u - u_h\|_{0,\Omega} \lesssim h^{\frac{5}{3}}\|u\|_{4,\Omega}.$$

For quadratic Lagrangian elements with the choice $\varepsilon \sim h^2$, it is possible to prove that the penalty method satisfies the following error estimates,

$$\|u - u_h\|_{1,\Omega} \lesssim h^2\|u\|_{5,\Omega}, \quad \|u - u_h\|_{0,\Omega} \lesssim h^2\|u\|_{5,\Omega},$$

which, under the strengthened regularity assumption $u \in H^5(\Omega)$, are optimal for the H^1-norm case, but suboptimal when the convergence is measured in the L^2-norm.

Conversely, the Lagrange multipliers method provides optimal convergence rates with respect to h. More precisely, we assume that the spaces $V_h,\ \Lambda_h$ satisfy the following approximation properties respectively,

$$\inf_{v_h \in V_h} \|v - v_h\|_{1,\Omega} \lesssim h^k \|v\|_{k+1,\Omega}, \qquad \inf_{\mu_h \in \Lambda_h} \|\mu - \mu_h\|_{0,\partial\Omega} \lesssim h^{l+1} \|\mu\|_{l+1,\partial\Omega},$$

for regular functions $v \in H^{k+1}(\Omega),\ \mu \in H^{l+1}(\partial\Omega)$. Then, provided that conditions (7)- (8) hold true for $V_h,\ \Lambda_h$, the following error estimates are satisfied, see [39, 40, 41],

$$\|u - u_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},h,\partial\Omega} \lesssim h^k \|u\|_{k+1,\Omega} + h^{l+\frac{3}{2}} \|\mu\|_{l+1,\partial\Omega}.$$

Thanks to the property $\lambda + \partial_n u = 0$, the Lagrange multipliers method has the advantage to simultaneously provide an approximation of the solution u and of its flux at the boundary. For Nitsche's method, the calculation of fluxes can be achieved after the solution of the problem determining u_h. It is interesting to observe that an accurate flux reconstruction involves both the normal gradient of the numerical solution and the penalty term. Indeed, multiplying equation (1) with homogeneous Dirichlet boundary data $g = 0$ by a test function $v_h \in V_h$, integrating over Ω and applying Green's formula, we straightforwardly obtain

$$(\nabla u, \nabla v_h)_\Omega - (\nabla u \cdot \mathbf{n}, v_h)_{\partial\Omega} = (f, v_h),\ \forall v_h \in V_h.$$

Subtracting Nitsche's scheme from previous equation we obtain,

$$\begin{aligned}(\nabla u \cdot \mathbf{n}, v_h)_{\partial\Omega} = (\nabla u_h \cdot \mathbf{n}, v_h)_{\partial\Omega} &+ s\,(\nabla v_h \cdot \mathbf{n}, u_h)_{\partial\Omega} \\ &- \gamma \sum_{E \in \mathscr{B}_h} h_E^{-1}\,(u_h, v_h)_E + (\nabla(u - u_h), \nabla v_h)_\Omega\end{aligned}$$

and by selecting $v_h = 1$ we obtain the following flux reconstruction formula,

$$\int_{\partial\Omega} \nabla u \cdot \mathbf{n} = \int_{\partial\Omega} \nabla u_h \cdot \mathbf{n} - \gamma \sum_{E \in \mathscr{B}_h} h_E^{-1} \int_E u_h.$$

1.3 Nitsche's Method for Interface Problems

The aim of this section is to briefly illustrate the application of Nitsche's method to a prototype of the interface problem. This subject has been and still is an active field of research, and the topics addressed here represent a summary of the seminal works by Hansbo et al, [8, 31].

Our simplified multi-domain problem consists of two non overlapping polygonal subdomains, $\Omega_i,\ i=1,2$, with interface $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$. We aim to find $u_i \in H^1(\Omega_i)$ that weakly satisfy,

$$\begin{cases} -\Delta u_i = f, & \text{in } \Omega_i, \\ u_i = 0, & \text{on } \partial\Omega \cap \partial\Omega_i, \\ u_1 - u_2 = 0, & \text{on } \Gamma, \\ \partial_n u_1 - \partial_n u_2 = 0, & \text{on } \Gamma, \end{cases} \tag{19}$$

where $\mathbf{n}$ denotes a unit normal vector associated to Γ and $\partial_n u := \nabla u \cdot \mathbf{n}$, where $\mathbf{n}$ on Γ can be either chosen as $\mathbf{n} := \mathbf{n}_1$ or equivalently $\mathbf{n} := \mathbf{n}_2$. Such ambiguity does not affect the application of Nitsche's method. To proceed, we define jumps and averages of quantities across the interface Γ. In particular, given a function $v : \overline{\Omega}_1 \cup \overline{\Omega}_2 \to \mathbb{R}$, its jump across the interface is defined as $[\![v]\!] := v_1 - v_2$, according to the sign of the vector $\mathbf{n}$, which is here selected as $\mathbf{n} = \mathbf{n}_1$, while the average is $\{v\} := \frac{1}{2}(v_1 + v_2)$. Problem (19) can be rewritten more conveniently as follows,

$$\begin{cases} -\Delta u_i = f, & \text{in } \Omega_i, \\ u_i = 0, & \text{on } \partial\Omega \cap \partial\Omega_i, \\ [\![u]\!] = 0, & \text{on } \Gamma, \\ [\![\partial_n u]\!] = 0, & \text{on } \Gamma. \end{cases} \tag{20}$$

As an instance of the rich family of mortar methods for interface problems, the peculiarity of Nitsche's scheme is to provide an approximation $u_h := [u_{h,1}, u_{h,2}]$ of (20) that is non conforming with $H^1(\Omega)$, as alternative to most popular domain decomposition techniques, such as Dirichlet-Neumann splitting.

For the discretisation of (20) let $\mathscr{T}_{h,i}$ be a family of shape-regular, quasi-uniform triangulations of Ω_i. Note that $\mathscr{T}_{h,i}$ with $i = 1,2$ may be non conforming at the interface. Let $\mathscr{B}_{h,i}$ and $\mathscr{G}_{h,i}$ the collections of the faces/edges at the boundary and at the interface respectively. We look for discrete functions $[u_{h,1}, u_{h,2}] \in V_h := V_{h,1} \times V_{h,2}$, where $V_{h,i}$ are Lagrangian finite element spaces on $\mathscr{T}_{h,i}$.

A weak formulation of the multi-domain problem that is prone to discretisation by Nitsche's method is obtained by multiplying $(20)_a$ with a test function $v_i \in H^1\Omega_i$ and applying integration by parts, such that

$$\begin{aligned} &\sum_{i=1,2} \Big(\int_{\Omega_i} \nabla u \cdot \nabla v - \int_{\partial\Omega_i} \nabla u \cdot \mathbf{n}_i v \Big) \\ &\qquad = \sum_{i=1,2} \Big(\int_{\Omega_i} \nabla u \cdot \nabla v - \int_{\partial\Omega_i \setminus \Gamma} \nabla u \cdot \mathbf{n}_i v \Big) - \int_{\Gamma} [\![\nabla u \cdot \mathbf{n} v]\!]. \end{aligned}$$

Interface conditions prescribing continuity of fluxes, i.e. $[\![\partial_n u]\!] = 0$, can be enforced in the bilinear form with the help of the following algebraic identity $[\![ab]\!] = [\![a]\!]\{b\} + [\![b]\!]\{a\}$, such that

$$[\![\nabla u \cdot \mathbf{n} v]\!] = [\![\nabla u \cdot \mathbf{n}]\!]\{v\} + \{\nabla u \cdot \mathbf{n}\}[\![v]\!] = \{\nabla u \cdot \mathbf{n}\}[\![v]\!] + \{\nabla v \cdot \mathbf{n}\}[\![u]\!]$$

where we exploit $[\![u]\!]=0$ owing to the strong consistency. For interface conditions prescribing continuity of the solution at the interface, we exploit penalty,

$$\sum_{i=1,2}\Big(\sum_{E\in\mathscr{G}_{h,i}}\frac{\gamma}{h_E}\int_E[\![u]\!][\![v]\!]+\sum_{E\in\mathscr{B}_{h,i}}\frac{\gamma}{h_E}\int_E uv\Big)$$

where γ is the penalty parameter already introduced for the approximation of Poisson's problem. Then, the extension of Nitsche's method to interface conditions consists in finding $u_h:=[u_{h,1},u_{h,2}]\in V_h:=V_{h,1}\times V_{h,2}$ such that

$$a_h(u_h,v_h)=F_h(v_h),\ \forall v_h\in V_h \tag{21}$$

with $a_i(u,v):=(\nabla u_i,\nabla v_i)_{\Omega_i}$ for any $u_i,v_i\in H^1(\Omega_i)$ and

$$\begin{aligned}a_h(u_h,v_h):=\sum_{i=1,2}\Big(&a_i(u_{h,i},v_{h,i})+\sum_{E\in\mathscr{G}_{h,i}}\gamma h_E^{-1}\left([\![u_h]\!],[\![v_h]\!]\right)_E\Big)\\&-(\{\nabla u_h\cdot\mathbf{n}\},[\![v_h]\!])_\Gamma-(\{\nabla v_h\cdot\mathbf{n}\},[\![u_h]\!])_\Gamma\\&+\sum_{i=1,2}\Big(\sum_{E\in\mathscr{B}_{h,i}}\gamma h_E^{-1}\left(u_h,v_h\right)_E-(\nabla u_h\cdot\mathbf{n}_i,v_h)_{\partial\Omega_i\setminus\Gamma}-(\nabla v_h\cdot\mathbf{n}_i,u_h)_{\partial\Omega_i\setminus\Gamma}\Big),\end{aligned}$$

$$F_h(v_h):=F(v_h)=\int_\Omega fv_h,\ \text{since } u=0 \text{ on } \partial\Omega,$$

where for simplicity we restrict the setting to the case $s=1$. This turns out to be a Galerkin method with an approximation space that is not H^1-conformal on Ω. Indeed, u_h belongs to the broken Sobolev space $H^1(\Omega_1\cup\Omega_2):=H^1(\Omega_1)\times H^1(\Omega_2)$ and the natural norms for the analysis of the problem read as follows,

$$\begin{aligned}&\|v\|^2_{\pm\frac12,h,\mathscr{G}_{h,i}}:=\sum_{E\in\mathscr{G}_{h,i}}h_E^{\mp1}\|v\|^2_{0,E},\quad \forall v\in L^2(\Gamma),\\&\|v\|^2_{1,h,\Omega_1\cup\Omega_2}:=\sum_{i=1,2}\Big(|v_i|^2_{1,\Omega_i}+\|v_i\|^2_{\frac12,h,\mathscr{B}_{h,i}}+\|[\![v]\!]\|^2_{\frac12,h,\mathscr{G}_{h,i}}\Big),\quad \forall v_i\in H^1(\Omega_i).\end{aligned}$$

Then, proceeding analogously to the case of a single domain, it is possible to verify that, if (20) admits a regular solution $u\in H^2(\Omega_1\cup\Omega_2)\cap H^1_0(\Omega)$, then $a_h(u,v_h)=F_h(v_h)$ for any $v_h\in V_h$ and $a_h(u-u_h,v_h)=0$ for any $v_h\in V_h$. Furthermore, $a_h(\cdot,\cdot)$ is bounded in the norm $\|\cdot\|_{1,h,\Omega_1\cup\Omega_2}$ and also stable with a constant uniformly independent on the mesh characteristic size h. As a result of that, following the lines of Cea's lemma, we obtain an a priori estimate equivalent to (17).

We finally notice that Nitsche's multi-domain scheme can be easily decomposed into local problems, relative to each subdomain, and coupling terms that transfer information from one subdomain to others. In particular we write,

$$a_h(u_h, v_h) = \sum_{i=1,2} \sum_{j \neq i} \Big[a_{h,i}(u_{h,i}, v_{h,i}) - c_{h,ij}(u_{h,j}, v_{h,i}) \Big],$$

where each single term is defined as follows,

$$a_{h,i}(u_{h,i}, v_{h,i}) := a_i(u_{h,i}, v_{h,i}) + c_{h,ii}(u_{h,i}, v_{h,i}) + b_{h,i}(u_{h,i}, v_{h,i}),$$

$$c_{h,ii}(u_{h,i}, v_{h,i}) := \sum_{E \in \mathscr{G}_{h,i}} \gamma h_E^{-1} \left(u_{h,i}, v_{h,i}\right)_E - \left(\tfrac{1}{2} \nabla u_{h,i} \cdot \mathbf{n}_i, v_{h,i}\right)_\Gamma - \left(\tfrac{1}{2} \nabla v_{h,i} \cdot \mathbf{n}_i, u_{h,i}\right)_\Gamma,$$

$$c_{h,ij}(u_{h,j}, v_{h,i}) := \sum_{E \in \mathscr{G}_{h,i}} \gamma h_E^{-1} \left(u_{h,j}, v_{h,i}\right)_E + \left(\tfrac{1}{2} \nabla u_{h,j} \cdot \mathbf{n}_i, v_{h,i}\right)_\Gamma - \left(\tfrac{1}{2} \nabla v_{h,i} \cdot \mathbf{n}_i, u_{h,j}\right)_\Gamma,$$

$$b_{h,i}(u_{h,i}, v_{h,i}) := \sum_{i=1,2} \Big[\sum_{E \in \mathscr{B}_{h,i}} \gamma h_E^{-1} (u_h, v_h)_E - (\nabla u_h \cdot \mathbf{n}_i, v_h)_{\partial\Omega_i \setminus \Gamma} - (\nabla v_h \cdot \mathbf{n}_i, u_h)_{\partial\Omega_i \setminus \Gamma} \Big].$$

Such decomposition suggests that, starting from problem (21), it is possible to devise an iterative splitting strategy that aims to decompose the solution of a multi-domain problem on Ω into a sequence of local problems on Ω_i. Indeed, owing to the introduction of the following relaxation operators, where the relaxation effect from one iteration to another is again obtained through a penalty term similar to the one of (11),

$$s_{h,i}^\sigma(u_{h,i}, v_{h,i}; u_{h,i}^{(\mathrm{old})}) := \sum_{E \in \mathscr{G}_{h,i}} \sigma h_E^{-1} \Big(u_{h,i} - u_{h,i}^{(\mathrm{old})}, v_{h,i} \Big)_E,$$

$$s_h^\sigma(u_h, v_h; u_h^{(\mathrm{old})}) := \sum_{i=1,2} s_{h,i}^\sigma(u_{h,i}, v_{h,i}; u_{h,i}^{(\mathrm{old})}).$$

The iterative method obtained by giving $u_{h,i}^0 \in V_{h,i}$ for $i = 1,2$ and looking for a sequence of approximations $u_{h,i}^k$ for any $k > 0$ such that,

$$a_{h,i}(u_{h,i}^k, v_{h,i}) + s_{h,i}^\sigma(u_{h,i}^k, v_{h,i}; u_{h,i}^{k-1}) - c_{h,ij}(u_{h,j}^{k-1}, v_{h,i}) = F_{h,i}(v_{h,i}), \ \forall v_{h,i} \in V_{h,i}, \quad (22)$$

turns out to be convergent to $[u_{h,1}, u_{h,2}]$ provided that the relaxation parameter σ is large enough. Such technique has already been profitably applied to the approximation of advection dominated elliptic problems in [17] as well as to mixed problems in [19]. The convergence analysis of the iterative scheme is more easily performed if we rewrite it as follows

$$a_h(u_h^k, v_h) + s_h^\sigma(u_h^k, v_h; u_h^{k-1}) = F_h(v_h) - r_h(u_h^k - u_h^{k-1}, v_h), \quad (23)$$

$$r_h(u_h^k - u_h^{k-1}, v_h) := \sum_{i=1,2} \sum_{j \neq i} c_{h,ij}(u_{h,j}^k - u_{h,j}^{k-1}, v_{h,i}),$$

which is obtained from (22) by summing up the equations for $i=1,2$ and introducing the new terms $\pm c_{h,ij}(u^k_{h,j}, v_{h,i})$. Notice that $r_h(u^k_h - u^{k-1}_h, v_h)$ plays the role of iteration residual and the interplay of s^σ_h with r_h is the key point to prove convergence of iterations. To this purpose, we look at the iteration error, that is $w^k_h := u_h - u^k_h$. By subtracting (23) from (21), we obtain an equation for w^k_h, precisely

$$a_h(w^k_h, v_h) + s^\sigma_h(w^k_h, v_h; w^{k-1}_h) = -r_h(w^k_h - w^{k-1}_h, v_h).$$

Then, convergence of u^k_h relies on the following inequality

$$\alpha \|w^k_h\|^2_{1,h,\Omega_1\cup\Omega_2} + s^\sigma_h(w^k_h, w^k_h; w^{k-1}_h) \leq |r_h(w^k_h - w^{k-1}_h, w^k_h)|,$$

combined with the following estimates for s^σ_h and r_h,

$$\begin{aligned} r_h(w^k_h - w^{k-1}_h, w^k_h) = \sum_{i=1,2;j\neq i} \Big[\Big(\nabla(w^k_{h,j} - w^{k-1}_{h,j})\cdot \mathbf{n}_i, w^k_{h,i}\Big)_\Gamma \\ - \Big(\nabla w^k_{h,i}\cdot \mathbf{n}_i, w^k_{h,j} - w^{k-1}_{h,j}\Big)_\Gamma + \sum_{E\in\mathscr{G}_{h,i}} \gamma h_E^{-1}\Big(w^k_{h,j} - w^{k-1}_{h,j}, w^k_{h,i}\Big)_\Gamma \Big], \end{aligned}$$

$$\begin{aligned} &\sum_{i=1,2} \Big[\Big(\nabla(w^k_{h,j} - w^{k-1}_{h,j})\cdot \mathbf{n}_i, w^k_{h,i}\Big)_\Gamma - \Big(\nabla w^k_{h,i}\cdot \mathbf{n}_i, w^k_{h,j} - w^{k-1}_{h,j}\Big)_\Gamma \Big] \\ &\lesssim \sum_{i=1,2;j\neq i} \Big[\delta(\|w^k_{h,i}\|^2_{1,h,\Omega_i} + \|w^{k-1}_{h,i}\|^2_{1,h,\Omega_i}) + \delta^{-1}\|w^k_{h,i} - w^{k-1}_{h,i}\|^2_{\frac12,h,\Gamma}\Big], \end{aligned}$$

$$\begin{aligned} &\sum_{i=1,2;j\neq i}\sum_{E\in\mathscr{G}_{h,i}} \gamma h_E^{-1}\Big(w^k_{h,j} - w^{k-1}_{h,j}, w^k_{h,i}\Big)_\Gamma \\ &\lesssim \sum_{i=1,2}\gamma\Big[\|w^k_{h,i} - w^{k-1}_{h,i}\|^2_{\frac12,h,\Gamma} + \|[\![w^k_h]\!]\|^2_{\frac12,h,\Gamma}\Big] + s^\gamma_h(w^k_h, w^k_h; w^{k-1}_h), \end{aligned}$$

$$s^\sigma_h(w^k_h, w^k_h; w^{k-1}_h) = \frac{\sigma}{2}\sum_{i=1,2}\Big[\|w^k_{h,i}\|^2_{\frac12,h,\Gamma} - \|w^{k-1}_{h,i}\|^2_{\frac12,h,\Gamma} + \|w^k_{h,i} - w^{k-1}_{h,i}\|^2_{\frac12,h,\Gamma}\Big].$$

Together with suitable choices of δ and γ, the previous estimates can be suitably applied to obtain the following inequality

$$\begin{aligned} &\beta\|w^k_h\|^2_{1,h,\Omega_1\cup\Omega_2} + \xi\sum_{i=1,2}\|w^k_{h,i} - w^{k-1}_{h,i}\|^2_{\frac12,h,\Gamma} \\ &\qquad \lesssim \zeta\sum_{i=1,2}\Big[\|w^{k-1}_{h,i}\|^2_{\frac12,h,\Gamma} - \|w^k_{h,i}\|^2_{\frac12,h,\Gamma} + \|w^{k-1}_{h,i}\|^2_{1,h,\Omega_i} - \|w^k_{h,i}\|^2_{1,h,\Omega_i}\Big]. \end{aligned}$$

Summing up over the index k we conclude that there exists a constant $C>0$ independent on k, but possibly depending on the initial state, such that

$$\sum_{k=1}^{\infty}\|w^k_h\|_{1,h,\Omega_1\cup\Omega_2} \leq C,$$

which implies convergence of the sequence u_h^k to u_h in the natural norm.

1.4 The Unfitted Version of Nitsche's Method

The increasing complexity of geometrical configurations in applications addressed by means of computational analysis has motivated the research of finite element schemes capable to handle the case where the computational mesh is not fitted to boundaries or interfaces. Instead, a physical domain with a possibly complex shape is embedded into a computational domain with simple shape that is easily partitioned into elements. Thanks to their flexibility in the treatment of Dirichlet boundary conditions, the method of Lagrange multipliers and Nitsche's scheme have been profitably applied to such purpose. We report here simple examples for such schemes, together with a brief discussion of their intrinsic drawbacks. We refer to Section 3 for a detailed development of suitable stabilisation techniques to obtain efficient and robust schemes for the approximation of problems on boundaries or interfaces that do not fit with the computational mesh.

For the set up of a finite element method with unfitted boundary, we denote by Ω the physical domain, embedded into a computational domain $\Omega_{\mathscr{T}}$ corresponding to a computational mesh $\mathscr{T}_h$. The basic restrictive assumption for the correct definition of unfitted boundary methods is the requirement that each element $K \in \mathscr{T}_h$ must have a non vanishing intersection with Ω and that the boundary $\partial\Omega$ is regular and intersects each element boundary ∂K at most twice and each open edge E at most once. We refer to Figure 5 for an example of physical and computational domains. The approximation space consists on linear Lagrangian finite elements on $\Omega_{\mathscr{T}}$,

$$V_h := \{v_h \in C^0(\Omega_{\mathscr{T}}) : \ v_h|_K \in \mathbb{P}^1(K) \ \forall K \in \mathscr{T}_h\}.$$

Because of its simplicity, the penalty method turns out to be very attractive to build up finite element approximations on meshes not fitting the boundary of the physical domain. Under the assumption $\mathrm{dist}(\Omega, \Omega_{\mathscr{T}}) \lesssim h^2$, it is shown in [7] that the application of the simple penalty term $h^{-2}\,(u_h - g, v_h)_{\partial\Omega_{\mathscr{T}}}$ to a linear finite element approximation without boundary constraints is sufficient to recover a discrete solution that satisfies the error estimates

$$\|u-u_h\|_{1,\Omega} \lesssim h\|u\|_{4,\Omega}, \quad \|u-u_h\|_{0,\Omega} \lesssim h^{\frac{3}{2}}\|u\|_{4,\Omega},$$

for Poisson's problem (or any other second-order self-adjoint variant) with regular solution $u \in H^4(\Omega)$.

To extend the method of Lagrange multipliers to unfitted meshes, a major difficulty is the construction of a suitable multiplier space for a boundary that does not coincide with the edges of the mesh. An effective and simple solution is studied in [26] where piecewise constant multipliers are applied on Ω. If the multiplier mesh is suitably coarser than the one relative to the primal unknown, the application of piecewise linear approximations with piecewise constant multipliers gives rise to a stable scheme. Unless the finite element spaces are chosen so that the discrete inf-sup condition is satisfied, some stabilisation must be introduced. One of the most popular stabilised methods was introduced by Hughes and Barbosa [5, 6]. In this case the difference between the discrete Lagrange multiplier and the discrete normal derivative is penalised. Such a method was proposed in the fictitious domain framework by Renard et al. [33]. Another recent stabilised method is based on the idea of interior penalty, where the stabilisation acts on the Lagrange multiplier alone and acts as a coarsening operator effectively penalising the distance of the discrete Lagrange multiplier to a stable subspace. We give an example of this later formulation taken from [15] below.

For the construction of such a space we assume that $\partial\Omega$ is a curved boundary without corners (for the extension to the polygonal case we refer to [15]) and we define the collection of all elements cut by the unfitted boundary as $\mathscr{C}_h := \{K \in \mathscr{T}_h : K \cap \partial\Omega \neq \emptyset\}$, then the space of multipliers is

$$\Lambda_h := \{v_h \in L^2(\mathscr{C}_h) : \; v_h|_K \in \mathbb{P}^0(K) \; \forall K \in \mathscr{C}_h\}.$$

For the approximation of problem (1) on an unfitted mesh we aim to find a couple $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\begin{cases} a(u_h, v_h) + b(\lambda_h, v_h) = F(v_h) & \forall v_h \in V_h, \\ b(\mu_h, u_h) - J(\lambda_h, \mu_h) = b(\mu_h, g) & \forall \mu_h \in \Lambda_h, \end{cases} \tag{24}$$

where the definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ do not change with respect to (6), while $J(\lambda_h, v_h)$ is a stabilisation term proposed in [15] and defined as follows,

$$J(\lambda_h, \mu_h) = \sum_{E \in \mathscr{E}_B} (\gamma h [\![\lambda_h]\!], [\![\mu_h]\!])_E \,,$$

where $\mathscr{E}_B$ is the set of edges or faces in $\mathscr{C}_h$ intersected by the boundary $\partial\Omega$, the jump of the piecewise constant function λ_h across such edges is denoted by $[\![\lambda_h]\!]$ and γ is a stabilisation parameter that should be selected large enough according to the analysis performed in [15]. Problem (24) features a remarkable advantage for unfitted boundaries, because the primal and the dual variables u_h and λ_h respectively, are defined on the same computational mesh $\mathscr{T}_h$, in contrast to a more classical choice for Lagrange multipliers that needs an independent partition of the boundary $\partial\Omega$.

The application of Nitsche's method to the case of unfitted boundary only requires a minor modification with respect to the standard case. We notice that the concept of edges or faces on the unfitted boundary is not properly defined yet. Then, instead of defining the penalty term on each edges, we simply consider $\gamma h^{-1}(u_h, v_h)_\Gamma$. As a result of that, Nitsche's method for an unfitted boundary requires to find $u_h \in V_h$ such that $a_h(u_h, v_h) = F_h(v_h)$ for any $v_h \in V_h$ with

$$a_h(u_h, v_h) := a(u_h, v_h) - (\partial_n u_h, v_h)_{\partial\Omega} - s\,(\partial_n v_h, u_h)_{\partial\Omega} + \gamma h^{-1}\,(u_h, v_h)_{\partial\Omega}\,,$$
$$F_h(v_h) := F(v_h) + h^{-1}\,(g, v_h)_{\partial\Omega} - s\,(\partial_n v_h, g)_{\partial\Omega}\,.$$

The main drawback of such a scheme is its lack of robustness with respect to the position of the boundary. Indeed, an unfitted boundary Γ may cut the computational mesh such that some intersections of elements with the physical domain are very small and/or feature very large aspect ratios. In such cases, as illustrated in [12], the linear system arising from Nitsche's discrete problem may be ill posed. Let $\mathbf{x}_k$ be the vertexes of the computational mesh $\mathscr{T}_h$ and let $\mathscr{P}_k$ be the patch of elements relative to the vertex $\mathbf{x}_k$. Given a generic function $v_h \in V_h$, let $\mathbf{v}$ the vector of its degrees of freedom endowed with the Euclidean norm $\|\mathbf{v}\|$, namely these are the values of v_h in the vertexes $\mathbf{x}_k$ for linear Lagrangian finite elements. We denote with ν a non-dimensional parameter that quantifies the size of the minimal intersection of finite element patches with the physical domain. Precisely, we define

$$\nu = \min_k \frac{|\mathscr{P}_k \cap \Omega|}{|\mathscr{P}_k|},$$

where from now on $|\Omega|$ will denote the d-dimensional volume of $\Omega \subset \mathbb{R}^d$. According to the analysis developed in [12, 43], there exists a function $v_h^* \in V_h$ such that

$$\|v_h^*\|^2_{1,\varepsilon,\Omega} \lesssim h^{d-2}\nu\|\mathbf{v}^*\|^2.$$

Denoting with A_h the stiffness matrix related to Nitsche's method, such an estimate directly implies that its spectral condition number admits the lower bound $K_2(A_h) \gtrsim \nu^{-1}h^{-2}$. For any boundary configuration such that $\nu \to 0$, matrix A_h becomes ill posed and almost singular. In conclusion, the development of stabilisation techniques to complement the unfitted Nitsche's scheme and make it fully robust with respect to any boundary configuration is a vivid field of research on which we will concentrate in Section 3.

2 A Modified Nitsche's Method for Large Contrast Problems

In the previous section we have studied how to apply Nitsche's method to enforce interface conditions to couple second order problems of the same type on adjacent domains. The purpose of this section is to extend this method to problems that vary in character form one part of the domain to another. To be more precise, we restrict to interface problems where the governing equations are similar on adjacent subdomains, but they may be characterized by heterogeneous coefficients. We refer to this large family of problems with the general name of large contrast problems and we remark that they are encountered in relevant applications such as computational mechanics, for the study of the deformation of heterogeneous bodies, or geosciences, for the analysis of flow and mass transport in soils or aquifers.

Several authors have already successfully applied Nitsche's method to the discretization of large contrast problems. For the case of computational mechanics we refer for instance to [31], while for the analysis of a generic singularly perturbed advection diffusion problem we refer to [17]. In this section we focus on the latter case, in particular we study the coupling of a second order scalar problem where one of the subproblems features a singularly perturbed behaviour. Typical model problems are advection / diffusion equations with heterogeneous diffusion coefficients between subregions, i.e.

$$-\nabla \cdot (\varepsilon \nabla u) + \beta \cdot \nabla u = f \text{ in } \Omega,$$

where ε denotes the diffusivity of a given medium and β is a given advective field, which for simplicity we assume to be solenoidal. Provided that ε is a positive and bounded function, the advection / diffusion problem turns out to be well posed owing to a straightforward application of Lax-Milgram's lemma. In the case of variable, possibly discontinuous diffusivity ε the interest in rewriting the problem as a multi-domain problem, subdividing regions with uniform properties, arises from the observation that internal layers of the solution may appear in the neighbourhood of the interfaces where coefficients are discontinuous. In several applications, such as heat or mass transfer problems, the configuration of such layers determine the fluxes exchanged between different bodies, and thus a correct approximation of them is necessary.

2.1 Approximation of Large Contrast Problems with Locally Vanishing Diffusion

We consider for simplicity two non overlapping polygonal subdomains, $\Omega_i,\ i = 1,2$, with interface $\Gamma := \overline{\Omega}_1 \cup \overline{\Omega}_2$ as an instance of a more general multi-material problem depicted in Figure 1. Furthermore, without significant loss of generality, we restrict to the case of uniform coefficients on each subregion. In particular, given two constant parameters $\varepsilon_i > 0,\ i = 1,2$ and $\beta \in [C^1(\Omega)]^d$ with $\nabla \cdot \beta = 0$, $|\beta| \simeq 1$, we aim to find u_i such that

$$\begin{cases} \nabla \cdot \big(-\varepsilon_i \nabla u_i + \beta u_i\big) = f_i, & \text{in } \Omega_i, \\ u_i = 0, & \text{on } \partial\Omega \cap \partial\Omega_i, \\ [\![u]\!] = 0, & \text{on } \Gamma, \\ [\![-\varepsilon \nabla u \cdot \mathbf{n} + \beta \cdot \mathbf{n} u]\!] = 0, & \text{on } \Gamma. \end{cases} \tag{25}$$

First of all, we notice that an internal layer may appear in the neighborhood of Γ when $\varepsilon_1 \neq \varepsilon_2$. This happens for instance if $\varepsilon_1 \ll \varepsilon_2$ and the interface Γ (or part of it) is an outflow region for the advective field β. In this case the internal layer is located upwind to the interface, in other words it is confined into the domain Ω_1. Moreover, in the singularly perturbed limit case, i.e. $\varepsilon_1 \to 0$, $\varepsilon_2 > 0$ the internal

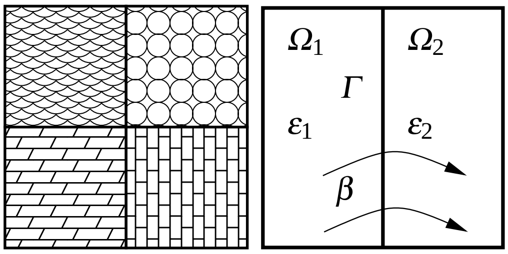

Fig. 1 A general multi-material problem (left) restricted to a two-domain case (right).

layer becomes thinner and stiffer, while the global solution u of (25) approaches a discontinuous function. In conclusion, under these particular conditions, the solution of the limit problem (25) with $\varepsilon_1 \to 0$ fails to be H^1-conformal. Thus, we focus on Nitsche's technique as a discretization method for interface problems pursuing the idea that only a H^1 non-conformal discretization technique can robustly approximate the problem under all possible conditions including the singularly perturbed limit.

For the discretization of problem (25) we could proceed in analogy with Poisson problem, already addressed in Section 1.3. Since problem (25) is written in divergence form, it is easy to extend the treatment of natural interface conditions of type $[\![\partial_n u]\!] = 0$ to the case of the conormal derivative $[\![-\varepsilon\nabla u \cdot \mathbf{n} + \beta \cdot \mathbf{n}u]\!]$. We will see later on that such an approach will only partially fulfill the objective to set up a robust discretization scheme for local singularly perturbed problems. To further improve the resulting scheme, we look at interface conditions with a bias to domain decomposition methods. Observing that in the limit case $\varepsilon_1 \to 0$ the sub-problem in Ω_1 tends to an hyperbolic problem coupled to an elliptic problem on Ω_2, we consider the set up of a new Nitsche method arising from a set of generalized interface conditions, introduced in [24] to couple both elliptic and hyperbolic problems, which give rise to the so called heterogeneous domain decomposition methods, see [42]. Our purpose is to obtain a weak coupling scheme that inherits the robustness of heterogeneous domain decomposition methods for the approximation of problems that vary in character form one part of the domain to another. As a result of that, such a method will turn out to be effective for problems whose solution features sharp internal layers.

The starting point of such a procedure is the definition and analysis of the coupling conditions between an advection / diffusion (elliptic) equation with a purely advective model (hyperbolic problem). Setting $\varepsilon_1 = 0,\ \varepsilon_2 > 0$ in (25), we identify Ω_1 as the hyperbolic and Ω_2 as the elliptic subregion. Let n_{hy} be the outward unit normal with respect to Ω_{hy} and let $\partial\Omega_{in} := \{x \in \partial\Omega_{hy} : \beta \cdot n_{hy} < 0\}$. According to this definition, the interface can be split in two parts $\Gamma_{in} := \Gamma \cap \partial\Omega_{in}$ and the complementary $\Gamma_{out} := \Gamma \setminus \Gamma_{in}$. We look for $u_{hy} = u_1,\ u_{el} = u_2$ such that

$$
\begin{cases}
\nabla\cdot(-\varepsilon\nabla u_{el}+\beta u_{el})=f & \text{in } \Omega_{el},\\
\nabla\cdot(\beta u_{hy})=f & \text{in } \Omega_{hy},\\
-\varepsilon\nabla u_{el}\cdot n+\beta\cdot n u_{el}=\beta\cdot n u_{hy} & \text{on } \Gamma,\\
u_{el}=u_{hy} & \text{on } \Gamma_{in},\\
u_{el}=0 & \text{on } \partial\Omega\cap\partial\Omega_{el},\\
u_{hy}=0 & \text{on } \partial\Omega\cap\partial\Omega_{in}.
\end{cases}
\tag{26}
$$

Comparing problem (25) with (26), we notice that interface conditions involving mass fluxes are naturally extended to the limit case $\varepsilon_1=0$, because the diffusive flux has disappeared from the right hand side of $(26)_c$. Conversely, interface conditions for the solution itself feature a singular behaviour in the vanishing viscosity case. Indeed, continuity of the solution is only enforced on the inflow part of the interface, referred to as the hyperbolic boundary, i.e. $\Gamma\cap\partial\Omega_{in}$, while on the complementary outflow interface the solutions u_{el} and u_{hy} do not conform; that is the global solution of the heterogeneous problem, u, can be discontinuous across this part of the interface. Nitsche's method turns out to be particularly effective to handle such conditions in a general setting. On the one hand, as for the aforementioned Poisson problem, the continuity of mass fluxes can be naturally handled by integrating by parts the local governing equations and exploiting the algebraic inequality $[\![ab]\!]=[\![a]\!]\{b\}+[\![b]\!]\{a\}$. On the other hand, the singular behaviour of the continuity of the solution u can be addressed by a suitable manipulation of the interface penalty term. Exploiting the flexibility of Nitsche's technique, we aim to set up a discrete interface problem that is strongly consistent with both problems (25), (26), resorting to a robust finite element scheme for the local singularly perturbed limit case. To perform this task we start from a unified formulation of continuity interface conditions for problems (25) and (26) in a sufficiently general setting that allows the extension of similar concepts to the case of fluid dynamics, like the case addressed in [19].

If ε_i were positive and quasi-uniform on Ω, the standard condition to enforce the continuity of the solution would be,

$$
\left[\tfrac{1}{2}|\beta\cdot n_\Gamma|+\{\varepsilon\}\right][\![u]\!]=0 \quad \text{on } \Gamma\setminus\partial\Omega.
$$

where we the factor $\left[\tfrac{1}{2}|\beta\cdot n_\Gamma|+\{\varepsilon\}\right]$ appears to modulate the intensity of the penalty term that weakly enforces the continuity requirement. In order to correct this condition in the case where ε_i significantly varies from region to region, we introduce an *heterogeneity factor*, which quantifies the variation of ε on the interface, $\lambda(x)|_\Gamma:\ \Gamma\to[-1,1]$ such that

$$
\lambda(x)|_\Gamma:=\begin{cases}
\frac{1}{2}\frac{[\![\varepsilon(x)]\!]}{\{\varepsilon(x)\}}, & \text{if } \{\varepsilon(x)\}>0,\\
0, & \text{if } \{\varepsilon(x)\}=0.
\end{cases}
$$

Then, starting from the case of uniform diffusivity considered above, we propose the following generalized interface conditions for the continuity of the solution,

$$\left[\tfrac{1}{2}|\beta\cdot n_\Gamma|\big(1-\mathrm{sign}(\beta\cdot n_\Gamma)\varphi_\Gamma(\lambda)\big)+\{\varepsilon\}\big(1-\chi_\Gamma(\lambda)\big)\right][\![u]\!]=0 \ \text{ on } \Gamma, \qquad (27)$$

where $\varphi_\Gamma(\lambda)$ and $\chi_\Gamma(\lambda)$ are scaling functions that must satisfy the following requirements in order to make sure that in the limit case the continuity of the solution is enforced on $\Gamma\cap\partial\Omega_{in}$ solely. Precisely, we assume that they satisfy $|\chi_\Gamma(\lambda)|\le 1$, $|\varphi_\Gamma(\lambda)|\le 1$ and,

$$\begin{array}{ll}\chi_\Gamma(\lambda)=0 & \text{if } \lambda|_\Gamma=0,\\ \chi_\Gamma(\lambda)=1 & \text{if } \lambda|_\Gamma=\pm 1,\\ \varphi_\Gamma(\lambda)=0 & \text{if } \lambda|_\Gamma=0,\\ \varphi_\Gamma(\lambda)=\mp 1 & \text{if } \lambda|_\Gamma=\pm 1.\end{array}$$

According to these properties, we further assume that $\chi_\Gamma(\lambda)$ is a symmetric function while $\varphi_\Gamma(\lambda)$ is skew-symmetric.

It is straightforward to verify that when $\varepsilon_1=\varepsilon_2$ and thus $\lambda(\varepsilon)=0$, condition (27) coincides with $\left[\tfrac{1}{2}|\beta\cdot n_\Gamma|+\{\varepsilon\}\right][\![u]\!]=0$. In the vanishing viscosity case let us fix $\mathbf{n}=\mathbf{n}_{hy}$ as reference orientation of the interface. Then, we obtain

$$\lambda=\frac{\varepsilon_{hy}-\varepsilon_{el}}{\varepsilon_{hy}+\varepsilon_{el}}=-1$$

and by consequence $\varphi_\Gamma(\lambda=-1)=1$. As a result of that, it turns out that, for the elliptic / hyperbolic coupling, condition (27) is equivalent to $\big(1-\mathrm{sign}(\beta\cdot\mathbf{n})\varphi_\Gamma(\lambda)\big)=2$ on Γ_{in} and $\big(1-\mathrm{sign}(\beta\cdot\mathbf{n})\varphi_\Gamma(\lambda)\big)=0$ on Γ_{out}, which coincides with the continuity condition of (26).

In conclusion, to set up Nitsche's method that suits problem (25) and (26) we start from the following general formulation that combines both,

$$\begin{cases}\nabla\cdot\big(-\varepsilon_i\nabla u_i+\beta u_i\big)=f_i & \text{in } \Omega_i,\\ \big(\tfrac{1}{2}|\beta\cdot\mathbf{n}_i|-\tfrac{1}{2}\beta\cdot\mathbf{n}_i+\varepsilon\big)u_i=0 & \text{on } \partial\Omega\cap\partial\Omega_i,\\ [\![-\varepsilon\nabla u\cdot\mathbf{n}+\beta\cdot\mathbf{n}u]\!]=0 & \text{on } \Gamma,\\ \left[\tfrac{1}{2}|\beta\cdot\mathbf{n}|\big(1-\mathrm{sign}(\beta\cdot\mathbf{n})\varphi_\Gamma(\lambda)\big)+\{\varepsilon\}\big(1-\chi_\Gamma(\lambda)\big)\right][\![u]\!]=0 & \text{on } \Gamma.\end{cases} \qquad (28)$$

2.1.1 Variational formulation and analysis

To proceed with the variational formulation of problem (28), propaedeutic to the application of Nitsche's coupling technique, we integrate the governing equations on each sub-region and applying Green's formula (including the advective terms) we obtain,

$$\begin{aligned}
&\sum_{i=1,2}\Big[\int_{\Omega_i}\big(\varepsilon_i\nabla u_i\cdot\nabla v_i-\beta u_i\cdot\nabla v_i\big)+\int_{\partial\Omega_i}\big(-\varepsilon_i\nabla u_i\cdot\mathbf{n}_i v_i+\beta\cdot\mathbf{n}_i u_i v_i\big)\Big]\\
&=\sum_{i=1,2}\Big[\int_{\Omega_i}\big(\varepsilon_i\nabla u_i\cdot\nabla v_i-\beta u_i\cdot\nabla v_i\big)+\int_{\partial\Omega_i\setminus\Gamma}\big(-\varepsilon_i\nabla u_i\cdot\mathbf{n}_i v_i+\beta\cdot\mathbf{n}_i u_i v_i\big)\Big]\\
&\quad+\int_{\Gamma}[\![-\varepsilon\nabla u\cdot\mathbf{n}v+\beta\cdot\mathbf{n}uv]\!]. \qquad (29)
\end{aligned}$$

The term $[\![-\varepsilon\nabla u\cdot\mathbf{n}v+\beta\cdot\mathbf{n}uv]\!]$ allows us to weakly enforce continuity of the conormal derivatives. However, to maintain strong consistency with problem (28), it is necessary to generalize the technique already described for Poisson's equation, to the case of weighted averages,

$$\{v(x)\}_w := w_i(x)v_i(x)+w_j(x)v_j(x),$$

$$\{v(x)\}^w := w_j(x)v_i(x)+w_i(x)v_j(x),$$

with $i=1,2,\ j\neq i$, where v is a regular function, $x\in\Gamma$ and the weights necessarily satisfy $w_1(x)+w_2(x)=1$. We say that these averages are conjugate, because they fulfill the following identity,

$$[\![ab]\!]=\{a\}_w[\![b]\!]+\{b\}^w[\![a]\!],$$

that can be exploited to obtain,

$$\begin{aligned}
[\![(-\varepsilon\nabla u\cdot\mathbf{n}+\beta\cdot\mathbf{n}u)v]\!]&=[\![-\varepsilon\nabla u\cdot\mathbf{n}+\beta\cdot\mathbf{n}u]\!]\{v\}^w+\{-\varepsilon\nabla u\cdot\mathbf{n}+\beta\cdot\mathbf{n}u\}_w[\![v]\!]\\
&=[\![-\varepsilon\nabla u\cdot\mathbf{n}+\beta\cdot\mathbf{n}u]\!]\{v\}^w-\{\varepsilon\nabla u\cdot\mathbf{n}\}_w[\![v]\!]+\{\beta\cdot\mathbf{n}u\}_w[\![v]\!].
\end{aligned}$$

First, the previous identity allows to weakly enforce the continuity of fluxes at the interface, by setting $[\![-\varepsilon\nabla u\cdot\mathbf{n}+\beta\cdot\mathbf{n}u]\!]=0$. Secondly, it shows that the choice of the averaging weights w_i is not completely arbitrary. Indeed, to reproduce the interface condition $-\varepsilon\nabla u_{el}\cdot n+\beta\cdot nu_{el}=\beta\cdot nu_{hy}$ at the level of the variational formulation, that is to maintain strong consistency with problem (26), we have to make sure that the term $\{\varepsilon\nabla u\cdot\mathbf{n}\}_w[\![v]\!]$ vanishes when $\varepsilon_1=\varepsilon_{hy}=0$ while $\{\beta\cdot\mathbf{n}u\}_w[\![v]\!]=\beta\cdot nu_{hy}[\![v]\!]$. Such requirements correspond to the following constraint:

$$w_1=1,\ w_2=0\quad\text{when}\quad\varepsilon_1\to 0$$

We conclude that, for a strongly consistent treatment of the interface conditions with Nitsche's method, not only the intensity of the penalty terms, but also the averaging weights must suitably depend on the coefficients of the problem, and in particular on their heterogeneity. We will discuss later on suitable expressions for these problem dependent parameters.

Defining the following problem dependent penalty factors that modulate the enforcement of interface and boundary conditions, respectively,

$$\xi_\Gamma(\varepsilon,\beta) := \tfrac{1}{2}\big(|\beta\cdot\mathbf{n}| - \beta\cdot\mathbf{n}\varphi_\Gamma(\lambda|_\Gamma)\big) + \{\varepsilon\}\big(1-\chi_\Gamma(\lambda|_\Gamma)\gamma h_E^{-1}\big),$$
$$\xi_{i,\partial\Omega}(\varepsilon,\beta) := \frac{1}{2}\big(|\beta\cdot\mathbf{n}_i| - \beta\cdot\mathbf{n}_i\big) + \varepsilon\gamma h_E^{-1},$$

where $\gamma > 0$ is a penalty parameter to be selected large enough in order to ensure stability of the resulting scheme, the bilinear form corresponding to Nitsche's method for the discretization of problem (28) is assembled adding the following penalty terms

$$\sum_{i=1,2}\Big(\sum_{E\in\mathscr{G}_{h,i}} \xi_\Gamma(\varepsilon,\beta)\int_E [\![u_h]\!][\![v_h]\!] + \sum_{E\in\mathscr{B}_{h,i}} \xi_{i,\partial\Omega}(\varepsilon,\beta)\int_E u_h v_h\Big),$$

to the equation arising from (29) after weak enforcement of flux continuity. Exploiting the same finite element approximation defined for Poisson's problem, see Section (1.3), we aim to find discrete functions $[u_{h,1},u_{h,2}] \in V_h := V_{h,1}\times V_{h,2}$, where $V_{h,i}$ are Lagrangian finite element spaces on $\mathscr{T}_{h,i}$ relative to each subregion Ω_i, such that

$$a_h(u_h,v_h) = F_h(v_h),\ \forall v_h \in V_h,$$

with

$$\begin{aligned}
a_h(u_h,v_h) := &\sum_{i=1,2}\big(\varepsilon_i\nabla u_{h,i} - \beta u_{h,i}, \nabla v_{h,i}\big)_{\Omega_i} \\
&+ \sum_{i=1,2}\Big[\sum_{E\in\mathscr{G}_{h,i}} \xi_\Gamma(\varepsilon,\beta)\,([\![u_h]\!],[\![v_h]\!])_E + \sum_{E\in\mathscr{B}_{h,i}} \xi_{i,\partial\Omega}(\varepsilon,\beta)\,\big(u_{h,i},v_{h,i}\big)_E\Big] \\
&- (\{\varepsilon\nabla u_h\cdot\mathbf{n}\}_w,[\![v_h]\!])_\Gamma - (\{\varepsilon\nabla v_h\cdot\mathbf{n}\}_w,[\![u_h]\!])_\Gamma + (\{\beta\cdot\mathbf{n}u_h\}_w,[\![v_h]\!])_\Gamma \\
&- \big(\varepsilon_i\nabla u_{h,i}\cdot\mathbf{n}_i, v_{h,i}\big)_{\partial\Omega_i\setminus\Gamma} - \big(\varepsilon_i\nabla v_{h,i}\cdot\mathbf{n}_i, u_{h,i}\big)_{\partial\Omega_i\setminus\Gamma} + \big(\beta\cdot\mathbf{n}_i u_{h,i}, v_{h,i}\big)_{\partial\Omega_i\setminus\Gamma},
\end{aligned}$$

$$F_h(v_h) := F(v_h) = \int_\Omega f v_h,\ \text{if } u=0 \text{ on } \partial\Omega.$$

Three remarks are in order. First, we restrict ourselves to homogeneous Dirichlet boundary conditions, but the corresponding schemes for non-homogeneous Dirichlet or Neumann conditions can be obtained similarly. Secondly, we have applied the symmetrization technique already addressed for Poisson problem. For non symmetric problems such as advection / diffusion equations, also skew symmetrization turns out to be an interesting option. We will not dwell here on a detailed comparison of the two possibilities, but for a detailed discussion on the benefits of the non symmetric option we refer the interested reader to [38, 44]. Finally, we remind that the bilinear form $a_h(\cdot,\cdot)$ is not yet completely determined, because the scaling functions $\varphi_\Gamma(\lambda)$, $\chi_\Gamma(\lambda)$ and the averaging weights w_i still require a precise definition. Since there are infinitely many expressions that satisfy the aforementioned consistency requirements, we propose some criteria that allow to identify an admissible and effective choice for such parameters.

According to the usual practice for advection / diffusion equations, we split the bilinear form into its diffusive and advective components, denoted with $a_h^{\varepsilon}(\cdot,\cdot)$ and $a_h^{\beta}(\cdot,\cdot)$ respectively,

$$
\begin{aligned}
a_h^{\varepsilon}(u_h,v_h) := \sum_{i=1,2} & \big(\varepsilon_i \nabla u_{h,i}, \nabla v_{h,i}\big)_{\Omega_i} \\
& + \sum_{i=1,2} \sum_{E \in \mathscr{G}_{h,i}} \left[\tfrac{1}{2}|\beta\cdot\mathbf{n}| + \{\varepsilon\}\big(1-\chi_\Gamma(\lambda|_\Gamma)\gamma h_E^{-1}\big)\right] ([\![u_h]\!],[\![v_h]\!])_E \\
& + \sum_{i=1,2} \sum_{E \in \mathscr{B}_{h,i}} \big(\tfrac{1}{2}|\beta\cdot\mathbf{n}_i| + \varepsilon\gamma h_E^{-1}\big)\big(u_{h,i},v_{h,i}\big)_E \\
& - (\{\varepsilon\nabla u_h\cdot\mathbf{n}\}_w,[\![v_h]\!])_\Gamma - (\{\varepsilon\nabla v_h\cdot\mathbf{n}\}_w,[\![u_h]\!])_\Gamma \\
& - \big(\varepsilon_i\nabla u_{h,i}\cdot\mathbf{n}_i, v_{h,i}\big)_{\partial\Omega_i\setminus\Gamma} - \big(\varepsilon_i\nabla v_{h,i}\cdot\mathbf{n}_i, u_{h,i}\big)_{\partial\Omega_i\setminus\Gamma},
\end{aligned}
$$

$$
\begin{aligned}
a_h^{\beta}(u_h,v_h) := \sum_{i=1,2} & \left[-\big(\beta u_{h,i},\nabla v_{h,i}\big)_{\Omega_i} + \tfrac{1}{2}\big(\beta\cdot\mathbf{n}_i u_{h,i}, v_{h,i}\big)_{\partial\Omega_i\setminus\Gamma}\right] \\
& + \big(\{\beta\cdot\mathbf{n}u_h\}_w - \tfrac{1}{2}\beta\cdot\mathbf{n}\varphi_\Gamma(\lambda|_\Gamma),[\![v_h]\!]\big)_\Gamma.
\end{aligned}
$$

The aforementioned assumption that $\chi_\Gamma(\lambda)$ is a symmetric function together with the choice of exploiting the symmetric Nitsche formulation, makes sure that the diffusion bilinear form $a_h^{\varepsilon}(\cdot,\cdot)$ respects the symmetry of the underlying operator. Correspondingly, we want to make sure that $a_h^{\beta}(\cdot,\cdot)$ is skew-symmetric, i.e. $a_h^{\beta}(u_h,v_h) = -a_h^{\beta}(v_h,u_h)$. Since the satisfaction of such property depends on $\varphi_\Gamma(\lambda)$, this is our criterion to determine the expression of this function. Exploiting integration by parts, we observe that $a_h^{\beta}(u_h,v_h)$ becomes skew-symmetric provided that the following equality holds true for any test function v_h,

$$
\{v_h\}^w + \tfrac{1}{2}\varphi_\Gamma(\lambda)[\![v_h]\!] = \{v_h\}_w - \tfrac{1}{2}\varphi_\Gamma(\lambda)[\![v_h]\!],
$$

which is equivalent to define

$$
\varphi_\Gamma(\lambda) := \big(w_i - w_j\big)
$$

in the particular case when the reference normal vector on the interface Γ, namely $\mathbf{n}$, points from Ω_i to Ω_j. Moreover, the following identity holds true,

$$
\beta\cdot\mathbf{n}\{v_h\}_w - \tfrac{1}{2}\beta\cdot\mathbf{n}\big(w_i - w_j\big)[\![v_h]\!] = \beta\cdot\mathbf{n}\{v_h\},
$$

and we notice that the advective bilinear form becomes,

$$
a_h^{\beta}(u_h,v_h) = \sum_{i=1,2}\left[-\big(\beta u_{h,i},\nabla v_{h,i}\big)_{\Omega_i} + \tfrac{1}{2}\big(\beta\cdot\mathbf{n}_i u_{h,i}, v_{h,i}\big)_{\partial\Omega_i\setminus\Gamma}\right] + (\{\beta\cdot\mathbf{n}u_h\},[\![v_h]\!])_\Gamma,
$$

which, together with the penalty term proportional to $\frac{1}{2}|\beta \cdot \mathbf{n}_i|$, corresponds to the treatment of advective fluxes through the interface by means of a standard upwind method.

For the identification of a suitable function $\chi_\Gamma(\lambda)$ and of the weights w_i in terms of ε and β, we formulate some technical requirements that will facilitate the proof of coercivity of $a_h^\varepsilon(\cdot,\cdot)$ in the forthcoming analysis of the scheme. First, we select $\chi_\Gamma(\lambda)$ such that,

$$\{\varepsilon\}\big(1-\chi_\Gamma(\lambda)\big) = \{\varepsilon\}_w.$$

Noticing that for any regular function v it holds $\{v\}_w = \{v\} - (w_j - w_i)[\![v]\!]$ and reminding of the definition of the heterogeneity factor $\lambda = [\![\varepsilon]\!]/(2\{\varepsilon\})$, we conclude that the aforementioned requirement for $\chi_\Gamma(\lambda)$ corresponds to set,

$$\chi_\Gamma(\lambda|_\Gamma) = 1 - \frac{\{\varepsilon\}_w}{\{\varepsilon\}} = (w_j - w_i)\lambda.$$

Finally, the weights w_i are conveniently selected in order to satisfy the following equality for any test function v,

$$\{\varepsilon v\}_w = \{\varepsilon\}_w \{v\},$$

that implies that $2\{\varepsilon\}_w = \varepsilon_i w_i = \varepsilon_j w_j$ being equivalent to set

$$w_i = \frac{\varepsilon_j}{\varepsilon_i + \varepsilon_j}, \quad w_j = \frac{\varepsilon_i}{\varepsilon_i + \varepsilon_j} \quad \text{and} \quad \{\varepsilon\}_w = \frac{2\varepsilon_i \varepsilon_j}{\varepsilon_i + \varepsilon_j}.$$

We observe that the aforementioned requirement $w_1 = 1,\ w_2 = 0$ when $\varepsilon_1 \to 0$ is satisfied and that the term $\{\varepsilon \nabla u \cdot \mathbf{n}\}_w [\![v]\!] = \{\varepsilon\}_w \{\varepsilon \nabla u \cdot \mathbf{n}\}[\![v]\!]$ vanishes together with the diffusivity parameter.

With these particular choices of scaling functions and weights, the stability of the discrete scheme, i.e. the consistency of its bilinear form, is readily proved. Indeed, it is sufficient to consider the diffusive (symmetric) part $a_h^\varepsilon(\cdot,\cdot)$, because we have shown that $a_h^\beta(\cdot,\cdot)$ is skew symmetric and it does not contribute to the energy of the system. First of all, we straightforwardly verify that,

$$\begin{aligned}
a_h^\varepsilon(v_h, v_h) = &\sum_{i=1,2} \|\varepsilon^{\frac{1}{2}} \nabla v_h\|_{0,\Omega_i}^2 \\
&+ \sum_{i=1,2} \sum_{E \in \mathscr{G}_{h,i}} \|\big(\tfrac{1}{2}|\beta \cdot \mathbf{n}| + \gamma \{\varepsilon\}_w h_E^{-1}\big)^{\frac{1}{2}} [\![v_h]\!]\|_{0,E}^2 \\
&+ \sum_{i=1,2} \sum_{E \in \mathscr{B}_{h,i}} \|\big(\tfrac{1}{2}|\beta \cdot \mathbf{n}| + \varepsilon \gamma h_E^{-1}\big)^{\frac{1}{2}} v_h\|_{0,E}^2 \\
&- 2\left(\{\varepsilon \nabla v_h\}_w \cdot \mathbf{n}, [\![v_h]\!]\right)_\Gamma - 2\left(\varepsilon \nabla v_h \cdot \mathbf{n}, v_h\right)_{\partial\Omega},
\end{aligned}$$

where the first three terms on the right hand side represent the energy norm that is applied for the stability and convergence analysis of the scheme. For the remaining terms of $a_h^\varepsilon(\cdot,\cdot)$, we exploit that $\{\varepsilon\}_w = 2w_i\varepsilon_i \leq 2\varepsilon_i$ to obtain the following upper bound,

$$
\begin{aligned}
&2\left(\{\varepsilon\nabla v_h\}_w\cdot\mathbf{n},[\![v_h]\!]\right)_\Gamma + 2\left(\varepsilon\nabla v_h\cdot\mathbf{n},v_h\right)_{\partial\Omega}\\
&=\sum_{i=1,2} 2\left(\varepsilon_i w_i\nabla v_{h,i}\cdot\mathbf{n},[\![v_h]\!]\right)_\Gamma + 2\left(\varepsilon\nabla v_h\cdot\mathbf{n},v_h\right)_{\partial\Omega}\\
&\leq \sum_{i=1,2}\sum_{E\in\mathscr{G}_{h,i}}\left[\delta h_E\|(\varepsilon_i)^{\frac12}\nabla v_{h,i}\cdot\mathbf{n}\|_{0,E}^2 + \frac{1}{\delta h_E}\|\{\varepsilon\}_w^{\frac12}[\![v_h]\!]\|_{0,E}^2\right]\\
&+\sum_{i=1,2}\sum_{E\in\mathscr{B}_{h,i}}\left[\delta h_E\|\varepsilon_i^{\frac12}\nabla v_{h,i}\cdot\mathbf{n}\|_{0,E}^2 + \frac{1}{\delta h_E}\|\varepsilon^{\frac12}v_h\|_{0,E}^2\right]\\
&\lesssim \sum_{i=1,2}\delta\|\varepsilon^{\frac12}\nabla v_h\|_{0,\Omega_i}^2 + \frac{1}{\delta}\|\{\varepsilon\}_w^{\frac12}[\![v_h]\!]\|_{\frac12,h,\Gamma}^2 + \frac{1}{\delta}\|\varepsilon^{\frac12}v_h\|_{\frac12,h,\partial\Omega}^2.
\end{aligned}
$$

Then, $a_h^\varepsilon(\cdot,\cdot)$ turns out to be coercive for a sufficiently small δ and large γ.

In [23] the scheme has been extended to Problem (25) with an anisotropic symmetric positive definite diffusion tensor $K:\Omega\to\mathbb{R}^{d\times d}$ replacing the scalar diffusivity ε, under the practical assumption that K is a constant on each subregion denoted with K_i. With the aforementioned choice of the scaling function $\chi_\Gamma(\lambda|_\Gamma) = (w_j - w_i)\lambda$, the diffusive part of the bilinear form becomes

$$
\begin{aligned}
a_h^K(u_h,v_h) :=& \sum_{i=1,2}\left(K_i\nabla u_{h,i},\nabla v_{h,i}\right)_{\Omega_i}\\
&+\sum_{i=1,2}\sum_{E\in\mathscr{G}_{h,i}}\left[\tfrac12|\beta\cdot\mathbf{n}| + \gamma\{\kappa\}_w h_E^{-1})\right]\left([\![u_h]\!],[\![v_h]\!]\right)_E\\
&+\sum_{i=1,2}\sum_{E\in\mathscr{B}_{h,i}}\left(\tfrac12|\beta\cdot\mathbf{n}_i| + \gamma\kappa_i h_E^{-1}\right)\left(u_{h,i},v_{h,i}\right)_E\\
&-\left(\{\mathbf{n}^TK\nabla u_h\cdot\mathbf{n}\}_w,[\![v_h]\!]\right)_\Gamma - \left(\{\mathbf{n}^TK\nabla v_h\cdot\mathbf{n}\}_w,[\![u_h]\!]\right)_\Gamma\\
&-\left(\mathbf{n}_i^TK_i\nabla u_{h,i}\cdot\mathbf{n}_i,v_{h,i}\right)_{\partial\Omega_i\setminus\Gamma} - \left(\mathbf{n}_i^TK_i\nabla v_{h,i}\cdot\mathbf{n}_i,u_{h,i}\right)_{\partial\Omega_i\setminus\Gamma},
\end{aligned}
$$

where the averaging weights are selected as follows

$$
\kappa_i := \mathbf{n}^TK_i\mathbf{n},\quad w_i = \frac{\kappa_j}{\kappa_i+\kappa_j},\quad w_j = \frac{\kappa_i}{\kappa_i+\kappa_j},\quad \text{and}\quad \{\kappa\}_w = \frac{2\kappa_i\kappa_j}{\kappa_i+\kappa_j}.
$$

2.1.2 Stabilized Galerkin Methods for Singularly Perturbed Equations

The aforementioned Nitsche technique allows to robustly enforce interface conditions among second order elliptic problems with discontinuous diffusion coefficients, but such technique does not cure the intrinsic instability of any standard Galerkin approximation applied to singularly perturbed equations. For this reason, the previously developed scheme should be complemented with a stabilisation technique acting on each subregion Ω_i where the local Péclét number is large.

It is not our aim to review here the wide area of numerical schemes devoted to stabilisation of Galerkin method for transport dominated problems. We will simply present two options that suitably fit the present discretisation framework and are also related to Nitsche's idea.

Following [17], the first stabilisation strategy that we consider is suited for locally H^1-conforming approximations. More precisely, we use standard Lagrangian finite elements on each subdomain and obtain stability for high Péclét numbers by adding a penalty term on the gradient jumps over element faces. Combined with the previously presented Nitsche interface conditions, it will result in a robust continuous / discontinuous approximation of large contrast problems, where the discontinuous approximation functions are localized only along the discontinuities of problem coefficients. Denoting by $\mathscr{E}_{h,i}$ the collection of interior edges belonging to elements of $\mathscr{T}_{h,i}$, the stabilisation effect is then obtained by complementing the bilinear form $a_h(\cdot,\cdot)$ with the following additional terms on each Ω_i,

$$J_i(u_h,v_h) := \sum_{E\in\mathscr{E}_{h,i}} \left(\gamma_{cip} h_E^2 \|\beta\cdot n\|_{L^\infty(E)} [\![\nabla u_h\cdot \mathbf{n}_E]\!], [\![\nabla v_h\cdot \mathbf{n}_E]\!]\right)_E ,$$

proposed and thoroughly analysed in [11, 13, 14], which consist of interior penalty forms controlling the jumps in the gradient over *interior* faces of each sub-domain Ω_i. Since the finite element approximation to which this stabilisation is applied involves continuous functions, the resulting scheme has been called *continuous interior penalty* (CIP). The main idea behind the stabilisation based on the jump in the gradient between adjacent elements is to introduce a least squares control over the part of the convective derivative that is not in the finite element space. A key result is the following property of the Oswald quasi-interpolant

$$\pi_h^*:\ \{v\in L^2(\Omega): v|_K\in\mathbb{P}^k(K),\ \forall K\in\mathscr{T}_h\}\to\{v\in C^0(\Omega): v|_K\in\mathbb{P}^1(K),\ \forall K\in\mathscr{T}_h\}$$
$$\pi_h^* v(x_j) := \frac{1}{n_j}\sum_{\{K: x_j\in K\}} v_{|K}(x_j), \quad \forall v\in\{v\in L^2(\Omega): v|_K\in\mathbb{P}^k(K)\},$$

where x_j are the nodes of the local finite element meshes $\mathscr{T}_{h,i}$, and n_j is the number of elements containing x_j as a node. Let β_h be the piecewise affine Lagrange interpolant of β and let $u_h\in V_{h,i}$. Then there exists a constant $\gamma_{cip}\geq c_0>0$, depending only on the local mesh geometry, such that

$$\|h^{\frac{1}{2}}(\beta_h\cdot\nabla u_h - \pi_h^*(\beta_h\cdot\nabla u_h))\|_{0,\Omega_i}^2 \leq J_i(u_h,u_h).$$

Assuming that $\beta\in[W^{1,\infty}(\Omega)]^d$ with $\nabla\cdot\beta=0$, $\varepsilon\in L^\infty(\Omega)$ and that the exact solution of the multi-domain problem satisfies $u\in H^s(\Omega_1\cup\Omega_2)\cap H_0^1(\Omega)$ with $s\geq k+1\geq 2$ it has been shown in [17] that the following error estimate holds true,

$$|||u-u_h|||_{1,h,\Omega_1\cup\Omega_2} \lesssim \left(\|\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{2}}\mathscr{H}(0,u) + \|\beta\|_{L^\infty(\Omega)}^{\frac{1}{2}}\mathscr{H}(1,u)\right)$$

where for any $v\in H^s(\Omega_1\cup\Omega_2)\cap H_0^1(\Omega)$

$$|||v|||^2_{1,h,\Omega_1\cup\Omega_2} := \sum_{i=1,2}\Big(|\varepsilon_i^{\frac{1}{2}} v_i|^2_{1,\Omega_i} + \|\varepsilon_i^{\frac{1}{2}} v_i\|^2_{\frac{1}{2},h,\mathscr{B}_{h,i}} + \|\{\varepsilon\}_w^{\frac{1}{2}}[\![v]\!]\|^2_{\frac{1}{2},h,\mathscr{G}_{h,i}} + J_i(v,v)\Big),$$

and

$$\mathscr{H}(\alpha,u) = \left(\sum_{i=1}^{N}\sum_{K\in\mathscr{T}_{h,i}} h_K^{2k+\alpha}\|u\|^2_{k+1,K}\right)^{\frac{1}{2}}.$$

The CIP stabilisation is a suitable method when heterogeneities of the diffusion coefficient appear at a scale that is much larger than the element size. Conversely, if the bulk is so fractured that the diffusivity varies at the scale of single elements, the following approach, based on a fully discontinuous approximation space, seems to be more appropriate. The main idea consists in exploiting the robustness of the proposed Nitsche's method for the enforcement of transmission conditions, combined with the observation that fully discontinuous finite elements provide stable approximation of transport problems. This turns out to transform the previous continuous / discontinuous approximation of multi-domain (25) into a fully discontinuous approximation where each element plays the role of a domain, giving rise to an instance of the so called interior penalty discontinuous Galerkin methods, [1]. Different variants of such a method have been applied to the discretization of elliptic, possibly singularly perturbed problems, [2]. Because of the application of weighted averages, we will denote the scheme proposed here as *weighted interior penalty method* (WIPG) and we will compare it with similar formulations such as the symmetric interior penalty (SIPG) and the non symmetric interior penalty (NIPG). We refer the interested reader to [2] for a broad review of literature and to [22, 23, 47] for further details about the present approach. Consistently with the fact that this new approximation scheme is stable also for transport dominated problems, we notice that the continuous interior penalty term on the gradient jumps vanishes since there are no interior faces in the element-based subdomains. To set up such discontinuous Galerkin scheme we reformulate problem (28) at the level of single elements $K\in\mathscr{T}_h$,

$$\begin{cases} -\varepsilon\Delta u + \beta\cdot\nabla u = f & \text{in } K,\\ [\![-\varepsilon\nabla u + \beta u]\!]_{\partial K}\cdot n_{\partial K} = 0 & \text{on } \partial K\setminus\partial\Omega,\\ \gamma_{h,E}(\varepsilon,\beta)[\![u]\!]_{\partial K} = 0 & \text{on } \partial K\setminus\partial\Omega,\\ \gamma_{h,\partial\Omega}(\varepsilon,\beta)u = 0 & \text{on } \partial K\cap\partial\Omega, \end{cases}$$

and proceeding as for Nitsche's method we look for $u_h\in V_h := \{v_h\in L^2(\Omega):\ v_h|_K\in \mathbb{P}^k,\ \forall K\in\mathscr{T}_h\}$ such that

$$
\begin{aligned}
a_h^{(DG)}(u_h, v_h) := \sum_{K \in \mathscr{T}_h} & \big(\big(\varepsilon \nabla u_h - \beta u_h\big), \nabla v_h\big)_K \\
+ \sum_{E \in \mathscr{E}_h} & \Big[(\{\beta u_h\}_w \cdot \mathbf{n}_E, [\![v_h]\!])_E \\
& - (\{\varepsilon \nabla u_h\}_w \cdot \mathbf{n}_E, [\![v_h]\!])_E - (\{\varepsilon \nabla v_h\}_w \cdot \mathbf{n}_E, [\![u_h]\!])_E \\
& + \big(\tfrac{1}{2}|\beta \cdot \mathbf{n}_E| - \tfrac{1}{2}\beta \cdot \mathbf{n}_E (w_E^- - w_E^+) + \gamma \{\varepsilon\}_w h_E^{-1}\big) ([\![u_h]\!], [\![v_h]\!])_E \Big] \\
+ \sum_{E \in \mathscr{B}_h} & \Big[\big(\tfrac{1}{2}\beta \cdot \mathbf{n}_E u_h, v_h\big)_E - (\varepsilon \nabla u_h \cdot \mathbf{n}_E, v_h)_E - (\varepsilon \nabla v_h \cdot \mathbf{n}_E, u_h)_E \\
& + \big(\tfrac{1}{2}|\beta \cdot n| + \varepsilon \gamma h_E^{-1}\big) (u_h, v_h)_E \Big] = F(v_h),
\end{aligned}
$$

where $\mathscr{E}_h$ is the collection of interior edges, $\mathbf{n}_E$ denotes the reference unit normal vector to each inter-element interface and w_E^-, w_E^+ represent the weights relative to the inner element $(-)$ and outer element $(+)$ neighbouring the edge E, with respect to the reference direction $\mathbf{n}_E$, as depicted in Figure 2.

For the numerical validation of the robustness of weighted Nitsche's transmission conditions in presence of locally singularly perturbed problems, we will apply the element-wise version.

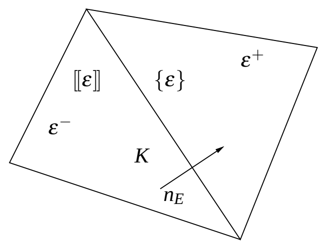

Fig. 2 A sketch of the element setting for the extension of Nitsche's method to a discontinuous Galerkin scheme.

2.1.3 Numerical Results and Discussion

To conclude this section, we will compare the efficiency of the proposed Nitsche technique for singularly perturbed problems (WIPG) with the symmetric interior penalty method (SIPG) and the non symmetric version (NIPG). Such methods are obtained from WIPG by setting $w_E^{\pm} = \frac{1}{2}$, not depending on the diffusivity parameter. The latter NIPG variant has the advantage that it only requires the condition $\gamma > 0$ to ensure stability. Consequently, we will set $\gamma = 2\ 10^{-2}$ for NIPG while $\xi = 2$ for SIPG and WIPG, to study how this parameter influences the accuracy when ε is vanishing.

To set up a test problem, featuring discontinuous coefficients, that allows us to analytically compute the exact solution we consider a domain $\Omega \subset \mathbb{R}^2$ corresponding to the rectangle $\hat{\Omega} = (0,\pi/2) \times (1-\pi/4,1)$ in polar coordinates (θ,r). We split $\hat{\Omega}$ into two subregions, $\hat{\Omega}_1 = (0,\pi/4) \times (1-\pi/4,1)$, $\Omega_2 = (\pi/4,\pi/2) \times (1-\pi/4,1)$. Then the domain Ω is split into Ω_1 and Ω_2, see Figure 3, owing to the mapping from polar to Cartesian coordinates. The viscosity $\varepsilon(x,y)$ is a discontinuous function across the interface between Ω_1 and Ω_2, namely the segment $x-y=0$ with $x \in ((1-\pi/4)\cos\pi/4, \cos\pi/4)$. Precisely, we will consider a constant $\varepsilon(x,y)$ in each subregion with several values of ε_1 in Ω_1 and a fixed $\varepsilon_2 = 1.0$ in Ω_2. Moreover, we set $\beta = [\beta_x = -y(x^2+y^2)^{-1}, \beta_y = x(x^2+y^2)^{-1}]$, $f=0$ and the boundary conditions $u(x,y=0)=1$, $u(x=0,y)=0$ and $\nabla u \cdot n = 0$ otherwise. Then, the exact solution of the problem on each subregion $\hat{\Omega}_1, \hat{\Omega}_2$ can be expressed in polar coordinates as an exponential function with respect to θ independently from r. The global solution $u(\theta,r)$ is provided by choosing the value at the interface $\theta = \pi/4$ in order to ensure the following matching conditions,

$$\lim_{\theta \to \frac{\pi}{4}^-} u(\theta,r) = \lim_{\theta \to \frac{\pi}{4}^+} u(\theta,r),$$
$$\lim_{\theta \to \frac{\pi}{4}^-} -\varepsilon(\theta,r)\partial_\theta u(\theta,r) = \lim_{\theta \to \frac{\pi}{4}^+} -\varepsilon(\theta,r)\partial_\theta u(\theta,r).$$

In the Cartesian coordinate system (x,y), this is a genuinely 2-dimensional test case, because the gradient of the solution is not constant along the interface where ε is discontinuous, and it decreases from the inner to the outer side of the domain Ω. Furthermore, it is easy to see that when $0 \simeq \varepsilon_1 \ll \varepsilon_2 = 1$ the global solution, u, features a sharp internal layer upwind to the discontinuity of ε.

The results, depicted in Figure 4 and also quantified in Table 1, give evidence that the WIPG scheme performs better than standard interior penalty methods, particularly in those cases where the solution is non smooth and at the same time the computational mesh with $h = 0.0654$ is not completely adequate to capture the singularities. From the analysis of Figure 4, it is possible to identify three regimens where the numerical methods behave differently. The first one consists of the diffusive region, where all methods provide similar results. For the intermediate value of ε a transition takes place, because the computational mesh is not adequate anymore to capture the sharp internal layer that originates upwind to the discontinuity of ε. Initially, the error relative to each method increases when ε is reduced, but this trend is inverted for the WIPG method solely, after the threshold $\varepsilon = 10^{-6}$, while the error monotonically increases for SIPG and NIPG. Finally, the smallest value of ε_1 corresponds to the hyperbolic regimen. In the limit case $\varepsilon_1 \to 0$, the discontinuities of the global solution u are aligned with those of ε. However, we observe that the standard interior penalty schemes (SIPG or NIPG equivalently) provide solutions that are almost continuous, as reported in Figure 4. This behaviour promotes the instability of the approximate solution in the neighborhood of the boundary layer, because the computational mesh is not adequate to smoothly approximate the high gradients across the interface. Conversely, the WIPG method is more effective, thanks to the

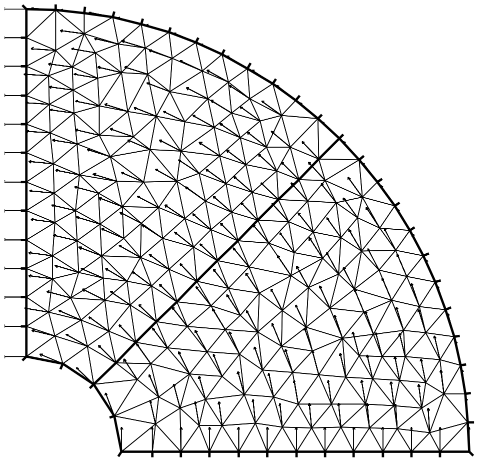

Fig. 3 The domain Ω and the subregions $\Omega_1,\ \Omega_2$ together with the computational mesh T_h and the advective field β.

consistency with the elliptic/hyperbolic limit case, because it replaces the part of the boundary layer with a jump that cannot be captured by the computational mesh.

$\|u-u_h\|_{L^2}$	i	$\varepsilon_1 = 2^{-i}$	SIPG	NIPG	WIPG
diffusive region	0	1	0.00101853	0.00123078	0.00101853
	-1	0.5	0.00123921	0.00134626	0.00121052
	-2	0.25	0.00200825	0.00167944	0.00182993
	-3	0.125	0.00393471	0.00333855	0.00315595
transition region	-4	0.0625	0.0079422	0.00703319	0.00532886
	-5	0.03125	0.0144257	0.0130603	0.00780319
	-6	0.015625	0.0224454	0.0207315	0.00908097
	-7	0.0078125	0.0307374	0.0289709	0.00831401
	-8	0.00390625	0.0380299	0.0363924	0.00655286
hyperbolic region	-9	0.00195312	0.0429129	0.0414616	0.0049148
	-10	0.000976562	0.0452834	0.0440218	0.00329726
	-11	0.000488281	0.0463316	0.0452286	0.00204598
	-12	0.000244141	0.0468732	0.0458791	0.00143603
	-13	0.00012207	0.0471628	0.0462332	0.00122399

Table 1 The L^2 norm error for WIPG, SIPG, NIPG for different values of $\varepsilon_1 = 2^{-i}$ and a fixed value of $\varepsilon_2 = 1$ in the test problem depicted in Figure 3.

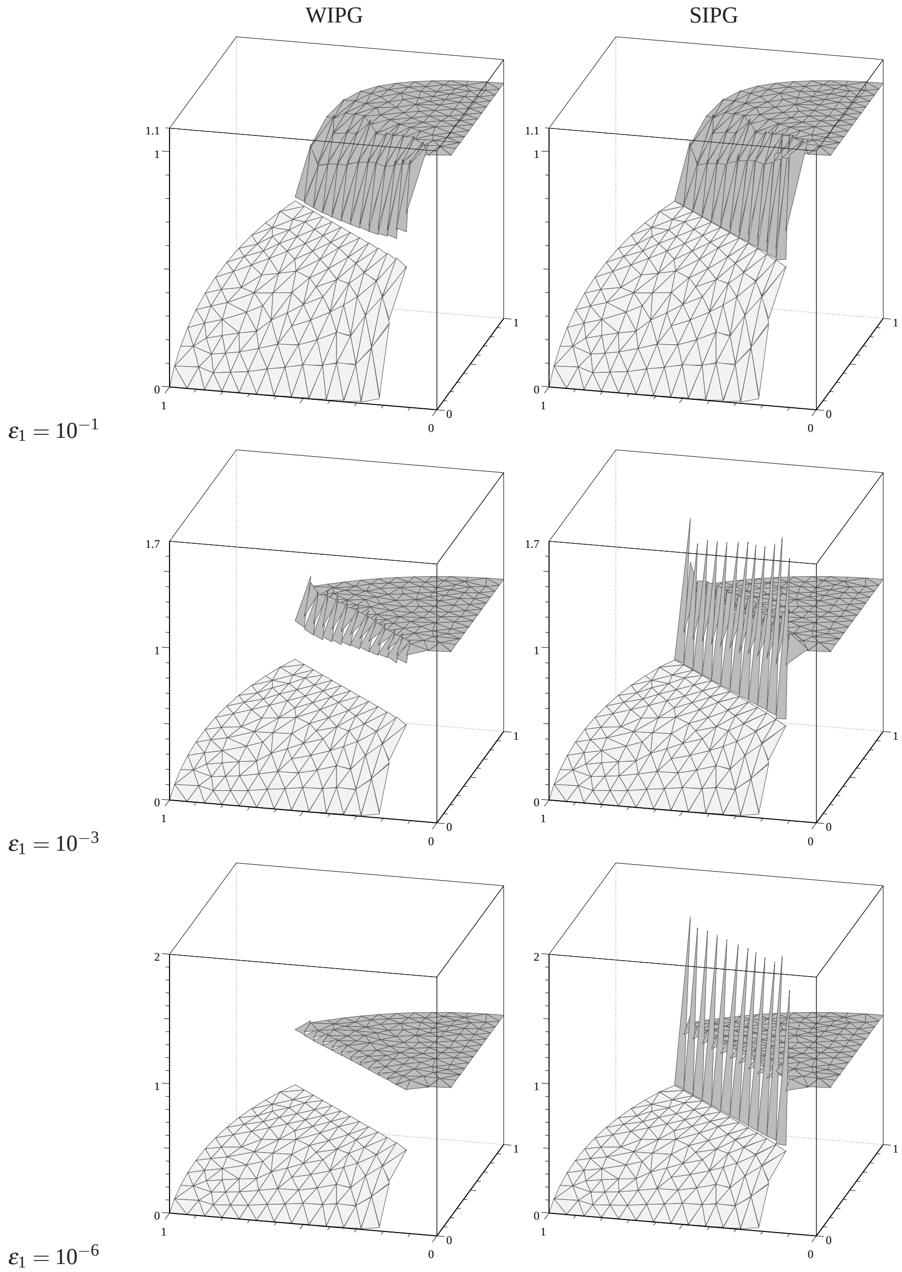

Fig. 4 A comparison of Nitsche's method with (WIPG) and without (SIPG) the application weighing technique.

3 Stabilized Nitsche's Method for Unfitted Boundaries and Interfaces

Fictitious domain methods turn out to be particularly effective for the approximation of boundary value problems on domains of complex shape and for free interface problems. The parametric description of the boundary with the subsequent mesh generation and the application of interface tracking techniques represent difficulties for the application of finite element methods. The idea of fictitious domain schemes consists in embedding the physical domain into a larger domain with reasonably simple shape. However, as discussed in [36], to preserve the accuracy of the selected finite element method, it is necessary to restrict the integration of the discrete variational formulation to the physical domain.

To illustrate the limitations of standard finite element approximations of unfitted interface problems, let us split the interval $\Omega := (0,1)$ in two parts $\Omega_1 := (0,\Gamma),\ \Omega_2 := (\Gamma,1)$ and look for $u(x)$ such that,

$$\begin{cases} -\varepsilon_i u_i'' = 1 & \text{in } \Omega_i, \\ u_1 = u_2 & \text{on } \Gamma, \\ \varepsilon_1 u_1' = \varepsilon_2 u_2' & \text{on } \Gamma, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us approximate u with piecewise linear finite elements on a uniform partition of width h. For any positive $\varepsilon_1 \neq \varepsilon_2$ we have $u \notin H^2(\Omega)$. Then optimal convergence cannot be expected. In particular, as confirmed by numerical results reported in Table 2, sub-optimal convergence is verified if we select Γ such that it never coincides with a vertex of the partitions underlying the finite element space.

h	$\varepsilon_1 = \varepsilon_2 = 1$	$\varepsilon_1 = 1,\ \varepsilon_2 = 10^{-2}$
5.00×10^{-2}	2.28×10^{-4}	4.60×10^{-2}
2.50×10^{-2}	5.70×10^{-5}	3.49×10^{-2}
1.25×10^{-2}	1.42×10^{-5}	3.23×10^{-2}
6.25×10^{-3}	3.56×10^{-6}	3.15×10^{-2}
3.12×10^{-3}	8.91×10^{-7}	2.92×10^{-2}
p	1.99	0.15

Table 2 Convergence rate of the error $\|u - u_h\|_{0,\Omega}$ of linear finite elements for an unfitted interface problem. The physical domain $\Omega = [0,1]$ is divided in two subdomains $\Omega_1 = [0, \frac{1}{\sqrt{5}}]$ and $\Omega_2 = [\frac{1}{\sqrt{5}}, 1]$. The exact solution is of the form $u_i(x) = -\frac{x^2}{2\varepsilon_i} + b_i x + c_i$. The coefficient b_i and c_i are chosen such that the functions u_i satisfy the boundary conditions and the continuity conditions at interface.

Since the boundary and the interface do not necessarily conform with the mesh, an optimally convergent finite element method must be defined on sub-elements. In the case of interface problems, this additional difficulty can be taken into account by enriching the approximation space with additional basis functions that lie on a portion of the mesh elements. Such a technique is often called the extended finite element method (XFEM) and has been successfully applied to different applications such as crack propagation problems [21] and free interface problems in fluid dynamics [43, 27].

The approximation of elliptic problems with unfitted boundary or interface has already been investigated in recent works, we mention for instance [25, 37, 20, 32, 18]. The discretisation schemes that we consider are closely related to [28, 30], where an extended finite element method has been combined with a Nitsche technique to enforce the matching conditions between contiguous sub-regions. However, the application of Nitsche's method for the treatment of boundary or interface conditions may give rise to numerical instabilities in presence of small element cuts. More precisely, it has been observed in [12, 15, 16, 43] that the stability and the condition number of the finite element scheme depend on how the interface cuts the computational mesh. To cure them, the application of interior penalty stabilisation techniques has been successfully considered in a sequel of papers [12, 15, 16]. The idea of such stabilisation methods is to introduce in the discrete formulation a minimum of artificial diffusion to ensure the positivity of the discrete bilinear form for any configuration of the boundary or interface.

For interface problems, the need to introduce additional finite element basis functions lying on sub-elements to restore optimal convergence represents a second source of instability. Following the approach proposed in [43], we study the H^1 stability of the extended finite element space in the case of piecewise linear approximation. We analyse the condition number of the corresponding mass and stiffness matrices in presence of small sub-elements and we conclude that their spectrum is affected by how elements are cut.

Finally, we will apply Nitsche's method to enforce transmission conditions in the extended finite element space for interface problems governed by symmetric elliptic equations with large contrast between diffusion coefficients. We aim to develop a scheme that is robust with respect to the configuration of sub-elements as well as the heterogeneity of the diffusion coefficients.

3.1 The Unfitted Nitsche Method for Boundary Conditions

We recall and analyse the Nitsche's method for the approximation of boundary conditions on a computational mesh that does not fit the physical domain. Let $\mathscr{T}_h^0$ be a given admissible computational mesh whose elements entirely cover the physical domain Ω. We also assume that all elements of $\mathscr{T}_h^0$ have non-empty intersection with Ω. Let $\Omega_{\mathscr{T}}$ be the domain covered by $\mathscr{T}_h^0$. To improve this possibly coarse approximation, we will also consider a family of shape regular, quasi-uniform triangulations, $\mathscr{T}_h$, built by recursive refinement of $\mathscr{T}_h^0$, omitting any elements whose intersection with Ω is empty. As previously mentioned in section 1.4, to keep the analysis of the schemes as simple as possible, we consider linear Lagrangian finite elements

$$V_h := \{v_h \in C^0(\Omega_{\mathscr{T}}) : \; v_h|_K \in \mathbb{P}^1(K) \; \forall K \in \mathscr{T}_h\}.$$

Referring to Poisson problem (1) with homogeneous boundary conditions i.e. $g = 0$, Nitsche's method requires to find $u_h \in V_h$ such that $a_h(u_h, v_h) = F_h(v_h)$ for any $v_h \in V_h$ with

$$\begin{aligned} a_h(u_h, v_h) &:= a(u_h, v_h) - (\partial_n u_h, v_h)_{\partial\Omega} - s\,(\partial_n v_h, u_h)_{\partial\Omega} + \gamma h^{-1}\,(u_h, v_h)_{\partial\Omega}\,, \\ F_h(v_h) &:= F(v_h). \end{aligned} \tag{30}$$

where $s = \pm 1$ gives rise to the symmetric or skew-symmetric formulations.

Although formally equivalent to the case of fitted boundary, the treatment of the unfitted case hides some additional difficulties for the set up of the discrete problem.

Firstly, for the assembly of mass and stiffness matrices, integrals over cut elements must be computed, such as

$$\int_{K\cap\Omega} u_h \cdot v_h, \quad \int_{K\cap\Omega} \nabla u_h \cdot \nabla v_h,$$

where $K \cap \Omega$ is a portion of a triangle or a tetrahedron. In two or three space dimensions, $K \cap \Omega$ may not be a simplex. For these reasons, the computation of these integrals requires particular attention, and the fact that $|K \cap \Omega|$ may vanish affects the condition number of mass and stiffness matrices, as it will be discussed in the forthcoming sections.

Secondly, the assembly of boundary terms involves integrals over manifolds that do not coincide with edges or faces. To automatically perform such calculations, some approximation of the boundary configuration is necessary. For instance, the boundary can be represented by means of the level set of a discrete distance function. This means that there exists a discrete implicit surface (or hyper-surface when $d = 3$) $\varphi_h \in V_h$ that defines $\partial\Omega$ as its zero level set. Coherently with the notation adopted in the previous sections, we apply here a two-dimensional notation and we denote quantities related to element edges of faces with E.

We denote with $\mathscr{C}_h := \{K \in \mathscr{T}_h : \; |K \cap \partial\Omega|_{\mathbb{R}^{d-1}} > 0\}$ a *crust* of elements with non vanishing intersection with the boundary, measured in $\mathbb{R}^{d-1}$ topology. Assuming that each element K is an open set, with the piecewise linear description of the boundary we observe that for all $K \in \mathscr{C}_h$ the set $\partial\Omega \cap \partial K$ consists on two points (in the two-dimensional case) and the portion of $\partial\Omega$ that connects them is a straight line (or a planar surface in three dimensions). An example is illustrated in Figure 5. If $\partial\Omega$ lies on an entire edge of an element K, then such element does not belong to $\mathscr{C}_h$. We denote with $E^{\partial\Omega} := K \cap \partial\Omega$ the cut edges and with $\mathscr{B}_h^{\partial\Omega}$ their collection, see Figure 5. For a fixed regular mesh $\mathscr{T}_h$, the size of any $E^{\partial\Omega} \in \mathscr{B}_h^{\partial\Omega}$ is upper bounded by the mesh characteristic size, h, but it can become arbitrarily small. For this reason, the penalty term cannot be scaled with respect of the size of edges or faces lying on $\partial\Omega$, but it has been taken inversely proportional to the characteristic mesh size.

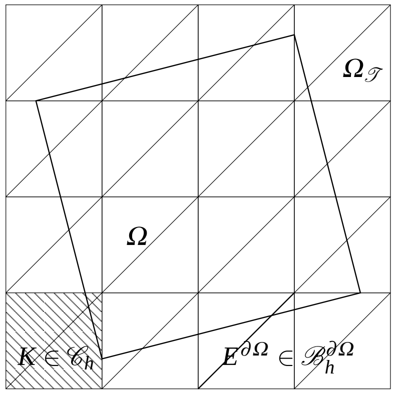

Fig. 5 A sketch of the physical domain, Ω, and the computational domain $\Omega_{\mathscr{T}}$, with the notation used to set up the fictitious domain method.

In order to analyse how the configuration of the boundary with respect to the mesh affects the stability of the scheme, we introduce the following indicator,

$$\nu' := \min_{K \in \mathscr{C}_h} \frac{|K \cap \Omega|}{|K|},$$

that corresponds to the minimum relative intersection of an element with the physical domain Ω. Since the unfitted Nitsche's method requires to evaluate integrals over cut elements or cut edges, we expect that the parameter ν' may affect the stability properties of the scheme.

Before addressing the analysis of the present unfitted Nitsche method, it is useful to recall some norms and related discrete inequalities as the basis for the forthcoming investigation. Concerning the norms, we notice that the definition of $\|\cdot\|_{\pm\frac{1}{2},h,\partial\Omega}$ should be adapted to the present scheme as follows,

$$\|v\|_{\pm\frac{1}{2},h,\partial\Omega} := h^{\mp\frac{1}{2}} \|v\|_{0,\partial\Omega},$$

while the definition of the energy norm is unchanged,

$$\|v\|_{1,h,\Omega}^2 := |v|_{1,\Omega}^2 + \|v\|_{\frac{1}{2},h,\partial\Omega}^2.$$

For the forthcoming analysis, it will be also useful to consider the augmented norm

$$|||v|||_{1,h,\Omega}^2 := \|v\|_{1,h,\Omega}^2 + \|\nabla v \cdot \mathbf{n}\|_{-\frac{1}{2},h,\partial\Omega}^2,$$

which provides additional control on boundary fluxes. Exploiting inverse inequalities we easily prove that,

$$\|v_h\|_{-\frac{1}{2},h,\partial\Omega} = \sum_{E^{\partial\Omega}\in\mathscr{B}_h^{\partial\Omega}} h\|v_h\|^2_{E^{\partial\Omega}} \lesssim \sum_{K\in\mathscr{C}_h} \|v_h\|^2_K \lesssim \|v_h\|^2_{0,\Omega_{\mathscr{T}}},$$

which is not satisfactory for our purpose, because the right hand side involves the entire computational domain and not the physical domain Ω solely. Proceeding similarly, the desired right hand side can be obtained,

$$\|v_h\|_{-\frac{1}{2},h,\partial\Omega} \lesssim \max_{K\in\mathscr{C}_h} \frac{|K|}{|K\cap\Omega|} \sum_{K\in\mathscr{C}_h} \|v_h\|^2_{0,K\cap\Omega} \lesssim (\nu')^{-1}\|v_h\|^2_{0,\Omega}. \tag{31}$$

Since, given Ω, it is possible to construct a triangulation $\mathscr{T}_h$ with an arbitrarily small ν, (31) shows that unfitted Nitsche's method is not robust with respect to the configuration of the boundary. Precisely, we say that a scheme is robust with respect to the parameter ν if the spectrum of the discrete problem admits lower and upper bounds that are independent on the parameter itself.

Our main purpose is to study how small cut elements affect the fundamental properties of the numerical scheme. We perform such analysis simultaneously for symmetric $s = 1$ and skew-symmetric schemes. To quantify the stability of the scheme, we look at the coercivity of the bilinear form and we exploit (31) to observe that

$$\begin{aligned} a_h(v_h,v_h) =& |v_h|^2_{1,\Omega} + \gamma\|v_h\|^2_{\frac{1}{2},h,\partial\Omega} - (s+1)\,(\nabla v_h\cdot\mathbf{n},v_h)_{\partial\Omega} \\ \gtrsim& \big(1-(\delta_1+\delta_2(s+1))(\nu')^{-1}\big)|v_h|^2_{1,\Omega} + \delta_1\|\nabla v_h\cdot\mathbf{n}\|^2_{-\frac{1}{2},h,\partial\Omega} \\ &+ \big(\gamma-(s+1)\delta_2^{-1}\big)\|v_h\|^2_{\frac{1}{2},h,\partial\Omega}, \end{aligned}$$

where δ_1, δ_2 are positive constants to be suitably chosen.

Three conclusions come out immediately. For the skew-symmetric case, i.e. $s = -1$, coercivity of $a_h(\cdot,\cdot)$ holds in the energy norm $\|\cdot\|_{1,h,\Omega}$ with $\delta_1 = 0$ and for any positive δ_2 and γ. As a result of that, the stability estimate of the skew-symmetric variant is robust with respect to the configuration of the interface. This is never true for the symmetric case because $s+1 = 2$. If we analyse coercivity in the norm $\|\cdot\|_{1,h,\Omega}$, we can set $\delta_1 = 0$, but to make sure that the first term on the right hand side is positive it is necessary to satisfy $\delta_2 \lesssim \nu'$. Such a restriction entails that $\gamma \gtrsim (\nu')^{-1}$, which is unsatisfactory because the penalty term depends on the interface configuration and it becomes arbitrarily large for small element cuts. Finally, neither the skew-symmetric nor the symmetric cases feature robust stability properties in the augmented norm $|||\cdot|||_{1,h,\Omega}$. Indeed, coercivity of $a_h(\cdot,\cdot)$ in this norm can be only proved under the condition $\delta_1 \lesssim \nu'$, but in this case the control on the additional term $\delta_1\|\nabla v_h\cdot\mathbf{n}\|^2_{-\frac{1}{2},h,\partial\Omega}$ is lost for small element cuts.

Concerning the boundedness of the bilinear form, the consistency term $(\partial_n u_h, v_h)_{\partial\Omega}$ must be controlled by means of the energy or the augmented norms. To this purpose, the choice of the norm makes a significant difference. Since $|||v_h|||_{1,h,\Omega}$ directly controls $\|\nabla v_h\cdot\mathbf{n}\|_{-\frac{1}{2},h,\partial\Omega}$, owing to a Cauchy-Schwarz inequality it is straightforward to conclude that

$$(\partial_n u_h, v_h)_{\partial\Omega} \leq \|\nabla u_h \cdot \mathbf{n}\|_{-\frac{1}{2},h,\partial\Omega} \, \|v_h \cdot \mathbf{n}\|_{+\frac{1}{2},h,\partial\Omega} \leq |||u_h|||_{1,h,\Omega} \, |||v_h|||_{1,h,\Omega}.$$

Conversely, if we perform our analysis in the energy norm $\|\cdot\|_{1,h,\Omega}$, resorting to inverse inequality (31) is necessary to obtain an upper bound of the consistency term,

$$\begin{aligned}(\partial_n u_h, v_h)_{\partial\Omega} &\leq \|\nabla u_h \cdot \mathbf{n}\|_{-\frac{1}{2},h,\partial\Omega} \, \|v_h\|_{+\frac{1}{2},h,\partial\Omega} \\ &\lesssim (\nu')^{-1} \|\nabla u_h\|_{0,\Omega} \, \|v_h\|_{+\frac{1}{2},h,\partial\Omega} \lesssim (\nu')^{-1} \|u_h\|_{1,h,\Omega} \, \|v_h\|_{1,h,\Omega}.\end{aligned}$$

In the latter case, however, the fact that the continuity constant is proportional to $(\nu')^{-1}$ spoils the robustness of the scheme.

In conclusion, such analysis shows that both the symmetric and skew-symmetric variants of the unfitted Nitsche's method are unsatisfactory if we aim to set up a scheme that is fully robust with respect to the configuration of the computational mesh with respect to the boundary and the possibility to produce small element cuts. For this reason, in the forthcoming section we will propose a stabilisation technique to override this limitation of Nitsche's method.

3.2 The Ghost Penalty Stabilisation Method

In order to design a fully robust fictitious domain method, stability must be obtained in a norm at least as strong as the norm $|||u_h|||_{1,h,\Omega}$. This can be achieved by modifying the bilinear form in the interface zone. The idea is to add a penalty term that improves the stability in the elements cut by the interface and distributes the coercivity to the parts of the triangulation outside the physical domain. This added term must guarantee stability but at the same time be weakly consistent to the right order. Since the nodes outside the physical domain are often referred to as ghost nodes, this term is called the ghost penalty term.

Below we will follow the approach proposed in [16] with the higher order generalisation of [12]. For the proofs of the results we refer to these references. Recalling the definitions of (30) we propose the formulation: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + g_h(u_h, v_h) = F_h(v_h), \; \forall v_h \in V_h. \tag{32}$$

where $g_h(\cdot,\cdot)$ is the ghost penalty stabilisation term. Define the set of element edges in the boundary zone by

$$\mathscr{E}_B := \{F = K \cap K', \text{ where either } K \in \mathscr{C}_h \text{ or } K' \in \mathscr{C}_h\}.$$

A possible ghost penalty term for piecewise affine approximations is then given by a penalty on the jumps of the gradients over the element edges of $\mathscr{E}_B$,

$$g_h(u_h, v_h) := \sum_{E \in \mathscr{E}_B} (\gamma_g h_E [\![\nabla u_h \cdot \mathbf{n}_E]\!], [\![\nabla v_h \cdot \mathbf{n}_E]\!])_E.$$

We also introduce the discrete $H^1(\Omega_{\mathscr{T}_h})$-norm

$$\|v_h\|^2_{1,h,\Omega_{\mathscr{T}}} := \|\nabla v_h\|^2_{0,\Omega_{\mathscr{T}}} + \|v_h\|^2_{\frac{1}{2},h,\partial\Omega} \text{ with } \|v_h\|^2_{0,\Omega_{\mathscr{T}}} := \sum_{K\in\mathscr{T}_h} \|v_h\|^2_{0,K}.$$

The enhanced stability obtained by adding $g_h(\cdot,\cdot)$, is reflected in the coercivity estimate

$$|||v_h|||^2_{1,h,\Omega} \lesssim \|v_h\|^2_{1,h,\Omega_{\mathscr{T}_h}} \lesssim a_h(v_h,v_h) + g_h(v_h,v_h), \quad \forall v_h \in V_h. \tag{33}$$

The first inequality is a consequence of the discrete trace inequality

$$\|\nabla v_h \cdot \mathbf{n}\|_{-\frac{1}{2},h,\partial\Omega} \lesssim \|\nabla v_h\|_{0,\mathscr{C}_h}$$

and the second holds thanks to the following fundamental property of the ghost penalty term

$$\|\nabla v_h\|^2_{0,\Omega_{\mathscr{T}}} \lesssim \|\nabla v_h\|^2_{0,\Omega} + g_h(v_h,v_h). \tag{34}$$

For piecewise affine Lagrangian finite element approximations, the idea in [16] to prove such an inequality is to observe that the gradient over any cut element $K \in \mathscr{C}_h$ is a piecewise constant function that is bounded from above by the gradient on another element $K' \notin \mathscr{C}_h$ plus the jumps of gradients across all elements that should be crossed to connect K with K'. Indeed, the ghost penalty stabilisation provides control on the additional terms involving jumps.

Under regularity assumptions on Ω, for all $v \in H^2(\Omega)$, we may introduce an extension operator $\mathbb{E} : H^2(\Omega) \mapsto H^2(\Omega_{\mathscr{T}})$ such that $\mathbb{E}v|_\Omega = v$ and $\|\mathbb{E}v\|_{H^2(\Omega_{\mathscr{T}})} \lesssim \|v\|_{H^2(\Omega)}$. It is then convenient to introduce an interpolation operator $i_h : H^2(\Omega) \mapsto V_h$ by $i_h v := I_h \mathbb{E}v$ where I_h is the standard nodal Lagrange interpolator. It is then straightforward to show that

$$|||v - i_h v|||_g := |||v - i_h v|||_{1,h,\Omega} + \sqrt{g_h(\mathbb{E}v - i_h v, \mathbb{E}v - i_h v)} \lesssim h\|v\|_{H^2(\Omega)}. \tag{35}$$

For the convergence analysis we need the following continuity result, that is a straightforward application of Cauchy-Schwarz inequalities and local trace inequalities. For all $v \in H^2(\Omega)$ and $w_h \in V_h$ there holds

$$\begin{aligned} &|a_h(v - i_h v, w_h) + g_h(v - i_h v, w_h)| \\ &\qquad \lesssim |||v - i_h v|||_g \Big(\sum_{K\in\mathscr{T}_h} \|\nabla w_h\|^2_K + \|w_h\|^2_{\frac{1}{2},h,\partial\Omega}\Big)^{1/2}. \end{aligned} \tag{36}$$

The optimal convergence estimate

$$|||u - u_h|||_{1,h,\Omega} \lesssim h\|u\|_{H^2(\Omega)}$$

is an immediate consequence of (33), (35) and (36). Using a duality argument one may also prove that

$$\|u - u_h\|_{0,\Omega} \lesssim h^2 \|u\|_{H^2(\Omega)}.$$

In a similar fashion exploiting the uniform upper and lower bounds of the bilinear form one may show that the condition number of the system matrix is robust with respect to the interface position. We will give some detail on this analysis in Section 3.3 in the case of multi-domain problems with large contrast.

In the case of high order approximations a penalty on the normal gradient is insufficient. Either one has to resort to a multi-penalty method or a stabilisation of local projection type, [12]. For example if we instead consider a Lagrangian finite element space where the polynomials are of degree k, the following multi-penalty operator will allow for a similar analysis in the high order case:

$$g_h(u_h,v_h) := \sum_{E\in\mathscr{E}_B}\sum_{i=1}^{k}(\gamma_g h_E^{2i-1}[\![\partial_n^i u_h]\!],[\![\partial_n^i v_h]\!])_E,$$

where $\partial_n^i u$ denotes the i-th order normal derivative of u across the edge E. Since such a quantity is combined with the jump across E, the orientation of the unit normal vector is irrelevant for the definition of $[\![\partial_n^i u_h]\!]$.

The role of this multi-penalty operator for the stabilisation of the unfitted method is better understood if we look at its connection with local projection operators. For any given element $K \in \mathscr{C}_h$, let $\mathscr{P}_K$ be the patch containing the shortest piecewise linear path connecting all the centres of mass to move from the centre of mass of K to the centre of $K' \notin \mathscr{C}_h$. Let $\mathscr{E}_K$ be the set of edges cut by such path. It is straightforward to verify that for any element $K \in \mathscr{C}_h$ and any corresponding patch $\mathscr{P}_K$ the ratio

$$\frac{|\mathscr{P}_K|}{|\mathscr{P}_K\cap\Omega|}$$

is uniformly bounded with respect to the position of the interface. Furthermore, for shape-regular and quasi-uniform meshes we expect that the number of individual elements contained in $\mathscr{P}_K$ is uniformly bounded from above. As a result of that, we have

$$\sum_{K\in\mathscr{C}_h}\|v_h\|^2_{0,\mathscr{P}_K} \simeq \sum_{K\in\mathscr{C}_h}\|v_h\|^2_{0,K},$$

where $a \simeq b$ means that there exist two constants c,C, uniformly independent of the mesh characteristic size h, such that $ca \le b \le Ca$.

Let us define by $\pi_h : L^2(\mathscr{P}_K) \to \mathbb{P}^k(\mathscr{P}_K)$ the local L^2 projection onto $\mathbb{P}^k(\mathscr{P}_K)$. The following equivalence can be proven, see [12] and references therein,

$$(v_h - \pi_h v_h, v_h)_{\mathscr{P}_K} \simeq \sum_{i=1}^{k}\sum_{E\in\mathscr{E}_K}\int_E [\![\partial_n^i v_h]\!]^2.$$

We will then use the local projection operator to prove that the following local counterpart of (34) holds true for any polynomial order $k \ge 1$,

$$\|v_h\|^2_{0,\mathscr{P}_K} \lesssim \|v_h\|^2_{0,\mathscr{P}_K\cap\Omega} + h^{-2}\int_{\mathscr{P}_K}\left(v_h - \pi_h v_h\right)^2. \tag{37}$$

To prove (37) we look at the restriction on $\mathscr{P}_K$ of any $v_h \in V_h$ and we split it as $v_h = \pi_h v_h + r_k$ where $r_k = v_h - \pi_h v_h$. We notice that either $\pi_h v_h = 0$ on the entire patch, or $\pi_h v_h \neq 0$ on any subset of $\mathscr{P}_k$ with non zero measure. As a result of that we obtain,

$$\|\nabla \pi_h v_h\|^2_{0,\mathscr{P}_K} \lesssim \frac{|\mathscr{P}_K|}{|\mathscr{P}_K \cap \Omega|} \|\nabla \pi_h v_h\|^2_{0,\mathscr{P}_K \cap \Omega}. \tag{38}$$

When the residual $r_k \neq 0$, exploiting (38), we notice that

$$\|\nabla v_h\|^2_{0,\mathscr{P}_K} \lesssim \|\nabla \pi_h v_h\|^2_{0,\mathscr{P}_K} + \|\nabla r_k\|^2_{0,\mathscr{P}_K} \lesssim \|\nabla \pi_h v_h\|^2_{0,\mathscr{P}_K \cap \Omega} + \|\nabla r_k\|^2_{0,\mathscr{P}_K}.$$

Owing to the inverse inequality we observe that,

$$\|\nabla r_k\|^2_{0,\mathscr{P}_K \cap \Omega} \lesssim \|\nabla r_k\|^2_{0,\mathscr{P}_K} \lesssim h^{-2} \|r_k\|^2_{0,\mathscr{P}_K}$$

and combining the previous estimates we conclude that,

$$\begin{aligned}
|\nabla v_h\|^2_{0,\mathscr{P}_K} &\lesssim \|\nabla \pi_h v_h\|^2_{0,\mathscr{P}_K \cap \Omega} - \|\nabla r_k\|^2_{0,\mathscr{P}_K \cap \Omega} + 2h^{-2} \|r_k\|^2_{0,\mathscr{P}_K} \\
&\lesssim \|\nabla \pi_h v_h + \nabla r_k\|^2_{0,\mathscr{P}_K \cap \Omega} + 2h^{-2} \|r_k\|^2_{0,\mathscr{P}_K} \\
&\lesssim \|\nabla v_h\|^2_{0,\mathscr{P}_K \cap \Omega} + 2h^{-2} \int_{\mathscr{P}_K} \big(v_h - \pi_h v_h\big)^2.
\end{aligned}$$

In conclusion, summing up over all elements $K \in \Omega_{\mathscr{T}}$, applying the previous estimate for any element cut by the interface $K \in \mathscr{C}_h$ and exploiting the equivalence between multi-penalty and the local projection we conclude that,

$$\|\nabla v_h\|^2_{\Omega_{\mathscr{T}}} \lesssim \|\nabla v_h\|^2_{\Omega} + \sum_{i=1}^{k} \sum_{E \in \mathscr{E}_B} \int_E [\![\partial_n^i v_h]\!]^2,$$

which generalizes (34) to high order Lagrangian finite elements.

3.3 The Unfitted Nitsche Method for Large Contrast Problems

We now place ourselves in the setting of Section 2, considering two non-overlapping subdomains, $\Omega_i,\ i = 1,2$, with interface $\Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2$. This time, however, the mesh will not be fitted to the interface. For simplicity we assume that Γ is a plane separating the two domains. The problem that we will study is (25), but this time we let $\beta = 0$. We recall the equations here for convenience

$$\begin{cases}
\nabla \cdot \big(-\varepsilon_i \nabla u_i\big) = f_i, & \text{in } \Omega_i, \\
u_i = 0, & \text{on } \partial\Omega \cap \partial\Omega_i, \\
[\![u]\!] = 0, & \text{on } \Gamma, \\
[\![-\varepsilon \nabla u \cdot \mathbf{n}]\!] = 0, & \text{on } \Gamma.
\end{cases} \tag{39}$$

We let $\mathscr{T}_{hi}$ denote a triangulation fitted to $\partial\Omega_i \setminus \Gamma$, but not to Γ. Let $\mathscr{T}_{h1}$ and $\mathscr{T}_{h2}$ match across the interface so that $\mathscr{T}_{h1} \cup \mathscr{T}_{h2}$ is a conforming triangulation of Ω. Let

$$V_{h,i} := \{v_h \in C^0(\overline{\Omega}_{\mathscr{T}_{hi}}) : v_h|_K \in \mathbb{P}_1(K),\ \text{for all } K \in \mathscr{T}_{hi}; v_h|_{\partial\Omega} = 0\},$$

with $\mathbb{P}_1(K)$ denoting the set of polynomials of degree less than or equal to 1 on K. We denote by $\mathscr{T}_h := \mathscr{T}_{h1} \cup \mathscr{T}_{h2}$ the triangulation of the physical domain and by V_h^{Ω} the corresponding piecewise affine finite element space. Here we have for simplicity included the boundary conditions in the approximation space.

We may then write the following formulation, similar to that of Section 2. Find $[u_{h,1}, u_{h,2}] \in V_h := V_{h,1} \times V_{h,2}$, such that

$$a_h(u_h, v_h) = F_h(v_h),\ \forall v_h \in V_h \tag{40}$$

where now

$$\begin{aligned} a_h(u_h, v_h) := & \sum_{i=1,2} \left(\varepsilon_i \nabla u_{h,i}, \nabla v_{h,i}\right)_{\Omega_i} + \gamma \xi(\varepsilon) h^{-1} \left([\![u_h]\!], [\![v_h]\!]\right)_\Gamma \\ & - \left(\{\varepsilon \nabla u_h \cdot \mathbf{n}\}_w, [\![v_h]\!]\right)_\Gamma - \left(\{\varepsilon \nabla v_h \cdot \mathbf{n}\}_w, [\![u_h]\!]\right)_\Gamma , \end{aligned} \tag{41}$$

$$F_h(v_h) := F(v_h) = (f, v_h)_\Omega .$$

The main advantage of method (40) consists in the fact that it restores the optimal convergence rate that is lost for the approximation of problem (39) with standard Lagrangian finite elements when the mesh does not fit with the interface. Indeed, for the test case already addressed for Table 2, we observe the convergence rates reported in Table 3, where $\|v\|^2_{1,h,\Omega} := \sum_{i=1,2} \|\varepsilon_i^{1/2} \nabla v\|^2_{0,\Omega} + \|\{\varepsilon\}_w^{1/2} [\![v]\!]\|^2_{\frac{1}{2},h,\Gamma}$ being $\|v\|_{\pm\frac{1}{2},h,\Gamma} := h^{\mp 1/2} \|v\|_{0,\Gamma}$ as for Nitsche's fictitious domain method.

	$\|u-u_h\|_{0,\Omega}$		$\|u-u_h\|_{1,h,\Omega}$	
h	$\varepsilon_1 = \varepsilon_2 = 1$	$\varepsilon_1 = 1,\ \varepsilon_2 = 10^{-2}$	$\varepsilon_1 = \varepsilon_2 = 1$	$\varepsilon_1 = 1,\ \varepsilon_2 = 10^{-2}$
5.00×10^{-2}	2.24×10^{-4}	1.69×10^{-2}	1.48×10^{-2}	1.08×10^{-1}
2.50×10^{-2}	5.65×10^{-5}	4.20×10^{-3}	7.30×10^{-3}	5.41×10^{-2}
1.25×10^{-2}	1.41×10^{-5}	1.00×10^{-3}	3.60×10^{-3}	2.70×10^{-2}
6.25×10^{-3}	3.55×10^{-6}	2.64×10^{-4}	1.80×10^{-3}	1.35×10^{-2}
3.12×10^{-3}	8.90×10^{-7}	6.62×10^{-5}	9.02×10^{-4}	6.70×10^{-3}
p	1.99	1.99	1.00	1.00

Table 3 Convergence rate of (40) with linear finite elements for the test case already considered for Table 2.

However, this method has two major drawbacks that will be discussed thoroughly. Firstly, the corresponding matrix may become ill conditioned in case $|K \cap \Omega_i|$ is small for all the triangles in the support of a basis function. Secondly, the scheme cannot be simultaneously robust with respect to small cut elements and large contrast problems. A partial remedy exploiting the arbitrary choice of the averaging weights w_i will be proposed below, but a robust stability estimate can be only achieved with the help of a stabilisation term.

3.3.1 Stability Analysis of the Discrete Space with Cut Elements

The objective of this section is to reformulate the definition of $V_h = V_{h,1} \times V_{h,2}$ as an approximation space of functions with support on the physical domain Ω. As discussed in [43, 48], this allows us to exhibit and analyse the instabilities arising from the presence of small cut elements.

We consider the alternative representation of V_h proposed in [43], which exploits a hierarchical representation in terms of a standard Lagrangian finite element space, enriched with additional basis functions over cut elements. We start by defining the following restriction operator :

$$R_i:\ L^2(\Omega) \to L^2(\Omega), \quad R_i v := \begin{cases} v|_{\Omega_i} & \text{in } \Omega_i, \\ 0 & \text{in } \Omega \setminus \Omega_i. \end{cases}$$

Mimicking the Nitsche's fictitious domain method, we denote by $\mathscr{C}_h := \{K \in \mathscr{T}_h : |K \cap \Gamma|_{\mathbb{R}^{d-1}} > 0\}$ the crust of elements with non vanishing intersection with the interface. Let $\mathscr{I}$ be the set of indexes numbering the nodes associated to V_h^Ω and let $\{x_k\}_{k\in\mathscr{I}}$ be the corresponding set of points on Ω. We define collections of nodes neighbouring the interface and we apply them to construct enrichment spaces,

$$\begin{aligned} &\mathscr{I}_i^\Gamma := \{k \in \mathscr{I} :\ x_k \in \Omega_j,\ \mathrm{supp}(\phi_k) \cap \mathscr{C}_h \neq \emptyset\},\ \forall i,j = 1,2,\ j \neq i \\ &V_{h,i}^\Gamma := \mathrm{span}\{R_i\phi_k :\ k \in \mathscr{I}_i^\Gamma\}, \end{aligned}$$

where ϕ_k denotes the hat basis function associated to the node x_k. Owing to Theorem 2 in [43], the following direct decomposition holds:

$$V_h = V_h^\Omega \oplus V_{h,1}^\Gamma \oplus V_{h,2}^\Gamma,$$

i.e. any function $v \in V_h$ can be uniquely decomposed as $v = v^\Omega + v_1^\Gamma + v_2^\Gamma$ with $v^\Omega \in V_h^\Omega$, $v_i^\Gamma \in V_{h,i}^\Gamma$. We notice that the spaces $V_{h,1}^\Gamma, V_{h,2}^\Gamma$ are L^2-orthogonal on Ω, because their basis functions have disjoint supports.

Owing to this decomposition, finite element matrices in V_h feature the following block structure that can be exploited in their analysis. Let us denote with $M \in \mathbb{R}^{N_h \times N_h}$ and $L \in \mathbb{R}^{N_h \times N_h}$ the standard mass and stiffness matrices in the finite element space V_h,

$$\mathbf{v}' M \mathbf{w} = (v,w)_{0,\Omega}, \quad \mathbf{v}' L \mathbf{w} = (\nabla v, \nabla w)_{0,\cup\Omega_i}, \quad \forall v,w \in V_h$$

which can be rearranged as follows

$$M = \begin{bmatrix} M^{\Omega} & M_1^{\Omega\Gamma} & M_2^{\Omega\Gamma} \\ (M_1^{\Omega\Gamma})' & M_1^{\Gamma} & 0 \\ (M_2^{\Omega\Gamma})' & 0 & M_2^{\Gamma} \end{bmatrix} \quad L = \begin{bmatrix} L^{\Omega} & L_1^{\Omega\Gamma} & L_2^{\Omega\Gamma} \\ (L_1^{\Omega\Gamma})' & L_1^{\Gamma} & 0 \\ (L_2^{\Omega\Gamma})' & 0 & L_2^{\Gamma} \end{bmatrix}.$$

To quantify how the presence of small cut elements affects the spectrum of finite element mass and stiffness matrices, we introduce the following mesh dependent indicators. Let $x_k \in \mathscr{I}_i^{\Gamma}$ be any vertex associated to the enrichment spaces $V_{h,i}^{\Gamma}$, let ϕ_k be the corresponding basis function and $\mathscr{P}_k$ be its patch. The indicators that affect the conditioning of a finite element method with respect to small sub-elements can be defined as

$$\underline{\nu}_i := \min_{k \in \mathscr{I}_i^{\Gamma}} \frac{|\mathscr{P}_k \cap \Omega_i|}{|\mathscr{P}_k|}, \quad \overline{\nu}_i := \max_{k \in \mathscr{I}_i^{\Gamma}} \frac{|\mathscr{P}_k \cap \Omega_i|}{|\mathscr{P}_k|},$$

$$\nu := \min_i \min_{k \in \mathscr{I}_i^{\Gamma}} \frac{|\mathscr{P}_k \cap \Omega_i|}{|\mathscr{P}_k|}.$$

Furthermore, we assume that for any index k, the corresponding patch satisfies $\mathscr{P}_k \cap \big(\Omega \setminus \Omega_{\Gamma}\big) \neq \emptyset$, i.e. there exists at least one element in the patch that is not cut by the interface.

Under the previous assumption on the mesh and owing to Lemma 2 of [43] the following strengthened Cauchy-Schwarz inequality hold true for any $v^{\Omega} \in V_h^{\Omega}$, $v^{\Gamma} \in V_{h,1}^{\Gamma} \oplus V_{h,2}^{\Gamma}$. There exist constants $0 < c_{cs}^0, c_{cs}^1 < 1$ such that

$$(v^{\Omega}, v^{\Gamma})_{\Omega} \le c_{cs}^0 \|v^{\Omega}\|_{0,\Omega} \|v^{\Gamma}\|_{0,\Omega},$$
$$(\nabla v^{\Omega}, \nabla v^{\Gamma})_{\cup \Omega_i} \le c_{cs}^1 \|\nabla v^{\Omega}\|_{0,\cup\Omega_i} \|\nabla v^{\Gamma}\|_{0,\cup\Omega_i}.$$

Then, exploiting the decomposition $v = v^{\Omega} + v_1^{\Gamma} + v_2^{\Gamma}$ together with Pythagoras' theorem, straightforward computations show that

$$\begin{aligned}(1-c_{cs}^0)(\|v^{\Omega}\|_{0,\Omega}^2 + \|v_1^{\Gamma}\|_{0,\Omega_1}^2 + \|v_2^{\Gamma}\|_{0,\Omega_2}^2) \\ \le \|v\|_{0,\Omega}^2 \le 2(\|v^{\Omega}\|_{0,\Omega}^2 + \|v_1^{\Gamma}\|_{0,\Omega_1}^2 + \|v_2^{\Gamma}\|_{0,\Omega_2}^2),\end{aligned}$$

$$\begin{aligned}(1-c_{cs}^1)(\|\nabla v^{\Omega}\|_{0,\Omega}^2 + \|\nabla v_1^{\Gamma}\|_{0,\Omega_1}^2 + \|\nabla v_2^{\Gamma}\|_{0,\Omega_2}^2) \\ \le \|\nabla v\|_{0,\cup\Omega_i}^2 \le 2(\|\nabla v^{\Omega}\|_{0,\Omega}^2 + \|\nabla v_1^{\Gamma}\|_{0,\Omega_1}^2 + \|\nabla v_2^{\Gamma}\|_{0,\Omega_2}^2).\end{aligned}$$

The previous inequalities directly imply that the mass and stiffness matrices are spectrally equivalent to their block diagonals,

$$\mathbf{v}' M \mathbf{v} \simeq (\mathbf{v}^{\Omega})' M^{\Omega} \mathbf{v}^{\Omega} + (\mathbf{v}_1^{\Gamma})' M_1^{\Gamma} \mathbf{v}_1^{\Gamma} + (\mathbf{v}_2^{\Gamma})' M_2^{\Gamma} \mathbf{v}_2^{\Gamma}, \tag{42}$$
$$\mathbf{v}' L \mathbf{v} \simeq (\mathbf{v}^{\Omega})' L^{\Omega} \mathbf{v}^{\Omega} + (\mathbf{v}_1^{\Gamma})' L_1^{\Gamma} \mathbf{v}_1^{\Gamma} + (\mathbf{v}_2^{\Gamma})' L_2^{\Gamma} \mathbf{v}_2^{\Gamma}. \tag{43}$$

Since the spectral properties of M^Ω and L^Ω are well known, we focus on the analysis of M_i^Γ, L_i^Γ. As shown in [43], Lemma 3, for any $v_i^\Gamma \in V_{h,i}^\Gamma$ there exist positive constants $\underline{c}_0^\Omega, \overline{c}_0^\Omega$, independent on how the interface Γ cuts the mesh $\mathscr{T}_h$, such that

$$\underline{c}_0^\Omega \sum_{k\in\mathscr{I}_i^\Gamma} \left(\beta_k^i\right)^2 \|R_i\phi_k\|_{0,\Omega_i}^2 \leq \|v_i^\Gamma\|_{0,\Omega_i}^2 \leq \overline{c}_0^\Omega \sum_{k\in\mathscr{I}_i^\Gamma} \left(\beta_k^i\right)^2 \|R_i\phi_k\|_{0,\Omega_i}^2. \tag{44}$$

The extension of this analysis to the H^1-norm holds true due to the fact that gradients of the local basis functions on $V_{h,i}^\Gamma$ are linearly independent functions. Indeed, for any $v_i^\Gamma \in V_{h,i}^\Gamma$ there exist positive constants $\underline{c}_1^\Omega, \overline{c}_1^\Omega$, independent on how the interface Γ cuts the mesh $\mathscr{T}_h$, such that

$$\underline{c}_1^\Omega \sum_{k\in\mathscr{I}_i^\Gamma} \left(\beta_k^i\right)^2 \|R_i\nabla\phi_k\|_{0,\Omega_i}^2 \leq \|\nabla v_i^\Gamma\|_{0,\Omega_i}^2 \leq \overline{c}_1^\Omega \sum_{k\in\mathscr{I}_i^\Gamma} \left(\beta_k^i\right)^2 \|R_i\nabla\phi_k\|_{0,\Omega_i}^2. \tag{45}$$

Let $\mathbf{v}$ denote the vector of degrees of freedom that identify a generic function $v \in V_h$ and let $\|\mathbf{v}\|$ be its Euclidean norm. Let $\mathbf{v}_i^\Gamma$ and $\mathbf{v}^\Omega$ be the vectors relative to $v_i^\Gamma \in V_{h,i}^\Gamma$ and $v^\Omega \in V_h^\Omega$, respectively. For any $v_i^\Gamma \in V_{h,i}^\Gamma$ there exist positive constants $\underline{c}_0^\Gamma, \overline{c}_0^\Gamma$, independent on ν, h, such that

$$\underline{c}_0^\Gamma h^d \underline{\nu}_i^{2/d+1} \|\mathbf{v}_i^\Gamma\|^2 \leq \|v_i^\Gamma\|_{0,\Omega_i}^2 \leq \overline{c}_0^\Gamma h^d \overline{\nu}_i^{2/d+1} \|\mathbf{v}_i^\Gamma\|^2, \tag{46}$$

and there exists $\underline{v}_i^\Gamma \in V_{h,i}^\Gamma$ such that

$$\|\underline{v}_i^\Gamma\|_{0,\Omega_i}^2 \leq \overline{c}_0^\Gamma h^d \underline{\nu}_i^{2/d+1} \|\underline{\mathbf{v}}_i^\Gamma\|^2. \tag{47}$$

To prove (46) we have to estimate the smallest $\|R_i\phi_k\|_{0,\Omega_i}^2$. We split the integrals over the elements that belong to the patch of $R_i\phi_k$ and we apply a suitable quadrature formula. We notice that the measure of the support where the integrals are evaluated is proportional to $h^d \underline{\nu}_i$ while the pointwise evaluations of the function to be integrated can be at most equivalent to $(\underline{\nu}_i^{1/d})^2$. The upper bound is obtained replacing the smallest $\|R_i\phi_k\|_{0,\Omega_i}^2$ with the largest. By the same argument, (47) holds true if we select $\underline{v}_i^\Gamma := R_i\phi_k$ corresponding to $\min_{k\in\mathscr{I}_i^\Gamma} \|R_i\phi_k\|_{0,\Omega}^2$.

By means of the same reasoning applied to (45), a similar result can be shown for gradients of discrete functions, with a different scaling with respect to ν, because $R_i\nabla\phi_k$ are constant functions proportional to h^{-1} and thus $\|R_i\nabla\phi_k\|_{0,K_i}^2 \simeq h^{d-2}\nu_i^K$. As a result of that, there exist $\underline{c}_1^\Gamma, \overline{c}_1^\Gamma > 0$, independent on ν, h such that

$$\underline{c}_1^\Gamma h^{d-2} \underline{\nu}_i \|\mathbf{v}_i^\Gamma\|^2 \leq \|\nabla v_i^\Gamma\|_{0,\Omega}^2 \leq \overline{c}_1^\Gamma h^{d-2} \overline{\nu}_i \|\mathbf{v}_i^\Gamma\|^2. \tag{48}$$

Furthermore, for the same $\underline{v}_i^\Gamma \in V_{h,i}^\Gamma$ of (47) we have

$$\|\nabla \underline{v}^\Gamma\|_{0,\Omega}^2 \leq \overline{c}_1^\Gamma h^{d-2} \underline{\nu}_i \|\underline{\mathbf{v}}^\Gamma\|^2. \tag{49}$$

Inequalities (47) and (49) show that minimal eigenvalues of $M_i^\Gamma L_i^\Gamma$ become arbitrarily small in presence of small element cuts. This clearly influences the conditioning of the finite element scheme, which will be affected by a factor ν^{-1}. However, the present analysis immediately points out a cure for this drawback. Indeed, combining (42) and (43) with (47) and (49) we conclude that the mass and stiffness matrices of the enriched finite element space V_h are spectrally equivalent to

$$M \simeq \begin{bmatrix} M^\Omega & 0 & 0 \\ 0 & \mathrm{diag}(M_1^\Gamma) & 0 \\ 0 & 0 & \mathrm{diag}(M_2^\Gamma) \end{bmatrix} \quad L \simeq \begin{bmatrix} L^\Omega & 0 & 0 \\ 0 & \mathrm{diag}(L_1^\Gamma) & 0 \\ 0 & 0 & \mathrm{diag}(L_2^\Gamma) \end{bmatrix}.$$

where, given a real square matrix B, we denote with $\mathrm{diag}(B)$ its diagonal. This shows that solving a finite element scheme in the enriched space V_h is computationally equivalent to solving it in the standard space V_h^Ω, because the only genuinely stiff block is L^Ω. Another way to formulate this conclusion is based on the *optimal condition number* of the problem, see [46]. More precisely, given $A \in \mathbb{R}^{N\times N}$ the optimal condition number is

$$K_{opt}(A) := \min_{D\in\mathbb{R}^{N\times N}} K_2(DAD)$$

and the previous analysis shows that $K_{opt}(\alpha_M M + \alpha_L L) = K_{opt}(\alpha_M M^\Omega + \alpha_L L^\Omega)$ for any positive constants α_M, α_L.

3.3.2 Stability Issues for the Unfitted Nitsche Method

An important question concerning the stability of the scheme is how to choose the averages in the interface terms. In [29], Hansbo and Hansbo proposed a method for which they could prove stability and optimal convergence. In their analysis they chose mesh dependent weights,

$$w_i|_K = \frac{|K \cap \Omega_i|}{|K|}, \quad \text{and} \quad \xi(\varepsilon) = \max\{\varepsilon_1, \varepsilon_2\}.$$

As a result of that, integrals of the normal derivative on the interface on elements with a very small fraction intersecting one of the physical domains will get a small weight, which will balance the factor of order $|K|$ appearing after taking the trace inequality. In Section 2 we showed that

$$w_1 = \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}, \quad w_2 = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} \quad \text{and} \quad \xi(\varepsilon) = \{\varepsilon\}_w = \frac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2}. \tag{50}$$

leads to robustness with respect to the jump in the diffusivities.

This poses a situation in which the unfitted character of the method requires a certain set of weights, and the large contrast character requires another set. For instance, these contradictory requirements clearly appear in the following estimate,

$$\int_{\partial K\cap\Gamma}\{\varepsilon\}_w(\nabla v_{h,i})^2 \lesssim h^{-1}w_i\Big(1+\frac{(\varepsilon_j w_j)|_K}{(\varepsilon_i w_i)|_K}\Big)\frac{|K|}{|K\cap\Omega_i|}\|\varepsilon_i^{\frac{1}{2}}\nabla v_{h,i}\|^2_{0,K\cap\Omega_i},$$

which is needed to quantify an upper bound for the spectrum of the discrete problem. Indeed, the constant

$$\Big(1+\frac{(\varepsilon_j w_j)|_K}{(\varepsilon_i w_i)|_K}\Big)\frac{|K|}{|K\cap\Omega_i|},$$

may become arbitrarily large for some configuration of the interface or highly heterogeneous weights. A partial remedy consists selecting the weights w_i to minimise the dominating effect. If the worse case comes from the way the interface is cut, then we define

$$w_i = \frac{|K\cap\Omega_i|}{|K|} \quad \text{satisfying} \quad w_1+w_2=1,$$

otherwise, when the heterogeneity of coefficients is dominating, we choose

$$w_i = \frac{\varepsilon_j}{\varepsilon_i+\varepsilon_j} \quad \text{such that} \quad \Big(1+\frac{(\varepsilon_j w_j)|_K}{(\varepsilon_i w_i)|_K}\Big)=2.$$

Nevertheless, this technique does not work in situations where both difficulties arise simultaneously. To handle both effects at the same time, we may draw from the fictitious domain formulation proposed in the previous section. The introduction of a ghost penalty term on the interface elements both in $\mathscr{T}_{h1}$ and $\mathscr{T}_{h2}$ gives the same extended coercivity as in the fictitious domain case and we are then allowed to choose the weights so as to control the large contrast in diffusivity.

3.3.3 The Stabilized Unfitted Nitsche Method

The stabilised method that we propose takes the form: find $[u_{h,1},u_{h,2}]\in V_h := V_{h,1}\times V_{h,2}$, such that

$$a_h(u_h,v_h)+g_h(u_h,v_h)=F_h(v_h),\ \forall v_h\in V_h, \tag{51}$$

where the weights have been chosen as in (50) and

$$g_h(u_h,v_h) := \sum_{i=1}^{2}\sum_{E\in\mathscr{E}_{B_i}}(\gamma_g\varepsilon_i h_E[\![\nabla u_{h,i}\cdot\mathbf{n}_E]\!],[\![\nabla v_{h,i}\cdot\mathbf{n}_E]\!])_E,$$

with

$$\mathscr{E}_{B_i} := \{E=K\cap K' : K\in\mathscr{T}_{hi}, K'\in\mathscr{T}_{hi} \text{ where either } K\cap\Gamma\neq\emptyset \text{ or } K'\cap\Gamma\neq\emptyset\}.$$

For the analysis of Nitsche's method for unfitted interfaces, we introduce the norms

$$\|v_h\|^2_{1,h,\Omega_{\mathscr{T}}} := \sum_{i=1}^{2}\sum_{K\in\mathscr{T}_{hi}}\|\varepsilon_i^{\frac{1}{2}}\nabla v_{h,i}\|^2_{0,K}+\|\{\varepsilon\}_w^{\frac{1}{2}}[\![v_{h,i}]\!]\|^2_{\frac{1}{2},h,\Gamma}$$

and

$$|||v_h|||^2_{1,h,\Omega} := \sum_{i=1}^{2} \|\varepsilon_i^{\frac{1}{2}} \nabla v_{h,i}\|^2_{0,\Omega} + \|\{\varepsilon\}_w^{\frac{1}{2}} \{\nabla v_h \cdot \mathbf{n}\}\|^2_{-\frac{1}{2},h,\Gamma} + \|\{\varepsilon\}_w^{\frac{1}{2}} [\![v_h]\!]\|_{+\frac{1}{2},h,\Gamma}.$$

To obtain a robust stability estimate, we use the extended coercivity obtained thanks to the ghost penalty term combined with the inverse inequality to conclude that,

$$|||v_h|||^2_{1,h,\Omega} \lesssim \|v_h\|^2_{1,h,\Omega_{\mathscr{T}}} \lesssim a_h(v_h,v_h) + g_h(v_h,v_h), \quad \forall v_h \in V_h. \tag{52}$$

This is obtained in the same fashion as the analogous result for the fictitious domain method. The boundedness of the stabilised bilinear form is also guaranteed by means of standard arguments, see [12],

$$\begin{aligned} a_h(u_h,v_h) + g_h(u_h,v_h) &\lesssim |||u_h|||_{1,h,\Omega}\, |||v_h|||_{1,h,\Omega} \\ &\lesssim \|u_h\|_{1,h,\Omega_{\mathscr{T}}}\, \|v_h\|_{1,h,\Omega_{\mathscr{T}}} \quad \forall u_h,\, v_h \in V_h. \end{aligned}$$

To proceed with the convergence analysis, we introduce extension operators $\mathbb{E}_i : H^2(\Omega_i) \mapsto H^2(\Omega_{\mathscr{T}i})$ such that $\mathbb{E}_i v|_{\Omega_i} = v|_{\Omega_i}$ and $\|\mathbb{E}_i v\|_{H^2(\Omega_{\mathscr{T}i})} \lesssim \|v\|_{H^2(\Omega_i)}$. In a similar fashion as above, we define an interpolation operator $i_h : H^2(\Omega_1) \times H^2(\Omega_2) \mapsto V_h$ by $i_h v := [I_h \mathbb{E}_1 v, I_h \mathbb{E}_2 v]$ where I_h is the standard nodal Lagrange interpolator. It is straightforward to show that

$$|||v - i_h v|||_g := |||v - i_h v|||_{1,h,\Omega} + \sqrt{g_h\left(\mathbb{E}v - i_h v, \mathbb{E}v - i_h v\right)} \lesssim h \|v\|_{H^2(\Omega_1 \cup \Omega_2)}.$$

For the convergence analysis we need the following continuity result, that is a straightforward application of Cauchy-Schwarz inequalities and local trace inequalities. For all $v \in H^2(\Omega)$ and $w_h \in V_h$ there holds

$$\begin{aligned} &|a_h(v - i_h v, w_h) + g_h(v - i_h v, w_h)| \\ &\qquad \lesssim |||v - i_h v|||_g \Big(\sum_{i=1,2} \sum_{K \in \mathscr{T}_{h,i}} \|\nabla w_{h,i}\|^2_K + \|w_h\|^2_{\frac{1}{2},h,\partial\Omega} \Big)^{1/2}. \end{aligned}$$

Then, the optimal convergence estimate $|||u - u_h|||_{1,h,\Omega} \lesssim h \|u\|_{H^2(\Omega)}$, is an immediate consequence of (52).

3.3.4 Bounded Condition Number

In this section we will show that the choice of weights (50) together with the use of ghost penalty term leads to a method with a system matrix whose condition number, after diagonal scaling with the diffusivity, has the same asymptotic scaling as the standard Galerkin method for the Poisson problem with fitted mesh and constant coefficients. This means that the conditioning is independent both of the interface configuration and the jump of the diffusivities. To fix the ideas let $\varepsilon_1 = 1$ and $0 < \varepsilon_2 < \varepsilon_1$. Other configurations can be obtained by scaling.

Let $\{\phi_{k,i}\}$ denote the nodal basis of $V_{h,i}$ with $i = 1,2$. Consequently we may write $u_{h,i} \in V_{h,i}$ in the form $u_{h,i} := \sum_{k=1}^{N_i} U_{k,i}\phi_{k,i}$. The formulation (41) may then be written as the linear system

$$\begin{bmatrix} \varepsilon_1 \mathbf{A}_{11} + \{\varepsilon\}_w \mathbf{A}_{11}^{\Gamma} & \{\varepsilon\}_w \mathbf{A}_{12}^{\Gamma} \\ \{\varepsilon\}_w \mathbf{A}_{21}^{\Gamma} & \varepsilon_2 \mathbf{A}_{22} + \{\varepsilon\}_w \mathbf{A}_{22}^{\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix}, \tag{53}$$

where for symmetry it holds that $\mathbf{A}_{12}^{\Gamma} = \left(\mathbf{A}_{21}^{\Gamma}\right)^T$ and the vectors are defined by

$$\mathbf{U}_i := \{U_{k,i}\}_{k=1}^{N_i}, \text{ and } \mathbf{F}_i := \{F_h(\phi_{k,i})\}_{k=1}^{N_i}$$

and the weight function $\{\varepsilon\}_w$ is given in (50). Examining formulation (40) we see that the matrices are given by

$$\mathbf{A}_{ii} := \{\left(\varepsilon_i \nabla\phi_{k,i}, \nabla\phi_{l,i}\right)_{\Omega_i} + \varepsilon_i g_i(\phi_{k,i}, \phi_{l,i})\}_{k,l=1}^{N_i}, \quad i = 1,2$$

where $g_i(\phi_{k,i}, \phi_{l,i})$ denotes a ghost penalty term on the subdomain Ω_i,

$$\mathbf{A}_{ii}^{\Gamma} := \{-\left(\nabla\phi_{k,i} \cdot \mathbf{n} + h^{-1}\gamma\phi_{k,i}, \phi_{l,i}\right)_{\Gamma}\}_{k,l=1}^{N_i} \quad i = 1,2$$

which is independent of ε_i, and

$$\mathbf{A}_{ij}^{\Gamma} := \left\{\frac{1}{2}\left(\nabla\phi_{k,i} \cdot \mathbf{n} + h^{-1}\gamma\phi_{k,i}, \phi_{l,j}\right)_{\Gamma} + \frac{1}{2}\left(\nabla\phi_{l,j} \cdot \mathbf{n} + h^{-1}\gamma\phi_{l,j}, \phi_{k,i}\right)_{\Gamma}\right\},$$

with $k = 1,\dots,N_i$, $l = 1,\dots,N_j$ and $i, j = 1,2$, $i \neq j$. After diagonal symmetric scaling, the system matrix takes the form

$$\mathbf{A}_{scal} := \begin{bmatrix} \mathbf{A}_{11} + \frac{\{\varepsilon\}_w}{\varepsilon_1}\mathbf{A}_{11}^{\Gamma} & \frac{\{\varepsilon\}_w}{\sqrt{\varepsilon_1\varepsilon_2}}\mathbf{A}_{12}^{\Gamma} \\ \frac{\{\varepsilon\}_w}{\sqrt{\varepsilon_1\varepsilon_2}}\mathbf{A}_{21}^{\Gamma} & \mathbf{A}_{22} + \frac{\{\varepsilon\}_w}{\varepsilon_2}\mathbf{A}_{22}^{\Gamma}. \end{bmatrix}.$$

To study the behaviour of the unfitted Nitsche method in the case of highly heterogeneous coefficients, we notice that the matrix $\mathbf{A}_{scal}$ converges to

$$\lim_{\varepsilon_2 \to 0} \mathbf{A}_{scal} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} + \mathbf{A}_{22}^{\Gamma}. \end{bmatrix}$$

in the limit $\varepsilon_2 \to 0$.

Under the assumption that Γ is a planar interface we know that for each subdomain Ω_i it holds $\partial\Omega_i \cap \partial\Omega \neq \emptyset$. Furthermore, since the homogeneous Dirichlet boundary conditions are strongly enforced in the finite element space, we conclude that the stiffness matrices $\mathbf{A}_{ii}$ are symmetric positive definite. Owing to an inverse inequality and with the stabilisation parameter γ large enough, this conclusion holds true even though we extend it to $\mathbf{A}_{ii}+\mathbf{A}_{ii}^{\Gamma}$. This illustrates that the stabilised Nitsche's method with symmetric diagonal scaling becomes robust with respect to the heterogeneity of coefficients provided that the averaging weights are selected as in (50).

To quantify the robustness of the scheme with respect to the configuration of the interface, we exploit the strengthened coercivity ensured by the ghost penalty term. Under the aforementioned assumption on the stiffness matrices, inverse and Poincaré inequalities imply that

$$\sum_{i=1}^{2}\sum_{K\in\mathscr{T}_{hi}}\|\varepsilon_i v_{h,i}\|_{0,K}^2 \lesssim \|v_h\|_{1,h,\Omega_{\mathscr{T}_h}}^2 \lesssim a_h(v_h,v_h)+g_h(v_h,v_h)$$

$$\lesssim \|v_h\|_{1,h,\Omega_{\mathscr{T}_h}}^2 \lesssim h^{-2}\sum_{i=1}^{2}\sum_{K\in\mathscr{T}_{hi}}\|\varepsilon_i v_{h,i}\|_{0,K}^2, \quad \forall v_h\in V_h.$$

The matrix of system (53) is thus spectrally equivalent to a block diagonal matrix that is uniformly independent of the configuration of the interface. Furthermore, when diagonal scaling is applied to such matrix, the equivalent system becomes independent of the heterogeneity for diffusion coefficients.

3.3.5 Asymptotic Convergence to the Fictitious Domain Method

The aim of this section is to show that unfitted Nitsche method for interface problems coincides with the corresponding unfitted boundary method in the case that the diffusion coefficient on one of the subdomains becomes arbitrarily large.

This property has two interesting consequences. On the one hand, it shows that the choice of the balancing weights proposed for large contrast problems is consistent with the unfitted boundary case. On the other hand, it allows to exploit the unfitted interface formulation as a fictitious domain method, where the choice of the computational domain $\Omega_{\mathscr{T}}$ is completely arbitrary with respect to the physical domain Ω, provided that in the complementary domain $\Omega_{\mathscr{T}}\setminus\Omega$ a sufficiently large diffusivity is applied.

To fix the ideas, we assume that $\varepsilon_1=1$ and study the case $\varepsilon_2\to\infty$. Accordingly, we denote the fictitious domain bilinear forms (30) defined on Ω_1 by $a_{fd}(\cdot,\cdot), g_{fd}(\cdot,\cdot)$ and denote the domain decomposition bilinear forms (41) by $a_{dd}(\cdot,\cdot), g_{dd}(\cdot,\cdot)$. Let $u_{h,fd}$ denote the solution of (32) and $u_{h,dd}$ the solution of (40) with *ghost penalty* stabilisation. We assume that the penalty parameters for both formulations are set to the same values. We are interested in the behaviour of the discrete error between the two formulations in the limit as $\varepsilon_2\to\infty$. We therefore define $e_h:=(u_{h,fd}-u_{h,dd})|_{\Omega_1}$. Using the coercivity of the formulation (32) we have

$$\|e_h\|_{1,h,\Omega_{\mathscr{T}_{h1}}}^2 \leq a_{fd}(e_h,e_h)+g_{dd}(e_h,e_h).$$

Note that since $e_h|_{\mathscr{T}_{h2}} = 0$, $g_{fd}(e_h, e_h) = g_{dd}(e_h, e_h)$. By the definition of the discrete problems we have

$$\begin{aligned}
\|e_h\|^2_{1,h,\Omega_{\mathscr{T}_{h1}}} &\leq F_h(e_h) - a_{fd}(u_{h,dd}, e_h) - g_{dd}(u_{h,dd}, e_h) \\
&= a_{dd}(u_{h,dd}, e_h) - a_{fd}(u_{h,dd}, e_h).
\end{aligned}$$

It is straightforward to show that

$$\begin{aligned}
a_{dd}(u_{h,dd}, e_h) - a_{fd}(u_{h,dd}, e_h) = &\left((1 - \frac{\varepsilon_2}{\varepsilon_2 + 1})\partial_n u_{h,dd}|_{\partial\Omega_1}, e_h|_{\partial\Omega_1}\right)_{\partial\Omega_1} \\
&+ \left((1 - \frac{\varepsilon_2}{\varepsilon_2 + 1})\partial_n e_h|_{\partial\Omega_1}, u_{h,dd}|_{\partial\Omega_1}\right)_{\partial\Omega_1} \\
&- \left((\frac{\varepsilon_2}{\varepsilon_2 + 1})\partial_n u_{h,dd}|_{\partial\Omega_2}, e_h|_{\partial\Omega_1}\right)_{\partial\Omega_1} \\
&- \left((\frac{\varepsilon_2}{\varepsilon_2 + 1})\partial_n e_h|_{\partial\Omega_1}, u_{h,dd}|_{\partial\Omega_2}\right)_{\partial\Omega_1} \\
&+ \left((1 - \frac{\varepsilon_2}{\varepsilon_2 + 1})\gamma_{bc} h^{-1} u_{h,dd}|_{\partial\Omega_1}, e_h|_{\partial\Omega_1}\right)_{\partial\Omega_1} \\
&- \left(\frac{\varepsilon_2}{\varepsilon_2 + 1}\gamma h^{-1} u_{h,dd}|_{\partial\Omega_2}, e_h|_{\partial\Omega_1}\right)_{\partial\Omega_1}.
\end{aligned}$$

By repeated application of the mesh weighted Cauchy-Schwarz inequality and trace inequalities in the right hand side we arrive at the bound

$$\begin{aligned}
&|a_{dd}(u_{h,dd}, e_h) - a_{fd}(u_{h,dd}, e_h)| \\
&\lesssim \left(1 - \frac{\varepsilon_2}{\varepsilon_2 + 1}\right)\Big(\sum_{K \in \mathscr{T}_{h1}} (1 + h_K^{-1})\|\nabla u_{h,dd}\|_K^2\Big)^{\frac{1}{2}} \|e_h\|_{1,h,\Omega_{\mathscr{T}_{h1}}} \\
&\quad + \frac{\varepsilon_2}{(\varepsilon_2 + 1)}\Big(\sum_{K \in \mathscr{T}_{h2}} (1 + h_K^{-1})\|\nabla u_{h,dd}\|_K^2\Big)^{\frac{1}{2}} \|e_h\|_{1,h,\Omega_{\mathscr{T}_{h1}}}.
\end{aligned}$$

Using the formulation (40), (41) with *ghost penalty* stabilisation, observing that there exists a positive constant C_F such that $F_h(v_h) \leq C_F \|v_h\|_{1,h,\Omega_{\mathscr{T}}}$ and exploiting the stability (52), we obtain for $u_{h,dd}|_{\Omega_2}$

$$\sqrt{\sum_{K \in \mathscr{T}_{h2}} \|\varepsilon_2^{\frac{1}{2}} \nabla u_{h,dd}\|_K^2} \leq C_F$$

and for $u_{h,dd}|_{\Omega_1}$

$$\sqrt{\sum_{K \in \mathscr{T}_{h1}} \|\nabla u_{h,dd}\|_K^2} \leq C_F.$$

We conclude that we have the bound

$$\|e_h\|_{1,h,\Omega_{\mathscr{T}_{h1}}} \leq \frac{1}{(\varepsilon_2+1)}(1+h^{-1})C_F$$

and that

$$\lim_{\varepsilon_2\to\infty} \|e_h\|_{1,h,\Omega_{\mathscr{T}_{h1}}} = 0.$$

Hence, the unfitted Nitsche interface method reduces to the fictitious domain method in the limit of infinite diffusivity.

Acknowledgements The authors acknowledge the support of the project *5 per Mille Junior "Computational models for heterogeneous media. Application to microscale analysis of tissue engineered constructs"*, CUP D41J10000490001, Politecnico di Milano.

References

1. Arnold, D.N.: An interior penalty finite element method with discontinuous elements. SIAM J. Numer. Anal. **19**(4), 742–760 (1982). DOI 10.1137/0719052. URL http://dx.doi.org/10.1137/0719052
2. Arnold, D.N., Brezzi, F., Cockburn, B., Marini, L.D.: Unified analysis of discontinuous Galerkin methods for elliptic problems. SIAM J. Numer. Anal. **39**(5), 1749–1779 (2001/02). DOI 10.1137/S0036142901384162. URL http://dx.doi.org/10.1137/S0036142901384162
3. Babuška, I.: The finite element method with Lagrangian multipliers. Numer. Math. **20**, 179–192 (1972/73)
4. Babuška, I.: The finite element method with penalty. Math. Comp. **27**, 221–228 (1973)
5. Barbosa, H.J.C., Hughes, T.J.R.: The finite element method with Lagrange multipliers on the boundary: circumventing the Babuška-Brezzi condition. Comput. Methods Appl. Mech. Engrg. **85**(1), 109–128 (1991). DOI 10.1016/0045-7825(91)90125-P. URL http://dx.doi.org/10.1016/0045-7825(91)90125-P
6. Barbosa, H.J.C., Hughes, T.J.R.: Boundary Lagrange multipliers in finite element methods: error analysis in natural norms. Numer. Math. **62**(1), 1–15 (1992). DOI 10.1007/BF01396217. URL http://dx.doi.org/10.1007/BF01396217
7. Barrett, J.W., Elliott, C.M.: Finite element approximation of the Dirichlet problem using the boundary penalty method. Numer. Math. **49**(4), 343–366 (1986). DOI 10.1007/BF01389536. URL http://dx.doi.org/10.1007/BF01389536
8. Becker, R., Hansbo, P., Stenberg, R.: A finite element method for domain decomposition with non-matching grids. M2AN Math. Model. Numer. Anal. **37**(2), 209–225 (2003). DOI 10.1051/m2an:2003023. URL http://dx.doi.org/10.1051/m2an:2003023
9. Bramble, J.H.: The Lagrange multiplier method for Dirichlet's problem. Math. Comp. **37**(155), 1–11 (1981). DOI 10.2307/2007496. URL http://dx.doi.org/10.2307/2007496
10. Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods, *Texts in Applied Mathematics*, vol. 15, third edn. Springer, New York (2008). DOI 10.1007/978-0-387-75934-0. URL http://dx.doi.org/10.1007/978-0-387-75934-0
11. Burman, E.: A unified analysis for conforming and nonconforming stabilized finite element methods using interior penalty. SIAM J. Numer. Anal. **43**(5), 2012–2033 (electronic) (2005). DOI 10.1137/S0036142903437374. URL http://dx.doi.org/10.1137/S0036142903437374
12. Burman, E.: Ghost penalty. Comptes Rendus Mathematique **348**(21-22), 1217 – 1220 (2010). DOI DOI: 10.1016/j.crma.2010.10.006
13. Burman, E., Ern, A.: Continuous interior penalty hp-finite element methods for advection and advection-diffusion equations. Math. Comp. **76**(259), 1119–1140 (electronic) (2007). DOI 10.1090/S0025-5718-07-01951-5. URL http://dx.doi.org/10.1090/S0025-5718-07-01951-5

14. Burman, E., Fernández, M.A., Hansbo, P.: Continuous interior penalty finite element method for Oseen's equations. SIAM J. Numer. Anal. **44**(3), 1248–1274 (electronic) (2006). DOI 10.1137/040617686. URL http://dx.doi.org/10.1137/040617686
15. Burman, E., Hansbo, P.: Fictitious domain finite element methods using cut elements: I. a stabilized lagrange multiplier method. Computer Methods in Applied Mechanics and Engineering **199**(41-44), 2680 – 2686 (2010). DOI DOI: 10.1016/j.cma.2010.05.011
16. Burman, E., Hansbo, P.: Fictitious domain finite element methods using cut elements: Ii. a stabilized nitsche method. Applied Numerical Mathematics **In Press, Corrected Proof**, – (2011). DOI DOI: 10.1016/j.apnum.2011.01.008
17. Burman, E., Zunino, P.: A domain decomposition method based on weighted interior penalties for advection-diffusion-reaction problems. SIAM J. Numer. Anal. **44**(4), 1612–1638 (electronic) (2006). DOI 10.1137/050634736. URL http://dx.doi.org/10.1137/050634736
18. Codina, R., Baiges, J.: Approximate imposition of boundary conditions in immersed boundary methods. Internat. J. Numer. Methods Engrg. **80**(11), 1379–1405 (2009). DOI 10.1002/nme.2662. URL http://dx.doi.org/10.1002/nme.2662
19. D'Angelo, C., Zunino, P.: A finite element method based on weighted interior penalties for heterogeneous incompressible flows. SIAM J. Numer. Anal. **47**(5), 3990–4020 (2009). DOI 10.1137/080726318. URL http://dx.doi.org/10.1137/080726318
20. Dolbow, J., Harari, I.: An efficient finite element method for embedded interface problems. Internat. J. Numer. Methods Engrg. **78**(2), 229–252 (2009). DOI 10.1002/nme.2486. URL http://dx.doi.org/10.1002/nme.2486
21. Dolbow, J., Moës, N., Belytschko, T.: An extended finite element method for modeling crack growth with frictional contact. Comput. Methods Appl. Mech. Engrg. **190**(51-52), 6825–6846 (2001). DOI 10.1016/S0045-7825(01)00260-2. URL http://dx.doi.org/10.1016/S0045-7825(01)00260-2
22. Dryja, M.: On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients. Comput. Methods Appl. Math. **3**(1), 76–85 (electronic) (2003). Dedicated to Raytcho Lazarov
23. Ern, A., Stephansen, A.F., Zunino, P.: A discontinuous Galerkin method with weighted averages for advection-diffusion equations with locally small and anisotropic diffusivity. IMA J. Numer. Anal. **29**(2), 235–256 (2009). DOI 10.1093/imanum/drm050. URL http://dx.doi.org/10.1093/imanum/drm050
24. Gastaldi, F., Quarteroni, A.: On the coupling of hyperbolic and parabolic systems: analytical and numerical approach. Appl. Numer. Math. **6**(1-2), 3–31 (1989/90). Spectral multi-domain methods (Paris, 1988)
25. Gerstenberger, A., Wall, W.A.: An embedded Dirichlet formulation for 3D continua. Internat. J. Numer. Methods Engrg. **82**(5), 537–563 (2010)
26. Girault, V., Glowinski, R.: Error analysis of a fictitious domain method applied to a Dirichlet problem. Japan J. Indust. Appl. Math. **12**(3), 487–514 (1995)
27. Groß, S., Reusken, A.: An extended pressure finite element space for two-phase incompressible flows with surface tension. J. Comput. Phys. **224**(1), 40–58 (2007). DOI 10.1016/j.jcp.2006.12.021. URL http://dx.doi.org/10.1016/j.jcp.2006.12.021
28. Hansbo, A., Hansbo, P.: An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. Comput. Methods Appl. Mech. Engrg. **191**(47-48), 5537–5552 (2002). DOI 10.1016/S0045-7825(02)00524-8. URL http://dx.doi.org/10.1016/S0045-7825(02)00524-8
29. Hansbo, A., Hansbo, P.: An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. Comput. Methods Appl. Mech. Engrg. **191**(47-48), 5537–5552 (2002). DOI 10.1016/S0045-7825(02)00524-8. URL http://dx.doi.org/10.1016/S0045-7825(02)00524-8
30. Hansbo, A., Hansbo, P.: A finite element method for the simulation of strong and weak discontinuities in solid mechanics. Comput. Methods Appl. Mech. Engrg. **193**(33-35), 3523–3540 (2004). DOI 10.1016/j.cma.2003.12.041. URL http://dx.doi.org/10.1016/j.cma.2003.12.041
31. Hansbo, P.: Nitsche's method for interface problems in computational mechanics. GAMM-Mitt. **28**(2), 183–206 (2005)

32. Harari, I., Dolbow, J.: Analysis of an efficient finite element method for embedded interface problems. Comput. Mech. **46**(1), 205–211 (2010). DOI 10.1007/s00466-009-0457-5. URL http://dx.doi.org/10.1007/s00466-009-0457-5
33. Haslinger, J., Renard, Y.: A new fictitious domain approach inspired by the extended finite element method. SIAM J. Numer. Anal. **47**(2), 1474–1499 (2009). DOI 10.1137/070704435. URL http://dx.doi.org/10.1137/070704435
34. Juntunen, M., Stenberg, R.: Nitsche's method for general boundary conditions. Math. Comp. **78**(267), 1353–1374 (2009). DOI 10.1090/S0025-5718-08-02183-2. URL http://dx.doi.org/10.1090/S0025-5718-08-02183-2
35. Ladyzhenskaya, O.A.: The boundary value problems of mathematical physics, *Applied Mathematical Sciences*, vol. 49. Springer-Verlag, New York (1985). Translated from the Russian by Jack Lohwater [Arthur J. Lohwater]
36. Maury, B.: Numerical analysis of a finite element/volume penalty method. SIAM J. Numer. Anal. **47**(2), 1126–1148 (2009). DOI 10.1137/080712799. URL http://dx.doi.org/10.1137/080712799
37. Moës, N., Béchet, E., Tourbier, M.: Imposing Dirichlet boundary conditions in the extended finite element method. Internat. J. Numer. Methods Engrg. **67**(12), 1641–1669 (2006). DOI 10.1002/nme.1675. URL http://dx.doi.org/10.1002/nme.1675
38. Oden, J.T., Babuška, I., Baumann, C.E.: A discontinuous hp finite element method for diffusion problems. J. Comput. Phys. **146**(2), 491–519 (1998). DOI 10.1006/jcph.1998.6032. URL http://dx.doi.org/10.1006/jcph.1998.6032
39. Pitkäranta, J.: Boundary subspaces for the finite element method with Lagrange multipliers. Numer. Math. **33**(3), 273–289 (1979). DOI 10.1007/BF01398644. URL http://dx.doi.org/10.1007/BF01398644
40. Pitkäranta, J.: Local stability conditions for the Babuška method of Lagrange multipliers. Math. Comp. **35**(152), 1113–1129 (1980). DOI 10.2307/2006378. URL http://dx.doi.org/10.2307/2006378
41. Pitkäranta, J.: The finite element method with Lagrange multipliers for domains with corners. Math. Comp. **37**(155), 13–30 (1981). DOI 10.2307/2007497. URL http://dx.doi.org/10.2307/2007497
42. Quarteroni, A., Pasquarelli, F., Valli, A.: Heterogeneous domain decomposition: principles, algorithms, applications. In: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations (Norfolk, VA, 1991), pp. 129–150. SIAM, Philadelphia, PA (1992)
43. Reusken, A.: Analysis of an extended pressure finite element space for two-phase incompressible flows. Comput. Vis. Sci. **11**(4-6), 293–305 (2008). DOI 10.1007/s00791-008-0099-8. URL http://dx.doi.org/10.1007/s00791-008-0099-8
44. Rivière, B., Wheeler, M.F., Girault, V.: A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. SIAM J. Numer. Anal. **39**(3), 902–931 (2001). DOI 10.1137/S003614290037174X. URL http://dx.doi.org/10.1137/S003614290037174X
45. Stenberg, R.: On some techniques for approximating boundary conditions in the finite element method. J. Comput. Appl. Math. **63**(1-3), 139–148 (1995). DOI 10.1016/0377-0427(95)00057-7. URL http://dx.doi.org/10.1016/0377-0427(95)00057-7. International Symposium on Mathematical Modelling and Computational Methods Modelling 94 (Prague, 1994)
46. Strang, G., Fix, G.J.: An analysis of the finite element method. Prentice-Hall Inc., Englewood Cliffs, N. J. (1973). Prentice-Hall Series in Automatic Computation
47. Zunino, P.: Discontinuous Galerkin methods based on weighted interior penalties for second order PDEs with non-smooth coefficients. J. Sci. Comput. **38**(1), 99–126 (2009). DOI 10.1007/s10915-008-9219-3. URL http://dx.doi.org/10.1007/s10915-008-9219-3
48. Zunino, P., Cattaneo, L., Colciago, C.M.: An unfitted interface penalty method for the numerical approximation of contrast problems. Tech. rep., MOX - Department of Mathematics, Politecnico di Milano (2011)