

1. IMMERSSED COBORDISMS, SURGERY KERNELS, AND THEIR INTERSECTION FORMS

Let A and B be compact manifold with identified boundary $\partial A = \Sigma = \partial B$, so we can form a closed manifold $A \cup_{\Sigma} B$. Equivalently, up to diffeomorphism at least,

$$(1.1) \quad A \cup_{\Sigma} B \cong A \cup_{\{0\} \times \Sigma} ([0, 1] \times \Sigma) \cup_{\{1\} \times \Sigma} B.$$

In this situation, a *cobordism from A to B* is a compact manifold W bounding (1.1), i.e. with a specified diffeomorphism

$$\partial W = A \cup_{\{0\} \times \Sigma} ([0, 1] \times \Sigma) \cup_{\{1\} \times \Sigma} B.$$

We will mostly consider such a cobordism to be a manifold with corners,

$$\angle W = (\{0\} \times \Sigma) \sqcup (\{1\} \times \Sigma)$$

and three smooth parts $A \subset \partial W$, $[0, 1] \times \Sigma \subset W$, and $B \subset W$ in the sense of Wall. Occasionally it may be convenient to straighten the corners to turn W into a manifold with smooth boundary $A \cup_{\Sigma} B$.

We will be particularly interested in the situation where the cobordism comes with an immersion

$$j : W \looparrowright [0, 2] \times A$$

satisfying $j(x) = (0, x)$ for $a \in A \subset \partial W$ and $j(t, \sigma) = (t, \sigma)$ for $(t, \sigma) \in [0, 1] \times \Sigma \subset \partial W$. The composition

$$(1.2) \quad A \hookrightarrow W \xrightarrow{j} [0, 2] \times A \xrightarrow{\pi} A$$

is the identity, where π denotes the projection to the second factor. This gives a splitting of chain complexes and of homology

$$\begin{aligned} C_*(A) \oplus C_*(A, W) &\xrightarrow{\cong} C_*(W) \\ H_n(A) \oplus K_n(W) &\xrightarrow{\cong} H_n(W), \end{aligned}$$

where

$$\begin{aligned} C_*(A, W) &= \text{Ker}((\pi \circ j)_* : C_*(W) \rightarrow C_*(A)) \subset C_*(W) \\ K_n(W) &= H_n(C_*(A, W)) \cong \text{Ker}(j_* : H_n(W) \rightarrow H_n([0, 2] \times A)). \end{aligned}$$

In this situation, the second factor in the splitting $H_n(W) = H_n(A) \oplus K_n(W)$ is called the *surgery kernel*. As an abstract group it is canonically isomorphic to the relative homology $H_n(W, A)$, which does not depend on the immersion j , but splitting the map $H_n(W) \rightarrow H_n(W, A)$ and realizing the surgery kernel as a summand of $H_n(W)$ does.

We will be interested in the composition

$$(1.3) \quad K_n(W) \hookrightarrow H_n(W) \rightarrow H_n(W, B)$$

and will relate it to the composition

$$(1.4) \quad (\pi \circ j|_B)_* : H_n(B) \rightarrow H_n(A)$$

Here, $\pi \circ j|_B$ is the composition

$$B \hookrightarrow W \xrightarrow{j} [0, 2] \times A \xrightarrow{\pi} A,$$

which is a map of manifolds with boundary and by construction it restricts to a diffeomorphism $\partial B \cong \partial A$. To detect whether it is a homotopy equivalence we may, at least if A and B are both simply connected, look for whether it induces an isomorphism on integral homology.

Proposition 1.1. *The homomorphisms (1.4) are isomorphisms for all n if and only if the compositions (1.3) are isomorphism for all n .*

Proof. Recall that $K_n(W)$ was defined as homology of the chain complex $C_*(A, W) = \text{Ker}((\pi \circ j)_* : C_*(W) \rightarrow C_*(A))$. We have a canonical comparison map

$$C_n(A, W) \rightarrow C((\pi \circ j)_*)_{n+1}$$

from the kernel to the mapping cone, with the indicated shift of degrees. Since the chain map $(\pi \circ j)_*$ is surjective, this comparison map induces an isomorphism on homology by the 5-lemma, and we deduce the isomorphism

$$K_n(W) \cong H_{n+1}(C((\pi \circ j)_*)).$$

Inside the mapping cone of $(\pi \circ j)_*$ we have the mapping cone of $(\pi \circ j|_B)_* : C_*(B) \rightarrow C_*(A)$, and there is the short exact sequence of chain complexes

$$C((\pi \circ j|_B)_*) \rightarrow C((\pi \circ j)_*) \rightarrow C_{*-1}(W, B)$$

inducing a long exact sequence

$$\cdots \rightarrow H_{n+1}(C((\pi \circ j|_B)_*)) \rightarrow K_n(W) \xrightarrow{(1.3)} H_n(W, B) \rightarrow H_n(C((\pi \circ j|_B)_*)) \rightarrow \cdots$$

Since the homology of the mapping cone also sit in a long exact sequence with the homomorphism (1.4), we get the asserted equivalence. \square

Given also an orientation $[W] \in H_d(W, \partial W)$ we get a Poincaré duality isomorphism

$$(1.5) \quad \begin{aligned} H^{d-n}(W, A) &\rightarrow H_n(W, B) \\ \alpha &\mapsto \alpha \cap [M] \end{aligned}$$

and it is convenient to compose (the inverse of) this isomorphism with (1.3) to get a map from homology to cohomology, which then no longer explicitly mentions B . We can also combine with the universal coefficient theorems

$$TH^{d-n}(W, A) \xrightarrow{\cong} \text{Ext}(H_{d-n-1}(W, A), \mathbb{Z}) \cong \text{Hom}(TH_{d-n-1}(W, A), \mathbb{Q}/\mathbb{Z})$$

$$H^{d-n}(W, A)/TH^{d-n}(W, A) \xrightarrow{\cong} \text{Hom}(H_{d-n}(W, A), \mathbb{Z})$$

and the isomorphism $H_*(W, A) \cong K_*(W)$ to get pairings

$$(1.6) \quad \begin{aligned} TK_n(W) \otimes TK_{d-n-1}(W) &\rightarrow \mathbb{Q}/\mathbb{Z} \\ (K_n(W)/TK_n(W)) \otimes (K_n(W)/TK_{d-n}(W)) &\rightarrow \mathbb{Z}. \end{aligned}$$

These pairings can be identified with the restrictions of the linking and intersection pairings on $H_*(W)$. Here we have written $TG = \text{Ker}(G \rightarrow G \otimes_{\mathbb{Z}} \mathbb{Q}) \subset G$ for the torsion subgroup of an abelian group G . (Exercise: construct a natural isomorphism $\text{Hom}(TG, \mathbb{Q}/\mathbb{Z}) \cong \text{Ext}(G, \mathbb{Z})$ for finitely generated abelian groups G .)

Corollary 1.2. *In the situation above, the map $j|_B : B \rightarrow A$ induces an isomorphism on integral homology if and only if both pairings (1.6) are unimodular for all n .* \square

Corollary 1.3. *If $K_n(W) \subset H_n(W)$ is a free abelian group for all n , then $(\pi \circ j|_B)_* : H_n(B) \rightarrow H_n(A)$ is an isomorphism for all n if and only if the intersection pairing on $H_*(W)$ restricts to a unimodular pairing*

$$K_n(W) \otimes K_{d-n}(W) \rightarrow \mathbb{Z}$$

for all n .

Definition 1.4. The *structure set* of the compact smooth manifold A is the set

$$S(A) = \{(M, f) \mid M \text{ compact smooth, } f : M \xrightarrow{\cong} A, f|_{\partial M} : \partial M \rightarrow \partial A \text{ diffeo.}\} / \sim,$$

where the equivalence relation $(M, f) \sim (M', f')$ when there exists a diffeomorphism $\phi : M \rightarrow M'$ such that $f' \circ \phi|_{\partial M} = f|_{\partial M}$ and $f' \sim \phi \circ f$ relative to ∂M .

As a strategy for producing elements of the structure set, we may first construct an immersed cobordism $j : W \looparrowright [0, 2] \times A$ and then restricting j to the outgoing boundary. The results above then give a criterion for this restriction to be a homology equivalence; if both A and the outgoing boundary are simply connected then homology equivalence implies homotopy equivalence by the Hurewicz theorem so we can use immersed cobordisms to produce new elements in the structure set.

2. LOCAL COEFFICIENTS

By the Hurewicz theorem, a map $A \rightarrow B$ between simply connected spaces, both homotopy equivalent to CW complexes, is a homotopy equivalence if and only if it induces an isomorphism on integral homology in all degrees. Any manifold is homotopy equivalent to a CW complex, but the assumption of simple-connectivity is a serious drawback. Fortunately, there is a stronger version of the Hurewicz theorem involving homology with *local coefficients*.

2.1. Summary of definitions. A local coefficient system on B is the assignment of an abelian group \mathcal{F}_b to each point $b \in B$ and an isomorphism $\gamma_* : \mathcal{F}_{b_0} \rightarrow \mathcal{F}_{b_1}$ to each path $\gamma : [0, 1] \rightarrow B$ with $\gamma(0) = b_0$ and $\gamma(1) = b_1$, such that paths that are homotopic relative to their endpoints give equal isomorphisms and such that concatenation of paths goes to composition of isomorphisms. In other words, a functor $\mathcal{F} : \pi_{\leq 1}(B) \rightarrow \text{Ab}$ from the fundamental groupoid of B to abelian groups. Given such an assignment, we also get an abelian group \mathcal{F}_σ for each continuous map $\sigma : \Delta^p \rightarrow B$, namely $\mathcal{F}_\sigma = \mathcal{F}_{\sigma(e_0)}$ where $e_0 = (1, 0, \dots, 0) \in \Delta^p$ is the initial vertex. If $\delta^i : \Delta^{p-1} \rightarrow \Delta^p$ is the i th face map, then we furthermore get canonical isomorphisms $\delta_i : \mathcal{F}_\sigma \cong \mathcal{F}_{\sigma \circ \delta^i}$ using any path in Δ^p between e_0 to $\delta^i(e_0)$.

Chains with coefficients in \mathcal{F} may be defined as

$$C_p(B; \mathcal{F}) = \bigoplus_{\sigma : \Delta^p \rightarrow B} \mathcal{F}_\sigma$$

$$\partial = \sum_{i=0}^p \delta_i : C_p(B; \mathcal{F}) \rightarrow C_{p-1}(B; \mathcal{F})$$

and similarly cochains

$$C^p(B; \mathcal{F}) = \prod_{\sigma : \Delta^p \rightarrow B} \mathcal{F}_\sigma$$

$$d = \sum_{i=0}^p (\delta_i)^{-1} : C^{p-1}(B; \mathcal{F}) \rightarrow C^p(B; \mathcal{F}),$$

and of course

$$H_p(B; \mathcal{F}) = H_p(C_*(B; \mathcal{F}), \partial)$$

$$H^p(B; \mathcal{F}) = H^p(C^*(B; \mathcal{F}), d)$$

Homology and cohomology relative to a subspace is defined similarly.

Functoriality works as follows. Given a coefficient system \mathcal{F} on A and a continuous map $f : B \rightarrow A$, the latter induces a morphism of fundamental groupoids $\pi_{\leq 1}(B) \rightarrow \pi_{\leq 1}(A)$ and hence a pulled-back coefficient system $f^*\mathcal{F}$ on B and there are induced homomorphisms

$$f_* : H_n(B; f^*\mathcal{F}) \rightarrow H_n(A; \mathcal{F})$$

$$f^* : H^n(A; \mathcal{F}) \rightarrow H^n(B; f^*\mathcal{F})$$

induced by analogous maps of chain complexes. In this case the Hurewicz theorem asserts

Theorem 2.1. *A continuous map $f : B \rightarrow A$ between topological spaces A and B is a weak homotopy equivalence if and only if*

- (i) *the induced map $\pi_0(B) \rightarrow \pi_0(A)$ is a bijection,*
- (ii) *the induced maps $\pi_1(B, b) \rightarrow \pi_1(A, f(b))$ is an isomorphism for all $a \in A$, and*
- (iii) *the induced maps $f_* : H_n(B; f^*\mathcal{F}) \rightarrow H_n(A; \mathcal{F})$ is an isomorphism for all coefficient systems on B .*

If A and B are homotopy equivalent to CW complexes, these conditions then also imply f is a homotopy equivalence. \square

2.2. Examples.

Example 2.2 (Orientation systems). If M is a manifold of dimension d , there is a local system ω^M

$$\omega_x^M = H_d(M, M \setminus \{x\})$$

for any $x \in M \setminus \partial M$.

Example 2.3 (Local systems from covering maps). If $f : X \rightarrow Y$ is a covering map (i.e., X is a covering space of Y), then there is local system \mathcal{F}^f with

$$\mathcal{F}_y^f = \bigoplus_{x \in f^{-1}y} \mathbb{Z}.x,$$

the free abelian group generated by the set $f^{-1}(y)$. Homology and cohomology of Y with coefficients in this local system is canonically isomorphic to homology/cohomology of Y with coefficients in \mathbb{Z} (exercise: use lifting theorems to prove that this holds even on the chain/cochain level).

Two types of covering maps deserve special mention:

- If Y is path connected and $f : X \rightarrow Y$ is a universal cover, then we obtain a “universal” local system on Y . Homology of Y with coefficients in this local system is then homology of the simply connected space X . This is the reason why the Hurewicz theorem generalizes as above: the generalization is an easy consequence of the usual Hurewicz theorem, applied to universal covers.
- If M is a manifold and $f : \widetilde{M} \rightarrow M$ is the “orientation double cover”, then there is a short exact sequence of coefficient systems

$$0 \rightarrow \omega^M \rightarrow \mathcal{F}^f \rightarrow \mathbb{Z} \rightarrow 0,$$

in which \mathbb{Z} denotes the constant coefficient system (assigning \mathbb{Z} to any point and the identity to any path).

Example 2.4 (Tensor products). If \mathcal{F} and \mathcal{G} are coefficient systems on a space B , then there is an induced coefficient system $\mathcal{F} \otimes \mathcal{G}$ whose value at $b \in B$ is $\mathcal{F}_b \otimes \mathcal{G}_b$. More generally we can form $\mathcal{F} \otimes_R \mathcal{G}$ for a ring R , when \mathcal{F} takes values in right R -modules and \mathcal{G} takes values in left R -modules.

Example 2.5 (Universal coefficients). Let X , $x \in X$ a basepoint, and write $\pi_{\leq 1}(X)(x, y)$ for the set of homotopy classes of paths in X from x to y (morphisms in the fundamental groupoid). Then there is a coefficient system $\mathcal{F}^{\text{univ}}$ whose value at $y \in X$ is $\mathbb{Z}[\pi_{\leq 1}(X)(x, y)]$, the free abelian group with basis $\pi_{\leq 1}(X)(x, y)$. It takes values in left modules over $\mathbb{Z}[\pi_1(X, x)]$, the group ring of the fundamental group. There is a canonical identification

$$H_n(X; \mathcal{F}^{\text{univ}}) \cong H_n(\widetilde{X}; \mathbb{Z}),$$

where $\tilde{X} \rightarrow X$ is the universal covering space of the path component containing $x \in X$. When X is compact, there is also a canonical identification

$$H^n(X; \mathcal{F}^{\text{univ}}) \cong H_c^n(\tilde{X}; \mathbb{Z}),$$

where the latter denotes compactly supported cohomology. Both isomorphisms can be established on the chain/cochain level.

To any right module M over $\mathbb{Z}[\pi_1(X, x)]$ there is an associated coefficient system $\mathcal{F}^M = M \otimes_{\mathbb{Z}[\pi_1(X, x)]} \mathcal{F}^{\text{univ}}$ whose value at $y \in X$ is

$$\mathcal{F}_y^M = M \otimes_{\mathbb{Z}[\pi_1(X, x)]} \mathbb{Z}[\pi_{\leq 1}(X)(x, y)].$$

When X is path connected, it can be shown that all coefficient systems arise this way.

Example 2.6 (Cup and cap products). For coefficient systems \mathcal{F} and \mathcal{G} on B , there are cup and cap products

$$\begin{aligned} H^p(B; \mathcal{F}) \otimes H^q(B; \mathcal{G}) &\rightarrow H^{p+q}(B; \mathcal{F} \otimes \mathcal{G}) \\ H^p(B; \mathcal{F}) \otimes H_q(B; \mathcal{G}) &\rightarrow H_{q-p}(B; \mathcal{F} \otimes \mathcal{G}), \end{aligned}$$

defined by analogues of the usual formulas (whenever those formulas multiply some elements in a ring, substitute tensor products).

2.3. Poincaré duality for local coefficients. Now Poincaré duality generalizes to arbitrary (not necessarily oriented) manifolds and arbitrary coefficient systems. If M is a compact d -manifold with boundary, there is a canonical fundamental class $[M] \in H_d(M, \partial M; \omega^M)$ inducing isomorphisms

$$\begin{aligned} H^p(M; \mathcal{F}) &\rightarrow H_{d-p}(M, \partial M; \mathcal{F} \otimes \omega^M) \\ H^p(M, \partial M; \mathcal{F}) &\rightarrow H_{d-p}(M; \mathcal{F} \otimes \omega^M) \\ \alpha &\mapsto \alpha \cap [M] \end{aligned}$$

and more generally

$$H^p(M, A; \mathcal{F}) \rightarrow H_{d-p}(M, B; \mathcal{F} \otimes \omega^M)$$

when $\partial M = A \cup_{\Sigma} B$ when A and B are compact submanifolds of ∂M with $\partial A = \Sigma = \partial B = A \cap B$.

Let us return to the situation studied in Section 1 with an immersed cobordism $W \looparrowright [0, 2] \times A$, and suppose given a coefficient system \mathcal{F} on A . We can pull it back along $\pi \circ j : W \rightarrow A$ to get a coefficient system on W and obtain splittings

$$\begin{aligned} H_n(W; (\pi \circ j)^* \mathcal{F}) &\cong H_n(A; \mathcal{F}) \oplus K_n(W; \mathcal{F}), \\ K_n(W; \mathcal{F}) &= \text{Ker}((\pi \circ j)_* : H_n(W; (\pi \circ j)^* \mathcal{F}) \rightarrow H_n(A; \mathcal{F})) \\ &\cong H_n(W, A; (\pi \circ j)^* \mathcal{F}) \end{aligned}$$

from the composition $A \hookrightarrow W \rightarrow A$ being the identity, by the same argument as in Section 1. Simplifying notation and writing the local system $(\pi \circ f)^* \mathcal{F}$ on W simply as \mathcal{F} , we then have the composition

$$(2.1) \quad K_n(W; \mathcal{F}) \hookrightarrow H_n(W; \mathcal{F}) \rightarrow H_n(W, B; \mathcal{F}) \xleftarrow{\cong} H^{d-n}(W, A; \mathcal{F} \otimes \omega^W),$$

completely analogous to (1.3).

Proposition 2.7. *Let $j : W \looparrowright [0, 2] \times A$ be an immersed cobordism from A to B as above, let \mathcal{F} be a coefficient system on A . Then the induced map*

$$(\pi \circ j|_B) : H_n(B; (\pi \circ j|_B)^* \mathcal{F}) \rightarrow H_n(A; \mathcal{F})$$

is an isomorphism for all n , if and only if the composition (2.1) is an isomorphism for all n .

If furthermore $\pi \circ j|_B : B \rightarrow A$ induces a bijection $\pi_0(A) \rightarrow \pi_0(B)$ and isomorphisms $\pi_1(B, b) \rightarrow \pi_1(A, \pi \circ j(b))$ for all basepoints $b \in B$, then it is a homotopy equivalence if and only if the composition (2.1) is an isomorphism for all n and all coefficient systems. (Equivalently, for the coefficient systems associated to universal covers of path components of A .)

Proof. Completely analogous to the simply-connected case, using the generalized Hurewicz theorem to deduce homotopy equivalence. \square

The assumption in the Proposition about π_0 and π_1 hold if W has dimension d and admits a handle presentation where all handles have index $\in \{3, \dots, d-3\}$, because both inclusions $A \subset W$ and $B \subset W$ will be 2-connected in this case. If there are 2-handles then it still holds provided these are trivially attached to A , and similarly if there are $(d-2)$ -handles.

2.4. Universal coefficient theorems. And now for the bad news: there is no especially good analogue of the universal coefficient theorem, and in general I don't know of a good way of rephrasing the condition that (2.1) be an isomorphism, as some pairings being unimodular. The problem is that there is in general no good formula for cohomology with local coefficients by “dualizing” homology, analogous to the universal coefficient theorem expressing $H^n(X; \mathbb{Z})$ in terms of $\text{Hom}(H_n(X), \mathbb{Z})$ and $\text{Ext}(H_{n-1}(X), \mathbb{Z})$. The weaker statement that $H^n(X; \mathbb{Z}) \cong \text{Hom}(H_n(X), \mathbb{Z})$ when $H_{n-1}(X)$ is projective as a \mathbb{Z} -module has the following analogue though.

Lemma 2.8. *If X is a topological space, $A \subset X$ is a subspace, $x \in A$ and M is a module over $\mathbb{Z}[\pi_1(X, x)]$, then there is a natural map for each $n \in \mathbb{N}$*

$$H^n(X, A; \mathcal{F}^M) \rightarrow \text{Hom}_{\mathbb{Z}[\pi_1(X, x)]}(H_n(X, A; \mathcal{F}^{\text{univ}}), M).$$

This map is an isomorphism for n , provided $H_t(X, A; \mathcal{F}^{\text{univ}})$ is a projective $\mathbb{Z}[\pi_1(X, x)]$ -module for all $t < n$.

Proof sketch. Then there is an isomorphism of cochain complexes

$$C^*(X; \mathcal{F}^M) \cong \text{Hom}_{\mathbb{Z}[\pi_1(X, x)]}(C_*(X; \mathcal{F}^{\text{univ}}), M)$$

and a corresponding spectral sequence

$$E_2^{s,t} = \text{Ext}_{\mathbb{Z}[\pi_1(X, x)]}^s(H_t(X; \mathcal{F}^{\text{univ}}), M) \Rightarrow H^{s+t}(X; \mathcal{F}^M),$$

with differentials as in the cohomology Serre spectral sequence. Here, $H_t(X; \mathcal{F}^{\text{univ}}) \cong H_t(\tilde{X}; \mathbb{Z})$ can be identified with integral homology of the universal cover of the path component containing x , on which the fundamental group acts by deck transformations.

The assumptions imply that the columns $E_2^{s,t}$ vanish for $t > 0$ and $s < n$, so we get

$$E_\infty^{0,n} = E_2^{0,n} = \text{Hom}_{\mathbb{Z}[\pi_1(X, x)]}(H_n(X; \mathcal{F}^{\text{univ}}), M),$$

and this is the only group contributing to $H^n(X; \mathcal{F}^M)$. Convergence of the spectral sequence in total degree n then gives the asserted isomorphism. \square

Using this lemma, the condition that (2.1) be an isomorphism in the case W is path connected and $a \in A \subset W$ is a basepoint, for $\mathcal{F} = \mathcal{F}^{\text{univ}}$ and all n , can be rephrased as a certain pairing on $K_*(W; \mathcal{F}^{\text{univ}})$ being unimodular, provided the $\mathbb{Z}[\pi_1(W, a)]$ -modules $K_n(W; \mathcal{F}^{\text{univ}})$ are projective for all n . This happens when $W = ([0, 1] \times A) \cup h^n \cup \dots \cup h^n \looparrowright [0, 2] \times A$ for instance: the immersion forces the attaching maps $S^{n-1} \times D^n \rightarrow \{1\} \times A$ to be null homotopic so $W \simeq A \vee \bigvee S^n$ as a CW complex, making $K_n(W; \mathcal{F}^{\text{univ}}) \cong H_n(W, A; \mathcal{F}^{\text{univ}})$ be a free module over $\mathbb{Z}[\pi_1(W, a)]$ on one generator for each n -handle.