1. Immersed cobordisms, surgery kernels, and their intersection forms

Let A and B be compact manifold with identified boundary $\partial A = \Sigma = \partial B$, so we can form a closed manifold $A \cup_\Sigma B$. Equivalently, up to diffeomorphism at least,

$$A \cup_\Sigma B \cong A \cup_{\{0\}\times\Sigma} ([0,1] \times \Sigma) \cup_{\{1\}\times\Sigma} B. \tag{1.1}$$

In this situation, a *cobordism from A to B* is a compact manifold W bounding (1.1), i.e. with a specified diffeomorphism

$$\partial W = A \cup_{\{0\}\times\Sigma} ([0,1] \times \Sigma) \cup_{\{1\}\times\Sigma} B.$$

We will mostly consider such a cobordism to be a manifold with corners,

$$\angle W = (\{0\} \times \Sigma) \sqcup (\{1\} \times \Sigma)$$

and three smooth parts $A \subset \partial W$, $[0,1] \times \Sigma \subset W$, and $B \subset W$ in the sense of Wall. Occasionally it may be convenient to straighten the corners to turn W into a manifold with smooth boundary $A \cup_\Sigma B$.

We will be particularly interested in the situation where the cobordism comes with an immersion

$$j : W \looparrowright [0,2] \times A$$

satisfying $j(x) = (0, x)$ for $a \in A \subset \partial W$ and $j(t, \sigma) = (t, \sigma)$ for $(t, \sigma) \in [0,1] \times \Sigma \subset \partial W$. The composition

$$A \hookrightarrow W \overset{j}{\looparrowright} [0,2] \times A \overset{\pi}{\twoheadrightarrow} A \tag{1.2}$$

is the identity, where π denotes the projection to the second factor. This gives a splitting of chain complexes and of homology

$$C_*(A) \oplus C_*(A, W) \overset{\cong}{\to} C_*(W)$$
$$H_n(A) \oplus K_n(W) \overset{\cong}{\to} H_n(W),$$

where

$$C_*(A, W) = \mathrm{Ker}((\pi \circ j)_* : C_*(W) \to C_*(A)) \subset C_*(W)$$
$$K_n(W) = H_n(C_*(A, W)) \cong \mathrm{Ker}(j_* : H_n(W) \to H_n([0,2] \times A)).$$

In this situation, the second factor in the splitting $H_n(W) = H_n(A) \oplus K_n(W)$ is called the *surgery kernel*. As an abstract group it is canonically isomorphic to the relative homology $H_n(W, A)$, which does not depend on the immersion j, but splitting the map $H_n(W) \to H_n(W, A)$ and realizing the surgery kernel as a summand of $H_n(W)$ does.

We will be interested in the composition

$$K_n(W) \hookrightarrow H_n(W) \to H_n(W, B) \tag{1.3}$$

and will relate it to the composition

$$(\pi \circ j_{|B})_* : H_n(B) \to H_n(A) \tag{1.4}$$

Here, $\pi \circ j_{|B}$ is the composition

$$B \hookrightarrow W \overset{j}{\looparrowright} [0,2] \times A \overset{\pi}{\twoheadrightarrow} A,$$

which is a map of manifolds with boundary and by construction it restricts to a diffeomorphism $\partial B \cong \partial A$. To detect whether it is a homotopy equivalence we may, at least if A and B are both simply connected, look for whether it induces an isomorphism on integral homology.

Proposition 1.1. *The homomorphisms (1.4) are isomorphisms for all n if and only if the compositions (1.3) are isomorphism for all n.*

Proof. Recall that $K_n(W)$ was defined as homology of the chain complex $C_*(A,W) = \mathrm{Ker}((\pi \circ j)_* : C_*(W) \to C_*(A))$. We have a canonical comparison map

$$C_n(A,W) \to C((\pi \circ j)_*)_{n+1}$$

from the kernel to the mapping cone, with the indicated shift of degrees. Since the chain map $(\pi \circ j)_*$ is surjective, this comparison map induces an isomorphism on homology by the 5-lemma, and we deduce the isomorphism

$$K_n(W) \cong H_{n+1}(C((\pi \circ j)_*)).$$

Inside the mapping cone of $(\pi \circ j)_*$ we have the mapping cone of $(\pi \circ j_{|B})_* : C_*(B) \to C_*(A)$, and there is the short exact sequence of chain complexes

$$C((\pi \circ j_{|B})_*) \to C((\pi \circ j)_*) \to C_{*-1}(W,B)$$

inducing a long exact sequence

$$\cdots \to H_{n+1}(C((\pi \circ j_{|B})_*)) \to K_n(W) \xrightarrow{(1.3)} H_n(W,B) \to H_n(C((\pi \circ j_{|B})_*)) \to \dots.$$

Since the homology of the mapping cone also sit in a long exact sequence with the homomorphism (1.4), we get the asserted equivalence. □

Given also an orientation $[W] \in H_d(W, \partial W)$ we get a Poincaré duality isomorphism

$$\begin{aligned} H^{d-n}(W,A) &\to H_n(W,B) \\ \alpha &\mapsto \alpha \cap [M] \end{aligned} \tag{1.5}$$

and it is convenient to compose (the inverse of) this isomorphism with (1.3) to get a map from homology to cohomology, which then no longer explicitly mentions B. We can also combine with the universal coefficient theorems

$$TH^{d-n}(W,A) \xrightarrow{\cong} \mathrm{Hom}(TH_{d-n-1}(W,A), \mathbb{Q}/\mathbb{Z})$$
$$H^{d-n}(W,A)/TH^{d-n}(W,A) \xrightarrow{\cong} \mathrm{Hom}(H_{d-n}(W,A), \mathbb{Z})$$

and the isomorphism $H_*(W,A) \cong K_*(W)$ to get pairings

$$\begin{aligned} TK_n(W) \otimes TK_{d-n-1}(W) &\to \mathbb{Q}/\mathbb{Z} \\ (K_n(W)/TK_n(W)) \otimes (K_n(W)/TK_{d-n}(W)) &\to \mathbb{Z}. \end{aligned} \tag{1.6}$$

These pairings can be identified with the restrictions of the linking and intersection pairings on $H_*(W)$. Here we have written $TG = \mathrm{Ker}(G \to G \otimes_{\mathbb{Z}} \mathbb{Q}) \subset G$ for the torsion subgroup of an abelian group G. (Exercise: construct a natural isomorphism $\mathrm{Ext}(G,\mathbb{Z}) \cong \mathrm{Hom}(TG, \mathbb{Q}/\mathbb{Z})$ for finitely generated abelian groups G, convenient for the universal coefficient theorem).

Corollary 1.2. *In the situation above, the map $j_{|B} : B \to A$ induces an isomorphism on integral homology if and only if both pairings (1.6) are unimodular for all n.* □

Corollary 1.3. *If $K_n(W) \subset H_n(W)$ is a free abelian group for all n, then $(\pi \circ j_{|B})_* : H_n(B) \to H_n(A)$ is an isomorphism for all n if and only if the intersection pairing on $H_*(W)$ restricts to a unimodular pairing*

$$K_n(W) \otimes K_{d-n}(W) \to \mathbb{Z}$$

for all n.

Definition 1.4. The *structure set* of the compact smooth manifold A is the set

$$S(A) = \{(M,f) \mid M \text{ compact smooth, } f : M \xrightarrow{\simeq} A,\ f_{|\partial M} : \partial M \to \partial A \text{ diffeo.}\}/\sim,$$

where the equivalence relation $(M,f) \sim (M',f')$ when there exists a diffeomorphism $\phi : M \to M'$ such that $f' \circ \phi_{|\partial M} = f_{|\partial M}$ and $f' \sim \phi \simeq f$ relative to ∂M.

As a strategy for producing elements of the structure set, we may first construct an immersed cobordism $j : W \looparrowright [0,2] \times A$ and then restricting j to the outgoing boundary. The results above then give a criterion for this restriction to be a homology equivalence; if both A and the outgoing boundary are simply connected then homology equivalence implies homotopy equivalence by the Hurewicz theorem so we can use immersed cobordisms to produce new elements in the structure set.

2. Local coefficients

By the Hurewicz theorem, a map $A \to B$ between simply connected spaces, both homotopy equivalent to CW complexes, is a homotopy equivalence if and only if it induces an isomorphism on integral homology in all degrees. Any manifold is homotopy equivalent to a CW complex, but the assumption of simple-connectivity is a serious drawback. Fortunately, there is a stronger version of the Hurewicz theorem involving homology with *local coefficients*.

2.1. **Summary of definitions.** A local coefficient system on B is the assignment of an abelian group $\mathcal{F}_b$ to each point $b \in B$ and an isomorphism $\gamma_* : \mathcal{F}_{b_0} \to \mathcal{F}_{b_1}$ to each path $\gamma : [0,1] \to B$ with $\gamma(0) = b_0$ and $\gamma(1) = b_1$, such that paths that are homotopic relative to their endpoints give equal isomorphisms and such that concatenation of paths goes to composition of isomorphisms. In other words, a functor $\mathcal{F} : \pi_{\leq 1}(B) \to \mathrm{Ab}$ from the fundamental groupoid of B to abelian groups. Given such an assignment, we also get an abelian group $\mathcal{F}_\sigma$ for each continuous map $\sigma : \Delta^p \to B$, namely $\mathcal{F}_\sigma = \mathcal{F}_{\sigma(e_0)}$ where $e_0 = (1, 0, \dots, 0) \in \Delta^p$ is the initial vertex. If $\delta^i : \Delta^{p-1} \to \Delta^p$ is the ith face map, then we furthermore get canonical isomorphisms $\delta_i : \mathcal{F}_\sigma \cong \mathcal{F}_{\sigma \circ \delta^i}$ using any path in Δ^p between e_0 to $\delta^i(e_0)$.

Chains with coefficients in $\mathcal{F}$ may be defined as

$$C_p(B; \mathcal{F}) = \bigoplus_{\sigma : \Delta^p \to B} \mathcal{F}_\sigma$$

$$\partial = \sum_{i=0}^{p} \delta_i : C_p(B; \mathcal{F}) \to C_{p-1}(B; \mathcal{F})$$

and similarly cochains

$$C^p(B; \mathcal{F}) = \prod_{\sigma : \Delta^p \to B} \mathcal{F}_\sigma$$

$$d = \sum_{i=0}^{p} (\delta_i)^{-1} : C^{p-1}(B; \mathcal{F}) \to C^p(B; \mathcal{F}),$$

and of course

$$H_p(B; \mathcal{F}) = H_p(C_*(B; \mathcal{F}), \partial)$$
$$H^p(B; \mathcal{F}) = H^p(C^*(B; \mathcal{F}), d)$$

Homology and cohomology relative to a subspace is defined similarly.

Functoriality works as follows. Given a coefficient system $\mathcal{F}$ on A and a continuous map $f : B \to A$, the latter induces a morphism of fundamental groupoids $\pi_{\leq 1}(B) \to \pi_{\leq 1}(A)$ and hence a pulled-back coefficient system $f^*\mathcal{F}$ on B and there are induced homomorphisms

$$f_* : H_n(B; f^*\mathcal{F}) \to H_n(A; \mathcal{F})$$
$$f^* : H^n(A; \mathcal{F}) \to H^n(B; f^*\mathcal{F})$$

induced by analogous maps of chain complexes. In this case the Hurewicz theorem asserts

Theorem 2.1. *A continuous map $f : B \to A$ between topological spaces A and B is a weak homotopy equivalence if and only if*

(i) the induced map $\pi_0(B) \to \pi_0(A)$ is a bijection,
(ii) the induced maps $\pi_1(B, b) \to \pi_1(A, f(b))$ is an isomorphism for all $a \in A$, and
(iii) the induced maps $f_* : H_n(B; f^*\mathcal{F}) \to H_n(A; \mathcal{F})$ is an isomorphism for all coefficient systems on B.

If A and B are homotopy equivalent to CW complexes, these conditions then also imply f is a homotopy equivalence. □

2.2. **Examples.**

Example 2.2 (Orientation systems)**.** If M is a manifold of dimension d, there is a local system ω^M

$$\omega^M_x = H_d(M, M \setminus \{x\})$$

for any $x \in M \setminus \partial M$.

Example 2.3 (Local systems from covering maps)**.** If $f : X \to Y$ is a covering map (i.e., X is a covering space of Y), then there is local system $\mathcal{F}^f$ with

$$\mathcal{F}^f_y = \bigoplus_{x \in f^{-1}y} \mathbb{Z}.x,$$

the free abelian group generated by the set $f^{-1}(y)$. Homology and cohomology of Y with coefficients in this local system is canonically isomorphic to homology/cohomology of Y with coefficients in $\mathbb{Z}$ (exercise: use lifting theorems to prove that this holds even on the chain/cochain level).

Two types of covering maps deserve special mention:

- If Y is path connected and $f : X \to Y$ is a universal cover, then we obtain a "universal" local system on Y. Homology of Y with coefficients in this local system is then homology of the simply connected space X. This is the reason why the Hurewicz theorem generalizes as above: the generalization is an easy consequence of the usual Hurewicz theorem, applied to universal covers.
- If M is a manifold and $f : \widetilde{M} \to M$ is the "orientation double cover", then there is a short exact sequence of coefficient systems

  $$0 \to \omega^M \to \mathcal{F}^f \to \mathbb{Z} \to 0,$$

  in which $\mathbb{Z}$ denotes the constant coefficient system (assigning $\mathbb{Z}$ to any point and the identity to any path).

Example 2.4 (Tensor products)**.** If $\mathcal{F}$ and $\mathcal{G}$ are coefficient systems on a space B, then there is an induced coefficient system $\mathcal{F} \otimes \mathcal{G}$ whose value at $b \in B$ is $\mathcal{F}_b \otimes \mathcal{G}_b$. More generally we can form $\mathcal{F} \otimes_R \mathcal{G}$ for a ring R, when $\mathcal{F}$ takes values in right R-modules and $\mathcal{G}$ takes values in left R-modules.

Example 2.5 (Universal coefficients)**.** Let X, $x \in X$ a basepoint, and write $\pi_{\leq 1}(X)(x, y)$ for the set of homotopy classes of paths in X from x to y (morphisms in the fundamental groupoid). Then there is a coefficient system $\mathcal{F}^{\text{univ}}$ whose value at $y \in X$ is $\mathbb{Z}[\pi_{\leq 1}(X)(x, y)]$, the free abelian group with basis $\pi_{\leq 1}(X)(x, y)$. It takes values in left modules over $\mathbb{Z}[\pi_1(X, x)]$, the group ring of the fundamental group. There is a canonical identification

$$H_n(X; \mathcal{F}^{\text{univ}}) \cong H_n(\widetilde{X}; \mathbb{Z}),$$

where $\tilde{X} \to X$ is the universal covering space of the path component containing $x \in X$. When X is compact, there is also a canonical identification

$$H^n(X; \mathcal{F}^{\text{univ}}) \cong H^n_c(\widetilde{X}; \mathbb{Z}),$$

where the latter denotes compactly supported cohomology. Both isomorphisms can be established on the chain/cochain level.

To any right module M over $\mathbb{Z}[\pi_1(X,x)]$ there is an associated coefficient system $\mathcal{F}^M = M \otimes_{\mathbb{Z}[\pi_1(X,x)]} \mathcal{F}^{\text{univ}}$ whose value at $y \in X$ is

$$\mathcal{F}^M_y = M \otimes_{\mathbb{Z}[\pi_1(X,x)]} \mathbb{Z}[\pi_{\leq 1}(X)(x,y)].$$

When X is path connected, it can be shown that all coefficient systems arise this way.

Example 2.6 (Cup and cap products)**.** For coefficient systems $\mathcal{F}$ and $\mathcal{G}$ on B, there are cup and cap products

$$\begin{aligned} H^p(B;\mathcal{F}) \otimes H^q(B;\mathcal{G}) &\to H^{p+q}(B;\mathcal{F}\otimes\mathcal{G}) \\ H^p(B;\mathcal{F}) \otimes H_q(B;\mathcal{G}) &\to H_{q-p}(B;\mathcal{F}\otimes\mathcal{G}), \end{aligned}$$

defined by analogues of the usual formulas (whenever those formulas multiply some elements in a ring, substitute tensor products).

2.3. **Universal coefficient theorems.** And now for the bad news: there is no especially good analogue of the universal coefficient theorem, and hence no general formula for cohomology with local coefficients by “dualizing” homology, analogous to the universal coefficient theorem expressing $H^n(X;\mathbb{Z})$ in terms of $\operatorname{Hom}(H_n(X),\mathbb{Z})$ and $\operatorname{Ext}(H_{n-1}(X),\mathbb{Z})$. The weaker statement that $H^n(X;\mathbb{Z}) \cong \operatorname{Hom}(H_n(X),\mathbb{Z})$ when $H_{n-1}(X)$ is projective as a $\mathbb{Z}$-module has the following analogue though.

Lemma 2.7. *If X is a topological space, $A \subset X$ is a subspace, $x \in A$ and M is a module over $\mathbb{Z}[\pi_1(X,x)]$, then there is a natural map for each $n \in \mathbb{N}$*

$$H^n(X,A;\mathcal{F}^M) \to \operatorname{Hom}_{\mathbb{Z}[\pi_1(X,x)]}(H_n(X,A;\mathcal{F}^{\text{univ}}), M).$$

This map is an isomorphism for n, provided $H_t(X,A;\mathcal{F}^{\text{univ}})$ is a projective $\mathbb{Z}[\pi_1(X,x)]$-module for all $t < n$. In particular this holds when the inclusion $A \hookrightarrow X$ is $(n-1)$-connected.

Proof sketch. Then there is an isomorphism of cochain complexes

$$C^*(X;\mathcal{F}^M) \cong \operatorname{Hom}_{\mathbb{Z}[\pi_1(X,x)]}(C_*(X;\mathcal{F}^{\text{univ}}), M)$$

and a corresponding spectral sequence

$$E_2^{s,t} = \operatorname{Ext}^s_{\mathbb{Z}[\pi_1(X,x)]}(H_t(X;\mathcal{F}^{\text{univ}}), M) \Rightarrow H^{s+t}(X;\mathcal{F}^M),$$

with differentials as in the cohomology Serre spectral sequence. Here, $H_t(X;\mathcal{F}^{\text{univ}}) \cong H_t(\widetilde{X};\mathbb{Z})$ can be identified with integral homology of the universal cover of the path component containing x, on which the fundamental group acts by deck transformations.

The assumptions imply that the columns $E_2^{s,t}$ vanish for $t > 0$ and $s < n$, so we get

$$E_\infty^{0,n} = E_2^{0,n} = \operatorname{Hom}_{\mathbb{Z}[\pi_1(X,x)]}(H_n(X;\mathcal{F}^{\text{univ}}), M),$$

and this is the only group contributing to $H^n(X;\mathcal{F}^M)$. Convergence of the spectral sequence in total degree n then gives the asserted isomorphism. □

3. The intersection pairing with local coefficients

3.1. **Poincaré duality for local coefficients.** Now Poincaré duality generalizes to arbitrary (not necessarily oriented) manifolds and arbitrary coefficient systems. If M is a compact d-manifold with boundary and ω is the orientation local system from Example 2.2, there is a canonical fundamental class $[M] \in H_d(M, \partial M; \omega)$ inducing isomorphisms

$$\begin{aligned} H^p(M; \mathcal{F}) &\to H_{d-p}(M, \partial M; \mathcal{F} \otimes \omega) \\ H^p(M, \partial M; \mathcal{F}) &\to H_{d-p}(M; \mathcal{F} \otimes \omega) \\ \alpha &\mapsto \alpha \cap [M] \end{aligned}$$

and more generally

$$H^p(M, A; \mathcal{F}) \xrightarrow{\cong} H_{d-p}(M, B; \mathcal{F} \otimes \omega) \tag{3.1}$$

when $\partial M = A \cup_\Sigma B$ when A and B are compact submanifolds of ∂M with $\partial A = \Sigma = \partial B = A \cap B$, and $\mathcal{F}$ is any local system on M.

3.2. **Poincaré duality and immersed cobordisms.** Let us return to the situation studied in Section 1 with an immersed cobordism $W \looparrowright [0,2] \times A$, and suppose given a coefficient system $\mathcal{F}$ on A. We can pull it back along $\pi \circ j : W \to A$ to get a coefficient system on W and obtain splittings

$$\begin{aligned} H_n(W; (\pi \circ j)^* \mathcal{F}) &\cong H_n(A; \mathcal{F}) \oplus K_n(W; \mathcal{F}), \\ K_n(W; \mathcal{F}) &= \mathrm{Ker}((\pi \circ j)_* : H_n(W; (\pi \circ j)^* \mathcal{F}) \to H_n(A; \mathcal{F}) \\ &\cong H_n(W, A; (\pi \circ j)^* \mathcal{F}) \end{aligned}$$

from the composition $A \hookrightarrow W \to A$ being the identity, by the same argument as in Section 1. Simplifying notation and writing the local system $(\pi \circ f)^* \mathcal{F}$ on W simply as $\mathcal{F}$, we then have the composition

$$K_n(W; \mathcal{F}) \hookrightarrow H_n(W; \mathcal{F}) \to H_n(W, B; \mathcal{F}) \xleftarrow{\cong} H^{d-n}(W, A; \mathcal{F} \otimes \omega^W), \tag{3.2}$$

completely analogous to (1.3).

Proposition 3.1. *Let $j : W \looparrowright [0,2] \times A$ be an immersed cobordism from A to B as above, let $\mathcal{F}$ be a coefficient system on A. Then the induced map*

$$(\pi \circ j_{|B}) : H_n(B; (\pi \circ j_{|B})^* \mathcal{F}) \to H_n(A; \mathcal{F})$$

is an isomorphism for all n, if and only if the composition (3.2) is an isomorphism for all n.

If furthermore $\pi \circ j_{|B} : B \to A$ induces a bijection $\pi_0(A) \to \pi_0(B)$ and isomorphisms $\pi_1(B, b) \to \pi_1(A, \pi \circ j(b))$ for all basepoints $b \in B$, then it is a homotopy equivalence if and only if the composition (3.2) is an isomorphism for all n and all coefficient systems. (Equivalently, for the coefficient systems associated to universal covers of path components of A.)

Proof. Completely analogous to the simply-connected case, using the generalized Hurewicz theorem to deduce homotopy equivalence. □

The assumption in the second part of the Proposition about π_0 and π_1 hold if W has dimension d and admits a handle presentation where all handles have index $\in \{3, \dots, d-3\}$, because both inclusions $A \subset W$ and $B \subset W$ will be 2-connected in this case. In the case of immersed cobordisms $W \looparrowright [0,2] \times A$, the immersion ensures that the attaching maps for the handles are null-homotopic and hence attaching them does not change the fundamental group either (i.e., the inclusion $A \hookrightarrow W$ induces an isomorphism on fundamental groups). In the immersed case it therefore suffices to know all handles have index $\in \{2, \dots, d-3\}$.

3.3. **The intersection pairing, cohomological interpretation.** Setting $B = \emptyset$ in (3.1) and replacing $\mathcal{F}$ by $\mathcal{F}^{\mathrm{univ}} \otimes \omega$ for some basepoint $m_0 \in M$, we can combine the inverse isomorphism

$$H_{d-p}(M; \mathcal{F}^{\mathrm{univ}}) \xrightarrow{\cong} H^p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega)$$

with the "universal coefficient" homomorphism in Lemma 2.7. The composition can be interpreted as a pairing

$$H_{d-p}(M; \mathcal{F}^{\mathrm{univ}}) \otimes H_p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \to \mathbb{Z}[\pi_1(M, m_0)]$$

which is the intersection pairing on homology with local coefficients. It can in turn be interpreted, using Poincaré duality again, as a pairing

$$H^p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \otimes H^{d-p}(M; \mathcal{F}^{\mathrm{univ}}) \to \mathbb{Z}[\pi_1(M, m_0)]. \tag{3.3}$$

The analogy of (3.3) for constant coefficients has another useful description, namely $\alpha \otimes \beta \mapsto (\alpha \cup \beta)([M])$, the cup product evaluated on the fundamental class. The pairing (3.3) has a similar formula as cup product evaluated on the fundamental class, which we now explain in a few steps. Start with the the composition in the fundamental groupoid

$$\begin{aligned} \pi_{\leq 1}(M)(m_0, m) \times \pi_{\leq 1}(M)(m_0, m) &\to \pi_1(M, m_0) \\ ([\lambda], [\rho]) &\mapsto [\lambda * \rho^{-1}]. \end{aligned} \tag{3.4}$$

Here we regard m_0 as fixed and $m \in \pi_{\leq 1}(M)$ as varying over objects. This can then be viewed as a defining a natural transformation of functors $\pi_{\leq 1}(M) \to \mathrm{Sets}$, whose codomain is the constant functor assigning $\pi_1(M, m_0)$ to every $m \in M$. Taking free abelian group then gives a map of coefficient systems

$$\mathcal{F}^{\mathrm{univ}} \otimes \mathcal{F}^{\mathrm{univ}} \to \mathbb{Z}[\pi_1(M, m_0)]$$

whose codomain is the constant coefficient system assigning $\mathbb{Z}[\pi_1(M, m_0)]$ to each $m \in M$. Now tensor with the orientation local system ω and combine with cup product to get

$$\begin{aligned} H^p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \times H^{d-p}(M; \mathcal{F}^{\mathrm{univ}}) &\xrightarrow{\cup} H^d(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \mathcal{F}^{\mathrm{univ}} \otimes \omega) \\ \to H^d(M, \partial M; \mathbb{Z}[\pi_1(M, m_0)] \otimes \omega) &\xleftarrow{\cong} H^d(M, \partial M; \omega) \otimes \mathbb{Z}[\pi_1(M, m_0)], \end{aligned} \tag{3.5}$$

where in the last step we used that $\mathbb{Z}[\pi_1(M, m_0)]$ is a free $\mathbb{Z}$-module to see that there is no Tor term. Finally, compose with the homomorphism

$$H^d(M, \partial M; \omega) \to \mathbb{Z}$$

given by evaluating on $[M] \in H_d(M, \partial M; \omega)$ to obtain the desired pairing

$$\begin{aligned} H^p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \otimes H^{d-p}(M; \mathcal{F}^{\mathrm{univ}}) &\to \mathbb{Z}[\pi_1(M, m_0)] \\ \alpha \otimes \beta &\mapsto (\alpha \cup \beta)([M]), \end{aligned}$$

which agrees with (3.3).

We finish this subsection with two remarks about the algebraic properties satisfied by this pairing.

Remark 3.2. The group $\pi = \pi_1(M, m_0)$ acts on "everything in sight" on this pairing, but some care with signs and left/right actions is necessary. Firstly, $\pi_1(M, m_0)$ acts as automorphisms of the coefficient system $\mathcal{F}^{\mathrm{univ}} : y \mapsto \mathbb{Z}\pi_{\leq 1}(M)(m_0, y)$ by pre-composing with elements of $\pi_1(M, m_0) = \pi_{\leq 1}(M)(m_0, m_0)$ in the fundamental groupoid. Since taking cohomology is functorial with respect to morphisms of coefficient systems, we obtain a π-action on cohomology (relative or absolute) with coefficients in $\mathcal{F}^{\mathrm{univ}}$. The behavior of (3.3) with respect to this action is inherited from (3.4): acting by $[\lambda] \in \pi_1(M, m_0)$ on the left factor in the domain of the pairing corresponds to left multiplication by the same element $[\lambda]$ in the codomain, but acting on the right factor corresponds to right multiplication by $[\lambda]^{-1}$ in the codomain.

If we write the pairing (3.3) as $\alpha \otimes \beta \mapsto \langle \alpha, \beta \rangle$, then we have the formulas

$$\langle g.\alpha, \beta \rangle = g.\langle \alpha, \beta \rangle$$
$$\langle \alpha, g.\beta \rangle = \langle \alpha, \beta \rangle.g^{-1}$$

Remark 3.3. The pairing (3.3) is between relative cohomology and absolute cohomology, and also with not quite equal coefficients. As it stands, it therefore does not quite make sense to ask about symmetry propertiesof the pairing—how $\langle \alpha, \beta \rangle$ relates to $\langle \beta, \alpha \rangle$—simply because these notations are not simultaneously defined. With some slight modification we can get a pairing in which α and β is the same type of object, let us briefly explain this.

Firstly, the local coefficient systems $\mathcal{F}^{\mathrm{univ}}$ and $\mathcal{F}^{\mathrm{univ}} \otimes \omega$ are in fact isomorphic: choosing once and for all a generator of $\omega_{m_0} = H_d(M, M \setminus \{m_0\}; \mathbb{Z})$, we can use a path $[\lambda] \in \pi_{\leq 1}(M)(m_0, m)$ from m_0 to $m \in M$ to propagate this generator to a generator of ω_m. This defines an isomorphism of local systems

$$\mathcal{F}^{\mathrm{univ}} \to \mathcal{F}^{\mathrm{univ}} \otimes \omega. \tag{3.6}$$

This isomorphism is not necessarily compatible with the $\pi_1(M, m_0)$ action, unless that action fixes the chosen generator of $H_d(M, M \setminus \{m_0\})$. In general, the action of $\pi_1(M, m_0)$ on $H_d(M, M \setminus \{m_0\}) \cong \mathbb{Z}$ is encoded by a homomorphism

$$\pi_1(M, m_0) \xrightarrow{w} \mathbb{Z}^\times,$$

sometimes called the orientation character (or the "first Stiefel–Whitney class"). If we temporarily denote the isomorphism (3.6) by ϕ, then its relationship with the action of $g \in \pi_1(M, m_0)$ is

$$\phi(g.\alpha) = w(g)g.\phi(\alpha)$$

The pairing (3.3) is asymmetric in that it pairs relative cohomology with absolute cohomology, and also with slightly different coefficients $\mathcal{F}^{\mathrm{univ}}$ versus $\mathcal{F}^{\mathrm{univ}} \otimes \omega$. If we use the isomorphism (3.6) and compose with the restriction map $H^*(M, \partial M; \mathcal{F}^{\mathrm{univ}}) \to H^*(M; \mathcal{F}^{\mathrm{univ}})$, we obtain a more symmetric looking pairing

$$H^p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \otimes H^{d-p}(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \to \mathbb{Z}[\pi_1(M, m_0)] \tag{3.7}$$

which will now satisfy

$$\langle g.\alpha, \beta \rangle = g.\langle \alpha, \beta \rangle$$
$$\langle \alpha, g.\beta \rangle = w(g)\langle \alpha, \beta \rangle.g^{-1}.$$

The symmetry properties of the pairing (3.7) can be phrased by equations reminiscent of Hermitian inner products on complex vector spaces, if we introduce the notation $x \mapsto \overline{x}$ for the anti-involution on the group ring $\Lambda = \mathbb{Z}[\pi_1(M, m_0)]$ given by

$$\overline{\sum_{g \in \pi_1(M, m_0)} a_g.g} = \sum_{g \in \pi_1(M, m_0)} w(g) a_g.g^{-1}.$$

(The "anti" in anti-involution refers to the property that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$, so it defines a ring isomorphism $\Lambda \to \Lambda^{\mathrm{op}}$.) With this notation, the pairing (3.7) on the Λ-modules $H^*(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \cong H_*(M; \mathcal{F}^{\mathrm{univ}})$ satisfy

$$\alpha \mapsto \langle \alpha, \beta \rangle \text{ is } \Lambda\text{-linear for fixed } \beta$$

and the symmetry property

$$\langle \beta, \alpha \rangle = (-1)^d \overline{\langle \alpha, \beta \rangle}.$$

For compatibility with Wall's book *Surgery on Compact Manifolds*, we may write $\lambda(\alpha, \beta) = \langle \beta, \alpha \rangle$. The above properties are then part of his theorem 5.2.

3.4. **The intersection pairing with local coefficients.** Returning to the situation of an immersed cobordism $j : W \looparrowright [0, 2] \times A$ from $\partial_- W = A$ to $\partial_+ W = B$, and a local coefficient system $\mathcal{F}$ on A, we have seen that the restriction

$$\pi \circ j_{|B} : B \to A$$

induces an isomorphism on homology with coefficients in $\mathcal{F}$ in all degrees, if and only if a certain homomorphism

$$H_n(W, A; \mathcal{F}) \cong K_n(W; \mathcal{F}) \hookrightarrow H^{d-n}(W, A; \mathcal{F}) \tag{3.8}$$

is an isomorphism for all n. As above, we have used the same notation $\mathcal{F}$ for the pullback of the local system to W, along $\pi \circ j : W \to A$.

Let us now choose a basepoint $a_0 \in A$ and consider the corresponding "universal" local system $\mathcal{F}^{\mathrm{univ}} : a \mapsto \mathbb{Z}[\pi_{\leq 1}(a_0, a)]$ on A. In this case we may compose (3.8), which is a homomorphism from homology to cohomology, with the homomorphism from Lemma 2.7 to get a pairing on homology of the form

$$H_n(W, A; \mathcal{F}^{\mathrm{univ}}) \times H_{d-n}(W, A; \mathcal{F}^{\mathrm{univ}}) \to \mathbb{Z}[\pi_1(A, a_0)],$$

which we can also interpret as a pairing on the "universal" surgery kernel $K_*(W; \mathcal{F}^{\mathrm{univ}})$. It is obtained by restricting a similar pairing on $H_*(W; \mathcal{F}^{\mathrm{univ}})$.

For a general cobordism W, the deficiencies of the universal coefficient theorem for local coefficients due to higher Ext groups mean that we cannot easily reinterpret the condition of (3.8) being an isomorphism as a property of this pairing on the surgery kernel. In case Lemma 2.7 applies,

Corollary 3.4. *Let $W \looparrowright [0, 2] \times A$ be an immersed cobordism from A to B, and assume $d = \dim(W) = 2n$ for some $n \geq 3$ and that*

$$W = ([0, 1] \times A) \cup h^h \cup \cdots \cup h^n.$$

Then the following are equivalent:

- *the restriction $\pi \circ j_{|B} : B \to A$ is a homotopy equivalence,*
- *the intersection pairing*
$$H_n(W, A; \mathcal{F}^{\mathrm{univ}}) \times H_n(W, A; \mathcal{F}^{\mathrm{univ}}) \to \mathbb{Z}[\pi_1(A, a)]$$
is unimodular for any $a \in A$ (or just one in each path component).

Proof sketch. In Proposition 3.1, we have seen in general that the first condition is equivalent to a certain homomorphism

$$H_k(W, A; \mathcal{F}^{\mathrm{univ}}) \to H^{2n-k}(W, A; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \tag{3.9}$$

being an isomorphism for all k. For W of the indicated form, both domain and codomain vanish for $k \neq n$, so this condition is non-vacuous only for $k = n$, in which case we can use the universal coefficient theorem to rewrite $H^n(W, A; \mathcal{F}^{\mathrm{univ}} \otimes \omega)$ as $\mathrm{Hom}_{\mathbb{Z}[\pi_1(A,a)]}(H_n(W, A; \mathcal{F}^{\mathrm{univ}} \otimes \omega), \mathbb{Z}[\pi_1(A, a)])$; in other words, "cohomology is linear dual to cohomology" in this case. As discussed above, the resulting homomorphism into $\mathrm{Hom}(\dots)$ is adjoint to the intersection pairing on the $\mathbb{Z}[\pi_1(A, a)]$-module $H_n(W, A; \mathcal{F}^{\mathrm{univ}})$, which is then unimodular if and only if (3.9) is an isomorphism. $\square$

The intended application of this Corollary is to produce interesting elements of the structure set of A, similar to the way the Milnor sphere is constructed starting from the $E8$ pairing on $\mathbb{Z}^8$. To carry this out in practice, we need some understanding of bilinear forms over the group ring $\mathbb{Z}[\pi]$, where $\pi = \pi_1(A, a_0)$, good enough that we can at least write down interesting unimodular pairings algebraically. On a free module, such pairings are given by an intersection matrix $M \in M_k(\mathbb{Z}[\pi_1(A,a)])$. We can then try to choose the attaching map

$$\coprod^k S^{n-1} \times D^n \hookrightarrow \{1\} \times A$$

for the n-handles of an immersed cobordism $W = ([0,1] \times A) \cup h^n \cup \cdots \cup h^n \looparrowright [0,2] \times A$ to have some specified intersection matrix $M \in M_k(\mathbb{Z}[\pi_1(A,a)])$. If this is possible and M is invertible, then $\pi \circ j_{|B} : B \to A$ will be a homotopy equivalence and hence give an element of the structure set of A.

4. The intersection pairing, geometric interpretation

In the case of constant coefficients, we have a geometric interpretation of the intersection pairing on the middle homology of a $2n$-dimensional oriented manifold W, at least for classes represented by maps $S^n \looparrowright W$: if $x, y \in H_n(W)$ are given as $x = f_*([S^n])$ and $y = g_*([S^n])$, then $x \bullet y \in \mathbb{Z}$ can be calculated by counting—with appropriate signs—the points in the finite set $f(S^n) \cap g(S^n)$ after homotoping f and g to transverse immersions. Then $z \in f(S^n) \cap g(S^n)$ corresponds to a point in the fiber product

$$S^n{}_f\times_g S^n = \{(z', z'') \in S^n \times S^n \mid f(z') = g(z'')\}, \tag{4.1}$$

which is a finite set. The sign associated to (z', z'') is determined by a local procedure: if $z = f(z') = g(z'')$ then we first choose a chart $c : \mathbb{R}^{2n} \to W$ sending $0 \in \mathbb{R}^{2n}$ to z, oriented in the sense that the generator $[W] \in H_{2n}(W, W \setminus \{z\})$ corresponds to the canonical generator of $H_{2n}(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \{0\})$, and such that the intersection of $f(S^n)$ and $g(S^n)$ looks like $\mathbb{R}^n \times \{0\}$ intersecting $\{0\} \times \mathbb{R}^n$. Then $f_*([S^n]) \in H_n(W) \cong H^n(W, \partial W)$ canonically lifts to a class in $H^n(W, W \setminus f(S^n))$ which under the chart pulls back to a class

$$c^*(x) \in H^n(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \mathbb{R}^n \times \{0\}) \xleftarrow{\cong} H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}),$$

where the isomorphism is induced by the projection $\mathbb{R}^{2n} \to \mathbb{R}^n$ onto the first n coordinates. This group $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ has a canonical generator, the nth power with respect to cup product of a chosen generator of $H^1(\mathbb{R}, \mathbb{R} \setminus \{0\})$, and $c^*(x)$ will agree with either that generator or its negative. That sign depends on whether the local diffeomorphism map $f^{-1} \circ c_{|\mathbb{R}^n \times \{0\}} : \mathbb{R}^n \to S^n$ is orientation preserving or not. Similarly, $c^*(y) \in H^n(\mathbb{R}^{2n}, \mathbb{R}^{2n} \setminus \{0\} \times \mathbb{R}^n)$ is either the canonical generator or its negative; the product of the two signs is then the sign attached to the intersection point $z \in f(S^n) \pitchfork g(S^n)$, and the sum of these signs gives the intersection product $x \bullet y \in \mathbb{Z}$.

For a general path connected, but not necessarily simply connected or oriented, $2n$-dimensional manifold W with chosen base point w_0, an entirely similar interpretation can be given of the intersection form

$$H_n(W; \mathcal{F}^{\text{univ}}) \otimes H_n(W; \mathcal{F}^{\text{univ}}) \to \mathbb{Z}[\pi_1(W, w_0)],$$

which after all is derived from the cup product in a very similar fashion as in the case of constant coefficients. In this case the classes to be intersected should be given by *tethered immersions*: an immersion $f : S^n \looparrowright W$ together with a specified homotopy class $[\lambda]$ of a path from $f(*) \in W$ to w_0, where $* \in S^n$ is some chosen base point. This path represents an isomorphism between $f(*)$ and w_0 in the fundamental groupoid, and gives rise to a morphism

$$\mathbb{Z} \to f^*(\mathcal{F}^{\text{univ}})$$

of coefficient systems on S^n, whose domain is the constant coefficient system. Therefore it makes sense to form the image of

$$[S^n] \in H_n(S^n;\mathbb{Z}) \to H_n(S^n; f^*\mathcal{F}^{\mathrm{univ}}) \xrightarrow{f_*} H_n(W;\mathcal{F}^{\mathrm{univ}}).$$

We say that $x \in H_n(W;\mathcal{F}^{\mathrm{univ}})$ is represented by the tethered immersion $(f,[\lambda])$ when it is given as $f_*([S^n])$ in this way.

If $x, y \in H_n(W;\mathcal{F}^{\mathrm{univ}})$ are given as $x = f_*([S^n])$ and $y = g_*([S^n])$ for tethered immersions f and g, then their image in

$$H_n(W;\mathcal{F}^{\mathrm{univ}}) \xleftarrow{\cong} H^n(W,\partial W;\mathcal{F}^{\mathrm{univ}} \otimes \omega)$$

canonically lift to cohomology relative to the complements of $f(S^n)$ and $g(S^n)$. If these maps are transverse immersions, then their intersection is a finite set and the cup product can be calculated as the sum of a local contribution from each intersection point.

The contribution from an intersection point $z = f(z') = g(z'')$ is of the form

$$\pm g \in \mathbb{Z}[\pi_1(W, w_0)],$$

where $g \in \pi_1(W, w_0)$ is the group element obtained by concatenating the following paths

- the tether from w_0 to $f(*)$,
- any path from $f(*)$ to $f(z')$ obtained by composing f with a path in S^n,
- any path from $g('') = f(z')$ to $g(*)$ obtained by composing g with a path in S^n,
- the tether from $g(*)$ to w_0.

5. Arranging specific intersection pairings

Assuming A is path connected but not necessarily simply connected, of dimension $2n-1$ for $n \geq 3$, we consider immersed cobordisms of the form

$$W = [0,1] \times A \cup h^n \cup \cdots \cup h^n \overset{j}{\looparrowright} [0,2] \times A$$

with k many n-handles attached along disjoint embeddings

$$e_i : S^{n-1} \times D^{d-n} \to \{1\} \times A, \quad i = 1, \ldots, k. \tag{5.1}$$

The immersion implies that the maps e_i are null homotopic, so the homotopy type of W relative to A is simply

$$W \simeq A \vee \bigvee^k S^n. \tag{5.2}$$

The relative homology $H_*(W, A)$ is therefore concentrated in degree n, also with local coefficients. Choosing a basepoint $a_0 \in A$, the inclusion $A \subset W$ induces an isomorphism in fundamental group, and we shall write $\pi = \pi_1(A, a_0) = \pi_1(W,(a_0)$. With coefficients in the associated universal coefficient system we get an isomorphism of $\mathbb{Z}[\pi_1(A, a_0)]$-modules

$$H_n(W, A;\mathcal{F}^{\mathrm{univ}}) \cong (\mathbb{Z}[\pi])^{\oplus k}.$$

The summands in this isomorphism of $\mathbb{Z}[\pi]$-modules correspond to the wedge dummands in (5.2). To get the corresponding $\mathbb{Z}[\pi]$-module basis vector in $H_n(W, A;\mathcal{F}^{\mathrm{univ}})$ we must also choose a (homotopy class of) a path between the basepoint a_0 and some point in the image of e_i.

Up to isomorphism of $\mathbb{Z}[\pi]$-modules, the surgery kernel is therefore independent of the attaching maps e_i. The intersection pairing will depend on the attaching maps though—just as in the construction of the Milnor spheres and the Kervaire spheres, the intersection pairing on the surgery kernel will measure the "linking pattern" of the disjoint embeddings $e_1, \dots, e_k$.

5.1. **Trivial intersection matrix.** We are mainly interested in choosing the $e_i : S^{n-1} \times D^n \hookrightarrow$ in a way that the intersection pairing is unimodular—that is what ensures the outgoing boundary maps by a homotopy equivalence to A and hence produces a (possibly new) element in the structure set. As in the construction of the Kervaire sphere and the Milnor sphere, we start by explaining how to realize the extreme opposite, where the intersection product of any two elements is zero (equivalently, that the intersection pairing of any two $\mathbb{Z}[\pi]$-module basis vectors vanishes).

Remark 5.1. In order to actually write down an immersion

$$W = ([0,1] \times A) \cup h^n \cup \dots \cup h^n \hookrightarrow [0,2] \times A$$

in practice, for given embeddings (5.1), it essentially suffices to specify immersions

$$E_i : D^n \times D^{d-n} \to [1,2] \times A, \quad i = 1, \dots, k$$

such that $E_i(x,y) = e_i(x,y)$ for $x \in \partial D^n = S^{n-1}$. Without making further choices, we then obtain a map

$$W \looparrowright [0,2] \times A$$

whose restriction to $[0,1] \times A \subset W$ is the inclusion and whose restriction to the ith handle $D^n \times D^{d-n} \hookrightarrow W$ is given by E_i. As it stands, this map $W \to [0,2] \times A$ is a *topological immersion*: any point in its domain admits an open neighborhood on which the map restricts to a topological embedding onto its image. It is not quite a smooth map because of the corners created in the outgoing boundary of $[0,1] \times A$ before gluing the handles, but it does restrict to a smooth immersion of the complement of the sets $e_i(S^{n-1} \times D^{d-n-1}) \subset \partial_+ W$. To obtain a smooth immersion of the entire cobordism, we simply compose with a suitable self-embedding

$$W \hookrightarrow W \setminus \bigcup_{i=1}^{k} e_i(S^{n-1} \times D^{d-n-1}),$$

which can be constructed using bump functions and a choice of collar of the outgoing boundary.

Example 5.2 (Attaching a handle trivially)**.** To "trivially" attach an n-handle to $[0,1] \times A$, compose a chart

$$\mathbb{R}^{d-1} = \mathbb{R}^n \times \mathbb{R}^{d-n-1} \xrightarrow{c} A$$

with the "standard" embedding

$$\begin{aligned} S^{n-1} \times D^{d-n} &\overset{e_0}{\hookrightarrow} \mathbb{R}^n \times \mathbb{R}^{d-n-1} \\ (x,y) &\mapsto \tfrac{2+y_1}{4}(x,0) + \tfrac{1}{4}(0, y_2, \dots, y_{d-n}). \end{aligned} \tag{5.3}$$

A distinguishing feature of the trivial handle attachment is that this cobordism not only immerses, but in fact *embeds* into $[0,2] \times A$. The most convincing way to see this is perhaps low-dimensional pictures, but let us try to explain how to construct an embedding using formulas.

Writing coordinates on S^n as $(x_0, x) \in S^n \subset \mathbb{R} \times \mathbb{R}^n$ with $x = (x_1, \dots, x_n)$, the embedding

$$\begin{aligned} S^n \times D^{d-n} &\overset{e_1}{\hookrightarrow} [0,2] \times \mathbb{R}^n \times \mathbb{R}^{d-n-1} \\ (x_0, x, y) &\mapsto (1,0,0) + \tfrac{2+y_1}{4}(x_0, x, 0) + \tfrac{1}{4}(0,0,y_2, \dots, y_{d-n}) \end{aligned} \tag{5.4}$$

extends (5.3) in the sense that $e_1(0,x,y) = (1, e_0(x,y)) \in [0,2] \times \mathbb{R}^{d-1}$. The subspace of the domain of (5.4) defined by $x_0 \le 0$ is diffeomorphic to $D^n \times D^{d-n}$, using e.g. stereographic projection to identify D^n with the upper hemisphere in S^n, so we can use this subspace as a model for an n-handle. The resulting embedding

$$\begin{aligned}\{(x_0,x,y)\in S^n\times D^{d-n}\mid x_0\geq 0\} &\to [1,2]\times A \\ (x_0,x,y) &\mapsto (\mathrm{id}_{[1,2]}\times c)\circ e_1(x_0,x,y)\end{aligned} \tag{5.5}$$

glues along (5.3) to the inclusion $[0,1]\times A\hookrightarrow[0,2]\times A$ to an embedding

$$W=([0,1]\times A)\cup_{e_0}h^n\to[0,2]\times A.$$

As in Remark 5.1, this glued map will strictly speaking not be smooth at the corners created before attaching the handle. To obtain a smooth embedding, pre-compose with a self-embedding as in that remark.

The formula (5.4) also gives a smooth embedding $f:S^n\times\mathring{D}^{d-n}\hookrightarrow W$, and the image $f_*([S^n])\in H_n(W)$ is a canonical generator of the surgery kernel

$$\mathbb{Z}\cong K_n(W)\subset H_n(W).$$

Example 5.3 (Attaching multiple handles trivially)**.** The standard embedding (5.3) has image contained in the unit disk in $\mathbb{R}^{d-1}$, so we may obtain disjoint embeddings simply by displacing it by some vector, e.g.

$$\begin{aligned}\{1,\dots,k\}\times S^{n-1}\times D^{d-n} &\hookrightarrow \mathbb{R}^{d-1}\\ (i,x,y) &\mapsto (i,0,0)+e_0(x,y).\end{aligned}$$

Composing also with a chart $c:\mathbb{R}^{d-1}\to A$ then gives the necessary data to form the cobordism

$$W=([0,1]\times A)\cup h^n\cup\dots\cup h^n.$$

The embedding explained in Example 5.2 evidently generalizes to this setting, giving an embedding $W\hookrightarrow[0,2]\times A$. As in the example, the formula (5.4) gives embeddings

$$f_i:S^n\times\mathring{D}^{d-n}\to W,\quad i=1,\dots,k,$$

and the classes $(f_i)_*([S^n])\in H_n(W)$ form a basis for the surgery kernel

$$\mathbb{Z}^{\oplus k}\cong K_n(W)\subset H_n(W).$$

In the case where $d=2n$ and A is oriented, we have the intersection form

$$K_n(W)\otimes K_n(W)\to\mathbb{Z}$$

defined, but it turns out to be zero when the handles are attached in the particular way described in this example. To see this we use that the image of the class $(f_i)_*([S^n])$ under Poincaré duality $H_n(W)\cong H^n(W,\partial W)$ lifts to a class $\alpha_i\in H^n(W,W\setminus f_i(S^n\times\{0\}))$. For $i\neq j$ the cup product $\alpha_i\smile\alpha_j$ then lifts to the zero group $H^{2n}(W,W)$.

A similar argument applies for the self-intersection $\alpha_i\smile\alpha_i$, using that α_i also has support $f_i(S^n\times\{y\})$ for any $y\in\mathring{D}^n$ and we may choose $y\neq 0$.

5.2. **Modifying intersection numbers.** We continue to consider immersed cobordisms of the form

$$W=([0,1]\times A)\cup h^n\cup\dots\cup h^n\looparrowright[0,2]\times A$$

with handles attached along some embedding

$$\coprod_{i=1}^k S^{n-1}\times D^{d-n}\overset{e}{\hookrightarrow}\{1\}\times A$$

and write $e_i : S^{n-1} \times D^{d-n} \hookrightarrow \{1\} \times A$ for the restriction to the ith copy of $S^{n-1} \times D^n$. Restricting the immersion to the ith handle gives an immersion

$$\{(x_0, x, y) \in S^n \times D^n \mid x_0 \geq 0\} \overset{f_i^+}{\looparrowright} [1,2] \times A$$

which may be glued to its "mirror"

$$\{(x_0, x, y) \in S^n \times D^n \mid x_0 \leq 0\} \overset{f_i^-}{\looparrowright} [0,1] \times A,$$

defined as $f_i^-(x_0, x, y) = R \circ f_i^+(-x_0, x, y)$ where $R : [0,2] \times A \to [0,2] \times A$ is defined by $R(t, a) = (2 - t, a)$. These maps $f_i^{\pm}$ glue to a single map

$$f_i : S^n \times D^n \to W.$$

Unlike in the trivially attached case, these maps need not be embeddings nor have disjoint image. They do give rise to a preferred basis of the surgery kernel though, namely

$$(f_i)_*([S^n \times \{0\}]) \in K_n(W; \mathcal{F}^{\mathrm{univ}}) \subset H_n(W; \mathcal{F}^{\mathrm{univ}}).$$

to be elaborated

Now, starting from the trivial handle attachments we can change the intersection pairing of $(f_i)_*([S^n \times \{0\}])$ and $(f_j)_*([S^n \times \{0\}])$ by isotoping e_j along an embedded path. The isotopy can be supported in an arbitrarily small neighborhood of the path, but may leave the chart $\mathbb{R}^{2n-1} \hookrightarrow A$. This way, we may change the intersection pairing of $(f_i)_*([S^n \times \{0\}])$ and $(f_j)_*([S^n \times \{0\}])$ by adding $\pm g$ for an arbitrary sign and arbitrary group element $g \in \pi$, but will simultaneously change the intersection pairing of $(f_j)_*([S^n \times \{0\}])$ and $(f_i)_*([S^n \times \{0\}])$ by adding $(-1)^n w(g) g^{-1}$, where $w : \pi : \mathbb{Z}^\times$ is the orientation character.

By this method, we may obtain any intersection matrix of the form

$$M + \overline{M}^t \in M_k(\mathbb{Z}[\pi]),$$

where $\overline{M}$ denotes the matrix obtained by applying $g \mapsto (-1)^n w(g) g^{-1}$ to each group element, and the superscript t denotes transpose.