1. Immersed cobordisms, surgery kernels, and their intersection forms

Let A and B be compact manifold with identified boundary $\partial A = \Sigma = \partial B$, so we can form a closed manifold $A \cup_\Sigma B$. Equivalently, up to diffeomorphism at least,

$$A \cup_\Sigma B \cong A \cup_{\{0\}\times\Sigma} ([0,1]\times\Sigma) \cup_{\{1\}\times\Sigma} B. \tag{1.1}$$

In this situation, a *cobordism from A to B* is a compact manifold W bounding (1.1), i.e. with a specified diffeomorphism

$$\partial W = A \cup_{\{0\}\times\Sigma} ([0,1]\times\Sigma) \cup_{\{1\}\times\Sigma} B.$$

We will mostly consider such a cobordism to be a manifold with corners,

$$\angle W = (\{0\}\times\Sigma) \sqcup (\{1\}\times\Sigma)$$

and three smooth parts $A \subset \partial W$, $[0,1]\times\Sigma \subset W$, and $B \subset W$ in the sense of Wall. Occasionally it may be convenient to straighten the corners to turn W into a manifold with smooth boundary $A \cup_\Sigma B$.

We will be particularly interested in the situation where the cobordism comes with an immersion

$$j : W \looparrowright [0,2]\times A$$

satisfying $j(x) = (0,x)$ for $a \in A \subset \partial W$ and $j(t,\sigma) = (t,\sigma)$ for $(t,\sigma) \in [0,1]\times\Sigma \subset \partial W$. The composition

$$A \hookrightarrow W \overset{j}{\looparrowright} [0,2]\times A \overset{\pi}{\twoheadrightarrow} A \tag{1.2}$$

is the identity, where π denotes the projection to the second factor. This gives a splitting of chain complexes and of homology

$$C_*(A) \oplus C_*(A,W) \overset{\cong}{\to} C_*(W)$$
$$H_n(A) \oplus K_n(W) \overset{\cong}{\to} H_n(W),$$

where

$$C_*(A,W) = \mathrm{Ker}((\pi\circ j)_* : C_*(W) \to C_*(A)) \subset C_*(W)$$
$$K_n(W) = H_n(C_*(A,W)) \cong \mathrm{Ker}(j_* : H_n(W) \to H_n([0,2]\times A)).$$

In this situation, the second factor in the splitting $H_n(W) = H_n(A) \oplus K_n(W)$ is called the *surgery kernel.* As an abstract group it is canonically isomorphic to the relative homology $H_n(W,A)$, which does not depend on the immersion j, but splitting the map $H_n(W) \to H_n(W,A)$ and realizing the surgery kernel as a summand of $H_n(W)$ does.

We will be interested in the composition

$$K_n(W) \hookrightarrow H_n(W) \to H_n(W,B) \tag{1.3}$$

and will relate it to the composition

$$(\pi \circ j_{|B})_* : H_n(B) \to H_n(A) \tag{1.4}$$

Here, $\pi \circ j_{|B}$ is the composition

$$B \hookrightarrow W \overset{j}{\looparrowright} [0,2]\times A \overset{\pi}{\twoheadrightarrow} A,$$

which is a map of manifolds with boundary and by construction it restricts to a diffeomorphism $\partial B \cong \partial A$. To detect whether it is a homotopy equivalence we may, at least if A and B are both simply connected, look for whether it induces an isomorphism on integral homology.

Proposition 1.1. *The homomorphisms (1.4) are isomorphisms for all n if and only if the compositions (1.3) are isomorphism for all n.*

Proof. Recall that $K_n(W)$ was defined as homology of the chain complex $C_*(A, W) = \mathrm{Ker}((\pi \circ j)_* : C_*(W) \to C_*(A))$. We have a canonical comparison map

$$C_n(A, W) \to C((\pi \circ j)_*)_{n+1}$$

from the kernel to the mapping cone, with the indicated shift of degrees. Since the chain map $(\pi \circ j)_*$ is surjective, this comparison map induces an isomorphism on homology by the 5-lemma, and we deduce the isomorphism

$$K_n(W) \cong H_{n+1}(C((\pi \circ j)_*)).$$

Inside the mapping cone of $(\pi \circ j)_*$ we have the mapping cone of $(\pi \circ j_{|B})_* : C_*(B) \to C_*(A)$, and there is the short exact sequence of chain complexes

$$C((\pi \circ j_{|B})_*) \to C((\pi \circ j)_*) \to C_{*-1}(W, B)$$

inducing a long exact sequence

$$\cdots \to H_{n+1}(C((\pi \circ j_{|B})_*)) \to K_n(W) \xrightarrow{(1.3)} H_n(W, B) \to H_n(C((\pi \circ j_{|B})_*)) \to \dots.$$

Since the homology of the mapping cone also sit in a long exact sequence with the homomorphism (1.4), we get the asserted equivalence. □

Given also an orientation $[W] \in H_d(W, \partial W)$ we get a Poincaré duality isomorphism

$$\begin{aligned} H^{d-n}(W, A) &\to H_n(W, B) \\ \alpha &\mapsto \alpha \cap [M] \end{aligned} \tag{1.5}$$

and it is convenient to compose (the inverse of) this isomorphism with (1.3) to get a map from homology to cohomology, which then no longer explicitly mentions B. We can also combine with the universal coefficient theorems

$$TH^{d-n}(W, A) \xrightarrow{\cong} \mathrm{Hom}(TH_{d-n-1}(W, A), \mathbb{Q}/\mathbb{Z})$$
$$H^{d-n}(W, A)/TH^{d-n}(W, A) \xrightarrow{\cong} \mathrm{Hom}(H_{d-n}(W, A), \mathbb{Z})$$

and the isomorphism $H_*(W, A) \cong K_*(W)$ to get pairings

$$\begin{aligned} TK_n(W) \otimes TK_{d-n-1}(W) &\to \mathbb{Q}/\mathbb{Z} \\ (K_n(W)/TK_n(W)) \otimes (K_n(W)/TK_{d-n}(W)) &\to \mathbb{Z}. \end{aligned} \tag{1.6}$$

These pairings can be identified with the restrictions of the linking and intersection pairings on $H_*(W)$. Here we have written $TG = \mathrm{Ker}(G \to G \otimes_{\mathbb{Z}} \mathbb{Q}) \subset G$ for the torsion subgroup of an abelian group G. (Exercise: construct a natural isomorphism $\mathrm{Ext}(G, \mathbb{Z}) \cong \mathrm{Hom}(TG, \mathbb{Q}/\mathbb{Z})$ for finitely generated abelian groups G, convenient for the universal coefficient theorem).

Corollary 1.2. *In the situation above, the map $j_{|B} : B \to A$ induces an isomorphism on integral homology if and only if both pairings (1.6) are unimodular for all n.* □

Corollary 1.3. *If $K_n(W) \subset H_n(W)$ is a free abelian group for all n, then $(\pi \circ j_{|B})_* : H_n(B) \to H_n(A)$ is an isomorphism for all n if and only if the intersection pairing on $H_*(W)$ restricts to a unimodular pairing*

$$K_n(W) \otimes K_{d-n}(W) \to \mathbb{Z}$$

for all n.

Definition 1.4. The *structure set* of the compact smooth manifold A is the set

$$S(A) = \{(M, f) \mid M \text{ compact smooth, } f : M \xrightarrow{\simeq} A,\ f_{|\partial M} : \partial M \to \partial A \text{ diffeo.}\}/\sim,$$

where the equivalence relation $(M, f) \sim (M', f')$ when there exists a diffeomorphism $\phi : M \to M'$ such that $f' \circ \phi_{|\partial M} = f_{|\partial M}$ and $f' \sim \phi \simeq f$ relative to ∂M.

As a strategy for producing elements of the structure set, we may first construct an immersed cobordism $j : W \looparrowright [0,2] \times A$ and then restricting j to the outgoing boundary. The results above then give a criterion for this restriction to be a homology equivalence; if both A and the outgoing boundary are simply connected then homology equivalence implies homotopy equivalence by the Hurewicz theorem so we can use immersed cobordisms to produce new elements in the structure set.

2. Local coefficients

By the Hurewicz theorem, a map $A \to B$ between simply connected spaces, both homotopy equivalent to CW complexes, is a homotopy equivalence if and only if it induces an isomorphism on integral homology in all degrees. Any manifold is homotopy equivalent to a CW complex, but the assumption of simple-connectivity is a serious drawback. Fortunately, there is a stronger version of the Hurewicz theorem involving homology with *local coefficients*.

2.1. **Summary of definitions.** A local coefficient system on B is the assignment of an abelian group $\mathcal{F}_b$ to each point $b \in B$ and an isomorphism $\gamma_* : \mathcal{F}_{b_0} \to \mathcal{F}_{b_1}$ to each path $\gamma : [0,1] \to B$ with $\gamma(0) = b_0$ and $\gamma(1) = b_1$, such that paths that are homotopic relative to their endpoints give equal isomorphisms and such that concatenation of paths goes to composition of isomorphisms. In other words, a functor $\mathcal{F} : \pi_{\leq 1}(B) \to \mathrm{Ab}$ from the fundamental groupoid of B to abelian groups. Given such an assignment, we also get an abelian group $\mathcal{F}_\sigma$ for each continuous map $\sigma : \Delta^p \to B$, namely $\mathcal{F}_\sigma = \mathcal{F}_{\sigma(e_0)}$ where $e_0 = (1, 0, \ldots, 0) \in \Delta^p$ is the initial vertex. If $\delta^i : \Delta^{p-1} \to \Delta^p$ is the ith face map, then we furthermore get canonical isomorphisms $\delta_i : \mathcal{F}_\sigma \cong \mathcal{F}_{\sigma \circ \delta^i}$ using any path in Δ^p between e_0 to $\delta^i(e_0)$.

Chains with coefficients in $\mathcal{F}$ may be defined as

$$C_p(B; \mathcal{F}) = \bigoplus_{\sigma : \Delta^p \to B} \mathcal{F}_\sigma$$

$$\partial = \sum_{i=0}^{p} \delta_i : C_p(B; \mathcal{F}) \to C_{p-1}(B; \mathcal{F})$$

and similarly cochains

$$C^p(B; \mathcal{F}) = \prod_{\sigma : \Delta^p \to B} \mathcal{F}_\sigma$$

$$d = \sum_{i=0}^{p} (\delta_i)^{-1} : C^{p-1}(B; \mathcal{F}) \to C^p(B; \mathcal{F}),$$

and of course

$$H_p(B; \mathcal{F}) = H_p(C_*(B; \mathcal{F}), \partial)$$
$$H^p(B; \mathcal{F}) = H^p(C^*(B; \mathcal{F}), d)$$

Homology and cohomology relative to a subspace is defined similarly.

Functoriality works as follows. Given a coefficient system $\mathcal{F}$ on A and a continuous map $f : B \to A$, the latter induces a morphism of fundamental groupoids $\pi_{\leq 1}(B) \to \pi_{\leq 1}(A)$ and hence a pulled-back coefficient system $f^*\mathcal{F}$ on B and there are induced homomorphisms

$$f_* : H_n(B; f^*\mathcal{F}) \to H_n(A; \mathcal{F})$$
$$f^* : H^n(A; \mathcal{F}) \to H^n(B; f^*\mathcal{F})$$

induced by analogous maps of chain complexes. In this case the Hurewicz theorem asserts

Theorem 2.1. *A continuous map $f : B \to A$ between topological spaces A and B is a weak homotopy equivalence if and only if*

(i) the induced map $\pi_0(B) \to \pi_0(A)$ is a bijection,
(ii) the induced maps $\pi_1(B, b) \to \pi_1(A, f(b))$ is an isomorphism for all $a \in A$, and
(iii) the induced maps $f_* : H_n(B; f^*\mathcal{F}) \to H_n(A; \mathcal{F})$ is an isomorphism for all coefficient systems on B.

If A and B are homotopy equivalent to CW complexes, these conditions then also imply f is a homotopy equivalence. □

2.2. **Examples.**

Example 2.2 (Orientation systems)**.** If M is a manifold of dimension d, there is a local system ω^M

$$\omega_x^M = H_d(M, M \setminus \{x\})$$

for any $x \in M \setminus \partial M$.

Example 2.3 (Local systems from covering maps)**.** If $f : X \to Y$ is a covering map (i.e., X is a covering space of Y), then there is local system $\mathcal{F}^f$ with

$$\mathcal{F}_y^f = \bigoplus_{x \in f^{-1} y} \mathbb{Z}.x,$$

the free abelian group generated by the set $f^{-1}(y)$. Homology and cohomology of Y with coefficients in this local system is canonically isomorphic to homology/cohomology of Y with coefficients in $\mathbb{Z}$ (exercise: use lifting theorems to prove that this holds even on the chain/cochain level).

Two types of covering maps deserve special mention:

- If Y is path connected and $f : X \to Y$ is a universal cover, then we obtain a "universal" local system on Y. Homology of Y with coefficients in this local system is then homology of the simply connected space X. This is the reason why the Hurewicz theorem generalizes as above: the generalization is an easy consequence of the usual Hurewicz theorem, applied to universal covers.
- If M is a manifold and $f : \widetilde{M} \to M$ is the "orientation double cover", then there is a short exact sequence of coefficient systems

  $$0 \to \omega^M \to \mathcal{F}^f \to \mathbb{Z} \to 0,$$

  in which $\mathbb{Z}$ denotes the constant coefficient system (assigning $\mathbb{Z}$ to any point and the identity to any path).

Example 2.4 (Tensor products)**.** If $\mathcal{F}$ and $\mathcal{G}$ are coefficient systems on a space B, then there is an induced coefficient system $\mathcal{F} \otimes \mathcal{G}$ whose value at $b \in B$ is $\mathcal{F}_b \otimes \mathcal{G}_b$. More generally we can form $\mathcal{F} \otimes_R \mathcal{G}$ for a ring R, when $\mathcal{F}$ takes values in right R-modules and $\mathcal{G}$ takes values in left R-modules.

Example 2.5 (Universal coefficients)**.** Let X, $x \in X$ a basepoint, and write $\pi_{\leq 1}(X)(x, y)$ for the set of homotopy classes of paths in X from x to y (morphisms in the fundamental groupoid). Then there is a coefficient system $\mathcal{F}^{\text{univ}}$ whose value at $y \in X$ is $\mathbb{Z}[\pi_{\leq 1}(X)(x, y)]$, the free abelian group with basis $\pi_{\leq 1}(X)(x, y)$. It takes values in left modules over $\mathbb{Z}[\pi_1(X, x)]$, the group ring of the fundamental group. There is a canonical identification

$$H_n(X; \mathcal{F}^{\text{univ}}) \cong H_n(\widetilde{X}; \mathbb{Z}),$$

where $\tilde{X} \to X$ is the universal covering space of the path component containing $x \in X$. When X is compact, there is also a canonical identification

$$H^n(X; \mathcal{F}^{\mathrm{univ}}) \cong H^n_c(\widetilde{X}; \mathbb{Z}),$$

where the latter denotes compactly supported cohomology. Both isomorphisms can be established on the chain/cochain level.

To any right module M over $\mathbb{Z}[\pi_1(X,x)]$ there is an associated coefficient system $\mathcal{F}^M = M \otimes_{\mathbb{Z}[\pi_1(X,x)]} \mathcal{F}^{\mathrm{univ}}$ whose value at $y \in X$ is

$$\mathcal{F}^M_y = M \otimes_{\mathbb{Z}[\pi_1(X,x)]} \mathbb{Z}[\pi_{\leq 1}(X)(x,y)].$$

When X is path connected, it can be shown that all coefficient systems arise this way.

Example 2.6 (Cup and cap products)**.** For coefficient systems $\mathcal{F}$ and $\mathcal{G}$ on B, there are cup and cap products

$$\begin{aligned} H^p(B;\mathcal{F}) \otimes H^q(B;\mathcal{G}) &\to H^{p+q}(B;\mathcal{F}\otimes\mathcal{G}) \\ H^p(B;\mathcal{F}) \otimes H_q(B;\mathcal{G}) &\to H_{q-p}(B;\mathcal{F}\otimes\mathcal{G}), \end{aligned}$$

defined by analogues of the usual formulas (whenever those formulas multiply some elements in a ring, substitute tensor products).

2.3. **Universal coefficient theorems.** And now for the bad news: there is no especially good analogue of the universal coefficient theorem, and hence no general formula for cohomology with local coefficients by "dualizing" homology, analogous to the universal coefficient theorem expressing $H^n(X;\mathbb{Z})$ in terms of $\mathrm{Hom}(H_n(X),\mathbb{Z})$ and $\mathrm{Ext}(H_{n-1}(X),\mathbb{Z})$. The weaker statement that $H^n(X;\mathbb{Z}) \cong \mathrm{Hom}(H_n(X),\mathbb{Z})$ when $H_{n-1}(X)$ is projective as a $\mathbb{Z}$-module has the following analogue though.

Lemma 2.7. *If X is a topological space, $A \subset X$ is a subspace, $x \in A$ and M is a module over $\mathbb{Z}[\pi_1(X,x)]$, then there is a natural map for each $n \in \mathbb{N}$*

$$H^n(X,A;\mathcal{F}^M) \to \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x)]}(H_n(X,A;\mathcal{F}^{\mathrm{univ}}), M).$$

This map is an isomorphism for n, provided $H_t(X,A;\mathcal{F}^{\mathrm{univ}})$ is a projective $\mathbb{Z}[\pi_1(X,x)]$-module for all $t < n$. In particular this holds when the inclusion $A \hookrightarrow X$ is $(n-1)$-connected.

Proof sketch. Then there is an isomorphism of cochain complexes

$$C^*(X;\mathcal{F}^M) \cong \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x)]}(C_*(X;\mathcal{F}^{\mathrm{univ}}), M)$$

and a corresponding spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{\mathbb{Z}[\pi_1(X,x)]}(H_t(X;\mathcal{F}^{\mathrm{univ}}), M) \Rightarrow H^{s+t}(X;\mathcal{F}^M),$$

with differentials as in the cohomology Serre spectral sequence. Here, $H_t(X;\mathcal{F}^{\mathrm{univ}}) \cong H_t(\widetilde{X};\mathbb{Z})$ can be identified with integral homology of the universal cover of the path component containing x, on which the fundamental group acts by deck transformations.

The assumptions imply that the columns $E_2^{s,t}$ vanish for $t > 0$ and $s < n$, so we get

$$E_\infty^{0,n} = E_2^{0,n} = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X,x)]}(H_n(X;\mathcal{F}^{\mathrm{univ}}), M),$$

and this is the only group contributing to $H^n(X;\mathcal{F}^M)$. Convergence of the spectral sequence in total degree n then gives the asserted isomorphism. ☐

3. The intersection pairing with local coefficients

3.1. **Poincaré duality for local coefficients.** Now Poincaré duality generalizes to arbitrary (not necessarily oriented) manifolds and arbitrary coefficient systems. If M is a compact d-manifold with boundary and ω is the orientation local system from Example 2.2, there is a canonical fundamental class $[M] \in H_d(M, \partial M; \omega)$ inducing isomorphisms

$$\begin{aligned} H^p(M; \mathcal{F}) &\to H_{d-p}(M, \partial M; \mathcal{F} \otimes \omega) \\ H^p(M, \partial M; \mathcal{F}) &\to H_{d-p}(M; \mathcal{F} \otimes \omega) \\ \alpha &\mapsto \alpha \cap [M] \end{aligned}$$

and more generally

$$H^p(M, A; \mathcal{F}) \xrightarrow{\cong} H_{d-p}(M, B; \mathcal{F} \otimes \omega) \tag{3.1}$$

when $\partial M = A \cup_\Sigma B$ when A and B are compact submanifolds of ∂M with $\partial A = \Sigma = \partial B = A \cap B$, and $\mathcal{F}$ is any local system on M.

3.2. **Poincaré duality and immersed cobordisms.** Let us return to the situation studied in Section 1 with an immersed cobordism $W \looparrowright [0, 2] \times A$, and suppose given a coefficient system $\mathcal{F}$ on A. We can pull it back along $\pi \circ j : W \to A$ to get a coefficient system on W and obtain splittings

$$\begin{aligned} H_n(W; (\pi \circ j)^*\mathcal{F}) &\cong H_n(A; \mathcal{F}) \oplus K_n(W; \mathcal{F}), \\ K_n(W; \mathcal{F}) &= \mathrm{Ker}((\pi \circ j)_* : H_n(W; (\pi \circ j)^*\mathcal{F}) \to H_n(A; \mathcal{F}) \\ &\cong H_n(W, A; (\pi \circ j)^*\mathcal{F}) \end{aligned}$$

from the composition $A \hookrightarrow W \to A$ being the identity, by the same argument as in Section 1. Simplifying notation and writing the local system $(\pi \circ f)^*\mathcal{F}$ on W simply as $\mathcal{F}$, we then have the composition

$$K_n(W; \mathcal{F}) \hookrightarrow H_n(W; \mathcal{F}) \to H_n(W, B; \mathcal{F}) \xleftarrow{\cong} H^{d-n}(W, A; \mathcal{F} \otimes \omega^W), \tag{3.2}$$

completely analogous to (1.3).

Proposition 3.1. *Let $j : W \looparrowright [0, 2] \times A$ be an immersed cobordism from A to B as above, let $\mathcal{F}$ be a coefficient system on A. Then the induced map*

$$(\pi \circ j_{|B}) : H_n(B; (\pi \circ j_{|B})^*\mathcal{F}) \to H_n(A; \mathcal{F})$$

is an isomorphism for all n, if and only if the composition (3.2) is an isomorphism for all n.

If furthermore $\pi \circ j_{|B} : B \to A$ induces a bijection $\pi_0(A) \to \pi_0(B)$ and isomorphisms $\pi_1(B, b) \to \pi_1(A, \pi \circ j(b))$ for all basepoints $b \in B$, then it is a homotopy equivalence if and only if the composition (3.2) is an isomorphism for all n and all coefficient systems. (Equivalently, for the coefficient systems associated to universal covers of path components of A.)

Proof. Completely analogous to the simply-connected case, using the generalized Hurewicz theorem to deduce homotopy equivalence. □

The assumption in the second part of the Proposition about π_0 and π_1 hold if W has dimension d and admits a handle presentation where all handles have index $\in \{3, \dots, d-3\}$, because both inclusions $A \subset W$ and $B \subset W$ will be 2-connected in this case. In the case of immersed cobordisms $W \looparrowright [0, 2] \times A$, the immersion ensures that the attaching maps for the handles are null-homotopic and hence attaching them does not change the fundamental group either (i.e., the inclusion $A \hookrightarrow W$ induces an isomorphism on fundamental groups). In the immersed case it therefore suffices to know all handles have index $\in \{2, \dots, d-3\}$.

3.3. **The intersection pairing, cohomological interpretation.** Setting $B = \emptyset$ in (3.1) and replacing $\mathcal{F}$ by $\mathcal{F}^{\text{univ}} \otimes \omega$ for some basepoint $m_0 \in M$, we can combine the inverse isomorphism

$$H_{d-p}(M; \mathcal{F}^{\text{univ}}) \xrightarrow{\cong} H^p(M, \partial M; \mathcal{F}^{\text{univ}} \otimes \omega)$$

with the "universal coefficient" homomorphism in Lemma 2.7. The composition can be interpreted as a pairing

$$H_{d-p}(M; \mathcal{F}^{\text{univ}}) \otimes H_p(M, \partial M; \mathcal{F}^{\text{univ}} \otimes \omega) \to \mathbb{Z}[\pi_1(M, m_0)]$$

which is the intersection pairing on homology with local coefficients. It can in turn be interpreted, using Poincaré duality again, as a pairing

$$H^p(M, \partial M; \mathcal{F}^{\text{univ}} \otimes \omega) \otimes H^{d-p}(M; \mathcal{F}^{\text{univ}}) \to \mathbb{Z}[\pi_1(M, m_0)]. \tag{3.3}$$

The analogy of (3.3) for constant coefficients has another useful description, namely $\alpha \otimes \beta \mapsto (\alpha \cup \beta)([M])$, the cup product evaluated on the fundamental class. The pairing (3.3) has a similar formula as cup product evaluated on the fundamental class, which we now explain in a few steps. Start with the the composition in the fundamental groupoid

$$\begin{aligned} \pi_{\leq 1}(M)(m_0, m) \times \pi_{\leq 1}(M)(m_0, m) &\to \pi_1(M, m_0) \\ ([\lambda], [\rho]) &\mapsto [\lambda * \rho^{-1}]. \end{aligned} \tag{3.4}$$

Here we regard m_0 as fixed and $m \in \pi_{\leq 1}(M)$ as varying over objects. This can then be viewed as a defining a natural transformation of functors $\pi_{\leq 1}(M) \to \text{Sets}$, whose codomain is the constant functor assigning $\pi_1(M, m_0)$ to every $m \in M$. Taking free abelian group then gives a map of coefficient systems

$$\mathcal{F}^{\text{univ}} \otimes \mathcal{F}^{\text{univ}} \to \mathbb{Z}[\pi_1(M, m_0)]$$

whose codomain is the constant coefficient system assigning $\mathbb{Z}[\pi_1(M, m_0)]$ to each $m \in M$. Now tensor with the orientation local system ω and combine with cup product to get

(3.5)

$$\begin{aligned} H^p(M, \partial M; \mathcal{F}^{\text{univ}} \otimes \omega) \times H^{d-p}(M; \mathcal{F}^{\text{univ}}) &\xrightarrow{\cup} H^d(M, \partial M; \mathcal{F}^{\text{univ}} \otimes \mathcal{F}^{\text{univ}} \otimes \omega) \\ \to H^d(M, \partial M; \mathbb{Z}[\pi_1(M, m_0)] \otimes \omega) &\xleftarrow{\cong} H^d(M, \partial M; \omega) \otimes \mathbb{Z}[\pi_1(M, m_0)], \end{aligned}$$

where in the last step we used that $\mathbb{Z}[\pi_1(M, m_0)]$ is a free $\mathbb{Z}$-module to see that there is no Tor term. Finally, compose with the homomorphism

$$H^d(M, \partial M; \omega) \to \mathbb{Z}$$

given by evaluating on $[M] \in H_d(M, \partial M; \omega)$ to obtain the desired pairing

$$\begin{aligned} H^p(M, \partial M; \mathcal{F}^{\text{univ}} \otimes \omega) \otimes H^{d-p}(M; \mathcal{F}^{\text{univ}}) &\to \mathbb{Z}[\pi_1(M, m_0)] \\ \alpha \otimes \beta &\mapsto (\alpha \cup \beta)([M]), \end{aligned}$$

which agrees with (3.3).

We finish this subsection with two remarks about the algebraic properties satisfied by this pairing.

Remark 3.2. The group $\pi = \pi_1(M, m_0)$ acts on "everything in sight" on this pairing, but some care with signs and left/right actions is necessary. Firstly, $\pi_1(M, m_0)$ acts as automorphisms of the coefficient system $\mathcal{F}^{\text{univ}} : y \mapsto \mathbb{Z}\pi_{\leq 1}(M)(m_0, y)$ by pre-composing with elements of $\pi_1(M, m_0) = \pi_{\leq 1}(M)(m_0, m_0)$ in the fundamental groupoid. Since taking cohomology is functorial with respect to morphisms of coefficient systems, we obtain a π-action on cohomology (relative or absolute) with coefficients in $\mathcal{F}^{\text{univ}}$. The behavior of (3.3) with respect to this action is inherited from (3.4): acting by $[\lambda] \in \pi_1(M, m_0)$ on the left factor in the domain of the pairing corresponds to left multiplication by the same element $[\lambda]$ in the codomain, but acting on the right factor corresponds to right multiplication by $[\lambda]^{-1}$ in the codomain.

If we write the pairing (3.3) as $\alpha \otimes \beta \mapsto \langle \alpha, \beta \rangle$, then we have the formulas

$$\langle g.\alpha, \beta \rangle = g.\langle \alpha, \beta \rangle$$
$$\langle \alpha, g.\beta \rangle = \langle \alpha, \beta \rangle.g^{-1}$$

Remark 3.3. The pairing (3.3) is between relative cohomology and absolute cohomology, and also with not quite equal coefficients. As it stands, it therefore does not quite make sense to ask about symmetry propertiesof the pairing—how $\langle \alpha, \beta \rangle$ relates to $\langle \beta, \alpha \rangle$—simply because these notations are not simultaneously defined. With some slight modification we can get a pairing in which α and β is the same type of object, let us briefly explain this.

Firstly, the local coefficient systems $\mathcal{F}^{\mathrm{univ}}$ and $\mathcal{F}^{\mathrm{univ}} \otimes \omega$ are in fact isomorphic: choosing once and for all a generator of $\omega_{m_0} = H_d(M, M \setminus \{m_0\}; \mathbb{Z})$, we can use a path $[\lambda] \in \pi_{\leq 1}(M)(m_0, m)$ from m_0 to $m \in M$ to propagate this generator to a generator of ω_m. This defines an isomorphism of local systems

$$\mathcal{F}^{\mathrm{univ}} \to \mathcal{F}^{\mathrm{univ}} \otimes \omega. \tag{3.6}$$

This isomorphism is not necessarily compatible with the $\pi_1(M, m_0)$ action, unless that action fixes the chosen generator of $H_d(M, M \setminus \{m_0\})$. In general, the action of $\pi_1(M, m_0)$ on $H_d(M, M \setminus \{m_0\}) \cong \mathbb{Z}$ is encoded by a homomorphism

$$\pi_1(M, m_0) \xrightarrow{w} \mathbb{Z}^\times,$$

sometimes called the orientation character (or the "first Stiefel–Whitney class"). If we temporarily denote the isomorphism (3.6) by ϕ, then its relationship with the action of $g \in \pi_1(M, m_0)$ is

$$\phi(g.\alpha) = w(g)g.\phi(\alpha)$$

The pairing (3.3) is asymmetric in that it pairs relative cohomology with absolute cohomology, and also with slightly different coefficients $\mathcal{F}^{\mathrm{univ}}$ versus $\mathcal{F}^{\mathrm{univ}} \otimes \omega$. If we use the isomorphism (3.6) and compose with the restriction map $H^*(M, \partial M; \mathcal{F}^{\mathrm{univ}}) \to H^*(M; \mathcal{F}^{\mathrm{univ}})$, we obtain a more symmetric looking pairing

$$H^p(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \otimes H^{d-p}(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \to \mathbb{Z}[\pi_1(M, m_0)] \tag{3.7}$$

which will now satisfy

$$\langle g.\alpha, \beta \rangle = g.\langle \alpha, \beta \rangle$$
$$\langle \alpha, g.\beta \rangle = w(g)\langle \alpha, \beta \rangle.g^{-1}.$$

The symmetry properties of the pairing (3.7) can be phrased by equations reminiscent of Hermitian inner products on complex vector spaces, if we introduce the notation $x \mapsto \overline{x}$ for the anti-involution on the group ring $\Lambda = \mathbb{Z}[\pi_1(M, m_0)]$ given by

$$\overline{\sum_{g \in \pi_1(M, m_0)} a_g.g} = \sum_{g \in \pi_1(M, m_0)} w(g)a_g.g^{-1}.$$

(The "anti" in anti-involution refers to the property that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$, so it defines a ring isomorphism $\Lambda \to \Lambda^{\mathrm{op}}$.) With this notation, the pairing (3.7) on the Λ-modules $H^*(M, \partial M; \mathcal{F}^{\mathrm{univ}} \otimes \omega) \cong H_*(M; \mathcal{F}^{\mathrm{univ}})$ satisfy

$$\alpha \mapsto \langle \alpha, \beta \rangle \text{ is } \Lambda\text{-linear for fixed } \beta$$

and the symmetry property

$$\langle \beta, \alpha \rangle = (-1)^d \overline{\langle \alpha, \beta \rangle}.$$

For compatibility with Wall's book *Surgery on Compact Manifolds*, we may write $\lambda(\alpha, \beta) = \langle \beta, \alpha \rangle$. The above properties are then part of his theorem 5.2.

3.4. **The intersection pairing with local coefficients.** Returning to the situation of an immersed cobordism $j : W \looparrowright [0, 2] \times A$ from $\partial_- W = A$ to $\partial_+ W = B$, and a local coefficient system $\mathcal{F}$ on A, we have seen that the restriction

$$\pi \circ j_{|B} : B \to A$$

induces an isomorphism on homology with coefficients in $\mathcal{F}$ in all degrees, if and only if a certain homomorphism

$$H_n(W, A; \mathcal{F}) \cong K_n(W; \mathcal{F}) \hookrightarrow H^{d-n}(W, A; \mathcal{F}) \tag{3.8}$$

is an isomorphism for all n. As above, we have used the same notation $\mathcal{F}$ for the pullback of the local system to W, along $\pi \circ j : W \to A$.

Let us now choose a basepoint $a_0 \in A$ and consider the corresponding "universal" local system $\mathcal{F}^{\text{univ}} : a \mapsto \mathbb{Z}[\pi_{\leq 1}(a_0, a)]$ on A. In this case we may compose (3.8), which is a homomorphism from homology to cohomology, with the homomorphism from Lemma 2.7 to get a pairing on homology of the form

$$H_n(W, A; \mathcal{F}^{\text{univ}}) \times H_{d-n}(W, A; \mathcal{F}^{\text{univ}}) \to \mathbb{Z}[\pi_1(A, a_0)],$$

which we can also interpret as a pairing on the "universal" surgery kernel $K_*(W; \mathcal{F}^{\text{univ}})$. It is obtained by restricting a similar pairing on $H_*(W; \mathcal{F}^{\text{univ}})$.

For a general cobordism W, the deficiencies of the universal coefficient theorem for local coefficients due to higher Ext groups mean that we cannot easily reinterpret the condition of (3.8) being an isomorphism as a property of this pairing on the surgery kernel. In case Lemma 2.7 applies,

Corollary 3.4. *Let $W \looparrowright [0, 2] \times A$ be an immersed cobordism from A to B, and assume $d = \dim(W) = 2n$ for some $n \geq 3$ and that*

$$W = ([0, 1] \times A) \cup h^h \cup \cdots \cup h^n.$$

Then the following are equivalent:

- *the restriction $\pi \circ j_{|B} : B \to A$ is a homotopy equivalence,*
- *the intersection pairing*

  $$H_n(W, A; \mathcal{F}^{\text{univ}}) \times H_n(W, A; \mathcal{F}^{\text{univ}}) \to \mathbb{Z}[\pi_1(A, a)]$$

  is unimodular for any $a \in A$ (or just one in each path component).

Proof sketch. In Proposition 3.1, we have seen in general that the first condition is equivalent to a certain homomorphism

$$H_k(W, A; \mathcal{F}^{\text{univ}}) \to H^{2n-k}(W, A; \mathcal{F}^{\text{univ}} \otimes \omega) \tag{3.9}$$

being an isomorphism for all k. For W of the indicated form, both domain and codomain vanish for $k \neq n$, so this condition is non-vacuous only for $k = n$, in which case we can use the universal coefficient theorem to rewrite $H^n(W, A; \mathcal{F}^{\text{univ}} \otimes \omega)$ as $\text{Hom}_{\mathbb{Z}[\pi_1(A,a)]}(H_n(W, A; \mathcal{F}^{\text{univ}} \otimes \omega), \mathbb{Z}[\pi_1(A, a)])$; in other words, "cohomology is linear dual to cohomology" in this case. As discussed above, the resulting homomorphism into $\text{Hom}(\dots)$ is adjoint to the intersection pairing on the $\mathbb{Z}[\pi_1(A, a)]$-module $H_n(W, A; \mathcal{F}^{\text{univ}})$, which is then unimodular if and only if (3.9) is an isomorphism. $\square$

4. The intersection pairing, geometric interpretation