1. Immersed cobordisms, surgery kernels, and their intersection forms

Let A and B be compact manifold with identified boundary $\partial A = \Sigma = \partial B$, so we can form a closed manifold $A \cup_\Sigma B$. Equivalently, up to diffeomorphism at least,

$$A \cup_\Sigma B \cong A \cup_{\{0\}\times\Sigma} ([0,1]\times\Sigma) \cup_{\{1\}\times\Sigma} B. \tag{1.1}$$

In this situation, a *cobordism from A to B* is a compact manifold W bounding (1.1), i.e. with a specified diffeomorphism

$$\partial W = A \cup_{\{0\}\times\Sigma} ([0,1]\times\Sigma) \cup_{\{1\}\times\Sigma} B.$$

We will mostly consider such a cobordism to be a manifold with corners,

$$\angle W = (\{0\}\times\Sigma) \sqcup (\{1\}\times\Sigma)$$

and three smooth parts $A \subset \partial W$, $[0,1]\times\Sigma \subset W$, and $B \subset W$ in the sense of Wall. Occasionally it may be convenient to straighten the corners to turn W into a manifold with smooth boundary $A \cup_\Sigma B$.

We will be particularly interested in the situation where the cobordism comes with an immersion

$$j : W \looparrowright [0,2]\times A$$

satisfying $j(x) = (0,x)$ for $a \in A \subset \partial W$ and $j(t,\sigma) = (t,\sigma)$ for $(t,\sigma) \in [0,1]\times\Sigma \subset \partial W$. The composition

$$A \overset{i}{\hookrightarrow} W \looparrowright [0,2]\times A \overset{\pi}{\twoheadrightarrow} A \tag{1.2}$$

is the identity, where π denotes the projection to the second factor. This gives a splitting of chain complexes and of homology

$$C_*(A) \oplus \mathrm{Ker}((\pi\circ j)_* : C_*(W) \to C_*(A)) \overset{\cong}{\to} C_*(W)$$
$$H_n(A) \oplus \mathrm{Ker}((\pi\circ j)_* : H_n(W) \overset{\cong}{\to} H_n(A))$$

and we shall write

$$C_*(A,W) = \mathrm{Ker}((\pi\circ j)_* : C_*(W) \to C_*(A)) \subset C_*(W)$$
$$K_n(W) = H_n(C_*(A,W)) \cong \mathrm{Ker}((\pi\circ j)_* : H_n(W) \to H_n(A)).$$

In this situation, the second factor in the splitting

$$H_n(W) = H_n(A) \oplus K_n(W)$$

is called the *surgery kernel*. As an abstract group it is canonically isomorphic to the relative homology $H_n(W,A)$, which does not depend on the immersion j, but splitting the map $H_n(W) \to H_n(W,A)$ and realizing the surgery kernel as a summand of $H_n(W)$ does.

Given an orientation $[W] \in H_d(W,\partial W)$ we get $(-1)^d$-symmetric pairings

$$(K_n(W)/TK_n(W)) \otimes (K_{d-n}(W)/TK_{d-n}(W)) \to \mathbb{Z} \tag{1.3}$$

where $TK_n(W) \subset K_n(W)$ denotes the subgroup consisting of finite-order elements, by restricting the intersection pairing on $H_*(W) \cong H^{d-*}(W,\partial W)$. It is at this point that we use that the surgery kernel is a subgroup of $H_n(W)$ instead of a quotient group.

Similarly, by restricting the linking pairing on $TH_*(W)$ we obtain a linking pairing

$$TK_n(W) \otimes TK_{d-n-1}(W) \to \mathbb{Q}/\mathbb{Z}. \tag{1.4}$$

On the remaining smooth part $B \subset \partial W$, we consider the composition

$$B \hookrightarrow \partial W \hookrightarrow W \looparrowright [0,2]\times A \twoheadrightarrow A \tag{1.5}$$

which is the identity on $\Sigma = \partial A = \partial B$. The following is a generalization of the discussion of when the boundary of a compact simply connected manifold is a homology sphere.

Proposition 1.1. *In the situation above, the map (1.5) induces an isomorphism on integral homology if and only if both pairings (1.3) and (1.4) are unimodular for all n.*

Proof. Recall that $K_n(W)$ was defined as homology of the chain complex given as the kernel of the surjection $j_*(W) \to j_*(A)$, and hence sits in a long exact sequence with $(\pi \circ j)_* : H_*(W) \to H_*(A)$. We may compare this to homology of the mapping cone of $j_* : C_*(W) \to C_*(A)$ which sits in a similar long exact sequence, and deduce a canonical isomorphism

$$K_n(W) \cong H_{n+1}(C((\pi \circ j)_*)).$$

Inside the mapping cone of $(\pi \circ j)_*$, we have the mapping cone of $(\pi \circ j_{|B})_* : C_*(B) \to C_*(A)$, and we have a short exact sequence of chain complexes

$$C((\pi \circ j_{|B})_*) \to C((\pi \circ j)_*) \to C_{*-1}(W, B)$$

inducing a long exact sequence

$$\cdots \to H_{n+1}(C((\pi \circ j_{|B})_*)) \to K_n(W) \to H_n(W, B) \to H_n(C((\pi \circ j_{|B})_*)) \to \ldots.$$

Since the outer terms also sit in a long exact sequence with $(\pi \circ j_{|B}) : H_n(A) \to H_n(B)$, we deduce that this map is an isomorphism for all n if and only it the middle map

$$K_n(W) \to H_n(W, B) \tag{1.6}$$

is an isomorphism for all n. This map can be idenfied with the composition

$$K_n(W) \hookrightarrow H_n(W) \to H_n(W, B)$$

of the inclusion as a summand, composed with the map from the long exact sequence of the pair (W, B).

Now by Poincaré duality in the form

$$\begin{aligned} H^{d-n}(W, A) &\to H_n(W, B) \\ \alpha &\mapsto [M] \frown \alpha \end{aligned} \tag{1.7}$$

this is in turn identified with a map

$$K_n(W) \hookrightarrow H_n(W) \xleftarrow{\cong} H^{d-n}(W, \partial W) \to H^{d-n}(W, A). \tag{1.8}$$

On torsion free quotients, this homomorphism can be identified with

$$K_n(W)/TK_n(W) \to \mathrm{Hom}_{\mathbb{Z}}(H_{d-n}(W, A), \mathbb{Z}),$$

which under the isomorphism $K_{d-n}(W) \cong H_{d-n}(W, A)$ is the adjoint of (1.3). Similarly, on the torsion subgroups the homomorphism (1.8) may be identified with

$$TK_n(W) \to \mathrm{Ext}_{\mathbb{Z}}(H_{d-n-1}(W, A), \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(TH_{d-n-1}(W, A), \mathbb{Q}/\mathbb{Z}),$$

adjoint to (1.4). □

Corollary 1.2. *If $K_n(W)$ is a free abelian group for all n, then (1.5) induces an isomorphism on integral homology if and only if the pairings*

$$K_n(W) \otimes K_{d-n}(W) \to \mathbb{Z}$$

are unimodular for all n.

Remark 1.3. The splitting $H_n(W) \cong H_n(A) \oplus K_n(W)$ has an analogue in cohomology, also constructed from the composition $A \hookrightarrow W \looparrowright [0,2] \times A \twoheadrightarrow A$ being the identity. We can write this as

$$H^n(W) \cong H^n(A) \oplus K^n(W),$$

where $K^n(W) \subset H^n(W)$ is the kernel of the (split) surjection $i^* : H^n(W) \to H^n(A)$ and therefore the image of the (split) injection $H^n(W, A) \to H^n(W)$. From (1.8) in the proof of the Proposition, we see that when W is oriented, Poincaré duality induces a homomorphism

$$K_n(W) \to K^{d-n}(W)$$

and that this map is an isomorphism if and only if (1.5) induces an isomorphism on integral homology.

Definition 1.4. The *structure set* of the compact smooth manifold A is the set

$$S(A) = \{(M, f) \mid M \text{ compact smooth, } f : M \overset{\simeq}{\to} A,\ f_{|\partial M} : \partial M \to \partial A \text{ diffeo.}\}/\sim,$$

where the equivalence relation $(M, f) \sim (M', f')$ when there exists a diffeomorphism $\phi : M \to M'$ such that $f' \circ \phi_{|\partial M} = f_{|\partial M}$ and $f' \sim \phi \simeq f$ relative to ∂M.

As a strategy for producing elements of the structure set, we may first construct an immersed cobordism $j : W \looparrowright [0,2] \times A$ and then restricting j to the outgoing boundary. The Proposition and the Corollary give a criterion for this restriction to be a homology equivalence; if both A and the outgoing boundary are simply connected then homology equivalence implies homotopy equivalence by the Hurewicz theorem.

2. Immersed cobordisms from handle attachments

If the immersed cobordism $W \looparrowright [0,2] \times A$ is obtained, as an abstract cobordism, as a handle attachment

$$W = ([0,1] \times A) \cup h^n \cup \cdots \cup h^n,$$

with k many n-handles attached along disjoint embeddings

$$e_i : S^{n-1} \times D^{d-n} \to \{1\} \times A, \quad i = 1, \ldots, k \tag{2.1}$$

then the condition in Corollary 1.2 that the surgery kernels be free holds. Indeed, $H_*(W, A) = 0$ for $* \neq n$ while $H_n(W, A)$ is the free abelian group with one generator for each n-handle. If the resulting cobordism is given an immersion into $[0,2] \times A$, then Corollary 1.2 gives a necessary and sufficient condition for the induced map from the outgoing boundary to A being a homology isomorphism.

For unimodularity to hold, it is necessary that the rank of $K_n(W)$ equals the rank of $K_{d-n}(W)$, so the most interesting case is $d = 2n$. In other words, given a compact $(2n-1)$-manifold A, possibly with boundary, we attach n-handles to the trivial cobordism $[0,1] \times A$.

Remark 2.1. In order to actually write down an immersion

$$W = ([0,1] \times A) \cup h^n \cup \cdots \cup h^n \hookrightarrow [0,2] \times A$$

in practice, for given embeddings (2.1), it essentially suffices to specify immersions

$$E_i : D^n \times D^{d-n} \to [1,2] \times A, \quad i = 1, \ldots, k$$

such that $E_i(x, y) = e_i(x, y)$ for $x \in \partial D^n = S^{n-1}$. Without making further choices, we then obtain a map

$$W \looparrowright [0,2] \times A$$

whose restriction to $[0,1] \times A \subset W$ is the inclusion and whose restriction to the ith handle $D^n \times D^{d-n} \hookrightarrow W$ is given by E_i. As it stands, this map $W \to [0,2] \times A$

is a *topological immersion*: any point in its domain admits an open neighborhood on which the map restricts to a topological embedding onto its image. It is not quite a smooth map because of the corners created in the outgoing boundary of $[0,1] \times A$ before gluing the handles, but it does restrict to a smooth immersion of the complement of the sets $e_i(S^{n-1} \times D^{d-n-1}) \subset \partial_+ W$. To obtain a smooth immersion of the entire cobordism, we simply compose with a suitable self-embedding

$$W \hookrightarrow W \setminus \bigcup_{i=1}^{k} e_i(S^{n-1} \times D^{d-n-1}),$$

which can be constructed using bump functions and a choice of collar of the outgoing boundary.

Example 2.2 (Attaching a handle trivially)**.** To "trivially" attach an n-handle to $[0,1] \times A$, compose a chart

$$\mathbb{R}^{d-1} = \mathbb{R}^n \times \mathbb{R}^{d-n-1} \xrightarrow{c} A$$

with the "standard" embedding

$$\begin{aligned} S^{n-1} \times D^{d-n} &\overset{e_0}{\hookrightarrow} \mathbb{R}^n \times \mathbb{R}^{d-n-1} \\ (x,y) &\mapsto \tfrac{2+y_1}{4}(x,0) + \tfrac{1}{4}(0, y_2, \dots, y_{d-n}). \end{aligned} \tag{2.2}$$

A distinguishing feature of the trivial handle attachment is that this cobordism not only immerses, but in fact *embeds* into $[0,2] \times A$. The most convincing way to see this is perhaps low-dimensional pictures, but let us try to explain how to construct an embedding using formulas.

Writing coordinates on S^n as $(x_0, x) \in S^n \subset \mathbb{R} \times \mathbb{R}^n$ with $x = (x_1, \dots, x_n)$, the embedding

$$\begin{aligned} S^n \times D^{d-n} &\overset{e_1}{\hookrightarrow} [0,2] \times \mathbb{R}^n \times \mathbb{R}^{d-n-1} \\ (x_0, x, y) &\mapsto (1,0,0) + \tfrac{2+y_1}{4}(x_0, x, 0) + \tfrac{1}{4}(0,0,y_2, \dots, y_{d-n}) \end{aligned} \tag{2.3}$$

extends (2.2) in the sense that $e_1(0,x,y) = (1, e_0(x,y)) \in [0,2] \times \mathbb{R}^{d-1}$. The subspace of the domain of (2.3) defined by $x_0 \leq 0$ is diffeomorphic to $D^n \times D^{d-n}$, using e.g. stereographic projection to identify D^n with the upper hemisphere in S^n, so we can use this subspace as a model for an n-handle. The resulting embedding

$$\begin{aligned} \{(x_0, x, y) \in S^n \times D^{d-n} \mid x_0 \geq 0\} &\to [1,2] \times A \\ (x_0, x, y) &\mapsto (\mathrm{id}_{[1,2]} \times c) \circ e_1(x_0, x, y) \end{aligned} \tag{2.4}$$

glues along (2.2) to the inclusion $[0,1] \times A \hookrightarrow [0,2] \times A$ to an embedding

$$W = ([0,1] \times A) \cup_{e_0} h^n \to [0,2] \times A.$$

As in Remark 2.1, this glued map will strictly speaking not be smooth at the corners created before attaching the handle. To obtain a smooth embedding, pre-compose with a self-embedding as in that remark.

The formula (2.3) also gives a smooth embedding $f : S^n \times \mathring{D}^{d-n} \hookrightarrow W$, and the image $f_*([S^n]) \in H_n(W)$ is a canonical generator of the surgery kernel

$$\mathbb{Z} \cong K_n(W) \subset H_n(W).$$

Example 2.3 (Attaching multiple handles trivially)**.** The standard embedding (2.2) has image contained in the unit disk in $\mathbb{R}^{d-1}$, so we may obtain disjoint embeddings simply by displacing it by some vector, e.g.

$$\begin{aligned} \{1, \dots, k\} \times S^{n-1} \times D^{d-n} &\hookrightarrow \mathbb{R}^{d-1} \\ (i, x, y) &\mapsto (i, 0, 0) + e_0(x, y). \end{aligned}$$

Composing also with a chart $c : \mathbb{R}^{d-1} \to A$ then gives the necessary data to form the cobordism
$$W = ([0,1] \times A) \cup h^n \cup \cdots \cup h^n.$$
The embedding explained in Example 2.2 evidently generalizes to this setting, giving an embedding $W \hookrightarrow [0,2] \times A$. As in the example, the formula (2.3) gives embeddings
$$f_i : S^n \times \mathring{D}^{d-n} \to W, \quad i = 1, \ldots, k,$$
and the classes $(f_i)_*([S^n]) \in H_n(W)$ form a basis for the surgery kernel
$$\mathbb{Z}^{\oplus k} \cong K_n(W) \subset H_n(W).$$

In the case where $d = 2n$ and A is oriented, we have the intersection form
$$K_n(W) \otimes K_n(W) \to \mathbb{Z}$$
defined, but it turns out to be zero when the handles are attached in the particular way described in this example. To see this we use that the image of the class $(f_i)_*([S^n])$ under Poincaré duality $H_n(W) \cong H^n(W, \partial W)$ lifts to a class $\alpha_i \in H^n(W, W \setminus f_i(S^n \times \{0\}))$. For $i \neq j$ the cup product $\alpha_i \smile \alpha_j$ then lifts to the zero group $H^{2n}(W, W)$.

A similar argument applies for the self-intersection $\alpha_i \smile \alpha_i$, using that α_i also has support $f_i(S^n \times \{y\})$ for any $y \in \mathring{D}^n$ and we may choose $y \neq 0$.