Shor’s Algorithms

Giovanni Galatro

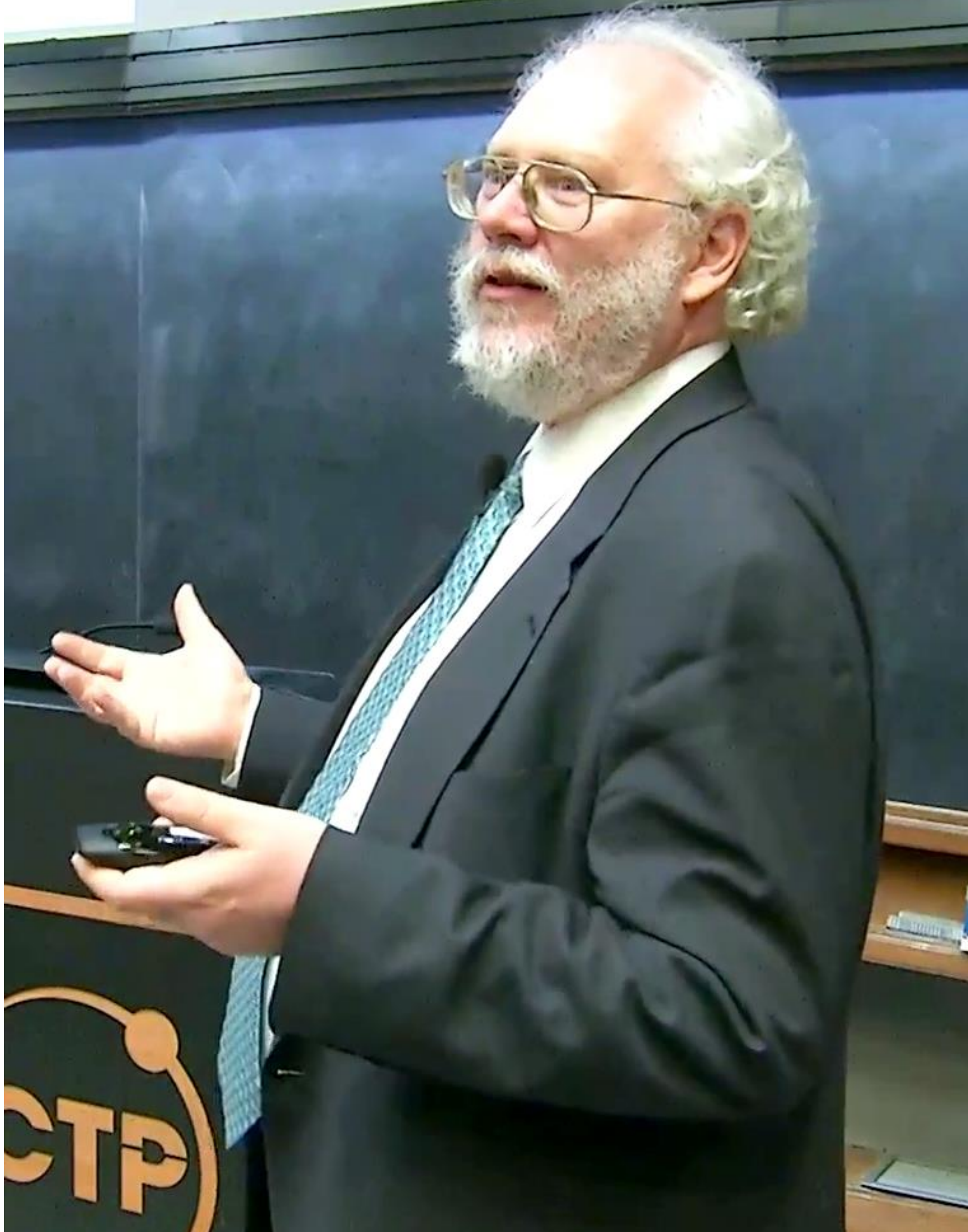
Shor’s Algorithm

Applications

- Factorization of large integers
- Cryptanalysis (breaking RSA encryption)

Generalization

- Discrete logarithm problem
- Period-finding problem



Reminder

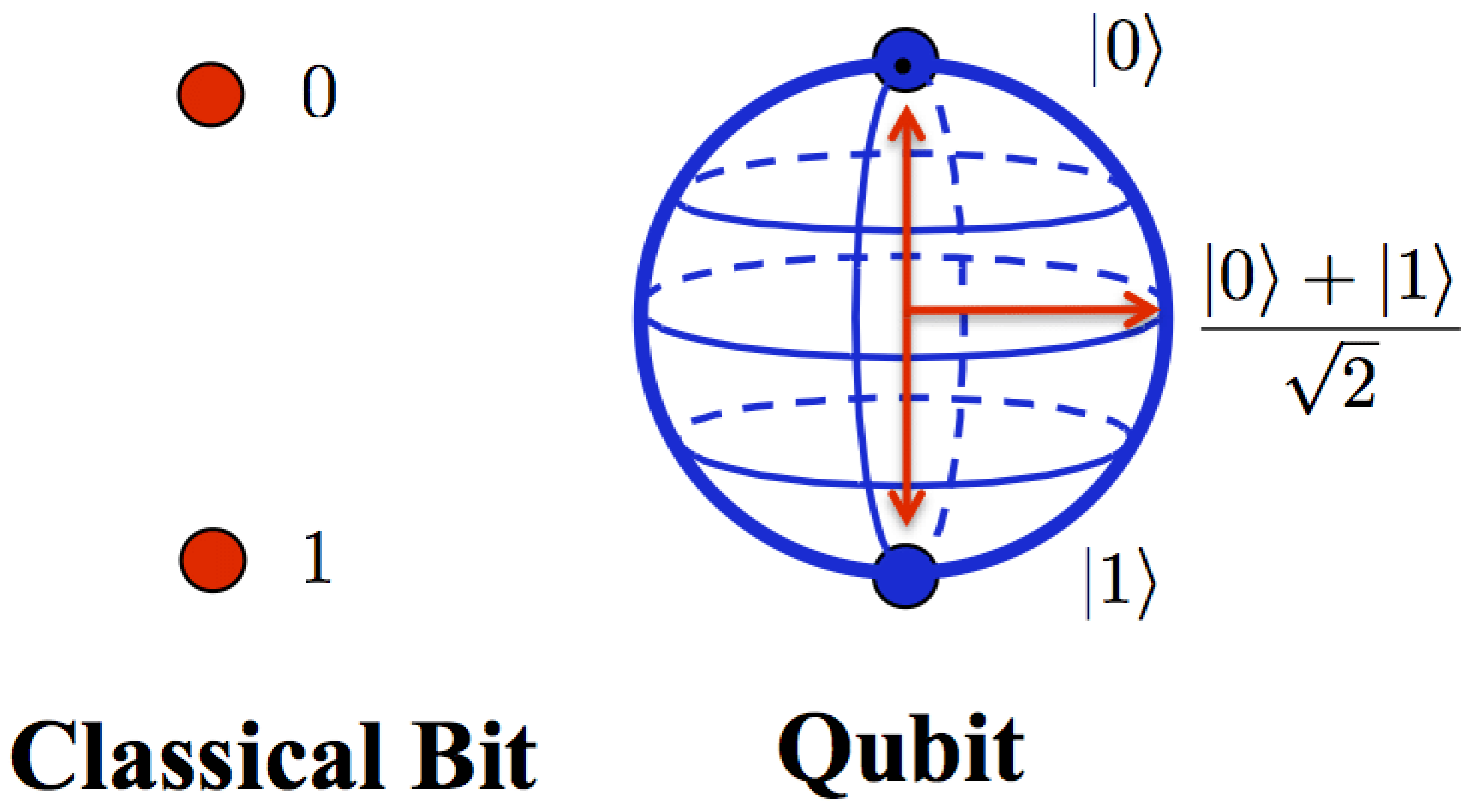
Classical bits have two states (0 or 1), but qubits can exist in a superposition of both simultaneously:

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle$$

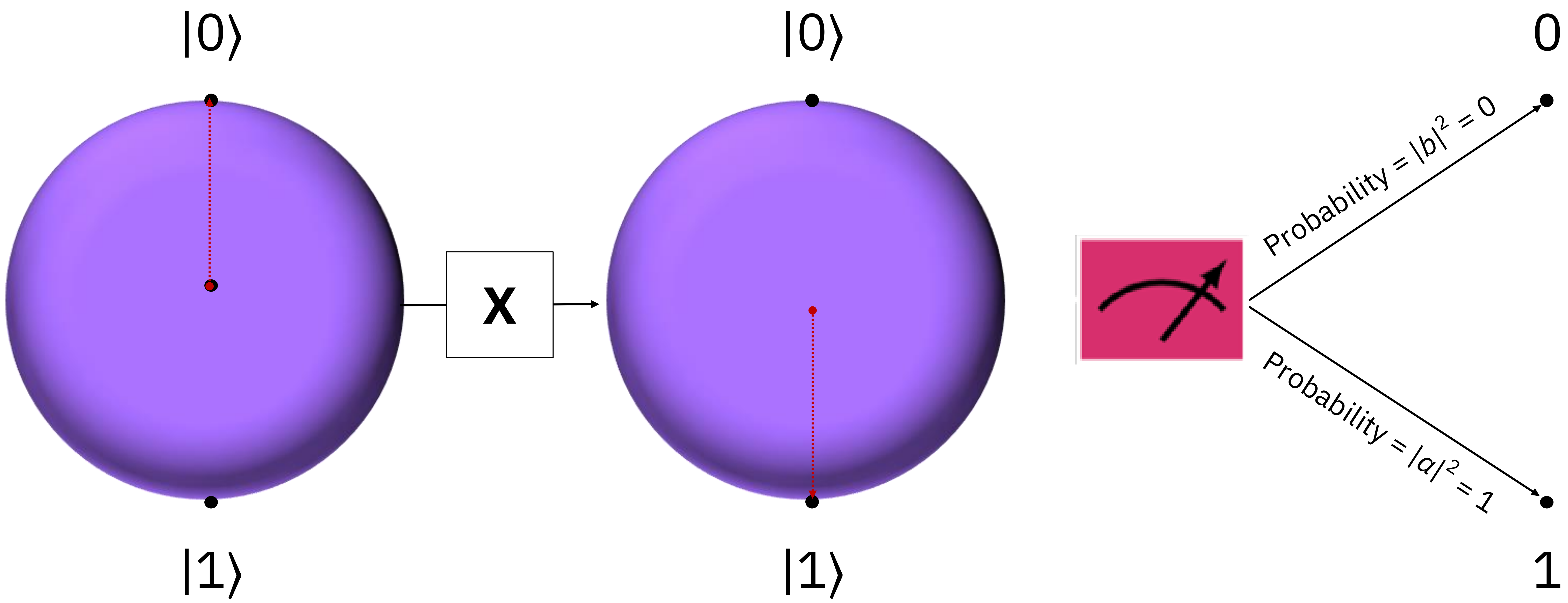
Quantum gates manipulate qubit states.

Es. Hadamard Transform:

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



Bits and qubits: the effect of the X gate on $|0\rangle$



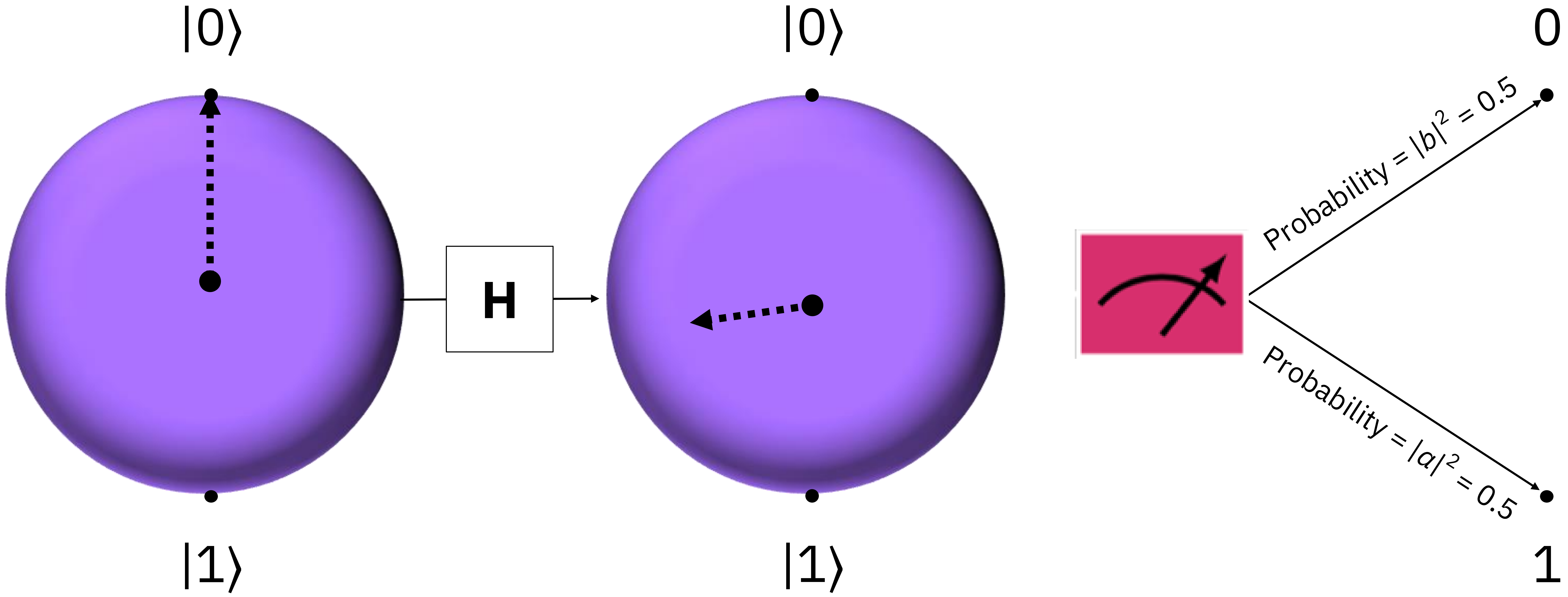
The **X** gate reverses $|0\rangle$ and $|1\rangle$:

$$a\,|0\rangle + b\,|1\rangle \;\mapsto\; b\,|0\rangle + a\,|1\rangle$$

$a = 1$ and $b = 0$, so $|0\rangle$ is mapped to $|1\rangle$.

When measured, the result is **1** with 100% probability.

Bits and qubits: the effect of the H gate on $|0\rangle$



The **H** gate maps $|0\rangle$ via

$|0\rangle \mapsto (1/\sqrt{2})\,|0\rangle + (1/\sqrt{2})\,|1\rangle = a\,|0\rangle + b\,|1\rangle$

Since $a = b = 1/\sqrt{2}$, $|a|^2 = |b|^2 = ½$.

When measured, the probability of getting **0** or **1** is the same, 0.5.
Quantum randomness!

Qiskit is the most preferred and performant quantum SDK

74%

Quantum computing programmers prefer the Qiskit SDK

(2024 Unitary Foundation Open Source Software Survey)

7400+

Dependent projects

(Next nearest: PennyLane, 1000+)

83x

Mean transpilation time improvement

(Qiskit 2.2.0 vs TKET 2.6.0)

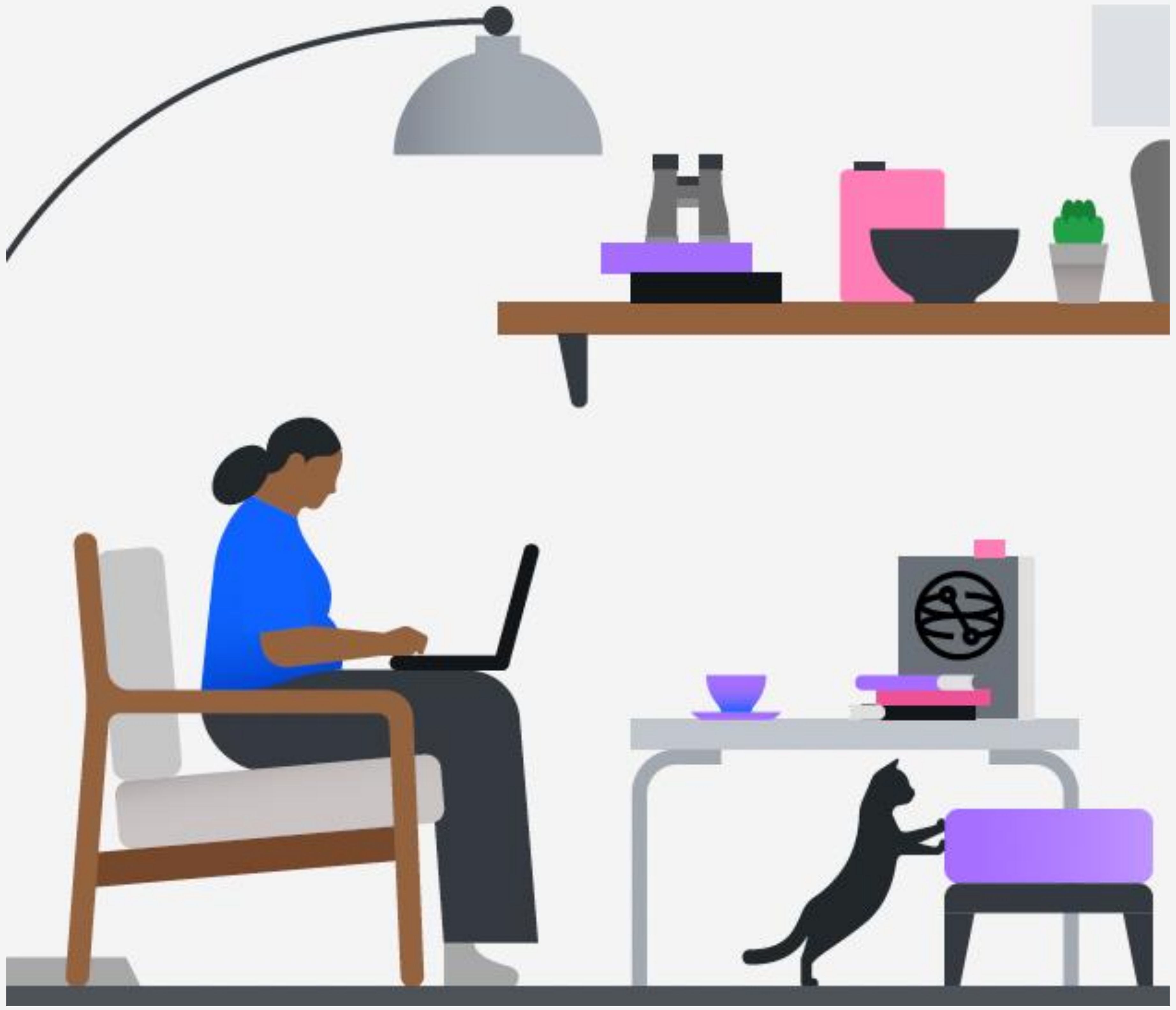
29%

Fewer 2Q gates

(Qiskit 2.2.0 vs TKET 2.6.0)

Results reported from Benchpress tests of ~1000 circuits (https://github.com/Qiskit/benchpress) and published in *Nature Computational Science* (2025): https://www.nature.com/articles/s43588-025-00792-y. Timing and quality measured using only completed tests. Dependencies from Github insights. Current as of 10/14/25.

IBM Certified Quantum Computation using Qiskit v2.X Developer



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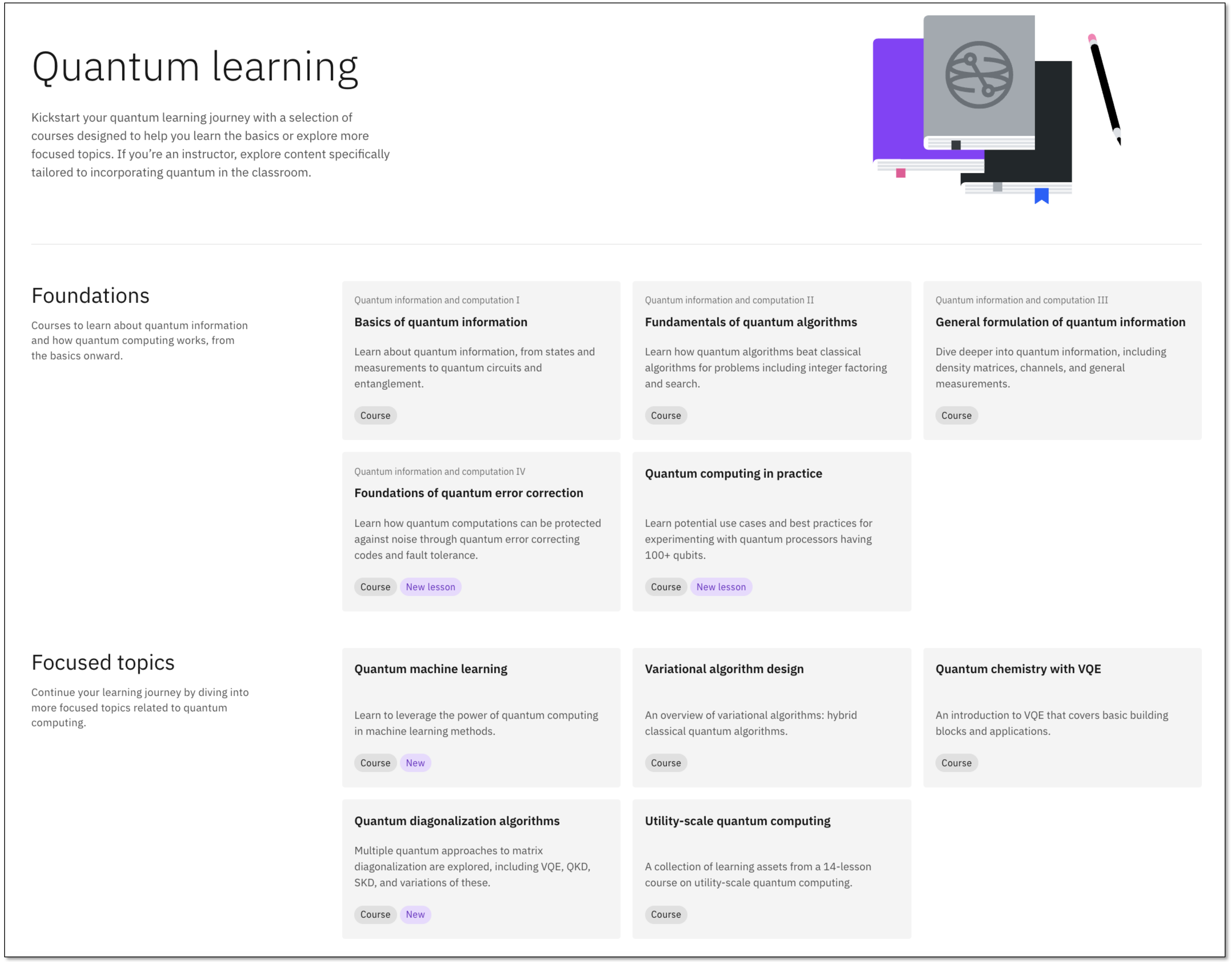
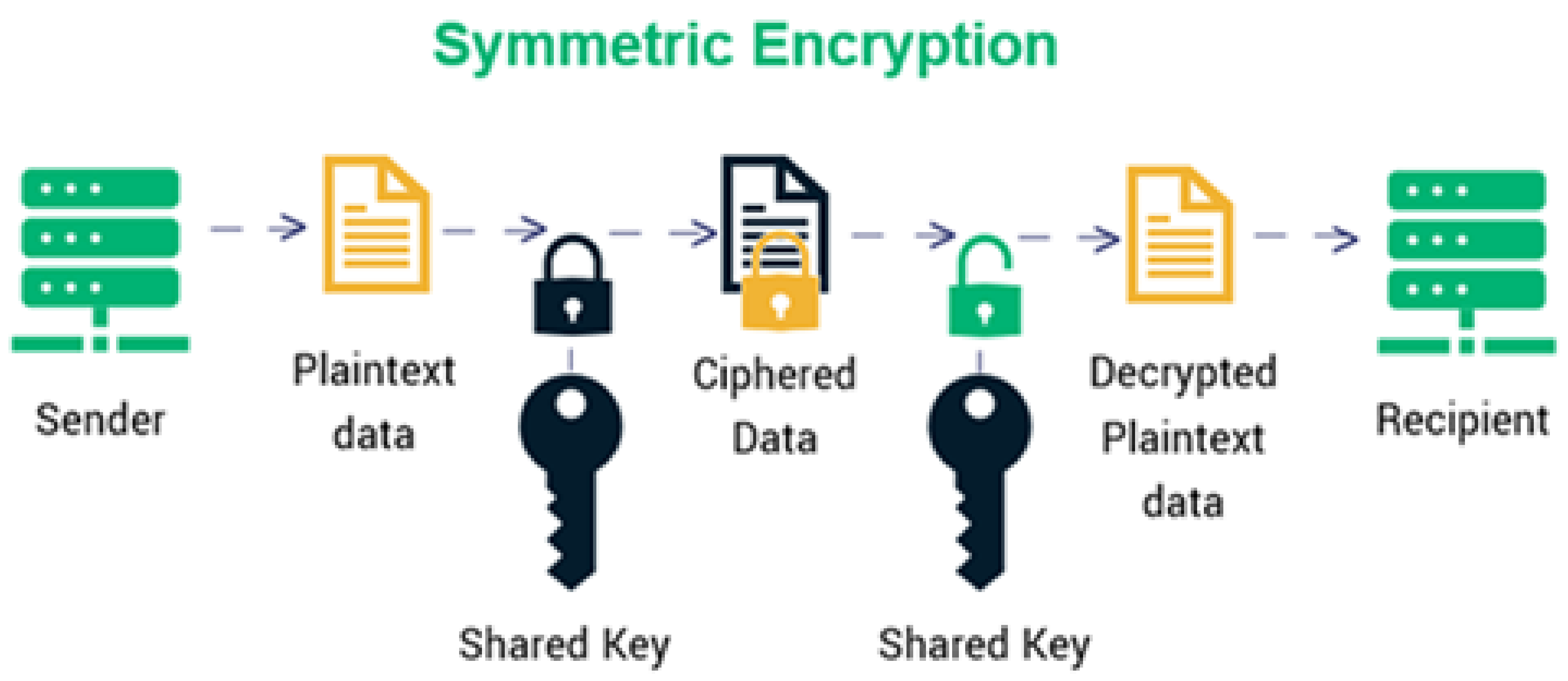
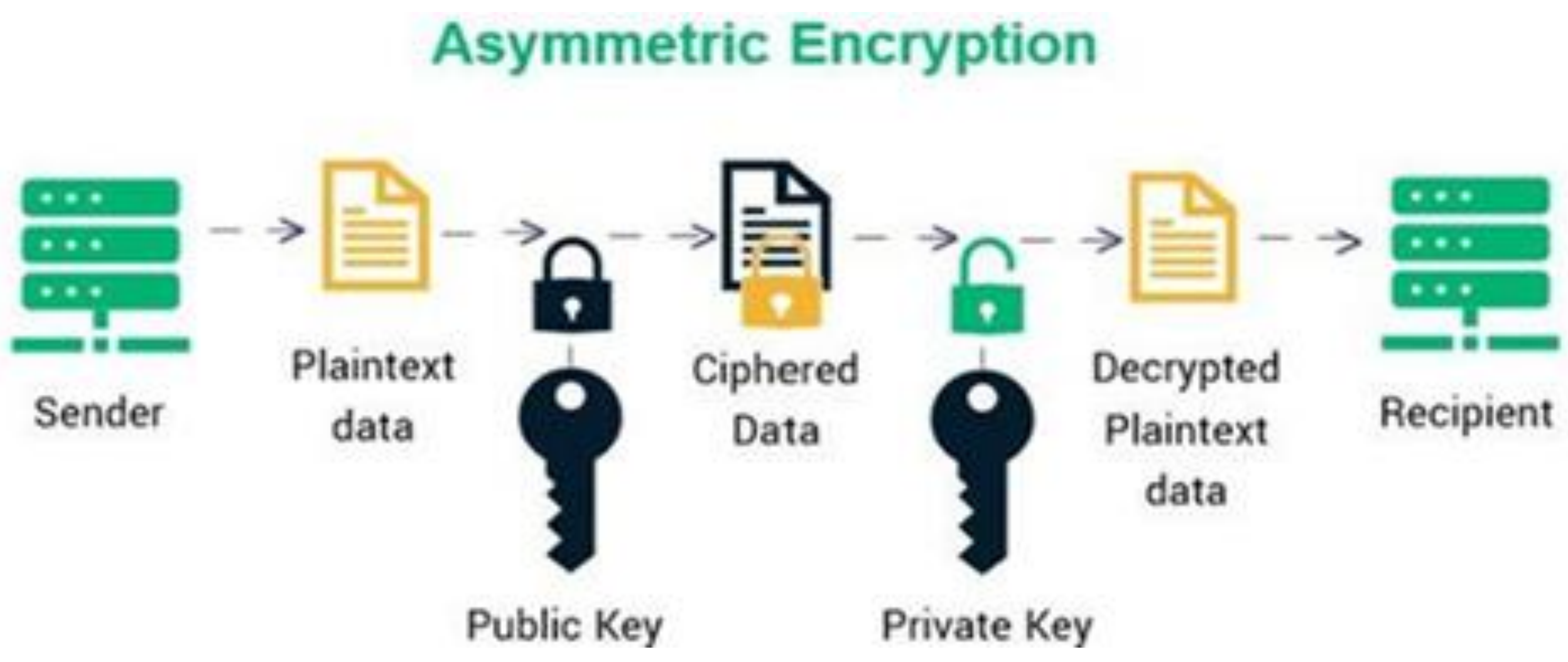


Shor's Algorithm

The fundamental crypto primitives





- Use **the same (secret) Shared key** for encryption and decryption.
- Simple and efficient
- Problem: the shared key need to be .... Shared
- Most common: Advanced Encryption Standard (AES), ChaCha20, (3DES, DES (deprecated))

- Uses two keys — a **public key** for encrypting data and a **private key** for decrypting it.
- Computationally more expensive
- Most common: Rivest-Shamir-Adleman (RSA), Elliptic Curve Cryptography (ECC), Digital Signature Algorithm (DSA)

Image source :https://sectigostore.com/blog/public-key-vs-private-key-how-do-they-work

How can we say they are safe?

- Both kinds of primitives are using varying degrees of mathematical structure. Breaking a primitive means solve an "hard" mathematical problem.
- Hardness assumptions: the hypothesis that a particular computational problem cannot be solved efficiently (in polynomial time).

Today's **asymmetric** encryption relies mostly on two mathematical problems with the hardness assumption:

Factoring

Given a number N, find a pair of prime numbers (p, q), $p < q$, $p \approx q$, whose product is equal to $N = p \cdot q$

Discrete Logarithms (DLOG)

Given a finite cyclic group G, a generator $\mathrm{g} \in G$ and an element $h \in G$ find the integer x where $g^x = h$

What changes in last 40 years?

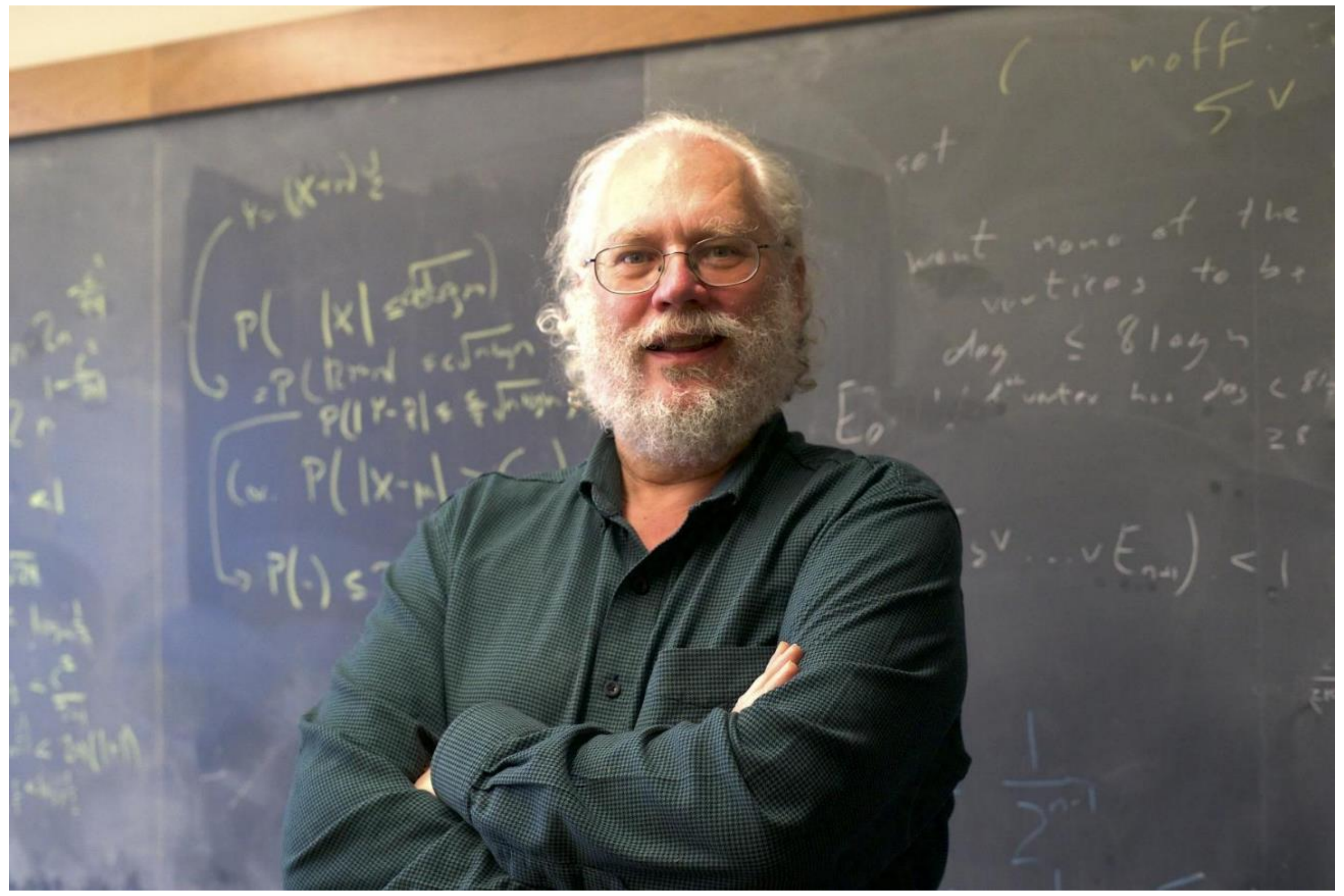
Shor's Algorithm

- General number field sieve (GNFS) is the most efficient classic algorithm for factoring integers larger than $\mathbf{10^{100}}$. Complexity:

$$\exp\left(\left((64/9)^{1/3}+o(1)\right)(\log n)^{1/3}(\log\log n)^{2/3}\right)$$

- Shor algorithm have a complexity of:

$$O\left((\log N)^2(\log\log N)(\log\log\log N)\right)$$

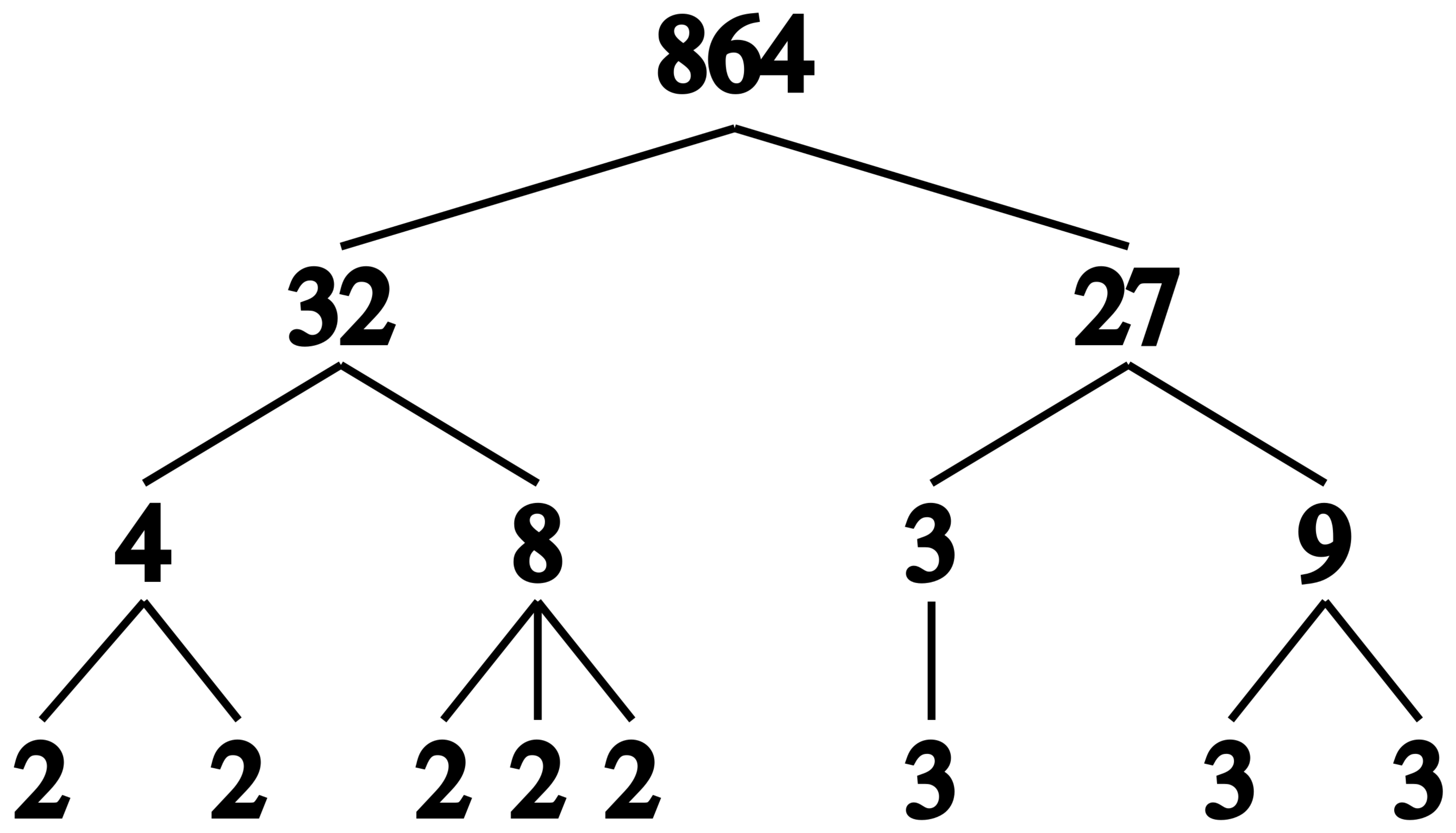


- We moved from subexponential in log N (hard) to poly-logarithmic (easy)
- Shor proposed multiple similar algorithms for solving the factoring_problem, the discret proposed e logarithm problem, and the period-finding problem
- In just one hit we lost two families of hardness assumptions widely used for PKC

Factorization problem

Fundamental theorem of arithmetic:

every integer greater than 1 can be represented uniquely as a product of prime numbers



How can we do it?

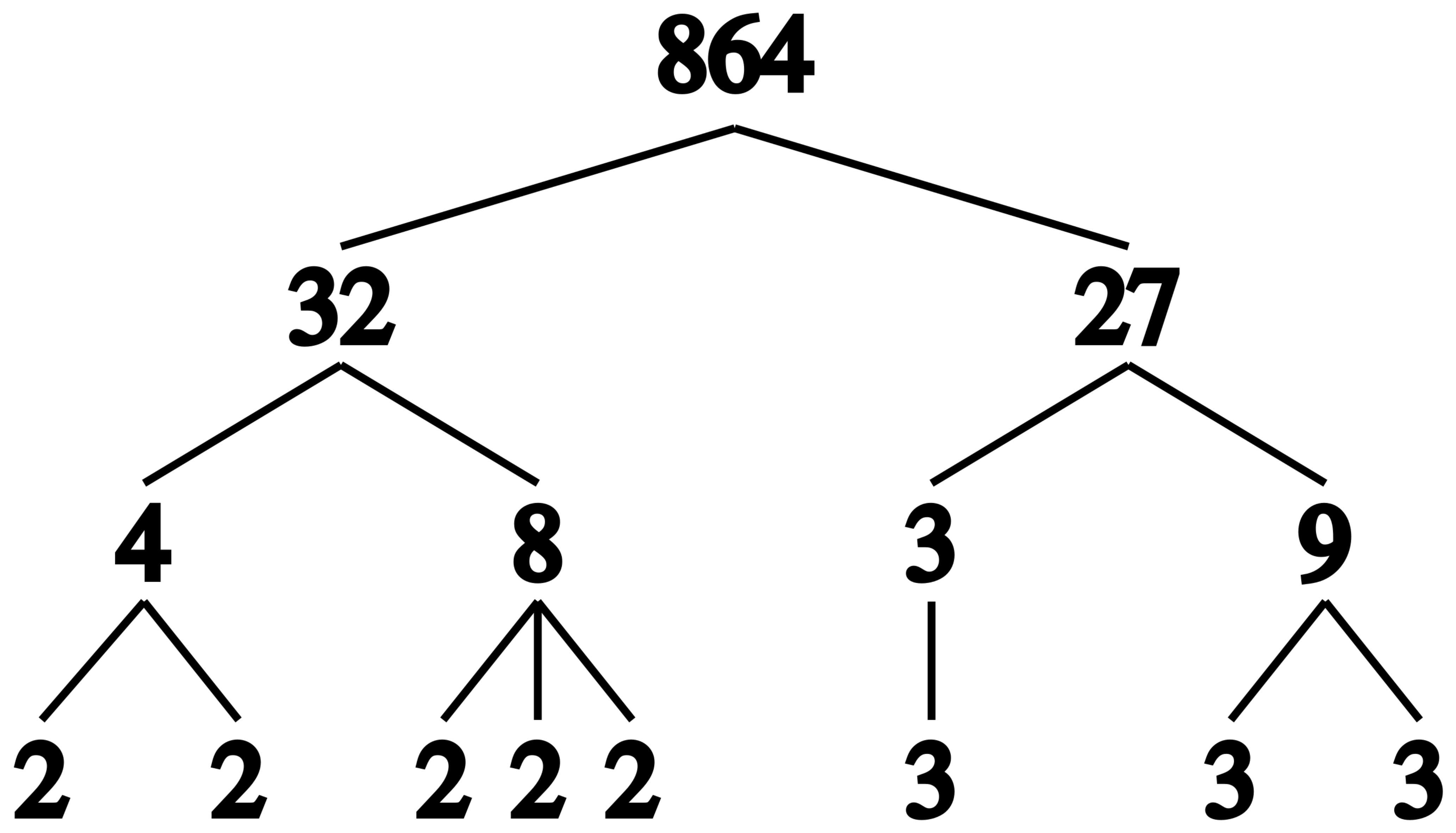
Base approach:

whether the number is divisible by the prime numbers, 2, 3, 5 and so on up to the square root of the number.

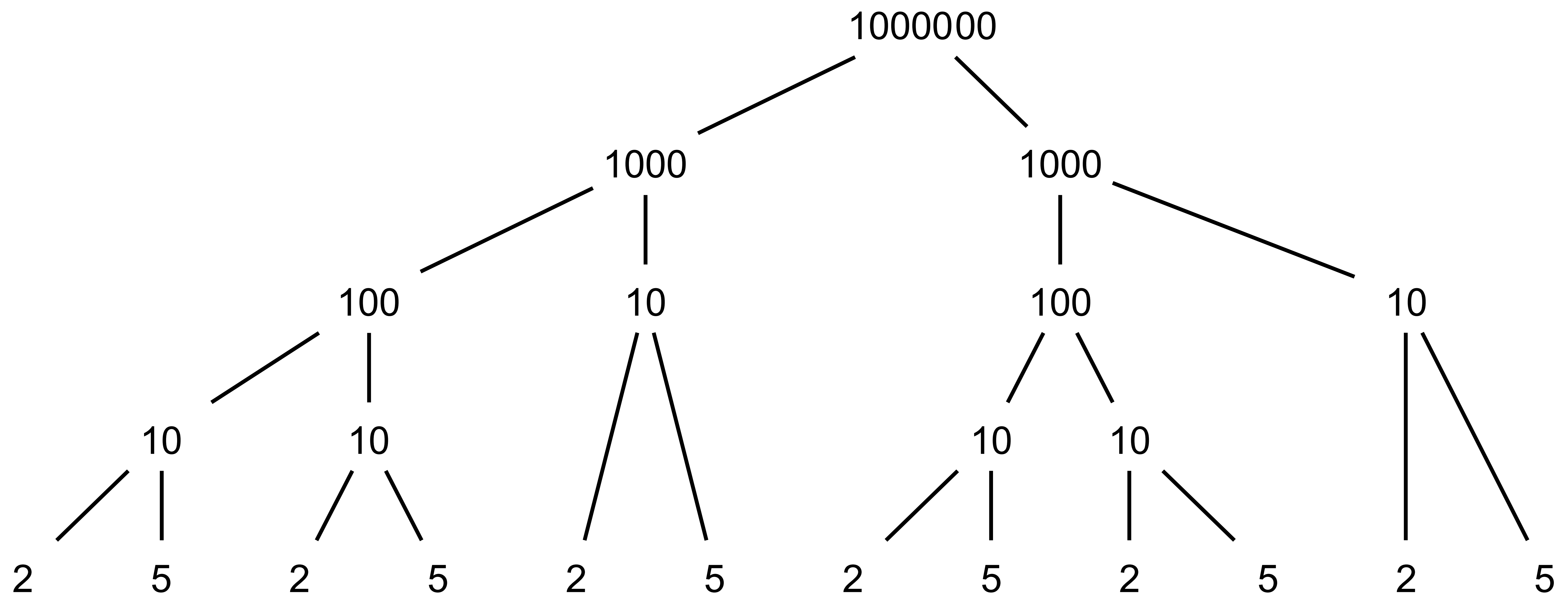
Best approach:

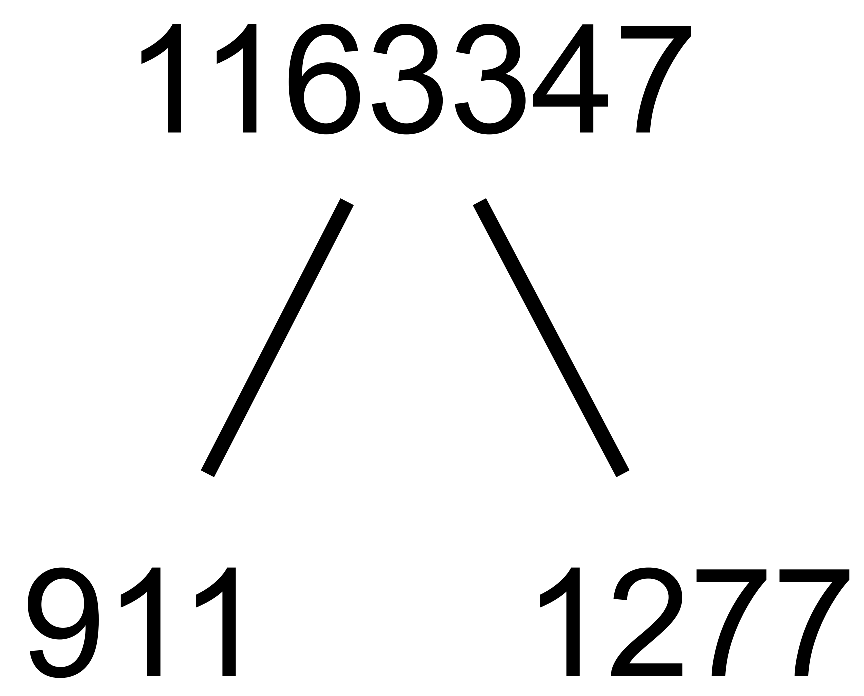
The General number field sieve (1993), for numbers > 10^{100}

$$\exp\left(\left(\left(\frac{8}{3}\right)^{\frac{2}{3}} + o(1)\right)(\log n)^{\frac{1}{3}}(\log\log n)^{\frac{2}{3}}\right).$$



Different difficulties





Factorization in practice

Factoring a number N coincides with choosing a random integer m that is coprime (i.e. has no divisor in common) with N and finding the multiplicative order P , i.e. finding P such that:

$$m^P \equiv 1(modN)$$

where the symbol ≡ stands for congruence in modulus.

Thus, ∃k ∈ N such that

$$m^P - 1 = kN.$$

Shor's solution

Principle of Operation

Shor's algorithm is a masterpiece.
It works by reducing the factorization problem to a period-finding problem. Here are the key steps:

- Transforms the factorization problem into a period-finding problem
- Uses the Quantum Fourier Transform to determine this period
- Utilizes the discovered period to compute the factors of the original number

The true magic lies in the **Quantum Fourier Transform**, which can be executed exponentially faster on a quantum computer compared to a classical one.

Implementation Challenges

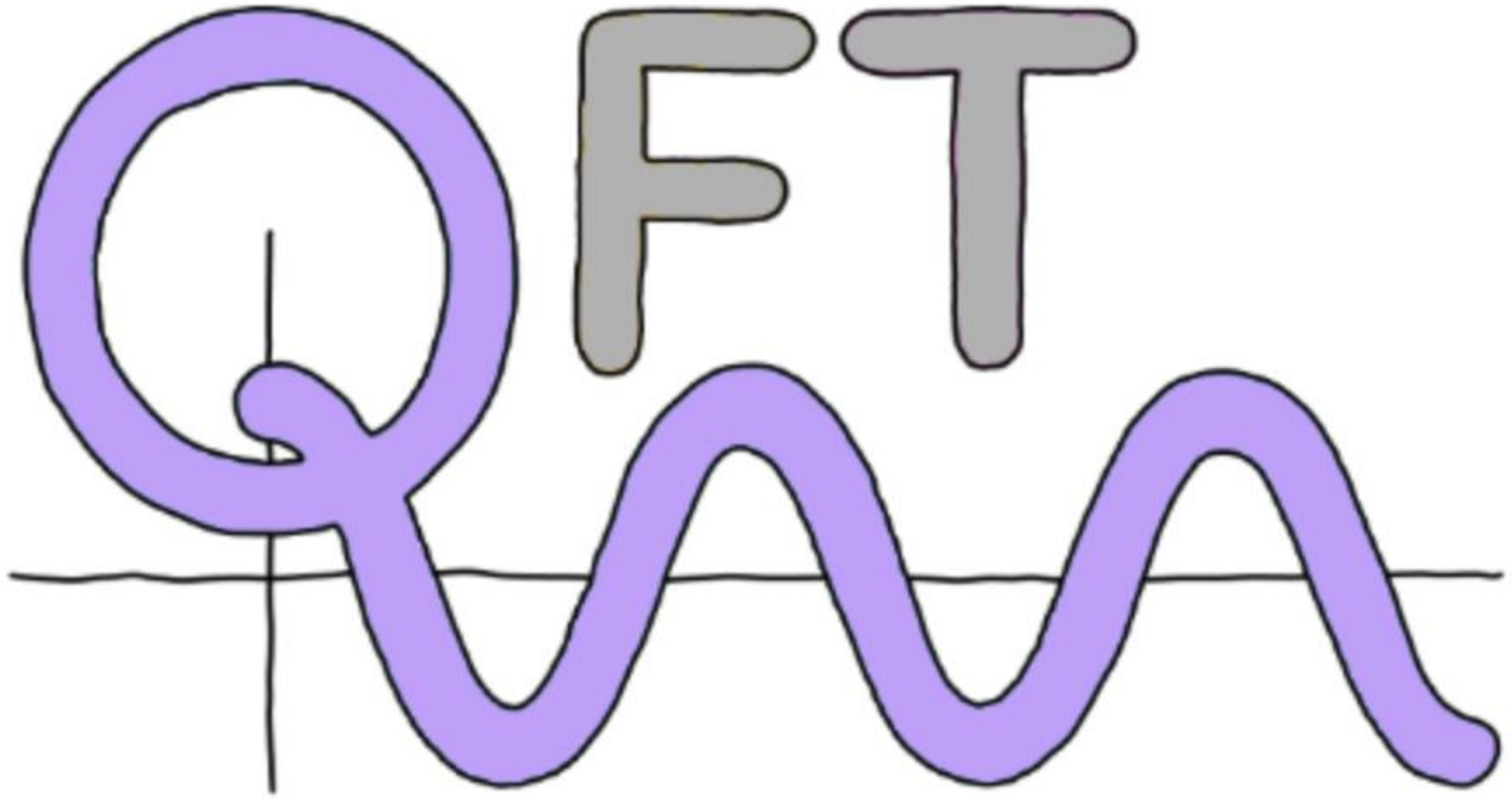
Despite its theoretical elegance, the practical implementation of Shor's algorithm presents significant challenges:

- Requires a large number of stable qubits
- Is highly sensitive to errors and decoherence
- Needs high-precision quantum gates

Quantum Fourier transform

The Quantum Fourier Transform (QFT) is a quantum analog of the discrete Fourier transform — the main tool of digital signal processing — which is used to analyze periodic functions by mapping between time and frequency representations.

The Discrete Fourier Transform (DFT) is a transform that converts a finite collection of N equispaced points of a function into a collection of coefficients of a linear combination of complex sinusoids, ordered with increasing frequency.



Quantum Fourier transform

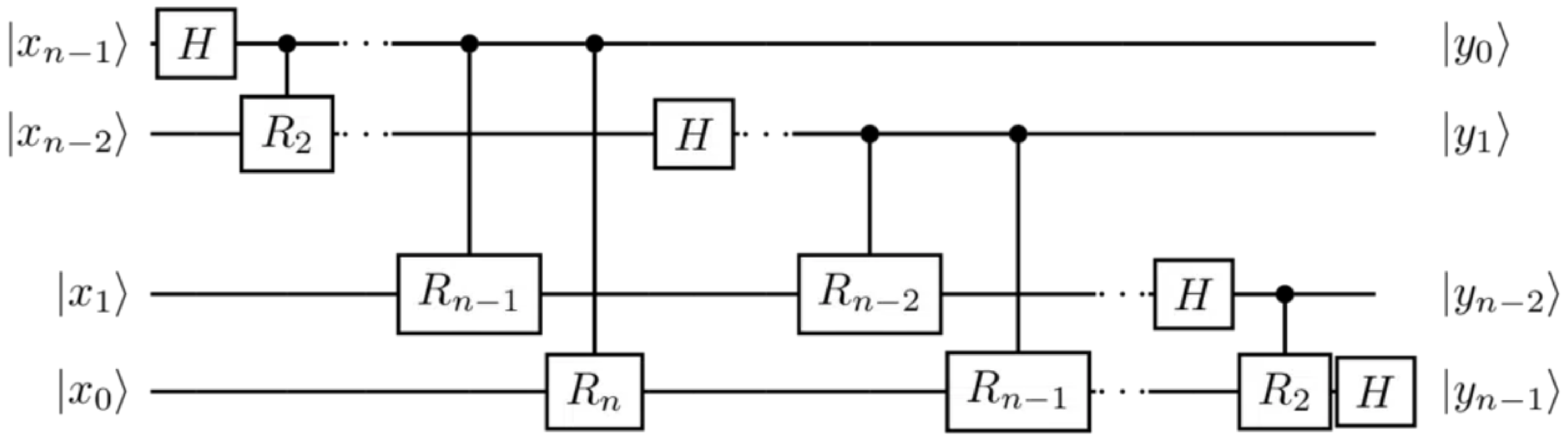
The Quantum Fourier Transform of the n-qubit |x⟩ is defined as

$$QFT(|x\rangle) = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} e^{\frac{2\pi xy}{2^n}} |y\rangle.$$

The definition is then extended by linearity to all n-qubits.
Considering $w_l := e^{\frac{2\pi i}{2^l}}$,

$$e^{\frac{2\pi ixy}{2^n}} = w_n^{xy} = w_1^{xy_{n-1}} w_2^{xy_{n-2}} \cdots w_n^{xy_0}.$$

$$\begin{aligned} QFT(|x\rangle) &= \frac{1}{\sqrt{2^n}} \sum_{y_0,\ldots y_{n-1}\in\{0,1\}} w_1^{xy_{n-1}} \cdots w_n^{xy_0} |y_{n-1} \ldots y_0\rangle \\ &= \frac{|0\rangle + w_1^x|1\rangle}{\sqrt{2}} \frac{|0\rangle + w_2^x|1\rangle}{\sqrt{2}} \cdots \frac{|0\rangle + w_n^x|1\rangle}{\sqrt{2}} \end{aligned}$$



Quantum Phase Estimation

Quantum Phase Estimation (QPE) is a fundamental quantum algorithm used to determine with high precision the phase φ associated with an eigenvalue of a unitary operator U. This algorithm is crucial in various quantum algorithms, including Shor's algorithm for factorization and quantum simulation.

Given a unitary operator U and an eigenstate |u⟩ with the corresponding eigenvalue

$$e^{2\pi i\theta}$$

the goal is to estimate the phase φ, which is a fractional number between 0 and 1.

$$U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$$

The QPE algorithm allows us to obtain an estimate of φ with increasing precision as the number of qubits used increases.

Shor's Algorithm

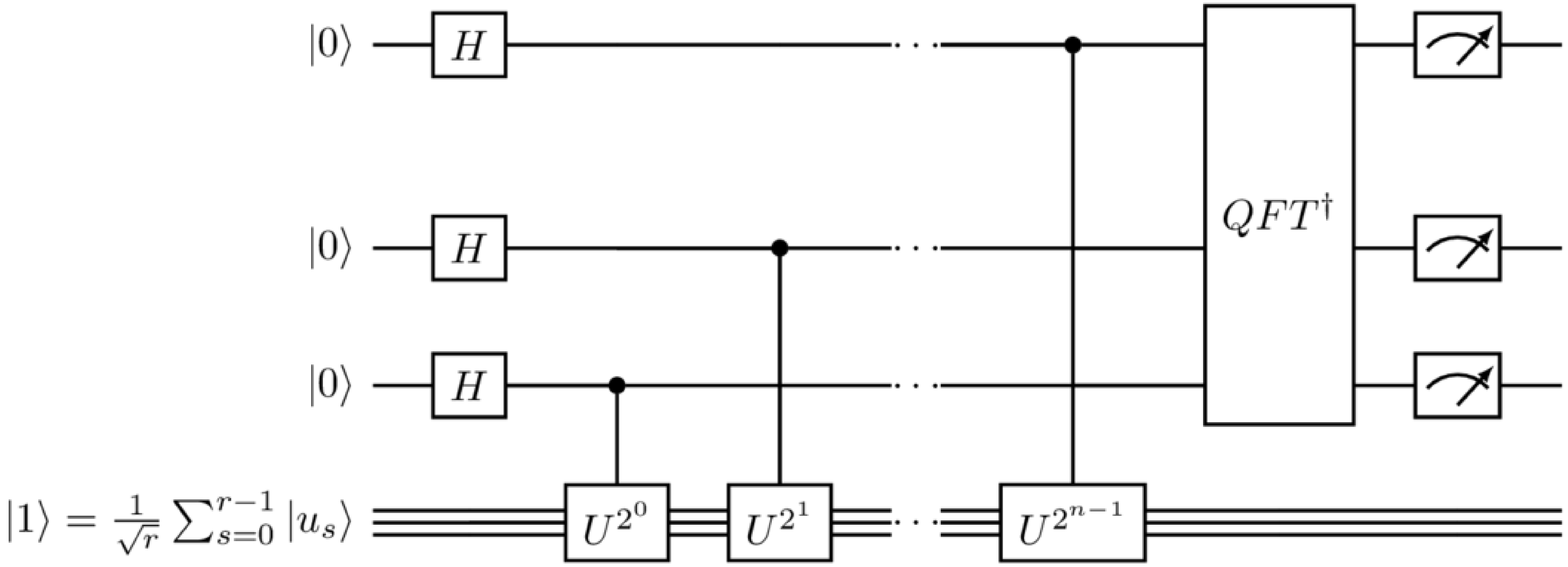
Let N be a positive integer and b a coprime integer with N, the objective of Shor's quantum algorithm is to find the period of the function

$$f : x \mapsto b^x \mod N$$

in a polynomial number of log N steps. We use a unitary operator

$$U|y\rangle = |ay \mod N\rangle$$

Thus, if one proceeds to the QPE of U using |1⟩ which is a superposition of eigenstates. If you do not have a solution, you only get an estimate of the desired value and proceed with continuous fractions to find it.



Shor's Algorithm in practice

N=15
b=2

The sequence $2^x mod\ 15$ produce 2, 4, 8, 1, 2, 4, 8, 1 ... a function with period r=4.

$$2^{\frac{4}{2}} - 1 = 3$$

$$2^{\frac{4}{2}}+1 = 5$$

3 and 5 are the factors of 15

Recap: From Hardness to Vulnerability

Classical World (today)

Security relies on hard mathematical problems

- Factoring large integers (RSA)
- Discrete logarithm (Diffie–Hellman, ECC)

Best classical algorithms: subexponential in logN

Practically secure — impossible to break in reasonable time

Quantum World (tomorrow)

Shor's algorithm solves factoring and discrete log in polynomial time

Grover's algorithm weakens symmetric crypto (square-root speed-up)

Public-key cryptography as we know it becomes obsolete

Implications

Transition to Post-Quantum Cryptography (PQC)

- Based on lattice, hash-based, and code-based assumptions
- NIST standardization in progress

"Harvest now, decrypt later" risk → migration must begin today

Shor’s Algorithm in practice