DSE210 - Yuan Hu - HW03

Worksheet 6 - Modeling dependence between variables.

1. (a) Uncorrelated.

(b) Uncorrelated.

(c) Uncorrelated.

2. (a) X and Y are not independent.

from the table, we get:

$P(X=-1)=1/4$, $P(X=0)=1/2$, $P(X=1)=1/4$

$P(Y=-1)=1/4$, $P(Y=0)=1/2$, $P(Y=-1)=1/4$

Simply, $P(X=-1, Y=-1)=\frac{1}{8} \neq P(X=-1)\cdot P(Y=-1)=\frac{1}{16}$

(b) $cov(X,Y) = E[XY] - E[X]E[Y]$

$E[X] = -1\times 1/4 + 0\times\frac{1}{2} + 1\times 1/4 = 0$ $E[Y] = -1\times\frac{1}{4} + 0\times\frac{1}{2} + 1\times\frac{1}{4} = 0$

$E[XY] = -1\times\frac{1}{4} + 0\times\frac{1}{2} + 1\times\frac{1}{4} = 0$

Hence. $cov(X,Y) = 0$.

$corr(X,Y) = \frac{cov(X,Y)}{std(X)\cdot std(Y)} = 0.$

3. (a) $E(X)=0$, $std(X)=10$, therefore $Var(X)=100$

$Cov(X,Y) = E[XY] - E[X]E[Y] = 2E[X^2] - 2(E[X])^2$

$= 2[E[X^2] - (E[X])^2] = 2Var(X) = 200$

(b) $Var(Y) = Var(2X) = 4\,Var(X) = 400$

$std(Y) = \sqrt{Var(Y)} = \sqrt{400} = 20$

$corr(X,Y) = \frac{cov(X,Y)}{std(X)\cdot std(Y)} = \frac{200}{10\times 20} = 1.$

4. (a) $\mu_X = 2$, $\sigma_X = 1$, $\mu_Y = 4$, $\sigma_Y = 0.5$ and $corr(X,Y) = -0.5$

$cov(X,Y) = \sigma_X \cdot \sigma_Y \cdot corr(X,Y) = 1 \times 0.5 \times -0.5 = -0.25$

Hence, scenario (a) is parametrized with a bivariate Gaussian by:

mean $\mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, covariance matrix $\Sigma = \begin{pmatrix} Var(X) & cov(X,Y) \\ cov(X,Y) & Var(Y) \end{pmatrix}$

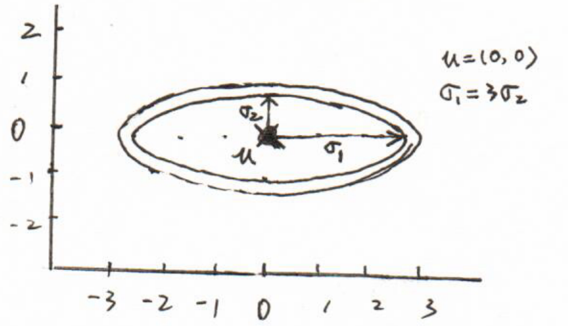
$= \begin{pmatrix} 1 & -0.25 \\ -0.25 & 0.25 \end{pmatrix}$

(b) $\mu_X = 1$, $\sigma_X = 1$, $\mu_Y = 1$, $\sigma_Y = 1$ and $cov(X,Y) = Var(X) = 1$ since $X = Y$.

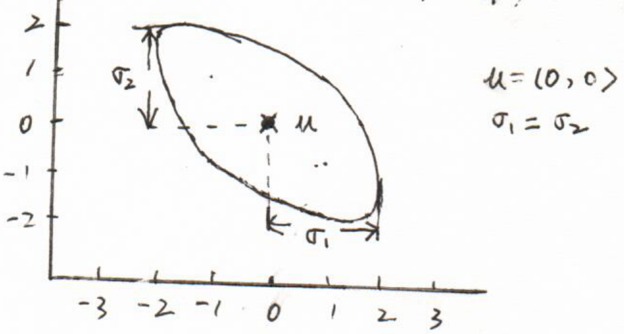
This scenario is parametrized with a bivariate Gaussian by:

mean $\mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, covariance Matrix $\Sigma = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

5. (a) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$



(b) $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

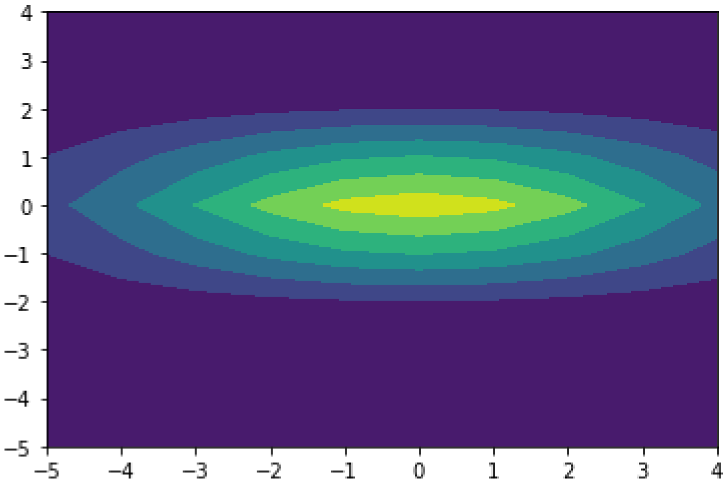


6. refer to next page.

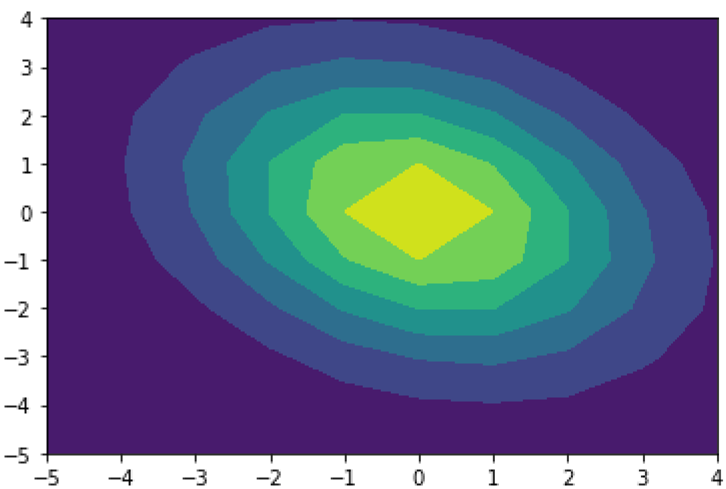
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import multivariate_normal
```

In [2]:
```
def draw_contour(mean, cov):
    x,y = np.mgrid[-5:5:1,-5:5:1]
    pos = np.dstack((x,y))
    rv = multivariate_normal(mean, cov)
    plt.contourf(x, y, rv.pdf(pos))
```

In [3]:
```
# (a)
mean_a = [0, 0]
cov_a = [[9,0],[0,1]]
draw_contour(mean_a, cov_a)
```



In [4]:
```
# (b)
mean_b = [0, 0]
cov_b = [[4,-1],[-1,4]]
draw_contour(mean_b, cov_b)
```



DSE 210 - Yuan Hu - HW03

Worksheet 7 - Linear Algebra Primer

1. $\vec{v} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{\sqrt{1+4+9}}$, $\underline{\vec{v} = (\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})}$

2. let $\vec{v} = (x, y)$, $\vec{u} = (1, 1)$ $\vec{u} \cdot \vec{v} = x + y = 0$, also $x^2 + y^2 = 1$ ($\vec{v}$ has norm 1)
   Hence, $\underline{\vec{v} = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) \text{ or } (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})}$

3. $\underline{\{x \in R^d : \|x\| = 5\}}$

4. $\underline{\vec{w} = (2, -1, 6)}$

5. $A \cdot B$ is 10×20, let A be $a \times 30$, B be $30 \times b$, hence <u>A is 10×30 , B is 30×20</u>

6. (a) Total number of rows is n, total number of columns is d.
   <u>The dimension of X is $(n \times d)$</u>.
   (b) X is $(n \times d)$, X^T is $(d \times n)$, hence <u>$X \cdot X^T$ is $(n \times n)$</u>
   (c) $(X \cdot X^T)_{ij} = X_i \cdot (X^T)^{(j)} = \underline{\sum_{l=1}^{d} X_{il} \cdot (X^T)_{lj}}$

7. the ~~dimension~~ length of x is ~~10~~ 10, hence $x^T x$ is $\|x\|^2 = 100$
   $x^T x x^T x x^T x$ is $\|x\|^6$, ~~which is a scalar~~ <u>which is 10^6</u>

8. $x = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$, $x^T = (1\ 3\ 5)$, $\underline{x^T x = 35}$, $\underline{x \cdot x^T = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 9 & 15 \\ 5 & 15 & 25 \end{pmatrix}}$

9. x, y both has length 2, $\|x\| = \|y\| = 2$
   $x^T \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_d y_d = 2$
   $\cos\theta = \frac{x \cdot y}{\|x\| \cdot \|y\|} = \frac{2}{2 \times 2} = \frac{1}{2}$, hence $\theta = 60°$, <u>the angle between x and Y is 60°</u>

10. $\underline{M = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 0 & 0 \\ -2 & 0 & 6 \end{pmatrix}}$, for $f(x) = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$

DSE 210 - Yuan Hu - HW03

11. (a) <u>Symmetric.</u>

$(AA^T)^T = (A^T)^T A^T = AA^T$

(b) <u>Symmetric</u>

$(A^TA)^T = A^T(A^T)^T = A^TA$

(c) <u>Symmetric</u>

$(A+A^T)^T = A^T + A$

(d) <u>Not Symmetric or Skew-Symmetric ($A^T = -A$)</u>

$(A-A^T)^T = -(A-A^T)$

12. (a) $|A| = |\mathrm{diag}(1,2,3,4,5,6,7,8)| = 1\times 2\times 3\times 4\times 5\times 6\times 7\times 8 = \underline{40320}$

(b) $\underline{A^{-1} = \mathrm{diag}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8})}$

13. (a)

$$U \cdot U^T = \begin{pmatrix} \|u_1\|^2 & & & 0 \\ & \|u_2\|^2 & & \\ & & \ddots & \\ 0 & & & \|u_d\|^2 \end{pmatrix} = \mathrm{diag}(\|u_1\|^2, \|u_2\|^2, \dots, \|u_d\|^2) = \mathrm{diag}(1, 1, 1, \dots, 1)_d = \underline{I_d}$$

(b) From inverse defination: $U \cdot U^{-1} = I_d$.

And from (a) we have $U \cdot U^T = I_d$

Hence $\underline{U^{-1} = U^T}$

14. $|A| = 1\times z - 2\times 3 = z - 6 = 0 \Rightarrow \underline{z = 6}$