WORKSHEET 3 MULTIPLE EVENTS, CONDITIONING, AND INDEPENDENCE

1. A fair (six-sided) die is rolled three times. What is the probability that at least one of the rolls is a 6?

Solution:

For each roll, the chance of not getting six is 5/6, and since three rolls are independent, the chance of getting that at least one of the rolls is a 6 is $\mathbf{1 - 5/6 * 5/6 * 5/6 = 31/156}$

2. A fair coin is tossed 10 times. What is the probability that there are at least two heads?

Solution:

The chance of getting no heads is $1/2^{10}$, the chance of getting only one head is $10/2^{10}$and the probability that there are at least two heads is $\mathbf{1 - 1/2^{10} - 10/2^{10} = 1013/1024}$.

6. A coin is tossed three times. What is the probability that there are exactly two heads, given that:

(a) the first outcome is a head?
(b) the first outcome is a tail?
(c) the first two outcomes are both heads?
(d) the first two outcomes are both tails?
(e) the first outcome is a head and the third outcome is a tail?

Solution:

(a) The sample space is $\Omega = \{(H, T)\}^3$, let event A denotes that there are exactly two heads, let B denotes the event that the first outcome is a head, $Pr(A \bigcap B) = Pr((HHT), (HTH)) = 1/4$, $Pr(B) = Pr((HHH), (HHT), (HTH), (HTT)) = 1/2$, $Pr(A|B) = Pr(A \bigcap B)/Pr(B) = 1/4/1/2 = \mathbf{1/2}$

(b) Let event A denotes that there are exactly two heads, let C denotes the event that the first out come is a tail, $Pr(A \bigcap C) = Pr(THH) = 1/8$, $Pr(C) = Pr((THH), (TTH), (THT), (TTT)) = 1/2$, $Pr(A|C) = Pr(A \bigcap C)/Pr(C) = 1/8/1/2 = \mathbf{1/4}$

(c) Let event A denotes that there are exactly two heads,let D denotes the event that the first two outcome are heads, $Pr(A \bigcap D) = Pr(HHT) = 1/8$, $Pr(D) = Pr((HHT), (HHH)) = 1/4$, $Pr(A|D) = Pr(A \bigcap D)/Pr(D) = 1/8/1/4 = \mathbf{1/2}$

(d) Let event A denotes that there are exactly two heads,let E denotes the event that the first two outcome are tails, $Pr(A \bigcap E) = 0$, $Pr(A|E) = Pr(A \bigcap E)/Pr(E) = \mathbf{0}$

(e) Let event A denotes that there are exactly two heads,let f denotes the event that the first outcome is a head and the third outcome is a tail, $Pr(A \bigcap D) = Pr(HHT) = 1/8$, $Pr(F) = Pr((HHT), (HTT)) = 1/4$, $Pr(A|F) = Pr(A \bigcap F)/Pr(F) = 1/8/1/4 = \mathbf{1/2}$

8. If $Pr(B^c) = 1/4$ and $Pr(A|B) = 1/2$, what is $Pr(A \bigcap B)$?

Solution:

$Pr(B) = 1 - Pr(B^c) = 3/4$, $\mathbf{Pr(A \bigcap B) = Pr(B)Pr(A|B) = 3/4 \times 1/2 = 3/8}$.

9. A die is rolled twice. What is the probability that the sum of the two rolls is > 7, given that:

(a) the first roll is a 4?
(b) the first roll is a 1?

(c) the first roll is > 3?

Solution:

(a) $Pr(sum > 7|first = 4) = Pr(second > 3) = \mathbf{1/2}$

(b) $Pr(sum > 7|first = 1) = \mathbf{0}$

(c)

$$\begin{aligned}
Pr(sum > 7|first > 3) =& \frac{Pr(sum > 7, first > 3)}{Pr(first > 3)} \\
&= \frac{Pr((4,4),(4,5),(4,6),(5,3),(5,4),(5,5),(5,6),(6,6))}{1/2} \\
&= \frac{8/36}{1/2} \approx \mathbf{0.444}
\end{aligned}$$

11. In a particular boarding school, every student is associated with exactly one of the four houses Gryffindor, Hufflepuff, Ravenclaw, and Slytherin. The fraction of students in these houses is 1/3, 1/4, 1/6, and 1/4, respectively. Suppose that 1/2 the Gryffindors are good at the Dark Arts, along with 1/3 of the Hufflepuffs 1/2 of the Ravenclaws and 2/3 of the Slytherins. What is the overall fraction of students at this school who are good at the Dark Arts?

Solution:

Suppose the overall fraction of students who are good at the Dark arts is $P(D)$, the probability of both a student comes from Gryffindor and he/she is good at Dark arts is $P(D \bigcap G)$, similarly we have $P(D \bigcap H)$, $P(D \bigcap R)$, $P(D \bigcap S)$, based on summation rule, we have:

$$\begin{aligned}
P(D) &= P(D \bigcap G) + P(D \bigcap H) + P(D \bigcap R) + P(D \bigcap S) \\
&= P(G)P(D|G) + P(H)P(D|H) + P(R)P(D|R) + P(S)P(D|S) \\
&= 1/3 * 1/2 + 1/4 * 1/3 + 1/6 * 1/2 + 1/4 * 2/3 \\
&= \mathbf{1/2}
\end{aligned}$$

12. A particular car manufacturer has three factories $F1$, $F2$, $F3$ making 25%, 35%, and 40%, respectively, of its cars. Of their output, 5%, 4%, and 2%, respectively, are defective. A car is chosen at random from the manufacturer's supply.
(a) What is the probability that the car is defective?
(b) Given that it is defective, what is the probability that it came from factory $F1$?

Solution:

(a) Let P(D) denotes the probability that the car is defective. $P(F1)$, $P(F2)$, $P(F3)$ denote the probability that the car is from $F1$, $F2$, $F3$ factories respectively, and $P(D|F1)$, $P(D|F2)$, $P(D|F3)$ denote their defective rate for each factory. Hence, the probability that the car is defective is:

$$\begin{aligned}
P(D) &= P(D \bigcap F1) + P(D \bigcap F2) + P(D \bigcap F3) \\
&= P(F1)P(D|F1) + P(F2)P(D|F2) + P(F3)P(D|F3) \\
&= 0.25 * 0.05 + 0.35 * 0.04 + 0.4 * 0.02 \\
&= \mathbf{0.0345}
\end{aligned}$$

(b) Let $P(F1|D)$ denotes the probability that the car came from factory F1 given that it is defective.

$$
\begin{aligned}
P(F1|D) &= \frac{P(D|F1)P(F1)}{P(D)} \\
&= \frac{0.05 * 0.25}{0.0345} \\
&\approx \mathbf{0.362}
\end{aligned}
$$

14. A doctor assumes that his patients has one of the three diseases $d1$, $d2$, or $d3$, each with probability 1/3. He carries out a test that will be positive with probability 0.8 if the patient has $d1$, with probability 0.6 if the patient has $d2$, and with probability 0.4 if the patient has $d3$.

(a) What is the probability that the test will be positive?

(b) Suppose that the outcome of the test is positive. What probabilities should the doctor now assign to the three possible diseases?

Solution:

(a) Let P(+) denotes the probability that the test will be positive, and $P(d1)$, $P(d2)$. $P(d3)$ denote the probability that the patients has diseases $d1$, $d2$, or $d3$ respectively, and $P(+|d1)$, $P(+|d2)$, $P(+|d3)$ denote the probability that the patient will be tested positive given that the patients have diseases $d1$, $d2$, or $d3$ respectively. Hence:

$$
\begin{aligned}
P(+) &= P(+\bigcap d1) + P(+\bigcap d2) + P(+\bigcap d3) \\
&= P(d1)P(+|d1) + P(d2)P(+|d2) + P(d3)P(+|d3) \\
&= 1/3 * 0.8 + 1/3 * 0.6 + 1/3 * 0.4 \\
&= \mathbf{0.6}
\end{aligned}
$$

(b) Let $P(d1|+)$, $P(d2|+)$, $P(d3|+)$ denote the probability that the doctor will assign to one of three diseases respectively given that the outcome of test is positive.

$$
\begin{aligned}
P(d1|+) &= \frac{P(+|d1)P(d1)}{P(+)} \\
&= \frac{0.8 * 1/3}{0.6} \\
&\approx \mathbf{0.444}
\end{aligned}
$$

$$
\begin{aligned}
P(d2|+) &= \frac{P(+|d1)P(d1)}{P(+)} \\
&= \frac{0.6 * 1/3}{0.6} \\
&\approx \mathbf{0.333}
\end{aligned}
$$

$$
\begin{aligned}
P(d3|+) &= \frac{P(+|d1)P(d1)}{P(+)} \\
&= \frac{0.4 * 1/3}{0.6} \\
&\approx \mathbf{0.222}
\end{aligned}
$$

16. You randomly shuffle a standard deck and deal two cards. Which of the following pairs of events are independent?

(i) A = {first card is a heart}, B = {second card is a heart}

(ii) A = {first card is a heart}, B = {first card is a 10}

(iii) A = {first card is a 10}, B = {second card is a 9}

(iv) A = {first card is a heart}, B = {second card is a 10}

Solution:

(i) **NOT independent**

$P(A) = 1/4$, $P(B) = 1/4$, $P(A \bigcap B) = 1/4 * 11/51 \neq P(A)P(B)$

(ii) **Independent**
$P(A) = 1/4$, $P(B) = 1/13$, $P(A \bigcap B) = P(A|B)P(B) = 1/4 * 1/13 = P(A)P(B)$

(iii)**NOT Independent**
$P(A) = 1/13$, $P(B) = 1/13$, $P(A \bigcap B) = 1/13 * 4/51 \neq P(A)P(B)$

(iv) **Independent**
$P(A) = 1/4$, $P(B) = 1/13$, $P(A \bigcap B) = 12/52 * 4/51 + 1/52 * 3/51 = P(A)P(B)$