WORKSHEET 4 RANDOM VARIABLES, EXPECTATION AND VARIANCE

1. A die is thrown twice. Let X_1 and X_2 denote the outcomes, and define random variable X to be the minimum of X_1 and X_2. Determine the distribution of X.

Solution:

$X = min(X_1, X_2) \in \{1, 2, 3, 4, 5, 6\}$, for Event (X_1, X_2), $|\Omega| = 36$
$Pr(X = 1) = Pr(\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (6, 1), (5, 1), (4, 1), (3, 1), (2, 1)\}) = \mathbf{11/36}$
$Pr(X = 2) = Pr(\{(2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (4, 2), (5, 2), (6, 2)\}) = 9/36 = \mathbf{1/4}$
$Pr(X = 3) = Pr(\{(3, 3), (3, 4), (3, 5), (3, 6), (4, 3), (5, 3), (6, 3)\}) = \mathbf{7/36}$
$Pr(X = 4) = Pr(\{(4, 5), (4, 6), (4, 4), (5, 4), (6, 4)\}) = \mathbf{5/36}$
$Pr(X = 5) = Pr(\{(5, 5), (5, 6), (6, 5)\}) = \mathbf{1/12}$
$Pr(X = 6) = Pr(\{(6, 6)\}) = \mathbf{1/36}$

2. A fair die is rolled repeatedly until a six is seen. What is the expected number of rolls?

Solution:

Let E be the expected number of rolls, we may split all the rollings into two cases, first case we rolled 1 time getting a 6, the probability of this case is 1/6. For the second case, we don't get a 6 at the first rolling, the probability of not getting 6 for one roll is 5/6, noticed in this case, the total number of rollings to get a 6 would be (E+1).
Hence, we will have: $E = 1/6 * 1 + 5/6 * (E + 1)$, solve this equation we get E = 6. **Therefore the expected number of rolls is 6**.

3. A die has six sides that come up with different probabilities:
$Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8, Pr(5) = Pr(6) = 1/4.$
(a) You roll the die; let Z be the outcome. What is $E(Z)$?
(b) What is $var(Z)$?
(c) You roll the die 10 times. What is the probability that exactly five of the rolls are sixes?
(d) You keep rolling the die until you get a six. What is expected number of rolls?
(e) You keep rolling until you get a second six (that is, roll until the first six, and then keep going until you get another six; they don't have to be consecutive). What is the expected number of rolls?

Solution:

(a) $E(Z) = 1 * Pr(1) + 2 * Pr(2) + 3 * Pr(3) + 4 * Pr(4) + 5 * Pr(5) + 6 * Pr(6) = 1 * 1/8 + 2 * 1/8 + 3 * 1/8 + 4 * 1/8 + 5 * 1/4 + 6 * 1/4 = \mathbf{4}$
(b) $Var(Z) = E(Z^2) - E(Z)^2$, $Z^2 = \{1,4,9,16,25,36\}$
$E(Z^2) = 1 * 1/8 + 4 * 1/8 + 9 * 1/8 + 16 * 1/8 + 25 * 1/4 + 36 * 1/4 = 30/8 + 61/4 = 19$
$Var(Z) = 19 - 4^2 = \mathbf{3}$
(c)$Pr(X = 6) = 1/4$, $\mathbf{Pr(Five6) = \binom{10}{5} * (1/4)^5 * (3/4)^5 = 0.058}$
(d)$E = 1/4 * 1 + 3/4 * (E + 1)$, $\mathbf{E = 4}$
(e)

8. After studying your favorite insect carefully, you determine that length X (in centimeters) of a member of that species has expected value 5, with a standard deviation of 2.
(a) What is the variance of X?
(b) Let Z denote the length of an insect in millimeters (so $Z = 10X$). What is $E(Z)$?
(c) What is $std(Z)$?
(d) What is $var(Z)$?

Solution:

(a) $E(X) = 5$, $Std(X) = 2$, $\mathbf{Var(X) = Std(X)^2 = 4}$
(b) $\mathbf{E(Z)} = E(10X) = 10E(x) = \mathbf{50}$

(c) $\mathbf{Std(Z)} = Std(10X) = 10Std(X) = \mathbf{20}$
(d) $\mathbf{Var(Z)} = Var(10X) = 100Var(X) = \mathbf{400}$

9. An elevator operates in a building with 10 floors. One day, n people get into the elevator, and each of them chooses to go to a floor selected uniformly at random from 1 to 10.
(a) What is the probability that exactly one person gets out at the ith floor? Give your answer in terms of n.
(b) What is the expected number of floors in which exactly one person gets out? Hint: let X_i be 1 if exactly one person gets out on floor i, and 0 otherwise. Then use linearity of expectation.

Solution:

(a) Let X denotes that exactly one person gets out at the ith floor.
$\mathbf{Pr(X)} = \binom{\mathbf{n}}{\mathbf{1}} * \mathbf{(1/10)(9/10)^{n-1}}$
(b) let X_i be 1 if exactly one person gets out on floor i, and 0 otherwise.
$E(X_i) = n/10 * (9/10)^{n-1}$, $\mathbf{E(X)} = E(10X_i) = 10E(X_i) = \mathbf{n(9/10)^{n-1}}$

11. In each of the following cases, say whether X and Y are independent.
(a) You randomly permute (1, 2,...,n). X is the number in the first position and Y is the number in the second position.
(b) You randomly pick a sentence out of Hamlet. X is the first word in the sentence and Y is the second word.
(c) You randomly pick a card from a pack of 52 cards. X is 1 if the card is a nine, and is 0 otherwise. Y is 1 if the card is a heart, and is 0 otherwise.
(d) You randomly deal a ten-card hand from a pack of 52 cards. X is 1 if the hand contains a nine, and is 0 otherwise. Y is 1 if all cards in the hand are hearts, and is 0 otherwise.

Solution:

(a) **Not Independent**.
(b) **Independent**.
(c) **Independent**.
(d) **Not Independent**.

12. Suppose the probability that a car accident occurs on Monday is 5%, Tuesday 10%, Wednesday 10%, Thursday 20%, Friday 20%, Saturday 30%, and Sunday 5%. Suppose a series of 200 accidents occur at random, independently. Let X denote the number of them that occur on Sunday.
(a) What is the expected value and variance of X?
(b) What is the probability that X is exactly its expected value?

Solution:

(a) Define X_i is 1 if the ith car accident occurs on Sunday, X_i is 0 otherwise. $E(X_i) = 0.05$, $E(X_i^2) = 0.05$, $\mathbf{E(X)} = E(200X_i) = 200E(X_i) = 200*0.05 = \mathbf{10}$, $Var(X_i) = E(X_i^2) - E(X_i)^2 = 0.05 - 0.05^2 = 0.0475$, since X_i are independent, $\mathbf{Var(X)} = 200Var(X_i) = 200 * 0.0475 = \mathbf{9.5}$
(b) $\mathbf{Pr(X = E(X))} = Pr(X = 10) = \binom{\mathbf{200}}{\mathbf{10}}\mathbf{0.05^{10}0.95^{190}} = \mathbf{0.1284}$

18. Random variables X and Y take values in $\{1, 2, 3\}$, and are known to be independent. The joint distribution of X and Y is given in the table below, but some of the entries are missing. Provide the missing entries.

Solution:

Let $Pr(X = 2, Y = 1) = A$, $Pr(X = 2, Y = 2) = B$, $Pr(X = 2, Y = 3) = C$, $Pr(X = 2) = A + B + C = 1 - 2(1/12 + 1/24 + 1/8) = 1/2$
Since X and Y are independent, $Pr(X = 2, Y = 1) = Pr(X = 2) * Pr(Y = 1) = 1/2 * (A + 2 * 1/12) = A$, we get $\mathbf{A = 1/6}$, similarly we can get $\mathbf{B = 1/12}$, $\mathbf{C = 1/4}$.

20. Biased random walk. A drunken man leaves a bar. At each time $t = 1, 2, ...$, he either takes one step to the left, with probability 1/3, or one step to the right, with probability 2/3. Let X denote his position after n steps. For instance, if he takes a total of 3n/4 steps to the left and n/4 steps to the right, then his final position is $X = -1(3n/4) + 1(n/4) = -n/2$.
(a) What is E(X)?
(b) What is var(X)?
(c) Use the standard deviation of X to summarize roughly where you would expect the man to be after n steps.

Solution:

Let $X_i \in \{-1, 1\}$ be his ith step. His position after n steps is $X = X_1 + X_2 + X_3 + .. + X_n$.
$E(X_i) = 1 * 2/3 + (-1) * 1/3 = 1/3$
(a) $\mathbf{E(X)} = n * E(X_i) = \mathbf{n/3}$
(b) $X_i^2 = 1$, $E(X_i^2) = 1$, $Var(X_i) = E(X_i^2) - E(X_i)^2 = 1 - 1/9 = 8/9$
Since X_i are independent, $\mathbf{Var(X)} = Var(nX_i) = nVar(X_i) = \mathbf{8n/9}$
(c) $\mathbf{Std(X)} = \sqrt{Var(X)} = \sqrt{8n/9} = \mathbf{2\sqrt{2n}/3}$, so after n steps, he would be $\mathbf{2\sqrt{2n}/3}$ steps far.