WORKSHEET 2 PROBABILITY SPACES

1. In each of the following experiments, define the sample space Ω.
(a) A two-sided coin is tossed, with heads on one side and tails on the other.
(b) You can choose a color for your new car's exterior (choices: red, black, silver, blue) and interior (choices: beige, black).
(c) A customer is asked for the month and day-of-week on which her birthday lies.
(d) A coin is tossed 100 times in a row.

Solution:

(a) $\mathbf{\Omega} = \{\mathbf{H}, \mathbf{T}\}$, H and T stands for heads and tails respectively. $|\Omega| = 2$.

(b) $\mathbf{\Omega} = \{(\{\mathbf{red}, \mathbf{black}, \mathbf{silver}, \mathbf{blue}\}, \{\mathbf{beige}, \mathbf{black}\})\}$, $|\Omega| = 4 \times 2 = 8$

(c) $\mathbf{\Omega} = \{(\{\mathbf{12months}\}, \{\mathbf{7daysofweek}\})\}$, $|\Omega| = 12 \times 7 = 84$

(d) $\mathbf{\Omega} = \{\mathbf{H}, \mathbf{T}\}^{\mathbf{100}}$, $|\Omega| = 2^{100}$, H and T stands for heads and tails respectively.

2. Let A, B, C be events defined on a sample space Ω. Use notation for union, intersection, and set difference to write expressions for each of the following combinations.
(a) All three events occur.
(b) At least one of the events occurs.
(c) A and B occur, but not C.

Solution:

(a) All three events occur: $\mathbf{A} \bigcap \mathbf{B} \bigcap \mathbf{C}$

(b) At least one of the events occurs: $\mathbf{A} \bigcup \mathbf{B} \bigcup \mathbf{C}$

(c) A and B occur, but not C: $\mathbf{A} \bigcap \mathbf{B} \bigcap \mathbf{C^c}$

5. Let A and B be events defined on a sample space Ω such that $Pr(A \bigcap B) = 1/4$, $Pr(A^c) = 1/3$, and $Pr(B) = 1/2$. Here $A^c = \Omega \setminus A$ is the event that A doesn't happen. What is $Pr(A \bigcup B)$?

Solution:

$$\mathbf{Pr(A \bigcup B) = Pr(A) + Pr(B) - Pr(A \bigcap B) = (1 - 1/3) + 1/2 - 1/4 = 11/12}$$

6. A pair of dice are rolled. What is the probability that they show the same value?

Solution:

The sample space $\Omega = \{(\{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\})\}$, $|\Omega| = 36$, there are 6 same values, hence the probability they show the same value is $\mathbf{6/|\Omega| = 1/6}$

7. Recall that a chessboard has 64 squares arranged in an 8 x 8 grid. A rook is a particular chess piece that is said to attack anything that shares either the same row or the same column. Suppose two rooks are placed at random on a chessboard(in distinct locations). What is the chance that they are attacking each other?

Solution:

The size of sample space is $|\Omega| = \binom{64}{2} = 2016$, there are 64 ways to place one rook on a random grid, and 14 ways to place another rook so that they attacking each other, and hence the chance that they are attacking each other is $\mathbf{64 \times 14/2016 = 896/2016 = 28/63 \approx 0.444}$.

9. A die is loaded in such a way that the probability of each face turning up is proportional to the number of dots on that face (for instance, a six is three times as probable as a two). What is the probability of getting an even number in one throw?

Solution:

Assume the probability of getting a one is $Pr(1)$, then $Pr(1)+2Pr(1)+3Pr(1)+4Pr(1)+5Pr(1)+6Pr(1)=1$, hence $Pr(1)=1/21$, $Pr(2)=2Pr(1)$, $Pr(4)=4Pr(1)$, $Pr(6)=6Pr(1)$, $\mathbf{Pr(even)=Pr(2)+Pr(4)+Pr(6)=2Pr(1)+4Pr(1)+6Pr(1)=12*1/21=4/7}$

10. Five people of different heights are lined up against a wall in random order. What is the probability that they just happen to be in increasing order of height (left-to-right)?

Solution:

The size of sample space is $|\Omega| = 5!$, and there is only one way to line up 5 people in increasing order of height, hence the probability is $\mathbf{1/5! = 1/120}$

11. A deck of ordinary cards is shuffled and a hand of 13 cards are dealt. What is the probab -ility that the first and second cards are of the same suit?

Solution:

The size of sample space is $|\Omega| = \binom{52}{13}$, there are 4 ways to select a suit for the first and second cards, and within that suit, the first draw has 13 ways, and the second draw has 12 ways, then $\binom{50}{11}$ ways for the rest of 11 draws. Hence the probability is $\mathbf{4*13*12*\binom{50}{11}/\binom{52}{13}}$

13. Assume that whenever a child is born, it is equally likely to be a girl or boy. What is the probability that a randomly-chosen family with six children has exactly three girls and three boys?

Solution:

The sample space is $\Omega = \{Boy, Girl\}^6$, $|\Omega| = 2^6$, and there are $\binom{6}{3}$ ways with six children has exactly three girls and three boys, hence the probability is $\mathbf{\binom{6}{3}/2^6}$.

15. How long does a sequence of random decimal digits (i.e. each digit is equally likely to be 0, 1, 2,..., 9) have to be in order for the probability of the digit 7 appearing to be at least 0.9?

Solution:

Suppose the length of the sequence is N, the probability of the digit 7 appearing to be at least 0.9 can be translated to the probability of the digit 7 not appearing to be at most 0.1, for the sequence of length N, the probability of single digit that 7 not showing up is 9/10, hence we need to find out N in $(9/10)^N < 0.1$, the answer is 22. **Hence, the sequence has to be length of 22 digits in order for the probability of the digit 7 appearing to be at least 0.9**.