DSE 210: Probability and Statistics using Python

Worksheet 4 — Random variables, expectation, and variance

1. A die is thrown twice. Let X_1 and X_2 denote the outcomes, and define random variable X to be the minimum of X_1 and X_2. Determine the distribution of X.

2. A fair die is rolled repeatedly until a six is seen. What is the expected number of rolls?

3. A die has six sides that come up with different probabilities:

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = 1/8, \ \Pr(5) = \Pr(6) = 1/4.$$

   (a) You roll the die; let Z be the outcome. What is $\mathbb{E}(Z)$?

   (b) What is $\text{var}(Z)$?

   (c) You roll the die 10 times. What is the probability that *exactly five* of the rolls are sixes?

   (d) You keep rolling the die until you get a six. What is expected number of rolls?

   (e) You keep rolling until you get a second six (that is, roll until the first six, and then keep going until you get another six; they don't have to be consecutive). What is the expected number of rolls?

4. You have two different dice, each with four sides. For the first die, the probabilities of the various outcomes are:

$$\Pr(1) = \Pr(2) = \frac{1}{4}, \ \ \Pr(3) = \frac{1}{3}, \ \ \Pr(4) = \frac{1}{6}.$$

   For the second die, they are:

$$\Pr(1) = \frac{1}{2}, \ \ \Pr(2) = \Pr(3) = \Pr(4) = \frac{1}{6}.$$

   Suppose you roll both dice. Let X_1 be the outcome of the first die, and X_2 the outcome of the second die. Moreover, define random variable Y to be the sum of the X_1 and X_2, and define random variable Z to be the larger of X_1 and X_2.

   (a) Specify the distribution of Y.

   (b) Specify the distribution of Z.

5. Random variable Z is uniformly distributed in $[-1, 1]$.

   (a) What is the density of Z?

   (b) What is the expected value of Z?

   (c) What is the median value of Z?

   (d) What is the variance of Z?

   (e) What is the standard deviation of Z?

6. You have a balloon that is perfectly spherical with radius 1. There is a particular gas molecule inside the balloon whose position is uniformly distributed (that is, equally likely to be anywhere in the balloon). Let random variable R denote the distance from this molecule to the center of the balloon.
   (a) What is the range of values that R can take?
   (b) The cumulative distribution function (cdf) of R is $F(r) = \Pr(R \leq r)$. What is $F(r)$?
   (c) What is the density of R?
   (d) What is the probability that $1/3 \leq R \leq 2/3$?
   (e) What is the expected value of R?
   (f) What is the median value of R?

7. A random variable X is uniformly distributed over the set $[1, 2] \cup [3, 4]$.
   (a) Specify the density of X.
   (b) What is $\mathbb{E}(X)$?
   (c) What is the median of X?
   (d) What is $\mathbb{E}(X^2)$?
   (e) What is $\text{var}(X)$?

8. After studying your favorite insect carefully, you determine that length X (in centimeters) of a member of that species has expected value 5, with a standard deviation of 2.
   (a) What is the variance of X?
   (b) Let Z denote the length of an insect in millimeters (so $Z = 10X$). What is $\mathbb{E}(Z)$?
   (c) What is $\text{std}(Z)$?
   (d) What is $\text{var}(Z)$?

9. An elevator operates in a building with 10 floors. One day, n people get into the elevator, and each of them chooses to go to a floor selected uniformly at random from 1 to 10.
   (a) What is the probability that exactly one person gets out at the ith floor? Give your answer in terms of n.
   (b) What is the expected number of floors in which exactly one person gets out? *Hint:* let X_i be 1 if exactly one person gets out on floor i, and 0 otherwise. Then use linearity of expectation.

10. You throw m balls into n bins, each independently at random. Let X be the number of balls that end up in bin 1.
    (a) Let X_i be the event that the ith ball falls in bin 1. Write X as a function of the X_i.
    (b) What is the expected value of X?

11. In each of the following cases, say whether X and Y are independent.
    (a) You randomly permute $(1, 2, \ldots, n)$. X is the number in the first position and Y is the number in the second position.
    (b) You randomly pick a sentence out of *Hamlet*. X is the first word in the sentence and Y is the second word.

(c) You randomly pick a card from a pack of 52 cards. X is 1 if the card is a nine, and is 0 otherwise. Y is 1 if the card is a heart, and is 0 otherwise.

(d) You randomly deal a ten-card hand from a pack of 52 cards. X is 1 if the hand contains a nine, and is 0 otherwise. Y is 1 if *all* cards in the hand are hearts, and is 0 otherwise.

12. Suppose the probability that a car accident occurs on Monday is 5%, Tuesday 10%, Wednesday 10%, Thursday 20%, Friday 20%, Saturday 30%, and Sunday 5%. Suppose a series of 200 accidents occur at random, independently. Let X denote the number of them that occur on Sunday.

    (a) What is the expected value and variance of X?

    (b) What is the probability that X is exactly its expected value?

13. In a sequence of coin tosses, a *run* is a series of consecutive heads or consecutive tails. For instance, the longest run in $HTHHHTTTHHTHH$ consists of three heads. We are interested in the following question: when a fair coin is tossed n times, how long a run is the resulting sequence likely to contain?

    To study this, pick any k between 1 and n, and let R_k denote the number of runs of length exactly k (for instance, a run of length $k+1$ doesn't count). In order to figure out $\mathbb{E}(R_k)$, we define the following random variables: $X_i = 1$ if a run of length exactly k begins at position i, where $i \leq n-k+1$.

    (a) What are $\mathbb{E}(X_1)$ and $\mathbb{E}(X_{n-k+1})$?

    (b) What is $\mathbb{E}(X_i)$ for $1 < i < n-k+1$?

    (c) What is $\mathbb{E}(R_k)$?

    (d) What is, roughly, the largest k for which $\mathbb{E}(R_k) \geq 1$?

14. Let $X_1, X_2, \ldots, X_{100}$ be the outcomes of 100 independent rolls of a fair die.

    (a) What are $\mathbb{E}(X_1)$ and $\text{var}(X_1)$?

    (b) Define the random variable X to be $X_1 - X_2$. What are $\mathbb{E}(X)$ and $\text{var}(X)$?

    (c) Define the random variable Y to be $X_1 - 2X_2 + X_3$. What is $\mathbb{E}(Y)$ and $\text{var}(Y)$?

    (d) Define the random variable $Z = X_1 - X_2 + X_3 - X_4 + \cdots + X_{99} - X_{100}$. What are $\mathbb{E}(Z)$ and $\text{var}(Z)$?

15. Pick a random permutation of $(1, 2, \ldots, n)$. Let X_i be the number that ends up in the ith position. For instance, if the permutation is $(3, 2, 4, 1)$ then $X_1 = 3$, $X_2 = 2$, $X_3 = 4$, and $X_4 = 1$.

    (a) What is the expected number of positions at which $X_i \neq i$?

    (b) What is the expected number of positions at which $X_i = i + 1$?

    (c) What is the expected number of positions at which $X_i \geq i$?

    (d) What is the expected number of positions at which $X_i > \max(X_1, \ldots, X_{i-1})$? You can give a rough approximation.

16. A set of n people are lined up against a wall in a random order. Among them are Alice, Bob, and Chet.

    (a) What is the probability that Alice appears somewhere to the left of Bob, and Bob appears somewhere to the left of Chet?

    (b) What is the expected number of people between Alice and Bob?

17. Suppose a fair coin is tossed repeatedly until the same outcome occurs twice in a row (that is, two heads in a row or two tails in a row). What is the expected number of tosses?

18. Random variables X and Y take values in $\{1, 2, 3\}$, and are known to be independent. The joint distribution of X and Y is given in the table below, but some of the entries are missing. Provide the missing entries.

		Y		
		1	2	3
	1	1/12	1/24	1/8
X	2	???	???	???
	3	1/12	1/24	1/8

19. Let $X, Y \in \{0, 1\}$ be independent fair coin flips. Let U be the exclusive-OR of X and Y, and let V be the AND of X and Y.

    (a) Are X and U independent? Why or why not?

    (b) Are X and V independent? Why or why not?

20. *Biased random walk.* A drunken man leaves a bar. At each time $t = 1, 2, \ldots$, he either takes one step to the left, with probability $1/3$, or one step to the right, with probability $2/3$. Let X denote his position after n steps. For instance, if he takes a total of $3n/4$ steps to the left and $n/4$ steps to the right, then his final position is $X = -1 \cdot (3n/4) + 1 \cdot (n/4) = -n/2$.

    (a) What is $\mathbb{E}(X)$?

    (b) What is $\text{var}(X)$?

    (c) Use the standard deviation of X to summarize roughly where you would expect the man to be after n steps.

21. *Stumbling race.* Mario and Luigi decide to race against each other. Each of them runs in an unusual manner: at each time $t = 1, 2, \ldots$,

    - Mario moves ahead two steps with probability $1/2$ and moves back one step with probability $1/2$
    - Luigi moves one step ahead if t is even; but if t is odd, then he moves forward one step with probability $1/3$ and backwards one step with probability $2/3$

    What are their expected positions after T steps, where T is an even number?