DSE 210: Probability and Statistics using Python

Worksheet 7 — Linear algebra primer

1. Find the unit vector in the same direction as $x = (1, 2, 3)$.
2. Find all unit vectors in $\mathbb{R}^2$ that are orthogonal to $(1, 1)$.
3. How would you describe the set of all points $x \in \mathbb{R}^d$ with $x \cdot x = 25$?
4. The function $f(x) = 2x_1 - x_2 + 6x_3$ can be written as $w \cdot x$ for $x \in \mathbb{R}^3$. What is w?
5. For a certain pair of matrices A, B, the product AB has dimension 10×20. If A has 30 columns, what are the dimensions of A and B?
6. We have n data points $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^d$ and we store them in a matrix X, one point per row.
   (a) What is the dimension of X?
   (b) What is the dimension of XX^T?
   (c) What is the (i, j) entry of XX^T, simply?
7. Vector x has length 10. What is $x^T x x^T x x^T x$?
8. For $x = (1, 3, 5)$ compute $x^T x$ and xx^T.
9. Vectors $x, y \in \mathbb{R}^d$ both have length 2. If $x^T y = 2$, what is the angle between x and y?
10. The quadratic function $f : \mathbb{R}^3 \to \mathbb{R}$ given by
$$f(x) = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$$
can be written in the form $x^T M x$ for some **symmetric** matrix M. What is M?
11. Which of the following matrices is necessarily symmetric?
   (a) AA^T for arbitrary matrix A.
   (b) $A^T A$ for arbitrary matrix A.
   (c) $A + A^T$ for arbitrary square matrix A.
   (d) $A - A^T$ for arbitrary square matrix A.
12. Let $A = \text{diag}(1, 2, 3, 4, 5, 6, 7, 8)$.
   (a) What is $|A|$?
   (b) What is A^{-1}?
13. Vectors $u_1, \ldots, u_d \in \mathbb{R}^d$ all have unit length and are orthogonal to each other. Let U be the $d \times d$ matrix whose rows are the u_i.
   (a) What is UU^T?
   (b) What is U^{-1}?
14. Matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & z \end{pmatrix}$ is singular. What is z?