Matrix Multiplication, 67826 - Exercise 1

Due: April 24, 2025

Question 1

(a) Write a U V W representation of the following $\langle 2, 1, 2; 4 \rangle$-algorithm:

$$\begin{aligned} M_1 &= a_{11}\left(b_{11}+b_{12}\right) & \qquad c_{11} &= M_1 - M_2 - M_4 \\ M_2 &= \left(a_{11}-a_{21}\right) b_{12} & c_{12} &= M_2 + M_4 \\ M_3 &= a_{21}\left(b_{11}-b_{12}\right) & c_{21} &= M_3 + M_4 \\ M_4 &= a_{21} b_{12} & c_{22} &= M_4 \end{aligned}$$

(b) What are the dimensions, exponent, and leading coefficient of this matrix multiplication algorithm?

$$\langle U, V, W \rangle = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \right\rangle$$

Question 2

In this question, you will prove that the complexity of matrix multiplication is preserved under permutations of dimensions. Let ALG be an $\langle a, b, c; t \rangle$-algorithm.

(a) Prove that there is a $\langle c, b, a; t \rangle$-algorithm. Hint: $(A \cdot B)^\top = B^\top \cdot A^\top$

(b) Prove that there is a $\langle c, a, b; t \rangle$-algorithm. Hint: Triple product.

(c) Conclude that for every permutation (d, e, f) of (a, b, c), there is a $\langle d, e, f; t \rangle$-algorithm.

Question 3

(a) Prove that de-Groote equivalence (slides 8, 9 in the second presentation) is an equivalence relation. That is, show that it is reflexive, symmetric, and transitive.

(b) Let $\langle U, V, W \rangle, \langle U', V', W' \rangle$ be two algorithms in the same de-Groote equivalence class. Prove that if U contains duplicate rows (that is, there are $i \neq j$ such that $U_{i,*} = U_{j,*}$) then U' contains duplicate rows up to a multiplication by a constant (that is, $U'_{i',*} = aU'_{j',*}$, for some constant a).

(c) Show that the following matrix multiplication algorithms are not de-Groote equivalent:

$$\mathrm{ALG}_1 = \left\langle \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\mathrm{ALG}_2 = \left\langle \begin{pmatrix} -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & 1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \right\rangle$$

(d) Describe the following de-Groote transformation as an alternative basis. That is, describe the 4×4 matrices ϕ, ψ, ν such that $\phi\vec{A} = \overrightarrow{(XAY)}, \psi\vec{B} = \overrightarrow{(Y^{-1}BZ)}, \nu\vec{C} = \overrightarrow{(X^{-1}CZ^{-1})}$.

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Question 4

Implement a code that verifies the triple product condition (use the template from Moodle).